POLYHEDRAL INCLUSION–EXCLUSION

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Abstract

Motivated by numerical computations to solve probabilistic constrained stochastic programming problems, we derive a new identity claiming that many terms are cancelled out in the inclusion–exclusion formula expressing the complement of a Euclidean polyhedron.

1 Introduction

The main theorem of the paper was motivated by the following problem in stochastic optimization. When solving probabilistic constrained stochastic programming problems it is necessary to determine the probability of the event that components of a random vector fulfill some linear inequalities. This corresponds to the probability content of a convex polyhedron in the m dimensional Euclidean space. As these problems are usually some kind of reliability type stochastic optimization problems, the most interesting cases

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are those when the probability content of the convex polyhedron is large, i.e. close to one. In this case it seemed to be a good idea to calculate the opposite event probability, i.e. the probability content of the space outside the convex polyhedron. Here after applying the De-Morgan identity one can use the well-known inclusion–exclusion formula in which many terms to be calculated are numerically irrelevant. Beyond that we realised that many terms of the inclusion–exclusion formula are equal to zero as the product of the events is the impossible event and what is more interesting, many terms are cancelling each other. This fact led us to the formulation of our main theorem.

The main theorem can be considered as a relative of the Gram–Brianchon–Sommerville identities (cf. [1] for details). However, our theorem shows new relations. In addition, our theorem is valid for unbounded polyhedra, as well. Further relatives of the inclusion–exclusion formula are studied in details by [3].

2 The main theorem

For $n = 2, 3, ...$, let $\mathcal{P}$ denote the power set of $\{1, ..., n\}$, i.e., $\mathcal{P}$ is the set of all subsets of $\{1, ..., n\}$. Consider an arbitrary nonempty set S_0. For any $S \subseteq S_0$, the *indicator function* $\chi_S : S_0 \to \{0, 1\}$ is defined such that $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ if $x \in S_0 \setminus S$. Instead of χ_S we use the more usual notation $\chi(S)$.

In S_0, consider arbitrary subsets $S_1, ..., S_n$. Concerning the indicator functions, the well-known inclusion–exclusion formula states that

$$\chi(S_1 \cup ... \cup S_n) = \sum_{J \in \mathcal{P}, J \neq \emptyset} (-1)^{|J|-1} \chi(\cap_{j \in J} S_j). \tag{1}$$

Note that there are $2^n - 1$ terms in the right hand side of (1).

In this paper we prove that if $S_1, ..., S_n$ are open halfspaces in the m-dimensional Euclidean space then (1) holds even if the summation is only for fewer nonempty elements of $\mathcal{P}$.

In the m-dimensional Euclidean space (where $m \geq 2$) consider $n \geq m+2$ distinct *hyperplanes* denoted by $H_1, ..., H_n$. Assume that the normal vectors of the hyperplanes have the property that any m of them are linearly independent. In addition assume that the intersection of more than m hyperplanes is always empty. Each hyperplane H_j defines two open halfspaces; let

S_j be defined as one of them, and let S_0 be the set of all points in the entire Euclidean space, that is $S_0 = \mathbb{R}^m$. For any $S \subseteq \mathbb{R}^m$, let $\overline{S}$ denote $\mathbb{R}^m \setminus S$, i.e., the complement of S, and let $M = \overline{S}_1 \cap ... \cap \overline{S}_n$. Consider 0, the origin in the Euclidean space. For notational convenience, we assume that $0 \in M$ and that $0 \notin H_1 \cup ... \cup H_n$. In words, M is a closed *polyhedron* which is not necessarily bounded, and the interior of M is nonempty because it contains 0. Furthermore, assume that no $H_j \cap M$ is empty, $j = 1, ..., n$. This means that each hyperplane is tangent to the polyhedron. To be comprehensive, we say that the open halfspaces are in general position if all the above conditions hold.

