Multistage Optimization

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Abstract

We provide a new identity for the multistage Average Value-at-Risk. The identity is based on the conditional Average Value-at-Risk at random level, which is introduced. It is of interest in situations, where the information available increases over time, so it is – among other applications – customized to multistage optimization. The identity relates to dynamic programming and is adapted to problems which involve the Average Value-at-Risk in its objective. We elaborate further dynamic programming equations for specific multistage optimization problems and derive a characterizing martingale property for the value function.

The concept solves a particular aspect of time consistency and is adapted for situations, where decisions are planned and executed consecutively in subsequent instants of time. We discuss the approach for other risk measures, which are in frequent use for decision making under uncertainty, particularly for financial decisions.

Keywords: Stochastic optimization, risk measure, Average Value-at-Risk, dynamic programming, time consistency
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1. Introduction

Risk averse stochastic programming has become increasingly important in recent years. The concept builds on risk measures, which have been introduced and initiated in the pioneering paper [ADEH99]. Since then risk measures have been investigated from theoretical perspectives, but their behavior in practical applications is of interest as well. Later the concept was generalized to account for problems formulated in several stages of time.

As regards practical perspectives planning, particularly long-time financial planning has to be mentioned as an important application area. Insurance and re-insurance companies as well incorporate risk measures in their decisions, or in the pricing of individual contracts.

A special question arises when comparing multistage stochastic programming with dynamic programming, which builds on the dynamic programming principle as a fundamental principle. This is known as the Bellman principle, or Pontryagin's minimum principle, cf. [FS06]. No such relation is known for multistage stochastic programming, which makes multistage stochastic programming considerably more difficult. However, significant efforts and investigations have been started in order to provide a similar principle, among them the papers [Sha09, RS06a, Rus10]. The results available are in such way that given some risk measures which account for consecutive times-periods, a risk measure for the entire period is built. Other efforts attempt to decompose a given risk measure such that some new risk measures, compounded, finally give the initial risk measure (cf. [RS06b]). Such a decomposition is of particular interest for multistage optimization, as a decomposition would allow to better characterize the strategy just for the next period, without having the entire time horizon in view –

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and this is expected to considerably reduce the time needed to finally compute the optimal solution for the entire problem.

We prove that the Average Valuer-at-Risk has this property, it can be decomposed. For this purpose one has to give up the constant risk level and accept a random risk level, which is adapted for each partial observation.

The Average Value-at-Risk, for various reasons, is the most prominent (maybe most important) risk measure and there exist various descriptions and representations for the Average Value-at-Risk. Moreover any version independent risk measure – by Kusuoka's representation provided in [Kus01] – is built just of convex combinations of Average Value-at-Risk functionals, they represent the extreme points in a reasonable class of risk measures. The recent paper [Sha11] further exposes the central role of the Average Value-at-Risk.

We shall introduce the Average Value-at-Risk in Section 2, and *conditionally at random level* in Section 3. Section 4 is devoted to the new decomposition for the Average Value-at-Risk. In the subsequent section we intend to generalize the concept and provide a method to decompose more general risk measures. However, it seems that just the Average Value-at-Risk can easily be handled. Section 7 provides the link to *dynamic programming*. The implications on multistage stochastic optimization, as well as the central *martingale property* of the value function, are discussed in the following Section 8. Selected applications and examples serve throughout the paper, and an intuitive reformulation of the results is provided in the Appendix.

Finally, in Section 9 we propose an algorithm which is based on the results developed.

Artzner et al. gave an initial example to study the multiperiodic character of the Average Value-at-Risk in [ADE+07]. We shall discuss and solve this example. However, as their example does not reveal all peculiarities of the multistage Average Value-at-Risk we have added another simple example in order to describe and foster the results in sufficient detail.

2. The Average Value-at-Risk and Representations

An outstanding example of a version independent acceptability functional, providing an idea of *how bad is bad* (cf. [ADE+07]), is the Average Value-at-Risk at level α, denoted AV@R_α[1].

Definition 1 (Average Value-at-Risk)**.** Let Y denote an integrable ($\mathbb{L}^1$) random variable and $0 \le \alpha \le 1$ a number. The *Average Value-at-Risk at level* α is

$$\text{AV@R}_\alpha(Y) = \frac{1}{\alpha}\int_0^\alpha \text{V@R}_q(Y)\,\mathrm{d}q, \tag{1}$$

where $\text{V@R}_\alpha(Y) := \inf\{y\colon \mathbb{P}(Y \le y) \ge \alpha\}$ is the *Value-at-Risk*, the random variable Y's (left-continuous) quantile function.

[1] The importance of the

- *Average Value-at-Risk*

is reflected by the fact that there are various names simultaneously in the literature for the same quantity. Other names include

- *conditional value-at-risk* (for the additional representation $\text{AV@R}_\alpha(Y) = \mathbb{E}[Y : Y \le \text{V@R}_\alpha(Y)]$, valid for probabilities without atoms),
- *expected shortfall,*
- *tail value-at-risk* or newly
- *super-quantile* (of course *sub-quantile* could be justified as well by simply changing the sign).
- Actuaries tend to use the term *Conditional Tail Expectation* (CTE).

We have chosen the definition of the Average Value-at-Risk in its *concave* setting to account for *profits* which we intend to maximize. For this we shall call the AV@R an acceptability functional rather than a risk functional. The converse setting – in frequent use as well, accounting for losses instead of profits representing the risk functional – is $\rho(Y) = -\,\text{AV@R}(-Y)$.

We list some alternative representation of the Average Value-at-Risk as they will be of later use.

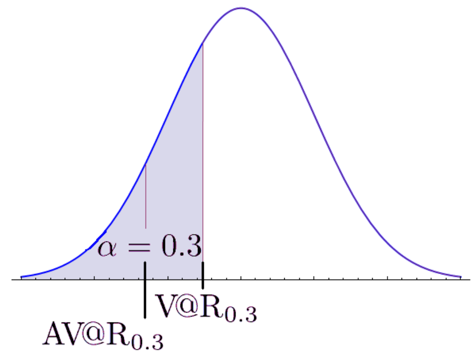

Figure 1: Density of Y and the Value-at-Risk, as well as the Average Value-at-Risk at level $\alpha = 0.3$

(i) The Average Value-at-Risk at level α is

$$\begin{aligned} &\text{AV@R}_\alpha(Y) \\ &= \inf\left\{\mathbb{E}[YZ]:\ 0 \le Z,\, \alpha Z \le 1 \text{ and } \mathbb{E}Z = 1\right\}. \end{aligned} \tag{2}$$

The infimum in (2) is among all positive random variables $Z \ge 0$ with expectation $\mathbb{E}Z = 1$ (densities), satisfying the additional truncation constraint $\alpha Z \le 1$, as indicated. This infimum is attained if $\alpha > 0$, and in this case the optimal random variable Z in (2) is coupled in a anti-monotone way with Y (cf. [Nel98]). (2) is often being referred to as the AV@R's *dual representation* or *dual formula*.

(ii) The change of numéraire

$$\text{AV@R}_\alpha(Y) = \min\left\{\mathbb{E}_{\mathbb{Q}}[Y]:\ \frac{d\mathbb{Q}}{d\mathbb{P}} \le \frac{1}{\alpha}\right\}$$

is an immediate reformulation of (2).

(iii) The Average Value-at-Risk at level $\alpha = 0$ is

$$\text{AV@R}_0(Y) = \operatorname{ess\,inf}(Y); \tag{3}$$

it holds that $\text{AV@R}_0(Y) = \lim_{\alpha\to 0} \text{AV@R}_\alpha(Y)$.

(iv) The relation[2]

$$\text{AV@R}_\alpha(Y) = \max_{q\in\mathbb{R}}\, q - \frac{1}{\alpha}\mathbb{E}(q-Y)^+ \tag{4}$$

was elaborated in [RU02]. Notably the maximum in (4) is attained at some $q^* \in \mathbb{R}$ satisfying the quantile like condition $\mathbb{P}[Y < q^*] \le \alpha \le \mathbb{P}[Y \le q^*]$ – cf. Figure 1.

For future reference we state the following – rather straight forward – property on continuity and convexity in the (adjusted) parameter α:

Lemma 2. *The mapping*

$$\begin{aligned} [0,1] &\to \mathbb{R} \\ \alpha &\mapsto \begin{cases} 0 & \textit{for } \alpha = 0, \\ \alpha \cdot \text{AV@R}_\alpha(Y) & \textit{whenever } \alpha > 0 \end{cases} \end{aligned} \tag{5}$$

is convex and continuous for $Y \in \mathbb{L}^1$.

[2] x^+ is the positive part of x, that is $x^+ = x$ if $x \ge 0$, and $x^+ = 0$ else.

Proof. From (1) it follows that $\alpha \cdot \text{AV@R}_\alpha(Y) = \int_0^\alpha \text{V@R}_q(Y)\,\mathrm{d}q$. As the function $q \mapsto \text{V@R}_q(Y)$ is increasing the assertion on convexity is immediate. Continuity in the interior of (0, 1) follows from convexity. As for continuity in the point 0 just observe that $\int_0^\alpha \text{V@R}_q(Y)\,\mathrm{d}q \le \int_0^1 \text{V@R}_q(Y)\,\mathrm{d}q = \mathbb{E}Y < \infty$, as $Y \in \mathbb{L}^1$, and whence $\alpha \cdot \text{AV@R}_\alpha(Y) = \int_0^\alpha \text{V@R}_q(Y)\,\mathrm{d}q \to 0$ for $\alpha \to 0$. □

Remark 3. Another way of accepting convexity in Lemma 2 is (cf. [RO02]) to define

$$F_Y^{(2)}(\eta) := \mathbb{E}(\eta - Y)^+ = \int_{-\infty}^{\eta} (\eta - \xi)\,\mathbb{P}(\mathrm{d}\xi)$$

and to observe that

$$\alpha \cdot \text{AV@R}_\alpha(Y) = \sup_\eta \eta\alpha - F_Y^{(2)}(\eta),$$

which immediately reveals convexity of (5) (cf. [Roc70] for more properties of the convex conjugate function).

3. The Conditional Average Value-at-Risk at Random Level

We intend to account for multiple stages in time by employing a sigma algebra for each stage of time, and we start with the situation of just two consecutive time steps. We denote the sigma algebra at the final stage T by $\mathcal{F}_T$, and at some previous stage $t < T$ by $\mathcal{F}_t$. As the information increases successively over time it is natural to assume that $\mathcal{F}_t \subset \mathcal{F}_T$. Moreover the initial time shall be 0 $(0 < t < T)$, and the respective sigma algebra $\mathcal{F}_0$ is typically trivial, that is $\mathcal{F}_0 = \{\emptyset, \Omega\}$ which indicates that at time $t = 0$ nothing is known about the future evolution. To abbreviate the notation we shall moreover write $Y \lhd \mathcal{F}$ to express that Y is measurable with respect to the sigma algebra $\mathcal{F}$.

The Average Value-at-Risk, as defined above, is an $\mathbb{R}-$valued function on $\mathbb{L}^1(\mathcal{F}_T)$, so AV@R quantifies the entire future risk at stage 0. Having multistage stochastic optimization in mind it is desirable to have an idea of the risk at the later stage $t > 0$ as well, dependent on observations $\mathcal{F}_t$ already available at time t. This is accomplished by the *conditional* Average Value-at-Risk at random level. A conditional Average Value-at-Risk for some previous sigma algebra $\mathcal{F}_t$ $(\mathcal{F}_t \subset \mathcal{F}_T)$ is introduced, among other attempts, in Pflug and Römisch, [PR07]. Based on the definition used there it will be necessary to *extend* its formulation in order to find a conditional Average Value-at-Risk, given a previous sigma algebra $\mathcal{F}_t \subset \mathcal{F}_T$, where we allow in addition the level parameter α to vary itself – the random level. That is to say we consider α not fixed any longer $(\alpha \lhd \mathcal{F}_0)$, but α a $\mathcal{F}_t-$measurable random variable $(\alpha \lhd \mathcal{F}_t)$ in the sequel.

