APPLICATIONS OF DIFFERENTIAL CALCULUS TO NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS WITH DISCONTINUOUS COEFFICIENTS

D.K. PALAGACHEV, L. RECKE AND L.G. SOFTOVA

Abstract. We deal with Dirichlet's problem for second order quasilinear non-divergence form elliptic equations with discontinuous coefficients. First we state suitable structure, growth, and regularity conditions ensuring solvability of the problem under consideration. Then we fix a solution u_0 such that the linearized in u_0 problem is non-degenerate, and we apply the Implicit Function Theorem: For all small perturbations of the coefficient functions there exists exactly one solution $u \approx u_0$, and u depends smoothly (in $W^{2,p}$ with p larger than the space dimension) on the data. For that no structure and growth conditions are needed, and the perturbations of the coefficient functions can be general L^∞-functions with respect to the space variable x. Moreover we show that the Newton Iteration Procedure can be applied to calculate a sequence of approximate (in $W^{2,p}$ again) solutions for u_0.

1. Introduction

This article concerns quasilinear elliptic boundary value problems in non-divergence form of the type

$$\begin{cases} a_{ij}(x,u,Du)D_{ij}u(x) + b(x,u,Du) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Throughout the paper $\Omega \subset \mathbb{R}^n$ will be a bounded domain (open and connected set) with $C^{1,1}$-smooth boundary $\partial\Omega$, $a_{ij} = a_{ji}$ and b are Carathéodory functions, and as usual, the summation over indices $i, j, k, \dots$ is understood from 1 to n, if these appear pairwise. Our assumptions will be, on the one side, general enough to include cases such that

- the functions $a_{ij}(\cdot, u, \xi)$ and $b(\cdot, u, \xi)$ can be discontinuous,

and, on the other side, strong enough to have

- existence of strong solutions $u \in W^{2,p}(\Omega)$ to (1.1) with $p > n$;
- applicability of the Implicit Function Theorem and the Newton Iteration Procedure to such solutions.

In Section 2 we summarize known results ensuring existence of solutions $u \in W^{2,p}(\Omega)$ to (1.1) with $p > n$. In the semilinear case, i.e. when the coefficients $a_{ij}(x, u, \xi)$ are independent of ξ, we suppose, among other conditions, that

$$a_{ij}(\cdot, u) \in VMO(\Omega) \cap L^\infty(\Omega) \text{ for all } i, j = 1, \dots, n \text{ and } u \in \mathbb{R}. \tag{1.2}$$

In the general case of quasilinear operators we have to suppose that, for a certain $p > n$,

$$a_{ij}(\cdot, u, \xi) \in W^{1,p}(\Omega) \text{ for all } i, j = 1, \dots, n,\ u \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^n. \tag{1.3}$$

If $n = 2$, the assumptions (1.2) and (1.3) can be weakened to

$$a_{ij}(\cdot, u) \in L^\infty(\Omega) \text{ for all } i, j = 1, \dots, n \text{ and } u \in \mathbb{R} \tag{1.4}$$

1991 *Mathematics Subject Classification.* 35J65, 35R05, 58C15.

and

$$a_{ij}(\cdot, u, \xi) \in L^\infty(\Omega) \text{ for all } i, j = 1, \ldots, n,\ u \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^n, \tag{1.5}$$

respectively.

Our main new results are presented in Sections 3 and 4. There we suppose that the functions a_{ij} are differentiable with respect to the variables $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Moreover, we fix a solution $u_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to (1.1) with $p > n$ and assume that the homogeneous linearized boundary value problem

$$\begin{cases} a_{ij}(x, u_0, Du_0) D_{ij} u & \\ + \left(D_{\xi_k} a_{ij}(x, u_0, Du_0) D_{ij} u_0 + D_{\xi_k} b(x, u_0, Du_0)\right) D_k u & \\ \quad + \left(D_u a_{ij}(x, u_0, Du_0) D_{ij} u_0 + D_u b(x, u_0, Du_0)\right) u = 0 & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

has no solution $u \not\equiv 0$. Then, in Section 3, a result of the type of the Implicit Function Theorem will be proved, which, roughly speaking, asserts the following: For all small perturbations of the coefficient functions a_{ij} and b there exists exactly one solution u to (1.1) close to u_0 in $W^{2,p}(\Omega)$, and this solution depends C^1-smoothly in the sense of $W^{2,p}(\Omega)$ on the perturbations. Remark that the perturbations of the coefficient functions a_{ij} do not have to satisfy (1.2) or (1.3), but only (1.4) or (1.5), respectively. Hence, as a byproduct of an application of the Implicit Function Theorem we get existence results for solutions $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $p > n$ for (1.1) with coefficient functions a_{ij}, which do not necessarily satisfy (1.2) or (1.3), but which are in a certain sense close to functions satisfying (1.2) or (1.3), respectively.

In Section 4 we consider the following sequence of linear non-homogeneous boundary value problems determining to Newton iteration u_{l+1} for given u_l ($l = 1, 2, \ldots$):

$$\begin{cases} a_{ij}(x, u_l, Du_l) D_{ij} u_{l+1} & \\ + D_u a_{ij}(x, u_l, Du_l)(u_{l+1} - u_l) D_{ij} u_l & \\ + D_{\xi_k} a_{ij}(x, u_l, Du_l) D_k(u_{l+1} - u_l) D_{ij} u_l & \\ + D_u b(x, u_l, Du_l)(u_{l+1} - u_l) & \\ + D_{\xi_k} b(x, u_l, Du_l) D_k(u_{l+1} - u_l) + b(x, u_l, Du_l) = 0 & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_{l+1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

We prove that, if the initial iteration u_1 is sufficiently close to u_0 in $W^{2,p}(\Omega)$, then there exists a unique sequence of solutions $u_2, u_3, \ldots \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to (1.7), and u_l converges to u_0 in $W^{2,p}(\Omega)$ as $l \to \infty$.

In Section 5 we state some remarks concerning similar results for

- other boundary conditions,
- quasilinear elliptic systems in non-divergence form,
- nonlinear elliptic equations and systems in divergence form.

For the results of Sections 3 and 4 we do not need any growth conditions on the functions $a_{ij}(x, \cdot, \cdot)$ and $b(x, \cdot, \cdot)$, but only some uniform boundedness and continuity of these functions and their derivatives, which ensures that the superposition operators

$$u \mapsto a_{ij}(\cdot, u(\cdot), Du(\cdot)) \text{ and } u \mapsto b(\cdot, u(\cdot), Du(\cdot))$$

are C^1 from $W^{1,\infty}(\Omega)$ into $L^\infty(\Omega)$. The corresponding proofs are presented in the Appendix of this paper. For the sake of simplicity of the formulations, in the Appendix we introduce the notion of C^k-Carathéodory functions and a norm in the space of those functions, which is just the norm measuring the smallness of the perturbations of the coefficient functions a_{ij} and b, which is used for the result of the type of the Implicit

Function Theorem in Section 3.

Finally, let us mention some notations commonly used in the paper. We write $|\cdot|$ for the absolute value in $\mathbb{R}$ and the Euclidean norm in $\mathbb{R}^n$, respectively, and Ω is a bounded and $C^{1,1}$-smooth domain in $\mathbb{R}^n$. For functions $u : \Omega \to \mathbb{R}$ we denote by $D_i u$ the partial derivative of u with respect to the i-th component of the independent variable $x \in \Omega$, $Du := (D_1 u, \ldots, D_n u)$ is the gradient of u, and $D_{ij}u$ is the second partial derivatives with respect to the i-th and the j-th components of x. For functions $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ we write $D_u b$ and $D_{\xi_k} b$ for the partial derivatives of b with respect to the variable $u \in \mathbb{R}$ and to the k-th component of the variable $\xi \in \mathbb{R}^n$, respectively. As usual, a function $a : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is called Carathéodory function, if $a(\cdot, v)$ is measurable for all $v \in \mathbb{R}^m$ and $a(x, \cdot)$ is continuous for almost all (a.a.) $x \in \Omega$.

