Integral Manifolds for Nonautonomous Slow-Fast Systems without Dichotomy

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Abstract

This work is devoted to nonautonomous slow-fast systems of ordinary differential equation without dichotomy. We are interested in the existence of a slow integral manifold in order to eliminate the fast variables.

The peculiarity of the problem under consideration is that the right hand side of the system depends on some parameter vector which can be considered as a control to be determined in order to guarantee the existence of an integral manifold consisting of canard trajectories. We call the vector function as gluing function. We prove that under some conditions on the right hand side of the system there exists a unique gluing function such that the system has a slow integral manifold. We investigate the problems of asymptotic expansions of the integral manifold and the gluing function, and study their smoothness.

Keywords: integral manifolds, slow-fast systems, canard-trajectories, missing dichotomy

Zusammenfassung

In der vorliegenden Arbeit betrachten wir ein System nichtautonomer gewhnlicher Differentialgleichungen, das aus zwei gekoppelten Teilsystemen besteht. Die Teilsysteme bestehen aus langsamen bzw. schnellen Variablen, wobei die Zeitskalierung durch Multiplikation der rechten Seite eines Teilsystems mit einem kleinen Faktor erzeugt wird.
Das Ziel unserer Untersuchungen besteht im Nachweis der Existenz einer Integralmannigfaltigkeit, mit deren Hilfe die schnellen Variablen eliminiert werden können. Dabei verzichten wir auf die übliche Annahme einer Dichotomiebedingung und ersetzen diese durch die Hinzunahme eines zusätzlichen Steuervektors. Wir beweisen, dass unter gewissen Voraussetzungen über die rechten Seiten der Teilsysteme ein eindeutiger Steuervektor existiert, der die Existenz der gewünschten Integralmannigfaltigkeit impliziert. Das Prinzip des Nachweises einer solchen beschränkten Integralmannigfaltigkeit basiert auf dem Zusammenkleben von anziehenden und abstossenden invarianten Mannigfaltigkeiten. In der Arbeit wird die Glattheit dieser Mannigfaltigkeit sowie deren asymptotische Entwicklung nach dem kleinen Parameter untersucht.

Schlagwörter: Integralmannigfaltigkeiten, langsamen und schnellen Variablen, Canard-Trajektorien, fehlende Dichotomie

Contents

Chapter 1

Introduction

Systems of differential equations with several time scales play an important role in modelling of processes of different nature studied in mechanics [28], reaction kinetics [6, 9], biophysics [26], modern technologies (e.g. dynamics of semiconductor lasers [31, 32, 35]). There are many well developed methods to study such systems including methods of the theory of singular perturbations, geometric methods, asymptotic methods. They allow one to study the problem of existence of solutions, longtime behavior of the system [7, 11, 33, 38], the phenomenon of delayed loss of stability [18, 19], existence of canard-type solutions [16], manifolds consisting of such solutions and other problems.

In this work we restrict ourself to systems of the type

$$\begin{aligned} \frac{dy}{dt} &= \varepsilon f(t, y, z, \varepsilon), \\ \frac{dz}{dt} &= g(t, y, z, \varepsilon), \end{aligned} \tag{1.1}$$

where $y \in \mathbb{R}^n$, $z \in \mathbb{R}^2$, ε is a small positive parameter, f, g are sufficiently smooth functions. The variable y is called a slow variable, the variable z is a fast variable.

One of the effective tools to study such type of systems is the method of integral manifolds. The method has been extensively developed by many authors, see for example [3, 2, 5, 8, 10, 17, 20, 22, 34, 39].

Definition 1.1 *A surface $S_\varepsilon \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^2$ is called an integral manifold of the system (1.1) if any trajectory $(t, y(t, \varepsilon; y_0, z_0),\ z(t, \varepsilon; y_0, z_0))$ with $(t_0, y_0, z_0) \in S_\varepsilon$ belongs to S_ε for all $t \in \mathbb{R}$.*

We are interested in the integral manifolds of the form $z = h(t, y, \varepsilon)$.

Thus, we can reduce the dimension of the system. Then, the dynamics of the system on this manifold is described by the equation

$$\frac{dy}{dt} = \varepsilon f(t, y, h(t, y, \varepsilon), \varepsilon).$$

Setting $\varepsilon = 0$ we get the degenerate problem

$$\begin{aligned} \frac{dy}{dt} &= 0, \\ \frac{dz}{dt} &= g(t, y, z, 0). \end{aligned} \tag{1.2}$$

the manifold of equilibria of (1.2) is a solution of the equation

$$g(t, y, z, 0) = 0, \tag{1.3}$$

of the form $z = \varphi(t, y)$. Here y is considered as a parameter.

Assume that (1.3) has a root $z = \varphi(t, y)$. Then by linearizing (1.1) in the small neighbourhood of $z = \varphi(t, y)$ we obtain the system

$$\begin{aligned} \frac{dy}{dt} &= \varepsilon f(t, y, z, \varepsilon), \\ \frac{dz}{dt} &= Bz + \tilde{g}(t, y, z, \varepsilon), \end{aligned} \tag{1.4}$$

where

$$B = \frac{\partial}{\partial z} g(t, y, \varphi(t, y), 0),$$

In the case (1.3) has multiple root the problem of existence of integral manifolds is not well developed. We would like to mention [25], where some cases of branching of integral manifolds have been studied.

Suppose that B in (1.4) is a constant matrix. Under the condition that B is hyperbolic the problem of existence of the integral manifold for (1.4) has been studied by many authors (see for example [5, 17, 23, 39]).

In the case that $B = B(t, y)$, the uniformly exponential dichotomy assumption implies the existence if the integral manifold we have to assume that the linear problem $\dot{z} = B(t)z$ possesses an exponential dichotomy [10, 17, 34].

In the present work we consider the nonautonomous slow-fast system

$$\begin{aligned}\frac{dy}{dt} &= \varepsilon f(t,y,z,\varepsilon),\\ \frac{dz}{dt} &= B(t)z+\tilde{g}(t,y,z,a,\varepsilon),\end{aligned} \tag{1.5}$$

in the case that the dichotomy assumption fails. More precisely, the matrix $B(t)$ has a pair of simple complex conjugate eigenvalues crossing the imaginary axis for increasing t at some moment $t=t_0$ from left to right, that is the dichotomy conditions fails. We study the problem of existence of an integral manifold, its asymptotics and smoothness.

The problem considered has an important feature compared to the case when the dichotomy condition is valid: The system contains a parameter a, in the simple case it is a vector, in more general case it is a function depending on the slow variables. This parameter we call gluing vector or gluing function, respectively. We prove that the system has an integral manifold $z=h(t,y,\varepsilon)$ for a unique a. The idea to use an additional parameter is similar to the method of functionalization of parameter [13]. The use of an additional parameter a to ensure the existence of integral manifolds and canard solutions in the cases of the absence of dichotomy has been known for some classes of singularly perturbed systems [6, 24, 27, 29].

The work is organized as follows. In chapter 2 we consider the system

$$\frac{dz}{dt} = B(t)z+Z(t,z)+a, \tag{1.6}$$

where B is defined as

$$B(t)=\begin{pmatrix}\alpha t & \beta\\ -\beta & \alpha t\end{pmatrix}, \qquad \alpha,\beta>0, \tag{1.7}$$

and a is a gluing vector.

We prove the existence of the uniformly bounded solution of (1.6) for a unique value of a. The proof is based on the gluing method: mainly, with the help of the parameter a we glue together solutions bounded on semiaxes. The results and methods of this chapter play an important role in the further study of slow-fast systems.

In the rest of the work we consider the system

$$\begin{aligned}\frac{dy}{dt} &= \varepsilon Y(t,y,z,\varepsilon),\\ \frac{dz}{dt} &= B(t)z+Z(t,y,z,a(y,\varepsilon),\varepsilon)+a(y,\varepsilon),\end{aligned} \tag{1.8}$$

where ε is a small positive parameter, $a(y, \varepsilon)$ is the gluing function, $B(t)$ is the matrix (1.7). We prove that under some conditions there exists a unique function $a(y, \varepsilon)$ such that system (1.8) has the integral manifold $z = h(t, y, \varepsilon)$, where h is a uniformly bounded function. This manifold is attractive for $t < 0$ and repulsive for $t > 0$.

Chapter 4 is devoted to the study of asymptotic approximations of the integral manifold and the gluing function. We derive an algorithm of finding the coefficients of the approximations and estimate the error of approximations. In the last chapter we give some differential properties of the manifold.

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Chapter 2

Bounded solutions for nonlinear systems

2.1 Problem statement

This chapter is devoted to the problem of existence of bounded solutions for systems of the form

$$\frac{dz}{dt} = B(t)z + Z(t,z), \tag{2.1}$$

where $z \in \Omega_z$, $\Omega_z := \{z \in \mathbb{R}^2 : \|z\| \leq \Delta\}$.

In the case $B(t)$ possesses an exponential dichotomy the problem of existence of bounded solutions for (2.1) was extensively studied (see e.g. [15, 21]).

In what follows we suppose that $B(t)$ is the matrix

$$B(t) = \begin{pmatrix} \alpha t & \beta \\ -\beta & \alpha t \end{pmatrix}. \tag{2.2}$$

We note that the eigenvalues of B have negative real parts for $t < 0$ and positive ones for $t > 0$.

Concerning the function $Z(t,z)$ we suppose

(A$_1$). $Z(t,y)$ is continuous on $\mathbb{R} \times \Omega_z$ and satisfies the following conditions

$$\begin{aligned} \|Z(t,z)\| &\leq M, \\ \|Z(t,z) - Z(t,\bar{z})\| &\leq \mu\|z - \bar{z}\|. \end{aligned} \tag{2.3}$$

Here and elsewhere, $\|\cdot\|$ denotes the Euclidean norm and the corresponding norm of matrices.

Let $W(t)$ be the matrix

$$W(t) = \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}. \tag{2.4}$$

Then

$$V(t,t_0) := e^{\frac{\alpha(t^2-t_0^2)}{2}} W(t-t_0) \tag{2.5}$$

is a fundamental matrix of the linear system

$$\frac{dz}{dt} = B(t)z. \tag{2.6}$$

Here, $z \equiv 0$ is the only bounded solution of (2.6). Other solutions satisfy

$$\|z(t)\| = \|z(t_0)\| e^{\frac{\alpha(t^2-t_0^2)}{2}}.$$

Since the matrix $B(t)$ is stable for $t < 0$ and unstable for $t > 0$, this relation shows that the behaviour of the trajectories of the system is similar to that, typical for problems on delayed loss of stability.

From the assumption (A_1) it follows that for any pair (t_0, z_0) the Cauchy problem for equation (2.1) with the initial condition $z(t_0) = z_0$ has a unique solution. This problem is equivalent to the integral equation

$$z(t) = V(t,t_0)\left(z_0 + \int_{t_0}^{t} V^{-1}(s,t_0) Z(s, z(s)) ds\right), \tag{2.7}$$

that can be rewritten as

$$V^{-1}(t,t_0)z(t) = z_0 + \int_{t_0}^{t} V^{-1}(s,t_0) Z(s, z(s)) ds. \tag{2.8}$$

If there exists a bounded solution $z(t)$ of (2.1), then from (2.8) it follows that

$$\|V^{-1}(t,t_0)z(t)\| \le c e^{\frac{\alpha(t_0^2-t^2)}{2}} \tag{2.9}$$

and we get

$$\lim_{t\to\pm\infty} \|V^{-1}(t,t_0)z(t)\| = 0. \tag{2.10}$$

Since $W(t-s) = W(t)W^{-1}(s)$, from (2.8) and (2.10) it follows that the initial value z_0 has to fulfil the conditions

$$
\begin{aligned}
z_0 &= \int_{-\infty}^{t_0} e^{\frac{\alpha(t_0^2 - s^2)}{2}} W(t_0 - s) Z(s, z(s)) ds, \\
z_0 &= -\int_{t_0}^{+\infty} e^{\frac{\alpha(t_0^2 - s^2)}{2}} W(t_0 - s) Z(s, z(s)) ds,
\end{aligned}
\tag{2.11}
$$

Substituting these formulas into (2.7) we get for a bounded solution of (2.1)

$$
z(t) = \begin{cases} \displaystyle\int_{-\infty}^{t} e^{\frac{\alpha(t^2 - s^2)}{2}} W(t-s) Z(s, z(s)) ds, & t < 0 \\ \displaystyle -\int_{t}^{+\infty} e^{\frac{\alpha(t^2 - s^2)}{2}} W(t-s) Z(s, z(s)) ds, & t \geq 0. \end{cases}
\tag{2.12}
$$

From the condition of continuity of the bounded solution we get the condition

$$
\int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) Z(s, z(s)) ds = 0 \tag{2.13}
$$

on the function Z. It is clear that (2.13) is not fulfilled for arbitrary function $Z(t, z)$.

Let us consider some examples.

Example 2.1 Consider the system

$$
\frac{dz}{dt} = B(t)z + a, \tag{2.14}
$$

where $a = (a_1, a_2)^T$ is a parameter vector.

For system (2.14)

$$
z^+(t) = -\int_{t}^{+\infty} e^{\frac{\alpha(t^2 - s^2)}{2}} W(t-s) a \, ds,
$$

represents the solution bounded for $t > 0$ and

$$z^-(t) = \int\limits_{-\infty}^{t} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)a\,ds, ,$$

is the solution bounded for $t < 0$.

Between these solutions there is a "step"

$$z^-(0) - z^+(0) = \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W(t-s)a\,ds = \frac{\sqrt{2\pi}}{\sqrt{\alpha}}\, e^{\frac{-\beta^2}{2\alpha}}\, a.$$

Taking the vector $a = 0$ we can remove this step and "glue" these solutions. Then under the condition that $a = 0$ system (2.14) has the solution $z \equiv 0$ bounded for all t.

In this example the vector a plays a role of a control or "gluing" parameter: by changing the value of a we are able to "glue" together solutions bounded on negative and positive semi-axes.

Example 2.2 Consider the system

$$\frac{dz}{dt} \quad = \quad B(t)z + f(t) + a, \tag{2.15}$$

where $f(t)$ is continuous and bounded for all $t \in \mathbb{R}$.

In order to have the uniformly bounded solution, we use (2.13) to get the equation for determining the vector a and arrive at

$$\int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)\,(f(s) + a)\; ds = 0. \tag{2.16}$$

Let us introduce the following notation

$$J := \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)\,ds = \frac{\sqrt{2\pi}}{\sqrt{\alpha}}\, e^{\frac{-\beta^2}{2\alpha}}\, I, \tag{2.17}$$

where I is the identity matrix. From (2.16), we get

$$a_0 := -J^{-1} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) f(s)\,ds.$$

Therefore system (2.15) with $a = a_0$ has a unique solution bounded for all t. This solution is defined by

$$z(t) = \begin{cases} \int\limits_{-\infty}^{t} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\left(f(s)+a_0\right) ds, & t<0, \\ -\int\limits_{t}^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\left(f(s)+a_0\right) ds, & t \geq 0. \end{cases}$$

For example, let us take in (2.15)

$$\alpha = \beta = 0,$$

$$f(t) = (\cos t,\ 0)^T. \tag{2.18}$$

Then

$$a_0 = -\frac{e^{1/2}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} W^{-1}(s) Z(s,y) ds = -\left(\frac{e^{1/2}}{2}(1+e^{-2}), 0\right)^T,$$

and the bounded solution is defined by

$$z(t) = \begin{cases} \int\limits_{-\infty}^{t} e^{\frac{t^2-s^2}{2}} (f(s)+a_0) ds & t<0, \\ -\int\limits_{t}^{+\infty} e^{\frac{t^2-s^2}{2}} (f(s)+a_0) ds & t \geq 0. \end{cases}$$

Its graph is shown on Figure 1.

The idea of gluing attracting and repelling parts is applied in [6, 27] for obtaining integral manifolds with variable attractivity and canard solutions.

Let us apply this approach to system (2.1). For this purpose we introduce a gluing parameter into the system. Thus, we consider a system of the form

$$\frac{dz}{dt} = B(t)z + Z(t,z) + a. \tag{2.19}$$

In the next section we establish conditions under which (2.19) has a global uniformly bounded solution.

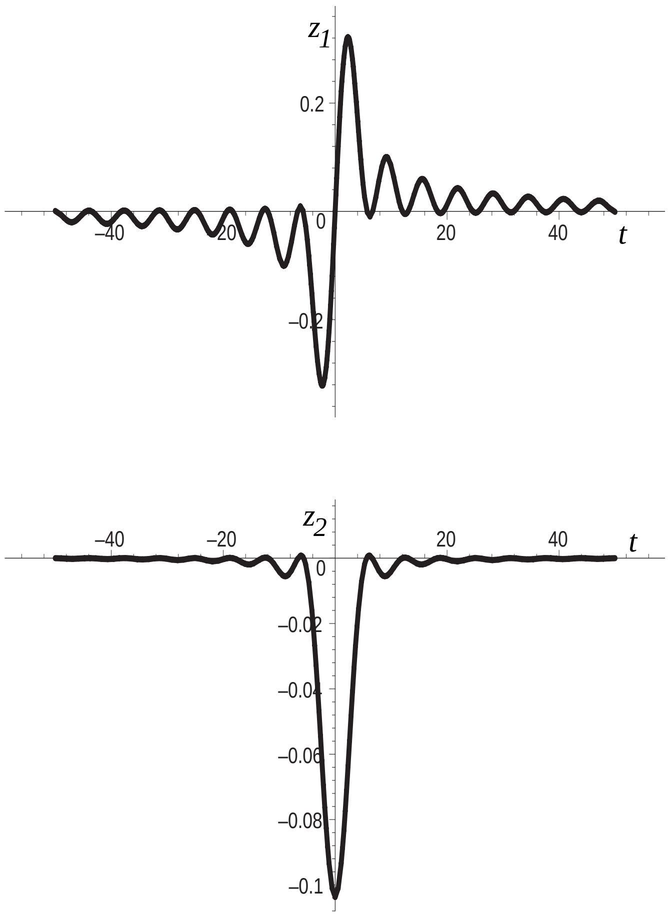

Figure 2.1: The two components of the bounded solution.

2.2 Existence of bounded solutions

We consider the system

$$\frac{dz}{dt} = B(t)z + Z(t,z) + a, \tag{2.20}$$

where $B(t)$ is defined by (2.2) and a is a vector of parameters.

Theorem 2.3 *Let the function $Z(t,z)$ in the r.h.s. of (2.20) satisfy the assumption (A_1). Let*

$$\frac{\sqrt{2\pi}}{\sqrt{\alpha}}\mu(1 + e^{\beta^2/2\alpha}) < 1. \tag{2.21}$$

Then there exists a unique vector a such that (2.20) has a global uniformly bounded solution.

Generally, solutions of (2.20) exhibit the same type of behaviour as that of (2.6). More precisely, the trajectory of system (2.20) starting for $t = t_0 < 0$ at any initial point z_0 enters after a short time interval a small neighbourhood of the uniformly bounded solution and stays in it until some time $t = t^*(t_0, z_0) > 0$, where t^* increases with respect to $|t_0|$. For $t > |t_0|$ the trajectory jumps away. This phenomenon is similar to the effect of delayed loss of stability for singularly perturbed systems [18, 19, 30].

2.3 Proof of Theorem 2.3

Let H be the complete metric space of functions $h(t)$ mapping continuously $\mathbb{R}$ into Ω_z, satisfying the inequality

$$\|h(t)\| \le N, \tag{2.22}$$

with $N \le \Delta$, and the uniform metric

$$\rho(h, \bar{h}) = \sup_{t\in\mathbb{R}} \|h(t) - \bar{h}(t)\|.$$

On the space H we define the operator T of the form

$$Th(t) = \begin{cases} -\int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\left[Z\left(s, h(s)\right) + a\right] ds, & t \ge 0, \\ \int\limits_{-\infty}^{t} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\left[Z\left(s, h(s)\right) + a\right] ds, & t < 0, \end{cases} \tag{2.23}$$

with $a = a(h)$. The formula for a will be given below explicitly.

In the upper line of (2.23) there is an operator for the existence of the bounded solution for $t \geq 0$, and in the lower line there is an operator for the existence of the bounded solution for $t < 0$. The vector $a = a(h)$ is for gluing these solutions. Following [6] we call a the gluing vector.

We shall prove that the operator T maps the space H into itself and is a contraction. The proof includes several steps. First, we show that the function Th is continuous, then we derive conditions under which $Th \in H$ for any $h \in H$. At the end we show that T is a contraction operator in H. Therefore, there exists a unique fixed point of T in H. The fixed point represents the solution of (2.20) bounded for all $t \in \mathbb{R}$.

2.3.1 Continuity of the function Th

It is easy to check that Th is continuous for $t < 0$ and $t > 0$ for arbitrary $h \in H$. The continuity of Th at $t = 0$ is considered in the following lemma.

Lemma 2.4 *For any function $h \in H$ there exist a unique vector a such that the function $T_a h$ is continuous.*

Proof.

The condition of continuity of the function $T_a h$ at the point $t = 0$ is equivalent to the following condition

$$\int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)[Z(s, h(s)) + a]\, ds = 0. \tag{2.24}$$

Let us rewrite (2.24) in the form

$$J_1 + J\, a = 0,$$

where J is defined by (2.17) and

$$J_1 := \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) Z(s, h(s))\, ds.$$

The integral J_1 converges due to the assumption (A_1) on the function Z. Therefore, $a := -J^{-1} J_1$, that is

$$a = -\frac{\sqrt{\alpha} e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) Z(s, h(s))\, ds. \tag{2.25}$$

It completes the proof.

For the following we need the next lemma.

Lemma 2.5 *The following estimates are valid*

$$\|a\| \leq e^{\beta^2/2\alpha} M,$$

$$\|a - \bar{a}\| \leq e^{\beta^2/2\alpha} \mu\rho(h, \bar{h}),$$

where $a = P(h)$ *and* $\bar{a} = P(\bar{h})$ *for any* $h, \bar{h} \in H$.

Proof. From (2.25) and the assumption (A_1) it follows

$$\|a\| \leq \|J^{-1}\| \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \|Z(s, h(s))\| \, ds \leq e^{\beta^2/2\alpha} M.$$

For the difference between a and $\bar{a}$ we have

$$\|a - \bar{a}\| \leq \|J^{-1}\| \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \|Z(s, h(s)) - Z\left(s, \bar{h}(s)\right)\| \, ds \leq$$

$$\leq \frac{\sqrt{\alpha} e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \mu \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \|h(s) - \bar{h}(s)\| \, ds \leq e^{\beta^2/2\alpha} \mu\rho(h, \bar{h}).$$

Thus,

$$\|a - \bar{a}\| \leq e^{\beta^2/2\alpha} \mu\rho(h, \bar{h}). \tag{2.26}$$

This completes the proof of Lemma 2.5.

2.3.2 Existence of the bounded solution

Now we derive the conditions guaranteeing that $Th(t)$ maps H into itself. Let $t \geq 0$. By the assumption (A_1) and Lemma 2.5 we have

$$\|Th(t)\| \leq \int_{t}^{+\infty} e^{\frac{\alpha(t^2 - s^2)}{2}} \left[\|Z(s, h(s))\| + \|a\|\right] ds \leq \frac{\sqrt{2\pi}}{\sqrt{\alpha}} M(1 + e^{\beta^2/2\alpha}).$$

Analogously, one sees that the same estimate holds for $t \leq 0$. It means that Th is uniformly bounded. Thus, under the condition

$$\frac{\sqrt{2\pi}}{\sqrt{\alpha}} M(1 + e^{\beta^2/2\alpha}) \leq N$$

the function Th belongs to the space H.

Under the assumption (A_1) and Lemma 2.5 we obtain for $t \geq 0$

$$\|Th(t) - T\bar{h}(t)\| \leq \int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} \left(\|Z(s,h(s)) - Z(s,\bar{h}(s))\| + \|a - \bar{a}\|\right) ds \leq$$

$$\leq \int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} [\mu\rho(h,\bar{h}) + e^{\beta^2/2\alpha}\mu\rho(h,\bar{h})]\, ds \;=\frac{\sqrt{2\pi}}{\sqrt{\alpha}}\mu(1+e^{\beta^2/2\alpha})\rho(h,\bar{h}).$$

The same estimate is valid for $t \leq 0$. Therefore,

$$\rho(Th, T\bar{h}) \leq \frac{\sqrt{2\pi}}{\sqrt{\alpha}}\mu(1+e^{\beta^2/2\alpha})\rho(h,\bar{h}),$$

and the condition (2.21) implies that T is a contraction in H. This completes the proof of Theorem 2.3.

Chapter 3

Integral manifolds for slow-fast systems

3.1 Problem statement

In this chapter we consider the system

$$\begin{aligned} \frac{dy}{dt} &= \varepsilon Y(t, y, z, \varepsilon), \\ \frac{dz}{dt} &= B(t)z + Z(t, y, z, \varepsilon), \end{aligned} \tag{3.1}$$

where $y \in \mathbb{R}^n$, $\varepsilon \in I_{\varepsilon_0} := \{\varepsilon \in \mathbb{R} : 0 \leq \varepsilon \leq \varepsilon_0 \ll 1\}$, $B(t)$ is the matrix

$$B(t) = \begin{pmatrix} \alpha t & \beta \\ -\beta & \alpha t \end{pmatrix}, \quad \alpha, \beta > 0.$$

For $\varepsilon = 0$ we get the system

$$\begin{aligned} y &= y_0, \\ \frac{dz}{dt} &= B(t)z + Z(t, y_0, z, 0). \end{aligned} \tag{3.2}$$

System (3.2) is a system of the same type as considered in chapter 2. Applying results of chapter 2 we can conclude that by adding a gluing parameter into the system

$$\begin{aligned} y &= y_0, \\ \frac{dz}{dt} &= B(t)z + Z(t, y_0, z, a, 0) + a, \end{aligned} \tag{3.3}$$

we can obtain a solution of the form $z = h(t; y_0)$ where h is bounded for all $t \in \mathbb{R}$. It is obvious that the value of a depends on y_0. Moreover, taking instead of y_0 any function $y = y(t, \varepsilon)$ we get that for the system

$$\frac{dz}{dt} = B(t)z + Z(t, y(t, \varepsilon), z, a, \varepsilon) + a, \tag{3.4}$$

there exists a unique value of a such that (3.4) has a global uniformly bounded solution $z = h(t; y, \varepsilon)$. It is obvious that the value of a is related to the choice of $y(t, \varepsilon)$. Then, by taking different values of a we can "glue" together the solutions bounded on semi-axes for different functions y.

Therefore, in order to obtain an integral manifold we must take a as a function depending on y and ε. This means we consider the system

$$\begin{aligned} \frac{dy}{dt} &= \varepsilon Y(t, y, z, \varepsilon), \\ \frac{dz}{dt} &= B(t)z + Z(t, y, z, a(y, \varepsilon), \varepsilon) + a(y, \varepsilon). \end{aligned} \tag{3.5}$$

In the next section we establish the conditions under which systems of the type (3.5) have an integral manifold $z = h(t, y, \varepsilon)$.