If we consider a point x of the Euclidean space as an m-dimensional column vector, and if we assume that the normal vector of the hyperplane H_j is the column vector a_j, then we may assume that the equation of the hyperplane is $a_j^T x = 1$. Since $0 \in M$, we have that $x \in S_j$ if and only if $a_j^T x > 1$.

Theorem. Given open halfspaces $S_1, ..., S_n$ with tangent hyperplanes $H_1, ..., H_n$ in the m-dimensional Euclidean space in general position, $n \geq m + 2 \geq 4$, if M is the polyhedron defined as the intersection of the complements of the halfspaces, then

$$\chi(S_1 \cup ... \cup S_n) = \sum_{J \in \mathcal{P}: (\cap_{j \in J} H_j) \cap M \neq \emptyset} (-1)^{|J|-1} \chi(\cap_{j \in J} S_j) \qquad (2)$$

The essence of the theorem is the fact that here the number of terms of the right hand side of (2) is usually much less than in the original inclusion–exclusion formula. From our assumption that the intersection of more than m hyperplanes is empty, it follows that in (2) only intersections of order $|J| \leq m$ occur. This observation is essential when applying formula (2) to the above-mentioned problem of probabilistic constraints. It allows, for instance, to reduce the calculation of regular multivariate normal distributions of polyhedra to a sum of values of the distribution function.

Example. In the Euclidean plane consider 4 straight lines, e.g., the thick lines in Figure 1. (The thin lines are the coordinate axes and the black dot is the origin.) Each thick line defines two open halfplanes. For line number j let S_j be the open halfplane whose boundary is the thick line number j such that S_j contains digit j in Figure 1. In this example S_0 is the Euclidean plane. The gray area illustrates the indicator function $\chi(S_1 \cup S_2 \cup S_3 \cup S_4)$ in the sense that a gray point means such an x where the indicator functions value is 0. In (1) we find this single term on the left hand side; however, the right hand side consists of 15 terms.

Observe that some tems are the same, e.g., $-\chi(S_1 \cap S_2)$ and $-\chi(S_1 \cap S_2 \cap S_3 \cap S_4)$. On the other hand, in (1) some terms kill each other, e.g., $-\chi(S_1 \cap S_3)$ and $+\chi(S_1 \cap S_3 \cap S_4)$.

In the example, the hyperplanes are the thick straight lines; $n = 4$, $m = 2$. The polyhedron M is the gray area. The interior of the polyhedron contains the origin (the black dot). The intersection of any two hyperplanes (thick lines) is exactly one point, and the intersection of any three hyperplanes (thick lines) is the empty set. Moreover, any two normal vectors are independent. Furthermore, observe that all thick lines are tangent to M.

Figure 1 shows that, in the example, $(\cap_{j\in J} H_j) \cap M \neq \emptyset$ holds if and only if J is one of the following sets: $\{1\}, \{2\}, \{3\}, \{4\}, \{1,4\}, \{2,3\}, \{3,4\}$. The corresponding terms are these:

$$+\chi(S_1) + \chi(S_2) + \chi(S_3) + \chi(S_4) - \chi(S_1 \cap S_4) - \chi(S_2 \cap S_3) - \chi(S_3 \cap S_4)$$

There are only 7 terms here instead of the original 15 terms. In addition, only intersections of order up to 2 occur, whereas (1) also contains triple intersections and the term $S_1 \cap S_2 \cap S_3 \cap S_4$.

