Topological analysis for consistent initialization in circuit simulation

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Abstract

One of the difficulties of the numerical integration methods for differential-algebraic equations (DAEs) is the computation of consistent initial values before starting the integration, i.e., to calculate values that satisfy the given algebraic constraints as well as the hidden constraints if higher index problems are considered.
This paper presents an algorithm that permits the consistent initialization of index 1 or 2 DAE-systems resulting from electric circuit simulation by means of modified nodal analysis (MNA). The presented approach arises from the topological properties of the network and holds for circuits that may contain some specific controlled sources.
The article starts up from [8]. Several denotations and results we use were introduced there in greater detail.

Key words: Consistent initial values; consistent initialization; differential-algebraic equation; DAE; index; circuit simulation; modified nodal analysis; MNA; structural properties; graph theory.

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1 Introduction

For ordinary differential equations initial values have to be prescribed for all variables in order to determine a uniquely defined solution. Since differential-algebraic systems consist precisely of differential equations coupled with algebraic equations, not all components appear in dynamic form. Indeed, some of them are determined by algebraic constraints. In Section 2 we briefly introduce the problems related to this fact for linear systems. The definition of consistent initial values for nonlinear systems is presented then in Section 3. For general nonlinear systems, there are many difficulties related to the determination of consistent initial values that have been handled making use of different approaches, cf. Pantelides (1988) , Leimkuhler (1991), Hansen (1992), Kröner et al. (1996), Lamour (1997), Gopal et al. (1999), etc. Fortunately, the equations obtained in circuit simulation by means of MNA present special structural properties which we can make use of to compute them. These equations are introduced in Section 4. In Section 5 we present an algorithm to construct a nonlinear system whose solution is a consistent initial value. This algorithm is based on a topological analysis of the network. To prove its correctness, it is necessary to realize a detailed analysis of the structural properties. Hence, in Section 6, we summarize the structural results obtained in [8] and extend them whith regard to consistent initialization. Thereupon, in Section 7, the equations are transcribed topologically. We conclude the article illustrating the algorithm by means of two examples in Section 8.

2 Considering linear DAEs

To briefly outline the difficulties associated with the computing of consistent initial values for differential-algebraic equations (DAEs), we present the tractability index for linear DAEs with constant coefficients and an approach for consistent initialization, which makes use of the projectors associated with it.

Consider a linear DAE of the form:

$$Ax' + Bx = q(t), \tag{2.1}$$

where A is singular. For the tractability index we define $N := \ker A$ and $S := \{z : Bz \in \operatorname{im} A\}$.

Definition 2.1 *A vector $x_0 \in R^m$ is a consistent initial value of (2.1) if there exists a solution of (2.1) that fulfils $x(t_0) = x_0$.*

Taking into account that the singularity of A implies that (2.1) contains some algebraic equations, a consistent initial value obviously has to fulfil them. Moreover, the differentiation of these algebraic equations can lead to further algebraic equations, called hidden constraints, which a consistent initial value has to fulfil, too. The following index definitions characterize this possibilities properly.

Definition 2.2 *The DAE (2.1) is called index-1-tractable* [1] *if the matrix $A_1 := A + BQ$ is regular for a constant projector Q onto N.*

Remarks:

1. The matrix A_1 is regular if and only if $N \cap S = \{0\}$.
2. The definition does not depend on the choice of the projector Q.

For the next definition we consider $N_1 := \ker A_1$ and $S_1 := \{z : BPz \in \operatorname{im} A_1\}$ for $P := (I - Q)$.

Definition 2.3 *The DAE (2.1) is called index-2-tractable*[2] *if*

[1]cf. [13].
[2]cf. [13].

1. *it is not index-1-tractable,*

2. $A_2 := A_1 + BPQ_1$ *is regular for a projector* Q_1 *onto* N_1.

Remarks:

1. The matrix A_2 is regular if and only if $N_1 \cap S_1 = \{0\}$.

2. The definition does not depend on the choice of the projector Q_1.

Definition 2.4 *In the following, the canonical projector* $Q_1 := Q_1 A_2^{-1} BP$ *onto* N_1 *along* S_1 *is considered.*

Note that this matrix chain can be continued to define higher index problems. For index-2 equations hidden constraints appear if we derive some of the equations of the system. This implies that an initial value has to be chosen in such a way that not only the system's equations, but, additionally, these hidden constraints have to be fulfilled.

With regard to the consistent initial value, we are also interested in the corresponding values of the derivatives appearing in the DAE, i. e., in Px'.

Definition 2.5 *A vector* (x_0, Py_0) *is a consistent initialization of (2.1) if* x_0 *is a consistent initial value and* (x_0, Py_0) *fulfils* $APy_0 + Bx_0 = q(t_0)$.

If index-1 DAEs are considered, then a consistent initialization can be calculated solving the system

$$\begin{aligned} Ay_0 + Bx_0 &= q(t_0), && (2.2)\\ P(x_0 - x^0) + Qy_0 &= 0, && (2.3) \end{aligned}$$

for an arbitrary x^0.

Note that $P(x_0 - x^0) = 0$ fixes values for the dynamic components and that $Qy_0 = 0$ precisely fixes the values we are not interested in, obtaining a regular system.

An efficient approach[3] to calculate consistent initial values at t_0 in the index-2 case consists in solving the system:

$$\begin{aligned} Ay_0 + Bx_0 &= q(t_0), && (2.4)\\ PP_1(x_0 - x^0) + PQ_1(y_0 - A_2^{-1} q'(t_0)) + Qy_0 &= 0, && (2.5) \end{aligned}$$

for $P_1 := I - Q_1$ and an arbitrary x^0.

Note that $PP_1(x_0 - x^0) = 0$ fixes values for the dynamic components, $PQ_1(y_0 - PQ_1 A_2^{-1} q'(t_0)) = 0$ describes the hidden constraints, and $Qy_0 = 0$ precisely

[3] cf. [20] ,[17]. Note that this approach can be extended to some nonlinear cases.

fixes the values we are not interested in, obtaining a regular system.

The results presented in [8] imply that for nonlinear circuits the projectors PP_1 and PQ_1 depend on the solution. In this paper we will set up a system similar to (2.4)-(2.5) with the aid of constant projectors, making use of the fact that the space S_1 is constant for the conventional MNA. The results can be adapted, afterwards, for the charge-oriented MNA.
Furthermore, the equations obtained in that way can be interpreted topologically. For instance, as the projector Q is known for the MNA equations, we know that, for certain components, we need not to calculate a value for y. Moreover, topological procedures that locate the hidden constraints and provide the information needed to fix values for the dynamic component will be developed. By means of these topological considerations we finally aim at constructing a system of equations that provides a consistent initialization and is, in general, considerably smaller than (2.4) -(2.5). This is of special importance because of the large dimension of many circuits (about 10^7 cirucit elements). In the present paper we will not consider how to choose a suitable x^0. This will be analyzed in [6].

3 About consistent initial values for nonlinear DAEs

Let us consider now more general differential-algebraic equations of the form

$$f(x', x, t) = 0, \tag{3.1}$$

where $\frac{df}{dx'}$ is singular. In this article, $\frac{df}{dx'}$ is assumed to have a constant nullspace.

Definition 3.1 *A vector $x_0 \in R^m$ is a consistent initial value of (3.1) if there exists a solution of (3.1) that fulfils $x(t_0) = x_0$.*

Again, the singularity of $\frac{df}{dx'}$ implies difficulties with regard to the calculation of consistent initial values, which can be described analogously as for the linear case.
Actually, we are also interested in the corresponding values of the derivatives appearing in the DAE, i.e., in the values of Px' if P is defined as $P := I - Q$ for a projector Q onto $ker \frac{df}{dx'}$.

Definition 3.2 *A vector (x_0, Py_0) is a consistent initialization of (3.1) if x_0 is a consistent initial value and (x_0, Py_0) fulfils the equation $f(Py_0, x_0, t_0) = 0$.*

In the following we will analyze how a consistent initialization can be computed for those DAEs that result in circuit simulation by modified nodal analysis (MNA). The index definitions and properties for the equations obtained by MNA have been exposed in detail in [8].

4 The MNA equations

Let us analyze the DAE system we obtain by the application of the MNA from lumped networks containing nonlinear and possibly time-variant resistances, capacitances, inductances, independent voltage and current sources, and some specific controlled sources.

We denote by q and ϕ the vectors of the charge associated with the capacitances and the fluxes associated with the inductances, by j_L and j_V the current vector of inductances and voltage sources and by e the vector of node potentials.
Let the vectors $i(\cdot)$, and $v(\cdot)$ represent functions of current and voltage sources. In this paper, for the controlled sources we assume the prerequisites from [8], which are exposed in the Tables 6.1 and 6.2.