By $L^p(\Omega)$ and $W^{k,p}(\Omega)$ we denote the usual Lebesgue and Sobolev spaces with their norms $\|\cdot\|_p$ and $\|\cdot\|_{k,p}$, respectively ($k = 1, 2, \ldots, 1 \le p \le \infty$). Finally, $VMO(\Omega)$ is the class of functions with vanishing mean oscillation in Ω (cf. [13], [21]), i.e., the space of all $f \in L^1_{\text{loc}}(\Omega)$ such that

$$\sup_r \gamma_f(r) < +\infty \text{ and } \lim_{r\to 0} \gamma_f(r) = 0.$$

Here $\gamma_f : (0, \infty) \to \mathbb{R}$ is the VMO-modulus of f defined by

$$\gamma_f(r) = \sup_{0<\rho\le r} \sup_{x\in\Omega} \frac{1}{|\Omega_{\rho,x}|} \int_{\Omega_{\rho,x}} |f(y) - f_{\Omega_{\rho,x}}| dy,$$

where $\Omega_{\rho,x} := \{y \in \Omega : |y - x| < \rho\}$, $f_{\Omega_{\rho,x}}$ is the average $|\Omega_{\rho,x}|^{-1} \int_{\Omega_{\rho,x}} f(y)dy$, and $|\Omega_{\rho,x}|$ stands for the Lebesgue measure of $\Omega_{\rho,x}$.

2. Selected Existence Theorems

This section collects known results regarding strong solvability of the Dirichlet problem for elliptic operators with discontinuous coefficients.

2.1. Linear equations with VMO coefficients.

Let us consider the linear Dirichlet problem

$$\left\{ \begin{array}{rcll} \mathcal{L}u \equiv a_{ij}(x)D_{ij}u(x) &=& f(x) & \text{a.e. in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{2.1}$$

Concerning the coefficient functions $a_{ij} : \Omega \to \mathbb{R}$ we suppose these are measurable, $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$, and that the following conditions are fulfilled:

- (2_1) *Uniform ellipticity of* $\mathcal{L}$: There exist positive constants λ and Λ such that for a.a. $x \in \Omega$ and all $\eta \in \mathbb{R}^n$
$$\lambda|\eta|^2 \le a_{ij}(x)\eta_i\eta_j \le \Lambda|\eta|^2.$$
- (2_2) *VMO property*: $a_{ij} \in VMO(\Omega)$ for all $i, j = 1, \ldots, n$.

Theorem 2.1. ([4, Theorem 4.4]) *Suppose* (2_1) *and* (2_2). *Then for all* $p \in (1, \infty)$ *and all* $f \in L^p(\Omega)$ *there exists a unique solution* $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ *of* (2.1).

Obviously, $\mathcal{L}$ is a linear bounded operator from $W^{2,p}(\Omega)$ into $L^p(\Omega)$. Hence, by Banach's inverse operator theorem, Theorem 2.1 claims that $\mathcal{L}$ is an isomorphism from $W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ onto $L^p(\Omega)$. This property will be used repeatedly in Sections 3 and 4 below.

2.2. **Semilinear equations with VMO coefficients.** In this subsection we consider the semilinear Dirichlet problem

$$\begin{cases} \mathcal{S}u \equiv a_{ij}(x,u)D_{ij}u + b(x,u,Du) &= 0 \quad \text{a.e. in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Suppose the coefficients $a_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions, $a_{ij} = a_{ji}$ for all $i, j = 1, \dots, n$, and that the following conditions are fulfilled:

(2_3) *Uniform ellipticity of $\mathcal{S}$*: There exists a non-increasing function $\lambda : [0,\infty) \to (0,\infty)$, such that for a.a $x \in \Omega$ and all $u \in \mathbb{R}, \eta \in \mathbb{R}^n$ it holds

$$\lambda(|u|)|\eta|^2 \le a_{ij}(x,u)\eta_i\eta_j \le \frac{1}{\lambda(|u|)}|\eta|^2.$$

(2_4) *Local uniform continuity of a_{ij} with respect to u*: For all $M > 0$ there exists a non-decreasing function $\mu_M : [0,\infty) \to (0,\infty)$ with $\lim_{t\downarrow 0} \mu_M(t) = 0$ such that for a.a. $x \in \Omega$ and all $u, u' \in [-M, M]$ it holds

$$|a_{ij}(x,u) - a_{ij}(x,u')| \le \mu_M(|u - u'|).$$

(2_5) *VMO property of a_{ij} with respect to x, locally uniformly in u*: For all $M > 0$ it holds

$$\lim_{r\downarrow 0}\Big(\sup_{|u|\le M} \sup_{0<\rho\le r} \sup_{x\in\Omega} \frac{1}{|\Omega_{\rho,x}|}\int_{\Omega_{\rho,x}} \Big|a_{ij}(y,u) - \frac{1}{|\Omega_{\rho,x}|}\int_{\Omega_{\rho,x}} a_{ij}(z,u)dz\Big|dy\Big) = 0.$$

($2_{6,p}$) *Quadratic gradient growth of b*: There exist $p > n$, $b_1 \in L^p(\Omega)$ and a non-decreasing function $\nu : [0,\infty) \to (0,\infty)$ such that

$$|b(x,u,\xi)| \le \nu(|u|)\big(b_1(x) + |\xi|^2\big)$$

for a.a. $x \in \Omega$, all $u \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$.

(2_7) *Monotonicity of b with respect to u*: There exists non-negative function $b_2 \in L^n(\Omega)$ such that

$$\operatorname{sign} u \cdot b(x,u,\xi) \le \lambda(|u|)b_2(x)\left(1 + |\xi|\right).$$

Theorem 2.2. ([18, Theorem 1.1], [16, Theorem 2.6.9]) *Suppose* (2_3)–(2_7). *Then there exists a solution $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ of* (2.2).

Since any $u \in W^{2,p}(\Omega)$ with $p > n$ is uniformly continuous, the assumptions (2_4) and (2_5) ensure that $a(\cdot, u(\cdot)) \in L^\infty(\Omega) \cap VMO(\Omega)$, and the corresponding VMO-modulus is bounded in terms of $\|u\|_{L^\infty(\Omega)}$ and of the continuity modulus of u (see [18, Lemma 2.1] or Lemma A.1 below). Further, assumptions (2_7) and ($2_{6,p}$) give a priori estimates for solutions u to (2.2) in $L^\infty(\Omega)$ and $W^{1,2p}(\Omega)$. Whence the existence result follows from the Leray–Schauder principle.

2.3. **Quasilinear equations with smooth coefficients.** Consider the general quasilinear Dirichlet problem

$$\begin{cases} \mathcal{Q}u \equiv a_{ij}(x,u,Du)D_{ij}u + b(x,u,Du) &= 0 \quad \text{a.e. in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Concerning the coefficient functions $a_{ij} : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ we suppose these are C^1 smooth and $a_{ij} = a_{ji}$ for all $i, j = 1, \dots, n$. Further, we suppose that $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function and that the following conditions are fulfilled:

(2_8) *Uniform ellipticity of $\mathcal{Q}$:* There exists a non-increasing function $\lambda : [0,\infty) \to (0,\infty)$, such that for a.a $x \in \Omega$ and all $u \in \mathbb{R}$, $\xi,\ \eta \in \mathbb{R}^n$ it holds

$$\lambda(|u|)|\eta|^2 \leq a_{ij}(x,u,\xi)\eta_i\eta_j \leq \frac{1}{\lambda(|u|)}|\eta|^2.$$

($2_{9,p}$) *Growth conditions for a_{ij}*: There exist $p > n$, $\Phi \in L^p(\Omega)$ and a non-decreasing function $\mu : [0,\infty) \to (0,\infty)$ such that for all $x \in \Omega$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ it holds

$$\begin{aligned} |D_u a_{ij}(x,u,\xi)| + |D_k a_{ij}(x,u,\xi)| &\leq \mu(|u|+|\xi|)\Phi(x), \\ |D_{\xi_k} a_{ij}(x,u,\xi)| &\leq \mu(|u|+|\xi|), \\ \big|D_{\xi_k} a_{ij}(x,u,\xi) - D_{\xi_j} a_{ik}(x,u,\xi)\big| &\leq \mu(|u|)\big(1+|\xi|^2\big)^{-1/2} \end{aligned}$$

and

$$\left|\sum_{k=1}^{n}\Big(D_u a_{ij}(x,u,\xi)\xi_k\xi_k - D_u a_{kj}(x,u,\xi)\xi_k\xi_i + D_k a_{ij}(x,u,\xi)\xi_k - D_k a_{kj}(x,u,\xi)\xi_i\Big)\right|$$
$$\leq \mu(|u|)(1+|\xi|^2)^{1/2}\big(|\xi|+\Phi(x)\big).$$