The theorem above is a relative of the *Gram–Brianchon–Sommerville* formula which is, however, valid only if M is bounded. If F is an arbitrary face of M such that the tangent halfspaces at F are S_j for $j \in J_F \in \mathcal{P}$,

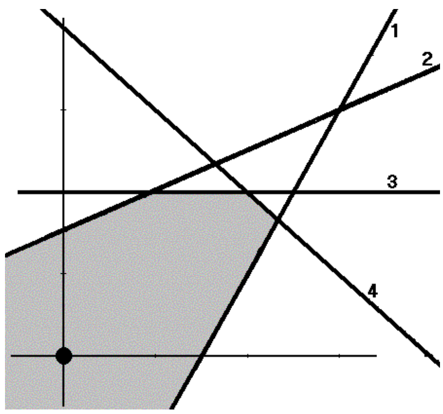

Figure 1: An illustrative example

then the Gram–Brianchon–Sommerville formula is the following where $\dim F$ denotes the dimension of F:

$$\chi(\overline{S_1 \cup ... \cup S_n}) = \sum_F (-1)^{\dim F} \chi(\overline{\cup_{j \in J_F} S_j})$$

Note that there is a $\cup$ in the right hand side, however, the right hand sides of (1) and (2) contain $\cap$.

3 Proof of the Theorem

Define a closed set $T \subseteq \mathbb{R}^m$ with nonempty interior as an *atom* if all indicator fuctions $\chi(S_j)$, $j = 1, ..., n$, are constant on the interior of T, and T is inclusion maximal with respect to the above conditions. (This means that T is not a proper subset of any closed set $T' \subseteq \mathbb{R}^m$ such that all indicator functions are contstant on the interior of T'.) Clearly, the union of all atoms is the entire space. Therefore it is enough to prove our theorem only for the points of an arbitrary but fixed atom T.

For a nonempty $J \in \mathcal{P}$ we use the notation $T_J = \cap_{j \in J}(S_j \cup H_j)$. In words, T_J is the topological closure of the intersection of those open halfspaces whose subscripts belong to J. We start the proof of our theorem by proving two claims for an arbitrary nonempty $J \in \mathcal{P}$. In their proofs we will apply the *strong duality* and *the weak complementary slackness* theorems of *linear programming*. Later we will apply some methods invented by [2].

Claim 1. If $(\cap_{j \in J} H_j) \cap M = \emptyset$, then $T_J \cap M = \emptyset$.

Proof of Claim 1. The straightforward proof is by contradiction. Assume that $(\cap_{j \in J} H_j) \cap M = \emptyset$ and that $s \in T_J \cap M$. So $s \in (S_j \cup H_j) \cap M$ for each $j \in J$. However, $(\cap_{j \in J} H_j) \cap M = \emptyset$ implies $s \notin H_j$ for at least one $j \in J$; for such a j we have $s \in S_j \cap M$. This contradicts to $M \subseteq \overline{S}_j$ proving Claim 1.

Claim 2. If T is a fixed atom, $T_J \neq \emptyset$ and $T_J \cap M = \emptyset$, then there exists at least one nonempty $R \subseteq J$ for which

$$T \cap (\cap_{r \in R} S_r) \subseteq S_k \ \textit{for at least one} \ \ k \notin R \tag{3}$$

holds.

Proof of Claim 2. We apply induction on $p := |J|$. If $p = 1$, then there is nothing to prove because each $H_j \cap M \neq \emptyset$; therefore $T_J \cap M = \emptyset$ condition in Claim 2 cannot hold.

As the induction step we assume that $p \geq 2$ and that Claim 2 has already been proved for any smaller p. Let d denote the distance between 0 and T_J. Clearly $d > 0$ because $0 \in M$ and $T_J \cap M = \emptyset$.

For any nonempty set U in the Euclidean space and for any nonnegative number λ we define λU as the set of all points λu where $u \in U$. Clearly, $U = 1 \cdot U$.

Consider positive numbers μ for which $(\mu M) \cap T_J \neq \emptyset$. For example, $\mu = d/\varepsilon$ is such a number where ε denotes the distance between 0 and $H_1 \cup ... \cup H_n$. Clearly $(\mu M) \cap T_J \neq \emptyset$ implies that $\mu > 1$ because if $\mu \leq 1$, then by the convexity of M and due to $0 \in M$ we would have $\mu M \subseteq M$, and so by $M \cap T_J = \emptyset$ we would have $(\mu M) \cap T_J = \emptyset$.