In order to describe the network composed by branches and nodes we make use of the reduced incidence matrix, which is defined by

$$a_{ik} := \begin{cases} +1 & \text{if branch k leaves node } i \\ \text{-1} & \text{if branch k enters node } i \\ 0 & \text{if branch } k \text{ is not incident with node } i \end{cases}$$

for all the nodes i but the datum node (cf. [5]).

To write down the MNA[4] equations, we split the reduced incidence matrix A into the element-related incidence matrices $A = (A_C A_L A_R A_V A_I)$, where A_C, A_L, A_R, A_V, and A_I describe the branch-current relation for capacitive branches, inductive branches, resistive branches, branches of voltage sources and branches of current sources, respectively.

If we define

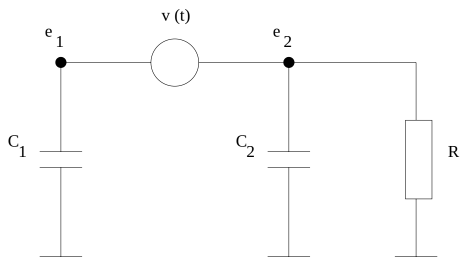
$$C(u,t) := \frac{\partial q(u,t)}{\partial u},\ q_t'(u,t) := \frac{\partial q(u,t)}{\partial t},\ L(j,t) := \frac{\partial \phi(j,t)}{\partial j},\ \phi_t'(j,t) := \frac{\partial \phi(j,t)}{\partial t},$$

the DAE system we obtain from networks by the conventional MNA reads

$$\begin{aligned} A_C C(A_C^T e,t) A_C^T \frac{de}{dt} + A_C q_t'(A_C^T e,t) + A_R r(A_R^T e,t) & & \\ + A_L j_L + A_V j_V + A_I i(\cdot) &= 0, & (4.1) \\ L(j_L,t)\frac{dj_L}{dt} + \phi_t'(j_L,t) - A_L^T e &= 0, & (4.2) \\ A_V^T e - v(\cdot) &= 0. & (4.3) \end{aligned}$$

Later on we will also need $G(u,t) := \frac{\partial r(u,t)}{\partial u}$.

[4] A detailed discussion on how we set up this equations can be found in [8].



Conventional MNA:

$$\begin{aligned} C_1 e_1' + j_V &= 0, \\ -j_V + C_2 e_2' + \frac{1}{R} e_2 &= 0, \\ e_1 - e_2 &= v(t_0). \end{aligned}$$

Figure 4.1: Circuit with C-V loop

In this article we first analyze the network with respect to the conventional MNA and, afterwards, extend the results to the systems obtained by charge-oriented MNA. These systems are (cf. again [8]):

$$A_C \frac{dq}{dt} + A_R r(A_R^T e, t) + A_L j_L + A_V j_V + A_I i(\cdot) = 0, \tag{4.4}$$

$$\frac{d\phi}{dt} - A_L^T e = 0, \tag{4.5}$$

$$A_V^T e - v(\cdot) = 0, \tag{4.6}$$

$$q - q_C(A_C^T e, t) = 0, \tag{4.7}$$

$$\phi - \phi_L(j_L, t) = 0. \tag{4.8}$$

Analogously to [8], we suppose that the capacitance matrix $C(A_C^T e, t)$, the inductance matrix $L(j_L, t)$, and the conductance matrix $G(A_R^T e, t)$ of all capacitances, inductances and resistances, respectively, are positive definite [5].
We will also make use of the fact that the reduced incidence matrix $(A_C A_L A_R A_V)$ has full row rank and that A_V has full column rank, because cutsets of current sources only and loops of voltage sources only are forbidden (cf. [23] , [8]).

Example: In Figure 4.1 we consider an index-2 example. The dynamical components seem to be e_1 and e_2. With regard to a consistent initialization, the problem arises from the fact that we cannot prescribe an arbitrary initial value for e_1 and e_2 . If we assign a value to one of them, say e_{10}, then the other one, i. e. e_{20}, is fixed by $e_{20} = e_{10} - v(t_0)$.

The problem becomes more complicated if many capacitances and voltage sources form several loops. The aim of this article is to develop topological criteria to fix values for the dynamic component and determine the hidden constraints in order to calculate a consistent initial value.

[5] Of course, the same restriction on the positive definiteness of the conductance matrix from Corollary 2.2 of [8] can be made here. Therefore, for the resistances with incidence nodes that are connected to each other by capacitances and/or voltage sources, no positive definiteness of the corresponding conductance matrix has to be assumed.

5 The resulting topological initialization

To characterize the structures of the network that are relevant for index-2 systems we introduce the following denotations.

Definition 5.1

1. *An* **L-I cutset** *is a cutset consisting of inductances and/or current sources only.*

2. *A* **C-V loop** *is a loop consisting of capacitances and voltage sources.*

3. *Consider the graph remaining when we delete all the branches except for those containing capacitances and all nodes that are not incident with capacitances. We call each of the connected subgraphs of the network obtained in this way a* **C-subgraph**. *Note that by this definition each C-subgraph is connected and the distinct C-subgraphs are disjoint.*

4. *Consider, analogously, the graph that remains when we delete all branches except for those containing capacitances, voltage sources or resistances, and the corresponding nodes. A* **CRV-subgraph** *is a connected subgraph obtained in this way.*

5. *We say that a C-subgraph forms a part of a C-V loop if one of the capacitances of that C-subgraph forms a part of a C-V loop.*

In [8] it was shown that the existence of L-I cutsets or C-V loops implicates index-2 systems (cf. Theorem 6.4). These two network properties affect, consequently, the determination of consistent initial values. The main result of this article is summarized in the following theorem.

Theorem 5.2 *We obtain a consistent initialization* (x_0, Py_0) *of the DAE system (4.1) - (4.3) for networks that may contain controlled sources like the ones specified in the Tables 6.1 and 6.2 solving the system consisting of:*

1. *the original DAE system*

2. *the equations obtained by the PROCEDURES 1 and 2*

3. *an arbitrary setting of the potential difference between each uncoloured node (uncoloured in the sense of PROCEDURE 1) and the corresponding minimal node of the C-subgraph which the node belongs to. For the nodes of the C-subgraph that contains the datum node, we set the node potentials.*

4. *an arbitrary setting for the currents through all inductances but the ones deleted in PROCEDURE 2.*

Theorem 5.3 *The values obtained in Theorem 5.2 can be used to determine a consistent initialization for the DAE system (4.4) - (4.8) calculating additionally the values of the charges and fluxes across capacitances and inductances* *that correspond to the values of the node potentials and currents through inductances, respectively, and computing the values of its corresponding derivatives analogously.*

The following two procedures analyze the structure of a given circuit with regard to the two topological properties described in Theorem 6.4, which lead to index-2 systems. Two examples illustrating them can be found in Section 8.

PROCEDURE 1

1. We find the independent C-V loops as follows:
   (a) Consider a C-V loop in the given network graph. If no loop is found, then go to 2 .
   (b) Save the voltage sources and the C-subgraphs that form part of the chosen C-V loop.
   (c) Write the equation resulting from the sum of the derivatives of the characteristic equations of the voltage sources contained in the C-V loop, taking into account the orientation of the loop and the reference direction of the considered branches.
   For instance, if the voltage sources $v_1, ..., v_k$ form part of the C-V loop and we define

   $$\alpha_j := \begin{cases} +1 & \text{if the orientation of the loop coincides with that of } v_j, \\ \text{-1} & \text{else,} \end{cases}$$

   then the equation we write in this step is $\sum_{j=1}^{k} \alpha_j((A_V^T e)'_j - v'_j) = 0$.
   (d) Form a new network graph by deleting the branch of one voltage source that forms a part of the loop, leaving the nodes unchanged.
   (e) Return to 1a, considering the new network graph.

2. To fix appropriate values in Theorem 5.2 we analyze again the C-subgraphs and the voltage sources that form part of the C-V loops more precisely:
   (a) Colour all nodes that are not incident with capacitances.
   (b) Colour one arbitrary node of each C-subgraph. But if a C-subgraph contains the datum node, then the datum node is coloured. We call each of these nodes the minimal node of the corresponding C-subgraph[6].
   (c) Denote by G the subgraph of the network composed only by the voltage sources that form part of the C-V loops. Note that G need not to be connected.
   (d) Consider a C-subgraph that forms part of a C-V loop.

[6] The notion of a minimal node will become clear in the course of the article. We need such a reference node because, in fact, we set branch voltages, not node potentials.