($2_{10,p}$) *A local uniform continuity property of b with respect to (u,ξ)*: There exists $p > n$ such that $b(\cdot,u,\xi) \in L^p(\Omega)$ for all $u \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$, and for all $M, \varepsilon > 0$ there exists $\delta > 0$ such that for a.a. $x \in \Omega$ and all $(u,\xi), (u',\xi') \in \mathbb{R}\times\mathbb{R}^n$ with $|u-u'| + |\xi-\xi'| < \delta$ and $|u|, |u'|, |\xi|, |\xi'| \leq M$ it holds

$$\int_\Omega |b(x,u,\xi) - b(x,u',\xi')|^p dx < \varepsilon.$$

Theorem 2.3. ([14, Theorem 7.1]) *Suppose* ($2_{6,p}$)–($2_{10,p}$)*. Then there exists a solution* $u \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ *of* (2.3)*.*

As in the case of semilinear operators, the monotonicity condition (2_7) and (2_8) ensure an $L^\infty(\Omega)$ a priori estimate for any solution to (2.3) (see [6, Theorems 10.4, 10.5]). Assumptions ($2_{6,p}$) and ($2_{9,p}$) provide for an a priori bound for a suitable Hölder norm of Du. Hence, Theorem 2.3 follows from ($2_{10,p}$) and the Leray–Schauder fixed point theorem.

2.4. **Planar quasilinear equations with L^∞ coefficients.** It the present subsection we consider the general quasilinear Dirichlet problem (2.3) in the case of two independent variables ($n = 2$). In this case the regularity assumptions on the coefficient functions a_{ij} can be significantly weakened. In fact, consider the Dirichlet problem

$$\left\{\begin{aligned} \mathcal{Q}_2 u \equiv \sum_{i,j=1}^{2} a_{ij}(x,u,Du)D_{ij}u + b(x,u,Du) &= 0 \quad \text{a.e. in } \Omega \subset \mathbb{R}^2, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right. \tag{2.4}$$

supposing that a_{ij} and b are Carathéodory functions and $a_{12} = a_{21}$.

Theorem 2.4. *Let $n = 2$ and Ω be convex. Suppose* (2_7) *and let $\mathcal{Q}_2$ be a uniformly elliptic operator, that is, there are positive constants λ and Λ such that*

$$\lambda|\eta|^2 \leq a_{ij}(x,u,\xi)\eta_i\eta_j \leq \Lambda|\eta|^2 \tag{2.5}$$

for a.a. $x \in \Omega$ and all $u \in \mathbb{R}$, $\xi,\ \eta \in \mathbb{R}^2$. Then there exists a number $p_0 > 2$ such that, whenever condition ($2_{6,p}$) *is fulfilled with a certain $p \in (2, p_0)$, there exists a solution* $u \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ *of* (2.4)*.*

Theorem 2.4 is a particular case of [16, Theorem 3.2.9]. In fact, each uniformly elliptic operator in two dimensions satisfies the Cordes condition ([16, Remark 1.2.17]), that is,

$$\frac{\sum_{i,j=1}^{2} a_{ij}^2(x,u,\xi)}{\big(a_{11}(x,u,\xi)+a_{22}(x,u,\xi)\big)^2} \le \frac{1}{1+\varepsilon} \quad \text{for all } u\in\mathbb{R}, \xi\in\mathbb{R}^2 \text{ and a.a. } x\in\Omega \tag{2.6}$$

for any $\varepsilon \in \big(0, 2\lambda\Lambda/(\lambda^2+\Lambda^2)\big)$. It is proved by Campanato in [2] (see also [16, Theorem 1.2.3]) that in case of a convex domain Ω there exists $p_0 > 2$ such that the *linear* Dirichlet problem

$$\begin{cases} \mathcal{L}u &= f\in L^q(\Omega) \quad \text{a.e. in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

is uniquely solvable in $W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega)$ $\forall q\in[2,p_0)$ for *any linear* operator $\mathcal{L}$ satisfying (2.6). The number p_0 depends on Ω and ε, i.e., on λ and Λ.

Take now $p\in(2,p_0)$ such that $(2_{6,p})$ is satisfied and let $v\in W^{1,2p}(\Omega)$. The *linear* Dirichlet problem

$$\begin{cases} \displaystyle\sum_{i,j=1}^{2} a_{ij}(x,v,Dv)D_{ij}(\mathcal{T}v)+b(x,v,Dv) &= 0 \quad \text{a.e. in } \Omega\subset\mathbb{R}^2,\\ \mathcal{T}v &= 0 \quad \text{on } \partial\Omega \end{cases}$$

admits a unique solution $\mathcal{T}v\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ as consequence of Campanato's result and of $(2_{6,p})$ (which gives $b(\cdot,v,Dv)\in L^p(\Omega)$). Thus, a nonlinear operator $\mathcal{T}\colon\ W^{1,2p}(\Omega)\to W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ is defined which, considered as a mapping from $W^{1,2p}(\Omega)$ into itself, is continuous and compact. This way, the Leray–Schauder theorem implies existence of a fixed point of $\mathcal{T}$, which is the desired solution of (2.4) (see [17], [22] or the proof of [16, Theorem 3.2.9] for details).

2.5. **Quasilinear operators satisfying the Campanato condition.** For $p\in(1,\infty)$ let us denote

$$C(p) := \sup\left\{ \frac{\Big(\sum_{i,j=1}^{n}\int_\Omega |D_{ij}u|^p dx\Big)^{1/p}}{\big(\int_\Omega|\Delta u|^p dx\big)^{1/p}} : u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega), \Delta u\not\equiv 0\right\}.$$

Because of the Calderón–Zygmund inequality, $C(p)$ is a finite number, and it is well known that $C(p)\ge 1$ for $p\ge 2$. Moreover, if Ω is convex then $\lim_{p\downarrow 2} C(p) = C(2) = 1$ as proved by C. Miranda and G. Talenti.

In this subsection we consider once again the general quasilinear Dirichlet problem (2.3) supposing that a_{ij} and b are Carathéodory functions and $a_{ij}=a_{ji}$ for all $i,j=1,\dots,n$. Moreover, we assume:

(2_{11}) *Campanato's ellipticity condition*: There exist positive constants α, γ and δ, with $\gamma+\delta<1$ such that

$$\big|\mathrm{Tr}\,\tau - \alpha\, a_{ij}(x,u,\xi)\tau_{ij}\big| \le \delta|\mathrm{Tr}\,\tau| + \frac{\gamma}{C(p)}\|\tau\|_{n\times n}$$

for a.a. $x\in\Omega$, all $u\in\mathbb{R}$, $\xi\in\mathbb{R}^n$, and all symmetric matrices $\tau\in\mathbb{R}^{n\times n}$. Here $\|\tau\|_{n\times n}$ is the Euclidean norm of the matrix τ and $\mathrm{Tr}\,\tau=\sum_{i=1}^{n}\tau_{ii}$.

Theorem 2.5. ([19, Theorem 1.1, Remark 1], [16, Proposition 3.2.18]) *Let conditions* $(2_{6,p})$, (2_7) *and* (2_{11}) *be satisfied. Then there exists a solution* $u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ *of* (2.3).

The proof makes essential use of (2_{11}) which ensures that the quasilinear operator $\mathcal{Q}$ is *near* (see [3], [16]) to the Laplacian both considered as mappings from $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ into $L^p(\Omega)$. A relevant example of a quasilinear operator $\mathcal{Q}$ satisfying condition (2_{11}) could be a uniformly elliptic one given by a coefficients matrix $\{a_{ij}\}_{i,j=1}^n$ with small enough difference between the highest and the lowest eigenvalue.