Definition 4. The *conditional* Average Value-at-Risk *at random level* $\alpha \lhd \mathcal{F}_t$ $(0 \le \alpha \le \mathbb{1})$ is the $\mathcal{F}_t-$random variable

$$\text{AV@R}_\alpha(Y|\mathcal{F}_t) := \operatorname{ess\,inf}\{\mathbb{E}(YZ|\mathcal{F}_t) : \mathbb{E}(Z|\mathcal{F}_t) = \mathbb{1},\, 0 \le Z,\, \alpha Z \le 1\}. \tag{6}$$

Remark 5. $\mathbb{E}(YZ|\mathcal{F}_t)$ is a random variable, the essential infimum in (6) thus is an infimum over a family of random variables, and the resulting $\text{AV@R}_\alpha(Y|\mathcal{F}_t)$ is an $\mathcal{F}_t-$random variable itself; for the definition of the essential infimum of random variables we refer – for example – to Appendix A in [KS98] or [DS57] for details.

Note that the accepted notation "ess inf" in (6) is in contrast to its use in (3), as in (3) the ess inf is just Y's biggest lower bound, which is a number in $\mathbb{R}$.

The next theorem elaborates that the *conditional* Average Value-at-Risk at random level basically preserves all properties of the usual Average Value-at-Risk introduced in (1).

Theorem 6. *For the conditional Average Value-at-Risk at random level $\alpha \lhd \mathcal{F}_t$ $(0 \le \alpha \le \mathbb{1})$ the following hold true:*

(i) Predictability: $\text{AV@R}_\alpha(Y|\mathcal{F}_t) = Y$ *if* $Y \lhd \mathcal{F}_t$*;*

(ii) Translation equivariance[3]: $\text{AV@R}_\alpha(Y + c|\mathcal{F}_t) = \text{AV@R}_\alpha(Y|\mathcal{F}_t) + c$ *if* $c \lhd \mathcal{F}_t$*;*

(iii) Positive homogeneity: $\text{AV@R}_\alpha(\lambda Y|\mathcal{F}_t) = \lambda\, \text{AV@R}_\alpha(Y|\mathcal{F}_t)$ *whenever* $\lambda \lhd \mathcal{F}_t$*,* $\lambda \ge 0$ *and bounded;*

(iv) Monotonicity: $\text{AV@R}_{\alpha_1}(Y_1|\mathcal{F}_t) \le \text{AV@R}_{\alpha_2}(Y_2|\mathcal{F}_t)$ *whenever* $Y_1 \le Y_2$ *and* $\alpha_1 \le \alpha_2$ *almost surely;*

(v) Concavity: $\text{AV@R}_\alpha((1-\lambda)Y_0 + \lambda Y_1|\mathcal{F}_t) \ge (1-\lambda)\,\text{AV@R}_\alpha(Y_0|\mathcal{F}_t) + \lambda\,\text{AV@R}_\alpha(Y_1|\mathcal{F}_t)$ *for* $\lambda \lhd \mathcal{F}_t$ *and* $0 \le \lambda \le 1$*, eventually;*

(vi) Lower and Upper bounds: $\text{AV@R}_0(Y) \le \text{AV@R}_0(Y|\mathcal{F}_t) \le \text{AV@R}_\alpha(Y|\mathcal{F}_t) \le \mathbb{E}(Y|\mathcal{F}_t)$.

Proof. As for the Predictability just observe that

$$\begin{aligned}\text{AV@R}_\alpha(Y|\mathcal{F}_t) &= \operatorname{ess\,inf}\{Y \cdot \mathbb{E}(Z|\mathcal{F}_t) : \mathbb{E}(Z|\mathcal{F}_t) = \mathbb{1},\, 0 \le Z,\, \alpha Z \le 1\},\\ &= \operatorname{ess\,inf}\{Y \cdot \mathbb{1} : \mathbb{E}(Z|\mathcal{F}_t) = \mathbb{1},\, 0 \le Z,\, \alpha Z \le 1\} = Y\end{aligned}$$

for $Y \lhd \mathcal{F}_t$, and Translation equivariance follows from

$$\begin{aligned}\text{AV@R}_\alpha(Y + c|\mathcal{F}_t) &= \operatorname{ess\,inf}\{\mathbb{E}(YZ|\mathcal{F}_t) + c\mathbb{E}(Z|\mathcal{F}_t) : \mathbb{E}(Z|\mathcal{F}_t) = \mathbb{1},\, 0 \le Z,\, \alpha Z \le 1\}\\ &= \operatorname{ess\,inf}\{\mathbb{E}(YZ|\mathcal{F}_t) + c : \mathbb{E}(Z|\mathcal{F}_t) = \mathbb{1},\, 0 \le Z,\, \alpha Z \le 1\}\\ &= \text{AV@R}_\alpha(Y|\mathcal{F}_t) + c.\end{aligned}$$

To accept that the conditional Average Value-at-Risk is positively homogeneous observe that the assertion is correct for $\lambda = \mathbb{1}_A$ $(A \in \mathcal{F}_t)$; by passing to the limit one gets the assertion for step-functions first, then for any positive function $\lambda \in \mathbb{L}^\infty(\mathcal{F}_t)$.

To prove Concavity as stated observe that

$$(1-\lambda)\,\mathbb{E}(Y_0 Z|\mathcal{F}_t) + \lambda \mathbb{E}(Y_1 Z|\mathcal{F}_t) = \mathbb{E}(((1-\lambda)\,Y_0 + \lambda Y_1)\,Z|\mathcal{F}_t)$$

by the measurability assumption $\lambda \lhd \mathcal{F}_t$, whence

$$\begin{aligned}\text{AV@R}_\alpha((1-\lambda)\,Y_0 + \lambda Y_1|\mathcal{F}_t) &= \operatorname*{ess\,inf}_Z (1-\lambda)\,\mathbb{E}(Y_0 Z|\mathcal{F}_t) + \lambda\mathbb{E}(Y_1 Z|\mathcal{F}_t)\\ &\ge \operatorname*{ess\,inf}_{Z_0, Z_1} (1-\lambda)\,\mathbb{E}(Y_0 Z_0|\mathcal{F}_t) + \lambda\mathbb{E}(Y_1 Z_1|\mathcal{F}_t)\\ &\ge (1-\lambda) \operatorname*{ess\,inf}_{Z_0} \mathbb{E}(Y_0 Z_0|\mathcal{F}_t) + \lambda \operatorname*{ess\,inf}_{Z_1} \mathbb{E}(Y_1 Z_1|\mathcal{F}_t)\\ &= (1-\lambda)\,\text{AV@R}_\alpha(Y_0|\mathcal{F}_t) + \lambda\,\text{AV@R}_\alpha(Y_1|\mathcal{F}_t),\end{aligned}$$

where $Z_0 \ge 0$ and $Z_1 \ge 0$ are chosen to satisfy $\mathbb{E}(Z_i|\mathcal{F}_t) = \mathbb{1}$ and $\alpha Z_i \le 1$ each.

To observe the monotonicity property recall that $\alpha_1 \le \alpha_2$, whence

$$\begin{aligned}\text{AV@R}_{\alpha_1}(Y_1|\mathcal{F}_t) &= \operatorname*{ess\,inf}_Z \{\mathbb{E}(ZY_1|\mathcal{F}_t) : Z \ge 0,\, \alpha_1 Z \le 1,\, \mathbb{E}(Z|\mathcal{F}_t) = \mathbb{1}\}\\ &\le \operatorname*{ess\,inf}_Z \{\mathbb{E}(ZY_2|\mathcal{F}_t) : Z \ge 0,\, \alpha_1 Z \le 1,\, \mathbb{E}(Z|\mathcal{F}_t) = \mathbb{1}\}\\ &\le \operatorname*{ess\,inf}_Z \{\mathbb{E}(ZY_2|\mathcal{F}_t) : Z \ge 0,\, \alpha_2 Z \le 1,\, \mathbb{E}(Z|\mathcal{F}_t) = \mathbb{1}\}\\ &= \text{AV@R}_{\alpha_2}(Y_2|\mathcal{F}_t).\end{aligned}$$

The Upper bound finally becomes evident because $Z = \mathbb{1}$ is feasible for (6), the lower bounds already have been used. □

[3] In an economic or monetary environment this is often called Cash invariance instead.

The next characterization, used in [PR07] to define the conditional Average Value-at-Risk in a simpler context, extends to the situation $\alpha \lhd \mathcal{F}_t$ but replaces the ess inf by a usual inf:

Theorem 7 (Characterization)**.** *Suppose that $\alpha \lhd \mathcal{F}_t$.*

(i) *The conditional Average Value-at-Risk at random level α is characterized by*

$$\mathbb{E}\left[\mathbb{1}_B \cdot \mathrm{AV@R}_\alpha\left(Y|\mathcal{F}_t\right)\right] = \inf\left\{\mathbb{E}\left[YZ\right]:\ 0 \le Z,\, \alpha Z \le \mathbb{1}_B,\, \mathbb{E}\left[Z|\mathcal{F}_t\right] = \mathbb{1}_B\right\} \tag{7}$$

for all sets $B \in \mathcal{F}_t$.

(ii) *Moreover the conjugate duality relation*

$$\mathrm{AV@R}_\alpha\left(Y|\mathcal{F}_t\right) = \operatorname*{ess\,inf}_Z \mathbb{E}\left[YZ|\mathcal{F}_t\right] - \mathrm{AV@R}^*_\alpha\left(Z|\mathcal{F}_t\right)$$

with

$$\mathrm{AV@R}^*_\alpha\left(Z|\mathcal{F}_t\right) = \begin{cases} 0 & \text{if } \mathbb{E}\left[Z|\mathcal{F}_t\right] = \mathbb{1},\ 0 \le Z \text{ and } \alpha Z \le 1 \\ -\infty & \text{else} \end{cases} \tag{8}$$

holds true.

Remark 8. Notably α, Z and $\mathbb{E}\left[Z|\mathcal{F}_t\right]$ may have various versions. The defining equation (8) is understood to provide a version of $\mathrm{AV@R}^*_\alpha$ for any such version.

Proof. The essential infimum ess inf, by the characterizing theorem (Appendix A in [KS98] or [DS57]), is a density provided by the Radon-Nikodym theorem satisfying

$$\int_B \mathrm{AV@R}_\alpha\left(Y|\mathcal{F}_t\right) \mathrm{d}\mathbb{P} = \inf\left\{\mathbb{E}\left[\sum_{k=1}^K \mathbb{1}_{B_k}\mathbb{E}\left(YZ_k|F_t\right)\right]:\ 0 \le Z_k,\, \alpha Z_k \le 1,\, \mathbb{E}\left[Z_k|\mathcal{F}_t\right] = \mathbb{1}\right\}$$

where the infimum is among all finite and pairwise disjoint partitions $B = \bigcup_{k=1}^K B_k$ ($B_j \cap B_k = \emptyset$), $B_k \in \mathcal{F}_t$ and Z_k feasible as above. Observe that $Z = Z \cdot \mathbb{1}_B$ for the random variable $Z := \sum_{k=1}^K \mathbb{1}_{B_k} Z_k$, and the latter equation thus rewrites as

$$\mathbb{E}\left[\mathbb{1}_B\, \mathrm{AV@R}_\alpha\left(Y|\mathcal{F}_t\right)\right] = \inf\left\{\mathbb{E}YZ\colon\ 0 \le Z,\, \alpha Z \le \mathbb{1}_B,\, \mathbb{E}\left[Z|\mathcal{F}_t\right] = \mathbb{1}_B\right\},$$

which is the desired assertion.

As for the second assertion recall the Fenchel-Moreau-Rockafellar duality theorem which states that

$$\mathrm{AV@R}_\alpha\left(Y|\mathcal{F}_t\right) = \operatorname*{ess\,inf}_Z \mathbb{E}\left[YZ|\mathcal{F}_t\right] - \mathrm{AV@R}^*_\alpha\left(Z|\mathcal{F}_t\right)$$

where

$$\mathrm{AV@R}^*_\alpha\left(Z|\mathcal{F}_t\right) = \operatorname*{ess\,inf}_Y \mathbb{E}\left[YZ|\mathcal{F}_t\right] - \mathrm{AV@R}_\alpha\left(Y|\mathcal{F}_t\right).$$

Thus

$$\begin{aligned} \mathrm{AV@R}^*_\alpha\left(Z|\mathcal{F}_t\right) &\le \operatorname*{ess\,inf}_{\gamma\in\mathbb{R}} \mathbb{E}\left[\left(\gamma\mathbb{1}\right)Z|\mathcal{F}_t\right] - \mathrm{AV@R}_\alpha\left(\gamma\mathbb{1}|\mathcal{F}_t\right) \\ &= \operatorname*{ess\,inf}_{\gamma\in\mathbb{R}} \gamma\left(\mathbb{E}\left[Z|\mathcal{F}_t\right] - \mathbb{1}\right) \end{aligned}$$

and whence $\mathrm{AV@R}^*_\alpha\left(Z|\mathcal{F}_t\right) = -\infty$ on the $\mathcal{F}_t-$set $\left\{\mathbb{E}\left[Z|\mathcal{F}_t\right] \ne \mathbb{1}\right\}$.