Let μ^* denote the infimum of the set of all those $\mu > 1$ for which $(\mu M) \cap T_J \neq \emptyset$. Clearly, $M \subseteq \mu^* M$ because $\mu^* \geq 1$ and $0 \in M$.

We claim that $(\mu^* M) \cap T_J \neq \emptyset$. We prove this by contradiction. Assume $(\mu^* M) \cap T_J = \emptyset$. Now we have two disjoint nonempty closed polyhedra: T_J and $\mu^* M$. (The latter one is nothing else but the intersection of the complements of the open halfspaces $\mu^* S_j$, $j = 1, ..., n$.) Since the two topologically closed nonempty polyhedra are disjoint, the distance between them (i.e., the minimal distance between two of their points), say α, is positive. Now we can choose such a positive β for which the distance of the hyperplanes $\mu^* H_j$ and $(\mu^* + \beta) H_j$ is less then α for any $j = 1, ..., n$. Therefore the distance of $(\mu^* + \beta) M$ and T_J is still positive, i.e. $(\mu M) \cap T_J \neq \emptyset$; this contradicts the definition of μ^*. As a consequence, we obtain that $\mu^* > 1$ because $1 \cdot M \cap T_j = M \cap T_j = \emptyset$.

Let a point $x^* \in M$ be chosen in such a way that $\mu^* x^* \in (\mu^* M) \cap T_J$. Observe x^* is in at least one hyperplane H_ℓ. We can choose x^* such that the number of hyperplanes H_j with $j \in J$ and containing $\mu^* x^*$ is maximal, moreover, we may assume that the distance between $\mu^* x^*$ and the union of the hyperplanes H_j, $j \in J$, $\mu^* x^* \notin H_j$ is minimum, and for such an x^* we can choose the largest such ℓ for which H_ℓ contains x^*. Without loss of generality, we put $J = \{1, ..., p\}$. If $T \cap (\cap_{j \in J} S_j) = \emptyset$, then Claim 2 is obvious because we have (3) for $R = J$ and for any $k > p$. Such a k exists as $p < n$ because of $T_J \cap M = \emptyset$. If the interior of T is contained by some S_k for $k > p$, then Claim 2 is obvious again because we have (3) for $R = J$ and this k. Therefore we may assume that $T \cap (\cap_{j \in J} S_j) \neq \emptyset$, i.e. as T is an atom, the interior of T is entirely in $\cap_{j \in J} S_j$, and T is not contained by any single S_k for $k > p$.

Now, inside the proof of Claim 2 we are going to prove a new claim:

Claim 2a. The vectors $a_1, ..., a_p$ are linearly dependent.

Proof of Claim 2a. The proof is by contradiction. Assume that the vectors $a_1, ..., a_p$ are linearly independent. We consider a special linear programming problem and we derive some duality results. The reader is referred to [4] for notions and basic results of linear programming. For an unknown ξ as a nonnegative real number and for an unknown point x in the Euclidean space, we maximize ξ subject to the constraints $x \in M$ and $x \in \xi T_J$. Here the constraint $x \in M$ means $a_j^T x \leq 1$ for $j = 1, ..., n$. On the other hand, the constraint $x \in \xi T_J$ means $a_j^T x \geq \xi$ for $j = 1, ..., p$.

For $q = 1, ..., n$, let $\mathbf{1}_q = (1, ..., 1)^T \in \mathbb{R}^q$ and consider the (q, m)-matrix A_q with rows a_i^T.

Now the constraint $x \in M$ can be rewritten as $A_n x \leq \mathbf{1}_n$. On the other hand, the constraint $x \in \xi T_J$ can be rewritten as $-A_p x + \xi \mathbf{1}_p \leq 0$. In addition we have $\xi \geq 0$ and we maximize ξ. This is in fact a linear programming problem where the decision variables are ξ and the m components of x. In matrix notation it can be written as

$$\begin{array}{rcrcl} (& & \xi) & \longrightarrow & \max \\ A_n x & & & \leq & \mathbf{1}_n \\ -A_p x & + & \mathbf{1}_p \xi & \leq & \mathbf{0}_p \\ & & \xi \geq 0 & & \end{array} \qquad (P)$$

We call (P) the *primal problem*.