More precisely, suppose that a_{ij} satisfies (2.5). Decomposing a_{ij} into $\lambda\delta_{ij} + (a_{ij} - \lambda\delta_{ij})$ with Kronecker's δ_{ij}, we get

$$\begin{aligned}
|\mathrm{Tr}\,\tau - \alpha a_{ij}(x,u,\xi)\tau_{ij}| &= |\mathrm{Tr}\,\tau - \alpha\lambda\mathrm{Tr}\,\tau - \alpha\,(a_{ij}-\lambda\delta_{ij})\tau_{ij}| \\
&\le |1-\alpha\lambda|.|\mathrm{Tr}\,\tau| + \alpha|a_{ij}-\lambda\delta_{ij}|.|\tau_{ij}| \\
&\le |1-\alpha\lambda|.|\mathrm{Tr}\,\tau| + \alpha\Big(\sum_{i=1}^n (a_{ii}-\lambda) + \sum_{\substack{i,j=1\\ i\neq j}}^n |a_{ij}|\Big)\|\tau\|_{n\times n} \\
&\le |1-\alpha\lambda|.|\mathrm{Tr}\,\tau| + \frac{\alpha n^2(\Lambda-\lambda)C(p)}{C(p)}\|\tau\|_{n\times n},
\end{aligned}$$

since $|a_{ij}| \le \Lambda - \lambda$ for $i \neq j$ and $\lambda \le a_{ii} \le \Lambda$ as it follows from (2.5). Let $\alpha \in (0, 1/\lambda)$. Then (2_{11}) will be satisfied with $\delta = 1 - \alpha\lambda$ and $\gamma = \alpha n^2(\Lambda - \lambda)C(p)$ if

$$n^2\left(\frac{\Lambda}{\lambda} - 1\right)C(p) < 1. \tag{2.7}$$

Remark 2.6. Global unicity of strong solutions to (2.2), (2.3) or (2.4) can be invoked under additional assumptions on the data which, roughly speaking, require a_{ij}'s to be independent of u and both $a_{ij}(x,\xi)$ and $b(x,u,\xi)$ to be Lipschitz continuous in ξ. The reader is referred to [6, Theorem 10.2] (cf. also [18, Theorem 1.4] and [16, Theorem 2.6.12]) for details.

3. Application of the Implicit Function Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded and $C^{1,1}$-smooth domain and consider the general quasilinear Dirichlet problem

$$\left\{\begin{array}{rcll} a_{ij}(x,u,Du)D_{ij}u + b(x,u,Du) &=& 0 & \text{a.e. in } \Omega,\\ u &=& 0 & \text{on } \partial\Omega \end{array}\right. \tag{3.1}$$

and its formal linearization at $u = u_0$

$$\left\{\begin{array}{rcll} a_{ij}(x,u_0,Du_0)D_{ij}v & & & \\ \quad + \left(D_{\xi_k}a_{ij}(x,u_0,Du_0)D_{ij}u_0 + D_{\xi_k}b(x,u_0,Du_0)\right)D_kv & & & \\ \qquad + \left(D_u a_{ij}(x,u_0,Du_0)D_{ij}u_0 + D_u b(x,u_0,Du_0)\right)v &=& 0 & \text{a.e. in } \Omega,\\ v &=& 0 & \text{on } \partial\Omega. \end{array}\right. \tag{3.2}$$

We impose the following hypotheses:

- (3_1) $a_{ij}, b : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ are C^1-Carathéodory functions and $a_{ij} = a_{ji}$ for all $i,j = 1,\dots,n$ (for the notion of C^1-Carathéodory functions see Definition A.2 in the Appendix).
- $(3_{2,p})$ $u_0 \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ is a solution to (3.1) with $p > n$.
- (3_3) There exists a positive constant λ such that for a.a $x \in \Omega$ and all $\eta \in \mathbb{R}^n$ it holds
$$a_{ij}(x,u_0(x),Du_0(x))\eta_i\eta_j \ge \lambda|\eta|^2.$$
- (3_4) The maps $x \in \Omega \mapsto a_{ij}(x,u_0(x),Du_0(x)) \in \mathbb{R}$ are in $VMO(\Omega)\cap L^\infty(\Omega)$ for all $i,j = 1,\dots,n$.

(3_5) There does not exist a non-zero solution $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to (3.2).

Theorem 3.1. *Suppose (3_1)–(3_5). Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be an open and bounded set and $K \subset U$ a compact such that $(u_0(x), Du_0(x)) \in K$ for a.a. $x \in \Omega$.*

Then there exist neighborhoods $V \subseteq \mathcal{C}^1(\Omega \times \overline{U})^{n^2} \times \mathcal{C}^1(\Omega \times \overline{U})$ of zero and $W \subseteq W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of u_0 and a C^1-map $\varphi : V \to W$ with $\varphi(0) = u_0$ such that for all

$$\left(\{\widetilde{a}_{ij}\}_{ij=1}^n, \widetilde{b}\right) \in V, \quad u \in W$$

we have

$$\left\{ \begin{array}{rcll} (a_{ij}(x,u,Du) + \widetilde{a}_{ij}(x,u,Du))\, D_{ij}u & & & \\ +b(x,u,Du) + \widetilde{b}(x,u,Du) & = & 0 & \text{in } \Omega, \\ u & = & 0, & \text{on } \partial\Omega \end{array} \right. \tag{3.3}$$

if and only if $u = \varphi\big(\{\widetilde{a}_{ij}\}_{i,j=1}^n, b\big)$.

Proof. For the sake of simplicity, let us denote

$$\widetilde{a} := \{\widetilde{a}_{ij}\}_{i,j=1}^n \quad \text{for} \quad \{\widetilde{a}_{ij}\}_{i,j=1}^n \in \mathcal{C}^1(\Omega \times \overline{U})^{n^2}.$$

Denote by $\mathcal{U}$ the set of all $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that there exists a compact $K \subset U$ with $(u(x), Du(x)) \in K$ for all $x \in \Omega$. Obviously, $\mathcal{U}$ is open in $W^{2,p}(\Omega)$. Because of assumption (3_1) and Lemma A.3, there exist C^1-maps

$$A_{ij} : \mathcal{C}^1(\Omega \times \overline{U})^{n^2} \times \mathcal{U} \to L^\infty(\Omega), \quad B : \mathcal{C}^1(\Omega, \overline{U}) \times \mathcal{U} \to L^\infty(\Omega)$$

such that

$$\begin{array}{rcl} \big(A_{ij}(\widetilde{a}, u)\big)(x) & = & a_{ij}(x, u(x), Du(x)) + \widetilde{a}_{ij}(x, u(x), Du(x)), \\ \big(B(\widetilde{b}, u)\big)(x) & = & b(x, u(x), Du(x)) + \widetilde{b}(x, u(x), Du(x)). \end{array}$$

Hence, the problem (3.3) is equivalent to

$$F(\widetilde{a}, \widetilde{b}, u) = 0, \tag{3.4}$$

where

$$F(\widetilde{a}, \widetilde{b}, u) := A_{ij}(\widetilde{a}, u) D_{ij} u + B(\widetilde{b}, u). \tag{3.5}$$

Obviously, the map F is C^1-smooth from $\mathcal{C}^1(\Omega \times \overline{U})^{n^2} \times \mathcal{C}^1(\Omega \times \overline{U}) \times \mathcal{U}$ into $L^p(\Omega)$. Moreover, $\widetilde{a} = 0$, $\widetilde{b} = 0$, $u = u_0$ is a solution to (3.4) because of $(3_{2,p})$. Let us solve (3.4) with respect to u nearby of this solution by means of the Implicit Function Theorem. In order to do this we have to check that

$$D_u F(0, 0, u_0) \in \operatorname{Iso}\big(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega); L^p(\Omega)\big). \tag{3.6}$$

Because of (3.5) we have

$$D_u F(0,0,u_0) v = A_{ij}(0, u_0) D_{ij} v + (D_u A_{ij}(0, u_0) v)\, D_{ij} u + D_u B(0, u_0) v$$

for all $u \in \mathcal{U}$ and $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Hence, the linear operator $D_u F(0,0,u_0)$ is the sum of the two linear operators

$$v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \mapsto A_{ij}(0, u_0) D_{ij} v \in L^p(\Omega), \tag{3.7}$$

$$v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \mapsto (D_u A_{ij}(0, u_0) v)\, D_{ij} u + D_u B(0, u_0) v. \tag{3.8}$$

By the definition of the map A_{ij}, the value of the right-hand side of (3.7) in a point $x \in \Omega$ is $a_{ij}(x, u_0, Du_0(x)) D_{ij} v(x)$. Hence, the assumptions (3_3) and (3_4) and Theorem 2.1 imply that (3.7) is an isomorphism.

Similarly, the definitions of A_{ij} and B imply that the right-hand side of (3.8) in a point $x \in \Omega$ is

$$\begin{aligned} D_u a_{ij}(x, u_0(x), Du_0(x))v(x)D_{ij}u_0(x) + D_{\xi_k} a_{ij}(x, u_0(x), Du_0(x))D_k v(x)D_{ij}u_0(x) \\ + D_u b(x, u_0(x), Du_0(x))v(x) + D_{\xi_k} b(x, u_0(x), Du_0(x))D_k v(x). \end{aligned}$$

Hence, because of the compact embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,p}(\Omega)$, the linear operator (3.8) is compact. Therefore, the linear operator $D_u F(0, 0, u_0)$ is Fredholm (index zero). In particular, it is an isomorphism if it is injective. Thus, assumption (3_5) yields that (3.6) is true.