Next suppose that $B := \{Z < 0\}$ has positive measure, then $\mathbb{E}\left[Z\mathbb{1}_B|\mathcal{F}_t\right] < 0$ on B. Thus

$$\begin{aligned} \mathrm{AV@R}^*_\alpha\left(Z|\mathcal{F}_t\right) &\le \operatorname*{ess\,inf}_{\gamma>0} \mathbb{E}\left[\gamma\mathbb{1}_B Z|\mathcal{F}_t\right] - \mathrm{AV@R}_\alpha\left(\gamma\mathbb{1}_B|\mathcal{F}_t\right) \\ &\le \operatorname*{ess\,inf}_{\gamma>0} \gamma\mathbb{E}\left[Z\mathbb{1}_B|\mathcal{F}_t\right] = -\infty \end{aligned}$$

on B. Finally suppose that $C := \{\alpha Z > 1\}$ has positive measure, so

$$\begin{aligned}
\mathrm{AV@R}_\alpha^*(Z|\mathcal{F}_t) &\leq \operatorname*{ess\,inf}_{\gamma>0} \mathbb{E}\left[-\gamma\alpha\mathbb{1}_C Z|\mathcal{F}_t\right] - \mathrm{AV@R}_\alpha\left(-\gamma\alpha\mathbb{1}_C|\mathcal{F}_t\right) \\
&\leq \operatorname*{ess\,inf}_{\gamma>0} -\gamma\mathbb{E}\left[\alpha Z\mathbb{1}_C|\mathcal{F}_t\right] + \gamma\mathbb{E}\left[\mathbb{1}_C|\mathcal{F}_t\right] \\
&= \operatorname*{ess\,inf}_{\gamma>0} -\gamma\left(\mathbb{E}\left[(\alpha Z - \mathbb{1})\,\mathbb{1}_C|\mathcal{F}_t\right]\right) = -\infty
\end{aligned}$$

on C by the same reasoning. Combining all three ingredients gives the statement, as they constitute all conditions for the Average Value-at-Risk in (6). □

Remark. It should be emphasized that the latter characterization (7) sequentially defines the $\mathrm{AV@R}_\alpha(Y|\mathcal{F}_t)$ just on the sets B, it is then compounded by appropriately arranging the sets in order to have the $\mathrm{AV@R}_\alpha(Y|\mathcal{F}_t)$ defined on the entire sample space.

Theorem 9. *The Average Value-at-Risk at random level has the additional representation*

$$\mathrm{AV@R}_\alpha(Y|\mathcal{F}_t) = \operatorname*{ess\,sup}\left\{Q - \frac{1}{\alpha}\mathbb{E}\left[(Q-Y)^+|\mathcal{F}_t\right] : Q \lhd \mathcal{F}_t\right\}$$

where the essential supremum is among all random variables $Q \in \mathbb{L}^\infty(\mathcal{F}_t)$ ($Q \lhd \mathcal{F}_t$).

Proof. without proof – the proof is rather technical, but along the lines as the classical equivalence (2) and (4). □

Other, and partially different attempts to define the same random variable $\mathrm{AV@R}_\alpha(Y|\mathcal{F}_t)$ for random α are elaborated and discussed in [Rus10] and [RS06a].

4. The Average Value-at-Risk in Multistage Evaluations

Given the Average Value-at-Risk conditional on $\mathcal{F}_t$, how can one compute the Average Value-at-Risk at time 0? This is the content of the next theorem, which contains a main result on the Average Value-at-Risk in multistage situations. The following observation is in the focus of this paper, and it will become central in our considerations on multistage stochastic optimization involving the Average Value-at-Risk.

Theorem 10 (Nested decomposition of the AV@R)**.** *Let $Y \in \mathbb{L}^1(\mathcal{F}_T)$, $\mathcal{F}_t \subset \mathcal{F}_\tau \subset \mathcal{F}_T$.*

(i) *For $\alpha \in [0,1]$ the Average Value-at-Risk obeys the decomposition*

$$\mathrm{AV@R}_\alpha(Y) = \inf \mathbb{E}\left[Z_t \cdot \mathrm{AV@R}_{\alpha\cdot Z_t}(Y|\mathcal{F}_t)\right], \tag{9}$$

where the infimum is among all densities $Z_t \lhd \mathcal{F}_t$ with $0 \leq Z_t$, $\alpha Z_t \leq \mathbb{1}$ and $\mathbb{E}Z_t = 1$. For $\alpha > 0$ the infimum in (9) is attained.

(ii) *Moreover if Z is the optimal dual density for (2), that is $\mathrm{AV@R}_\alpha(Y) = \mathbb{E}YZ$ with $Z \geq 0$, $\alpha Z \leq \mathbb{1}$ and $\mathbb{E}Z = 1$, then $Z_t = \mathbb{E}[Z|\mathcal{F}_t]$ is the best choice in (9).*

(iii) *The* conditional *Average Value-at-Risk at random level $\alpha \lhd \mathcal{F}_t$ ($0 \leq \alpha \leq \mathbb{1}$) has the recursive (nested) representation*

$$\mathrm{AV@R}_\alpha(Y|\mathcal{F}_t) = \operatorname*{ess\,inf} \mathbb{E}\left[\,Z_\tau \cdot \mathrm{AV@R}_{\alpha\cdot Z_\tau}(Y|\mathcal{F}_\tau)|\,\mathcal{F}_t\right], \tag{10}$$

where the infimum is among all densities $Z_\tau \lhd \mathcal{F}_\tau$ with $0 \leq Z$, $\alpha Z_\tau \leq \mathbb{1}$ and $\mathbb{E}[Z_\tau|\mathcal{F}_t] = \mathbb{1}$.

Remark. Note that $\alpha \cdot Z_t$ in the index of the inner $\mathrm{AV@R}_{\alpha\cdot Z_t}$ is a $\mathcal{F}_t$ random variable satisfying $0 \leq \alpha Z_t \leq \mathbb{1}$, which means that $\mathrm{AV@R}_{\alpha\cdot Z_t}(Y|\mathcal{F}_t)$ is indeed available and well-defined, $\mathbb{P}-$a.e..

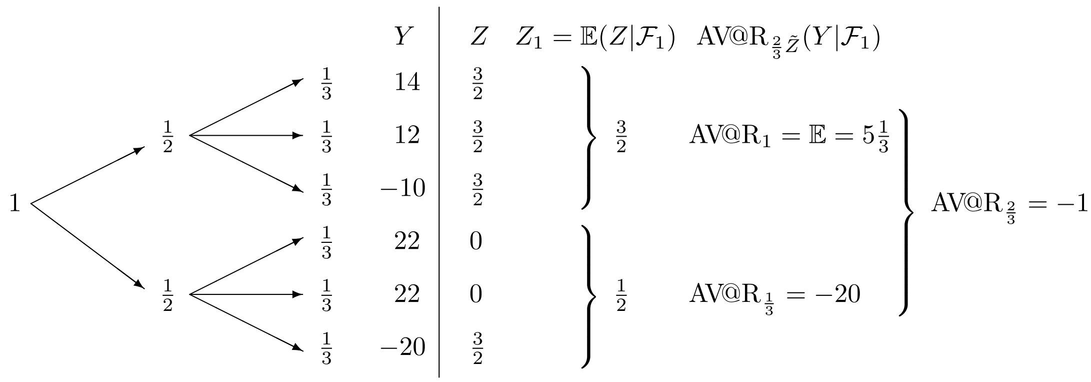

Figure 2: (cf. Artzner et al, [ADE+07]) Nested computation of $\text{AV@R}_\alpha(Y)$ with $\alpha = \frac{2}{3}$ and outcomes with equal probabilities. The intriguing and misleading fact is that the *conditional* Average Value-at-Risk, computed with the *initial,* constant $\alpha = \frac{2}{3}$, is $\text{AV@R}_{\frac{2}{3}\mathbb{1}}(Y|\mathcal{F}_t) = +\mathbb{1}$, but $\text{AV@R}_{\frac{2}{3}}(Y) = -1$.

Example. Both, Figure 2 and 3, depict a typical, simple situation with two stages in time, the increasing sigma algebras are visualized via the tree structure.

The statement of the latter Theorem 10 is contained in the figures, they include Z (the optimal dual for (2)) and Z_1 (the optimal dual for (9) at $t=1$).

The example in Figure 2 is due to Artzner et al., [ADE+07]. The Average Value-at-Risk of the random variable Y is $\text{AV@R}_{\frac{2}{3}}(Y) = -1$: The intriguing fact here is that the *conditional* Average Valuer-at-Risk, computed with the *initial* $\alpha = \frac{2}{3}$, is $\text{AV@R}_{\frac{2}{3}\mathbb{1}}(Y|\mathcal{F}_t) = +\mathbb{1}$, which seems in conflicting contrast to $\text{AV@R}_{\frac{2}{3}}(Y) = -1$. The following Corollary 11 elaborates on this gap.

However, by defining the conditional Average Value-at-Risk for *random* $\alpha \lhd \mathcal{F}_t$ – as we did in (6) – the discrepancy is eliminated and corrected.

Corollary 11. *For any level* $0 \le \alpha \le 1$*,*

$$\text{AV@R}_\alpha(Y) \le \mathbb{E}\,\text{AV@R}_\alpha(Y|\mathcal{F}_t) \le \mathbb{E}Y;$$

for any $\alpha \lhd \mathcal{F}_t$ *(*$0 \le \alpha \le \mathbb{1}$*),*

$$\text{AV@R}_\alpha(Y|\mathcal{F}_t) \le \mathbb{E}\left[\text{AV@R}_\alpha(Y|\mathcal{F}_\tau)\right|\mathcal{F}_t] \le \mathbb{E}(Y|\mathcal{F}_t).$$

Proof. The first inequality is immediate by choosing the feasible random variable $Z = \mathbb{1}$ in Theorem 10. The second inequality follows from the monotonicity property in Theorem 6, as

$$\mathbb{E}\,\text{AV@R}_\alpha(Y|\mathcal{F}_t) \le \mathbb{E}\,\text{AV@R}_1(Y|\mathcal{F}_t) = \mathbb{E}\mathbb{E}(Y|\mathcal{F}_t) = \mathbb{E}Y$$

and AV@R_1 has just one feasible dual variable, $Z = \mathbb{1}$. □

Proof of Theorem 10. We shall assume first that $\alpha > 0$.

Let $Z \lhd \mathcal{F}_t$ be a simple step function with $Z \ge 0$ and $\mathbb{E}Z = 1$, ie. $Z = \sum_i b_i \mathbb{1}_{B_i}$ where $b_i \ge 0$ and $B_i \cap B_j = \emptyset$. Then

$$\begin{aligned}\mathbb{E}\left[Z\,\text{AV@R}_{\alpha\cdot Z}(Y|\mathcal{F}_t)\right] &= \sum_i b_i \mathbb{E}\left[\mathbb{1}_{B_i}\,\text{AV@R}_{\alpha\cdot Z}(Y|\mathcal{F}_t)\right] = \\ &= \sum_i b_i \inf\left\{\mathbb{E}[YX_i] :\ 0 \le X_i,\ \alpha b_i \mathbb{1}_{B_i} X_i \le \mathbb{1}_{B_i},\ \mathbb{E}[X_i|\mathcal{F}_t] = \mathbb{1}_{B_i}\right\} \\ &= \inf\left\{\sum_i b_i \mathbb{E}[YX_i] :\ 0 \le X_i,\ \alpha b_i \mathbb{1}_{B_i} X_i \le \mathbb{1}_{B_i},\ \mathbb{E}[X_i|\mathcal{F}_t] = \mathbb{1}_{B_i}\right\}.\end{aligned}$$

As $\mathbb{E}[X_i|\mathcal{F}_t] = \mathbb{1}_{B_i}$, together with the additional constraint $X_i \geq 0$, one infers that $X_i = 0$ on the complement of B_i, that is to say $X_i \mathbb{1}_{B_i} = X_i$.