3.2 Assumptions. Notations

We consider a system of the type

$$\begin{aligned} \frac{dy}{dt} &= \varepsilon Y(t, y, z, \varepsilon), \\ \frac{dz}{dt} &= B(t)z + Z(t, y, z, a(y, \varepsilon), \varepsilon) + a(y, \varepsilon), \end{aligned} \tag{3.6}$$

where $y \in \mathbb{R}^n$, $z \in \Omega_z$, $a \in \Omega_a$, $\Omega_a := \{a \in \mathbb{R}^2 : \|a\| \leq \delta\}$, $\varepsilon \in I_{\varepsilon_0}$, and $B(t)$ is the matrix

$$B(t) = \begin{pmatrix} \alpha t & \beta \\ -\beta & \alpha t \end{pmatrix}, \qquad \alpha, \beta > 0. \tag{3.7}$$

Concerning the functions Y, Z we suppose

(H$_1$). The function Y is continuous on $\mathbb{R} \times \mathbb{R}^n \times \Omega_z \times I_{\varepsilon_0}$, and satisfies for $t \in \mathbb{R}$, $y, \bar{y} \in \mathbb{R}^n, z, \bar{z} \in \Omega_z, \varepsilon \in I_{\varepsilon_0}$ the inequalities:

$$\|Y(t, y, z, \varepsilon)\| \leq K, \tag{3.8}$$

$$\|Y(t, y, z, \varepsilon) - Y(t, \bar{y}, \bar{z}, \varepsilon)\| \leq \mu \left(\|y - \bar{y}\| + \|z - \bar{z}\|\right). \tag{3.9}$$

(H$_2$). The function Z is continuous on $\mathbb{R} \times \mathbb{R}^n \times \Omega_z \times \Omega_a \times I_{\varepsilon_0}$, and satisfy for $t \in \mathbb{R}$, $y, \bar{y} \in \mathbb{R}^n, z, \bar{z} \in \Omega_z, a, \bar{a} \in \Omega_a, \varepsilon \in I_{\varepsilon_0}$ the inequalities:

$$\|Z(t, y, z, a, \varepsilon)\| \leq M \left(\varepsilon + \varepsilon\|z\| + \|z\|^2\right), \tag{3.10}$$

$$\|Z(t, y, z, a, \varepsilon) - Z(t, \bar{y}, \bar{z}, \bar{a}, \varepsilon)\| \leq$$
$$D \left((\varepsilon + \varepsilon\|\tilde{z}\| + \|\tilde{z}\|^2)\|y - \bar{y}\| + (\varepsilon + \|\tilde{z}\|)\|z - \bar{z}\| + \varepsilon\|a - \bar{a}\|\right), \tag{3.11}$$

where $\|\tilde{z}\| := \max\{\|z\|, \|\bar{z}\|\}$.

Here K, M, D, μ, some positive numbers which will be specified below.

From (3.10) it follow that

$$Z(t, y, 0, a, 0) \equiv 0.$$

Hence, for $\varepsilon = 0$, $a = 0$ system (3.6) coincides with the linear system (2.6) and has the integral manifold $z \equiv 0$, which attracts all trajectories for $t < 0$ and repels them for $t > 0$. Moreover, the trajectories of system (3.6) starting for $t = t_0 < 0$ at any initial point after a short time interval enter a small neighbourhood of the integral manifold $z \equiv 0$ and stays in it until some time $t = t^* > 0$, where t^* increases with respect to $|t_0|$. For $t > |t_0|$ the trajectory jumps away. This effect is similar to the phenomenon of delayed loss of stability in the theory of singularly perturbed systems [4, 18, 30]. We call the manifolds with this property as the manifolds loosing their attractivity in time.

Let F be the complete metric space of continuous functions a mapping $\mathbb{R}^n \times I_{\varepsilon_0} \to \Omega_a$, satisfying the inequalities

$$\begin{aligned} \|a(y, \varepsilon)\| &\leq \varepsilon L, \\ \|a(y, \varepsilon) - a(\bar{y}, \varepsilon)\| &\leq \varepsilon\nu\|y - \bar{y}\|, \end{aligned} \tag{3.12}$$

where $\varepsilon L \leq \delta$, with the metric defined by

$$\rho\,(a, \bar{a}) = \sup_{y \in \mathbb{R}^n,\, \varepsilon \in I_{\varepsilon_0}} \|a(y, \varepsilon) - \bar{a}(y, \varepsilon)\|.$$

Let H be the complete metric space of continuous functions h mapping $\mathbb{R} \times \mathbb{R}^n \times I_{\varepsilon_0}$ into Ω_z satisfying the inequalities

$$\begin{aligned} \|h(t, y, \varepsilon)\| &\leq \varepsilon N, \\ \|h(t, y, \varepsilon) - h(t, \bar{y}, \varepsilon)\| &\leq \varepsilon\xi\|y - \bar{y}\|, \end{aligned} \tag{3.13}$$

for $t \in \mathbb{R}, y, \bar{y} \in \mathbb{R}^n, \varepsilon \in I_{\varepsilon_0}$, where N, ξ are some positive numbers such that $\varepsilon N \leq \Delta$, with the metric

$$\rho(h, \bar{h}) = \sup_{t \in \mathbb{R},\, y \in \mathbb{R}^n, \varepsilon \in I_{\varepsilon_0}} \| h(t, y, \varepsilon) - \bar{h}(t, y, \varepsilon) \|.$$

The functions $y(t, \varepsilon)$, $z = h(t, y(t, \varepsilon), \varepsilon)$ are the solution of (3.6) if they satisfy system (3.6).

Consider the equation

$$\frac{dy}{ds} = \varepsilon Y(s, y, h(s, y, \varepsilon), \varepsilon). \tag{3.14}$$

From the conditions (3.9), (3.13) it follows

$$\| Y(s, y, h(s, y, \varepsilon), \varepsilon) - Y(s, \bar{y}, h(s, \bar{y}, \varepsilon), \varepsilon) \| \leq \mu(1 + \varepsilon \xi) \| y - \bar{y} \|.$$

The function Y is uniformly bounded and Lipschitzian for all $s \in \mathbb{R}$, $y, \bar{y} \in \mathbb{R}^n$, therefore the Cauchy problem for (3.14) with the initial condition $y(t) = y_0$ has a global solution for all $y_0 \in \mathbb{R}^n$. We denote the solution by $\Phi_{s,t}(y_0, h, \varepsilon)$.

The function $z = h(t, \Phi_{s,t}(y_0, h, \varepsilon), \varepsilon)$ is a uniformly bounded solution of the equation

$$\frac{dz}{dt} = B(t)z + Z(t, \Phi_{s,t}(y_0, h, \varepsilon), z, a(\Phi_{s,t}(y_0, h, \varepsilon), \varepsilon), \varepsilon) + a(\Phi_{s,t}(y_0, h, \varepsilon), \varepsilon). \tag{3.15}$$

Therefore, applying results of chapter 2, the function z satisfies the integral relation

$$z(t; y_0, \varepsilon) = \begin{cases} -\int\limits_t^{+\infty} e^{\frac{\alpha(t^2 - s^2)}{2}} W(t-s) \left[Z(\cdot) + a(\, \Phi_{s,t}(y_0, h, \varepsilon), \varepsilon \,) \right] ds, & t \geq 0, \\ \int\limits_{-\infty}^{t} e^{\frac{\alpha(t^2 - s^2)}{2}} W(t-s) \left[Z(\cdot) + a\, (\Phi_{s,t}(y_0, h, \varepsilon), \varepsilon) \right] ds, & t < 0, \end{cases} \tag{3.16}$$

where

$$Z(\cdot) = Z(s, \Phi_{s,t}(y_0, h, \varepsilon), z, a(\Phi_{s,t}(y_0, h, \varepsilon), \varepsilon), \varepsilon).$$

Taking instead of y_0 an arbitrary function y, from (3.16) we get that the function $h(t, y, \varepsilon)$ describing the integral manifold satisfies the integral equation

$$h(t, y, \varepsilon) = \begin{cases} -\int\limits_t^{+\infty} e^{\frac{\alpha(t^2 - s^2)}{2}} W(t-s) \left[Z(\cdot) + a(\, \Phi_{s,t}(y, h, \varepsilon), \varepsilon \,) \right] ds, & t \geq 0, \\ \int\limits_{-\infty}^{t} e^{\frac{\alpha(t^2 - s^2)}{2}} W(t-s) \left[Z(\cdot) + a\, (\Phi_{s,t}(y, h, \varepsilon), \varepsilon) \right] ds, & t < 0, \end{cases} \tag{3.17}$$

here

$$Z(\cdot) = Z(s, \Phi_{s,t}(y,h,\varepsilon), h(t, \Phi_{s,t}(y,h,\varepsilon),\varepsilon), a(\Phi_{s,t}(y,h,\varepsilon),\varepsilon),\varepsilon).$$

On the other hand, if (3.17) has a solution satisfying (3.13) then it represents an integral manifold of (3.6). Indeed, for any fixed $\varepsilon \in I_{\varepsilon_0}$ and any point (t_0, y_0, z_0) belonging to the integral manifold ($z_0 = h(t_0, y_0, \varepsilon)$) equation (3.14) has a solution $y(t,\varepsilon) = \Phi_{t,t_0}(y_0, h, \varepsilon)$. From (3.17) it follows that $z = h(t, \Phi_{t,t_0}(y_0,h,\varepsilon),\varepsilon)$ is a solution of (3.15).

Thus, on the space H we define the operator T of the form

$$(Th)(t,y,\varepsilon) = \begin{cases} -\int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\left[Z(\cdot) + a(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\right] ds, & t \geq 0, \\ \int\limits_{-\infty}^{t} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\left[Z(\cdot) + a(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\right] ds, & t < 0, \end{cases} \tag{3.18}$$

with a depending on h (this dependence will be described implicitly below), and

$$Z(\cdot) = Z(s, \Phi_{s,t}(y,h,\varepsilon), h(s, \Phi_{s,t}(y,h,\varepsilon),\varepsilon), a(\Phi_{s,t}(y,h,\varepsilon),\varepsilon),\varepsilon).$$

In the upper line in the definition of T there is an operator for the existence of the bounded integral manifold for $t > 0$, and in the lower line there is an operator for the existence of the bounded integral manifold for $t < 0$. The function $a(y,\varepsilon)$ is for gluing these manifolds.

The following statement is true.

Theorem 3.1 *Let the functions Y, Z in the r.h.s. of (3.6) satisfy the assumptions (H_1), (H_2). Then, there is an $\varepsilon^* \in I_{\varepsilon_0}$ such that for $0 < \varepsilon \leq \varepsilon^*$ there exists a function $a \in F$ such that system (3.6) has an integral manifold $z = h(t,y,\varepsilon)$, $h \in H$.*

If for sufficiently small ε system (3.6) has an integral manifold $z = h(t,y,\varepsilon)$ with $h \in H$, then we know that for $\varepsilon = 0$ the integral manifold $z \equiv 0$ is attractive for $t < 0$ and repulsive for $t > 0$. Therefore, it follows from the continuous dependence of the trajectories of (3.6) on the parameter ε that also the integral manifold $z = h(t,y,\varepsilon)$ loses its attractivity for increasing t.

3.3 Proof of Theorem 3.1

The proof of Theorem 3.1 consists of several steps. First, we derive some auxiliary results. Then, we prove that the operator T defined by (3.18) maps the space H into itself and has a unique fixed point. We do it in the following way. In the beginning we show that the element Th is continuous for all $t \in \mathbb{R}$, and then that $Th \in H$ and the operator T is a contraction in H.

3.3.1 Auxiliary estimates

At first, we derive a lemma describing the dependence of the solution $\Phi_{s,t}(y, h, \varepsilon)$ of (3.14) on the initial value y and the function $h \in H$.

Lemma 3.2 *Let the function Y satisfy the assumption (H_1). Then the following inequalities are valid*

$$\|\Phi_{s,t}(y, h, \varepsilon) - \Phi_{s,t}(\bar{y}, h, \varepsilon)\| \leq \|y - \bar{y}\| e^{\varepsilon\mu(1+\varepsilon\xi)|s-t|},$$

$$\|\Phi_{s,t}(y, h, \varepsilon) - \Phi_{s,t}(y, \bar{h}, \varepsilon)\| \leq \frac{1}{1+\varepsilon\xi}\rho(h, \bar{h}) \left(e^{\varepsilon\mu(1+\varepsilon\xi)|s-t|} - 1\right),$$

where $h, \bar{h} \in H$.

Proof. By (3.14) it holds

$$\begin{aligned}
\Phi_{s,t}(y, h, \varepsilon) &= y + \varepsilon \int_t^s Y(\eta, \Phi_{\eta,t}(y, h, \varepsilon), h(\eta, \Phi_{\eta,t}(y, h, \varepsilon), \varepsilon), \varepsilon) d\eta, \\
\Phi_{s,t}(\bar{y}, h, \varepsilon) &= \bar{y} + \varepsilon \int_t^s Y(\eta, \Phi_{\eta,t}(\bar{y}, h, \varepsilon), h(\eta, \Phi_{\eta,t}(\bar{y}, h, \varepsilon), \varepsilon), \varepsilon) d\eta, \quad (3.19) \\
\Phi_{s,t}(y, \bar{h}, \varepsilon) &= y + \varepsilon \int_t^s Y(\eta, \Phi_{\eta,t}(y, \bar{h}, \varepsilon), \bar{h}(\eta, \Phi_{\eta,t}(y, \bar{h}, \varepsilon), \varepsilon), \varepsilon) d\eta.
\end{aligned}$$

Using (3.19) and inequalities (3.8), (3.9) and (3.13) we obtain for $s \geq t$

$$\|\Phi_{s,t}(y, h, \varepsilon) - \Phi_{s,t}(\bar{y}, h, \varepsilon)\| \leq \|y - \bar{y}\| + \int_t^s \varepsilon \|Y(\eta, \Phi_{\eta,t}(y, h, \varepsilon), h(\eta, \Phi_{\eta,t}(y, h, \varepsilon), \varepsilon), \varepsilon)$$

$$-Y(\eta, \Phi_{\eta,t}(\bar{y}, h, \varepsilon), h(\eta, \Phi_{\eta,t}(\bar{y}, h, \varepsilon), \varepsilon), \varepsilon)\| d\eta$$

$$\leq \|y-\bar{y}\| + \int\limits_t^s \varepsilon\mu\left(\|\Phi_{\eta,t}(y,h,\varepsilon) - \Phi_{\eta,t}(\bar{y},h,\varepsilon)\|\right.$$

$$\left. + \|h(\eta, \Phi_{\eta,t}(y,h,\varepsilon),\varepsilon) - h(\eta, \Phi_{\eta,t}(\bar{y},h,\varepsilon),\varepsilon)\|\right) d\eta$$

$$\leq \|y-\bar{y}\| + \int\limits_t^s \varepsilon\mu(1+\varepsilon\xi)\|\Phi_{\eta,t}(y,h,\varepsilon) - \Phi_{\eta,t}(\bar{y},h,\varepsilon)\| d\eta.$$

Using the Gronwall-Bellman inequality we have

$$\|\Phi_{s,t}(y,h,\varepsilon) - \Phi_{s,t}(\bar{y},h,\varepsilon)\| \leq \|y-\bar{y}\| e^{\varepsilon\mu(1+\varepsilon\xi)(s-t)},\ s \geq t. \tag{3.20}$$

For the difference $\|\Phi_{s,t}(y,h,\varepsilon) - \Phi_{s,t}(y,\bar{h},\varepsilon)\|$ we get

$$\|\Phi_{s,t}(y,h,\varepsilon) - \Phi_{s,t}(y,\bar{h},\varepsilon)\| \leq \int\limits_t^s \varepsilon\|Y(\eta, \Phi_{\eta,t}(y,h,\varepsilon), h(\eta, \Phi_{\eta,t}(y,h,\varepsilon),\varepsilon),\varepsilon)$$

$$-Y(\eta, \Phi_{\eta,t}(y,\bar{h},\varepsilon), \bar{h}(\eta, \Phi_{\eta,t}(y,\bar{h},\varepsilon),\varepsilon),\varepsilon)\| d\eta$$

$$\leq \int\limits_t^s \varepsilon\mu\Big((1+\varepsilon\xi)\|\Phi_{\eta,t}(y,h,\varepsilon) - \Phi_{\eta,t}(y,\bar{h},\varepsilon)\| + \rho(h,\bar{h})\Big) d\eta.$$

Using the Gronwall-Bellman inequality we obtain

$$\|\Phi_{s,t}(y,h,\varepsilon) - \Phi_{s,t}(y,\bar{h},\varepsilon)\| \leq \frac{1}{1+\varepsilon\xi}\rho(h,\bar{h})\left(e^{\varepsilon\mu(1+\varepsilon\xi)(s-t)} - 1\right),\ s \geq t. \tag{3.21}$$

In the same way we get for $s \leq t$

$$\|\Phi_{s,t}(y,h,\varepsilon) - \Phi_{s,t}(\bar{y},h,\varepsilon)\| \leq \|y-\bar{y}\| e^{\varepsilon\mu(1+\varepsilon\xi)(t-s)},$$

$$\|\Phi_{s,t}(y,h,\varepsilon) - \Phi_{s,t}(y,\bar{h},\varepsilon)\| \leq \frac{1}{1+\varepsilon\xi}\rho(h,\bar{h})\left(e^{\varepsilon\mu(1+\varepsilon\xi)(t-s)} - 1\right).$$

This completes the proof.

In the sequel, the error integral

$$\mathrm{erf}(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \int\limits_0^x e^{-x^2} dx$$

will be used. In the next lemma we give some estimate for it.

Lemma 3.3 *For $0 \le x \le \frac{1}{2}$ the following estimate is valid*

$$e^{x^2}\mathrm{erf}(x) \le 1.$$

Proof. For $x < 1$ the error integral can be approximated as [1]

$$\mathrm{erf}(x) = \frac{\sqrt{2}}{\sqrt{\pi}} e^{x^2} x \left(1 + \frac{2x^2}{1\cdot 3} + \frac{(2x^2)^2}{1\cdot 3\cdot 5} + \cdots\right) = \frac{\sqrt{2}}{\sqrt{\pi}} e^{x^2} x \sum_{n=0}^{\infty} \frac{(2x^2)^n}{(2n+1)!!}. \tag{3.22}$$

Since $0 \le x \le \frac{1}{2}$, we get

$$\frac{(2x^2)^n}{(2n+1)!!} \le \frac{1}{2^n(2n+1)!!} \le \frac{1}{2^n 3^n}.$$

Thus, the series in (3.22) can be estimated by the geometric series $\sum_{n=0}^{\infty} \frac{1}{6^n}$. For the geometric series we have

$$\sum_{n=0}^{\infty} \frac{1}{6^n} = \frac{6}{5}.$$

Therefore,

$$e^{x^2}\mathrm{erf}(x) = \frac{\sqrt{2}}{\sqrt{\pi}} x \sum_{n=0}^{\infty} \frac{(2x^2)^n}{(2n+1)!!} \le \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{6^n} = \frac{6}{5\sqrt{\pi}} < 1.$$

This completes the proof.

3.3.2 Continuity of the function Th at $t = 0$

It is easy to check that the function Th is continuous for $t < 0$ and $t > 0$ for any $h \in H$. From the definition of the operator T (3.18) it follows that the continuity of Th at $t = 0$ is equivalent to the following equation

$$\int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) \left[Z\left(s, \Phi_{s,0}(y,h,\varepsilon), h(s, \Phi_{s,0}(y,h,\varepsilon), \varepsilon), a(\Phi_{s,0}(y,h,\varepsilon), \varepsilon), \varepsilon\right) \right.$$
$$\left. + a(\Phi_{s,0}(y,h,\varepsilon), \varepsilon) \right] ds \; = \; 0. \tag{3.23}$$

This equation will be used to determine the function $a(y, \varepsilon)$.

We rewrite equation (3.23) in the form

$$Aa(y,\varepsilon) = Qa(y,\varepsilon),$$

where the operators A and Q are defined by

$$Aa(y,\varepsilon) := \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) a(\Phi_{s,0}(y,h,\varepsilon),\varepsilon)\, ds,$$

$$Qa(y,\varepsilon) := -\frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) Z(\cdot)\, ds,$$

here

$$Z(\cdot) = Z(s, \Phi_{s,0}(y,h,\varepsilon), h(s, \Phi_{s,0}(y,h,\varepsilon),\varepsilon), a(\Phi_{s,0}(y,h,\varepsilon),\varepsilon),\varepsilon).$$

It is convenient to represent the operator A in the form $A = I + R$, where I is the identity and R is defined by

$$Ra(y,\varepsilon) := \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)[a(\Phi_{s,0}(y,h,\varepsilon),\varepsilon) - a(y,\varepsilon)] ds.$$

The inequalities (3.8), (3.12) imply

$$\|Ra(y,\varepsilon)\| \le \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \| \, a(\Phi_{s,0}(y,h,\varepsilon),\varepsilon) - a(y,\varepsilon)\| ds \le$$

$$\le \frac{\varepsilon\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \nu \|\Phi_{s,0}(y,h,\varepsilon) - y\| ds \le$$

$$\le \frac{2\varepsilon^2\nu\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} e^{\frac{-\alpha s^2}{2}} \int\limits_{0}^{s} \|Y(\eta, \Phi_{\eta,0}(y,h,\varepsilon), h(\eta, \Phi_{\eta,0}(y,h,\varepsilon),\varepsilon),\varepsilon)\| d\eta\, ds \le$$

$$\le \frac{2\varepsilon^2\sqrt{\alpha}e^{\beta^2/2\alpha}\nu K}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} e^{\frac{-\alpha s^2}{2}} s ds = \frac{\varepsilon^2\sqrt{2}e^{\beta^2/2\alpha}\nu K}{\sqrt{\alpha\pi}}.$$

For $\frac{\varepsilon^2\sqrt{2}e^{\beta^2/2\alpha}\nu K}{\sqrt{\alpha\pi}} < 1$ there exists the linear operator $(I+R)^{-1}$ and the following inequality is true [14]

$$\|(I+R)^{-1}\| \le \frac{1}{1-\varepsilon^2\sqrt{2}e^{\beta^2/2\alpha}\nu K/\sqrt{\alpha\pi}}. \tag{3.24}$$

Let us introduce the operator P on the space F by

$$Pa := (I+R)^{-1}Qa. \tag{3.25}$$

In the sequel we prove that the operator P maps F into itself and is a contraction. By (3.10), we get for Q

$$\|Qa(y,\varepsilon)\| \le \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}}\|Z(\cdot)\|ds \le$$

$$\le \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}}M(\varepsilon+\varepsilon\|h\|+\|h\|^2)ds \le e^{\beta^2/2\alpha}M(\varepsilon+\varepsilon^2N+\varepsilon^2N^2).$$

Using the last inequality and the inequality (3.24), we obtain

$$\|Pa(y,\varepsilon)\| \le \frac{\varepsilon M e^{\beta^2/2\alpha}(1+\varepsilon N+\varepsilon N^2)}{1-\varepsilon^2\sqrt{2}e^{\beta^2/2\alpha}\nu K/\sqrt{\alpha\pi}}.$$

Under the condition

$$\frac{\varepsilon^2\sqrt{2}\nu K e^{\beta^2/2\alpha}}{\sqrt{\alpha\pi}} \le \frac{1}{2} \tag{3.26}$$

the inequality

$$\|Pa(y,\varepsilon)\| \le 2\varepsilon M e^{\beta^2/2\alpha}(1+\varepsilon N+\varepsilon N^2)$$

is true.

By Lemma 3.2 and the inequality (3.11), it is easy to verify the estimate

$$\|Qa(y,\varepsilon)-Qa(\bar{y},\varepsilon)\| \le$$

$$\le \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} De^{\frac{-\alpha s^2}{2}}\left[(\varepsilon+\varepsilon\|h\|+\|h\|^2)\|\Phi_{s,0}(y,h,\varepsilon)-\Phi_{s,0}(\bar{y},h,\varepsilon)\|+\right.$$

$$+(\varepsilon+\|h\|)\,\|h(s,\Phi_{s,0}(y,h,\varepsilon),\varepsilon)-h(s,\Phi_{s,0}(\bar{y},h,\varepsilon),\varepsilon)\|+$$

$$+\varepsilon\|a(\Phi_{s,0}(y,h,\varepsilon),\varepsilon)-a(\Phi_{s,0}(\bar{y},h,\varepsilon),\varepsilon)\|]\,ds\le$$

$$\le\frac{\varepsilon\sqrt{\alpha}e^{\beta^2/2\alpha}DS}{\sqrt{2\pi}}\int\limits_{\infty}^{+\infty}e^{\frac{-\alpha s^2}{2}}\|\Phi_{s,0}(y,h,\varepsilon)-\Phi_{s,0}(\bar{y},h,\varepsilon)\|ds\le$$

$$\le\frac{\varepsilon\sqrt{\alpha}e^{\beta^2/2\alpha}DS}{\sqrt{2\pi}}\|y-\bar{y}\|\int\limits_{-\infty}^{+\infty}e^{\frac{-\alpha s^2}{2}}e^{\varepsilon\mu(1+\varepsilon\xi)|s|}ds=$$

$$=\frac{2\varepsilon\sqrt{\alpha}e^{\beta^2/2\alpha}DS}{\sqrt{2\pi}}\|y-\bar{y}\|\int\limits_{0}^{+\infty}e^{\frac{-\alpha s^2}{2}}e^{\varepsilon\mu(1+\varepsilon\xi)s}ds,$$

where $S=1+\varepsilon N+\varepsilon N^2+\varepsilon\xi(1+N)+\varepsilon\nu$.

Using the error integral $\operatorname{erf}(\lambda)$ it is possible to estimate the last integral:

$$\frac{\sqrt{2\alpha}}{\sqrt{\pi}}\int\limits_{0}^{+\infty}e^{\frac{-\alpha s^2}{2}}e^{\varepsilon\mu(1+\varepsilon\xi)s}ds=\frac{\sqrt{2\alpha}}{\sqrt{\pi}}\int\limits_{0}^{+\infty}e^{\lambda^2-(\frac{\sqrt{\alpha}}{\sqrt{2}}s-\lambda)^2}ds=$$

$$=\frac{2}{\sqrt{\pi}}\int\limits_{-\lambda}^{+\infty}e^{\lambda^2-s_1^2}ds_1=\frac{2}{\sqrt{\pi}}\left[2\int\limits_{0}^{|\lambda|}e^{\lambda^2-s_1^2}ds_1+\int\limits_{|\lambda|}^{+\infty}e^{\lambda^2-s_1^2}ds_1\right]\le$$

$$\le 1+2e^{\lambda^2}\operatorname{erf}(\lambda),$$

where $\lambda=\frac{1}{\sqrt{2\alpha}}\varepsilon\mu(1+\varepsilon\xi)$. From Lemma 3.3 for $\lambda\le 1/2$ or

$$\frac{\sqrt{2}}{\sqrt{\alpha}}\varepsilon\mu(1+\varepsilon\xi)\le 1, \tag{3.27}$$

we get

$$e^{\lambda^2}\operatorname{erf}(|\lambda|)\le 1.$$

Consequently,

$$\frac{\sqrt{2\alpha}}{\sqrt{\pi}}\int\limits_{0}^{+\infty}e^{\frac{-\alpha s^2}{2}}e^{\varepsilon\mu(1+\varepsilon\xi)s}ds\le 3. \tag{3.28}$$

Therefore, we obtain the estimate

$$\|Qa(y,\varepsilon)-Qa(\bar{y},\varepsilon)\|\le 3\varepsilon e^{\beta^2/2\alpha}DS\|y-\bar{y}\|.$$

According to the definition of the operator P (3.25) and the inequalities (3.24), (3.26) the following estimate is true

$$\|Pa(y,\varepsilon)-Pa(\bar{y},\varepsilon)\| \le \frac{3\varepsilon e^{\beta^2/2\alpha}DS}{1-\varepsilon^2\sqrt{2}e^{\beta^2/2\alpha}\nu K/\sqrt{\alpha\pi}}\|y-\bar{y}\| \le 6\varepsilon e^{\beta^2/2\alpha}DS\|y-\bar{y}\|.$$

Therefore, if the inequality (3.26) and the inequalities

$$2Me^{\beta^2/2\alpha}(1+\varepsilon N+\varepsilon N^2) \le L, \tag{3.29}$$

$$\frac{\sqrt{2}}{\sqrt{\alpha}}\varepsilon\mu(1+\varepsilon\xi) \le 1, \tag{3.30}$$

$$6e^{\beta^2/2\alpha}DS \le \nu \tag{3.31}$$

hold, then P maps F into itself.