(e) If it forms different loops with G, then we consider only C-V loops that enter the C-subgraph once and proceed as follows:

   i. Check if the minimal node or an already coloured node is incident with G, forming a C-V loop. If such a loop is found, then we consider it first. Otherwise we consider an arbitrary one.

   ii. Look at the nodes of the loop at which G is incident with the C-subgraph and color one of them which is neither the minimal node nor has been colored before. Delete from G the voltage source of the loop that is incident with the node colored last.

   iii. If the C-subgraph forms further loops with G, then go to 2(e)i.

(f) Add to G the C-subgraph considered in the two previous steps.

(g) Consider a C-subgraph that forms a part of C-V loops and has not been considered yet, and go to 2e. If no such C-subgraph exists, then end.

The following procedure starts up again from the initial graph.

PROCEDURE 2

1. Search an L-I cutset. If one is found, then pick an arbitrary inductance of this cutset. Realize that we can always find such an inductance because cutsets of current sources only are forbidden. If no cutset is found, then end.

2. Write the equation resulting from the derivation of the cutset equation arising from 1.
   For instance, if the current sources $i_1, ..., i_k$ and the inductances $j_{L1}, ..., j_{L\tilde{k}}$ form part of the L-I cutset and we define

$$\alpha_j \quad := \quad \begin{cases} +1 & \text{if the orientation of the cutset coincides with that of } i_j, \\ \text{-1} & \text{else,} \end{cases}$$
$$\tilde{\alpha}_{\tilde{j}} \quad := \quad \begin{cases} +1 & \text{if the orientation of the cutset coincides with that of } j_{L\tilde{j}}, \\ \text{-1} & \text{else,} \end{cases}$$

   then the equation obtained in this step reads $\sum_{j=1}^{k} \alpha_j i'_j + \sum_{\tilde{j}=1}^{\tilde{k}} \tilde{\alpha}_{\tilde{j}} j'_{L\tilde{j}} = 0$.

3. Delete the chosen inductance from the network contracting the nodes it is incident with.

4. Return to 1.

In the next section we present the theoretical results which the above procedures are based on.

6 Index and consistent initialization for MNA

6.1 Some definitions and results

In the following we present some of the results from [8] concerning the index of the DAE system and the expressions for the hidden constraints in terms of appropriate projectors. To this end, we need the following definitions and results.

Definition 6.1 *To characterize the topological properties of the network, we define the projectors Q_C, Q_{V-C}, $\bar{Q}_{V-C}$ and Q_{R-CV} onto $kerA_C^T$, $kerA_V^T Q_C$, $kerQ_C^T A_V$, and $kerA_R^T Q_C Q_{V-C}$, respectively.*
Note that $Q_{CRV} := Q_C Q_{V-C} Q_{R-CV}$ is a projector onto $ker(A_C A_R A_V)^T$.
The complementary projectors will be denoted by $P_* := I - Q_*$, with the corresponding subindices.

In [23] the following was shown to hold:

Lemma 6.2
1. If the network does not contain L-I cutsets, then $Q_{CRV} = 0$.
2. If the network does not contain C-V loops, then $\bar{Q}_{V-C} = 0$.

In this article, we suppose that the controlled sources that form part of the network fulfil the conditions exposed in [8]. We summarize their properties in the Tables 6.1 and 6.2.

If we consider the element-related splitting of $\bar{Q}_{V-C}$, i. e.,

$$\bar{Q}_{V-C} = \begin{pmatrix} (\bar{Q}_{V-C})_t \\ (\bar{Q}_{V-C})_{contr.} \end{pmatrix},$$

then we can summarize the prerequisites we assume for the controlled voltage sources as follows:

$$\bar{Q}_{V-C}^T v(A^T e, \frac{dq(A_C^T e, t)}{dt}, j_L, j_V, t) = \bar{Q}_{V-C}^T v_t(t), \tag{6.1}$$

$$v(A^T e, \frac{dq(A_C^T e, t)}{dt}, j_L, j_V, t) = v_*(A_C^T e, j_L, t) \tag{6.2}$$

for a suitable function v_* and for a vector $v_t(t)$ that contains the functions of independent voltage sources and zeros instead of the functions of controlled voltage sources. Analogously as in [8], in the following we will drop the index *.

Table 6.1: Condition for controlled voltage sources

For controlled current sources we suppose that at least one of the following characterizations holds:

(a)

$$Q_{CRV}^T A_I i(A^T e, \frac{dq(A_C^T e,t)}{dt}, j_L, j_V, t) = Q_{CRV}^T A_{It} i_t, \tag{6.3}$$

$$i(A^T e, \frac{dq(A_C^T e,t)}{dt}, j_L, j_V, t) = i_a(A_C^T e, A_V^T e, j_L, t) \tag{6.4}$$

for a suitable function i_a.

(b)

$$Q_C^T A_{Ib} = 0, \tag{6.5}$$

$$i(A^T e, \frac{dq(A_C^T e,t)}{dt}, j_L, j_V, t) = i_b(A^T e, j_L, \bar{P}_{V-C} j_V, t) \tag{6.6}$$

for a suitable function i_b.

(c)

$$Q_{V-C}^T Q_C^T A_{Ic} = 0, \tag{6.7}$$

$$i(A^T e, \frac{dq(A_C^T e,t)}{dt}, j_L, j_V, t) = i_c(A^T e, j_L, t) \tag{6.8}$$

for a suitable function i_c.

Table 6.2: Conditions for controlled current sources

Regarding (6.3), (6.5), and (6.7), the assumptions made for the controlled current sources imply that

$$Q_{CRV}^T A_I i(A^T e, j_L, j_V, t) = Q_{CRV}^T A_{It} i_t \tag{6.9}$$

is always fulfilled. Thus, we generally assume that the controlled sources do not form part of the C-V loops or L-I cutsets.
To shorten denotations we write

$$i(A^T e, j_L, \bar{P}_{V-C} j_V, t) \tag{6.10}$$

when we do not distinguish between (6.4), (6.6) and (6.8). Furthermore, we will also write a dot instead of the argument sometimes.

Lemma 6.3 *The matrices*

$$
\begin{aligned}
H_1(A_C^T e,t) &:= A_C C(A_C^T e,t) A_C^T + Q_C^T Q_C,\\
H_2 &:= Q_C^T A_V A_V^T Q_C + Q_{V-C}^T Q_{V-C},\\
H_3 &:= A_V^T Q_C Q_C^T A_V + \bar{Q}_{V-C}^T \bar{Q}_{V-C},\\
H_4(A_C^T e,t) &:= \bar{Q}_{V-C}^T A_V^T H_1^{-1} A_V \bar{Q}_{V-C} + \bar{P}_{V-C}^T \bar{P}_{V-C},\\
H_5(j_L,t) &:= Q_{CRV}^T A_L L^{-1}(j_L,t) A_L^T Q_{CRV} + P_{CRV}^T P_{CRV},\\
H_6 &:= \bar{Q}_{V-C}^T A_V^T A_V \bar{Q}_{V-C} + \bar{P}_{V-C}^T \bar{P}_{V-C},\\
H_7 &:= Q_{CRV}^T A_L A_L^T Q_{CRV} + P_{CRV}^T P_{CRV}
\end{aligned}
$$

are regular.

The regularity of $H_1 - H_5$ has already been proved in [8]. The proof for H_6 and H_7 is analogous.

In [8] we obtained the following result.

Theorem 6.4 *Consider lumped electric circuits satisfying the assumptions of the Tables 6.1 and 6.2. Then it holds:*

1. *For the conventional MNA:*

   (a) *If the network contains neither L-I cutsets nor controlled C-V loops, then the conventional MNA leads to an index-1 DAE and the constraints are only the explicit ones:*

$$
\begin{aligned}
Q_C^T \Big[A_R r(A_R^T e,t) + A_L j_L + A_V j_V + \\
A_{Ia,c} i_{a,c}(A^T e, j_L, t) \Big] &= 0, \qquad (6.11)\\
A_V^T e - v(A_C^T e, j_L, t) &= 0. \qquad (6.12)
\end{aligned}
$$

   (b) *If the network contains L-I cutsets or C-V loops, then the conventional MNA leads to an index-2 DAE. With regard to the constraints, we distinguish the following three possibilities.*

      i. *If the network does not contain an L-I cutset (but contains controlled C-V loops), then the constraints are the explicit ones, (6.11) and (6.12), and, additionally, the hidden constraint:*

$$
\begin{aligned}
\bar{Q}_{V-C}^T A_V^T H_1^{-1}(A_C^T e,t) P_C^T \Big[A_C q_t'(A_C^T e,t) + A_R r(A_R^T e,t) + A_L j_L \\
+ A_V j_V + A_I i(A^T e, j_L, \bar{P}_{V-C} j_V, t) \Big] + \bar{Q}_{V-C}^T \frac{dv_t}{dt} = 0. (6.13)
\end{aligned}
$$

ii. *If the network does not contain controlled C-V loops, but contains L-I cutsets, the constraints are the explicit ones, (6.11) and (6.12), and, additionally, the hidden constraint:*

$$Q_{CRV}^T \left(A_L L^{-1}(j_L, t) \left(A_L^T e - \phi_t'(j_L, t) \right) + A_{I_t} \frac{di_t}{dt} \right) = 0. \quad (6.14)$$

iii. *If the network contains L-I cutsets and C-V loops, then the constraints are the explicit ones, (6.11) and (6.12), and the hidden ones (6.13) and (6.14).*

2. *For the charge-oriented MNA the explicit constraints are (6.11), (6.12), (4.7) and (4.8). The hidden constraints are (6.13) and (6.14), appearing under the same topological conditions as considering the conventional MNA. The index statements coincide for the conventional and the charge-oriented MNA.*

For the sake of simplicity, we will sometimes drop the arguments of the matrices H in the following and write a dot if they are not constant.