Hence, the Implicit Function Theorem can be applied to (3.5) in the described way and this gives the assertion of Theorem 3.1. □

4. Application of the Newton Iteration Procedure

In this section we again suppose the domain Ω to have a $C^{1,1}$–smooth boundary, and consider the general quasilinear Dirichlet problem

$$\text{(4.1)} \qquad \begin{cases} a_{ij}(x, u, Du)D_{ij}u + b(x, u, Du) &= 0 \quad \text{a.e. in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

and its formal linearization in $u = u_0$

$$\text{(4.2)} \qquad \begin{cases} a_{ij}(x, u_0, Du_0)D_{ij}v \\ \quad + \left(D_{\xi_k} a_{ij}(x, u_0, Du_0)D_{ij}u_0 + D_{\xi_k} b(x, u_0, Du_0)\right) D_k v \\ \qquad + \left(D_u a_{ij}(x, u_0, Du_0)D_{ij}u_0 + D_u b(x, u_0, Du_0)\right) v = 0 \quad \text{a.e. in } \Omega, \\ \qquad\qquad v = 0 \quad \text{on } \partial\Omega, \end{cases}$$

but this time together with the following sequence of linear non-homogeneous boundary value problems determining to Newton iteration u_{l+1} for given u_l ($l = 1, 2, \ldots$):

$$\text{(4.3)} \qquad \begin{cases} a_{ij}(x, u_l, Du_l)D_{ij}u_{l+1} \\ \quad + D_u a_{ij}(x, u_l, Du_l)(u_{l+1} - u_l)D_{ij}u_l \\ \quad + D_{\xi_k} a_{ij}(x, u_l, Du_l)D_k(u_{l+1} - u_l)D_{ij}u_l \\ \quad + D_u b(x, u_l, Du_l)(u_{l+1} - u_l) \\ \quad + D_{\xi_k} b(x, u_l, Du_l)D_k(u_{l+1} - u_l) + b(x, u_l, Du_l) = 0 \quad \text{in } \Omega, \\ \qquad\qquad u_{l+1} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Definition 4.1. *Denote by $\mathcal{A}_p$ the set of all symmetric matrix functions $\{a_{ij}\}_{i,j=1}^n \in L^\infty(\Omega)^{n^2}$, for which there exists $\lambda > 0$ such that*

$$\text{(4.4)} \qquad a_{ij}(x)\eta_i\eta_j \geq \lambda|\eta|^2 \text{ for all } \eta \in \mathbb{R}^n \text{ and a.a } x \in \Omega$$

and for which the map $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \mapsto a_{ij}D_{ij}u \in L^p(\Omega)$, is an isomorphism.

Obviously, any of the symmetric matrix functions $\{a_{ij}\}_{i,j=1}^n \in L^\infty(\Omega)^{n^2}$, considered in Section 2 (e.g., with $a_{ij} \in VMO(\Omega)$, or a_{ij}'s satisfying the Cordes condition (2.7)) is in $\mathcal{A}_p$, and any symmetric matrix function, which is close to them in $L^\infty(\Omega)^{n^2}$ and which satisfies (4.4) is in $\mathcal{A}_p$ as well.

We impose the following conditions:

(4_1) $a_{ij}, b : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are $C^{1,1}$-Carathéodory functions and $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$ (for the notion of $C^{1,1}$-Carathéodory functions see Definition A.2 in the Appendix below).

($4_{2,p}$) $u_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is a solution to (4.1) with $p > n$.

($4_{3,p}$) $\{a_{ij}(\cdot, u_0(\cdot), Du_0(\cdot))\}_{i,j=1}^n \in \mathcal{A}_p$.

(4_4) There does not exist a non-zero solution $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to (4.2).

Theorem 4.2. *Suppose* (4_1)–(4_4). *Then there exists a neighborhood* $W \subset W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *of* u_0 *such that for any* $u_1 \in W$ *there exists a unique sequence of solutions* $u_2, u_3, \ldots \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *to* (4.3), *and* u_l *converges to* u_0 *in* $W^{2,p}(\Omega)$ *as* $l \to \infty$.

Proof. We proceed as in the proof of Theorem 3.1. Writing $F(u)$ for $F(0,0,u)$, the problem (4.1) is equivalent to

$$F(u) = 0 \tag{4.5}$$

with

$$(F(u))(x) := a_{ij}(x, u(x), Du(x))D_{ij}u(x) + b(x, u(x), Du(x)). \tag{4.6}$$

Lemma A.3 implies that (4.6) defines a map $F \in C^1\big(W^{2,p}(\Omega); L^p(\Omega)\big)$. Assumption ($4_{2,p}$) yields that u_0 is a solution to (4.5). Finally, ($4_{3,p}$) and (4_4) imply (as in the proof of Theorem 3.1) that

$$F'(u_0) \in \mathrm{Iso}\big(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega); L^p(\Omega)\big).$$

Hence, all conditions for the applicability of the abstract Newton iteration procedure (see [24, Proposition 5.1]) to (4.5) in the solution u_0 are checked up to the following one:

$$F' \text{ is Lipschitz continuous in a neighborhood of } u_0. \tag{4.7}$$

For proving (4.7), we use the quasilinear structure of F. Because of (4.6) we have

$$F(u) = A_{ij}(u)D_{ij}u + B(u),$$

where $A_{ij}, B \in C^2\big(W^{1,\infty}(\Omega); L^\infty(\Omega)\big)$ are the superposition operators generated by a_{ij} and b. Hence

$$F'(u)w = A_{ij}(u)D_{ij}w + \big(A'_{ij}(u)w\big)\, D_{ij}u + B'(u)w.$$

Therefore $(F'(u) - F'(v))\, w$ is a sum of the following terms:

$$(A_{ij}(u) - A_{ij}(v))\, D_{ij}w, \tag{4.8}$$
$$\big(A'_{ij}(u) - A'_{ij}(v)\big)\, wD_{ij}u, \tag{4.9}$$
$$A'_{ij}(v)wD_{ij}(u-v), \tag{4.10}$$
$$(B'(u) - B'(v))\, w. \tag{4.11}$$

The L^p-norm of (4.8) can be estimated by

$$\text{const}\ \|u - v\|_{L^p(\Omega)} \|w\|_{W^{2,p}(\Omega)} \tag{4.12}$$

in view of the mean value theorem and because A'_{ij} is locally bounded from $W^{2,p}(\Omega)$ into $\mathcal{L}\left(W^{2,p}(\Omega), L^\infty(\Omega)\right)$ (as a locally Lipschitz continuous map, cf. Lemma A.3). The L^p-norms of (4.9) and (4.11) can be estimated by (4.12) because A'_{ij} and B' are locally Lipschitz continuous from $W^{2,p}(\Omega)$ into $\mathcal{L}\left(W^{2,p}(\Omega), L^\infty(\Omega)\right)$. Finally, the L^p-norm of (4.10) can be estimated by (4.12) again, because A'_{ij} is locally bounded from $W^{2,p}(\Omega)$ into $\mathcal{L}\left(W^{2,p}(\Omega), L^\infty(\Omega)\right)$. □

5. Concluding Remarks

Results of the type of Sections 3 and 4 are true also for other boundary conditions, in particular for the regular oblique derivative problem

$$\text{(5.1)} \qquad \begin{cases} \mathcal{Q}u \equiv a_{ij}(x,u,Du)D_{ij} + b(x,u,Du) = 0 & \text{a.e. in } \Omega, \\ \dfrac{\partial u}{\partial \ell} + \sigma(x)u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\ell(x) = \big(\ell_1(x), \dots, \ell_n(x)\big)$ is a unit vector field defined on $\partial\Omega$ which is never tangential to $\partial\Omega$, $\sigma(x) < 0$ and ℓ_i, $\sigma \in C^{0,1}(\partial\Omega)$. We dispose of various existence results for (5.1) under the set of hypotheses given in Section 2. Precisely, we refer the reader to [15] when $\mathcal{Q}$ is a linear operator, to [5] in case $\mathcal{Q}$ is semilinear, to [23] for general quasilinear operators with smooth coefficients and to [7] in the situation considered in Theorem 2.4.