Now we derive conditions assuring P to be a contraction operator. At first let us estimate the difference $\|Qa - Q\bar{a}\|$. Under the assumption (3.12) we have

$$\|Qa(y,\varepsilon)-Q\bar{a}(y,\varepsilon)\| \le \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}}\varepsilon D\rho(a,\bar{a})ds = \varepsilon De^{\beta^2/2\alpha}\rho(a,\bar{a}).$$

Consequently, by (3.24) and (3.25) we get

$$\|Pa(y,\varepsilon)-P\bar{a}(y,\varepsilon)\| \le \frac{\varepsilon De^{\beta^2/2\alpha}}{1-\varepsilon^2\sqrt{2}e^{\beta^2/2\alpha}\nu K/\sqrt{\alpha\pi}}\rho(a,\bar{a}) \le 2\varepsilon De^{\beta^2/2\alpha}\rho(a,\bar{a}).$$

If ε is sufficiently small, then the condition

$$2\varepsilon De^{\beta^2/2\alpha} < 1 \tag{3.32}$$

holds. It means that P is contraction operator in F. Therefore, the equation $a = Pa$, which is equivalent to (3.23), possesses a unique solution in F. Thus we have proved

Lemma 3.4 *Suppose the functions Y, Z in the r.h.s. of (3.6) satisfy (H_1), (H_2). Then there is $\varepsilon^* \in I_{\varepsilon_0}$ such that for all $\varepsilon \in (0,\varepsilon^*]$ and for any function $h \in H$ there exists a function $a \in F$ guaranteeing that the function Th defined by (3.18) is continuous.*

Now we study the dependence of the fixed point a of P on h. Let $a(y,\varepsilon)$ and $\bar{a}(y,\varepsilon)$ be the solutions of (3.23) corresponding to the functions h and $\bar{h}$ respectively. Then we have

$$Aa = Qa \qquad \text{or} \qquad (I+R)a = Qa,$$

$$\bar{A}\bar{a} = \bar{Q}\bar{a} \qquad \text{or} \qquad (I+\bar{R})\bar{a} = \bar{Q}\bar{a},$$

where

$$\bar{R}\bar{a}(y,\varepsilon) := \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)[\bar{a}(\Phi_{s,0}(y,\bar{h},\varepsilon),\varepsilon) - \bar{a}(y,\varepsilon)]ds,$$

$$\bar{Q}\bar{a}(y,\varepsilon) := -\frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)Z(\cdot)ds,$$

with

$$Z(\cdot) = Z(s,\Phi_{s,0}(y,\bar{h},\varepsilon),\bar{h}(s,\Phi_{s,0}(y,\bar{h},\varepsilon),\varepsilon),\bar{a}(\Phi_{s,0}(y,\bar{h},\varepsilon),\varepsilon),\varepsilon).$$

After some elementary transformations we obtain

$$(I+R)(a-\bar{a}) = Qa - \bar{Q}\bar{a} + (\bar{R}-R)\bar{a}$$

or

$$a - \bar{a} = (I+R)^{-1}[Qa - \bar{Q}\bar{a} + (\bar{R}-R)\bar{a}]. \tag{3.33}$$

The expression in the square brackets will be estimated at first. By inequalities (3.12), (3.13) and Lemma 3.2 we have

$$\|Qa(y,\varepsilon) - \bar{Q}\bar{a}(y,\varepsilon)\| \le$$

$$\le \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}D}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \Big[(\varepsilon + \varepsilon\|\tilde{h}\| + \|\tilde{h}\|^2)\|\Phi_{s,0}(y,h,\varepsilon) - \Phi_{s,0}(y,\bar{h},\varepsilon)\| +$$

$$+(\varepsilon + \|\tilde{h}\|)\|h(s,\Phi_{s,0}(y,h,\varepsilon),\varepsilon) - \bar{h}(s,\Phi_{s,0}(y,\bar{h},\varepsilon),\varepsilon)\| +$$

$$+\varepsilon\|a(\Phi_{s,0}(y,h,\varepsilon),\varepsilon) - \bar{a}(\Phi_{s,0}(y,\bar{h},\varepsilon),\varepsilon)\|\Big]ds \le$$

$$\le \frac{\varepsilon\sqrt{\alpha}e^{\beta^2/2\alpha}D}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \Big[\Big(1+\varepsilon N+\varepsilon N^2+\varepsilon\xi(1+N)+\varepsilon\nu\Big)\|\Phi_{s,0}(y,h,\varepsilon)-\Phi_{s,0}(y,\bar{h},\varepsilon)\| +$$

$$+(1+N)\rho(h,\bar{h})+\rho(a,\bar{a})\Big]ds \le$$

$$\le \varepsilon e^{\beta^2/2\alpha} D\Big[\rho(a,\bar{a})+(1+N)\rho(h,\bar{h})+$$

$$+\frac{2\sqrt{\alpha}S}{\sqrt{2\pi}}\int\limits_0^{+\infty} e^{\frac{-\alpha s^2}{2}}\frac{1}{1+\varepsilon\xi}(e^{\varepsilon\mu(1+\varepsilon\xi)s}-1)\rho(h,\bar{h})ds\Big] \le$$

$$\le \varepsilon e^{\beta^2/2\alpha} D\left(\rho(a,\bar{a})+\left(1+N+\frac{3S}{1+\varepsilon\xi}\right)\rho(h,\bar{h})\right),$$

and

$$\|(\bar{R}-R)\bar{a}\| \le \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}}\|\bar{a}(\Phi_{s,0}(y,\bar{h},\varepsilon),\varepsilon)-\bar{a}(\Phi_{s,0}(y,h,\varepsilon),\varepsilon)\|ds \le$$

$$\le \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}}\varepsilon\nu\|\Phi_{s,0}(y,h,\varepsilon)-\Phi_{s,0}(y,\bar{h},\varepsilon)\|ds \le$$

$$\le \frac{2\varepsilon\sqrt{\alpha}e^{\beta^2/2\alpha}\nu}{\sqrt{2\pi}(1+\varepsilon\xi)}\int\limits_0^{+\infty} e^{\frac{-\alpha s^2}{2}}(e^{\varepsilon\mu(1+\varepsilon\xi)s}-1)\rho(h,\bar{h})ds \le$$

$$\le \frac{3\varepsilon e^{\beta^2/2\alpha}\nu}{1+\varepsilon\xi}\rho(h,\bar{h}).$$

Then we get from (3.24), (3.33)

$$\|a(y,\varepsilon)-\bar{a}(y,\varepsilon)\| \le \frac{\varepsilon e^{\beta^2/2\alpha}}{1-\varepsilon^2\sqrt{2}e^{\beta^2/2\alpha}\nu K/\sqrt{\alpha\pi}}\Big[D\rho(a,\bar{a})+$$

$$+\left(D(1+N)+\frac{3(DS+\nu)}{1+\varepsilon\xi}\right)\rho(h,\bar{h})\Big].$$

From this inequality and the assumption (3.26) we obtain the following result

Lemma 3.5 *Suppose the conditions in Lemma 3.4 hold and the inequality (3.26) is valid. Then the following estimate is true*

$$\rho(a,\bar{a}) \le \frac{2\varepsilon e^{\beta^2/2\alpha}}{1-2\varepsilon e^{\beta^2/2\alpha}D}\left(D(1+N)+\frac{3(DS+\nu)}{1+\varepsilon\xi}\right)\rho(h,\bar{h}). \qquad (3.34)$$

3.3.3 Existence of the integral manifold

In this part we derive the conditions guaranteeing that $Th(t, y, \varepsilon)$ satisfies the inequalities (3.13). For $t \geq 0$ we have

$$\|Th(t,y,\varepsilon)\| \leq \int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} \Big[\|Z\,(\cdot)\,\| + \|a(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\| \Big]\, ds \leq$$

$$\leq \frac{\varepsilon\sqrt{\pi}}{\sqrt{2\alpha}} \left(M(1+\varepsilon N+\varepsilon N^2)+L \right).$$

The same estimate is valid for $t \leq 0$. Therefore, Th is bounded for all $t \in \mathbb{R}$. In order to show the Lipschitz continuity of Th we consider the difference $\|Th(t,y,\varepsilon) - Th(t,\bar{y},\varepsilon)\|$.

$$\|Th(t,y,\varepsilon) - Th(t,\bar{y},\varepsilon)\| \leq$$

$$\leq \int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} \Big[\|Z\left(s, \Phi_{s,t}(y,h,\varepsilon), h(s,\Phi_{s,t}(y,h,\varepsilon),\varepsilon), a(\Phi^{\varepsilon}_{s,t}(y;h),\varepsilon),\varepsilon\right) -$$

$$-Z\left(s, \Phi_{s,t}(\bar{y},h,\varepsilon), h(s,\Phi_{s,t}(\bar{y},h,\varepsilon),\varepsilon), a(\Phi_{s,t}(\bar{y},h,\varepsilon),\varepsilon),\varepsilon\right)\|+$$

$$+\|a(\Phi_{s,t}(y,h,\varepsilon),\varepsilon) - a(\Phi_{s,t}(\bar{y},h,\varepsilon),\varepsilon)\| \Big] ds \leq$$

$$\leq \int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} \Big[\varepsilon D(1+\varepsilon N+\varepsilon N^2)\|\Phi_{s,t}(y,h,\varepsilon) - \Phi_{s,t}(\bar{y},h,\varepsilon)\|+$$

$$+\varepsilon D(1+N)\|h(s,\Phi_{s,t}(y,h,\varepsilon),\varepsilon) - h(s,\Phi_{s,t}(\bar{y},h,\varepsilon),\varepsilon)\|+$$

$$+(\varepsilon D+1)\|a(\Phi_{s,t}(y,h,\varepsilon),\varepsilon) - a(\Phi_{s,t}(\bar{y},h,\varepsilon),\varepsilon)\| \Big] ds \leq$$

$$\leq \varepsilon(DS+\nu) \int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} \|\Phi_{s,t}(y,h,\varepsilon) - \Phi_{s,t}(\bar{y},h,\varepsilon)\| ds \leq$$

$$\leq \varepsilon(DS+\nu) \int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} e^{\varepsilon\mu(1+\varepsilon\xi)(s-t)} \|y-\bar{y}\| ds.$$

Using the error integral erf(λ) and Lemma 3.3 it is possible to estimate the last integral

$$\int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} e^{\varepsilon\mu(1+\varepsilon\xi)(s-t)} ds = \frac{\sqrt{2}}{\sqrt{\alpha}} \int_{t_1-\lambda}^{+\infty} e^{(t_1-\lambda)^2-(s_1-\lambda)^2} d(s_1-\lambda) \leq$$

$$\leq \frac{\sqrt{2}}{\sqrt{\alpha}} \left(2 \int_0^{|t_1-\lambda|} e^{(t_1-\lambda)^2-(s_1-\lambda)^2} d(s_1-\lambda) + \int_{|t_1-\lambda|}^{+\infty} e^{(t_1-\lambda)^2-(s_1-\lambda)^2} d(s_1-\lambda) \right) =$$

$$= \frac{\sqrt{2}}{\sqrt{\alpha}} \left(\frac{\sqrt{\pi}}{2} + \sqrt{\pi} e^{(t_1-\lambda)^2} \operatorname{erf}(|t_1-\lambda|) \right) < \frac{3\sqrt{\pi}}{\sqrt{2\alpha}},$$

where $\lambda = \frac{1}{\sqrt{2\alpha}} \varepsilon\mu(1+\varepsilon\xi)$. The last inequality is valid since we can choose ε small enough such that $\lambda \leq \frac{1}{2}$.

Consequently, for $t \geq 0$ we get

$$\|Th(t,y,\varepsilon) - Th(t,\bar{y},\varepsilon)\| < \frac{3\varepsilon\sqrt{\pi}}{\sqrt{2\alpha}}(DS+\nu)\|y-\bar{y}\|.$$

The same is true for $t \leq 0$.

Thus, under the assumptions of Lemma 3.4 and the following inequalities

$$\frac{\sqrt{\pi}}{\sqrt{2\alpha}} \big(M(1+\varepsilon N + \varepsilon N^2) + L \big) \leq N, \tag{3.35}$$

$$\frac{3\sqrt{\pi}}{\sqrt{2\alpha}}(DS+\nu) \leq \xi \tag{3.36}$$

T maps H into itself.

Now we prove that T is a strictly contractive operator in H

$$\|Th(t,y,\varepsilon) - T\bar{h}(t,y,\varepsilon)\| \leq$$

$$\leq \int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} \Big[\|Z\left(s, \Phi_{s,t}(y,h,\varepsilon), h(s,\Phi_{s,t}(y,h,\varepsilon),\varepsilon), a(\Phi_{s,t}(y,h,\varepsilon),\varepsilon),\varepsilon\right) -$$

$$-Z\left(s, \Phi_{s,t}(y,\bar{h},\varepsilon), \bar{h}(s,\Phi_{s,t}(y,\bar{h},\varepsilon),\varepsilon), \bar{a}(\Phi_{s,t}(y,\bar{h},\varepsilon),\varepsilon),\varepsilon\right)\|+$$

$$+\|a(\Phi_{s,t}(y,h,\varepsilon),\varepsilon) - \bar{a}(\Phi_{s,t}(y,\bar{h},\varepsilon),\varepsilon)\| \Big] ds \leq$$

$$\leq \int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} \big(\varepsilon D(1+\varepsilon N+\varepsilon N^2)(\|\Phi_{s,t}(y,h,\varepsilon)-\Phi_{s,t}(y,\bar{h},\varepsilon)\|+$$

$$+\varepsilon D(1+N)\|h(s,\Phi_{s,t}(y,h,\varepsilon),\varepsilon)-\bar{h}(s,\Phi_{s,t}(y,\bar{h},\varepsilon),\varepsilon)\|)+$$

$$+(1+\varepsilon D)\|a(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)-\bar{a}(\Phi_{s,t}(y,\bar{h},\varepsilon),\varepsilon)\|\big)\, ds \leq$$

$$\leq \int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} \big(\varepsilon(DS+\nu)\|\Phi_{s,t}(y,h,\varepsilon)-\Phi_{s,t}(y,\bar{h},\varepsilon)\|+$$

$$+\varepsilon D(1+N)\rho(h,\bar{h})+(1+\varepsilon D)\rho(a,\bar{a})\big)\, ds \leq$$

$$\leq \frac{\sqrt{\pi}}{\sqrt{2\alpha}} \big(\varepsilon D(1+N)\rho(h,\bar{h})+(1+\varepsilon D)\rho(a,\bar{a})\big)+$$

$$+\frac{\varepsilon(DS+\nu)}{1+\varepsilon\xi}\,\rho(h,\bar{h}) \int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} \big(e^{\varepsilon\mu(1+\varepsilon\xi)(s-t)}-1\big)\, ds \leq$$

$$\leq \frac{\varepsilon\sqrt{\pi}}{\sqrt{2\alpha}} \left[D(1+N)+\frac{2e^{\beta^2/2\alpha}(1+\varepsilon D)}{1-2\varepsilon e^{\beta^2/2\alpha}D}\left(D(1+N)+\frac{3(DS+\nu)}{1+\varepsilon\xi}\right)+\right.$$

$$\left.+\frac{3(DS+\nu)}{1+\varepsilon\xi}\right]\rho(h,\bar{h}).$$

Under the conditions (3.32), (3.36) and the inequality

$$\frac{\varepsilon\sqrt{\pi}}{\sqrt{2\alpha}} \left[D(1+N)+\frac{2e^{\beta^2/2\alpha}(1+\varepsilon D)}{1-2\varepsilon e^{\beta^2/2\alpha}D}\left(D(1+N)+\frac{3(DS+\nu)}{1+\varepsilon\xi}\right)+\right.$$

$$\left.+\frac{3(DS+\nu)}{1+\varepsilon\xi}\right]<1 \tag{3.37}$$

T is a contraction operator in H.

Thus, we have proved that the operator T has a unique fixed point in the space H. This fixed point represents an integral manifold of the system (3.6). It completes the proof of Theorem 3.1.

3.3.4 Examples

Example 3.6 Consider the system

$$\begin{aligned} \frac{dy}{dt} &= \varepsilon Y(t,y,z,\varepsilon), \\ \frac{dz}{dt} &= B(t)z + Z(t,y,z,a(y,\varepsilon),\varepsilon) + a(y,\varepsilon), \end{aligned} \tag{3.38}$$

where $y \in \mathbb{R}^n$, $\alpha = \beta = 1$ and Z has the form

$$Z(t,y,z,a(y,\varepsilon),\varepsilon) = Z(t,\varepsilon) = (\varepsilon \cos t,\ 0)^T. \tag{3.39}$$

The function Z satisfies the assumption (H_2). Therefore, we can apply the results of Theorem 3.1.

Using (3.23) we have the following equation for the function $a(y,\varepsilon)$

$$\int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} W^{-1}(s) a(y,\varepsilon) ds = -\int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} W^{-1}(s) Z(s,\varepsilon) ds.$$

Calculating the integral in the l.h.s we get

$$a^* := -\frac{e^{1/2}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} W^{-1}(s) Z(s,y) ds = -\left(\frac{\varepsilon e^{1/2}}{2}(1+e^{-2}), 0 \right)^T.$$

Then, substituting $a^*(y,\varepsilon)$ into (3.38) we obtain the system which has the integral manifold $z = h(t,y,\varepsilon)$ given by

$$h(t,\varepsilon) = \begin{cases} \displaystyle\int_{-\infty}^{t} e^{\frac{t^2-s^2}{2}} (Z(s,\varepsilon) + a^*) ds, & t < 0, \\ \displaystyle -\int_{t}^{+\infty} e^{\frac{t^2-s^2}{2}} (Z(s,\varepsilon) + a^*) ds, & t \geq 0. \end{cases}$$

In this example the function Z does not depend on y. Therefore we get that a does not depend on y, too.

Example 3.7 Consider system (3.38) under the assumption that $y \in \mathbb{R}$ and the function Z has the form

$$Z(t,y,z,a(y,\varepsilon),\varepsilon) = \begin{pmatrix} \varepsilon \cos t \cos y \\ 0 \end{pmatrix}. \tag{3.40}$$

The function Z satisfies the conditions of the Theorem 3.1. Therefore, there exist a unique function $a(y,\varepsilon)$, $a \in F$, such that this system has an integral manifold $z = h(t,y,\varepsilon)$, $h \in H$.

From equation (3.23) for a we have

$$a^*(y,\varepsilon) = -\frac{e^{1/2}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} Z(s,y)ds = -\left(\frac{\varepsilon e^{1/2}}{2}(1+e^{-2})\cos y, 0\right)^T.$$

Substituting $a^*(y,\varepsilon)$ into the system we get the integral manifold $z = h(t,y,\varepsilon)$ given by

$$h(t,y,\varepsilon) = \begin{cases} \int\limits_{-\infty}^{t} e^{\frac{t^2-s^2}{2}} (Z(s,y,\varepsilon) + a^*(y,\varepsilon))ds \quad t < 0, \\ -\int\limits_{t}^{+\infty} e^{\frac{t^2-s^2}{2}} (Z(s,y,\varepsilon) + a^*(y,\varepsilon))ds \quad t \geq 0. \end{cases}$$

Chapter 4

Asymptotic approximations

The method of integral manifolds is a very effective tool for studying the qualitative problems of differential equations. Therefore, the problem of finding the function describing the integral manifold is important. Usually, this function can not be found in explicit form. There are many papers devoted to the problem of approximation of solutions and integral manifolds (see e.g. [15, 34, 37, 36]).

In this chapter we will show that under the assumption that Y and Z in the r. h. s. of (3.6) are sufficiently smooth, the integral manifold $z = h(t, y, \varepsilon)$ and the gluing function $a(y, \varepsilon)$ can be represented in the form

$$\begin{aligned} h(t, y, \varepsilon) &= \sum_{i \geq 0} \varepsilon^i h_i(t, y), \\ a(y, \varepsilon) &= \sum_{i \geq 0} \varepsilon^i a_i(y), \end{aligned} \tag{4.1}$$

where h_i, a_i are continuous and uniformly bounded functions. In what follows we will establish an algorithm of finding the coefficients h_i, a_i, and then we shall estimate the error of the approximations.

In order to find the functions h_i and a_i we substitute (4.1) into equations (3.6).

$$
\begin{aligned}
\frac{dy}{dt} &= \varepsilon Y(t,y,\sum_{i\geq 0}\varepsilon^i h_i(t,y),\varepsilon),\\
\frac{\partial}{\partial t}\sum_{i\geq 0}\varepsilon^i h_i(t,y) &+ \left(\frac{\partial}{\partial y}\sum_{i\geq 0}\varepsilon^i h_i(t,y)\right)\varepsilon Y(t,y,\sum_{i\geq 0}\varepsilon^i h_i(t,y),\varepsilon) = \qquad (4.2)\\
B(t)\sum_{i\geq 0}\varepsilon^i h_i(t,y) &+ Z(t,y,\sum_{i\geq 0}\varepsilon^i h_i(t,y),\sum_{i\geq 0}\varepsilon^i a_i(y),\varepsilon)+\sum_{i\geq 0}\varepsilon^i a_i(y).
\end{aligned}
$$

In addition to (H_1), (H_2) we suppose that the functions Y and Z in the r. h. s. of (3.6) have continuous, uniformly bounded and globally Lipschitzian partial derivatives with respect to y, z, a, ε of order k. Then, the function Y can be represented as

$$
\begin{aligned}
&Y(t,y,\sum_{i\geq 0}\varepsilon^i h_i,\varepsilon) = Y(t,y,h_0,0)+\\
&+\varepsilon\left(\frac{\partial Y(t,y,h_0,0)}{\partial\varepsilon}+\frac{\partial Y(t,y,h_0,0)}{\partial z}h_1\right)+\\
&+\frac{1}{2}\varepsilon^2\left(\frac{\partial^2 Y(t,y,h_0,0)}{\partial\varepsilon^2}+2\frac{\partial Y(t,y,h_0,0)}{\partial z}h_2+\right.\\
&\left.+\frac{\partial^2 Y(t,y,h_0,0)}{\partial z^2}h_1^2+2\frac{\partial^2 Y(t,y,h_0,0)}{\partial\varepsilon\partial z}h_1\right)+\cdots=\\
&= Y(t,y,h_0,0)+\sum_{i\geq 1}\varepsilon^i\frac{\partial Y(t,y,h_0,0)}{\partial z}h_i+\sum_{i\geq 1}\varepsilon^i Y_i(t,y,h_0,\ldots,h_{i-1}). \qquad (4.3)
\end{aligned}
$$

In the same way for the function Z we obtain

$$
\begin{aligned}
&Z(t,y,\sum_{i\geq 0}\varepsilon^i h_i(t,y),\sum_{i\geq 0}\varepsilon^i a_i(y),\varepsilon) = Z(t,y,h_0,a_0,0)+\\
&+\sum_{i\geq 1}\varepsilon^i\left[\frac{\partial Z(t,y,h_0,a_0,0)}{\partial z}h_i+\frac{\partial Z(t,y,h_0,a_0,0)}{\partial a}a_i\right]+ \qquad (4.4)\\
&+\sum_{i\geq 1}\varepsilon^i Z_i(t,y,h_0,\ldots,h_{i-1},a_0,\ldots,a_{i-1}).
\end{aligned}
$$

Substituting (4.3)–(4.4) into equation (4.2) we get the following equation

$$
\sum_{i\geq 0}\frac{\partial h_i}{\partial t}+\sum_{i\geq 0}\frac{\partial h_i}{\partial y}\cdot\varepsilon\Big[Y(t,y,h_0,0)+
$$

$$+\sum_{i\geq 1}\varepsilon^i\frac{\partial Y(t,y,h_0,0)}{\partial z}h_i+\sum_{i\geq 1}\varepsilon^i Y_i(t,y,h_0,\ldots,h_{i-1})\Bigg]=$$

$$=B(t)\sum_{i\geq 0}\varepsilon^i h_i+Z(t,y,h_0,a_0,0)+$$

$$+\sum_{i\geq 1}\varepsilon^i\left[\frac{\partial Z(t,y,h_0,a_0,0)}{\partial z}h_i+\frac{\partial Z(t,y,h_0,a_0,0)}{\partial a}a_i\right]+$$

$$+\sum_{i\geq 1}\varepsilon^i Z_i(t,y,h_0,\ldots,h_{i-1},a_0,\ldots,a_{i-1}).$$

Letting $\varepsilon=0$ we have

$$\frac{\partial h_0}{\partial t}=B(t)h_0+Z(t,y,h_0,a_0,0)+a_0(y).$$

Under the inequalities (3.10) and (3.11)–(3.13) and from the definitions of the spaces H,F we have

$$\begin{gathered} h_0(t,y)\equiv 0, \qquad a_0(y)\equiv 0,\\ \frac{\partial Z(t,y,0,0,0)}{\partial z}=0, \qquad \frac{\partial Z(t,y,0,0,0)}{\partial a}=0. \end{gathered} \tag{4.5}$$

Equating the coefficients corresponding to the same powers of ε we get

$$\frac{\partial h_1}{\partial t}+\frac{\partial h_0}{\partial y}Y(t,y,0,0)=B(t)h_1+\frac{\partial Z(t,y,h_0,a_0,0)}{\partial \varepsilon}+a_1(y),$$

or by (4.5) we obtain

$$\frac{\partial h_1}{\partial t}=B(t)h_1+\frac{\partial Z(t,y,0,0,0)}{\partial \varepsilon}+a_1(y).$$

From this equation we can find the function $h_1(t,y)$ as the solution bounded for all $t\in\mathbb{R}$

$$h_1(t,y)=\begin{cases} -\displaystyle\int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}}W(t-s)\left(\frac{\partial Z(t,y,0,0,0)}{\partial\varepsilon}+a_1(y)\right)ds, \ t\geq 0,\\ \displaystyle\int_{-\infty}^{t} e^{\frac{\alpha(t^2-s^2)}{2}}W(t-s)\left(\frac{\partial Z(t,y,0,0,0)}{\partial\varepsilon}+a_1(y)\right)ds, \ t<0, \end{cases} \tag{4.6}$$

where the function $a_1(y)$ can be found from the condition that $h_1(t, y)$ is continuous at $t = 0$

$$\int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) \left(\frac{\partial Z(t, y, 0, 0, 0)}{\partial \varepsilon} + a_1(y) \right) ds = 0. \tag{4.7}$$

For a_1 we have the following equation

$$\int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) a_1(y) ds = - \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) \frac{\partial Z(t, y, 0, 0, 0)}{\partial \varepsilon} ds,$$

or

$$J a_1(y) = - \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) \frac{\partial Z(t, y, 0, 0, 0)}{\partial \varepsilon} ds,$$

where

$$J = \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)\, ds = \frac{\sqrt{2\pi} e^{-\beta^2/2\alpha}}{\sqrt{\alpha}} I,$$

where I is the identity matrix. Then

$$a_1(y) = -J^{-1} \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) \frac{\partial Z(t, y, 0, 0, 0)}{\partial \varepsilon} ds. \tag{4.8}$$