6.2 The conventional MNA

Let us now analyze the consequences Theorem 6.4 for the consistent initialization.

Corollary 6.5 *Assume that the network contains controlled sources like the ones described in the Tables 6.1 and 6.2. Let (e^0, j_L^0, j_V^0) be an arbitrary initial vector. Then we can fix values for the dynamic component of a consistent initial vector (e_0, j_{L0}, j_{V0}) for the system (4.1)-(4.3) in the following way:*

$$\left(P_C - P_C A_V \bar{Q}_{V-C} H_6^{-1} \bar{Q}_{V-C}^T A_V^T \right) \left(e_0 - e^0 \right) = 0, \quad (6.15)$$

$$\left(I - A_L^T Q_{CRV} H_7^{-1} Q_{CRV}^T A_L \right) \left(j_{L0} - j_L^0 \right) = 0. \quad (6.16)$$

Note that, if the network does not contain C-V loops, then $\bar{Q}_{V-C} \equiv 0$ holds and equation (6.15) reads $P_C(e_0 - e^0) = 0$. Correspondingly, if the network does not contain L-I cutsets, then $Q_{CRV} \equiv 0$ and equation (6.16) reads $j_{L0} - j_L^0 = 0$.

Corollary 6.6 *If the network contains controlled sources that fulfil the conditions of the Tables 6.1 and 6.2, we can, according to Corollary 6.5, gradually determine consistent initial values for the system (4.1)-(4.3).*
Considering the splitting $e_0 = P_C e_0 + Q_C P_{V-C} e_0 + Q_C Q_{V-C} P_{R-CV} e_0 + Q_{CRV} e_0$ and $j_{V0} = \bar{Q}_{V-C} j_{V0} + \bar{P}_{V-C} j_{V0}$, we obtain a consistent initial value as follows:

$$\begin{aligned}
P_C e_0 &:= P_C e^0 + P_C A_V \bar{Q}_{V-C} H_6^{-1} \bar{Q}_{V-C}^T \left(v_t(t_0) - A_V^T P_C e^0 \right), \\
j_{L0} &:= j_L^0 + A_L^T Q_{CRV} H_7^{-1} Q_{CRV}^T \left(-A_{I_t} i_t(t_0) - A_L j_L^0 \right), \\
Q_C P_{V-C} e_0 &:= Q_C H_2^{-1} Q_C^T A_V \bar{P}_{V-C}^T \left(-A_V^T P_C e_0 + v(A_C^T e_0, j_{L0}, t_0) \right),
\end{aligned}$$

whereas the value of $Q_CQ_{V-C}P_{R-CV}e_0$ can be obtained by solving the equation

$$P_{R-CV}^TQ_{V-C}^TQ_C^T\Big[A_Rr(A_R^T(P_C+Q_CP_{V-C}+Q_CQ_{V-C}P_{R-CV})e_0)+ \\ A_Lj_{L0}+A_Ii(A_C^Te_0,A_V^Te_0,j_{L0},t_0)\Big]=0.$$

With these values we can then figure out gradually that

$$\begin{aligned}
Q_{CRV}e_0 &:= -\left(Q_{CRV}H_5^{-1}(\cdot)Q_{CRV}^T\right)\cdot \\
&\quad \Big(A_LL^{-1}(j_{L0},t_0)A_L^T\left(P_C+Q_CP_{V-C}+Q_CQ_{V-C}P_{R-CV}\right)e_0 \\
&\quad -A_LL^{-1}(j_{L0},t_0)\phi_t'(j_{L0},t_0)+A_{I_t}\frac{di_t}{dt}(t_0)\Big), \\
\bar{P}_{V-C}j_{V0} &:= -H_3^{-1}A_V^TQ_CP_{V-C}^TQ_C^T\left[A_Rr(A_R^Te_0,t_0)\right. \\
&\quad \left.+A_Lj_{L0}+A_Ii(A^Te_0,j_{L0},t_0))\right], \\
\bar{Q}_{V-C}j_{V0} &:= -H_4^{-1}(\cdot)\bar{Q}_{V-C}^TA_V^TH_1^{-1}(\cdot)P_C^T\Big(A_C^Tq_t'(A_C^Te_0,t_0) \\
&\quad +A_Rr(A_R^Te_0,t_0)+A_Lj_{L_0}+A_V\bar{P}_{V-C}j_{V_0} \\
&\quad A_Ii(A^Te_0,j_{L0},\bar{P}_{V-C}j_{V0},t_0)\Big)-H_4^{-1}(\cdot)\bar{Q}_{V-C}^T\frac{dv_t}{dt}(t_0).
\end{aligned}$$

Note that each time $H_1^{-1}(\cdot)=H_1^{-1}(A_C^Te_0,t_0)$, $H_4^{-1}(\cdot)=H_4^{-1}(A_C^Te_0,t_0)$ or $H_5^{-1}(\cdot)=H_5^{-1}(j_{L0},t_0)$ appear, we already know the corresponding values $A_C^Te_0$ or j_{L0} and can thus insert them into the expressions. On the other hand, the Tables 6.1 and 6.2 imply that this holds analogously for the controlled sources. Observe further that, if the network contains no C-V loops or no L-I cutsets, then all expressions containing $\bar{Q}_{V-C}$ or Q_{CRV}, respectively, do not appear.

Remark: Corollary 6.6 implies that the choice of (e^0,j_L^0,j_V^0) is arbitrary as long as the nonlinear equation that leads to the expression for $Q_CQ_{V-C}P_{R-CV}e_0$ is solvable.

Proof of Corollary 6.5:
Let us split the constraints as follows:

$$\bar{P}_{V-C}^TA_V^Te=\bar{P}_{V-C}^Tv(\cdot), \tag{6.17}$$

$$\bar{Q}_{V-C}^TA_V^Te=\bar{Q}_{V-C}^Tv_t, \tag{6.18}$$

$$P_{V-C}^TQ_C^T[A_Rr(A_R^Te,t)+A_Lj_L+A_Vj_V+A_Ii(\cdot)]=0, \tag{6.19}$$

$$P_{R-CV}^T Q_{V-C}^T Q_C^T [A_R r(A_R^T e, t) + A_L j_L + A_I i(\cdot)] = 0, \tag{6.20}$$

$$Q_{CRV}^T [A_L j_L + A_{I_t} i_t] = 0, \tag{6.21}$$

$$\bar{Q}_{V-C}^T A_V^T H_1^{-1}(\cdot) P_C^T \Bigg(A_C q_t'(A_C^T e, t) + A_R r(A_R^T e, t) + A_L j_L \\ + A_V j_V + A_I i(\cdot) \Bigg) + \bar{Q}_{V-C}^T \frac{dv_t}{dt} = 0, \tag{6.22}$$

$$Q_{CRV}^T \left(A_L L^{-1}(j_L, t) \left(A_L^T e - \phi_t'(j_L, t) \right) + A_{I_t} \frac{di_t}{dt} \right) = 0. \tag{6.23}$$

Observe that, of course, if the network does not contain L-I cutsets, we have

$$Q_{V-C}^T Q_C^T [A_R r(A_R^T e, t) + A_L j_L + A_I i(\cdot)] = 0$$

instead of equation (6.20) , and the equations (6.23) and (6.21) do not appear. On the other hand, if the network does not contain C-V loops, then (6.17) reads

$$A_V^T e = v(\cdot),$$

whereas (6.18) and (6.22) do not appear.

We observe that each equation leads to a constraint for a particular component of the value (e_0, j_{L0}, j_{V0}), but that not all its components are constrained. Therefore, we have a certain degree of freedom for the choice of a consistent initial vector. In particular, we note that $P_C e_0$ can be chosen arbitrarily, if equation (6.18) does not constrain it. Analogously, j_{L0} is only constrained by (6.21) if this equation appears.