The results of Sections 3 and 4 can be generalized to weakly coupled systems of the type

$$\text{(5.2)} \qquad a^\alpha_{ij}(x,u^1,\dots,u^N,Du^1,\dots,Du^N)D_{ij}u^\alpha + b^\alpha(x,u^1,\dots,u^N,Du^1,\dots,Du^N) = 0.$$

In (5.2) the index α varies from 1 to N, but there is no summation over α. If ellipticity conditions of the type (3_3) are fulfilled for each α, then the main part of the linearization in a solution $(u^1_0, \dots, u^N_0)$ generates, in the case of homogeneous Dirichlet boundary conditions, for example, an isomorphism

$$v \in \left(W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)\right)^N \mapsto \left[a^\alpha_{ij}(\cdot, u^1_0, \dots, u^N_0, Du^1_0, \dots, Du^N_0)D_{ij}v^\alpha\right]^N_{\alpha=1} \in (L^p(\Omega))^N.$$

Hence, the whole linearization of (5.2) generates a Fredholm operator (index zero) from $\left(W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)\right)^N$ into $(L^p(\Omega))^N$, and it is an isomorphism iff it is injective.

Results of the type of Sections 3 and 4 are also true for boundary value problems for elliptic equations and systems in divergence form, see [20] for the case $N = 2$ and [12] for $N \geq 2$. In comarison with the results of the present paper for non-divergence type equations, in those papers some of the assumptions are weaker (arbitrary Lipschitz domains and arbitrary discontinuities in x, mixed boundary conditions), some stronger (the equations have to be linear with respect to the gradient Du). In the case $N > 2$ there are involved other function spaces (Sobolev-Campanato spaces), and the maximal regularity theory for the linear problems, used in [12], is developed in [8, 9, 11]. The maximal regularity theory for the linear problems, used in [20], is developed in [10].

Appendix: Superposition Operators

In this section Ω is a bounded domain in $\mathbb{R}^n$, and we consider superposition operators of the type

$$\text{(A.1)} \qquad (A(u))(x) = a(x, u(x), Du(x)) \text{ for a.a. } x \in \Omega.$$

Our first result proposes sufficient conditions in order that the superposition operator A maps functions $u \in C(\overline{\Omega})$ with $Du \in (VMO(\Omega) \cap L^\infty(\Omega))^n$ into $VMO(\Omega) \cap L^\infty(\Omega)$. It generalizes Lemma 2.1 in [18] and Lemma 2.6.2 in [16].

Lemma A.1. *Let $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function satisfying the following conditions:*

(A_1) *$a(\cdot,u,\xi)\in VMO(\Omega)$ locally uniformly in (u,ξ): For all $M>0$ it holds*

$$\gamma_M(r):=\sup_{|u|,|\xi|\le M}\sup_{0<\rho\le r}\sup_{x\in\Omega}\frac{1}{|\Omega_{\rho,x}|}\int_{\Omega_{\rho,x}}\Big|a(y,u,\xi)-\frac{1}{|\Omega_{\rho,x}|}\int\limits_{\Omega_{\rho,x}}a(z,u,\xi)dz\Big|dy$$

tends to zero as r tends to zero.

(A_2) *Continuity properties of $a(x,\cdot,\cdot)$: For all $M>0$ there exist $c_M>0$ and a non-decreasing function $\mu_M:[0,\infty)\to(0,\infty)$ with $\lim_{t\to 0}\mu_M(t)=0$ such that for a.a. $x\in\Omega$, all $u,\ u'\in\mathbb{R}$ and all $\xi,\ \xi'\in\mathbb{R}^n$ it holds*

$$|a(x,u,\xi)-a(x,u',\xi')|\le\mu_M(|u-u'|)+c_M|\xi-\xi'|.$$

(A_3) *$a(x,0,0)\in L^\infty(\Omega)$.*

Then $A(u)\in VMO(\Omega)\cap L^\infty(\Omega)$ for any $u\in C(\overline{\Omega})$ with $Du\in(VMO(\Omega)\cap L^\infty(\Omega))^n$.

Proof. Let $u\in C(\overline{\Omega})$ with $Du\in(VMO(\Omega)\cap L^\infty(\Omega))^n$, and take $M\ge\|u\|_{W^{1,\infty}(\Omega)}$. Then for a.a. $x\in\Omega$ we have

$$\begin{aligned}|a(x,u(x),Du(x))|\le&|a(x,0,0)|+|a(x,u(x),Du(x))-a(x,0,0)|\\ \le&\|a(\cdot,0,0)\|_{L^\infty(\Omega)}+\mu_M\left(\|u\|_{L^\infty(\Omega)}\right)+c_M\|Du\|_{L^\infty(\Omega)^n}.\end{aligned}$$

Hence, $A(u)\in L^\infty(\Omega)$.

Now, take $x\in\Omega$ and $0<\rho\le r$. Then

$$\begin{aligned}I(\rho,x):=&\frac{1}{|\Omega_{\rho,x}|}\int\limits_{\Omega_{\rho,x}}\Big|a(y,u(y),Du(y))-\frac{1}{|\Omega_{\rho,x}|}\int\limits_{\Omega_{\rho,x}}a(z,u(z),Du(z))dz\Big|dy\\ \le&2I_1(\rho,x)+I_2(\rho,x)\end{aligned}$$

with

$$\begin{aligned}I_1(\rho,x):=&\frac{1}{|\Omega_{\rho,x}|}\int\limits_{\Omega_{\rho,x}}\Big|a(y,u(y),Du(y))-a(y,u(x),(Du)_{\Omega_{\rho,x}})\Big|dy,\\ I_2(\rho,x):=&\frac{1}{|\Omega_{\rho,x}|}\int\limits_{\Omega_{\rho,x}}\Big|a(y,u(x),(Du)_{\Omega_{\rho,x}})-\frac{1}{|\Omega_{\rho,x}|}\int\limits_{\Omega_{\rho,x}}a(z,u(x),(Du)_{\Omega_{\rho,x}})dz\Big|dy,\\ (Du)_{\Omega_{\rho,x}}:=&\frac{1}{|\Omega_{\rho,x}|}\int\limits_{\Omega_{\rho,x}}Du(y)dy.\end{aligned}$$

It follows from (A_2) that

$$I_1(\rho,x)\le\mu_M(\omega_u(r))+c_M\gamma_{Du}(r)$$

with ω_u being the modulus of continuity of u and γ_{Du} the VMO modulus of Du. Further, (A_1) yields

$$I_2(\rho,x)\le\gamma_M(r).$$

Hence $\sup_{\rho\le r}\sup_{x\in\Omega}I(\rho,x)\to 0$ as $r\to 0$, and this completes the proof. □

The second result of this section describes conditions which imply that the superposition operator A is a C^1–smooth map from $W^{1,\infty}(\Omega)$ into $L^\infty(\Omega)$. Moreover, we show that the corresponding evaluation map

$$(a,u)\mapsto a(\cdot,u(\cdot),Du(\cdot))$$

is C^1 on suitable function spaces. The smoothness of evaluation maps depends on the choice of the function spaces (see, e.g., [1, Proposition 2.4.17]). In order to introduce our function space of the Carathéodory functions a let us use the following terminology:

Definition A.2. *Let $U \subseteq \mathbb{R} \times \mathbb{R}^n$, and let $a : \Omega \times U \to \mathbb{R}$ be a Carathéodory function.*

(i) The function a is called C^1-Carathéodory function on $\Omega \times U$ if the following conditions are fulfilled:

(A$_4$) *For almost all $x \in \Omega$ the function $a(x, \cdot)$ is continuously differentiable.*

(A$_5$) *For all compact sets $K \subset U$ there exists $c_K > 0$ such that for a.a. $x \in \Omega$ and all $(u, \xi) \in K$ it holds*

$$|a(x, u, \xi)| + |D_u a(x, u, \xi)| + \sum_{j=1}^{n} |D_{\xi_j} a(x, u, \xi)| \leq c_K.$$

(A$_6$) *For all compact sets $K \subset U$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that for a.a. $x \in \Omega$ and all $(u, \xi), (u', \xi') \in K$ with $|u - u'| + \|\xi - \xi'\| < \delta$ it holds*

$$\begin{aligned} |a(x, u, \xi) - a(x, u', \xi')| &+ |D_u a(x, u, \xi) - D_u a(x, u', \xi')| \\ &+ \sum_{j=1}^{n} |D_{\xi_j} a(x, u, \xi) - D_{\xi_j} a(x, u', \xi')| \leq \varepsilon. \end{aligned}$$