Comparing the functions multiplied by ε^2 we get

$$\frac{\partial h_2}{\partial t} + \frac{\partial h_1}{\partial y} Y(t, y, h_0, 0) + \frac{\partial h_0}{\partial y} \frac{\partial Y(t, y, h_0, 0)}{\partial z} h_1 + \frac{\partial h_0}{\partial y} Y_1(t, y, h_0) =$$

$$= B(t) h_2 + Z_2(t, y, h_0, h_1, a_0, a_1) + a_2(y).$$

Under the equalities (4.5) we can write

$$\frac{\partial h_2}{\partial t} + \frac{\partial h_1}{\partial y} Y(t, y, 0, 0) = B(t) h_2 + Z_2(t, y, 0, h_1, 0, a_1) + a_2(y). \tag{4.9}$$

The function $h_2(t, y)$ can be found as the uniformly bounded solution of

(4.9)

$$h_2(t,y) = \begin{cases} -\int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\,(Z_2(t,y,0,h_1,0,a_1) - \dfrac{\partial h_1}{\partial y} Y(t,y,0,0) \\ \qquad\qquad +a_2(y))\,ds, \quad t \geq 0, \\ \int\limits_{-\infty}^{t} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\,(Z_2(t,y,0,h_1,0,a_1) - \dfrac{\partial h_1}{\partial y} Y(t,y,0,0) \\ \qquad\qquad +a_2(y))\,ds, \quad t < 0, \end{cases}$$

where the function $a_2(y)$ is defined from the condition of continuity $h_2(t,y)$ at $t=0$

$$\int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) \left(Z_2(t,y,0,h_1,0,a_1) - \frac{\partial h_1}{\partial y} Y(t,y,0,0) + a_2(y) \right) ds = 0.$$

Thus, we obtain the equation for determining a_2

$$a_2(y) = -J^{-1} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) \left(Z_2(t,y,0,h_1,0,a_1) - \frac{\partial h_1}{\partial y} Y(t,y,0,0) \right) ds.$$

For ε in the k-th power we have

$$\frac{\partial h_k}{\partial t} + \frac{\partial h_{k-1}}{\partial y} Y(t,y,0,0) + \frac{\partial h_{k-2}}{\partial y} \left(\frac{\partial Y(t,y,0,0)}{\partial z} h_1 + Y_1(t,y,0) \right) + \dots =$$

$$= B(t)h_k + Z_k(t,y,0,h_1,\dots,h_{k-1},0,a_1,\dots,a_{k-1}) + a_k(y). \tag{4.10}$$

The function $h_k(t,y)$ can be found as a solution of equation (4.10) bounded for all $t \in \mathbb{R}$

$$h_k(t,y) = \begin{cases} -\int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\,(Z_k(t,y,0,h_1,\dots,h_{k-1},0,a_1,\dots,a_{k-1}) \\ \qquad\qquad -\frac{\partial h_{k-1}}{\partial y} Y(t,y,0,0) - \ldots + a_k(y) \Big)\,ds, \quad t \geq 0, \\ \int\limits_{-\infty}^{t} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\,(Z_k(t,y,0,h_1,\dots,h_{k-1},0,a_1,\dots,a_{k-1}) + \\ \qquad\qquad -\frac{\partial h_{k-1}}{\partial y} Y(t,y,0,0) - \ldots + a_k(y) \Big)\,ds, \quad t < 0, \end{cases}$$

where $a_k(y)$ is defined from the condition of continuity of $h_k(t,y)$ at $t=0$

$$\int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)\Big(Z_k(t,y,0,h_1,\dots,h_{k-1},0,a_1,\dots,a_{k-1}) -$$

$$- \frac{\partial h_{k-1}}{\partial y} Y(t,y,0,0) - \ldots + a_k(y)\Big)\, ds = 0,$$

or

$$a_k(y) = -J^{-1}\int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)\Big(Z_k(t,y,0,h_1,\dots,h_{k-1},0,a_1,\dots,a_{k-1})$$

$$- \frac{\partial h_{k-1}}{\partial y} Y(t,y,0,0) - \ldots\Big)\, ds.$$

Thus, we obtained the recurrent formulas for the coefficients h_i, a_i. Let us introduce the following notations

$$H_k(t,y,\varepsilon) = \sum_{i=1}^{k} \varepsilon^i h_i(t,y), \;\; A_k(y,\varepsilon) = \sum_{i=1}^{k} \varepsilon^i a_i(y). \qquad (4.11)$$

Then the question arises: What is the error of approximations (4.11)? In what follows we shall prove the following statement

Theorem 4.1 *Let the conditions of Theorem 3.1 hold and let the functions Y, Z in r. h. s. of system (3.6) have continuous and uniformly bounded partial derivatives with respect to y, z, a, ε up to the order $k+1$. Then the integral manifold $z = h(t,y,\varepsilon)$ of system (3.6) and the gluing function $a(y,\varepsilon)$ can be represented in the form*

$$z(t,y,\varepsilon) = \sum_{i=0}^{k} \varepsilon^i h_i(t,y) + O(\varepsilon^{k+1}),$$

$$a(y,\varepsilon) = \sum_{i=0}^{k} \varepsilon^i a_i(y) + O(\varepsilon^{k+1}),$$

where h_i and a_i are bounded and Lipschitz continuous functions.

4.1 Proof of Theorem 4.1

The boundedness and Lipschitz continuity of h_i, a_i follow from the definitions of these functions. We put

$$\begin{aligned} z &= H_k + u, \\ a &= A_k + v, \end{aligned} \tag{4.12}$$

u and v are the remainder terms. Substituting (4.12) into (3.6) we get

$$\frac{dy}{dt} = \varepsilon Y(t, y, H_k + u, \varepsilon),$$

$$\frac{dH_k}{dt} + \frac{du}{dt} = B(t)H_k + B(t)u + Z(t, y, H_k + u, A_k + v, \varepsilon) + A_k + v.$$

From the properties of the functions H_k, A_k we obtain the system

$$\begin{aligned} \frac{dy}{dt} &= \varepsilon\widetilde{Y}(t, y, u, \varepsilon), \\ \frac{du}{dt} &= B(t)u + \widetilde{Z}(t, y, u, v(y, \varepsilon), \varepsilon) + v(y, \varepsilon), \end{aligned} \tag{4.13}$$

where

$$\widetilde{Y}(t, y, u, \varepsilon) = Y(t, y, H_k + u, \varepsilon),$$

$$\widetilde{Z}(t, y, u, v, \varepsilon) = Z(t, y, H_k + u, A_k + v, \varepsilon) - Z(t, y, H_k, A_k, \varepsilon)$$
$$+\varepsilon^{k+1}\varphi(t, y, h_0, \dots, h_k, a_0, \dots, a_k),$$

with φ uniformly bounded

$$\|\varphi(t, y, h_0, \dots, h_k, a_0, \dots, a_k)\| \le C.$$

From the definition it follows that the functions $\widetilde{Y}, \widetilde{Z}$ are continuous on $\mathbb{R} \times \mathbb{R}^n \times \Omega_z \times I_{\varepsilon_0}$, $\mathbb{R} \times \mathbb{R}^n \times \Omega_z \times \Omega_a \times I_{\varepsilon_0}$, respectively, and satisfy for $t \in \mathbb{R}, y, \bar{y} \in \mathbb{R}^n, u, \bar{u} \in \Omega_z, v, \bar{v} \in \Omega_a, \varepsilon \in I_{\varepsilon_0}$ the inequalities

$$\|\widetilde{Y}(t, y, u, \varepsilon)\| \le K, \tag{4.14}$$

$$\|\widetilde{Y}(t, y, u, \varepsilon) - \widetilde{Y}(t, \bar{y}, \bar{u}, \varepsilon)\| \le \mu_1\|y - \bar{y}\| + \mu\|u - \bar{u}\|, \tag{4.15}$$

$$\|\widetilde{Z}(t, y, u, v, \varepsilon)\| \le \|Z(t, y, H_k{+}u, A_k{+}v, \varepsilon) - Z(t, y, H_k, A_k, \varepsilon)\| + \varepsilon^{k+1}\|\varphi\| \le$$
$$\le D_1\left(\varepsilon\|u\| + \varepsilon\|v\| + \|u\|^2 + \varepsilon^{k+1}\right), \tag{4.16}$$

$$\|\widetilde{Z}(t, y, u, v, \varepsilon) - \widetilde{Z}(t, \bar{y}, \bar{u}, \bar{v}, \varepsilon)\| \le$$

$$D_1\left((\varepsilon+\varepsilon\|\widetilde{u}\|+\|\widetilde{u}\|^2)\|y-\bar{y}\|+(\varepsilon+\|\widetilde{u}\|)\|u-\bar{u}\|+\varepsilon\|v-\bar{v}\|\right), \quad (4.17)$$

where $\|\widetilde{u}\| := \max\{\|u\|, \|\bar{u}\|\}$. These inequalities, except (4.16), are analogous to (3.8)-(3.11). Thus, system (4.13) is a system of the same type as (3.6).

In order to prove Theorem 4.1 we shall show that there is a function $v(y,\varepsilon)$, $\|v(y,\varepsilon)\| \leq \varepsilon^{k+1}q$ such that system (4.13) possesses an integral manifold $u = g(t,y,\varepsilon)$, $\|g(t,y,\varepsilon)\| \leq \varepsilon^{k+1}p$, where q,p are some positive number.

We shall use the same approach as in the previous sections.

Let us consider the complete metric space V of functions v, continuous on $\mathbb{R}^n \times I_{\varepsilon_0}$, satisfying the inequalities

$$\|v(y,\varepsilon)\| \leq \varepsilon^{k+1}q, \quad \|v(y,\varepsilon)-v(\bar{y},\varepsilon)\| \leq \varepsilon\varkappa\|y-\bar{y}\|. \quad (4.18)$$

with the metric defined by

$$\rho\,(v,\bar{v}) = \sup_{y\in\mathbb{R}^n,\varepsilon\in I_{\varepsilon_0}} \|v(y,\varepsilon)-\bar{v}(y,\varepsilon)\|$$

and the complete metric space G of functions g, continuous on $\mathbb{R}\times\mathbb{R}^n\times I_{\varepsilon_0}$, satisfying the inequalities

$$\begin{aligned}\|g(t,y,\varepsilon)\| &\leq \varepsilon^{k+1}p,\\ \|g(t,y,\varepsilon)-g(t,\bar{y},\varepsilon)\| &\leq \varepsilon\gamma\|y-\bar{y}\|,\end{aligned} \quad (4.19)$$

for $t\in\mathbb{R}, y,\bar{y}\in\mathbb{R}^n, \varepsilon\in I_{\varepsilon_0}$ with the metric

$$\rho(g,\bar{g}) = \sup_{t\in\mathbb{R},y\in\mathbb{R}^n,\varepsilon\in I_{\varepsilon_0}} \|g(t,y,\varepsilon)-\bar{g}(t,y,\varepsilon)\|.$$

The aim is to prove that there exists a function $v \in V$ guaranteeing that the modified system (4.13) has an integral manifold $u = g(t,y,\varepsilon)$, where $g \in G$.

We define on G the operator $\widetilde{T}$ of the form

$$(\widetilde{T}g)(t,y,\varepsilon) = \begin{cases} -\int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\left[\widetilde{Z}(\cdot)+v\Big(\Psi_{s,t}(y,g,\varepsilon),\varepsilon\,)\right] ds, & t\geq 0,\\ \int\limits_{-\infty}^{t} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\left[\widetilde{Z}(\cdot)+v\left(\Psi_{s,t}(y,g,\varepsilon),\varepsilon\right)\right] ds, & t<0,\end{cases}$$

with v depending on $g\in G$, and

$$\widetilde{Z}(\cdot) = \widetilde{Z}\left(s,\Psi_{s,t}(y,g,\varepsilon), g(s,\Psi_{s,t}(y,g,\varepsilon),\varepsilon), v(\Psi_{s,t}(y,g,\varepsilon),\varepsilon),\varepsilon\right).$$

Here $\Psi_{s,t}(y,g,\varepsilon)$ is the solution of the initial value problem for any given function $g \in G$

$$\begin{aligned} \frac{d\psi}{ds} &= \widetilde{Y}(s,y,g,\varepsilon), \\ \psi(t) &= y. \end{aligned} \tag{4.20}$$

As it was done in the Section 3.3 we shall show that for every $g \in G$ we can define a unique function $v(y,\varepsilon)$ such that $\widetilde{T}g$ is continuous, and then we shall show that the operator $\widetilde{T}$ maps the space G into itself and has a fixed point. The fixed point represents an integral manifold of (4.13).

4.1.1 Continuity of $\widetilde{T}g$ at $t = 0$

It is obvious that $\widetilde{T}g$ is continuous for $t < 0$ and $t > 0$ for any $g \in G$. Consider the following equation

$$\int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W(-s) \left[\widetilde{Z}(\cdot) + v(\Psi_{s,0}(y,g,\varepsilon),\varepsilon)\right] ds = 0 \tag{4.21}$$

with respect to a function $v(y,\varepsilon)$. This equation is obtained from the condition of continuity of $\widetilde{T}g(t,y,\varepsilon)$ at $t = 0$. In the same way as in Section 3.3.2, we shall prove that there exists a unique function $v \in V$ such that the element $\widetilde{T}g$ is continuous.

In the sequel we shall use the following Lemma which describes the depends of the solution $\Psi_{s,t}(y,g,\varepsilon)$ of (4.20) on the initial value y and the function $g \in G$.

Lemma 4.2 *The following inequalities are valid*

$$\|\Psi_{s,t}(y,g,\varepsilon) - \Psi_{s,t}(\bar{y},g,\varepsilon)\| \le \|y - \bar{y}\| e^{\varepsilon(\mu_1 + \varepsilon\gamma\mu)|s-t|},$$

$$\|\Psi_{s,t}(y,g,\varepsilon) - \Psi_{s,t}(y,\bar{g},\varepsilon)\| \le \frac{\mu}{\mu_1 + \varepsilon\gamma\mu} \rho(g,\bar{g}) \left(e^{\varepsilon(\mu_1+\varepsilon\gamma\mu)|s-t|} - 1\right).$$

The proof is the same as it was for Lemma 3.2.

As it was done in Section 3.3.2 equation (4.21) can be represented in the form

$$(I + \widetilde{R})v(y,\varepsilon) = \widetilde{Q}v(y,\varepsilon),$$

where I is the identity operator, and $\widetilde{R}$ and $\widetilde{Q}$ are given by the relations

$$\widetilde{R}v(y,\varepsilon) := \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)[v(\Psi_{s,0}(y,g,\varepsilon),\varepsilon) - v(y,\varepsilon)]ds, \tag{4.22}$$

$$\widetilde{Q}v(y,\varepsilon) := -\frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)\widetilde{Z}(\cdot)ds, \tag{4.23}$$

here

$$\widetilde{Z}(\cdot) = \widetilde{Z}(s, \Psi_{s,0}(y,g,\varepsilon), g(s,\Psi_{s,0}(y,g,\varepsilon),\varepsilon), v(\Psi_{s,0}(y,g,\varepsilon),\varepsilon),\varepsilon).$$

The inequalities (4.14), (4.15) imply

$$\|\widetilde{R}v(y,\varepsilon)\| \le \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \ \varepsilon\varkappa\|\Psi_{s,0}(y,g,\varepsilon) - y\|ds \le$$

$$\le \frac{2\varepsilon^2\sqrt{\alpha}e^{\beta^2/2\alpha}\varkappa}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} e^{\frac{-\alpha s^2}{2}} \int\limits_{0}^{s} \|\widetilde{Y}(\eta, \Psi_{s,0}(y,g,\varepsilon), g(\eta, \Psi_{s,0}(y,g,\varepsilon),\varepsilon))\|d\eta\, ds \le$$

$$\le \frac{\varepsilon^2\sqrt{2}e^{\beta^2/2\alpha}\varkappa K}{\sqrt{\alpha\pi}}.$$

For $\frac{\varepsilon^2\sqrt{2}e^{\beta^2/2\alpha}\varkappa K}{\sqrt{\alpha\pi}} < 1$ there exists the linear operator $(I+\widetilde{R})^{-1}$ and the following inequality is true

$$\|(I+\widetilde{R})^{-1}\| \le \frac{1}{1-\varepsilon^2\sqrt{2}e^{\beta^2/2\alpha}\varkappa K/\sqrt{\alpha\pi}}. \tag{4.24}$$

We introduce the operator $\widetilde{P}$ on V by

$$\widetilde{P}v = (I+\widetilde{R})^{-1}\widetilde{Q}v. \tag{4.25}$$

In the sequel we prove that the operator $\widetilde{P}$ maps V into itself and is a contraction.

For $\widetilde{Q}$ we get from (4.16), (4.23)

$$\|\widetilde{Q}v(y,\varepsilon)\| \le \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \|\widetilde{Z}(\cdot)\|ds \le$$

$$\leq e^{\beta^2/2\alpha}\varepsilon^{k+1}D_1(1+\varepsilon q+\varepsilon p+\varepsilon^{k+1}p^2).$$

Using the last inequality and (4.24), we obtain

$$\|\widetilde{P}v(y,\varepsilon)\| \leq \frac{\varepsilon^{k+1}e^{\beta^2/2\alpha}D_1(1+\varepsilon q+\varepsilon p+\varepsilon^{k+1}p^2)}{1-\varepsilon^2\sqrt{2}e^{\beta^2/2\alpha}\varkappa K/\sqrt{\alpha\pi}}.$$

Under the condition

$$\frac{\varepsilon^2\sqrt{2}e^{\beta^2/2\alpha}\varkappa K}{\sqrt{\alpha\pi}} \leq \frac{1}{2}, \tag{4.26}$$

the estimate

$$\|\widetilde{P}v(y,\varepsilon)\| \leq 2\varepsilon^{k+1}D_1(1+\varepsilon q+\varepsilon p+\varepsilon^{k+1}p^2)$$

is true.

From the inequalities (4.17)-(4.19) and Lemma 4.2 it follows

$$\|\widetilde{Q}v(y,\varepsilon)-\widetilde{Q}v(\bar{y},\varepsilon)\| \leq$$

$$\leq \frac{\alpha e^{1/4}}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}}\|\widetilde{Z}(s,\Psi_{s,0}(y,g,\varepsilon),g(s,\Psi_{s,0}(y,g,\varepsilon),\varepsilon),v(\Psi_{s,0}(y,g,\varepsilon),\varepsilon),\varepsilon)-$$

$$-\widetilde{Z}(s,\Psi_{s,0}(\bar{y},g,\varepsilon),g(s,\Psi_{s,0}(\bar{y},g,\varepsilon),\varepsilon),v(\Psi_{s,0}(\bar{y},g,\varepsilon),\varepsilon),\varepsilon)\|ds \leq$$

$$\leq \frac{\varepsilon\sqrt{\alpha}e^{\beta^2/2\alpha}D_1}{\sqrt{2\pi}}\int\limits_{\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}}(1+\varepsilon^{k+1}p+\varepsilon^{2k+1}p^2+\varepsilon\gamma(1+\varepsilon^k p)+\varepsilon\varkappa)\times$$

$$\|\Psi_{s,0}(y,g,\varepsilon)-\Psi_{s,0}(\bar{y},g,\varepsilon)\|ds \leq$$

$$\leq \frac{2\varepsilon\sqrt{\alpha}e^{\beta^2/2\alpha}D_1S_1}{\sqrt{2\pi}}\int\limits_{0}^{+\infty} e^{\frac{-\alpha s^2}{2}}e^{\varepsilon(\mu_1+\varepsilon\gamma\mu)s}ds\|y-\bar{y}\|, \tag{4.27}$$

where $S_1 = 1+\varepsilon(1+\varepsilon^k p)(\gamma+\varepsilon^k p)+\varepsilon\varkappa$.

Under the condition

$$\frac{\sqrt{2}}{\sqrt{\alpha}}\varepsilon(\mu_1+\varepsilon\gamma\mu) \leq 1, \tag{4.28}$$

the integral in (4.27) can be estimated due to Lemma 3.3

$$\frac{\sqrt{2\alpha}}{\sqrt{\pi}}\int\limits_{0}^{+\infty} e^{\frac{-\alpha s^2}{2}}e^{\varepsilon(\mu_1+\varepsilon\gamma\mu)s}ds \leq 3.$$

Consequently, we obtain

$$\|\widetilde{Q}v(y,\varepsilon) - \widetilde{Q}v(\bar{y},\varepsilon)\| \leq 3\varepsilon e^{\beta^2/2\alpha} D_1 S_1 \|y - \bar{y}\|. \tag{4.29}$$

Thus, by (4.24), (4.25), (4.29) we have

$$\|\widetilde{P}v(y,\varepsilon) - \widetilde{P}v(\bar{y},\varepsilon)\| \leq \frac{3\varepsilon e^{\beta^2/2\alpha} D_1 S_1}{1 - \varepsilon^2\sqrt{2}e^{\beta/2\alpha}\varkappa K/\sqrt{2\pi}} \|y - \bar{y}\| \leq 6\varepsilon e^{\beta^2/2\alpha} D_1 S_1 \|y - \bar{y}\|.$$

For sufficiently small ε the inequality (4.26) and the inequalities

$$2e^{\beta^2/2\alpha} D_1 (1 + \varepsilon p + \varepsilon q + \varepsilon^{k+1} p^2) \leq q, \tag{4.30}$$

$$\frac{\sqrt{2}\varepsilon(\mu_1 + \varepsilon\gamma\mu)}{\sqrt{\alpha}} \leq 1, \tag{4.31}$$

$$6e^{\beta^2/2\alpha} D_1 S_1 \leq \varkappa \tag{4.32}$$

hold, therefore $\widetilde{P}$ maps V into itself.

Now we derive conditions assuring $\widetilde{P}$ to be a contraction operator in V. For the difference $\|\widetilde{Q}v - \widetilde{Q}\bar{v}\|$ by (4.17), (4.23) we have

$$\|\widetilde{Q}v(y,\varepsilon) - \widetilde{Q}\bar{v}(y,\varepsilon)\| \leq \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \varepsilon D_1 \rho(v,\bar{v}) ds = \varepsilon e^{\beta^2/2\alpha} D_1 \rho(v,\bar{v}).$$

Consequently, for $\widetilde{P}v - \widetilde{P}\bar{v}$ we get

$$\|\widetilde{P}v(y,\varepsilon) - \widetilde{P}\bar{v}(y,\varepsilon)\| \leq \frac{\varepsilon e^{\beta^2/2\alpha} D_1}{1 - \varepsilon^2\sqrt{2}e^{\beta^2/2\alpha}\varkappa K/\sqrt{\alpha\pi}} \rho(v,\bar{v}) \leq 2\varepsilon e^{\beta^2/2\alpha} D_1 \rho(v,\bar{v}).$$

If $2\varepsilon D_1 e^{\beta^2/2\alpha} < 1$ holds then $\widetilde{P}$ is a contraction operator in V. Therefore, the equation $v = \widetilde{P}v$, which is equivalent to (4.21), has a unique solution in V. Thus, we have proved

Lemma 4.3 *Suppose the functions $\widetilde{Y}$, $\widetilde{Z}$ are continuous on $\mathbb{R} \times \mathbb{R}^n \times \Omega_z \times I_{\varepsilon_0}$, $\mathbb{R} \times \mathbb{R}^n \times \Omega_z \times \Omega_a \times I_{\varepsilon_0}$,respectively, and satisfy the conditions (4.14)-(4.17). Then for sufficiently small $\varepsilon \in I_{\varepsilon_0}$ there exists a unique function $v \in V$ guaranteeing that the function $\widetilde{T}g$ is continuous.*

Now we derive some auxiliary estimates which we shall use in estimating the Lipschitz constant for the function $\widetilde{T}h$. Let $v(y,\varepsilon)$ and $\bar{v}(y,\varepsilon)$ be the solutions of (4.21) corresponding to the functions g and $\bar{g}$ respectively. Then we have

$$(I+\widetilde{R})v=\widetilde{Q}v,$$
$$(I+\overline{\widetilde{R}})\bar{v}=\overline{\widetilde{Q}}\bar{v},$$

where

$$\overline{\widetilde{R}}\bar{v}(y,\varepsilon):=\frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}e^{\frac{-\alpha s^2}{2}}W^{-1}(s)[\bar{v}(\Psi_{s,0}(y,\bar{g},\varepsilon),\varepsilon)-\bar{v}(y,\varepsilon)]ds,$$

$$\overline{\widetilde{Q}}\bar{v}(y,\varepsilon):=-\frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}e^{\frac{-\alpha s^2}{2}}W^{-1}(s)\widetilde{Z}(\cdot)ds,$$

here

$$\widetilde{Z}(\cdot)=\widetilde{Z}(s,\Psi_{s,0}(y,\bar{g},\varepsilon),\bar{g}(s,\Psi_{s,0}(y,\bar{g},\varepsilon),\varepsilon),\bar{v}(\Psi_{s,0}(y,\bar{g},\varepsilon),\varepsilon),\varepsilon).$$

After some elementary transformations we obtain

$$v-\bar{v}=(I+\widetilde{R})^{-1}[\widetilde{Q}v-\overline{\widetilde{Q}}\bar{v}+(\overline{\widetilde{R}}-\widetilde{R})\bar{v}]. \tag{4.33}$$

The expression in the square brackets will be estimated at first. By (4.16)-(4.18) we have

$$\|\widetilde{Q}v(y,\varepsilon)-\overline{\widetilde{Q}}\bar{v}(y,\varepsilon)\|\leq$$

$$
\begin{aligned}
&\leq \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \Big[D_1(\varepsilon + \varepsilon\|\widetilde{g}\| + \|\widetilde{g}\|^2)\|\Psi_{s,0}(y,g,\varepsilon) - \Psi_{s,0}(y,\bar{g},\varepsilon)\| \\
&\quad + D_1(\varepsilon + \|\widetilde{g}\|)\|g(s,\Psi_{s,0}(y,g,\varepsilon),\varepsilon) - \bar{g}(s,\Psi_{s,0}(y,\bar{g},\varepsilon),\varepsilon)\| \\
&\quad + \varepsilon D_1\|v(\Psi_{s,0}(y,g,\varepsilon),\varepsilon) - \bar{v}(\Psi_{s,0}(y,\bar{g},\varepsilon),\varepsilon)\|\Big] ds \\
&\leq \frac{\varepsilon D_1\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \Big[(1 + \varepsilon^{k+1}p + \varepsilon^{2k+1}p^2 + \varepsilon\gamma(1+\varepsilon^k p) + \varepsilon\varkappa) \times \\
&\quad \|\Psi_{s,0}(y,g,\varepsilon) - \Psi_{s,0}(y,\bar{g},\varepsilon)\| + (1+\varepsilon^k p)\rho(g,\bar{g}) + \rho(v,\bar{v})\Big] ds \\
&\leq \varepsilon D_1 e^{\beta^2/2\alpha}\left[\rho(v,\bar{v}) + (1+\varepsilon^k p)\rho(g,\bar{g})\right] \\
&\quad + \frac{\varepsilon D_1 S_1\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} e^{\frac{-\alpha s^2}{2}} \frac{\mu}{\mu_1 + \varepsilon\gamma\mu}(e^{\varepsilon(\mu_1+\varepsilon\gamma\mu)s} - 1)\rho(g,\bar{g}) ds \\
&\leq \varepsilon e^{\beta^2/2\alpha} D_1 \left(\rho(v,\bar{v}) + \left(1 + \varepsilon^k p + \frac{3S_1\mu}{\mu_1 + \varepsilon\gamma\mu}\right)\rho(g,\bar{g})\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\|(\overline{\widetilde{R}} - \widetilde{R})\bar{v}\| &\leq \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \|\bar{v}(\Psi_{s,0}(y,\bar{g},\varepsilon),\varepsilon) - \bar{v}(\Psi_{s,0}(y,g,\varepsilon),\varepsilon)\| ds \\
&\leq \frac{\varepsilon\sqrt{\alpha}e^{\beta^2/2\alpha}\varkappa}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \|\Psi_{s,0}(y,g,\varepsilon) - \Psi_{s,0}(y,\bar{g},\varepsilon)\| ds \\
&\leq \frac{2\varepsilon\sqrt{\alpha}e^{\beta^2/2\alpha}\mu\varkappa}{\sqrt{2\pi}(\mu_1 + \varepsilon\gamma\mu)} \int\limits_{0}^{+\infty} e^{\frac{-\alpha s^2}{2}} e^{\varepsilon(\mu_1+\varepsilon\gamma\mu)s}\rho(g,\bar{g}) ds \\
&\leq \frac{3\varepsilon e^{\beta^2/2\alpha}\varkappa\mu}{\mu_1 + \varepsilon\gamma\mu}\rho(g,\bar{g}).
\end{aligned}
$$