Using the projectors

$$\left(P_C - P_C A_V \bar{Q}_{V-C} H_6^{-1} \bar{Q}_{V-C}^T A_V^T\right) \quad \text{and} \quad \left(I - A_L^T Q_{CRV} H_7^{-1} Q_{CRV}^T A_L\right)$$

we can set the values of $P_C e_0$ and j_{L0}, which can be chosen arbitrarily, precisely. This holds because, on the one hand,

$$\mathrm{im} P_C = \\ \mathrm{im}\left(P_C - P_C A_V \bar{Q}_{V-C} H_6^{-1} \bar{Q}_{V-C}^T A_V^T\right) \oplus \mathrm{im}\left(P_C A_V \bar{Q}_{V-C} H_6^{-1} \bar{Q}_{V-C}^T A_V^T\right)$$

holds. Moreover, since rank $P_C A_V \bar{Q}_{V-C} H_6^{-1} \bar{Q}_{V-C}^T A_V^T$= rank $\bar{Q}_{V-C}$, the relation rank $\left(P_C - P_C A_V \bar{Q}_{V-C} H_6^{-1} \bar{Q}_{V-C}^T A_V^T\right)$= rank P_C - rank $\bar{Q}_{V-C}$ is fulfilled. On the other hand, if n_L denotes the number of inductive branches of the network, we obtain analogously

$$R^{n_L} = \mathrm{im}\left(I - A_L^T Q_{CRV} H_7^{-1} Q_{CRV}^T A_L\right) \oplus \mathrm{im}\left(A_L^T Q_{CRV} H_7^{-1} Q_{CRV}^T A_L\right).$$

Furthermore, since rank $A_L^T Q_{CRV} H_7^{-1} Q_{CRV}^T A_L$ = rank Q_{CRV} is satisfied, rank $\left(I - A_L^T Q_{CRV} H_7^{-1} Q_{CRV}^T A_L\right) = n_L -$ rank Q_{CRV} is given.

q.e.d.

Proof of Corollary 6.6
The representations follow directly from Corollary 6.5, by transformations analogous to those we made in the proof of Theorem 2.1 in [8]. Observe that the controlled sources that may appear do not form part of the C-V loops nor L-I cutsets, by assumption. Therefore, it is easy to verify Corollary 6.6, because, according to the Tables 6.1 and 6.2, the sources lead to expressions on the right-hand side that already have a fixed value thanks to the preceding steps.

q.e.d.

Remark: Let us compare the previous approach to set values for the dynamic component with the one presented in Section 2. To this end, we interpret the setting of values for the dynamic component from Corollary 6.5 as a projector Π of the form:

$$\Pi = \begin{pmatrix} P_C - P_C A_V \bar{Q}_{V-C} H_6^{-1} \bar{Q}_{V-C}^T A_V^T & 0 & 0 \\ 0 & I - A_L^T Q_{CRV} H_7^{-1} Q_{CRV}^T A_L & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Like this, the statements of Corollary 6.5 can be expressed for $x^0 = (e^0, j_L^0, j_V^0)$ and $x_0 = (e_0, j_{L0}, j_{V0})$ in terms of $\Pi(x_0 - x^0) = 0$, in analogy to the expression $PP_1(x_0 - x^0) = 0$ (cf. Section 2).
Analyzing these projectors by making use of the expressions presented in [8], we note that im PP_1 = im Π is satisfied.
Indeed, it holds that $\Pi := P(I - \Omega)$ if Ω is a projector along $\ker(PQ_1) = S_1$.This may be of special interest with regard to projected methods like the one presented in [10].

6.3 The charge-oriented MNA

For the charge-oriented MNA the results from the previous section can be adapted.

Corollary 6.7 *Corollary 6.5 holds for the charge-oriented MNA, too.*

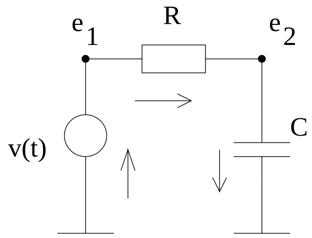
Proof: Theorem 6.4 shows that the constraints that appear for the conventional MNA are constraints for the charge-oriented MNA, too. This implies that we can also set the initial values analogously as we did in Corollary 6.5.

q.e.d.

Corollary 6.8 *The settings of Corollary 6.6 can be realized for the charge-oriented MNA analogously, if we set*

$$\begin{aligned} q_0 &:= q_C(A_C^T e_0, t_0), \\ \phi_0 &:= \phi_L(j_{L0}, t_0) \end{aligned}$$

additionally.



Conventional MNA

$$\begin{aligned} \frac{1}{R}(e_1 - e_2) - j_V &= 0, \\ -\frac{1}{R}(e_1 - e_2) + Ce_2' &= 0, \\ -e_1 - v(t) &= 0. \end{aligned}$$

Fixing a value for the dynamic component:

$$e_{2_0} := e_2^0.$$

Charge-oriented MNA

$$\begin{aligned} \frac{1}{R}(e_1 - e_2) - j_V &= 0, \\ -\frac{1}{R}(e_1 - e_2) + q' &= 0, \\ -e_1 - v(t) &= 0, \\ q - Ce_2 &= 0. \end{aligned}$$

Fixing a value for the dynamic component:

Explicitly: $q_0 := q^0$,

Implicitly: $e_{2_0} := e_2^0$

and, therefore, $q_0 = Ce_2^0$.

Figure 6.1: Fixing a value for the dynamic component for the conventional and the charge-oriented MNA for an index-1 example.

Proof: If we regard the equations arising from the charge-oriented MNA it becomes clear that we can fix the values $P_C e_0$ and j_{L0} to fix the values of q_0 and ϕ_0 by means of the equations (4.7) and (4.8).

q.e.d.

Remarks:

- Comparing the previous results for the initial values with those obtained by the approach presented in Section 2 we note that, if the index-1 case is considered, we do not fix Px explicitely, but a part of Qx that fixes Px. Figure 6.1 illustrates the difference. In the index-2 case, this can be interpreted correspondingly.
- Note that the Corollaries 6.5, 6.6, 6.7 and 6.8 hold for index-1 and index-2 systems.

7 Topological consistent initialization by means of the PROCEDURES 1 and 2

7.1 Introduction

In this section we will analyze the equations described in Corollary 6.5

$$\left(P_C - P_C A_V \bar{Q}_{V-C} H_6^{-1} \bar{Q}_{V-C}^T A_V^T\right)\left(e_0 - e^0\right) = 0, \tag{7.1}$$

$$\left(I - A_L^T Q_{CRV} H_7^{-1} Q_{CRV}^T A_L\right)\left(j_{L0} - j_L^0\right) = 0, \tag{7.2}$$

and the equations that lead to the hidden constraints from Theorem 6.4, i.e.

$$\bar{Q}_{V-C}^T A_V^T P_C \frac{de}{dt} - \bar{Q}_{V-C}^T \frac{dv_t}{dt} = 0, \tag{7.3}$$

$$Q_{CRV}^T A_L \frac{dj_L}{dt} + Q_{CRV}^T A_{I_t} \frac{di_t}{dt} = 0, \tag{7.4}$$

from a topological point of view to set up a system that allows us to calculate a consistent initialization.

Observe that $A_V \bar{Q}_{V-C} H_6^{-1} \bar{Q}_{V-C}^T A_V^T$ is a projector onto $\operatorname{im}(A_V \bar{Q}_{V-C})$ and that $A_L^T Q_{CRV} H_7^{-1} Q_{CRV}^T A_L$ is a projector onto $\operatorname{im}(A_L^T Q_{CRV})$.

In the following our aim is to guarantee that the equations (7.1) - (7.4) are fulfilled without requiring the inversion of the matrices H_6 and H_7. The approach makes use of the topological properties of the network and the considerations from above will be used for the proof. We start analyzing the projectors Q_C, Q_{CRV} and $\bar{Q}_{V-C}$.

7.2 The projectors that describe the network properties

In fact, there are several possibilities to choose the projectors from Definition 6.1. Since the index conditions and the manifold defined by the constraints are independent of their choice, we will define some specific projectors and explain their meaning in terms of the network topology.

Construction of Q_C
To build a projector onto $\ker A_C^T$ we do not have to write down the matrix A_C^T explicitly. By the following procedure, we can construct this projector with the aid of topological considerations.

Step 1
Consider all C-subgraphs of the network[7].