Define $X := \sum_i \mathbb{1}_{B_i} X_i$, thus

$$ZX = \sum_{i,j} b_i \mathbb{1}_{B_i} \mathbb{1}_{B_j} X_j = \sum_i b_i \mathbb{1}_{B_i} X_i = \sum_i b_i X_i$$

and

$$\mathbb{E}[XYZ] = \sum_i b_i \mathbb{E}[YX_i],$$

such that we further obtain by assembling on the mutually disjoint sets B_i

$$\mathbb{E}[Z \cdot \text{AV@R}_{\alpha \cdot Z}(Y|\mathcal{F}_t)] = \inf\{\mathbb{E}[YZX] :\ 0 \leq X,\, \alpha ZX \leq \mathbb{1},\, \mathbb{E}[X|\mathcal{F}_t] = \mathbb{1}\}. \tag{11}$$

Note next that $\mathbb{E}[XZ] = \mathbb{E}[Z \cdot \mathbb{E}[X|\mathcal{F}_t]] = \mathbb{E}[Z \cdot \mathbb{1}] = 1$, and hence (associate $\tilde{Z}$ with XZ)

$$\begin{aligned}\mathbb{E}[Z \cdot \text{AV@R}_{\alpha \cdot Z}(Y|\mathcal{F}_t)] &\geq \inf\left\{\mathbb{E}\left[Y\tilde{Z}\right] :\ 0 \leq \tilde{Z},\, \alpha\tilde{Z} \leq \mathbb{1},\, \mathbb{E}\left[\tilde{Z}\right] = 1\right\}\\ &= \text{AV@R}_\alpha(Y).\end{aligned}$$

It follows by semi-continuity that $\mathbb{E}[Z \cdot \text{AV@R}_{\alpha \cdot Z}(Y|\mathcal{F}_t)] \geq \text{AV@R}_\alpha(Y)$ for all $Z \geq 0$ with $\mathbb{E}Z = 1$ and $\alpha Z \leq \mathbb{1}$.

To obtain equality it remains to be shown that there is an $Z_t \lhd \mathcal{F}_t$ such that $\text{AV@R}_\alpha(Y) = \mathbb{E}[Z_t\, \text{AV@R}_{\alpha Z_t}]$. For this let Z be the optimal dual variable in equation (2), that is $\text{AV@R}_\alpha(Y) = \mathbb{E}[YZ]$ with $Z \geq 0$, $\alpha Z \leq 1$ and $\mathbb{E}Z = 1$, and define

$$Z_t := \mathbb{E}[Z|\mathcal{F}_t].$$

$Z_t \lhd \mathcal{F}_t$ is feasible, as $0 \leq Z_t$, $\alpha Z_t \leq \mathbb{1}$ and $\mathbb{E}Z_t = 1$. From the fact that $X := \begin{cases} \frac{Z}{Z_t} & \text{if } Z_t > 0 \\ 1 & \text{if } Z_t = 0 \end{cases}$ is $\mathbb{P}-$a.e. well-defined and feasible for (11) one deduces further that

$$\begin{aligned}\mathbb{E}[Z_t \cdot \text{AV@R}_{\alpha \cdot Z_t}(Y|\mathcal{F}_t)] &= \inf\{\mathbb{E}[YZ_tX] :\ 0 \leq X,\, \alpha Z_t X \leq \mathbb{1},\, \mathbb{E}[X|\mathcal{F}_t] = \mathbb{1}\}\\ &\leq \mathbb{E}\left[YZ_t\frac{Z}{Z_t}\right] = \mathbb{E}[YZ] = \text{AV@R}_\alpha(Y).\end{aligned}$$

This is the converse inequality such that assertion (9) follows. The minimum is thus indeed attained for $Z_t = \mathbb{E}[Z|\mathcal{F}_t]$, where Z is the optimal dual variable for the AV@R_α, which exists for $\alpha > 0$.

As for $\alpha = 0$ recall that $\text{AV@R}_0(Y) = \operatorname{ess\,inf} Y$ and $\text{AV@R}_0(Y) \leq \text{AV@R}_0(Y|\mathcal{F}_t)$ and thus

$$\text{AV@R}_0(Y) = \mathbb{E}Z_t\, \text{AV@R}_0(Y) \leq \mathbb{E}Z_t\, \text{AV@R}_0(Y|\mathcal{F}_t) = \mathbb{E}Z_t\, \text{AV@R}_{0 \cdot Z_t}(Y|\mathcal{F}_t).$$

As for the converse inequality choose $Z^\varepsilon \geq 0$ with $\mathbb{E}Z^\varepsilon Y \leq \text{AV@R}_0(Y) + \varepsilon$. By the conditional $L^1 - L^\infty-$Hölder inequality

$$\begin{aligned}\text{AV@R}_0(Y) + \varepsilon \geq \mathbb{E}Z_\varepsilon Y &\geq \mathbb{E}(\mathbb{E}[Z^\varepsilon|\mathcal{F}_t]\, \text{AV@R}_0(Y|\mathcal{F}_t))\\ &\geq \mathbb{E}(\mathbb{E}[Z^\varepsilon|\mathcal{F}_t]\, \text{AV@R}_0(Y)) = \text{AV@R}_0(Y),\end{aligned}$$

whence

$$\text{AV@R}_0(Y) \geq \mathbb{E}Z_t^\varepsilon\, \text{AV@R}_{0 \cdot Z_t^\varepsilon}(Y|\mathcal{F}_t) - \varepsilon$$

for $Z_t^\varepsilon := \mathbb{E}[Z^\varepsilon|\mathcal{F}_t]$.

As for the remaining statement the proof reads along the same lines as above, but conditioned on $\mathcal{F}_t$. $\square$

Convexity and Concavity

The Average Value-at-Risk – in its respective variable – is convex *and* concave:

- Concavity of the Average Value-at-Risk
$$Y \mapsto \text{AV@R}_{\alpha}\left(Y|\mathcal{F}_t\right)$$
was elaborated in Theorem 6.
- Convexity of the Average Value-at-Risk – for the level parameter α – was elaborated in Lemma 2. The following Theorem 12 extends this observation for measurable level parameters, the map
$$Z \mapsto Z \cdot \text{AV@R}_{\alpha\cdot Z}\left(Y|\mathcal{F}_t\right)$$
is convex:

Theorem 12 (Convexity of the AV@R in its dual parameter)**.** *Let α, Z_0, Z_1 and $\lambda \lhd \mathcal{F}_t$ with $0 \leq \lambda \leq 1$, then*

$$Z_\lambda \cdot \text{AV@R}_{\alpha\cdot Z_\lambda}\left(Y|\mathcal{F}_t\right) \leq \left(1-\lambda\right) Z_0 \cdot \text{AV@R}_{\alpha\cdot Z_0}\left(Y|\mathcal{F}_t\right) + \lambda Z_1 \cdot \text{AV@R}_{\alpha\cdot Z_1}\left(Y|\mathcal{F}_t\right)$$

where $Z_\lambda = \left(1-\lambda\right) Z_0 + \lambda Z_1$.

Proof. Recall that by the definition of the conditional Average Value-at-Risk we have that

$$\begin{aligned}\left(1-\lambda\right) Z_0 \cdot & \text{AV@R}_{\alpha\cdot Z_0}\left(Y|\mathcal{F}_t\right) + \lambda Z_1 \cdot \text{AV@R}_{\alpha\cdot Z_1}\left(Y|\mathcal{F}_t\right)\\ &= \operatorname{ess\,inf} \mathbb{E}\left(Y\left(1-\lambda\right) Z_0 f_0 |\mathcal{F}_t\right) + \operatorname{ess\,inf} \mathbb{E}\left(Y \lambda Z_1 f_1 |\mathcal{F}_t\right)\\ &= \operatorname{ess\,inf} \mathbb{E}\left(Y\left(\left(1-\lambda\right) Z_0 f_0 + \lambda Z_1 f_1\right)|\mathcal{F}_t\right)\end{aligned}$$

where $f_0 \geq 0$, $f_1 \geq 0$, $\mathbb{E}\left(f_0|\mathcal{F}_t\right) = \mathbb{1}$, $\mathbb{E}\left(f_1|\mathcal{F}_t\right) = \mathbb{1}$, and moreover $\alpha Z_0 f_0 \leq \mathbb{1}$ and $\alpha Z_1 f_1 \leq \mathbb{1}$. It follows that $\alpha\left(\left(1-\lambda\right) Z_0 f_0 + \lambda Z_1 f_1\right) \leq \mathbb{1}$ and whence $\alpha Z_\lambda f \leq \mathbb{1}$ for $f := \frac{(1-\lambda) Z_0 f_0 + \lambda Z_1 f_1}{Z_\lambda}$. Notice that f is positive as well, and $\mathbb{E}\left(f|\mathcal{F}_t\right) = \mathbb{E}\left(\frac{(1-\lambda) Z_0 f_0 + \lambda Z_1 f_1}{Z_\lambda}\middle|\mathcal{F}_t\right) = \frac{(1-\lambda) Z_0 + \lambda Z_1}{Z_\lambda} = \mathbb{1}$, whence the latter display continues as

$$\begin{aligned}\left(1-\lambda\right) Z_0 \cdot & \text{AV@R}_{\alpha\cdot Z_0}\left(Y|\mathcal{F}_t\right) + \lambda Z_1 \cdot \text{AV@R}_{\alpha\cdot Z_1}\left(Y|\mathcal{F}_t\right)\\ &\geq \operatorname{ess\,inf} \mathbb{E}\left(Y Z_\lambda f|\mathcal{F}_t\right),\end{aligned}$$

the essential infimum being among all random variables $f \geq 0$ with $\mathbb{E}\left(f|\mathcal{F}_t\right) = \mathbb{1}$ and $\alpha Z_\lambda f \leq \mathbb{1}$. Whence

$$\begin{aligned}\left(1-\lambda\right) Z_0 \cdot & \text{AV@R}_{\alpha\cdot Z_0}\left(Y|\mathcal{F}_t\right) + \lambda Z_1 \cdot \text{AV@R}_{\alpha\cdot Z_1}\left(Y|\mathcal{F}_t\right)\\ &\geq \text{AV@R}_{\alpha Z_\lambda}\left(Y Z_\lambda|\mathcal{F}_t\right)\\ &= Z_\lambda \, \text{AV@R}_{\alpha Z_\lambda}\left(Y|\mathcal{F}_t\right)\end{aligned}$$

by positive homogeneity. □

A comparable decomposition as elaborated in Theorem 10 for other risk measures than the Average Value-at-Risk is significantly more complicated. As a notable example we give the following

Corollary 13. *Let $\mathcal{A}$ be a version independent, positively homogenous acceptability functional and $Y \in \mathbb{L}^1\left(\mathcal{F}_T\right)$, $\mathcal{F}_t \subset \mathcal{F}_T$. Then there is a measure m on $[0,1]$ such that*

$$\mathcal{A}\left(Y\right) \leq \int \mathbb{E}\left[Z_\alpha \cdot \text{AV@R}_{\alpha\cdot Z_\alpha}\left(Y|\mathcal{F}_t\right) m\left(\mathrm{d}\alpha\right)\right]$$

for any (measurable) family $\left(Z_\alpha\right)$, where $Z_\alpha \lhd \mathcal{F}_t$ with $0 \leq Z_\alpha$, $\alpha Z_\alpha \leq 1$ and $\mathbb{E} Z_\alpha = 1$. Moreover

$$\mathcal{A}\left(Y\right) = \inf_U \int \mathbb{E}\left[h_\alpha\left(U\right) \cdot \text{AV@R}_{\alpha h_\alpha(U)}\left(Y|\mathcal{F}_t\right) m\left(\mathrm{d}\alpha\right)\right]$$

where the infimum is among all uniform random variable U, ie. $\mathbb{P}\left[U < u\right] = u$ for all $0 \leq u \leq 1$.