Thus, we get by (4.33)

$$
\|v(y,\varepsilon) - \bar{v}(y,\varepsilon)\| \leq \frac{\varepsilon e^{\beta^2/2\alpha}}{1 - \varepsilon^2\sqrt{2}e^{\beta^2/2\alpha}\varkappa K/\sqrt{\alpha\pi}} \Big[D_1\rho(v,\bar{v}) +
$$

$$+\left(D_1(1+\varepsilon^k p)+\frac{3\mu(D_1S_1+\varkappa)}{\mu 1+\varepsilon\gamma\mu}\right)\rho(g,\bar g)\Bigg].$$

From this inequality it follows

Lemma 4.4 *Suppose the conditions of Lemma 4.3 and the inequality (4.26) are valid. Then the following estimate is true*

$$\rho(v,\bar v)\le \frac{2\varepsilon e^{\beta^2/2\alpha}}{1-2\varepsilon e^{\beta^2/2\alpha}D_1}\left(D_1(1+\varepsilon^k p)+\frac{3\mu(D_1S_1+\varkappa)}{\mu_1+\varepsilon\gamma\mu}\right)\rho(g,\bar g). \quad (4.34)$$

4.1.2 Order of the approximation

In this section we estimate the order of approximation (4.12). To do this we derive conditions under which the operator $\widetilde{T}$ maps the space G into itself and is a contraction. By (4.16) for $t\ge 0$ we have

$$\|\widetilde{T}g(t,y,\varepsilon)\|\le \int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}}\left[\|\widetilde{Z}(\cdot)\|+\|v(\Psi_{s,t}(y,g,\varepsilon),\varepsilon)\|\right]ds\le$$

$$\le \frac{\sqrt{\pi}}{\sqrt{2\alpha}}\varepsilon^{k+1}(D_1(1+\varepsilon p+\varepsilon q+\varepsilon^{k+1}p^2)+q).$$

The same estimate is true for $t\le 0$. Therefore, we have that $\widetilde{T}h$ is bounded. To prove the Lipschitz continuity of $\widetilde{T}h$ consider the difference $\|\widetilde{T}g(t,y,\varepsilon)-\widetilde{T}g(t,\bar y,\varepsilon)\|$. Then by (4.17) and Lemma 4.2 for $t\ge 0$ we get

$$\|\widetilde{T}g(t,y,\varepsilon)-\widetilde{T}g(t,\bar y,\varepsilon)\|\le$$

$$\le \int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}}\Big[\|\widetilde{Z}\left(s,\Psi_{s,t}(y,g,\varepsilon),g(s,\Psi_{s,t}(y,g,\varepsilon),\varepsilon),v(\Psi_{s,t}(y,g,\varepsilon),\varepsilon),\varepsilon\right)$$

$$-\widetilde{Z}\left(s,\Psi_{s,t}(\bar y,g,\varepsilon),g(s,\Psi_{s,t}(\bar y,g,\varepsilon),\varepsilon),v(\Psi_{s,t}(\bar y,g,\varepsilon),\varepsilon),\varepsilon\right)\|$$

$$+\|v(\Psi_{s,t}(y,g,\varepsilon),\varepsilon)-v(\Psi_{s,t}(\bar y,g,\varepsilon),\varepsilon)\|]\,ds$$

$$\le \int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}}\left(\varepsilon D_1(1+\varepsilon^{k+1}p+\varepsilon^{2k+1}p^2+\varepsilon\nu(1+\varepsilon^k p))\right.$$

$$\left.+\varepsilon\varkappa(\varepsilon D_1+1)\right)\|\Psi_{s,t}(y,g,\varepsilon)-\Psi_{s,t}(\bar y,g,\varepsilon)\|ds$$

$$\leq \varepsilon(D_1S_1+\varkappa)\int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}}\|\Psi_{s,t}(y,g,\varepsilon)-\Psi_{s,t}(\bar{y},g,\varepsilon)\|ds$$

$$\leq \varepsilon(D_1S_1+\varkappa)\int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}}e^{\varepsilon(\mu_1+\varepsilon\gamma\mu)(s-t)}\|y-\bar{y}\|ds.$$

Under the condition (4.28), using the error integral $\mathrm{erf}(\lambda)$ and Lemma 3.3 the last integral can be estimated

$$\int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}}e^{\varepsilon(\mu_1+\varepsilon\gamma\mu)(s-t)}ds=\frac{\sqrt{2}}{\sqrt{\alpha}}\int\limits_{t_1-\lambda}^{+\infty} e^{(t_1-\lambda)^2-(s_1-\lambda)^2}d(s_1-\lambda)\leq$$

$$\leq \frac{\sqrt{2}}{\sqrt{\alpha}}\left(2\int\limits_0^{|t_1-\lambda|} e^{(t_1-\lambda)^2-(s_1-\lambda)^2}d(s_1-\lambda)+\int\limits_{|t_1-\lambda|}^{+\infty} e^{(t_1-\lambda)^2-(s_1-\lambda)^2}d(s_1-\lambda)\right)=$$

$$=\frac{\sqrt{2}}{\sqrt{\alpha}}\left(\frac{\sqrt{\pi}}{2}+\sqrt{\pi}e^{(t_1-\lambda)^2}\mathrm{erf}(|t_1-\lambda|)\right)<3\frac{\sqrt{\pi}}{\sqrt{2\alpha}}.$$

Consequently, for $t\geq 0$ we get

$$\|\widetilde{T}g(t,y,\varepsilon)-\widetilde{T}g(t,\bar{y},\varepsilon)\|<\frac{3\varepsilon\sqrt{\pi}}{\sqrt{2\alpha}}(D_1S_1+\varkappa)\|y-\bar{y}\|.$$

Analogously, one sees that the same estimate is valid for $t\leq 0$.

Thus, if the inequalities

$$\frac{\sqrt{\pi}}{\sqrt{2\alpha}}(q+D_1(1+\varepsilon p+\varepsilon q+\varepsilon^{k+1}p^2))\leq p, \tag{4.35}$$

$$3\frac{\sqrt{\pi}}{\sqrt{2\alpha}}(D_1S_1+\varkappa)\leq\gamma \tag{4.36}$$

hold then $\widetilde{T}$ maps G into itself.

Now we prove that $\widetilde{T}$ is a contraction operator in the space G. According to the inequalities (4.17)-(4.19) and Lemmas 4.2, 4.4 we get

$$\|\widetilde{T}g(t,y,\varepsilon)-\widetilde{T}\bar{g}(t,y,\varepsilon)\|\leq$$

$$\int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}}\Big[\Big\|\widetilde{Z}\left(s,\Psi_{s,t}(y,g,\varepsilon),g(s,\Psi_{s,t}(y,g,\varepsilon),\varepsilon),v(\Psi_{s,t}(y,g,\varepsilon),\varepsilon),\varepsilon\right)-$$

$$-\widetilde{Z}\left(s,\Psi_{s,t}(y,\bar{g},\varepsilon),\bar{g}(s,\Psi_{s,t}(y,\bar{g},\varepsilon),\varepsilon),\bar{v}(\Psi_{s,t}(y,\bar{g},\varepsilon),\varepsilon),\varepsilon\right)\Big\|+$$

$$+\left\|v(\Psi_{s,t}(y,g,\varepsilon),\varepsilon)-\bar{v}(\Psi_{s,t}(y,\bar{g},\varepsilon),\varepsilon)\right\|\Big]\,ds\ \le$$

$$\le\int\limits_{t}^{+\infty}e^{\frac{\alpha(t^2-s^2)}{2}}\left(\varepsilon D_1(1+\varepsilon^{k+1}p+\varepsilon^{2k+1}p^2+\varepsilon\gamma(1+\varepsilon^{k+1}p)+\varepsilon\varkappa)+\varepsilon\varkappa\right)\times$$

$$\|\Psi_{s,t}(y,g,\varepsilon)-\Psi_{s,t}(y,\bar{g},\varepsilon)\|ds+$$

$$+\frac{\sqrt{\pi}}{\sqrt{2\alpha}}\left((1+\varepsilon D_1)\rho(v,\bar{v})+\varepsilon D_1(1+\varepsilon^k p)\rho(g,\bar{g})\right)\le$$

$$\le\frac{\varepsilon\sqrt{\pi}}{2\alpha}\Bigg[D_1(1+\varepsilon^k p)+(1+\varepsilon D_1)\frac{2e^{\beta^2/2\alpha}}{1-2\varepsilon e^{\beta^2/2\alpha}D_1}\times$$

$$\left(D_1(1+\varepsilon^k p)+\frac{3\mu(D_1S_1+\varkappa)}{\mu_1+\varepsilon\gamma\mu}\right)\Bigg]\rho(g,\bar{g})+$$

$$+\frac{\varepsilon\mu(D_1S_1+\varkappa)}{\mu_1+\varepsilon\gamma\mu}\int\limits_{t}^{+\infty}e^{\frac{\alpha(t^2-s^2)}{2}}\left(e^{\varepsilon(\mu_1+\varepsilon\gamma\mu)(s-t)}-1\right)\rho(g,\bar{g})ds\le$$

$$\le\frac{\varepsilon\sqrt{\pi}}{\sqrt{2\alpha}}\Bigg[D_1(1+\varepsilon^k p)+(1+\varepsilon D_1)\frac{2e^{\beta^2/2\alpha}}{1-2\varepsilon D_1e^{\beta^2/2\alpha}}\left(D_1(1+\varepsilon^k p)+\frac{3\mu(D_1S_1+\varkappa)}{\mu_1+\varepsilon\gamma\mu}\right)$$

$$+\frac{3\varepsilon\mu(D_1S_1+\varkappa)}{(\mu_1+\varepsilon\gamma\mu)}\Bigg]\rho(g,\bar{g}).$$

Taking into account the relations (4.35), (4.36) and the inequality

$$\frac{\varepsilon\sqrt{\pi}}{\sqrt{2\alpha}}\Bigg[D_1(1+\varepsilon^k p)+(1+\varepsilon D_1)\frac{2e^{\beta^2/2\alpha}}{1-2\varepsilon D_1e^{\beta^2/2\alpha}}\left(D_1(1+\varepsilon^k p)+\frac{3\mu(D_1S_1+\varkappa)}{\mu_1+\varepsilon\gamma\mu}\right)$$

$$+\frac{3\varepsilon\mu(D_1S_1+\varkappa)}{(\mu_1+\varepsilon\gamma\mu)}\Bigg]<1 \qquad (4.37)$$

we can conclude that $\widetilde{T}$ is a contraction operator in G. Therefore the operator $\widetilde{T}$ has a fixed point in the space G. This fixed point represents an integral manifold of system (4.13). This completes the proof of the Theorem 4.1.

Example 4.5 Consider the system

$$\begin{aligned} \frac{dy}{dt} &= \varepsilon Y(t,y,z,\varepsilon), \\ \frac{dz_1}{dt} &= tz_1 + z_2 + (z_1)^2 + \varepsilon \cos t \cos y + a_1(y,\varepsilon), \\ \frac{dz}{dt} &= -z_1 + tz_2 + a_2(y,\varepsilon), \end{aligned} \tag{4.38}$$

where $y \in \mathbb{R}$, $z = (z_1, z_2)^T$, $a = (a_1, a_2)^T$.

We shall look for the integral manifold $z = h(t,y,\varepsilon)$, $h = (h^1, h^2)^T$ and the function $a(y,\varepsilon)$ in the form

$$\begin{aligned} h^i(t,y,\varepsilon) &= h_0^i(t,y) + \varepsilon h_1^i(t,y) + \varepsilon^2 h_2^i(t,y) + \cdots, \\ a^i(y,\varepsilon) &= a_0^i(y) + \varepsilon a_1^i(y) + \varepsilon^2 a_2^i(y) + \cdots, \quad i = 1,2. \end{aligned} \tag{4.39}$$

Substituting the expansions (4.39) into (4.38) and equating the coefficients we get the equations for the determinig the functions h_j^i, a_j^i.

From the definition of the spaces H, F we get

$$h_0 \equiv 0, \qquad a_0 \equiv 0.$$

Then, equating the coefficients with ε we get

$$\begin{aligned} \frac{dh_1^1}{dt} &= th_1^1 + h_1^2 + \cos t \cos y + a_1^1, \\ \frac{dh_1^2}{dt} &= -h_1^1 + th_1^2 + a_1^2. \end{aligned} \tag{4.40}$$

The function $h_1 = (h_1^1, h_1^2)^T$ is the uniformly bounded solution of this equation (see (4.6)) and $a_1(y)$ is defined from the condition of continuity of $h_1(t,y)$ at $t = 0$ (4.7)

$$\int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} W^{-1}(s)\,(Z_1(s,y) + a_1(y))\,ds = 0. \tag{4.41}$$

From equation (4.41) we get

$$a_1^*(y) := -\frac{e^{1/2}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} Z_1(s,y) ds = -\left(\frac{e^{1/2}}{2}(1 + e^{-2}) \cos y, 0\right)^T. \tag{4.42}$$

Thus the function $a = \varepsilon a_1^*(y)$ is the first order approximation of the gluing function.

Substituting $a_1^*(y)$ into (4.40) we get the system

$$\tfrac{dh_1}{dt} = B(t)h_1 + Z_1(t, y) + a_1^*(y). \tag{4.43}$$

The function h_1 is a uniformly bounded solution of this system, and is given by

$$h_1^*(t, y) = \begin{cases} \int\limits_{-\infty}^{t} e^{\frac{t^2-s^2}{2}} W(t-s)\left(Z_1(s, y, h_1(t, y), \varepsilon) + a_1^*(y)\right) ds & t < 0, \\ -\int\limits_{t}^{+\infty} e^{\frac{t^2-s^2}{2}} W(t-s)\left(Z_1(s, y, h_1(t, y), \varepsilon) + a_1^*(y)\right) ds & t \geq 0. \end{cases} \tag{4.44}$$

Thus, the function $z = \varepsilon h_1^*(t, y)$ is the first-order approximation of the integral manifold of (4.38).

Chapter 5

Smoothness of the integral manifold

This chapter is devoted to the problem of smoothness of the integral manifold of system (3.6) and the gluing function. We shall study the dependence of the existence of the partial derivatives of the function $h(t, y, \varepsilon)$ representing the integral manifold and the function $a(y, \varepsilon)$ on the smoothness of the functions in the r. h. s. of (3.6). To this end, we consider the subspaces of the spaces H, F of smooth functions and show that the restrictions of the operators T, P defined in Sections 3.2, 3.3.2, map these subspaces into themselves and are contractions.

We shall use the induction principle. First, we prove the existence of the first derivatives. Then we show that under some assumptions it is possible to prove the existence of further derivatives.

5.1 Existence of the first derivative

5.1.1 Assumptions

Here and elsewhere f_x denotes the function

$$f_x = \frac{\partial f}{\partial x}.$$

We suppose that the functions Y, Z in the r.h.s. of (3.6) satisfy the assumptions (H_1), (H_2), and have first derivatives with respect to y, z, a that are continuous on $\mathbb{R} \times \mathbb{R}^n \times \Omega_z \times I_{\varepsilon_0}$, $\mathbb{R} \times \mathbb{R}^n \times \Omega_z \times \Omega_a \times I_{\varepsilon_0}$, and satisfy for $t \in \mathbb{R}$, $y, \bar{y} \in \mathbb{R}^n, z, \bar{z} \in \Omega_z, a, \bar{a} \in \Omega_a, \varepsilon \in I_{\varepsilon_0}$ the inequalities:

$$\begin{aligned} \|Y_x(t,y,z,\varepsilon)\| &\le \mu, \\ \|Y_x(t,y,z,\varepsilon) - Y_x(t,\bar{y},\bar{z},\varepsilon)\| &\le \mu_1 \left(\|y-\bar{y}\| + \|z-\bar{z}\|\right), \end{aligned} \tag{5.1}$$

$$\begin{aligned} \|Z_y(t,y,z,a,\varepsilon)\| &\le D\left(\varepsilon + \varepsilon\|z\| + \|z\|^2\right) \\ \|Z_z(t,y,z,a,\varepsilon)\| &\le D\left(\varepsilon + \|z\|\right), \\ \|Z_a(t,y,z,a,\varepsilon)\| &\le \varepsilon D \end{aligned} \tag{5.2}$$

$$\|Z_x(t,y,z,a,\varepsilon) - Z_x(t,\bar{y},\bar{z},\bar{a},\varepsilon)\| \le D_1\left(\|y-\bar{y}\| + \|z-\bar{z}\| + \|a-\bar{a}\|\right), \tag{5.3}$$

where $\|\tilde{z}\| := \max\{\|z\|, \|\bar{z}\|\}$, and $f_x = \frac{\partial f}{\partial x}$, x is a placeholder.

In what what follows we use the following inequality

$$\sum_{i=1}^{n} u \le \left(n\sum_{i=1}^{n} u_i^2\right)^{1/2}.$$

This inequality is obtained from the Hölder inequality

$$\sum_{i=1}^{n} u_i v_i \le \left(\sum_{i=1}^{n} u_i^p \sum_{i=1}^{n} v_i^p\right)^{1/p},$$

by taking $p = 2$, $v_i = 1$.

We consider the subspace $F^{(1)} \in F$ of functions $a(y,\varepsilon)$ differentiable with respect to y, satisfying the inequalities

$$\|a_{y_i}(y,\varepsilon)\| \le \varepsilon l_1,$$

$$\|a_{y_i}(y,\varepsilon) - a_{y_i}(\bar{y},\varepsilon)\| \le l_2\|y-\bar{y}\|, \quad i = 1,\dots,n.$$

Setting $\nu = \sqrt{n} l_1$, $\nu_1 = \sqrt{n} l_l$, we get

$$\begin{aligned} \|a_y(y,\varepsilon)\| &\le \varepsilon\nu, \\ \|a_y(y,\varepsilon) - a_y(\bar{y},\varepsilon)\| &\le \nu_1\|y-\bar{y}\|. \end{aligned} \tag{5.4}$$

We equip the space $F^{(1)}$ with the generalized metric $d(a,\bar{a}) = \mathrm{col}\left(\rho(a,\bar{a}), \rho(a_y,\bar{a}_y)\right)$, where $\rho(a_y,\bar{a}_y)$ is defined by

$$\rho(a_y,\bar{a}_y) = \max_{1\le i\le n} \sup_{y\in\mathbb{R}^n, \varepsilon\in I_{\varepsilon_0}} \|a_{y_i}(y,\varepsilon) - \bar{a}_{y_i}(y,\varepsilon)\|.$$

Then $F^{(1)}$ is a complete metric space [12].

Let $H^{(1)}$ be the subspace of H consisting of functions $h(t,y,\varepsilon)$ that have continuous partial derivatives with respect to y satisfying for $t \in \mathbb{R}, y, \bar{y} \in \mathbb{R}^n, \varepsilon \in I_{\varepsilon_0}$ the inequalities

$$
\begin{aligned}
\|h_{y_i}(t,y,\varepsilon)\| &\le \varepsilon n_1, \\
\|h_{y_i}(t,y,\varepsilon)-h_{y_i}(t,\bar y,\varepsilon)\| &\le n_2\|y-\bar y\|,\ i=1,\dots,n,
\end{aligned} \tag{5.5}
$$

with the generalized metric $d(h,\bar h)=\mathrm{col}\left(\rho(h,\bar h),\rho(h_y,\bar h_y)\right)$, where $\rho(h_y,\bar h_y)$ is defined by

$$
\rho(h_y,\bar h_y)=\max_{1\le i\le n}\ \sup_{t\in\mathbb{R},y\in\mathbb{R}^n,\varepsilon\in I_{\varepsilon_0}}\|h_{y_i}(t,y,\varepsilon)-\bar h_{y_i}(t,y,\varepsilon)\|.
$$

From the inequalities (5.5) it follows

$$
\begin{aligned}
\|h_y(t,y,\varepsilon)\| &\le \varepsilon\xi, \\
\|h_y(t,y,\varepsilon)-h_y(t,\bar y,\varepsilon)\| &\le \xi_1\|y-\bar y\|.
\end{aligned} \tag{5.6}
$$

In what follows we use the notation $\Phi^{(1)}_{s,t}(y,h,\varepsilon)=\frac{\partial\Phi_{s,t}(y,h,\varepsilon)}{\partial y_i}$.

Let us find the partial derivative with respect to y_i of the element Th. The functions $Z(t,y,z,a(y,\varepsilon),\varepsilon)$ and $a(y,\varepsilon)$ depend continuously on y and under our assumptions there exist continuous derivatives of Z and a with respect to y. Moreover, the integrals

$$
I_1=\int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}}W(t-s)\left[Z_y(\cdot)+Z_z(\cdot)\,h_y(s,\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\right.
$$

$$
\left.+Z_a(\cdot)\,a_y(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)+a_y(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\right]\Phi^{(1)}_{s,t}(y,h,\varepsilon)\,ds,
$$

$$
I_2=\int_{-\infty}^{t} e^{\frac{\alpha(t^2-s^2)}{2}}W(t-s)\left[Z_y(\cdot)+Z_z(\cdot)\,h_y(s,\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\right.
$$

$$
\left.+Z_a(\cdot)\,a_y(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)+a_y(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\right]\Phi^{(1)}_{s,t}(y,h,\varepsilon)\,ds,
$$

converge uniformly with respect to y. Therefore, we can write

$$
\frac{\partial}{\partial y_i}Th(t,y,\varepsilon)=\begin{cases}
-\int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}}W(t-s)\left[Z_y(\cdot)+Z_z(\cdot)\,h_y(s,\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\right. \\
\quad \left.+Z_a(\cdot)\,a_y(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)+a_y(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\right]\times \\
\quad \Phi^{(1)}_{s,t}(y,h,\varepsilon)\,ds,\quad t\ge 0, \\
\int_{-\infty}^{t} e^{\frac{\alpha(t^2-s^2)}{2}}W(t-s)\left[Z_y(\cdot)+Z_z(\cdot)\,h_y(s,\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\right. \\
\quad \left.+Z_a(\cdot)\,a_y(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)+a_y(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\right]\times \\
\quad \Phi^{(1)}_{s,t}(y,h,\varepsilon)\,ds,\quad t<0,
\end{cases}
$$

where

$$f(\cdot) = f\left(s, \Phi_{s,t}(y,h,\varepsilon), h(s, \Phi_{s,t}(y,h,\varepsilon),\varepsilon), a(\Phi_{s,t}(y,h,\varepsilon),\varepsilon),\varepsilon\right).$$

In the sequel we show that the element Th together with $\frac{\partial}{\partial y_i}Th(t,y,\varepsilon)$ belong to the space $H^{(1)}$ and the operator T is a contraction in $H^{(1)}$. Therefore, $\frac{\partial}{\partial y_i}Th(t,y,\varepsilon)$ represents the partial derivative of the integral manifold $z = h(t,y,\varepsilon)$ with respect to y.

5.1.2 Auxiliary estimates

From the definition of the functions $\Phi_{s,t}(y,h,\varepsilon)$, it follows $\Phi^{(1)}_{s,t}(y,h,\varepsilon)$ can be represented in the form

$$\Phi^{(1)}_{s,t}(y,h,\varepsilon) = e_i + \varepsilon \int_t^s \left[Y_y(\cdot) + Y_z(\cdot)\; h_y(\eta, \Phi_{\eta,t}(y,h,\varepsilon),\varepsilon)\right] \Phi^{(1)}_{\eta,t}(y,h,\varepsilon)\; d\eta,$$

where $f(\cdot) = f(\eta, \Phi_{\eta,t}(y,h,\varepsilon), h(\eta, \Phi_{\eta,t}(y,h,\varepsilon),\varepsilon),\varepsilon)$ and e_i is a vector which has ith component equals to 1 and all other components are 0. Then, the following results are valid.

Lemma 5.1 *Let the inequalities (5.1) be valid. Then for the norm of* $\Phi^{(1)}_{s,t}(y,h,\varepsilon)$ *we have the following estimate*

$$\|\Phi^{(1)}_{s,t}(y,h,\varepsilon)\| \le e^{\varepsilon\mu(1+\varepsilon\xi)|s-t|}.$$

Proof. For $s \ge t$ we have

$$\begin{aligned} \|\Phi^{(1)}_{s,t}(y,h,\varepsilon)\| &\le 1 + \varepsilon \int_t^s \left[\|Y_y(\cdot)\| + \|Y_z(\cdot)\| \|h_y(*)\|\right]\; \|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon)\| d\eta \\ &\le 1 + \varepsilon\mu\left(1+\varepsilon\xi\right) \int_t^s \|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon)\| d\eta. \end{aligned}$$

In the same way we can estimate the norm for $s \le t$. Then, using the Gronwall-Bellman inequality we obtain the result.