Step 2
Without loss of generality, we suppose that the nodes of the network are numbered in such a way that, for each C-subgraph, the numbers of the nodes contained in it are successive. We denote each set of nodes of a C-subgraph by $C(i)$ if i is the minimal node-number that appears in this set. Note that, for this purpose, we denote the datum-node by 0 and the corresponding set by $C(0)$.

Step 3
Consider $C(0)$ if it exists. For all $j \in C(0)$, $j \neq 0$, we set the j-th column of Q_C

[7] For the definition of a C-subgraph cf. Section 5.

equal to 0.

Step 4
For the remaining node-sets $C(i)$ we do the following successively:
For the i-th column of Q_C we set the elements $(Q_C)_{ki}$, $k = i, ..., (\max_{j \in C(i)} j)$, equal to 1, and the remaining elements equal to 0.
For all the other $j \in C(i)$, $j \neq i$, we set the j-th column of Q_C equal to 0.

Step 5
For the remaining columns we set the diagonal element equal to 1 and the others equal to 0.

Theorem 7.1 *The matrix defined above is a projector onto* $kerA_C^T$*, if the capacitive incidence matrix* A_C *is defined considering the same numbering of the nodes.*

Proof:
1. Q_C is a projector.
This follows directly from the diagonal block form of Q_C =diag$(B_1, ..., B_n)$, whereas:

$$B_i \equiv 0 \quad \text{ or } \quad B_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} \quad \text{for} \quad i = 1, ..., n. \tag{7.5}$$

2. im Q_C = kerA_C^T .
If n denotes the number of nodes of the network, it holds that

$$\begin{aligned} \text{im } Q_C &= \{(z_1, ..., z_n), z_i = z_j \ \forall j \in C(i), i \neq 0 \text{ and } , z_j = 0 \ \forall j \in C(0), j \neq 0\} \\ &= \ker \ A_C^T. \end{aligned}$$

q.e.d.

Remark: The same procedure can be used to construct a projector Q_{CRV} onto $\ker(A_C A_R A_V)^T$ if CRV-subgraphs[8] are considered instead of C-subgraphs.
Note that the assumptions on the numbering of the nodes made for both cases do not come into conflict with each other. To simplify further considerations, we assume that the nodes of all CRV-subgraphs are numbered successively in a way that the nodes of each C-subgraph are numbered successively, too. In this way the corresponding matrices Q_C and Q_{CRV} are lower triangular matrices and fulfil our purposes.

Concerning $Q_C^T A_V$. To get an idea of what the projector $\bar{Q}_{V-C}$ may look like, we consider the entries of the matrix $Q_C^T A_V$ if Q_C is defined as above.

[8]For a definition of CRV-subgraphs cf. Section 5.

- For the rows of $Q_C^T A_V$ that correspond to nodes that are not incident with capacitances, the entries $+1$ and -1 remain unchanged.
- For the rows of $Q_C^T A_V$ that correspond to nodes that are incident with capacitances, the multiplication of Q_C^T by A_V results in the summation of the 1 and -1 that indicate the incidence of the voltage sources with each C-subgraph, i.e., the i-th row of $Q_C^T A_V$ describes the incidence of the voltage sources with nodes of $C(i)$, while the j-th rows that do not correspond to minimal elements are trivial. Of course, if the C-subgraph $C(0)$ is considered, then all entries become zero.

Hence, each column of $Q_C^T A_V$ describes the incidence of one voltage source. The columns are ordered in the same way as A_V, describing the same voltage sources.

- If one column of $Q_C^T A_V$ becomes trivial, then the corresponding voltage source is, at both ends, incident with the same C-subgraph, forming a part of a relative simple C-V loop.
- If the column is not trivial, then we distinguish the following cases:
  - If, in the i-th row of the considered column, there appears a $+1$ or a -1 and the node i is not incident with capacitances, then the corresponding voltage source leaves or enters the node, respectively.
  - If, in the i-th row of the considered column, there is no zero and the node i is the minimal node of the C-subgraph $C(i)$, then $+1$ means that the node which the voltage source leaves belongs to $C(i)$, and -1 means that the node the voltage source enters belongs to $C(i)$.

Theorem 7.2 *Let $v_1, ..., v_r$, $1 \leq r$, be arbitrary voltage sources of the network. These voltage sources form part of at least one C-V loop, if and only if there exists a linear combination of the columns of $Q_C^T A_V$ with factors $(+1)$ and (-1) that corresponds to these voltage sources.*

Proof:
Without loss of generality, we assume that we can always choose the factor $(+1)$, i.e., that the orientation of the branches coincides with the orientation of the loop.

Again, we have to analyze separately the case that the datum node is incident with one of the voltage sources or that one of the C-subgraphs is C(0).

1. Let us first suppose, for reasons of simplicity, that the datum node is not incident with one of the voltage sources.
   (a) If a voltage source v_k is incident with the same C-subgraph at both ends, forming a part of a C-V loop, then the respective column $Q_C^T A_V$ is trivial by definition.

(b) If several voltage sources form a loop with some C-subgraphs, then, for each C-subgraph considered, there is a voltage source that enters and another one that leaves it. If we consider any kind of loop, it may also occur that several voltage sources of the loop are incident with the same C-subgraph, but there will always be as many leaving it as entering it.
Therefore, the sum of all columns corresponding to those voltage sources that form a part of one C-V loop is equal to zero.

2. Suppose that one of the C-subgraphs is $C(0)$. Then the corresponding columns of Q_C consist if zeros, and the respective rows of $Q_C^T A_V$ are trivial, too.

   (a) On the one hand, if a voltage source is incident with $C(0)$ at both ends, then the corresponding column of $Q_C^T A_V$ is trivial again.

   (b) On the other hand, if some elements of $C(0)$ belong to a C-V loop, then the same number of voltage sources that form part of the considered C-V loop will enter and leave $C(0)$. Consequently, also in this case the sum of the concerned columns becomes zero.

3. Finally, we consider the case that the datum node lies between two voltage sources that form part of one of the considered loops, but that it is not incident with a capacitance that forms part of that C-V loop. Then, multiplication by Q_C^T results in zeros instead of the corresponding entries $+1$ and -1. Hence, as both voltage sources are incident with the same node, the sum of the concerned columns becomes zero again.

By construction, the inverse is true, too.

q.e.d.

Lemma 7.3 *Consider an arbitrary network containing at least one C-V loop and the projector $\bar{Q}_{V-C}$ associated with it. If we consider the network that results when deleting a voltage source that forms part of a C-V loop, then the corresponding projector associated to this new network, say $\bar{Q}_{V_{-1}-C}$, fulfils:*

$$\textit{rank } \bar{Q}_{V-C} = (\textit{rank } \bar{Q}_{V_{-1}-C}) + 1.$$

Proof: The deletion of a voltage source of the network can be interpreted as the deletion of the corresponding column of A_V. We denote the new incidence matrix by $A_{V_{-1}}$. Obviously, we thus think of $Q_{V_{-1}}$ as the projector onto $Q_C^T A_{V_{-1}}$. On the one hand, as we just delete a voltage source that forms part of a C-V loop, Theorem 7.2 implies that rank $Q_C^T A_V =$ rank $Q_C^T A_{V_{-1}}$ is fulfilled. Therefore, dim (ker $Q_C^T A_V$) $=$ dim (ker $Q_C^T A_{V_{-1}}$) $+$ 1 holds. On the other hand, $Q_C^T A_{V_{-1}}$ has one column less than $Q_C^T A_V$.
Consequently, if n_V denotes the number of voltage sources of the network, then $\bar{Q}_{V-C}$ is an n_V-square matrix, $\bar{Q}_{V_{-1}-C}$ an $(n_V - 1)$-square matrix, and the asserted relation holds.

Summarizing, the deletion of the voltage source reduces the size and rank of $\bar{Q}_{V-C}$ by one.

q.e.d.

Lemma 7.4 *Consider an arbitrary network containing at least one L-I cutset and the projector Q_{CRV} associated with it. If we consider the network we obtain when deleting an inductance that forms part of an L-I cutset and contracting the corresponding incident nodes, then the corresponding projector associated with this new network, say $Q_{CRV_{-1}}$, fulfils:*

$$\textit{rank } Q_{CRV} = (\textit{rank } Q_{CRV_{-1}}) + 1.$$

Proof: Note that by the contraction of two nodes the resulting network graph contains one node less and, therefore, the incidence matrix of this new network graph, say $(A_C A_R A_V A_L A_I)_{-1}$, has one row less than the original one. With respect to the position of the deleted inductance we distinguish two possible cases:

1. Consider the case that the inductance is incident with a node that does not form a part of a CRV-subgraph. The deletion of such a node implies the deletion of a column of $(A_C A_R A_V A_L)^T$ that is trivial for the submatrix $(A_C A_R A_V)^T$. Therefore, this deletion reduces the rank and the size of Q_{CRV} by one.