(ii) The function a is called $C^{1,1}$-Carathéodory function on $\Omega \times U$ if (A$_4$) *and* (A$_5$) *hold and the following condition is fulfilled:*

(A$_7$) *For all compact sets $K \subset U$ there exists $L_K > 0$ such that for a.a. $x \in \Omega$ and all $(u, \xi), (u', \xi') \in K$ it holds*

$$\begin{aligned} |a(x, u, \xi) - & a(x, u', \xi')| + |D_u a(x, u, \xi) - D_u a(x, u', \xi')| \\ &+ \sum_{j=1}^{n} |D_{\xi_j} a(x, u, \xi) - D_{\xi_j} a(x, u', \xi')| \leq L_K \left(|u - u'| + |\xi - \xi'|\right). \end{aligned}$$

(iii) Let $K \subset \mathbb{R} \times \mathbb{R}^n$ be a compact. The vector space of all C^1-Carathéodory functions on $\Omega \times K$, equipped with the norm

$$\begin{aligned} \|a\| := \sup_{(u,\xi) \in K} \operatorname*{ess\,sup}_{x \in \Omega} |a(x, u, \xi)| &+ \sup_{(u,\xi) \in K} \operatorname*{ess\,sup}_{x \in \Omega} |D_u a(x, u, \xi)| \\ &+ \sum_{j=1}^{n} \sup_{(u,\xi) \in K} \operatorname*{ess\,sup}_{x \in \Omega} |D_{\xi_j} a(x, u, \xi)| \end{aligned}$$

will be denoted by $\mathcal{C}^1(\Omega \times K)$.

Lemma A.3. *Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be bounded and open. Denote by $\mathcal{U}$ the set of all $u \in W^{1,\infty}(\Omega)$ such that there exists a compact $K \subset U$ with $(u(x), Du(x)) \in K$ for a.a. $x \in \Omega$. Then the following is true:*

(i) $\mathcal{U}$ is open in $W^{1,\infty}(\Omega)$;

(ii) Let $a \in \mathcal{C}^1(\Omega \times \overline{U})$. Then there exists a C^1-map $A : \mathcal{U} \to L^\infty(\Omega)$ such that for a.a. $x \in \Omega$, all $a \in \mathcal{C}^1(\Omega \times \overline{U})$ and all $u \in \mathcal{U}$ it holds (A.1). *If, moreover, a is a $C^{1,1}$-Carathéodory function, then the derivative A' is locally Lipschitz continuous.*

(iii) There exists a C^1-map $E : \mathcal{C}^1(\Omega \times \overline{U}) \times \mathcal{U} \to L^\infty(\Omega)$ such that for a.a. $x \in \Omega$, all $a \in \mathcal{C}^1(\Omega \times \overline{U})$ and all $u \in \mathcal{U}$ it holds

$$(E(a, u))(x) = a(x, u(x), Du(x)) \tag{A.2}$$

Proof. Assertion (i) is obvious. Let us show that assertion (ii) is true. We have

$$D_u a(x,u,\xi) = \lim_{v\to 0} \frac{a(x,u+v,\xi)-a(x,u,\xi)}{v}$$

for a.a. $x\in\Omega$, all $u\in\mathbb{R}$ and all $\xi\in\mathbb{R}^n$. Thus, $D_u a(\cdot,u,\xi)$ is the limit almost everywhere of a sequence of measurable functions and, hence, measurable. Analogously we get that the functions $D_{\xi_j} a(\cdot,u,\xi)$ are measurable.

Now, let us fix a function $u\in\mathcal{U}$. By definition there exists a compact $K\subset U$ with $(u(x),Du(x))\in K$ for a.a. $x\in\Omega$. Hence, because of assumption (A_5), we get that

$$a(\cdot,u(\cdot),Du(\cdot)),\ D_u a(\cdot,u(\cdot),Du(\cdot)),\ D_{\xi_j} a(\cdot,u(\cdot),Du(\cdot))\in L^\infty(\Omega). \tag{A.3}$$

If the superposition operator A is differentiable in u then its derivative can be calculated pointwise for a.a. $x\in\Omega$, i.e.

$$(A'(u)v)(x) = D_u a(x,u(x),Du(x))v(x) + D_{\xi_k} a(x,u(x),Du(x))D_k v(x) \tag{A.4}$$

Thus, the right hand side of (A.4) is a candidate for the derivative $A'(u)$. Because of (A.3) the map

$$v\mapsto D_u a(\cdot,u(\cdot),Du(\cdot))v(\cdot) + D_{\xi_k} a(\cdot,u(\cdot),Du(\cdot))D_k v(\cdot) \tag{A.5}$$

is linear and bounded from $W^{1,\infty}(\Omega)$ into $L^\infty(\Omega)$. Let us show that (A.5) is indeed the derivative of A in u. For a.a $x\in\Omega$ and all $v\in W^{1,\infty}(\Omega)$ we have

$$\begin{aligned}
&a(x,u(x)+v(x),Du(x)+Dv(x)) - a(x,u(x),Du(x))\\
&\qquad -D_u a(x,u(x),Du(x))v(x) - D_{\xi_k} a(x,u(x),Du(x))D_k v(x)\\
&\quad = \int_0^1 \Big(D_u a(x,u(x)+tv(x),Du(x)+tDv(x))v(x) - D_u a(x,u(x),Du(x))v(x)\\
&\qquad +D_{\xi_k} a(x,u(x)+tv(x),Du(x)+tDv(x))D_k v(x) - D_{\xi_k} a(x,u(x),Du(x))D_k v(x)\Big)dt.
\end{aligned}$$

Take $\varepsilon>0$. There exist a compact set $K\subset U$ and $\delta>0$ such that for all $v\in W^{1,\infty}(\Omega)$ with $\|v\|_{W^{1,\infty}(\Omega)}<\delta$ it holds $(u(x)+v(x),Du(x)+Dv(x))\in K$ for a.a. $x\in\Omega$. Taking δ small enough we can assume that it is the δ corresponding to K and ε from (A_6). Hence, we have for a.a. $x\in\Omega$ and all $v\in W^{1,\infty}(\Omega)$ with $\|v\|_{W^{1,\infty}(\Omega)}<\delta$ that

$$\begin{aligned}
&\Big|a(x,u(x)+v(x),Du(x)+Dv(x)) - a(x,u(x),Du(x))\\
&\qquad - D_u a(x,u(x),Du(x))v(x) - D_{\xi_k} a(x,u(x),Du(x))D_k v(x)\Big| \le \varepsilon\|v\|_{W^{1,\infty}(\Omega)}.
\end{aligned}$$

Now, let us show that the derivative A' is continuous in u. Take ε, K and δ as above. Then, again by (A_6), for a.a $x\in\Omega$ and all $v,w\in W^{1,\infty}(\Omega)$ with $\|v\|_{W^{1,\infty}(\Omega)}<\delta$ we have

$$\begin{aligned}
&|(A'(u+v)-A'(u)w)(x)|\\
&\quad = \Big|\left(D_u a(x,u(x)+v(x),Du(x)+Dv(x)) - D_u a(x,u(x),Du(x))\right)w(x)\\
&\qquad + \left(D_{\xi_k} a(x,u(x)+v(x),Du(x)+Dv(x)) - D_{\xi_k} a(x,u(x),Du(x))\right)D_{\xi_k} w(x)\Big|\\
&\quad \le \varepsilon\|w\|_{W^{1,\infty}(\Omega)}.
\end{aligned}$$

Analogously, one shows that the derivative A' is locally Lipschitz continuous if condition (A_7) is satisfied.

(iii) In order to show that the evaluation map E is continuously differentiable we show that its partial derivatives with respect to a and to u exist and are continuous. Obviously, the map $E(\cdot, u)$ is linear. Hence, the partial derivative D_aE of E with respect to a exists everywhere, and for a.a. $x \in \Omega$, all $a, b \in \mathcal{C}^1(\Omega \times \overline{U})$ and all $u \in \mathcal{U}$ we have

$$(D_aE(a,u)b)\,(x) = b(x, u(x), Du(x)). \tag{A.6}$$

Moreover, as above one shows that the partial derivative D_uE of E with respect to u exists everywhere, and for a.a. $x \in \Omega$ and all $a \in \mathcal{C}^1(\Omega \times \overline{U})$, $u \in \mathcal{U}$ and $v \in W^{1,\infty}(\Omega)$ we have

$$(D_uE(a,u)v)(x) = D_ua(x, u(x), Du(x))v(x) + D_{\xi_k}a(x, u(x), Du(x))D_kv(x). \tag{A.7}$$

Let $a \in \mathcal{C}^1(\Omega \times \overline{U})$ and $u \in \mathcal{U}$ be fixed. We are going to show that D_aE and D_uE are continuous in the point (A, u).