Lemma 5.2 *Under the conditions (5.1), the following estimates are valid*

$$\|\Phi^{(1)}_{s,t}(y,h,\varepsilon)-\Phi^{(1)}_{s,t}(\bar{y},h,\varepsilon)\|\leq$$

$$\leq \frac{\|y-\bar{y}\|}{\mu(1+\varepsilon\xi)}\left(\mu_1(1+\varepsilon\xi)^2+\xi_1\mu\right)\left(e^{\varepsilon\mu(1+\varepsilon\xi)|s-t|}-1\right)e^{2\varepsilon\mu(1+\varepsilon\xi)|s-t|},$$

$$\|\Phi^{(1)}_{s,t}(y,h,\varepsilon)-\Phi^{(1)}_{s,t}(y,\bar{h},\varepsilon)\|\leq$$

$$\leq \frac{1}{\mu(1+\varepsilon\xi)}\left[\left(\frac{1}{1+\varepsilon\xi}\left(\mu_1(1+\varepsilon\xi)^2+\mu\xi_1\right)\left(e^{\varepsilon\mu(1+\varepsilon\xi)|s-t|}-1\right)\right.\right.$$

$$\left.\left.+\,\mu_1(1+\varepsilon\xi)\right)\rho(h,\bar{h})+\mu\rho(h_y,\bar{h}_y)\right]\left(e^{\varepsilon\mu(1+\varepsilon\xi)|s-t|}-1\right)e^{\varepsilon\mu(1+\varepsilon\xi)|s-t|}.$$

Proof. The proof is similar to the proof of Lemma 3.2. Using the inequalities (5.1) and Lemma 5.1, we obtain for the difference $\|\Phi^{(1)}_{s,t}(y,h,\varepsilon)-\Phi^{(1)}_{s,t}(\bar{y},h,\varepsilon)\|$ for $s\geq t$

$$\|\Phi^{(1)}_{s,t}(y,h,\varepsilon)-\Phi^{(1)}_{s,t}(\bar{y},h,\varepsilon)\|\leq$$

$$\leq \varepsilon\int_t^s [\|Y_y(\eta,\Phi_{\eta,t}(y,h,\varepsilon),h(\eta,\Phi_{\eta,t}(y,h,\varepsilon),\varepsilon),\varepsilon)$$

$$-Y_y(\eta,\Phi_{\eta,t}(\bar{y},h,\varepsilon),h(\eta,\Phi_{\eta,t}(\bar{y},h,\varepsilon),\varepsilon),\varepsilon)\|\|\Phi^{(1)}_{\eta,t}(\bar{y},h,\varepsilon)\|$$

$$+\|Y_y(\eta,\Phi_{\eta,t}(\bar{y},h,\varepsilon),h(\eta,\Phi_{\eta,t}(\bar{y},h,\varepsilon),\varepsilon),\varepsilon)\|\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon)-\Phi^{(1)}_{\eta,t}(\bar{y},h,\varepsilon)\|$$

$$+\|Y_z(\eta,\Phi_{\eta,t}(y,h,\varepsilon),h(\eta,\Phi_{\eta,t}(y,h,\varepsilon),\varepsilon),\varepsilon)$$

$$-Y_z(\eta,\Phi_{\eta,t}(\bar{y},h,\varepsilon),h(\eta,\Phi_{\eta,t}(\bar{y},h,\varepsilon),\varepsilon),\varepsilon)\|\times$$

$$\|h_y(\eta,\Phi_{\eta,t}(y,h,\varepsilon),\varepsilon)\|\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon)\|$$

$$+\|Y_z(\eta,\Phi_{\eta,t}(\bar{y},h,\varepsilon),h(\eta,\Phi_{\eta,t}(\bar{y},h,\varepsilon),\varepsilon),\varepsilon)\|\times$$

$$\|h_y(\eta,\Phi_{\eta,t}(y,h,\varepsilon),\varepsilon)-h_y(\eta,\Phi_{\eta,t}(\bar{y},h,\varepsilon),\varepsilon)\|\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon)\|$$

$$+\|Y_z(\eta,\Phi_{\eta,t}(\bar{y},h,\varepsilon),h(\eta,\Phi_{\eta,t}(\bar{y},h,\varepsilon),\varepsilon),\varepsilon)\|\times$$

$$\left.\|h_y(\eta,\Phi_{\eta,t}(\bar{y},h,\varepsilon),\varepsilon)\|\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon)-\Phi^{(1)}_{\eta,t}(\bar{y},h,\varepsilon)\|\right]d\eta$$

$$\leq \varepsilon\int_t^s\left[\left(\mu_1(1+\varepsilon\xi)^2+\mu\xi_1\right)e^{2\varepsilon\mu(1+\varepsilon\xi)(\eta-t)}\|y-\bar{y}\|\right.$$

$$\left.+\,\mu(1+\varepsilon\xi)\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon)-\Phi^{(1)}_{\eta,t}(\bar{y},h,\varepsilon)\|\right]d\eta.$$

Using the Gronwall-Bellman inequality we have

$$\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon) - \Phi^{(1)}_{\eta,t}(\bar{y},h,\varepsilon)\| \le$$

$$\le \frac{\|y-\bar{y}\|}{\mu(1+\varepsilon\xi)} \left(\mu_1(1+\varepsilon\xi)^2 + \mu\xi_1\right) \left(e^{\varepsilon\mu(1+\varepsilon\xi)(s-t)} - 1\right) e^{2\varepsilon\mu(1+\varepsilon\xi)(s-t)}. \quad (5.7)$$

For the difference $\|\Phi^{(1)}_{s,t}(y,h,\varepsilon) - \Phi^{(1)}_{s,t}(y,\bar{h},\varepsilon)\|$ we get from (5.1) and Lemma 5.1

$$\|\Phi^{(1)}_{s,t}(y,h,\varepsilon) - \Phi^{(1)}_{s,t}(y,\bar{h},\varepsilon)\| \le \int_t^s \varepsilon\,[\|Y_y(\eta,\Phi_{\eta,t}(y,h,\varepsilon),h(\eta,\Phi_{\eta,t}(y,h,\varepsilon),\varepsilon),\varepsilon)$$

$$-Y_y(\eta,\Phi_{\eta,t}(y,\bar{h},\varepsilon),\bar{h}(\eta,\Phi_{\eta,t}(y,\bar{h},\varepsilon),\varepsilon),\varepsilon)\|\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon)\|$$

$$+\|Y_y(\eta,\Phi_{\eta,t}(y,\bar{h},\varepsilon),\bar{h}(\eta,\Phi_{\eta,t}(y,\bar{h},\varepsilon),\varepsilon),\varepsilon)\|\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon) - \Phi^{(1)}_{\eta,t}(y,\bar{h},\varepsilon)\|$$

$$+\|Y_z(\eta,\Phi_{\eta,t}(y,h,\varepsilon),h(\eta,\Phi_{\eta,t}(y,h,\varepsilon),\varepsilon),\varepsilon)$$

$$-Y_z(\eta,\Phi_{\eta,t}(\bar{y},h,\varepsilon),\bar{h}(\eta,\Phi_{\eta,t}(y,\bar{h},\varepsilon),\varepsilon),\varepsilon)\|\times$$

$$\|h_y(\eta,\Phi_{\eta,t}(y,h,\varepsilon),\varepsilon)\|\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon)\|$$

$$+\|Y_z(\eta,\Phi^{\varepsilon}_{\eta,t}(y;\bar{h}),\bar{h}(\eta,\Phi_{\eta,t}(y,\bar{h},\varepsilon),\varepsilon),\varepsilon)\|\times$$

$$\|h_y(\eta,\Phi_{\eta,t}(y,h,\varepsilon),\varepsilon) - \bar{h}_y(\eta,\Phi_{\eta,t}(y,\bar{h},\varepsilon),\varepsilon)\|\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon)\|$$

$$+\|Y_z(\eta,\Phi_{\eta,t}(y,\bar{h},\varepsilon),\bar{h}(\eta,\Phi_{\eta,t}(y,\bar{h},\varepsilon),\varepsilon),\varepsilon)\|\times$$

$$\|\bar{h}_y(\eta,\Phi_{\eta,t}(y,\bar{h},\varepsilon),\varepsilon)\|\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon) - \Phi^{(1)}_{\eta,t}(y,\bar{h},\varepsilon)\|\Big]\,d\eta \le$$

$$\le \varepsilon\int_t^s \Big[\Big(\left(\mu_1(1+\varepsilon\xi)^2 + \mu\xi_1\right)\|\Phi_{\eta,t}(y,h,\varepsilon) - \Phi_{\eta,t}(y,\bar{h},\varepsilon)\|$$

$$+\mu_1(1+\varepsilon\xi)\rho(h,\bar{h}) + \mu\rho(h_y,\bar{h}_y)\Big)\,e^{\varepsilon\mu(1+\varepsilon\xi)(s-\eta)}$$

$$+\mu(1+\varepsilon\xi)\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon) - \Phi^{(1)}_{\eta,t}(y,\bar{h},\varepsilon)\|\Big]\,d\eta \le$$

$$\le \varepsilon\int_t^s \Bigg[\Bigg(\bigg(\frac{1}{1+\varepsilon\xi}\left(e^{\varepsilon\mu(1+\varepsilon\xi)(\eta-t)} - 1\right)\left(\mu_1(1+\varepsilon\xi)^2 + \mu\xi_1\right)$$

$$+\mu_1(1+\varepsilon\xi)\bigg)\rho(h,\bar{h}) + \mu\rho(h_y,\bar{h}_y)\Bigg)e^{\varepsilon\mu(1+\varepsilon\xi)(\eta-t)}$$

$$+\mu(1+\varepsilon\xi)\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon)-\Phi^{(1)}_{\eta,t}(y,\bar{h},\varepsilon)\|\Big]d\eta.$$

Applying the Gronwall-Bellman inequality we obtain

$$\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon)-\Phi^{(1)}_{\eta,t}(y,\bar{h},\varepsilon)\| \le \frac{1}{\mu(1+\varepsilon\xi)}\left[\left(\frac{1}{1+\varepsilon\xi}\left(\mu_1(1+\varepsilon\xi)^2+\mu\xi_1\right)\times\right.\right.$$
$$\left.\left(e^{\varepsilon\mu(1+\varepsilon\xi)(s-t)}-1\right)+\mu_1(1+\varepsilon\xi)\right)\rho(h,\bar{h})+\mu\rho(h_y,\bar{h}_y)\Big]\times$$
$$\left(e^{\varepsilon\mu(1+\varepsilon\xi)(s-t)}-1\right)e^{\varepsilon\mu(1+\varepsilon\xi)(s-t)}. \tag{5.8}$$

In the same way we get for $s \le t$

$$\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon)-\Phi^{(1)}_{\eta,t}(\bar{y},h,\varepsilon)\| \le$$
$$\le \frac{\|y-\bar{y}\|}{\mu(1+\varepsilon\xi)}\left(\mu_1(1+\varepsilon\xi)^2+\mu\xi_1\right)\left(e^{\varepsilon\mu(1+\varepsilon\xi)(t-s)}-1\right)e^{2\varepsilon\mu(1+\varepsilon\xi)(t-s)}, \tag{5.9}$$

$$\|\Phi^{(1)}_{\eta,t}(y,h,\varepsilon)-\Phi^{(1)}_{\eta,t}(y,\bar{h},\varepsilon)\| \le \frac{1}{\mu(1+\varepsilon\xi)}\left[\left(\frac{1}{1+\varepsilon\xi}\left(\mu_1(1+\varepsilon\xi)^2+\mu\xi_1\right)\times\right.\right.$$
$$\left.\left(e^{\varepsilon\mu(1+\varepsilon\xi)(t-s)}-1\right)+\mu_1(1+\varepsilon\xi)\right)\rho(h,\bar{h})+\mu\rho(h_y,\bar{h}_y)\Big]\times$$
$$\left(e^{\varepsilon\mu(1+\varepsilon\xi)(t-s)}-1\right)e^{\varepsilon\mu(1+\varepsilon\xi)(t-s)}. \tag{5.10}$$

This completes the proof of the lemma.

5.1.3 Continuity of the function $\frac{\partial}{\partial y_i}Th$ at $t=0$

In what follows we show that under the condition (5.1)-(5.2) the operator P defined in Section 3.3.2 maps $F^{(1)}$ into itself and is a contraction. Namely, the function $a_y(y,\varepsilon)$ satisfies conditions (5.4), and the function $a(y,\varepsilon)$ together with its derivative $a_y(y,\varepsilon)$ belong to the space $F^{(1)}$.

From the condition of continuity of $\frac{\partial}{\partial y_i}Th$ at $t=0$ we obtain the following equation to determine the function $a_y(y,\varepsilon)$

$$\int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}}W^{-1}(s)\left[Z_y(\cdot)+Z_z(\cdot)h_y(s,\Phi_{s,0}(y,h,\varepsilon),\varepsilon)\right.$$
$$\left.+\,Z_a(\cdot)a_y(\Phi_{s,0}(y,h,\varepsilon),\varepsilon)+a_y(\Phi_{s,0}(y,h,\varepsilon),\varepsilon)\right]\Phi^{(1)}_{s,0}(y,h,\varepsilon)=0. \tag{5.11}$$

Here $f(\cdot)=f(s,\Phi_{s,0}(y,h,\varepsilon),h(s,\Phi_{s,0}(y,h,\varepsilon),\varepsilon),a(\Phi_{s,0}(y,h,\varepsilon),\varepsilon),\varepsilon)$.

Let us introduce the following operators

$$R^{(1)}a_y(y,\varepsilon) := \frac{\partial}{\partial y_i} Ra(y,\varepsilon)$$

$$= \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) \left[a_y(\Phi_{s,0}(y,h,\varepsilon),\varepsilon)\Phi^{(1)}_{s,0}(y,h,\varepsilon) - a_y(y,\varepsilon)e_i\right] ds,$$

$$Q^{(1)}a_y(y,\varepsilon) := \frac{\partial}{\partial y_i} Qa(y,\varepsilon) = -\frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) \left[Z_y(\cdot) + \right.$$

$$\left. + Z_z(\cdot)h_y(s,\Phi_{s,0}(y,h,\varepsilon),\varepsilon) + Z_a(\cdot)a_y(\Phi_{s,0}(y,h,\varepsilon),\varepsilon)\right] \Phi^{(1)}_{s,0}(y,h,\varepsilon) ds.$$

Then equation (5.11) can be rewritten in the form

$$(I + R^{(1)})a_y(y,\varepsilon) = Q^{(1)}a_y(y,\varepsilon),$$

where I is the identity.

The inequalities (5.1)-(5.3), (5.4) imply

$$\begin{aligned} \|R^{(1)}a_y(y,\varepsilon)\| &\le \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \Big(\| a_y(\Phi_{s,0}(y,h,\varepsilon),\varepsilon)\| \|\Phi^{(1)}_{s,0}(y,h,\varepsilon)\| \\ &\quad + \|a_y(y,\varepsilon)\|\Big) ds \\ &\le \frac{\varepsilon\sqrt{\alpha}e^{\beta^2/2\alpha}\nu}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \left(e^{\varepsilon\mu(1+\varepsilon\xi)|s|} + 1\right) ds \\ &\le \varepsilon\nu e^{\beta^2/2\alpha} + \frac{2\varepsilon\nu\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} e^{\frac{-\alpha s^2}{2}} e^{\varepsilon\mu(1+\varepsilon\xi)s} ds. \end{aligned}$$

The last integral can be estimated using the error integral (Lemma 3.3). Hence, we have

$$\|R^{(1)}a_y(y,\varepsilon)\| \le 4\varepsilon\nu e^{\beta^2/2\alpha}.$$

If $4\varepsilon\nu e^{\beta^2/2\alpha} < 1$, then there exists the operator $(I + R^{(1)})^{-1}$ and we have

$$\|(I + R^{(1)})^{-1}\| \le \frac{1}{1 - 4\varepsilon\nu e^{\beta^2/2\alpha}}. \tag{5.12}$$

We introduce the operator $P^{(1)}$ by

$$P^{(1)}a_y = (I + R^{(1)})^{-1}Q^{(1)}a_y \tag{5.13}$$

In the sequel we shall show that $P^{(1)}a_y$ satisfies the condition (5.4) and $P^{(1)}$ is a contraction on the space $F^{(1)}$. To do this let us estimate $Q^{(1)}a_y$. By (5.3) and Lemmas 5.1, 5.2 we have

$$\|Q^{(1)}a_y(y,\varepsilon)\| \le \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} [\|Z_y(\cdot)\| + \|Z_z(\cdot)\|\|h_y(s,\Phi_{s,0}(y,h,\varepsilon),\varepsilon)\| +$$

$$+\|Z_a(\cdot)\|\|a_y(\Phi_{s,0}(y,h,\varepsilon),\varepsilon)]\,\|\Phi^{(1)}_{s,0}(y,h,\varepsilon)\|ds$$

$$\le \frac{\varepsilon\sqrt{\alpha}e^{\beta^2/2\alpha}D}{\sqrt{2\pi}}\left[(1+N+\varepsilon N^2)+\varepsilon\xi(1+N)+\varepsilon\nu\right]\int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} e^{\varepsilon\mu(1+\varepsilon\xi)|s|}ds$$

$$\le 3\varepsilon e^{\beta^2/2\alpha}DS,$$

where $S = 1 + \varepsilon N + \varepsilon N^2 + \varepsilon\xi(1+N) + \varepsilon\nu$.

Using the estimates on $(I + R^{(1)})^{-1}$ and $Q^{(1)}$ we obtain

$$\|P^{(1)}a_y(y,\varepsilon)\| \le \frac{3\varepsilon DSe^{\beta^2/2\alpha}}{1-4\varepsilon\nu e^{\beta^2/2\alpha}}.$$

Under the condition

$$4\varepsilon\nu e^{\beta^2/2\alpha} \le \frac{1}{2}, \tag{5.14}$$

the inequality

$$\|P^{(1)}a_y(y,\varepsilon)\| \le 6\varepsilon e^{\beta^2/2\alpha}DS$$

holds.

By the inequalities (5.2) and Lemmas 3.2, 5.1, 5.2, it is easy to verify the estimate

$$\|Q^{(1)}a_y(y,\varepsilon) - Q^{(1)}a_y(\bar{y},\varepsilon)\| \le$$

$$\le \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}}\Big[\big(D_1(1+\varepsilon\xi+\varepsilon\nu)^2 + \varepsilon D(\xi_1(1+N)+\nu_1)\big)\times$$

$$\|\Phi_{s,0}(y,h,\varepsilon) - \Phi_{s,0}(\bar{y},h,\varepsilon)\|\|\Phi^{(1)}_{s,0}(y,h,\varepsilon)\|$$

$$+\,\varepsilon D(1+\varepsilon N+\varepsilon N^2+\varepsilon\xi(1+N)+\varepsilon\nu)\|\Phi^{(1)}_{s,0}(y,h,\varepsilon) - \Phi^{(1)}_{s,0}(\bar{y},h,\varepsilon)\|\Big]\,ds$$

$$\leq \left(3D_1(1+\varepsilon\xi+\varepsilon\nu)^2 e^{\beta^2/2\alpha} + \varepsilon\frac{C_1}{2}\right) \|y-\bar{y}\|, \tag{5.15}$$

where C_1 is some constant depending on $D, D_1, N, \xi, \xi_1, \nu, \nu_1$. Thus, by (5.12), (5.14), (5.15) it holds

$$\|P^{(1)}a_y(y,\varepsilon) - P^{(1)}a_y(\bar{y},\varepsilon)\| \leq \left(6D_1(1+\varepsilon\xi+\varepsilon\nu)^2 e^{\beta^2/2\alpha} + \varepsilon C_1\right) \|y-\bar{y}\|. \tag{5.16}$$

If we choose ε sufficiently small then we have

$$6\varepsilon e^{\beta^2/2\alpha} DS \leq \nu,$$

$$6D_1(1+\varepsilon\xi+\varepsilon\nu)^2 e^{\beta^2/2\alpha} + \varepsilon C_1 \leq \nu_1$$

and the function $a_y(y,\varepsilon)$ satisfies the conditions (5.4). Therefore, the operator P maps $F^{(1)}$ into itself.

Let us estimate the difference $\|Q^{(1)}a_y(y,\varepsilon) - Q^{(1)}\bar{a}_y(y,\varepsilon)\|$. By the inequalities (5.2), (5.3), and Lemmas 5.1, 5.2 we have

$$\|Q^{(1)}a_y(y,\varepsilon) - Q^{(1)}\bar{a}_y(y,\varepsilon)\| \leq \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-s^2} \left[D_1\left(1+\varepsilon\xi+\varepsilon\nu\right)\rho(a,\bar{a})\right.$$

$$\left. + \varepsilon D\rho(a_y,\bar{a}_y)\right] e^{\varepsilon\mu(1+\varepsilon\xi)|s|} ds$$

$$\leq 3e^{\beta^2/2\alpha}\left[D_1\left(1+\varepsilon\xi+\varepsilon\nu\right)\rho(a,\bar{a}) + \varepsilon D\rho(a_y,\bar{a}_y)\right].$$

Then by (5.12)-(5.14) it holds

$$\|P^{(1)}a_y(y,\varepsilon) - P^{(1)}\bar{a}_y(y,\varepsilon)\| \leq 6e^{\beta^2/2\alpha}\left[D_1\left(1+\varepsilon\xi+\varepsilon\nu\right)\rho(a,\bar{a}) + \varepsilon D\rho(a_y,\bar{a}_y)\right].$$

Under the estimate (3.32) in Section 3.3.2 we have

$$d\left(Pa, P\bar{a}\right) \leq Ud(a,\bar{a}),$$

where U is the matrix

$$U = \begin{pmatrix} 2\varepsilon De^{\beta^2/2\alpha} & 0 \\ 6e^{\beta^2/2\alpha}D_1\left(1+\varepsilon\nu+\varepsilon\xi\right) & 6\varepsilon De^{\beta^2/2\alpha} \end{pmatrix}.$$

For sufficiently small ε the spectral radius of U is less than 1, therefore P is a contraction operator in $F^{(1)}$ [12].

Thus we have proved

Lemma 5.3 *Suppose the conditions of Lemma 3.2 are valid and the functions Y, Z in the r. h. s. of (3.6) have continuous and bounded partial derivatives with respect to y that satisfy the conditions (5.1)-(5.3). Then for sufficiently small ε the gluing function $a(y, \varepsilon)$ belongs to the space $F^{(1)}$.*

Now we study the dependence of the fixed point a of the operator P defined in (3.25) on h. This relation we shall need later in the proof of the fact that T is strictly contractive in $H^{(1)}$. Let $a(y, \varepsilon)$ and $\bar{a}(y, \varepsilon)$ be the solutions of (3.23) corresponding to the functions h and $\bar{h}$ respectively. The difference $\|a(y, \varepsilon) - \bar{a}(y, \varepsilon)\|$ has been estimated (Lemma 3.5). In the sequel we estimate the difference between first derivatives $a_y(y, \varepsilon)$, $\bar{a}_y(y, \varepsilon)$. We have $(I + R^{(1)})a_y = Q^{(1)}a_y$, and $(I + \bar{R}^{(1)})\bar{a}_y = \bar{Q}^{(1)}\bar{a}_y$, where

$$\bar{R}^{(1)}\bar{a}_y(y, \varepsilon) := \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)[\bar{a}_y(\Phi_{s,0}(y, \bar{h}, \varepsilon), \varepsilon)\Phi_{s,0}^{(1)}(y, \bar{h}, \varepsilon)$$

$$-\bar{a}_y(y, \varepsilon)e_i]ds,$$

$$\bar{Q}^{(1)}\bar{a}_y(y, \varepsilon) := -\frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) \left[\bar{Z}_y(\cdot) + \right.$$

$$\left. + \bar{Z}_z(\cdot)h_y(s, \Phi_{s,0}(y, \bar{h}, \varepsilon), \varepsilon) + \bar{Z}_a(\cdot)a_y(\Phi_{s,0}(y, \bar{h}, \varepsilon), \varepsilon)\right] \Phi_{s,0}^{(1)}(y, \bar{h}, \varepsilon)ds.$$

Here

$$\bar{Z}(\cdot) = Z(s, \Phi_{s,0}(y, \bar{h}, \varepsilon), \bar{h}(s, \Phi_{s,0}(y, \bar{h}, \varepsilon), \varepsilon), \bar{a}(\Phi_{s,0}(y, \bar{h}, \varepsilon), \varepsilon), \varepsilon).$$

After some elementary transformations we obtain

$$(I + R^{(1)})(a_y - \bar{a}_y) = Q^{(1)}a_y - \bar{Q}^{(1)}\bar{a}_y + (\bar{R}^{(1)} - R^{(1)})\bar{a}_y$$

or

$$a_y - \bar{a}_y = (I + R^{(1)})^{-1}[Q^{(1)}a_y - \bar{Q}^{(1)}\bar{a}_y + (\bar{R}^{(1)} - R^{(1)})\bar{a}_y].$$

The expression in the square brackets will be estimated at first. By applying (5.1)-(5.6) and Lemmas 3.2, 3.5, 5.2 we have

$$\|Q^{(1)}a(y, \varepsilon) - \bar{Q}^{(1)}\bar{a}_y(y, \varepsilon)\| \leq$$

$$\leq \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \left[\left(\left(D_1(1 + \varepsilon\xi + \varepsilon\nu)(\rho(h, \bar{h}) + \rho(a, \bar{a})\right)\right.\right.$$

$$+\varepsilon D((1+N)\rho(h_y,\bar{h}_y)+\rho(a_y,\bar{a}_y))$$
$$+\left(D_1(1+\varepsilon\xi+\varepsilon\nu)^2+\varepsilon D(\xi_1(1+N)+\nu_1)\right)\big)\times$$
$$\|\Phi_{s,0}(y,h,\varepsilon)-\Phi_{s,0}(y,\bar{h},\varepsilon)\|\big)\,\|\Phi^{(1)}_{s,0}(y,h,\varepsilon)\|$$
$$+\varepsilon D\left(1+\varepsilon N+\varepsilon N^2+\varepsilon\xi(1+N)+\varepsilon\nu\right)\|\Phi^{(1)}_{s,0}(y,h,\varepsilon)-\Phi^{(1)}_{s,0}(y,\bar{h},\varepsilon)\|\Big]\,ds$$
$$\leq 3e^{\beta^2/2\alpha}\Big((D_1+\frac{D_1}{1+\varepsilon\xi}+\varepsilon C_2)\rho(h,\bar{h})+\varepsilon D(1+N+\frac{S}{1+\varepsilon\xi})\rho(h_y,\bar{h}_y)+$$
$$+\varepsilon D\rho(a_y,\bar{a}_y)\Big);$$
$$\|(\bar{R}^{(1)}-R^{(1)})\bar{a}_y\|\leq$$
$$\leq\frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}e^{\frac{-\alpha s^2}{2}}\Big(\nu_1\|\Phi_{s,0}(y,h,\varepsilon)-\Phi_{s,0}(y,\bar{h},\varepsilon)\|\|\Phi^{(1)}_{s,0}(y,h,\varepsilon)\|$$
$$+\varepsilon\nu\|\Phi^{(1)}_{s,0}(y,h,\varepsilon)-\Phi^{(1)}_{s,0}(y,\bar{h},\varepsilon)\|\Big)ds$$
$$\leq 3e^{\beta^2/2\alpha}\left(\left(\frac{\nu_1}{1+\varepsilon\xi}+\varepsilon C_3\right)\rho(h,\bar{h})+\frac{\varepsilon}{1+\varepsilon\xi}\rho(h_y,\bar{h}_y)\right).$$

Combining the last estimates and (5.12), (5.14), we get

$$\|a_y(y,\varepsilon)-\bar{a}_y(y,\varepsilon)\|\leq 6e^{\beta^2/2\alpha}\Big((D_1+\frac{D_1+\nu_1}{1+\varepsilon\xi}+\varepsilon C_4)\rho(h,\bar{h})+$$
$$+\varepsilon(D(1+N)+\frac{DS+1}{1+\varepsilon\xi})\rho(h_y,\bar{h}_y)+\varepsilon D\rho(a_y,\bar{a}_y)\Big).$$

From the last inequality we get

Lemma 5.4 *Under the conditions of Lemma 5.3 the following estimate is true*

$$\rho(a_y,\bar{a}_y)\leq\frac{6De^{\beta^2/2\alpha}}{1-6\varepsilon De^{\beta^2/2\alpha}}\Big((D_1+\frac{D_1+\nu_1}{1+\varepsilon\xi}+\varepsilon C_4)\rho(h,\bar{h})+$$
$$+\varepsilon(D(1+N)+\frac{DS+1}{1+\varepsilon\xi})\rho(h_y,\bar{h}_y)\Big),$$

where $a_y(y,\varepsilon)$ and $\bar{a}_y(y,\varepsilon)$ are the partial derivatives with respect to y of the solutions of (3.23) corresponding to the functions h and $\bar{h}$ respectively.