2. Let now the inductance we contract be incident with two different CRV-subgraphs. Then its contraction means that we join two of the CRV-subgraphs, and thus have, for example, $m-1$ instead of m CRV-subgraphs. Further, the new network obtained contains one less node than the original one.
   Recall that the definition of Q_{CRV} depends directly on the cardinality and form of the CRV-subgraphs, and that its size coincides with the number of nodes of the network and is lowered by one if we delete one node. We consider the following two possible cases:

   (a) Suppose that we delete an inductance that is incident with the CRV-subgraph containing the datum node, i.e., $CRV(0)$.
   Then we remove exactly that node which is incident with this inductance and the CRV-subgraph that does not contain the datum node. In this case the new projector Q_{CRV-1}, which projects onto $(A_C A_R A_V)^T_{-1}$, results from Q_{CRV} by deleting, on the one hand, the row that corresponds to the minimal node of the CRV-subgraph which the deleted node corresponds to and, on the other hand, the symmetric column. Therefore, taking into account the block form of Q_{CRV}, we note that we delete a column with some nonzero elements, and that, because of the block form (7.5), it holds that rank $Q_{CRV}=$ rank $Q_{CRV-1} + 1$.

(b) Let us explain now what will happen if we remove an inductance that is not incident with $CRV(0)$. Consider then the two nodes we want to contract and which belong to two different CRV-subgraphs. To obtain a suitable projector, we may then proceed as follows:
First we construct a new column of the size of Q_{CRV} starting from the two columns i_1 and i_2 that correspond to the two minimal nodes of the CRV-subgraphs $CRV(i_1)$ and $CRV(i_2)$ which the chosen inductance is incident with, defining its elements as 1 if one of these two columns has a 1 in that row, and 0 else, i.e., the new column is the sum of the two others.
Then, we introduce this new column into Q_{CRV} instead of the column i_1 if $i_1 < i_2$. Finally, we delete the column i_2 and the symmetric row to obtain Q_{CRV-1}.
Note that Q_{CRV-1} is only of a form analogous to the projectors described in Theorem 7.1 if the CRV-subgraphs were successive, but that it is a projector in any case.

q.e.d.

7.3 Topological determination of the hidden constraints

With the aid of the projectors Q_C and Q_{CRV} defined in the last section we are able to show how the equations (7.3) and (7.4) can be interpreted in terms of the network topology.

Theorem 7.5 *The equations (7.3) and (7.4) can be written down making use of topological considerations:*

1. *PROCEDURE 1, Step 1 leads to the constraints (7.3) locating the linear independent C-V loops.*

2. *PROCEDURE 2 leads to the constraints (7.4) searching the linear independent L-I cutsets.*

Proof:
1. We already know that the number of linear independent constraints obtained from the C-V loops coincides exactly with the rank of $\bar{Q}_{V-C}$. Note that in the (j+1)-th run of PROCEDURE 1, the algorithm deletes a column of the successively resulting matrix $A_{V_{-j}}$ and that the procedure continues until there are no more linear dependent columns of $Q_C^T A_{V_{-j}}$, i. e., until $\bar{Q}_{V_{-j}-C}$ becomes trivial. Therefore, Lemma 7.3 implies that each loop considered in the procedure defines exactly one constraint, and that there are no further constraints derived from C-V loops.
Since the columns of $\bar{Q}_{V-C}$ that are relevant for the rank describe loops made of voltage sources and the C-subgraphs, each of such columns leads to an equation like the ones described in PROCEDURE 1.

2. Since $(A_C A_R A_V A_L)^T$ has full column rank, the number of linear independent constraints that result from (7.4) coincides with the rank of Q_{CRV}^T.
Therefore, Lemma 7.4 implies that each L-I cutset which is considered by PROCEDURE 2 leads to one constraint, and that there are no further constraints resulting from L-I cutsets. When the PROCEDURE 2 is unable to find further L-I cutsets, then the network that is considered contains only one single CRV-subgraph that contains the datum node. Hence, the corresponding nullspace becomes, by definition, trivial. In that moment the procedure stops.
Since Q_{CRV}^T multiplied by the incidence matrix describes the incidences of the network that results form the original one by contraction of its CRV-subgraphs, the constraints (7.4) can be interpreted as the node equations of this new network graph, or, of course, linear combinations of them. Therefore, the constraints (7.4) correspond to the derivatives of the cutset equations described in PROCEDURE 2.

q.e.d.

7.4 Topological fixing of values for the dynamic component

Let us now interpret the equations (7.1) and (7.2) in terms of the network topology.

Theorem 7.6 *Instead of solving the system (7.1) and (7.2) to set a consistent initial value, we can make use of the following topological considerations:*

1. *Set an initial value for the potential difference between each uncoloured node (uncoloured in the sense of PROCEDURE 1) and the corresponding minimal node of the C-subgraph which the node belongs to.*
2. *Set an initial value for the currents through all inductances except for those deleted in PROCEDURE 2.*

Proof:
For reasons of simplicity, we start considering the assertion for the currents through inductances.

2. Recall that, if there appear no L-I cutsets, we can set $j_{L0} - j_L^0 = 0$ arbitrarily. On the other hand, if they appear, then (7.4) is not trivial and, in particular, the equation

$$Q_{CRV}^T \left(A_L j_L + A_{I_t} i_t\right) = 0 \tag{7.6}$$

restricts the arbitrary setting of j_{L0}. Each equation of this kind fixes one inductive current in terms of the currents through the current sources and the remaining inductances that form the cutset, because it is of the form:

$$\sum_{j=1}^{k} \alpha_j j_{L_j} + \sum_{\tilde{j}=1}^{\tilde{k}} \alpha_{\tilde{j}} i_{\tilde{j}} = 0. \tag{7.7}$$

Note that by PROCEDURE 2 we obtain the derivative of such equations. Therefore, the setting performed by (7.2) may be interpreted as the setting of precisely the currents through the inductances we have not deleted with PROCEDURE 2. The currents through the deleted indunctances are then fixed by equations like (7.7). The solvability can easily be checked, because each deleted inductance does not appear in the forthcoming equations.

1. In order to understand the setting of the node potentials, let us first have a look at the meaning of the variable setting we undertake in case no C-V loops appear in the network. Considering the definition of P_C, we note that $P_C(e_0 - e^0) = 0$ sets node potential differences between the minimal nodes i of each C-subgraph and the remaining nodes of that C-subgraph $C(i)$.
Of course, if the C-subgraph is $C(0)$, then it fixes its node potential.
Note that, if C-V loops appear in the network, then (7.3) is not trivial, and, hence,

$$\bar{Q}_{V-C}^T A_V^T P_C e = \bar{Q}_{V-C}^T v \tag{7.8}$$

restricts the setting of $P_C e_0$.

Recall that the equations obtained by PROCEDURE 1 are sums of the derivatives of the characteristic equations of the voltage sources that appear in each independent loop, i.e., each of them is of the form:

$$\sum_{j \in loop} \alpha_j \left(\frac{de_j}{dt} - \frac{de_{-j}}{dt} \right) = \sum_{j \in loop} \alpha_j \frac{dv_j}{dt},$$

where e_j denotes the node potential of the node that the voltage source j leaves, e_{-j} denotes the node potential of the node that the voltage source j enters, and $\alpha_j = +1$ or -1, depending on the orientation of the loop and on the orientation of the branches.
Thus, the corresponding underived equations, which arise from (7.8), are of the form

$$\sum_{j \in loop} \alpha_j (e_j - e_{-j}) = \sum_{j \in loop} \alpha_j v_j. \tag{7.9}$$

These equations obviously restrict the choice of the initial value we set, if no C-V loops appear. Note that for each equation like (7.9) we can pick a node m that is incident with a capacitance, that is not a minimal node and that forms part of the loop. If we assume that e_i is the node potential of the minimal node of the C-subgraph which m belongs to, then we can write (7.9) in the following way:

$$\alpha_m (e_m - e_i) = \alpha_m (e_{-m} - e_i) - \sum_{\substack{j \in loop \\ j \neq m}}^{k} \alpha_j e_j + \sum_{\substack{j \in loop \\ j \neq m}}^{k} \alpha_j e_{-j} + \sum_{j \in loop}^{k} \alpha_j v_j.$$

As the loop enters and leaves the C-subgraph the same number of times, we can rearrange the summands of the right-hand side with regard to the incidence of the voltage sources with the C-subgraphs. Thus, $e_m - e_i$ is fixed if we have values for the node potential differences between the remaining nodes of the loop and the corresponding minimal nodes of the C-subgraphs which they belong to. Considering that each equation of that form fixes one of such node potential differences, it makes sense to take a different one for each loop. This explains the colouring in PROCEDURE 1. Note that Step 2 d (i) guarantees that we can always colour a suitable node.
It remains to show the solvability of the obtained equations. For an arbitrary colouring of as many nodes as C-V-loops appearing, equation (7.8) may not be solvable. To show that there exists a unique solution for the coloring performed in PROCEDURE 1, we first note that the order in which we consider the loops does not influence the solution space of the system (7.8). Therefore, we can suppose that the loops we find and the voltage sources we delete have the same order in the Steps 1 and 2. In this way it is easy to verify that we can successively obtain the values for the node potential differences if we consider the C-subgraphs in the same order as we did in Step 2. For this, we make use of the fact that each C-subgraph contains its corresponding minimal node and, additionally, as many colored nodes as C-V-loops it forms with the corresponding subgraph G. Note that, by construction, the successively defined subgraphs G never form C-V loops themselves. Therefore, there always exists a voltage source that forms part of the loops considered in the Steps 2(e)ii -2(e)iii and that is incident with a node k of the C-subgraph for which an expression $(e_k - e_i)$ is given. In this way, we can calculate values $(e_k - e_i)$ for all colored nodes of that C-subgraph successively and go on with the next one.

q.e.d.