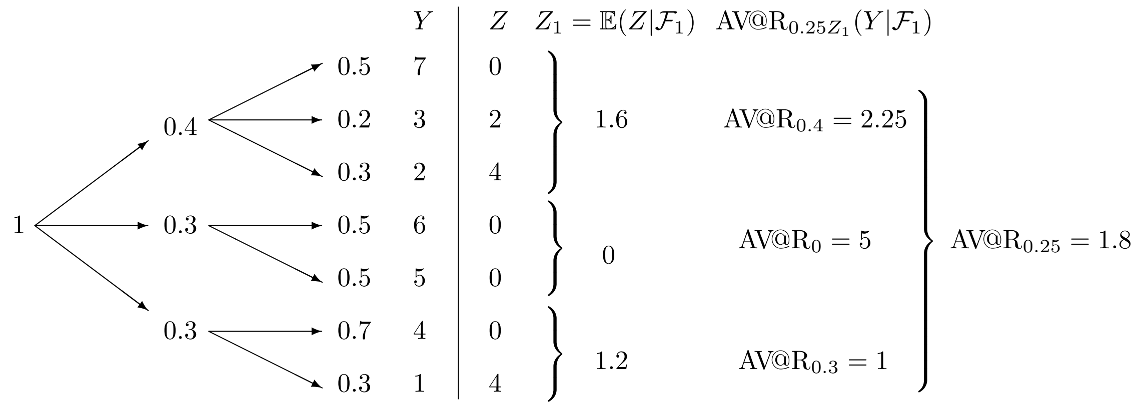

Figure 3: Nested computation of $\text{AV@R}_\alpha(Y)$ with $\alpha = 0.25$; in this tree-example with 11 nodes and 7 leaves the transitional probabilities are indicated. It holds true that $\text{AV@R}_{0.25}(Y) = 1.8 = \mathbb{E}\, Y Z = \mathbb{E}\left[Z_1 \cdot \text{AV@R}_{\alpha Z_1}(Y|F_t)\right]$.

Proof. Recall that by Kusuoka's representation ([Kus01], [JST06])

$$\mathcal{A}(Y) = \inf_{m \in \mathscr{P}} \int \text{AV@R}_\alpha(Y)\, m(\mathrm{d}\alpha)$$

where the infimum is among an adapted set of probability measures $\mathscr{P}$. By the previous theorem and (9) it follows that

$$\text{AV@R}_\alpha(Y) \le \mathbb{E}\left[Z_\alpha \cdot \text{AV@R}_{\alpha \cdot Z_t}(Y|\mathcal{F}_t)\right],$$

for any α provided that Z_α is feasible. Taking expectation with respect to $m(\mathrm{d}\alpha)$ gives

$$\mathcal{A}(Y) = \int \text{AV@R}_\alpha(Y)\, m(\mathrm{d}\alpha) \le \int \mathbb{E}\left[Z_\alpha \cdot \text{AV@R}_{\alpha \cdot Z_t}(Y|\mathcal{F}_t)\right] m(\mathrm{d}\alpha),$$

which is the first assertion.

As for the second observe that there is Z^*_α by (2) for any α such that $\mathbb{E}\left[Y Z^*_\alpha\right] < \text{AV@R}_\alpha(Y) + \varepsilon$. As Z^*_α is a distortion (cf. [PR07]) it has the special representation $Z^*_\alpha = h_\alpha(U_\alpha)$ for some uniform random variable U_α and $h_\alpha(x) = \frac{1}{\alpha}\mathbb{1}_{[0,\alpha]}(x)$. However, all Z^*_α are coupled in an anti-monotone way with Y, so U_α and $U_{\alpha'}$ are coupled in an co-monotone way and $\{U_\alpha \le \min(\alpha, \alpha')\} = \{U_{\alpha'} \le \min(\alpha, \alpha')\}$. Thus one may fix a single uniform variable U such that $Z^*_\alpha := h(U)$ where $h(x) = \int_x^1 \frac{1}{\alpha} m(\mathrm{d}\alpha) = \int_0^1 h_\alpha(x)\, m(\mathrm{d}\alpha)$. Define $Z_\alpha := \mathbb{E}\left[h(U)|\mathcal{F}_t\right]$ and choose an $\varepsilon-$optimal probability measure m. With this choice the second assertion is immediate as well. □

5. Composition of risk measures

There is an eye-catching similarity of the outer form between (9) and (2). To catch the analogy it would be desirable to establish an $\tilde{\alpha}$ such that

$$\text{AV@R}_\alpha(Y) = \text{AV@R}_{\tilde{\alpha}}\left(\text{AV@R}_{\alpha \cdot Z_t}(Y|\mathcal{F}_t)\right) \tag{12}$$

where Z_t may already be chosen to be optimal.

The first observation in this direction is that $\text{AV@R}_{\alpha \cdot Z_t}(Y|\mathcal{F}_t)$ and Z_t are *not* coupled in an anti-monotone way, as the example in Figure 3 reveals.

Next, in this example (Fig. 3) equality in (12) holds indeed for $\tilde{\alpha} = \frac{23}{32}$, the dual variable for this choice $\tilde{\alpha}$ is $Z = \left(\frac{32}{23}, \frac{2}{23}, \frac{32}{23}\right)$. This means that a positive weight $\left(\frac{2}{23}\right)$ is assigned to $\text{AV@R}_0 = 5$, although $\{Y > 4\}$ is not relevant for $\text{AV@R}_{0.25}$. We take both points as indicators that there is probably no insightful, general relation as suggested in (12).

To escape one may define $\mathcal{A}_{Z_t}(Y) := \mathbb{E}[Y \cdot Z_t]$ (Z_t the optimal dual), which is a risk measure. By the latter theorem thus

$$\text{AV@R}_\alpha(Y) = \mathcal{A}_{Z_t}\left(\text{AV@R}_{\alpha \cdot Z_t}(Y|\mathcal{F}_t)\right),$$

such that the AV@R is indeed a composition of risk-measures. However, $\mathcal{A}_{Z_t}$ already incorporates information from Y and $\mathcal{A}_{Z_t}$ is not version independent.

This can be avoided by the setting (cf. [ES11])

$$\tilde{\mathcal{A}}_{Z_t}(Y) := \inf_{\tilde{Z}_t \sim Z_t} \mathbb{E}\left[Y \cdot \tilde{Z}_t\right],$$

where the infimum is among all random variables $\tilde{Z}_t$ having the same law as Z_t (ie. $\mathbb{P}[Z_t \le z] = \mathbb{P}\left[\tilde{Z}_t \le z\right]$ for all $z \in \mathbb{R}$). This allows at least the inequality $\text{AV@R}_\alpha(Y) \ge \tilde{\mathcal{A}}_{Z_t}\left(\text{AV@R}_{\alpha \cdot Z_t}(Y|\mathcal{F}_t)\right)$.

6. Application to Multistage Optimization

Problem Formulation

The optimal investment problem (optimal portfolio allocation) is often chosen to elaborate and discuss stochastic optimization problems. It is the objective of these problems to maximize the future return, but taking the potential risk into account as well in some appropriate way.

Several problem formulations are in use to account for this purpose, for example

$$\begin{array}{rl} \text{maximize} & \mathbb{E}Y \\ \text{subject to} & \mathcal{A}(Y) \ge q, \\ & Y \in \mathscr{Y}, \end{array}$$

where a strategy for the holdings Y is accepted if its acceptability $\mathcal{A}(Y)$ at least exceeds a prior fixed q. Another frequent formulation takes the opposite point of view and intends to solve

$$\begin{array}{rl} \text{maximize} & \mathcal{A}(Y) \\ \text{s.t.} & \mathbb{E}Y \ge r, \\ & Y \in \mathscr{Y}, \end{array}$$

where the best strategy Y, maximizing the acceptability, is required to have at least return r on average. Both have restrictions imposed by some external parameter, q and r (resp.).

The alternative, in the sense of an integrated risk management and avoiding external parameters and empty feasibility sets, consists in incorporating acceptability functionals in the objective such as

$$\begin{array}{rl} \text{maximize} & \mathbb{E}Y + \gamma \cdot \mathcal{A}(Y) \\ \text{s.t.} & Y \in \mathscr{Y}, \end{array} \tag{13}$$

where γ is just a positive parameter to account for the emphasis that should be given to risk: γ is the risk appetite, the degree of uncertainty the investor is willing to accept in respect of negative changes to its assets.

Remark 14. For positively homogeneous acceptability functionals $\mathcal{A}$ the problem maximize $\mathbb{E}(1-\gamma)Y + \gamma \cdot \mathcal{A}(Y)$ is equivalent. This setting is often more appropriate for the Average Value-at-Risk, as the objective value – for the inequality $(1-\gamma)\mathbb{E}Y + \gamma \cdot \text{AV@R}_\alpha(Y) \le \mathbb{E}Y$ – provides a lower bound for the expected return.

The problem formulation (13), which does not have a priori constraints, applies for optimal investment problems, and it can be found in multistage decision models for electricity management as well. Multistage problems thus naturally can be formulated as

$$\begin{array}{rl} \text{maximize} & \mathbb{E}H(\xi, x(\xi)) + \gamma \cdot \mathcal{A}(H(\xi, x(\xi))) \\ \text{s.t.} & x \lhd \mathcal{F} \\ & x \in \mathscr{X}, \end{array} \tag{14}$$

where at each stage $t \in \mathbf{T}$ a decision x_t such that $x = (x_t) \in \mathscr{X}$.

A natural choice for the filtration in the present context is $\mathcal{F}_t := \sigma(\xi_0, \dots \xi_t)$: Note that $x \lhd \mathcal{F}$ forces x_t to be a function of all previous observations, $x_t = x_t(\xi_0, \dots \xi_t)$ (cf. [Shi96, Theorem II.4.3] for the respective measurability), which reflects the fact that the decisions x_t have to be fixed *without* knowledge of the future outcomes $\xi_{t+1}, \dots \xi_T$: the respective measurability constraint $x \lhd \mathcal{F}$ ($x_t \lhd \mathcal{F}_t$, $t \in \mathbf{T}$) is called *nonanticipativity.*

We shall require the $\mathbb{R}-$valued function H concave, defined on a convex set, such that (ξ any fixed state)

$$H(\xi, (1-\lambda) x_0 + \lambda x_1) \geq (1-\lambda) H(\xi, x_0) + \lambda H(\xi, x_1).$$

By the monotonicity property and concavity of the acceptability functional $\mathcal{A}$ thus

$$\begin{aligned} \mathcal{A}(H(\xi, (1-\lambda) x_0(\xi) + \lambda x_1(\xi))) &\geq \mathcal{A}((1-\lambda) H(\xi, x_0(\xi)) + \lambda H(\xi, x_1(\xi))) \\ &\geq (1-\lambda) \mathcal{A}(H(\xi, x_0(\xi))) + \lambda \mathcal{A}(H(\xi, x_1(\xi))), \end{aligned} \tag{15}$$

which means that $\mathcal{A} \circ H$ is concave as well. Notably concavity and the other assumptions above hold for the Average Value-at-Risk and the *conditional* Average Value-at-Risk in particular.

Remark 15. To put the ingredients – the observations $\xi = (\xi_0, \xi_1, \dots \xi_T)$, which are sequentially revealed, and the decisions $x = (x_0, x_1, \dots x_T)$ – in chronological order recall that $x_t = x_t(\xi_0, \dots \xi_t)$. The sequential arrangement using the arrow of time $\rightsquigarrow$ thus is

$$\xi_0 \rightsquigarrow x_0 \rightsquigarrow \xi_1 \rightsquigarrow x_1 \quad \dots \quad \rightsquigarrow \xi_t \rightsquigarrow x_t \quad \dots \quad \rightsquigarrow \xi_T \rightsquigarrow x_T.$$

We are hence interested in a decision x_t *after* the observation ξ_t, and x_t thus may be interpreted as a response to the observations $(\xi_0, \dots \xi_t)$.