5.1.4 Existence of the first derivative

Now we derive conditions guaranteeing that the operator T maps $H^{(1)}$ into itself. For this purpose we show that the element $\frac{\partial}{\partial y_i}Th(t,y,\varepsilon)$ satisfies the conditions (5.6). By applying (5.2), (5.4), (5.6) and Lemma 5.1 we have for $t \geq 0$

$$\|\frac{\partial}{\partial y_i}Th(t,y,\varepsilon)\| \leq$$

$$\leq \int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} \Big[\|Z_y(\cdot)\| + \|Z_z(\cdot)\| \|h_y(s,\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\| + \|Z_a(\cdot)\| \|a_y(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\|$$

$$+ \|a_y(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\| \Big] \|\Phi^{(1)}_{s,t}(y,h,\varepsilon)\|\, ds$$

$$\leq \varepsilon \int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} D \left(1+\varepsilon N + \varepsilon N^2 + \varepsilon\xi(1+N) + \varepsilon\nu + \nu\right) e^{\varepsilon\mu(1+\varepsilon\xi)(s-t)} ds$$

$$\leq \varepsilon D(S+\nu) \int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} e^{\varepsilon\mu(1+\varepsilon\xi)(s-t)} ds.$$

Using the error integral we finally have

$$\|\frac{\partial}{\partial y_i}Th(t,y,\varepsilon)\| \leq 3\frac{\sqrt{\pi}}{\sqrt{2\alpha}}\varepsilon D(S+\nu).$$

This means that the derivative $\frac{\partial}{\partial y_i}Th(t,y,\varepsilon)$ is uniformly bounded.

For the difference $\|\frac{\partial}{\partial y_i}Th(t,y,\varepsilon) - \frac{\partial}{\partial y_i}Th(t,\bar{y},\varepsilon)\|$ it follows from (5.2), (5.3), (5.4), (5.6)

$$\|\frac{\partial}{\partial y_i}Th(t,y,\varepsilon) - \frac{\partial}{\partial y_i}Th(t,\bar{y},\varepsilon)\| \leq$$

$$\leq \int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} \left[\left(D_1(1+\varepsilon\xi+\varepsilon\nu)^2 + \varepsilon D(\xi_1(1+N)+\nu_1) + \nu_1 \right) \times \right.$$

$$\|\Phi_{s,t}(y,h,\varepsilon) - \Phi_{s,t}(\bar{y},h,\varepsilon)\| \|\Phi^{(1)}_{s,t}(y,h,\varepsilon)\|$$

$$\left. + \varepsilon D(S+\nu) \|\Phi^{(1)}_{s,t}(y,h,\varepsilon) - \Phi^{(1)}_{s,t}(\bar{y},h,\varepsilon)\| \right] ds.$$

Applying Lemmas 3.2, 5.1, 5.2 and the error integral we have

$$\|\frac{\partial}{\partial y_i}Th(t,y,\varepsilon)-\frac{\partial}{\partial y_i}Th(t,\bar{y},\varepsilon)\| \le (\frac{3D_1\sqrt{\pi}}{\sqrt{2\alpha}}+\varepsilon C_5)\|y-\bar{y}\|,$$

where C_5 is a constant depending on $D, D_1, \xi, \xi_1, \mu, \mu_1, N$. This inequality gives us that $\frac{\partial}{\partial y_i}Th(t,y,\varepsilon)$ is Lipschitz continuous with respect to y.

For sufficiently small ε the inequalities

$$3\frac{\sqrt{\pi}}{\sqrt{2\alpha}}(DS+\nu)\le\xi, \tag{5.17}$$

$$\frac{3D_1\sqrt{\pi}}{\sqrt{2\alpha}}+\varepsilon C_5\le\xi_1 \tag{5.18}$$

hold. Then the element $\frac{\partial}{\partial y_i}Th(t,y,\varepsilon)$ satisfies conditions (5.6). Therefore, T maps $H^{(1)}$ into itself.

Now we prove that T is a contraction operator in $H^{(1)}$. By applying (5.2)-(5.6) and Lemmas 3.2, 5.1, 5.2 we can estimate the difference $\|\frac{\partial}{\partial y_i}Th(t,y,\varepsilon)-\frac{\partial}{\partial y_i}T\bar{h}(t,y,\varepsilon)\|$ as follows

$$\begin{aligned}\left\|\frac{\partial}{\partial y_i}Th(t,y,\varepsilon)-\frac{\partial}{\partial y_i}T\bar{h}(t,y,\varepsilon)\right\| \le \int\limits_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}}\Big[\big(D_1(1+\varepsilon\xi+\varepsilon\nu)(\rho(h,\bar{h})\\
+\rho(a,\bar{a}))+\varepsilon D(1+N)\rho(h_y,\bar{h}_y)+(\varepsilon D+1)\rho(a_y,\bar{a}_y)\\
+\big(D_1(1+\varepsilon\xi+\varepsilon\nu)^2+\varepsilon D\xi_1(1+N)+\nu_1(\varepsilon D+1)\big)\times\\
\|\Phi_{s,t}(y,h,\varepsilon)-\Phi_{s,t}(y,\bar{h},\varepsilon)\|\big)\,\|\Phi^{(0)}_{s,t}(y,h,\varepsilon)\|\Big]\,ds\\
\le(C_6+\varepsilon C_7)\rho(h,\bar{h})+\varepsilon C_8\rho(h_y,\bar{h}_y).\end{aligned}$$

Using the results of Section 3.3.3 for $\rho(Th,T\bar{h})$, from the definition of the generalized metric $d\left(Th,T\bar{h}\right)$ we have

$$d\left(Th,T\bar{h}\right)\le U_1 d(h,\bar{h}),$$

where U_1 is a matrix

$$U_1=\begin{pmatrix}\varepsilon C & 0\\ C_6+\varepsilon C_7 & \varepsilon C_8\end{pmatrix},$$

and C is the constant defined in (3.37). For sufficiently small ε spectral radius of this matrix is less than 1. Thus, T is a contraction operator in $H^{(1)}$. This means that the fixed point of the operator T, that represents an integral manifold of system (3.6), has first derivatives with respect to the components of y. Therefore, we can formulate the following result

Theorem 5.5 *Suppose that conditions of Theorem 3.1 are valid and the functions* Y, Z *have first derivatives with respect to* y, z, a*, continuous on* $\mathbb{R} \times \mathbb{R}^n \times \Omega_z \times I_{\varepsilon_0}$ *and* $\mathbb{R} \times \mathbb{R}^n \times \Omega_z \times \Omega_a \times I_{\varepsilon_0}$*, respectively, and that satisfy the inequalities (5.1)-(5.3). Then for sufficiently small* ε *the gluing function* $a(y, \varepsilon) \in F^{(1)}$ *and the integral manifold* $z = h(t, y, \varepsilon) \in H^{(1)}$

5.2 Higher derivatives

In this section we shall study the dependence of the existence of the higher derivatives of the integral manifold $z = h(t, y, \varepsilon)$ and the gluing function $a(y, \varepsilon)$ on the number of derivatives of the functions in the right hand side.

We use the induction principle. The base of induction is established in the previous section.

5.2.1 Assumptions

Let us introduce the following notation

$$Y^{(\alpha,\beta)}(t, y, z, \varepsilon) = \frac{\partial^{|\alpha|+|\beta|} Y(t, y, z, \varepsilon)}{\partial y_1^{\alpha_1} \ldots \partial y_n^{\alpha_n} \partial z_1^{\beta_1} \partial z_2^{\beta_2}},$$

$$Z^{(\alpha,\beta,\gamma)}(t, y, z, a, \varepsilon) = \frac{\partial^{|\alpha|+|\beta|+|\gamma|} Z(t, y, z, a, \varepsilon)}{\partial y_1^{\alpha_1} \ldots \partial y_n^{\alpha_n} \partial z_1^{\beta_1} \partial z_2^{\beta_2} \partial a_1^{\gamma_1} \partial a_2^{\gamma_2}},$$

$$|\alpha| = \alpha_1 + \ldots + \alpha_n, \quad |\beta| = \beta_1 + \beta_2, \quad |\gamma| = \gamma_1 + \gamma_2,$$

$$h^{(\sigma)}(t, y, \varepsilon) = \frac{\partial^{|\sigma|} h(t, y, \varepsilon)}{\partial y_1^{\sigma_1} \ldots \partial y_n^{\sigma_n}}, \quad |\sigma| = \sigma_1 + \ldots + \sigma_n,$$

$$\Phi_{s,t}^{(\sigma)}(y, h, \varepsilon) = \frac{\partial^{|\sigma|} \Phi_{s,t}(y, h, \varepsilon)}{\partial y_1^{\sigma_1} \ldots \partial y_n^{\sigma_n}}, \quad |\sigma| = \sigma_1 + \ldots + \sigma_n,$$

where α, β, γ, σ are multi-indices.

We assume that the functions Y, Z in the right hand side of (3.6) have continuous partial derivatives with respect to y, z, a up to the order r, and in addition to (5.1)-(5.3) the following inequalities are valid for all $y, \bar{y} \in \mathbb{R}^n$, $z, \bar{z} \in \Omega_z$, $a, \bar{a} \in \Omega_a$

$$\|Y^{(\alpha,\beta)}(t, y, z, \varepsilon)\| \le K_{(\alpha,\beta)}, \tag{5.19}$$

$$\|Y^{(\alpha,\beta)}(t, y, z, \varepsilon) - Y^{(\alpha,\beta)}(t, \bar{y}, \bar{z}, \varepsilon)\| \le \mu_{(\alpha,\beta)} \left(\|y - \bar{y}\| + \|z - \bar{z}\|\right), \ |\alpha| + |\beta| \le r, \tag{5.20}$$

$$\|Z^{(\alpha,\beta,\gamma)}(t, y, z, a, \varepsilon)\| \le M_{(\alpha,\beta,\gamma)}, \tag{5.21}$$

$$\begin{aligned}&\|Z^{(\alpha,\beta,\gamma)}(t,y,z,a,\varepsilon)-Z^{(\alpha,\beta,\gamma)}(t,\bar y,\bar z,\bar a,\varepsilon)\|\le\\ &\le D_{(\alpha,\beta,\gamma)}\left(\|y-\bar y\|+\|z-\bar z\|+\|a-\bar a\|\right),\quad |\alpha|+|\beta|+|\gamma|\le r,\end{aligned}\tag{5.22}$$

with $K_{(\alpha,\beta)}, \mu_{(\alpha,\beta)}, M_{(\alpha,\beta,\gamma)}, D_{(\alpha,\beta,\gamma)}$ some positive numbers.

Consider the subspace $F^{(r)}$ of the space F. The subspace $F^{(r)}$ consists of functions having r continuous and bounded derivatives with respect to the components of y with the following properties

$$\begin{aligned}\|a^{(k)}(y,\varepsilon)\| &\le L_k,\\ \|a^{(k)}(y,\varepsilon)-a^{(k)}(\bar y,\varepsilon)\| &\le \nu_k\|y-\bar y\|,\quad 1\le |k|\le r.\end{aligned}\tag{5.23}$$

We define on $F^{(r)}$ the generalized metric

$$d(a,\bar a)=\mathrm{col}(\rho(a,\bar a),\rho(a_y,\bar a_y),\ldots,\rho(a^{(r)},\bar a^{(r)})),$$

where

$$\rho(a^{(k)},\bar a^{(k)})=\max\sup_{y\in\mathbb{R}^n,\varepsilon\in I_{\varepsilon_0}}\|a^{(k)}(y,\varepsilon)-\bar a^{(k)}(y,\varepsilon)\|,\quad |k|=1,\ldots,r,$$

and max being taken with respect to all partial derivatives of order k. Then $F^{(r)}$ is a complete metric space.

Let us consider the space $H^{(r)}$. This space is a subspace of the space H consisting of the functions $h(t,y,\varepsilon)$ having continuous and bounded derivatives with respect to y of order r with the following properties

$$\begin{aligned}\|h^{(k)}(t,y,\varepsilon)\| &\le N_k,\\ \|h^{(k)}(t,y,\varepsilon)-h^{(k)}(t,\bar y,\varepsilon)\| &\le \xi_k\|y-\bar y\|,\quad 1\le |k|\le r.\end{aligned}\tag{5.24}$$

We equip the space $H^{(r)}$ with the generalized metric

$$d(h,\bar h)=\mathrm{col}(\rho(h,\bar h),\rho(h_y,\bar h_y),\ldots,\rho(h^{(r)},\bar h^{(r)})),$$

where

$$\rho(h^{(k)},\bar h^{(k)})=\max\sup_{t\in\mathbb{R},y\in\mathbb{R}^n,\varepsilon\in I_{\varepsilon_0}}\|h^{(k)}(t,y,\varepsilon)-\bar h^{(k)}(t,y,\varepsilon)\|,\quad |k|=1,\ldots,r.$$

Then $H^{(r)}$ is a complete metric space.

Suppose that under our assumptions the integral manifold $h(t,y,\varepsilon)$ and the gluing function $a(y,\varepsilon)$ have partial derivatives up to the order $r-1$ that are continuous on $\mathbb{R}\times\mathbb{R}^n\times I_{\varepsilon_0}$, $\mathbb{R}^n\times I_{\varepsilon_0}$, respectively, satisfying conditions (5.23)-(5.24) for all $|k|\le r-1$. The aim is to show that under assumptions

(5.19)-(5.22) there exist the derivatives of order r continuous on the same domains, satisfying (5.23)-(5.24), too. It means that the operators P, T map the spaces $F^{(r)}$, $H^{(r)}$ into themselves.

It is easy to show that the following equality is valid

$$\frac{\partial^r}{\partial y_1^{r_1}\dots\partial y_n^{r_n}} Th(t,y,\varepsilon) =$$

$$= \begin{cases} -\int\limits_t^{+\infty} \\ \int\limits_{-\infty}^{t} \end{cases} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\left[\left(Z_y(\cdot) + Z_z(\cdot)h_y(*) + Z_a(\cdot)a_y(\diamond) + a_y(\diamond)\right)\Phi^{(r)}(y,h,\varepsilon)\right.$$

$$\left. + \left(Z_z(\cdot)h_y^{(r)}(*) + Z_a(\cdot)a_y^{(r)}(\diamond) + a_y^{(r)}(\diamond)\right)\prod \Phi^{(1)} + \sum G_{(\alpha+\beta+\gamma)}(\cdot)\right] ds$$

where

$$Z(\cdot) = Z\left(s, \Phi_{s,t}(y,h,\varepsilon), h(s,\Phi_{s,t}(y,h,\varepsilon),\varepsilon), a(\Phi_{s,t}(y,h,\varepsilon),\varepsilon),\varepsilon\right),$$

$$h(*) = h(s,\Phi_{s,t}(y,h,\varepsilon),\varepsilon), \quad a(\diamond) = a(\Phi_{s,t}(y,h,\varepsilon),\varepsilon),$$

and the sum of the functions $G_{\alpha+\beta+\gamma}$ contains derivatives of Z, h, a, Φ only up to the order $r-1$. Indeed, all the functions under the integral depend continuously on y and under our assumptions there exist continuous derivatives of Z and a with respect to y. Moreover, the integrals converge uniformly with respect to y.

We have to show that the element

$$\frac{\partial^r}{\partial y_1^{r_1}\dots\partial y_n^{r_n}} Th(t,y,\varepsilon)$$

is continuous for all $t \in \mathbb{R}$ and satisfies conditions (5.24). Then we show that the operator T is a contraction in the space $H^{(r)}$.

5.2.2 Auxiliary estimates

In this section we prove the existence of the derivatives $\Phi_{s,t}^{(r)}$ and show that the following estimates are valid

$$\|\Phi_{s,t}^{(k)}(y,h,\varepsilon)\| \le C_k e^{\varepsilon(k+1)\mu(1+\varepsilon\xi|s-t|}, \tag{5.25}$$

$$\|\Phi_{s,t}^{(k)}(y,h,\varepsilon) - \Phi_{s,t}^{(k)}(\bar{y},\bar{h},\varepsilon)\| \le e^{\varepsilon\mu(1+\varepsilon\xi)|s-t|}\times$$

$$\left(\sum_{j=1}^{2k}\left(C_k^i\|y-\bar{y}\|+\sum_{i=0}^{k}C_k^{j,i}\rho(h^{(i)},\bar{h}^{(i)})\right)e^{\varepsilon j\mu(1+\varepsilon\xi)|s-t|}\right),\quad |k|\le r,\quad (5.26)$$

where $h^{(0)}=h$.

The proof is due to mathematical induction. The existence of the first derivative and the corresponding estimates have been established in the Section 5.1.2. Suppose that the function $\Phi_{s,t}(y,h,\varepsilon)$ has partial derivatives with respect to the components of the vector y up to the order $r-1$, and the conditions (5.25)-(5.26) are valid. Then from (3.19) it follows that the vector $\Phi^{(r)}_{s,t}(y,h,\varepsilon)$ can be represented in the form

$$\Phi^{(r)}_{s,t}(y,h,\varepsilon)=\varepsilon\int\limits_t^s\Big[[Y_y(\cdot)+Y_z(\cdot)h_y(*)]\,\Phi^{(r)}_{\eta,t}(y,h,\varepsilon)$$

$$+\,Y_z(\cdot)R_r(\eta)+P_r(\eta)]\,d\eta,\quad (5.27)$$

where

$$f(\cdot)=f(\eta,\Phi_{\eta,t}(y,h,\varepsilon),h(\eta,\Phi_{\eta,t}(y,h,\varepsilon),\varepsilon),\varepsilon),$$
$$h(*)=h(\eta,\Phi_{\eta,t}(y,h,\varepsilon),\varepsilon).$$

The function R_r consists of the sum of the products

$$h^{(r)}\prod\Phi^{(1)},$$

where

$$\prod\Phi^{(1)}=\prod\frac{\partial}{\partial y_i}\Phi_{\eta,t}(y,h,\varepsilon),$$

is the product of the first derivatives of the function $\Phi_{\eta,t}(y,h,\varepsilon)$ with the number of components equals to r, and the function P_r consists of the derivatives with respect to y of the functions $Y(t,y,z,\varepsilon)$, $h(t,y,\varepsilon)$ and $\Phi_{\eta,t}(y,h,\varepsilon)$ up to the order $r-1$.

From the definitions of the functions R_r and P_r and from the assumptions (5.19), (5.20) it follows that

$$\|R_r(\eta)\|\le\Psi_r e^{\varepsilon r\mu(1+\varepsilon\xi)|\eta-t|},\quad (5.28)$$

$$\|P_r(\eta)\|\le\Upsilon_r e^{\varepsilon r\mu(1+\varepsilon\xi)|\eta-t|},\quad (5.29)$$

where Ψ_r, Υ_r are some positive numbers depending on the constants $K_{(\alpha,\beta)}$, $\mu_{(\alpha,\beta)}$, $|\alpha|+|\beta|\le r-1$. Then we have

$$\|\Phi^{(r)}_{s,t}(y,h,\varepsilon)\|\le\varepsilon\int\limits_t^s\big[(\|Y_y\|+\|Y_z\|\|h_y\|)\,\|\Phi^{(r)}\|+\|Y_z\|\|R_r\|+\|P_r\|\big]\,d\eta$$

$$\leq \varepsilon \int_t^s \left(\mu(1+\varepsilon\xi)\|\Phi_{\eta,t}^{(r)}(y,h,\varepsilon)\| + (\mu\Psi_r + \Upsilon)e^{\varepsilon r\mu(1+\varepsilon\xi)|\eta-t|}\right) d\eta.$$

Applying the Gronwall-Bellman inequality we get

$$\|\Phi_{s,t}^{(r)}(y,h,\varepsilon)\| \leq \frac{\varepsilon(\mu\Psi_r + \Upsilon_r)}{\mu(1+\varepsilon\xi)} e^{\varepsilon\mu(r+1)(1+\varepsilon\xi)|s-t|} \leq C_r e^{\varepsilon\mu(r+1)(1+\varepsilon\xi)|s-t|}. \tag{5.30}$$

Let us estimate the difference $R_r - \bar{R}_r$. In what follows the bar above the function denotes that we consider the function depending on $\bar{y}$ and $\bar{h}$. Let us denote by ζ_i the ith component in the product $\prod \Phi^{(1)}$. Then we get

$$\|h^{(r)} \prod_{i=1}^{r} \zeta_i - \bar{h}^{(r)} \prod_{i=1}^{r} \bar{\zeta}_i\| \leq$$

$$\leq \|h^{(r)} - \bar{h}^{(r)}\| \prod_{i=1}^{r} \|\zeta_i\| + \|\bar{h}^{(r)}\| \sum_{i=1}^{r} \|\zeta_i - \bar{\zeta}_i\| \prod_{j=1}^{i} \|\zeta_j\| \prod_{j=i+1}^{r} \|\bar{\zeta}_j\|$$

$$\leq \left(\xi_r\|y-\bar{y}\| + \rho(h^{(r)}, \bar{h}^{(r)}) + rN_r\|\Phi^{(1)} - \bar{\Phi}^{(1)}\|\right) e^{\varepsilon r\mu(1+\varepsilon\xi)|\eta-t|}.$$

Therefore

$$\|R_r(\eta) - \bar{R}_r(\eta)\| \leq C\left(\xi_r\|y-\bar{y}\| + \rho(h^{(r)}, \bar{h}^{(r)}) + N_r\|\Phi^{(1)} - \bar{\Phi}^{(1)}\|\right) e^{\varepsilon r\mu(1+\varepsilon\xi)|\eta-t|}. \tag{5.31}$$

In order to have the estimate for the difference $P_r(\eta) - \overline{P}_r(\eta)$ we rewrite the functions under the integral in the r. h. s. of (5.27) in the form $Y^{(\alpha,\beta)} \prod_{p=1}^{p_1} \theta_p \prod_{q=1}^{q1} \upsilon_q$. Here θ_p is the pth component of the product $h^{(\beta)}$ and υ_q is the qth component of the product of $\Phi^{(\sigma)}$.

$$\|Y^{(\alpha,\beta)} \prod_{p=1}^{p_1} \theta_p \prod_{q=1}^{q_1} \upsilon_q - \overline{Y}^{(\alpha,\beta)} \prod_{p=1}^{p_1} \bar{\theta}_p \prod_{q=1}^{q_1} \bar{\upsilon}_q\| \leq$$

$$\leq \|Y^{(\alpha,\beta)} - \overline{Y}^{(\alpha,\beta)}\| \prod_{p=1}^{p_1} \|\theta_p\| \prod_{q=1}^{q_1} \|\upsilon_q\|$$

$$+\|\overline{Y}^{(\alpha,\beta)}\| \left[\left(\sum_{p=1}^{p_1} \|\theta_p - \bar{\theta}_p\| \prod_{p=1}^{k} \|\theta_p\| \prod_{p=k}^{p_1} \|\bar{\theta}_p\|\right) \prod_{q=1}^{q_1} \|\upsilon_q\|\right.$$

$$\left. + \prod_{p=1}^{p_1} \|\theta_p\| \left(\sum_{q=1}^{q_1} \|\upsilon_q - \bar{\upsilon}_q\| \prod_{q=1}^{k} \|\upsilon_q\| \prod_{p=k}^{p_1} \|\bar{\upsilon}_q\|\right)\right].$$

From the last relation and from (5.25), (5.26) we obtain

$$\|P_r(\eta) - \overline{P}_r(\eta)\| \le \sum_{j=1}^{2r}\left[C_P^j\|y-\bar y\| + \sum_{i=0}^{r} C_P^{j,i}\rho(h^{(i)},\bar h^{(i)})\right] e^{\varepsilon j\mu(1+\varepsilon\xi)|\eta-t|}. \tag{5.32}$$

Under the inequalities (5.31),(5.32) for the difference $\|\Phi^{(r)} - \overline{\Phi}^{(r)}\|$ we have

$$\|\Phi^{(r)} - \overline{\Phi}^{(r)}\| \le \varepsilon \int_t^s \left[\left(\|Y_y - \overline{Y}_y\| + \|Y_z - \bar Y_z\|\|h_y\| + \|\overline{Y}_z\|\|h_y - \bar h_y\|\right)\|\Phi^{(r)}\|\right.$$

$$+ \left(\|Y_y\| + \|Y_z\|\|h_y\|\right)\|\Phi^{(r)} - \bar\Phi^{(r)}\|$$

$$\left. + \|Y_z - \overline{Y}_z\|\|R_r\| + \|\overline{Y}_z\|\|R_r - \bar R_r\| + \|P_r - \bar P_r\|\right] d\eta$$

$$\le \int_t^s \left[\left((\mu_2(1+\varepsilon\xi) + \mu\xi_2)\|\Phi - \bar\Phi\| + (\mu(1+\varepsilon\xi) + \mu_2)\rho(h,\bar h)\right)\|\Phi^{(r)}\|\right.$$

$$+\mu(1+\varepsilon\xi)\|\Phi^{(r)} - \bar\Phi^{(r)}\|$$

$$+\mu_2(\rho(h,\bar h) + \varepsilon\xi)\|\Phi - \bar\Phi\|)\Psi_r e^{\varepsilon r\mu(1+\varepsilon\xi)|\eta-t|}$$

$$\left. + \sum_{j=1}^{2r}\left(C_P^j\|y-\bar y\| + \sum_{i=0}^{r} C_P^{j,i}\rho(h^{(i)},\bar h^{(i)})\right) e^{\varepsilon j\mu(1+\varepsilon\xi)|\eta-t|}\right] d\eta.$$

Under Lemma 3.2 and the inequality (5.30) we get

$$\|\Phi^{(r)} - \Phi^{(r)}\| \le e^{\varepsilon\mu(1+\varepsilon\xi)|s-t|}\times$$
$$\left[\sum_{j=1}^{2r}\left(C_r^j\|y-\bar y\| + \sum_{i=0}^{r} C_k^{j,i}\rho(h^{(i)},\bar h^{(i)})\right)\right] e^{\varepsilon j\mu(1+\varepsilon\xi)|s-t|}, \tag{5.33}$$

where C_r^j, $C_k^{j,i}$ are some constants depending on $K_{(\alpha,\beta)}$, $\mu_{(\alpha,\beta)}$. This completes the proof.