7.5 Recapitulation

Theorem 5.2 follows directly from the ones proved above and resumes the results with regard to the consistent initialization. Theorem 5.3 results by a straight forward computation.

We conclude this section with a few comments on the meaning of the fixing of values for the dynamic component. Remember that we set up the circuit equations (4.1) - (4.3) making use of KCL for the nodes and writing down the characteristic equations of the voltage defining elements. When we initialize index 0 or index 1 systems, we just give values to its dynamic variables. Additionally, the input functions, i. e., the independent voltage and current sources, assign values, too. Taking this into account, the consistent initialization exactly guarantees that Kirchhoff's Current and Voltage Laws are fulfilled for all cutset and loops, respectively, in the index-2 case. In the index 0 and index 1 cases, this occurs by construction, but in the index-2 case we have to take care for it, i. e., we cannot assign an initial value to all dynamic variables. That means, for

each independent L-I cutset the current of one of the inductances results from KCL and for each independent C-V loop one node potential results from the KVL.
Note that capacitive loops do not lead to initialization problems because the dynamic variables are the voltages across the branches, i.e., the node potential differences of the capacitive nodes.

8 Examples

Example 1: Consider the NAND-Gate[9] from Figure 9.1. PROCEDURE 1 colours the nodes 11 and 12 because no capacitances are incident with them, and then finds three C-V loops and colours the nodes 6, 9 and 10. For the remaining nodes, we can prescribe an arbitrary initial node potential.
$\bar{Q}_{V-C}^T A_V^T e + \bar{Q}_{V-C}^T v(\cdot) = 0$ is given by:

$$\begin{aligned} -e_6 &= V_1, \\ -e_9 &= V_2, \\ -e_{10} &= V_{BB}. \end{aligned}$$

Example 2: Consider the tree of an oscillator[10] from Figure 9.2. PROCEDURE 2 could delete, for instance, L_d, L_S and L_{G2}. Therefore, if we prescribe initial values for the currents through L_{G1}, L_g and L_{S1}, a consistent initial value can be computed.
$Q_{CRV}^T A_L j_L + Q_{CRV}^T A_I i(\cdot) = 0$ is given by:

$$\begin{aligned} -j_{L_{G1}} + j_{L_{G2}} + j_{Lg} &= 0, \\ +j_{L_S} - j_{Lg} + j_{Ld} &= 0, \\ +j_{L_{S1}} - j_{L_S} &= 0. \end{aligned}$$

9 Conclusion

In this article we show how to make use of the special structure of the equations obtained by means of the MNA in electric circuit simulation with regard to consistent initialization. The structural properties have been interpreted topologically to provide a selection of variables to fix values for the dynamic component similarly to [22].

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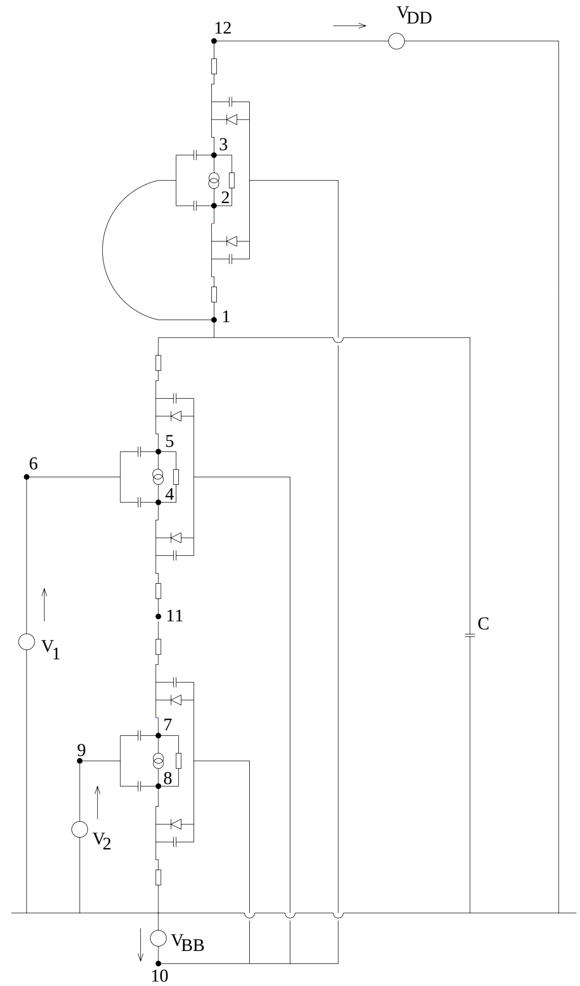
[9] cf. [24]
[10] cf. [1]

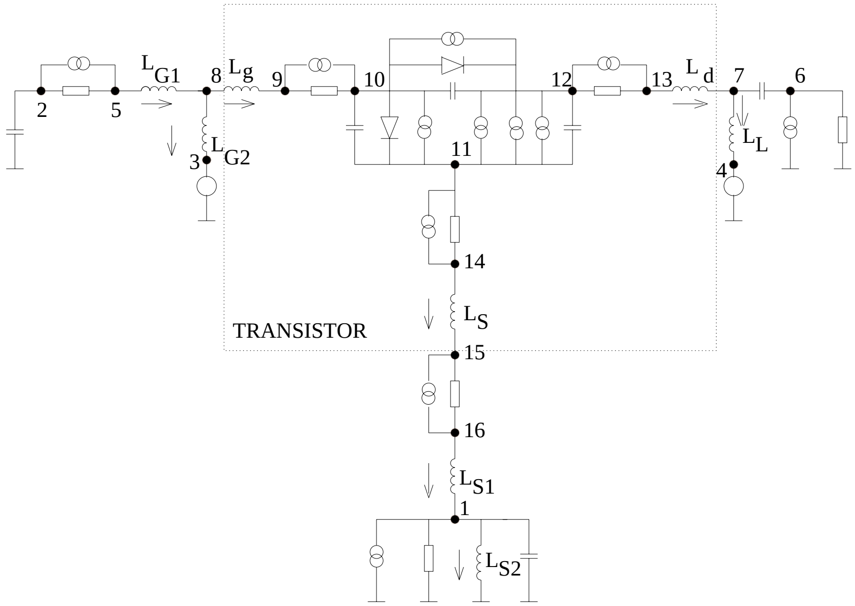


$$
A_V = \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_{BB} & V_{DD} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \end{array}
$$

$$
Q_C = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \qquad \bar{Q}_{V-C} = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{pmatrix}
$$

$$
\bar{Q}^T_{V-C} A^T_V = \begin{pmatrix} 0&0&0&0&0&-1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&-1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}
$$

Figure 9.1: NAND-Gate.



$$
Q_{CRV} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{pmatrix}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} L_{S1} & L_S & L_{G1} & L_{G2} & L_g & L_d & L_L & L_{S2} \end{array} \\
A_L = & \begin{pmatrix}
-1&0&0&0&0&0&0&+1\\
0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0\\
0&0&0&0&0&0&-1&0\\
0&0&+1&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&+1&0\\
0&0&-1&+1&+1&0&0&0\\
0&0&0&0&-1&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&+1&0&0\\
0&+1&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0\\
+1&0&0&0&0&0&0&0
\end{pmatrix}
\end{array},
\quad
Q_{CRV}^T A_L = \begin{pmatrix}
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&-1&+1&+1&0&0&0\\
0&+1&0&0&-1&+1&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
+1&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0
\end{pmatrix}
$$

Figure 9.2: Oscillator

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