Remark 16 (Notational convention). We shall write $H(x)$ for the random variable $H(x)(\xi) := H(\xi, x(\xi))$. For notational convenience we shall use the straight forward abbreviation $\xi_{i:j}$ for the fractional part $\xi_{i:j} = (\xi_i, \xi_{i+1}, \dots \xi_j)$; in particular $\xi_{i:i} = (\xi_i)$, $x_{0:T} = (x_{0:t-1}, x_{t:T})$ and $x_{t:t-1} = ()$ for consistency.

7. Dynamic Programming Formulation

The *dynamic programming principle* is the basis of the solution technique developed by Bellman [Bel57] in the 1950's for deterministic optimal control problems. They have been extended later to account for stochastic problems as well, where typically

(i) the objective is an expectation and
(ii) the transition does not depend on the history, but just on the current state of the system – that is to say for Markov chains.

Remark 17 (Multiperiod acceptability functionals). Some papers exclusively treat the function $H(x) = \sum_{t=0}^{T} H_t(x_{0:t})$ in the present setting. This setting is just a special case and included in our general formulation and framework of problem (14).

Moreover the theory developed below applies for more general risk functions as the Average Value-at-Risk – for example $\mathcal{A} = \sum_k \gamma_k \,\mathrm{AV@R}_{\alpha_k}$ – so in particular includes all approximations of law invariant acceptability functionals by Kusuoka's theorem. Above that an acceptability functional of the type

$$\mathcal{A}(Y) = \sum_k \mathbb{E}\gamma_k \,\mathrm{AV@R}_{\alpha_k}(Y|\mathcal{F}_{t_k}) \tag{16}$$

for some $\mathcal{F}_{t_k}-$measurable α_t and γ_t ($\alpha_t, \gamma_t \lhd \mathcal{F}_{t_k}$) are included as well in the following discussion (multiperiod acceptability functional).

However, as these more general acceptability functionals are to be treated in analogous way we continue with the simple Average Value-at-Risk in lieu of the more general setting (16).

The decomposition of the Average Value-at-Risk elaborated in Theorem 10 is the key which allows to define – in line with the classical dynamic programming principle – a value function with properties analogous to the classical theory.

For Markov processes the value function naturally is a function of time and the current status of the system. In order to derive dynamic programming equations for the general multistage problem it is necessary to carry the entire history of earlier decisions. To trigger the decision x_t *after* the observation ξ_t we define the *value function*

$$\mathscr{V}_t\left(x_{0:t-1},\alpha,\gamma\right):=\operatorname*{ess\,sup}_{x_{t:T}}\mathbb{E}\left[H\left(x_{0:T}\right)\middle|\,\mathcal{F}_t\right]+\gamma\cdot\mathrm{AV@R}_\alpha\left(H\left(x_{0:T}\right)|\mathcal{F}_t\right). \tag{17}$$

The value function (17) depends on

- the decisions up to time $t-1$, $x_{0:t-1}$, where $x_{t:T}$ in (17) is chosen such that $(x_{0:T})=(x_{0:t-1},\,x_{t,T})\in\mathscr{X}$,
- the random model parameters $\alpha\lhd\mathcal{F}_t$ and $\gamma\lhd\mathcal{F}_t$ and
- the current status of the system due to the filtration $\mathcal{F}_t$.

Evaluated at initial time $t=0$ and assuming the sigma-algebra $\mathcal{F}_0$ trivial the value function relates to the initial problem (14), as

$$\begin{aligned}\sup_{x_{0:T}}\mathbb{E}H\left(x_{0:T}\right)+\gamma\cdot\mathrm{AV@R}_\alpha\left(H\left(x_{0:T}\right)\right)=&\\ &=\operatorname*{ess\,sup}_{x_{0:T}}\mathbb{E}\left[H\left(x_{0:T}\right)|\mathcal{F}_0\right]+\gamma\cdot\mathrm{AV@R}_\alpha\left(H\left(x_{0:T}\right)|\mathcal{F}_0\right)\\ &=\mathscr{V}_0\left(\left(\right),\alpha,\gamma\right)\end{aligned} \tag{18}$$

when employing the Average Value-at-Risk.

The decomposition theorem (Theorem 10) above allows to formulate the following

Theorem 18 (Dynamic Programming Principle)**.** *Assume H random upper semi-continuous with respect to x and ξ valued in some convex, compact subset of $\mathbb{R}^n$.*

(i) *The value function evaluates to*

$$\mathscr{V}_T\left(x_{0:T-1},\alpha,\gamma\right)=\left(1+\gamma\right)\operatorname*{ess\,sup}_{x_T}H\left(x_{0:T}\right)$$

at terminal time T.

(ii) *For any $t<\tau$, $(t,\tau\in\mathbf{T})$ the recursive relation*

$$\mathscr{V}_t\left(x_{0:t-1},\alpha,\gamma\right)=\operatorname*{ess\,sup}_{x_{t:\tau-1}}\operatorname*{ess\,inf}_{Z_{t:\tau}}\mathbb{E}\left[\mathscr{V}_\tau\left(x_{0:\tau-1},\alpha\cdot Z_{t:\tau},\gamma\cdot Z_{t:\tau}\right)\middle|\,\mathcal{F}_t\right], \tag{19}$$

where $Z_{t:\tau}\lhd\mathcal{F}_\tau$, $0\le Z_{t:\tau}$, $\alpha Z_{t:\tau}\le\mathbb{1}$ and $\mathbb{E}\left[Z_{t:\tau}|F_t\right]=\mathbb{1}$, holds true.

Proof. A direct evaluation at terminal time $t=T$ gives

$$\begin{aligned}\mathscr{V}_T\left(x_{0:T-1},\alpha,\gamma\right)&=\operatorname*{ess\,sup}_{x_{T:T}}\mathbb{E}\left[H\left(x_{0:T}\right)|\mathcal{F}_T\right]+\gamma\cdot\mathrm{AV@R}_\alpha\left(H\left(x_{0:T}\right)|\mathcal{F}_T\right)\\ &=\operatorname*{ess\,sup}_{x_T}H\left(x_{0:T}\right)+\gamma\cdot H\left(x_{0:T}\right)\\ &=\left(1+\gamma\right)\operatorname*{ess\,sup}_{x_T}H\left(x_{0:T}\right),\end{aligned}$$

because the random variables, conditionally on the entire observations $\xi_{0:T}$, are constant. The final maximizations over $x_{T:T}=x_T\left(\xi_{0:T}\right)$ moreover are deterministic, because all stochastic observations are available.

As for an intermediate time ($t<T$) observe that

$$\begin{aligned}\mathscr{V}_t\left(x_{0:t-1},\alpha,\gamma\right) &= \operatorname*{ess\,sup}_{x_{t:T}} \mathbb{E}\left[H\left(x_{0:T}\right)|\mathcal{F}_t\right]+\gamma\cdot \mathrm{AV@R}_\alpha\left(H\left(x_{0:T}\right)|\mathcal{F}_t\right)\\ &= \operatorname*{ess\,sup}_{x_{t:T}} \operatorname*{ess\,inf}_{Z_{t:t+1}} \mathbb{E}\left[\begin{array}{l}\mathbb{E}\left[H\left(x_{0:T}\right)|\mathcal{F}_{t+1}\right]\\ +\gamma\cdot Z_{t:t+1}\cdot \mathrm{AV@R}_{\alpha\cdot Z_{t:t+1}}\left(H\left(x_{0:T}\right)|\mathcal{F}_{t+1}\right)\end{array}\middle|\mathcal{F}_t\right]\end{aligned}$$

due to the nested decomposition (10) of the Average Value-at-Risk at random level, elaborated in Theorem 10; the ess inf is among all random variables $Z_{t:t+1}\lhd\mathcal{F}_{t+1}$ satisfying $0\le\alpha$, $\alpha Z_{t:t+1}\le 1$ and $\mathbb{E}\left[Z_{t:t+1}|\mathcal{F}_t\right]\equiv\mathbb{1}$. By the discussions in the preceding sections the inner expression is concave in $x_{0:T}$ and convex in $Z_{t:t+1}$, by Sion's minimax theorem (cf. [Sio58] and [Kom88]) one may thus interchange the min and max to obtain

$$\mathscr{V}_t\left(x_{0:t-1},\alpha,\gamma\right)=\operatorname*{ess\,sup}_{x_t}\operatorname*{ess\,inf}_{Z_{t:t+1}}\operatorname*{ess\,sup}_{x_{t+1:T}}\mathbb{E}\left[\begin{array}{l}\mathbb{E}\left[H\left(x_{0:T}\right)|\mathcal{F}_{t+1}\right]\\ +\gamma\cdot Z_{t:t+1}\cdot \mathrm{AV@R}_{\alpha\cdot Z_{t:t+1}}\left(H\left(x_{0:T}\right)|\mathcal{F}_{t+1}\right)\end{array}\middle|\mathcal{F}_t\right]$$

As H is upper semi-continuous by assumption one may further apply the interchangeability principle [RW97, Theorem 14.60] (cf. also [SDR09, p. 405]) such that

$$\begin{aligned}\mathscr{V}_t\left(x_{0:t-1},\alpha,\gamma\right) &= \operatorname*{ess\,sup}_{x_t}\operatorname*{ess\,inf}_{Z_{t:t+1}}\mathbb{E}\left[\operatorname*{ess\,sup}_{x_{t+1:T}}\begin{array}{l}\mathbb{E}\left[H\left(x_{0:T}\right)|\mathcal{F}_{t+1}\right]\\ +\gamma\cdot Z_{t:t+1}\cdot \mathrm{AV@R}_{\alpha\cdot Z_{t:t+1}}\left(H\left(x_{0:T}\right)|\mathcal{F}_{t+1}\right)\end{array}\middle|\mathcal{F}_t\right]\\ &= \operatorname*{ess\,sup}_{x_t}\operatorname*{ess\,inf}_{Z_{t:t+1}}\mathbb{E}\left[\mathscr{V}_{t+1}\left(x_{0:t},\alpha\cdot Z_{t:t+1},\gamma\cdot Z_{t:t+1}\right)\middle|\mathcal{F}_t\right],\end{aligned}$$

which is the desired relation for $\tau=t+1$. Repeating the computation from above $t-\tau$ times, or conditioning on $\mathcal{F}_\tau$ instead of $\mathcal{F}_{t+1}$ reveals the general result. □

8. Martingale Representation and Verification Theorem

The value function $\mathscr{V}_t$ introduced in (17) is a function of some general $\alpha\lhd\mathcal{F}_t$ and $\gamma\lhd\mathcal{F}_t$. To specify for the right parameters assume that the optimal policy $\mathbf{x}=x_{0:T}$ of problem (14) exists. Theorem 18 then gradually reveals the optimal dual variables $\mathbf{Z}_T,\mathbf{Z}_{T-1},\dots$ and finally $\mathbf{Z}_0$ (assuming again that the respective argmins of the essential infimum ess inf exist). The conditions $\mathbb{E}\left[\mathbf{Z}_\tau|\mathcal{F}_t\right]=\mathbb{1}$ ($\tau>t$) imposed on the dual variables suggest to compound the densities and to consider the densities $\mathbf{Z}_{t:\tau}:=\mathbf{Z}_t\cdot\mathbf{Z}_{t+1}\cdot\ldots\mathbf{Z}_\tau$ such that $\mathbb{E}\left[\mathbf{Z}_{t:\tau}|\mathcal{F}_t\right]=\mathbf{Z}_t$ and $\mathbb{E}\left[\mathbf{Z}_{0:\tau}|\mathcal{F}_t\right]=\mathbf{Z}_{0:t}$. With this setting the process $\mathbf{Z}:=\left(\mathbf{Z}_{0:t}\right)_{t\in\mathbf{T}}$ is a martingale, satisfying moreover $0\le\mathbf{Z}_t$ and $\alpha\mathbf{Z}_t\le\mathbb{1}$ during all times $t\in\mathbf{T}$. The optimal pair $(\mathbf{x},\mathbf{Z})$ is a saddle point for the respective Lagrangian.

This gives rise for the following definition.

Definition 19. Let $\alpha\in[0,1]$ be a fixed level.

(i) $Z=\left(Z_t\right)_{t\in\mathbf{T}}$ is a feasible (for the nonanticipativity constraints) process of densities if
  (a) Z_t is a martingale with respect to the filtration $\mathcal{F}_t$ and
  (b) $0\le Z_t$, $\alpha Z_t\le\mathbb{1}$ and $\mathbb{E}Z_t=1$ ($t\in\mathbf{T}$).