5.2.3 Smoothness of the function $a(y,\varepsilon)$

For any fixed $h \in H^{(r)}$ we consider the following equation

$$\int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)\left[(Z_y(\cdot) + Z_z(\cdot)h_y(*) + Z_a(\cdot)a_y(\diamond) + a_y(\diamond))\,\Phi^{(r)}(y,h,\varepsilon)\right.$$

$$+\left(Z_z(\cdot)h_y^{(r)}(*)+Z_a(\cdot)a_y^{(r)}(\diamond)+a_y^{(r)}(\diamond)\right)\prod\Phi^{(1)}+\sum G_{\alpha+\beta+\gamma}\Big]\,ds=0.$$

This equation represents the condition of continuity of the element

$$\frac{\partial^k}{\partial y_1^{k_1}\dots\partial y_n^{k_n}}Th(t,y,\varepsilon)$$

at the point $t=0$. We can rewrite this equation in the form

$$(I+R^{(r)})a^{(r)}=Q^{(r)}a^{(r)},$$

where I is the identity, and $R^{(r)}$ and $Q^{(r)}$ defined by

$$R^{(r)}a^{(r)}(y,\varepsilon):=\frac{\partial^r}{\partial y_1^{r_1}\dots\partial y_n^{r_n}}Ra(y,\varepsilon)=$$

$$=\frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}e^{\frac{-\alpha s^2}{2}}W^{-1}(s)\left(a^{(r)}(\Phi_{s,t}(y,h,\varepsilon),\varepsilon)\prod\Phi^{(1)}-a^{(r)}(y,\varepsilon)\prod e_i\right)ds,$$

$$Q^{(r)}a^{(r)}(y,\varepsilon):=\frac{\partial^r}{\partial y_1^{r_1}\dots\partial y_n^{r_n}}Qa(y,\varepsilon)=$$

$$=-\frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}e^{\frac{-\alpha s^2}{2}}W^{-1}(s)\left[(Z_y(\cdot)+Z_z(\cdot)h_y(*)+Z_a(\cdot)a_y(\diamond)\right.$$

$$+a_y(\diamond))\,\Phi^{(r)}(y,h,\varepsilon)+\left(Z_z(\cdot)h_y^{(r)}(*)+Z_a(\cdot)a_y^{(r)}(\diamond)\right)\prod\Phi^{(1)}+\sum G_{(\alpha+\beta+\gamma)}\Big]\,ds.$$

From the assumptions (5.19)-(5.22), and using the error integral we can write the following estimate

$$\|R^{(r)}a^{(r)}(y,\varepsilon)\|\le\frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}e^{\frac{-\alpha s^2}{2}}\left(\|a^{(r)}\|\left(\prod\|\Phi^{(1)}\|+1\right)\right)ds$$

$$\le\frac{e^{\beta^2/\alpha}L_r}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}e^{\frac{-\alpha s^2}{2}}\left(e^{\varepsilon r\mu(1+\varepsilon\xi)|s-t|}+1\right)ds\le 4e^{\beta^2/2\alpha}L_r.$$

If $4e^{\beta^2/2\alpha}L_r<\frac{1}{2}$ the operator $(I+R^{(r)})^{-1}$ exists and the following inequality is true

$$\|(I+R^{(r)})^{-1}\|\le\frac{1}{1-4e^{\beta^2/2\alpha}L_r}\le 2. \tag{5.34}$$

By (5.2), (5.4), (5.6), (5.19)-(5.22) for $Q^{(r)}$ we have

$$\|Q^{(r)}a^{(r)}(y,\varepsilon)\| \leq \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}L_r}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \Big[\|Z_a\|\|a^{(r)}\| \prod \|\Phi^{(1)}\|$$

$$+ \left(\|Z_y\| + \|Z_z\|\|h_y\| + \|Z_a\|\|a_y\| + \|a_y\|\right)\|\Phi^{(r)}\|$$
$$+ \|Z_z\|\|h^{(r)}\| \prod \|\Phi^{(1)}\| + \sum \|G_{\alpha+\beta+\gamma}\|\Big]\, ds$$
$$\leq \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \big(D_2(L_r+N_r)e^{\varepsilon r\mu(1+\varepsilon\xi)|s-t|}$$
$$+ (D_2(1+\varepsilon\xi+\varepsilon\nu)+\varepsilon\nu+C_G)e^{\varepsilon(r+1)\mu(1+\varepsilon\xi)|s-t|}\big)\, ds$$
$$\leq (\varepsilon C_a L_k + C^Q_{r,1} + \varepsilon C^Q_{r,2}), \tag{5.35}$$

where $C^Q_{r,1}, C^Q_{r,2}$ do not depend on L_k.

On the space $F^{(r)}$ we can introduce the operator $P^{(r)}$

$$P^{(r)}a^{(r)}(y,\varepsilon) = (I+R^{(r)})^{-1}Q^{(r)}a^{(r)}(y,\varepsilon).$$

From (5.34), (5.35) it follows

$$\|P^{(r)}a^{(r)}(y,\varepsilon)\| \leq 2(C^Q_{r,1} + \varepsilon C^Q_{r,2}).$$

This implies that $P^{(r)}a^{(r)}(y,\varepsilon)$ is uniformly bounded.

For the difference $\|Q^{(r)}a^{(r)} - \bar{Q}^{(r)}\bar{a}^{(r)}\|$

$$\|Q^{(r)}a^{(r)}(y,\varepsilon) - \bar{Q}^{(r)}\bar{a}^{(r)}(y,\varepsilon)\| \leq \frac{\sqrt{\alpha}e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \big[\big(\|Z_y - \bar{Z}_y\| +$$

$$+\|Z_z - \bar{Z}_z\|\|h_y\| + \|\bar{Z}_z\|\|h_y - \bar{h}_y\| + \|Z_a - \bar{Z}_a\|\|a_y\| + \|\bar{Z}_a\|\|a_y - \bar{a}_y\| +$$
$$\|a_y - \bar{a}_y\|\big)\|\Phi^{(r)}\| + \left(\|Z_y\| + \|Z_z\|\|h_y\| + \|Z_a\|\|a_y\| + \|a_y\|\right)\|\Phi^{(r)} - \bar{\Phi}^{(r)}\|$$
$$+ \big(\|Z_z - \bar{Z}_z\|\|h^{(r)}\| + \|Z_z\|\|h^{(r)} - \bar{h}^{(r)}\| + \|Z_a - \bar{Z}_a\|\|a^{(r)}\|$$
$$+ \|Z_a\|\|a^{(r)} - \bar{a}^{(r)}\|\big) \prod \|\Phi^{(1)}\|$$
$$+ \left(\|Z_z\|\|h^{(r)}\| + \|Z_a\|\|a^{(r)}\|\right)\Big(\sum \|\Phi^{(1)} - \bar{\Phi}^{(1)}\| \prod \|\Phi^{(1)}\| \prod \|\bar{\Phi}^{(1)}\|\Big)$$
$$+ \|G_{\alpha+\beta+\gamma} - \bar{G}_{\alpha+\beta+\gamma}\|\big]\, ds$$

$$\leq \big((\varepsilon C_a\nu_k + C_{r,Q})\|y-\bar y\| + C^0_{r,Q}\rho(h,\bar h) + \cdots + \varepsilon C^r_{r,Q}\rho(h^{(r)},\bar h^{(r)})$$
$$+ C^0_{r,a}\rho(a,\bar a) + \cdots + \varepsilon C^r_{r,a}\rho(a^{(r)},\bar a^{(r)})\big)\,, \tag{5.36}$$

where $h^{(0)} = h$, $a^{(0)} = a$, and $C_{r,Q}$ does not depend on ν_k. So, if we put $\bar h = h$ and $\bar a = a$ then we have

$$\|P^{(r)}a^{(r)}(y,\varepsilon) - P^{(r)}a^{(r)}(\bar y,\varepsilon))\| \leq C_{r,Q}\|y-\bar y\|.$$

This inequality implies that $P^{(r)}a^{(r)}(y,\varepsilon)$ is Lipschitzian with respect to y. Thus, for sufficiently small ε the inequalities

$$2(\varepsilon C_a L_k + C^Q_{r,1} + \varepsilon C^Q_{r,2}) \leq L_k,$$

$$\varepsilon C_a\nu_k + C_{r,Q} \leq \nu_k$$

are valid, then the element $a^{(r)}(y,\varepsilon)$ satisfies the conditions (5.23).

Then, if in (5.36) we put $\bar y = y$ and $\bar h = h$, we get

$$\|P^{(r)}a^{(r)}(y,\varepsilon) - P^{(r)}\bar a^{(r)}(y,\varepsilon)\| \leq C^0_{r,a}\rho(a,\bar a) + \ldots + \varepsilon C^r_{r,a}\rho(a^{(r)},\bar a^{(r)}).$$

Then on the space $F^{(r)}$ for the operator P we have the following estimate

$$d(Pa(y,\varepsilon), P\bar a(y,\varepsilon)) \leq U d(a,\bar a), \tag{5.37}$$

where U is the matrix

$$U = \begin{pmatrix} 6\varepsilon e^{\beta^2/2\alpha}DS & 0 & 0 & \ldots & 0 \\ C_{1,a}0 & \varepsilon C^2_{1,a} & 0 & \ldots & 0 \\ C^0_{2,a} & C^1_{2,a} & \varepsilon C^2_{2,a} & \ldots & 0 \\ . & . & . & \ldots & 0 \\ C^0_{r,a} & C^1_{r,a} & C^2_{r,a} & \ldots & \varepsilon C^r_{r,a} \end{pmatrix}$$

For sufficiently small ε the spectral radius of U is less then 1. Therefore, the operator P is a contraction operator on the space $F^{(r)}$. It means that for any $h \in H^{(r)}$ there exists a unique function $a \in F^{(r)}$ such that the function Th is continuous.

If in (5.36) we put $\bar y = y$, $\bar a = a$ we obtain the inequality which will be useful in the sequel

$$\rho(a^{(r)},\bar a^{(r)}) \leq \big(k_r\|y-\bar y\| + k^0_{r,h}\rho(h,\bar h) + \ldots + \varepsilon k^r_{r,h}\rho(h^{(r)},\bar h^{(r)})$$
$$+ k^0_{r,a}\rho(a,\bar a) + \cdots + k^{r-1}_{r,a}\rho(a^{(r-1)},\bar a^{(r-1)})\big)\,. \tag{5.38}$$

5.2.4 Smoothness of the integral manifold

In this section we show that the operator T maps $H^{(r)}$ into itself and is a contraction. Consider the case $t>0$. Under the conditions (5.1)-(5.4), (5.19)-(5.24), Lemmas 3.2, 5.1, 5.2 and the estimates (5.25), (5.26), (5.38), we get

$$\|\frac{\partial^r}{\partial y_1^{r_1}\dots\partial y_n^{r_n}}Th(t,y,\varepsilon)\| \le$$

$$\le \int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} [(\|Z_y(\cdot)\| + \|Z_z(\cdot)\|\|h_y(*)\| + \|Z_a(\cdot)\|\|a_y(\diamond)\| + \|a_y(\diamond)\|) \times$$

$$\|\Phi^{(r)}(y,h,\varepsilon)\| + \left(\|Z_z(\cdot)\|\|h_y^{(r)}(*)\| + \|Z_a(\cdot)\|\|a_y^{(r)}(\diamond)\| + \|a_y^{(r)}(\diamond)\|\right)\prod\|\Phi^{(1)}\|$$

$$+\sum G_{\alpha+\beta+\gamma}\|\Big] ds \le (\varepsilon\tilde{C}_h N_r + \tilde{C})\int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} e^{\varepsilon r\mu(1+\varepsilon\xi)}ds \le \varepsilon C_h N_r + \bar{C}_{r,T},$$

where $\bar{C}_{r,T}$ does not depend on N_r. The last inequality gives us the boundedness of the derivative $\frac{\partial^r}{\partial y_1^{r_1}\dots\partial y_n^{r_n}}Th(t,y,\varepsilon)$.

$$\|\frac{\partial^r}{\partial y_1^{r_1}\dots\partial y_n^{r_n}}Th(t,y,\varepsilon) - \frac{\partial^r}{\partial y_1^{r_1}\dots\partial y_n^{r_n}}T\bar{h}(t,\bar{y},\varepsilon)\| \le$$

$$\le \int_{-\infty}^{+\infty} e^{-s^2}\Big[\big(\|Z_y-\bar{Z}_y\| + \|Z_z-\bar{Z}_z\|\|h_y\| + \|\bar{Z}_z\|\|h_y-\bar{h}_y\| + \|Z_a-\bar{Z}_a\|\|a_y\|$$

$$+\|\bar{Z}_a\|\|a_y-\bar{a}_y\| + \|a_y-\bar{a}_y\|\big)\|\Phi^{(r)}\|$$

$$+\left(\|Z_y\| + \|Z_z\|\|h_y\| + \|Z_a\|\|a_y\| + \|a_y\|\right)\|\Phi^{(r)}-\bar{\Phi}^{(r)}\|$$

$$+\big(\|Z_z-\bar{Z}_z\|\|h_y^{(r)}\| + \|Z_z\|\|h_y^{(r)}-\bar{h}_y^{(r)}\| + \|Z_a-\bar{Z}_a\|\|a_y^{(r)}\|$$

$$+\|Z_a\|\|a_y^{(r)}-\bar{a}_y^{(r)}\|\big)\prod\|\Phi^{(1)}\|$$

$$+\left(\|Z_z\|\|h_y^{(r)}\| + \|Z_a\|\|a_y^{(r)}\|\right)\left(\sum\|\Phi^{(1)}-\bar{\Phi}^{(1)}\|\prod\|\Phi^{(1)}\|\prod\|\bar{\Phi}^{(1)}\|\right)$$

$$+\|\left(\sum G^{(\alpha+\beta+\gamma)} - \sum\bar{G}^{(\alpha+\beta+\gamma)}\right)\|ds$$

$$\le (\varepsilon C_h\xi_r + C_{r,T})\|y-\bar{y}\| + C_{r,T}^0\rho(h,\bar{h}) + \dots + \varepsilon C_{r,T}^r\rho(h^{(r)},\bar{h}^{(r)}), \quad (5.39)$$

with $C_{r,T}$ not depending on ε.

If we put $h = \bar{h}$ then we get

$$\|\frac{\partial^r}{\partial y_1^{r_1} \dots \partial y_n^{r_n}} Th(t, y, \varepsilon) - \frac{\partial^r}{\partial y_1^{r_1} \dots \partial y_n^{r_n}} Th(t, \bar{y}, \varepsilon)\| \leq (\varepsilon C_h \xi_r + C_{r,T})\|y - \bar{y}\|,$$

that is $\frac{\partial^r}{\partial y_1^{r_1} \dots \partial y_n^{r_n}} Th(t, y, \varepsilon)$ is Lipschitzian with respect to y.

Thus, under the inequalities

$$\begin{aligned} \varepsilon C_h N_r + \bar{C}_{r,T} &\leq N_r, \\ (\varepsilon C_h \xi_r + C_{r,T}) &\leq \xi_r, \end{aligned}$$

the function $\frac{\partial^r}{\partial y_1^{r_1} \dots \partial y_n^{r_n}} Th(t, y, \varepsilon)$ satisfies conditions (5.24). It means that the operator T maps the space $H^{(r)}$ into itself.

If in (5.39) we put $y = \bar{y}$ we get

$$\|\frac{\partial^r}{\partial y_1^{r_1} \dots \partial y_n^{r_n}} Th(t, y, \varepsilon) - \frac{\partial^r}{\partial y_1^{r_1} \dots \partial y_n^{r_n}} T\bar{h}(t, y, \varepsilon)\| \leq C_{r,T}^0 \rho(h, \bar{h}) +$$

$$+ \dots + C_{r,T}^r \rho(h^{(r)}, \bar{h}^{(r)}).$$

Therefore, for the operator T on the space $H^{(r)}$ we obtain

$$d(Th, T\bar{h}) \leq \tilde{U} d(h, \bar{h}),$$

where $\tilde{U}$ is the matrix

$$\tilde{U} = \begin{pmatrix} \varepsilon C_0 & 0 & 0 & \dots & 0 \\ C_{1,T}^0 & \varepsilon C_{1,T}^1 & 0 & \dots & 0 \\ C_{2,T}^0 & C_{2,T}^1 & \varepsilon C_{2,T}^2 & \dots & 0 \\ . & . & . & \dots & 0 \\ C_{r,T}^0 & C_{r,T}^1 & C_{r,T}^2 & \dots & \varepsilon C_{r,T}^r \end{pmatrix}.$$

For sufficiently small ε the spectral radius of $\tilde{U}$ is less then 1. It means that T is a contraction operator in $H^{(r)}$. Therefore, the fixed point of it has r continuous and bounded partial derivatives with respect to y.

Thus, we have showed that the following theorem is true.

Theorem 5.6 *Let the conditions of the Theorems 3.1, 5.5 are satisfied and the functions Y, Z in the r. h. s. of (3.6) have partial derivatives with respect to y up to the order r that are continuous on $\mathbb{R} \times \mathbb{R}^n \times \Omega_z \times I_{\varepsilon_0}$, $\mathbb{R} \times \mathbb{R}^n \times \Omega_z \times \Omega_a \times I_{\varepsilon_0}$, respectively, and satisfy the conditions (5.19)-(5.22). Then the integral manifold $h(t, y, \varepsilon)$ of system (3.6) belongs to $H^{(r)}$ and the gluing function $a(y, \varepsilon)$ belongs to $F^{(r)}$.*

Appendix A

Contraction operator in metric spaces

In this Appendix we present some results from the theory of generalized metric spaces. We state some theorems on contraction operators in metric spaces which can be found in [12, 14].

Definition A.1 *A metric space is a pair* (X, ρ) *which consists of the space* X *and the function* $\rho(x_1, x_2)$, $\rho : X^2 \to [0; +\infty)$ *and satisfies the following conditions for all* $x_1, x_2, x_3 \in X$

- $\rho(x_1, x_2) = 0$ *iff* $x_1 = x_2$*;*
- $\rho(x_1, x_2) = \rho(x_2, x_1)$*;*
- $\rho(x_1, x_2) \leq \rho(x_1, x_3) + \rho(x_3, x_2)$.

Usually, the metric space is designates by X without mentioning the distance function ρ. An example of a metric space is the space H defined on p. 20 with $\rho(h, \bar{h}) = \sup_{t \in \mathbb{R}, y \in \mathbb{R}^n, \varepsilon \in I_{\varepsilon_0}} \|h(t, y, \varepsilon) - \bar{h}(t, y, \varepsilon)\|$.

In a metric space we can do analysis since the fundamental operation of analysis, that of finding limits of a sequence, becomes meaningful. If $\{x_n\}$ is a sequence in the metric space X, we say that x_n converges to x_0 and $x_0 = \lim x_n$ if

$$\lim_{n \to \infty} \rho(x_n, x_0) = 0.$$

We say that $\{x_n\}$ is a *Cauchy sequence* if, given any ε, there exists an N such that

$$\rho(x_m, x_n) < \varepsilon \quad \text{for } m, n > N. \tag{1.1}$$

If (1.1) holds, it follows that $\{x_n\}$ is a Cauchy sequence, but the converse is not necessarily true, for there may be gaps in the space. A metric space X is called a *complete metric space* if any fundamental sequence $\{x_n\}, x_n \in X$ is convergent in X.

It is a classical result that the space H is a complete metric space (see e.g. [14]).

An operator A mapping a metric space X into itself is said to be *continuous at a point* x_0 in X if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\rho(x, x_0) < \delta$ implies $\rho(Ax, Ax_0) < \varepsilon$.

A continuous operator A defined on a complete metric space X is called a *contraction operator* if the following inequality valid

$$\rho(Ax_1, Ax_2) \leq q\,\rho(x_1, x_2),$$

with

$$q < 1.$$

Theorem A.2 (Fixed point theorem) *Let the operator A maps the space X into itself and be a contraction operator. Then the equation*

$$x = Ax$$

has a unique solution in the space X.

Consider the Banach space E. The notion of partial ordering is one of the form of algebraic structure of the space. We say that E is *partially ordered* if for some pairs e_1, $e_2 \in E$ there is an ordering relation $e_1 \leq e_2$ which is reflexive, proper and transitive, that is

- $e_1 \leq e_1$ for all $e_1 \in E$,
- $e_1 \leq e_2$ and $e_2 \leq e_1$ imply $e_1 = e_2$,
- $e_1 \leq e_2$ and $e_2 \leq e_3$ imply $e_1 \leq e_3$.

In the case E is linear as well as partially ordered, we should say

$$e_1 \leq e_2 \text{ implies } e_1 + a \leq e_2 + a \text{ for all } a \in E,$$

$$e_1 \leq e_2 \text{ implies } \alpha e_1 \leq \alpha e_2 \text{ for all } \alpha > 0.$$

In this case E has a *positive cone* E^+, defined as the set of all elements $e \in E$ such that $0 \leq e$. This positive cone is invariant under addition and multiplication by positive scalars. It contains 0, the neutral element, usually referred as the zero element.

Definition A.3 *The pair* (X, ρ) *is called a generalized metric space if for any pair* $x_1, x_2 \in X$ *there exists a map* $\rho : X^2 \to E^+$ *that satisfies the following properties for all* $x_1, x_2, x_3 \in X$

- $\rho(x_1, x_2) \geq 0$, $\rho(x_1, x_2) = 0$ *iff* $x_1 = x_2$*;*
- $\rho(x_1, x_2) = \rho(x_2, x_1)$*;*
- $\rho(x_1, x_2) \leq \rho(x_1, x_3) + \rho(x_3, x_2)$.

Consider an operator A mapping the space X into itself. The operator A satisfies the *generalized Lipschitz condition* if the following inequality holds

$$\rho(Ax_1, Ax_2) \leq B\rho(x_1, x_2), \tag{1.2}$$

where B is a positive defined operator on the space E.

Theorem A.4 *Let the space* X *be complete with respect to the metric* $\rho(x_1, x_2)$*. Let the operator* A *maps the space* X *into itself and satisfies condition (1.2) where* B *is a positive defined operator with the spectral radius less then 1. Then the equation*

$$x = Ax$$

has a unique solution in the space X*.*

The proof of this theorem can be found in [12].

Bibliography

[1] M. Abramovitz and I.A. Stegun, editors. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover, New York, 9 edition, 1972.

[2] N.N. Bogolyubov and Yu.A. Mitropol'skij. The method of integral manifolds in nonlinear mechanics. In *Tr. Mezhdunarod. Simpoz. Nelinejnym Kolebaniam.*

[3] N.N. Bogolyubov and Yu.A Mitropol'skij. *Asymptotic methods in the theory of non-linear oscillations.* Gordon and Breach Science Publishers, New York, 1961.

[4] V.F. Butuzov, N.N. Nefedov, and K.R. Schneider. Singularly perturbed problems in case of exchange of stabilities. *J. Math. Sci.*, 21:1973–2079, 2004.

[5] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *Journal of Differential Equations*, 31:53–98, 1979.

[6] G.N. Gorelov, V.A. Sobolev, and E.A. Shchepakina. *Singularly perturbed combustion models.* SamWen, Samara, 1999. (in Russian).

[7] J Hale and A. Stokes. Behavior of solutions near integral manifolds. *Arch. ration. mech. and anal.*, 6(2):133–170, 1960.

[8] J.K. Hale. *Oscillations in nonlinear systems.* McGraw-Hill Series in Advanced Mathematics with Applications. New York, Toronto, London, 1963.

[9] S. Handrock-Meyer, L.V. Kalachev, and K.R. Schneider. A method to determine the dimension of long-time dynamics in multi-scale systems. *Journal of Math. Chem.*, 30(2):133–160, 2001.

[10] D. Henry. *Geometric Theory of Semilinear Parabolic Equations*. Number 840 in Lecture Notes in Math. Springer, Berlin- New York, 1981.

[11] L.I. Kononenko and V.A. Sobolev. Asymptotic decomposition of slow integral manifolds. *Sib. Math. Journal*, 35(6):1132–1119, 1994.

[12] M.A. Krasnosel'skij, G.M. Vainikko, P.P. Zabreiko, Ya. B. Rutitskij, and V.Ya. Stetsenko. *Approximate solution of operator equations*. Wolters-Noordhoff Series of Monographs and Textbooks on Pure and Applied Mathematics, 1972.

[13] M.A. Krasnosel'skij and P.P. Zabreiko. *Geometrical methods of nonlinear analysis*, volume 263 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1984.

[14] L.A. Ljusternik and W.I. Sobolew. *Elemente der Funktionalanalysis*. Akademie-Verlag, Berlin, 1976.

[15] O.B. Lykova and Ya.S. Baris. *Approximate integral manifolds*. Naukova Dumka, Kiev, 1993. (in Russian).

[16] E.F. Mishchenko, A.Yu. Kolesov, Yu.S.and Kolesov, and N.Kh. Rozov. *Asymptotic methods in singularly perturbed systems*. Plenum Publ. Corp., New York, 1994.

[17] Yu.A. Mitropol'skii and O.B. Lykova. *Integral manifolds in nonlinear mechanics*. Nauka, Moscow, 1975. (in Russian).

[18] A.I. Neishtadt. Persistence loss of stability for dynamical bifurcations, I. *Differents. Uravn.*, 23(12):2060–2067, 1987. (in Russian).

[19] A.I. Neishtadt. Persistence loss of stability for dynamical bifurcations, II. *Differents. Uravn.*, 24(2):226–233, 1988. (in Russian).

[20] V.A. Pliss. *Integral sets of periodic systems of differential equations*. Nauka, Moscow, 1977. (in Russian).

[21] A.M. Samoilenko. On the exponential dichotomy on $\mathbb{R}$ of linear differential equations in $\mathbb{R}^n$. *Ukr. Math. Zhur.*, 53(3):356–371, 2001. (in Russian).

[22] K.R. Schneider. Hopf bifurcation and center manifolds. In *Colloquia Mathematica Soc. Jànos Bolyai.*

[23] K.R. Schneider. On the existence of perturbed center submanifolds of a class of autonomous differential systems. Preprint P-Math-26/80 des KWI der AdW, 1980.

[24] K.R. Schneider, E.A. Shchepakina, and V.A. Sobolev. A new type of travelling wave solutions. *Math. Methods Appl. Sci.*, 26, 1349-1361 2003.

[25] K.R. Schneider and V.A. Sobolev. Existence and approximation of slow integral manifolds in some degenerate cases. WIAS Preprint No. 782, 2002.

[26] K.R. Schneider and Th. Wilhelm. Model reduction by extended quasi-steady-state approximation. *J. Math. Biol.*, 408, 443-450 2000.

[27] E. Shchepakina and V. Sobolev. Integral manifolds, canards and black swans. *Nonlinear Anal., Theory Methods Appl.*, 44A(7):897–908, 200.

[28] E.A. Shchepakina, E.V. Shchetinina, and V.A. Sobolev. Loss of stability scenario in the ziegler system. Preprint Series for Boole Centre for Research in Informatics, No. 09/2003, March 2003.

[29] E.V. Shchetinina. On existence of a bounded solution in a problem with a control parameter. WIAS Preprint No. 918, 2004.

[30] M.A. Shishkova. A discussion of a certain system of differential equations with a small parameter multiplying the highest derivatives. *Dokl. Akad. Nauk SSSR*, 209, 576-579 1973. (in Russian).

[31] J. Sieber. *Longitudinal dynamics of semiconductor lasers.* PhD thesis, Humboldt University, Berlin, 2001.

[32] J. Sieber, L. Recke, and K.R. Schneider. Dynamics of multisection semiconductor lasers. *Contemp. Mathematics*, 3, 70-82 2003.

[33] V.A. Sobolev. Integral manifolds and decomposition of singularly perturbes systems. *System and Control Lett.*, 209(5), 169-179 1984.

[34] V.V. Strygin and V.A. Sobolev. *Separation of motions by the integral manifolds method.* Nauka, Moscow, 1988. (in Russian).

[35] D. Turaev. Fundamental obstacles to self-pulsations in low-intensity lasers. WIAS Preprint No. 629, 2001.

[36] A.B. Vasil'eva and V.F. Butuzov. *Asymptotic expansion of solutions for singularly pertubed equations.* Nauka, Moscow, 1973. (in Russian).

[37] A.B. Vasil'eva and V.F. Butuzov. *Asymptotic methods in the theory of singular perturbations.* Vysshaya Shkola, Moskva, 1990. (in Russian).

[38] N.V. Voropava and V.A. Sobolev. Decomposition of singularly perturbed differential systems. *Comput. Math. Model.*, 9(1):28–37, 1998.

[39] K.V. Zadiraka. On nonlocal intergal manifold for a singularly perturbed differential system. *Ukr. Mat. Zhur.*, 1965(1):47–63, 1965. (in Russian).

Erklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig und ohne fremde Hilfe verfaßt und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

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