From (P) we can derive the *dual linear programming problem* (D) as follows (cf. e.g., Section 1.2 in [4]): For a nonnegative n-dimensional column vector y and for a nonnegative p-dimensional column vector z, we consider the constraint $A_n^T y - A_p^T z = 0$ and the constraint $\mathbf{1}_p^T z \geq 1$. With respect to these constraints we minimize $\mathbf{1}_n^T y$. In matrix notation it can be written as

$$\begin{array}{rcrcl} (\mathbf{1}_n^T y & &) & \longrightarrow & \min \\ A_n^T y & - & A_p^T z & = & \mathbf{0}_m \\ & & \mathbf{1}_p^T z & \geq & 1 \\ y \geq 0 & & z \geq 0 & & \end{array} \qquad (D)$$

We call this linear programming problem the *dual problem*.

We know that there exists at least one optimal solution to the primal problem, namely $x = x^*$ and $\xi = 1/\mu^*$. By the *strong duality theorem* (cf. e.g., Section 1.2 in [4]) we have that there exists at least one optimal solution to the dual problem, say y^* and z^*, furthermore the optimum value $\mathbf{1}_n^T y^*$ equals to the optimum value of the primal problem, that is to $1/\mu^*$.

Now we claim that $\ell \le p$. To the contrary of this, assume that $\ell > p$. From the independence of $a_1, ..., a_p$, we have $H_1 \cap ... \cap H_p \neq \emptyset$. Observe that $\mu^* x^* \in H_1 \cap ... \cap H_p$ because if $\mu^* x^* \notin H_i$ for $i \le p$ (that is $a_i^T(\mu^* x^*) > 1$), then, by using the fact that $a_j^T(\mu^* x^*) \le \mu^*$, $j = 1, ..., n$, and by using the other fact that the interior of $\mu^* M$ is full-dimensional, we could move $\mu^* x^*$ a little bit into the interior of $\mu^* M$ in a direction independent of the set $\{a_1, ..., a_{i-1}, a_{i+1}, ..., a_p\}$ without changing $a_j^T(\mu^* x^*)$ for $j = 1, ..., i-1, i+1, ..., p$ and with decreasing $a_i^T(\mu^* x^*)$. This would lead to a contradiction because of the choice of x^*.

Having $\mu^* x^* \in H_1 \cap ... \cap H_p$ and $\mu^* x^* \in \mu^* H_\ell$, for any point $u \in S_1 \cap ... \cap S_p$ (obviously, $u \notin \mu^* M$), we have that any inner point of the segment from u to $\mu^* x^*$ is also in $(S_1 \cap ... \cap S_p)$. Let w be chosen as an inner point of the segment from u to $\mu^* x^*$, and assume that w tends to $\mu^* x^*$. Since $\mu^* > 1$ and since, from the maximality of ℓ, $a_j^T(\mu^* x^*) < \mu^*$ for $j > \ell$ and $a_j^T(\mu^* x^*) = 1$ for $j \le p$, we gain such a w for which $a_j^T w < \mu^*$ holds for any $j \le p$ and $a_j^T w < \mu^*$ also holds for any $j > p$ with $a_j^T(\mu^* x^*) < \mu^*$.

Since $w \notin \mu^* M$, we have at least one k with $a_k^T w > \mu^*$; we obtain that $p < k \le \ell$ and $a_k^T(\mu^* x^*) = \mu^*$. Since $a_k^T w > \mu^*$ and since w is an inner point of the segment from u to $\mu^* x^*$, we gain that $a_k^T u > \mu^*$. In short, for any point in $S_1 \cap ... \cap S_p$ we have at least one $k \in \{p+1, p+2, ..., \ell\}$ with $\mu^* x^* \in \mu^* H_k$ for which $\mu^* S_k$ contains the point.