(ii) The intermediate densities are $Z_{t:\tau}:=\frac{Z_\tau}{Z_{t-1}}$ ($0<t<\tau$) and $Z_{0:\tau}:=Z_\tau$.

For feasible x and Z we consider the stochastic process

$$\mathbf{M}_t\left(x,Z\right):=\mathscr{V}_t\left(x_{0:t-1},\,\alpha Z_{0:t},\,\gamma Z_{0:t}\right)\quad\left(t\in\mathbf{T}\right)$$

where α and γ – in contrast to (17) – are simple *numbers*.

Recall from (18) that $\mathbf{M}_0$ is a constant (if $\mathcal{F}_0$ is trivial) solving the original problem (14) if $(\mathbf{x},\mathbf{Z})$ are optimal. Above that we shall prove in the next theorem that $\mathbf{M}_t\left(\mathbf{x},\mathbf{Z}\right)$ is a martingale in this case.

Theorem 20 (Martingale property)**.** *Given that* $\mathbf{x}$ *and* $\mathbf{Z}$ *are optimal, then the process* $\mathbf{M}_t\left(\mathbf{x},\mathbf{Z}\right)$ *is a martingale with respect to the filtration* $\mathcal{F}_t$*.*

Conversely, if $\mathbf{M}_t\left(x,Z\right)$ *is a martingale and the* argmax *sets (for x) and* argmin *sets (for Z) in* (19) *are non-empty, then x and Z are optimal.*

Proof. By the dynamic programming equation (19) and the respective maximality of $\mathbf{Z}_{t+1}$ and $\mathbf{x}_{t+1}$ we have that

$$\begin{aligned}\mathbf{M}_t(\mathbf{x},\mathbf{Z}) &= \mathscr{V}_t(\mathbf{x}_{0:t-1},\,\alpha\mathbf{Z}_{0:t},\,\gamma\mathbf{Z}_{0:t})\\ &= \operatorname*{ess\,sup}_{x_{t:t}}\operatorname*{ess\,inf}_{Z_{t+1}}\mathbb{E}\left[\mathscr{V}_{t+1}\left((\mathbf{x}_{0:t-1},x_t),\,\alpha\cdot\mathbf{Z}_{0:t}Z_{t+1},\,\gamma\cdot\mathbf{Z}_{0:t}Z_{t+1}\right)\middle|\,\mathcal{F}_t\right]\\ &= \operatorname*{ess\,sup}_{x_{t:t}}\mathbb{E}\left[\mathscr{V}_{t+1}\left((\mathbf{x}_{0:t-1},x_t),\,\alpha\cdot\mathbf{Z}_{0:t+1},\,\gamma\cdot\mathbf{Z}_{0:t+1}\right)\middle|\,\mathcal{F}_t\right]\\ &= \mathbb{E}\left[\mathscr{V}_{t+1}\left(\mathbf{x}_{0:t},\,\alpha\cdot\mathbf{Z}_{0:t+1},\,\gamma\cdot\mathbf{Z}_{0:t+1}\right)\middle|\,\mathcal{F}_t\right]\\ &= \mathbb{E}\left[\mathbf{M}_{t+1}(\mathbf{x},\mathbf{Z})\middle|\,\mathcal{F}_t\right]\end{aligned}$$

again by the interchangeable principle. $\mathbf{M}_t$, hence, is a martingale with respect to the filtration $\mathcal{F}_t$.

The converse follows from the following corollary. □

Corollary 21 (Verification Theorem)**.** *Let x and Z be feasible for* (14)*.*

(i) *Suppose that $\mathscr{W}$ satisfies*

$$\begin{aligned}\mathscr{W}_T(x_{0:T-1},\alpha Z_{0:T},\gamma Z_{0:T}) &\geq (1+\gamma Z_{0:T})\,H(x_{0:T}(\xi_{0:T}))\quad\text{and}\\ \mathscr{W}_t(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t}) &\geq \operatorname*{ess\,sup}_{x_t}\mathbb{E}\left[\mathscr{W}_{t+1}(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1})\middle|\,\mathcal{F}_t\right],\end{aligned}$$

then the process $\mathscr{W}_t(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t})$ ($t\in\mathbf{T}$) is a super-martingale dominating $\mathscr{V}(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t})$, $\mathscr{V}\leq\mathscr{W}$.

(ii) *Let $\mathscr{U}$ satisfy*

$$\begin{aligned}\mathscr{U}_T(x_{0:T-1},\alpha Z_{0:T},\gamma Z_{0:T}) &\leq (1+\gamma Z_{0:T})\,H(x_{0:T}(\xi_{0:T}))\quad\text{and}\\ \mathscr{U}_t(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t}) &\leq \operatorname*{ess\,inf}_{Z_{t+1}}\mathbb{E}\left[\mathscr{U}_{t+1}(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1})\middle|\,\mathcal{F}_t\right],\end{aligned}$$

then the process $\mathscr{U}_t(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t})$ is a sub-martingale dominated by $\mathscr{V}(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t})$, $\mathscr{U}\leq\mathscr{V}$.

Proof. The proof is by induction on t, starting at the final stage T. Observe first that $\mathscr{U}_T\leq\mathscr{V}_T\leq\mathscr{W}_T$ by assumption and (15). Then

$$\begin{aligned}\mathscr{U}_t(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t}) &\leq \operatorname*{ess\,inf}_{Z_{t+1}}\mathbb{E}\left[\mathscr{U}_{t+1}(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1})\middle|\,\mathcal{F}_t\right]\\ &\leq \operatorname*{ess\,sup}_{x_t}\operatorname*{ess\,inf}_{Z_{t+1}}\mathbb{E}\left[\mathscr{V}_{t+1}(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1})\middle|\,\mathcal{F}_t\right]\\ &= \mathscr{V}_t(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t}),\end{aligned}$$

and thus $\mathscr{U}\leq\mathscr{V}$.

As for $\mathscr{W}_t$ observe that

$$\begin{aligned}\mathscr{W}_t(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t}) &\geq \operatorname*{ess\,sup}_{x_t}\mathbb{E}\left[\mathscr{W}_{t+1}(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1})\middle|\,\mathcal{F}_t\right]\\ &\geq \operatorname*{ess\,inf}_{Z_{t+1}}\operatorname*{ess\,sup}_{x_t}\mathbb{E}\left[\mathscr{V}_{t+1}(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1})\middle|\,\mathcal{F}_t\right]\\ &\geq \operatorname*{ess\,sup}_{x_t}\operatorname*{ess\,inf}_{Z_{t+1}}\mathbb{E}\left[\mathscr{V}_{t+1}(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1})\middle|\,\mathcal{F}_t\right]\\ &= \mathscr{V}_t(x_{0:t-1},\alpha\cdot Z_{0:t},\gamma\cdot Z_{0:t}),\end{aligned}$$

because it always holds true that $\inf_z\sup_x L(x,z)\geq\sup_x\inf_z L(x,z)$.

$\mathscr{W}$ is a super-martingale, because

$$\begin{aligned}\mathscr{W}_t(x_{0:t-1},\alpha Z_{0:t},\gamma Z_{0:t}) &\geq \operatorname*{ess\,sup}_{x_t}\mathbb{E}\left[\mathscr{W}_{t+1}(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1})\middle|\,\mathcal{F}_t\right]\\ &\geq \mathbb{E}\left[\mathscr{W}_{t+1}(x_{0:t},\alpha\cdot Z_{0:t+1},\gamma\cdot Z_{0:t+1})\middle|\,\mathcal{F}_t\right],\end{aligned}$$

which is the characterizing property; the proof that $\mathscr{U}$ is a sub-martingale is analogous. □

9. Algorithm for a sequential improvement

The latter theorem and its subsequent corollary (verification theorem) allow to verify if a given policy x and dual Z are optimal to solve the multistage problem (14). In general, unfortunately, it may not be easy to verify even existence of such a saddle point.

However, if $\mathbf{x}$ is given, then the verification theorem allows to compute $\mathbf{Z}$ by looking up the respective argmin sets, and conversely it is possible to find the optimal $\mathbf{x}$ to given $\mathbf{Z}$ by computing the respective argmax sets.

Moreover, given a policy x and a dual density Z it is possible to improve on these choices by taking the respective maximizers from the verification theorem. Iterating the optimization procedures in an alternating way, however, may not necessarily give a sequence converging to the saddle point, but an improvement can always be achieved.

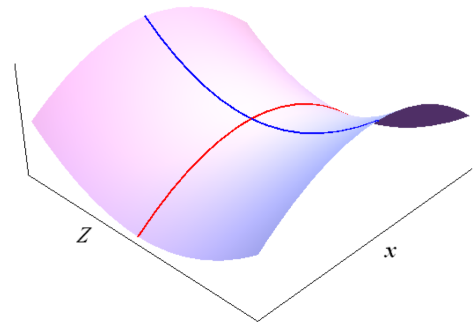

Figure 4: Saddle of the Lagrangian depending on the optimal decision x and the dual density Z.

The representation (4) is often and successively used to solve the problem (14), as it rewrites as

$$\begin{aligned} \text{maximize}_{x,q} \quad & \mathbb{E}\left(H(x) + \gamma q - \tfrac{\gamma}{\alpha}\left(q - H(x)\right)^+\right) \\ \text{s.t.} \quad & x \lhd \mathcal{F} \\ & x \in \mathscr{X}, \end{aligned}$$

and this problem just involves maximization, which can often be implemented in a straight forward way.

The decomposition theorem (Theorem 10) can be applied to specify a local problem, typically leading to a considerable acceleration. For this recall that

$$\mathbb{E}H(x) + \gamma\,\mathrm{AV@R}_\alpha\left(H(x)\right) = \inf_{Z_{0:t}} \mathbb{E}\left(\mathbb{E}\left(H(x)\middle|\mathcal{F}_t\right) + \gamma Z_{0:t}\,\mathrm{AV@R}_{\alpha Z_{0:t}}\left(H(x)\middle|\mathcal{F}_t\right)\right);$$

given the optimizing random variable $Z_{0:t}$ this suggests to improve x locally, that is to choose

$$\begin{aligned} x_t &\in \operatorname*{argmax}_{x_t} \mathbb{E}\left(H(x)\middle|\mathcal{F}_t\right) + \gamma Z_{0:t}\,\mathrm{AV@R}_{\alpha Z_{0:t}}\left(H(x)\middle|\mathcal{F}_t\right) \\ &= \operatorname*{argmax}_{q\lhd\mathcal{F}_t,\,x_t} \mathbb{E}\left(H(x)\middle|\mathcal{F}_t\right) + \gamma Z_{0:t} q - \frac{\gamma}{\alpha}\mathbb{E}\left(\left(q-H(x)\right)^+\middle|\mathcal{F}_t\right) \end{aligned}$$

or to find at least a local improvement.

This strategy is indicated in Algorithm 1 and indeed gives an improvement.

Monotonicity of Algorithm 1.

The Algorithm 1 generates decisions $x^k_{0:T}$ which are successive improvements, that is $\mathcal{U}\left(x^{k+1}_{0:T}\right) \geq \mathcal{U}\left(x^k_{0:T}\right)$ for all k.

Indeed, observe that

$$\begin{aligned} &\operatorname*{ess\,inf}_{Z_{t:t+1}} \mathbb{E}\left[H\left(x^{k+1}_{0:T}\right)\middle|\mathcal{F}_t\right] + \gamma Z_{t:t+1}\,\mathrm{AV@R}_{\alpha Z_{t:t+1}}\left(H\left(x^{k+1}_{0:T}\right)\middle|\mathcal{F}_t\right) \\ &\quad= \operatorname*{ess\,inf}_{Z_{t:t+1}} \operatorname*{ess\,sup}_{x_t} \mathbb{E}\left[H(x_{0:T})\middle|\mathcal{F}_t\right] + \gamma Z_{t:t+1}\,\mathrm{AV@R}_{\alpha Z_{t:t+1}}\left(H(x_{0:T})\middle|\mathcal{F}_t\right) \end{aligned} \tag{22}$$

by the particular assignment rule (21) for $x^{k+1}_{0:T}$. Moreover

$$(22) \geq \operatorname*{ess\,sup}_{x_t} \operatorname*{ess\,inf}_{Z_{t:t+1}} \mathbb{E}\left[H(x_{0:T})\middle|\mathcal{F}_t\right] + \gamma Z_{t:t+1}\,\mathrm{AV@R}_{\alpha Z_{t:t+1}}\left(H(x_{0:T})\middle|\mathcal{F}_t\right)$$

Algorithm 1
Sequential improvement of the strategy $x_{0:T}$ by exploiting the nested structure of the acceptability functional.