There exists a $\delta > 0$ such that for all $v \in W^{1,\infty}(\Omega)$ with $\|v\|_{W^{1,\infty}(\Omega)} < \delta$ it holds $(u(x) + v(x), Du(x) + Dv(x)) \in \overline{U}$ for a.a. $x \in \Omega$. Hence, for a.a. $x \in \Omega$, all $b, c \in \mathcal{C}^1(\Omega \times \overline{U})$ and all $v \in W^{1,\infty}(\Omega)$ with $\|v\|_{W^{1,\infty}(\Omega)} < \delta$ we have

$$\begin{aligned}
&|\,(D_aE(a+b, u+v)c - D_aE(a,u)c)\,(x)| \\
&\quad = \big|c(x, u(x)+v(x), Du(x)+Dv(x)) - c(x, u(x), Du(x))\big| \\
&\quad = \Big|\int_0^1 \big(D_uc(x, u(x)+tv(x), Du(x)+tDv(x))v(x) \\
&\qquad\quad + D_{\xi_k}c(x, u(x)+tv(x), Du(x)+tDv(x))D_kv(x)\big)dt\Big| \\
&\quad \le \|c\|_{\mathcal{C}^1(\Omega\times\overline{U})}\|v\|_{W^{1,\infty}(\Omega)}.
\end{aligned}$$

Finally, in order to show that D_uE is continuous in (A, u), we take an arbitrary $\varepsilon > 0$ and the δ from above. Choosing δ small enough we can assume that it is the δ corresponding to $\overline{U}$ and ε from (A_6). Hence, we have for a.a. $x \in \Omega$, all $b \in \mathcal{C}^1(\Omega \times \overline{U})$ and all $v, w \in W^{1,\infty}(\Omega)$ with $\|b\|_{\mathcal{C}^1(\Omega\times\overline{U})} + \|v\|_{W^{1,\infty}(\Omega)} < \delta$ that

$$\begin{aligned}
&|(D_uE(a+b, u+v)w - D_uE(a,u)w)\,(x)| \\
&\quad = \Big|\,(D_ua(x, u(x)+v(x), Du(x)+Dv(x)) - D_ua(x, u(x), Du(x)))\,w(x) \\
&\qquad + (D_{\xi_k}a(x, u(x)+v(x), Du(x)+Dv(x)) - D_{\xi_k}a(x, u(x), Du(x)))\,D_{\xi_k}w(x) \\
&\qquad + D_ub(x, u(x)+v(x), Du(x)+Dv(x))w(x) + D_{\xi_k}b(x, u(x)+v(x), Du(x)D_{\xi_k}w(x)\Big| \\
&\quad \le (\varepsilon + \delta)\|w\|_{W^{1,\infty}(\Omega)}.
\end{aligned}$$

□

Acknowledgements. This paper was written during a visit of the second author in Bari. He likes to thank the Istituto Nazionale di Alta Matematica "F. Severi" for the support and the Department of Mathematics, Technical University of Bari for the kind hospitality.

References

[1] R. Abraham, J. E. Marsden and T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, 2nd ed., Springer–Verlag, Berlin 1988.

[2] S. Campanato, Un risultato relativo ad equazioni ellittiche del secondo ordine di tipo non variazionale, *Ann. Scuola Norm. Sup. Pisa* **21** (1967), 701–707.

[3] S. Campanato, On the condition of nearness between operators, *Ann. Mat. Pura Appl.* **167** (1994), 243–256.

[4] F. Chiarenza, M. Frasca and P. Longo, $W^{2,p}$-solvability of the Dirichlet problem for non divergence elliptic equations with VMO coefficients, *Trans. Amer. Math. Soc.* **336** (1993), 841–853.

[5] G. Di Fazio and D.K. Palagachev, Oblique derivative problem for quasilinear elliptic equations with VMO coefficients, *Bull. Aust. Math. Soc.* **53** (1996), 501–513.
[6] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 3rd ed., Springer–Verlag, Berlin 1997.
[7] S. Giuffrè, Oblique derivative problem for nonlinear elliptic discontinuous operators in the plane with quadratic growth, *C. R. Acad. Sci., Paris, Sér. I, Math.* **328** (1999), 859–864.
[8] J. A. Griepentrog, Linear elliptic boundary value problems with non-smooth data: Campanato spaces of functionals, *Mathem. Nachr.* **243** (2002), 19–42.
[9] J. A. Griepentrog and L. Recke, Linear elliptic boundary value problems with non-smooth data: Normal solvability in Sobolev-Campanato spaces, *Mathem. Nachr.* **225** (2001), 39–74.
[10] K. Gröger, A $W^{1,p}$–estimate for solutions to mixed boundary value problems for second order elliptic differential equations, *Math. Ann.* **283** (1989), 679–687.
[11] K. Gröger and L. Recke, Preduals of Campanato and Sobolev-Campanato spaces: A general approach, *Mathem. Nachr.* **230** (2001), 45–72.
[12] K. Gröger and L. Recke, Applications of differential calculus to quasilinear elliptic boundary value problems with non-smooth data, to appear in *Nonl. Differ. Equat. Appl. (NoDEA)* (2004).
[13] F. John and L. Nirenberg, On functions of bounded mean oscillation, *Commun. Pure Appl. Math.* **14** (1961), 415–426.
[14] O.A. Ladyzhenskaya and N.N. Ural'tseva, A survey of results on the solubility of boundary-value problems for second-order uniformly elliptic and parabolic quasi-linear equations having unbounded singularities, *Russian Math. Surveys* **41** (1986), no. 5, 1–31.
[15] A. Maugeri and D.K. Palagachev, Boundary value problem with an oblique derivative for uniformly elliptic operators with discontinuous coefficients, *Forum Math.* **10** (1998), 393–405.
[16] A. Maugeri, D.K. Palagachev and L.G. Softova, *Elliptic and Parabolic Equations with Discontinuous Coefficients*, Math. Res., Vol. **109**, Wiley-VCH, Berlin, 2000.
[17] D.K. Palagachev, Global strong solvability of Dirichlet problem for a class of nonlinear elliptic equations in the plane, *Le Matematiche* **48** (1993), 311–321.
[18] D.K. Palagachev, Quasilinear elliptic equations with VMO coefficients, *Trans. Amer. Math. Soc.* **347** (1995), 2481–2493.
[19] D.K. Palagachev, Dirichlet problem for a class of second order nonlinear elliptic equations, in: *Proc. Fifth Int. Coll. Differ. Equations, Plovdiv'94,* VSP International Science Publishers, Zeist, The Netherlands 1995, pp. 273–282.
[20] L. Recke, Applications of the Implicit Function Theorem to quasilinear elliptic boundary value problems with non-smooth data, *Commun. Partial Differ. Equat.* **20** (1995), 1457–1479.
[21] D. Sarason, Functions of vanishing mean oscillation, *Trans. Amer. Math. Soc.* **207** (1975), 391–405.
[22] L.G. Softova, An integral estimate for the gradient for a class of nonlinear elliptic equations in the plane, *Z. Anal. Anwend.* **17** (1998), 57–66.
[23] K.S. Tulenbaev and N.N. Ural'tseva, A nonlinear boundary value problem for elliptic equations of general form, *Partial Differential Equations, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk*, **133** (1987), 95–112 (in Russian).
[24] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Volume I: Fixed-Points Theorems*, Springer-Verlag, Berlin, 1993.

Dian K. Palagachev, Politecnico di Bari, Dipartimento di Matematica, Via E. Orabona, 4, 70 125 Bari, Italy
E-mail address: dian@dm.uniba.it

Lutz Recke, Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, D-10099 Berlin, Germany
E-mail address: recke@mathematik.hu-berlin.de

Lubomira G. Softova, Bulgarian Academy of Sciences, Institute of Mathematics and Informatics, Sofia, Bulgaria
E-mail address: luba@dm.uniba.it