Observe that $\mu^* T \subseteq T_J$ and that the interior of $\mu^* T$ is contained by $(\mu^* S_1) \cap ... \cap (\mu^* S_p)$. By considering an inner point of $\mu^* T$ and by using the fact that T is an atom, we gain that the entire interior of $\mu^* T$ is contained by some $\mu^* S_k$ for $k \in \{p+1, p+2, ..., \ell\}$. In other words, the interior of T is a subset of S_k. This is a contradiction because before Claim 2a we assumed that the interior of T is not a subset of S_k for $k > p$.

This contradiction proves that $\ell > p$ is impossible. Therefore, by the (*weak*) *complementary slackness theorem* (cf. e.g., Section 1.7 in [4]), we have that the jth component of y^* must be zero for $j = p+1, ..., n$ since $x^* \notin H_j$ for $j = p+1, ..., n$. Therefore $A_n^T y^* = A_p^T y_p^*$ where y_p^* is defined as the p-dimensional column vector whose components are the first p components of y^*.

Since $\mathbf{1}_p^T z^* \ge 1$, and since $1 > 1/\mu^* = \mathbf{1}_n^T y^* = \mathbf{1}_p^T y_p^*$, we have that $z^* \neq y_p^*$. Therefore, by $0 = A_n^T y^* - A_p^T z^* = A_p^T y_p^* - A_p^T z^*$, the rows of A_p are linearly dependent. This completes the proof of Claim 2a.

Returning to the proof of Claim 2, by Claim 2a we have that $p > m$ because we assumed that the normal vectors of any m hyperplanes are linearly independent. Let $q \in \{1, ..., p\}$ be arbitrary. We may assume that

$$\bigcap_{j \in \{1,...,p\} \setminus \{q\}} (H_j \cap M) \neq \emptyset \tag{4}$$

because otherwise the proof of Claim 2 can be completed by induction and by Claim 1. Therefore by our original assumptions, any $p-1$ out of the hyperplanes H_j, $j = 1, ..., p$, have linearly independent normal vectors. Since $p > m$, we derive that $p = m+1$. Therefore, by the linear independence of any m normal vectors, the intersection of any $p-1$ out of the hyperplanes $H_1, ..., H_p$ is a singleton. Since q is arbitrary in (4), we gain $p = m+1$ distinct points in M. The convexity of M implies that the convex hull of these points is a subset of M. However, this convex hull is nothing else but the simplex determined by the hyperplanes $H_1, ..., H_{m+1}$. So M is a subset of this simplex and the simplex is a subset of M. However, this contradicts the assumptions that $n \geq m+2$ and $H_{m+2} \cap M \neq \emptyset$; the contradiction completes the proof of Claim 2.

We are going to finish the proof of our theorem by a method which is similar to the results of [2]. Consider an arbitrary but fixed atom T. For any nonempty $J \in \mathcal{P}$ with $T \cap (\cap_{j \in J} S_j) \neq \emptyset$ we define another set $J^* \in \mathcal{P}$:

$$J^* := \{k \in \{1, ..., n\} | T \cap (\cap_{j \in J} S_j) \subseteq T \cap S_k\}.$$

Clearly, $J \subseteq J^* = J^{**} \in \mathcal{P}$, and $T \cap (\cap_{j \in J} S_j) = T \cap (\cap_{j \in J^*} S_j)$. Moreover, if $\emptyset \neq J \subseteq K \subseteq J^*$, then $T \cap (\cap_{j \in K} S_j) = T \cap (\cap_{j \in J} S_j) \neq \emptyset$ and $J^* = K^*$.