Step 0
Let $x_{0:T}^0$ be any feasible, initial solution of the problem (14),
Set $k \leftarrow 0$.

Step 1
Find Z^k, such that $0 \le Z^k \le \frac{1}{\alpha}$, $\mathbb{E}Z^k = 1$ and

$$\mathbb{E}Z^k H\left(x_{0:T}^k\right) = \text{AV@R}_\alpha\left(H\left(x_{0:T}^k\right)\right) \tag{20}$$

and define $Z_t^k := \mathbb{E}\left(Z^k|\mathcal{F}_t\right)$.

Step 2
Choose

$$\begin{aligned} x_t^{k+1} &\in \underset{x_t \lhd \mathcal{F}_t}{\operatorname{argmax}}\, \mathbb{E}\left[H\left(x_{0:T}^k\right)\middle|\mathcal{F}_t\right] + \gamma Z_t^k\, \text{AV@R}_{\alpha Z_t^k}\left(H\left(x_{0:T}^k\right)\middle|\mathcal{F}_t\right) \\ &= \underset{q,\, x_t \lhd \mathcal{F}_t}{\operatorname{argmax}}\, \mathbb{E}\left[H\left(x_{0:T}^k\right) + \gamma Z_t^k q - \frac{\gamma}{\alpha}\left(q - H\left(x_{0:T}^k\right)\right)^+\middle|\mathcal{F}_t\right] \end{aligned} \tag{21}$$

at any arbitrary stage $t \in \mathbf{T}$ and node specified by $\mathcal{F}_t$.

Step 3
Set

$$\mathcal{U}\left(x_{0:T}^{k+1}\right) := \mathbb{E}H\left(x_{0:T}^{k+1}\right) + \gamma\, \text{AV@R}_\alpha\left(H\left(x_{0:T}^{k+1}\right)\right),$$

increase $k \leftarrow k+1$ and continue with **Step 1** unless

$$\mathcal{U}\left(x_{0:T}^{k+1}\right) - \mathcal{U}\left(x_{0:T}^k\right) < \varepsilon,$$

where ε is the desired improvement in each cycle k.

by the usual $\sup_x \inf_y L \le \inf_y \sup_x L$ inequality, and it finally follows that

$$\begin{aligned} (22) \quad &\ge \underset{Z_{t:t+1}}{\operatorname{ess\,inf}}\, \mathbb{E}\left[H\left(x_{0:T}^k\right)\middle|\mathcal{F}_t\right] + \gamma Z_{t:t+1}\, \text{AV@R}_{\alpha Z_{t:t+1}}\left(H\left(x_{0:T}^k\right)\middle|\mathcal{F}_t\right) \\ &= \mathbb{E}\left[H\left(x_{0:T}^k\right)\middle|\mathcal{F}_t\right] + \gamma Z_{t:t+1}^k\, \text{AV@R}_{\alpha Z_{t:t+1}^k}\left(H\left(x_{0:T}^k\right)\middle|\mathcal{F}_t\right) \end{aligned}$$

for the particular choice $x_{0:T}^k$ and due to the optimal choice $Z_{t:t+1}^k$ in (20).

Whence, irrespective of the next assignment for Z, the assignment x^{k+1} is better than x^k:

$$\mathcal{U}\left(x_{0:T}^k\right) \le \mathcal{U}\left(x_{0:T}^{k+1}\right) \le \dots \mathcal{V}_0,$$

the algorithm thus produces assignments x^k which sequentially improve the value function at each node specified by $\mathcal{F}_t$ and at each time $t \in \mathbf{T}$.

10. Summary and Outlook

In this paper we introduce the conditional Average Value-at-Risk *at random risk level*. The central result is a decomposition, which allows to reconstruct the Average Value-at-Risk given just the conditional risk observations. For this purpose one has to give up the constant risk level and accept a random risk level, adapted for each partial observation. The random risk level reflects the fact that risk has to be quantified by adapted means, whenever some information already is available.

Further it is the purpose of this paper to compound the ingredients developed and demonstrate, how they can be arranged for stochastic programming: Dynamic programming equations are being developed, which characterize the optimal policy.

Among influential papers and different attempts to get hold of dynamic programming equations for multistage programming are the papers by Römisch and Guiges [RG10], then Shapiro [Sha09], addressing the time consistency aspect, and in particular [ADE+07], addressing Bellman's principle.

We believe that the nested decomposition, which we have derived for the Average Value-at-Risk, can be extended to more general risk measures. Although there are some concepts to define the conditional versions of a risk measure in *general*, it seems difficult do describe or characterize their conditional versions in such way, that they can be compounded in some way to represent again the initial, general acceptability functional: For the average value at risk the conditional version is an Average Value-at-Risk again, but for other risk measures their shape considerably differs. Do polyhedral risk measures (cf. [ER05]) allow a reasonable decomposition? We take these observations and questions as a main intention and driver for future and further investigations on the subject.

A further discussion of approximations for the problem (14) with relation to the specific nested distance, which was introduced in [Pfl09], discussed in [PP11a] and properly elaborated in [PP11b], needs a clarification and is of particular interest for us as well.

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AppendixA. The "ω by ω" formulation of the decomposition

Disintegration And the Nesting Property

The random variable Z_T minimizing (9) was observed in the proof of the decomposition theorem (Theorem 10) to naturally decompose into

$$Z_T = Z_t \cdot Z_{t:T} \tag{A.1}$$

where $Z_t = \mathbb{E}\left[Z_T|\mathcal{F}_t\right] \lhd \mathcal{F}_t$ and $Z_{t:T} = \frac{Z_T}{\mathbb{E}[Z|\mathcal{F}_t]} \lhd \mathcal{F}_T$.

Suppose that $\mathcal{F}_t$ results from a continuous projection, then one may apply the disintegration theorem (cf. [AGS05] or [DM88]) to obtain the decomposition

$$\mathbb{P}\left[\,\cdot\,\right] = \int \mathbb{P}_{t:T}\left[\,\cdot\,|\omega_t\right]\mathbb{P}_t\left[\mathrm{d}\omega_t\right] \tag{A.2}$$

of the measure $\mathbb{P}$. The pushforward measure $\mathbb{P}_t$ is a measure on $\mathcal{F}_t$, and the map $\omega_t \mapsto \mathbb{P}_{t:T}\left[A|\omega_t\right]$ is $\mathcal{F}_t-$measurable for all $A \in \mathcal{F}_T$.

Combing both decompositions (A.1) and (A.2), the decomposition (9) rewrites as

$$\mathrm{AV@R}_\alpha\left(Y\right) = \int Z_t\left(\omega_t\right)\int Y\left(\omega\right)Z_{t:T}\left(\omega\right)\mathbb{P}_{t:T}\left[\mathrm{d}\omega|\omega_t\right]\,\mathbb{P}_t\left[\mathrm{d}\omega_t\right].$$

This gives evidence to rewrite the conditional Average Value-at-Risk as

$$\mathrm{AV@R}_{\alpha Z_t(\omega_t)}\left(Y|\omega_t\right) = \mathbb{E}_{\omega_t}\left[Z_{\omega_t}\left(\omega_T\right)Y\left(\omega_T\right)\right]$$

in an obvious "ω *by* ω" notation, such that

$$\mathrm{AV@R}_\alpha\left(Y\right) = \min_{Z(\cdot|\omega_0)\in\mathrm{AV@R}^*_\alpha(\omega_0)}\mathbb{E}\left[Z\left(\omega_t|\omega_0\right)\cdot\min_{Z(\cdot|\omega_t)\in\mathrm{AV@R}^*_{\alpha\cdot Z(\omega_t|\omega_0)}(\omega_t)}\mathbb{E}_{\omega_t}\left[Z\left(\omega_T|\omega_t\right)\cdot Y\left(\omega_T\right)\right]\right]$$

where $\mathrm{AV@R}^*_\alpha\left(\omega_t\right) := \left\{Z\left(.|\omega_t\right):\ \mathbb{E}_{\omega_t}Z\left(.|\omega_t\right) = 1,\ 0 \le Z\left(.|\omega_t\right) \text{ and } \alpha Z\left(.|\omega_t\right) \le 1\right\}$.

In order to provide the relations in form of a dynamic program one may define

$$\begin{aligned}\mathscr{V}_T\left(\omega_T,\alpha_T\right) &:= Y\left(\omega_T\right) \text{ and}\\ \mathscr{V}_t\left(\omega_t,\alpha_t\right) &:= \min_{Z\in\mathrm{AV@R}^*_{\alpha_t}(\omega_t)}\mathbb{E}_{\omega_t}\left[Z\left(\omega_T\right)\cdot\mathscr{V}_T\left(\omega_T,\alpha_t\left(\omega_t\right)\cdot Z\left(\omega_T\right)\right)\right],\end{aligned} \tag{A.3}$$

in a backwards recursive way such that

$$\mathrm{AV@R}_\alpha\left(Y\right) = \mathscr{V}_0\left(\omega_0,\alpha\right) = \min_{Z\in\mathrm{AV@R}^*_\alpha(\omega_0)}\mathbb{E}\left[Z\left(\omega_t\right)\cdot\mathscr{V}_t\left(\omega_t,\alpha\cdot Z\left(\omega_t\right)\right)\right],$$

as in (A.3).

This is a recursive formulation for the value function $\mathscr{V}$, which is in line with the principles of dynamic programming.

Tower Property and Dynamic Programming

Given not just two sigma algebras, but a finite sequence $\mathcal{F} := \left(\mathcal{F}_t\right)_{t\in\mathbf{T}}$ ($\mathbf{T} = \{t_1 = 0,\dots t_n = T\}$) with the property $\mathcal{F}_t \subset \mathcal{F}_\tau$ ($t < \tau$, a filtration), one may apply the ingredients developed above successively to obtain the recursive – or nested – representation

$$\begin{aligned}\mathrm{AV@R}_\alpha\left(Y\right) = &\int Z_{t_1}\left(\omega_{t_1}\right)\int Z_{t_2}\left(\omega_{t_2}\right)\dots\\ &\int Y\left(\omega\right)Z_{t:T}\left(\omega\right)\mathbb{P}_{t_n}\left[\mathrm{d}\omega_{t_n}|\omega_{t_{n-1}},\dots\omega_{t_1}\right]\dots\mathbb{P}_{t_2}\left[\mathrm{d}\omega_{t_2}|\omega_{t_1}\right]\mathbb{P}_{t_1}\left[\mathrm{d}\omega_{t_1}\right].\end{aligned}$$

This representation is of particular interest if the probabilities are given as trees, as indicated in the Figures 3 and 2.

Again this reduces to the recursive settings

$$\begin{aligned} \mathscr{V}_T\left(\omega_T, \alpha_T\right) &:= Y\left(\omega_T\right) \\ \mathscr{V}_t\left(\omega_t, \alpha_t\right) &:= \min_{Z \in \mathrm{AV@R}^*_{\alpha_t}} \mathbb{E}_{\omega_t}\left[Z\left(\omega_{t+1}\right) \cdot \mathscr{V}_{t+1}\left(\omega_{t+1}, \alpha_t Z\left(\omega_{t+1}\right)\right)\right] \end{aligned}$$

as above such that

$$\mathrm{AV@R}_\alpha\left(Y\right) = \mathscr{V}_0\left(\omega_0, \alpha\right) := \min_{Z \in \mathrm{AV@R}^*_\alpha} \mathbb{E}\left[Z\left(\omega_t\right) \cdot \mathscr{V}_t\left(\omega_t, \alpha \cdot Z\left(\omega_t\right)\right)\right].$$

The definition of $\mathcal{V}_t$ just involves the next $\mathcal{V}_{t+1}$ and not later times, which is again in line with dynamic programming.