Now, understood on T, we consider the right hand side of (1) for nonempty sets $J \in \mathcal{P}$. If $T \cap (\cap_{j \in J} S_j) = \emptyset$, then the corresponding term $\chi(\cap_{j \in J} S_j)$ is constant 0 on T; therefore, on T, (1) can be rewritten as

$$\begin{aligned} \chi(S_1 \cup ... \cup S_n) &= \sum_{\{K : K = K^*\}} \left(\Sigma_{\{J : J^* = K\}} (-1)^{|J|-1} \chi(\cap_{j \in J} S_j)\right) \\ &= \sum_{\{K : K = K^*\}} \left(\Sigma_{\{J : J^* = K\}} (-1)^{|J|-1} \chi(\cap_{j \in K} S_j)\right) \\ &= \sum_{\{K : K = K^*\}} \left(\Sigma_{\{J : J^* = K\}} (-1)^{|J|-1}\right) \chi(\cap_{j \in K} S_j). \end{aligned}$$

Now for a fixed K with $K = K^*$, we study the coefficient

$$\Sigma_{\{J : J^* = K\}} (-1)^{|J|-1}. \tag{5}$$

First assume that there exist at least two terms here. Since $K^* = K$ and since $J^* = K$ implies $J \subseteq K$, we gain a $k \in K$ and a nonempty $L \subseteq K \setminus \{k\}$ such that $L^* = K$. Here we may assume that L is *inclusion minimal* which means that for any $\ell \in L$ we have $(L \setminus \{\ell\})^* \neq K$. Observe, that this inclusion minimal L is uniquely determined by K via the relations $\emptyset \neq L \subseteq K \setminus \{k\}$ and $T \cap (\cap_{j \in L} S_j) = T \cap (\cap_{j \in K \setminus \{k\}} S_j) = T \cap (\cap_{j \in K} S_j) \neq \emptyset$. (To support the uniqueness, cf. Theorem 8.2 in [5]). Now observe that for any nonempty $J \in \mathcal{P}$, the relation $J^* = K$ is equivalent to $L \subseteq J \subseteq K$. Therefore (5) is equal to 0 because for $t = |K \setminus L|$ we have that (5) is equal to $(-1)^{|K|-1}$ times $\binom{t}{0} - \binom{t}{1} + \binom{t}{2} - ...$, and the latter factor is 0.

Now we turn our attention to the case where there is only one J with $J^* = K = K^*$. Since $\emptyset \neq T \cap (\cap_{j \in J} S_j) \subseteq T \cap T_J$, we have that $T_J \neq \emptyset$. Assume that $T_J \cap M = \emptyset$. By Claim 2 we obtain a nonempty $R \subseteq J$ and a $k \notin R$ for which (3) holds. We may assume that $|R|$ is the maximum. Since $\emptyset \neq T \cap (\cap_{j \in J} S_j) \subseteq T \cap (\cap_{r \in R} S_r)$, we obtain that $k \in J^* = J$. Therefore $R = J \setminus \{k\}$, and so $R^* = J$. This contradicts the choice of J.

We obtained that the only case where (5) is nonzero is the case where there is only one J with $J^* = K = K^*$ and for this J we have $T_J \cap M \neq \emptyset$, that is by Claim 1 we have that $(\cap_{j \in J} H_j) \cap M \neq \emptyset$. This fact completes the proof of our theorem. $\square$

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References

[1] A. Barvinok, A course in convexity, Graduate Studies in Mathematics, 54, Providence, RI, 2002.

[2] K. Dohmen, Improved inclusion–exclusion identities via closure operators, *Discrete Mathematics and Computer Science*, **4** (2000) 61–66.

[3] L. Lovász, Combinatorial Problems and Exercises, Akadémiai Kiadó; North Holland, Budapest, 1979.

[4] Prékopa, A., Stochastic Programming, Akadémiai Kiadó; Kluwer Academic Publishers, Budapest; Dordrecht, 1995.

[5] A. Schrijver, Theory of linear and integer programming, Wiley, New York, 1986.