Stochastic Control in Limit Order Markets

Curve Following, Portfolio Liquidation and Derivative Valuation

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Ich widme
diese Arbeit
meiner Familie.

Abstract

Traditional literature on financial markets assumes perfectly liquid markets, so that an arbitrary number of shares can be traded at any time, and trading has no impact on market prices. If only limited liquidity is available for trading, these assumptions are not always satisfied. In this thesis we study a new class of stochastic control problems and analyse optimal trading strategies in continuous time in *illiquid* markets, with a focus on limit order markets.

The first chapter addresses the problem of *curve following* in a limit order market. We consider an investor who wants to keep his stock holdings close to a given stochastic target function. Applications include portfolio liquidation, hedging and algorithmic trading. We construct the optimal strategy which balances the penalty for deviating and the cost of trading. Trading strategies comprise both (absolutely continuous) market and passive orders. We first establish a priori estimates on the trading strategy which allow to prove existence and uniqueness of an optimal control. The optimal trading strategy is then characterised in terms of the solution to a coupled forward backward stochastic differential equation (FBSDE) involving jumps via a stochastic maximum principle. Analysing the FBSDE, we give a second characterisation in terms of buy and sell regions. In the case of quadratic penalty functions the FBSDE admits an explicit solution. The important application of portfolio liquidation is studied in detail. Finally, we discuss some counterexamples where market and passive orders have different signs.

In the second chapter, we allow for a larger class of admissible controls including the economically more realistic case of *discrete* market orders. Using techniques of singular stochastic control, the results of the first chapter are extended to a two-sided limit order market with temporary market impact and resilience, where the bid ask spread is also controlled. We now face an optimisation problem with constraints, since passive buy and sell orders are modelled separately and both are nonnegative processes. We first show existence and uniqueness of an optimal control. In a second step, a suitable version of the stochastic maximum principle is derived which yields a characterisation of the optimal trading strategy in terms of a nonstandard coupled FBSDE. We show that the optimal control can be characterised via buy, sell and no-trade regions. Unlike in the first part, we now get a nondegenerate no-trade region, which implies that market orders are only used when the spread is small. Specifically, we construct a threshold for the spread in terms of the adjoint process. This allows to describe precisely when it is optimal to cross the bid ask spread, a fundamental problem of algorithmic trading. We also show that the controlled system can be described in terms of a reflected BSDE. As an application, we solve the portfolio liquidation problem with passive orders.

When markets are illiquid, option holders may have an incentive to increase their portfolio value by using their impact on the dynamics of the underlying. This problem is addressed in the third chapter in the framework of strategically interacting market participants. We provide a mathematical framework to construct optimal trading strategies under market impact in a multi-player extension of the model of Chapter 1. Specifically, we consider a financial market model with several players that hold European contingent claims and whose trading has an impact on the price of the underlying. We establish existence and uniqueness of equilibrium results for risk-neutral and CARA investors and show that the equilibrium dynamics can be

characterised in terms of a coupled system of non-linear PDEs. For the linear cost function, we obtain a (semi) closed form solution. Analysing this solution, we show how market manipulation can be reduced.
Keywords: Stochastic control, Maximum principle, BSDEs, Illiquid markets.

Zusammenfassung

Eine implizite Annahme vieler klassischer Modelle der Finanzmathematik ist, dass jederzeit beliebige Mengen eines Wertpapiers ohne Preiseinfluß gehandelt werden können. Ist die Menge der zum Handeln verfügbaren Liquidität beschränkt, so ist diese Annahme nicht immer erfüllt. In dieser Dissertation lösen wir eine neue Klasse stochastischer Kontrollprobleme und konstruieren optimale zeitstetige Handelsstrategien in *illiquiden* Märkten, insbesondere Limit-Order-Märkten. Wir benutzen Methoden der stochastischen und singulären Kontrolltheorie.

Im ersten Kapitel betrachten wir einen Investor in einem Limit-Order-Markt, der sein Portfolio möglichst nahe an einer gegebenen stochastischen Zielfunktion halten möchte. Jede Transaktion ist mit Liquiditätskosten verbunden, gesucht ist also diejenige Handelsstrategie, die gleichzeitig die Abweichung vom Zielportfolio und die Handelskosten minimiert. Typische Anwendung sind Portfolioliquidierung, Hedging und algorithmisches Handeln. Die Klasse der zulässigen Strategien umfasst aktive und passive ("market" und "limit") Orders. Wir zeigen zunächst eine a-priori Abschätzung an die Kontrolle und anschließend Existenz und Eindeutigkeit einer optimalen Strategie. Wir beweisen eine Version des stochastischen Maximumprinzips und leiten damit eine notwendinge und hinreichende Bedingung für Optimalität mittels einer gekoppelten stochastischen Vorwärts-Rückwärtsgleichung her. Anschließend beweisen wir eine zweite Charakterisierung der optimalen Strategie mittels Kauf- und Verkaufregionen. Die Form dieser Regionen in Abhängigkeit von der Zielfunktion wird im Detail analysiert. Den Spezialfall quadratischer Straffunktionen lösen wir explizit, dies liefert insbesondere eine Lösung des Portfolioliquidierungsproblems. Abschließend zeigen wir mittels dreier Gegenbeispiele, dass passive und aktive Orders verschiedene Vorzeichen haben können.

Im zweiten Kapitel verallgemeinern wir die Klasse der zulässigen Strategien und erlauben insbesondere *diskrete* Marktorders. Mittels Methoden und Techniken der singulären Kontrolltheorie erweitern wir die Resultate des ersten Kapitels auf zweiseitige Limit-Order-Märkte, in denen der Preiseinfluß einer Order nur langsam abnimmt. Insbesondere modellieren wir den Spread und seine Abhängigkeit von der Handelsstrategie explizit. Dies führt zu einem Kontrollproblem mit Nebenbedingungen, da passive Kauf- und Verkauforders separat als nichtnegative Prozesse modelliert werden. Wie zuvor zeigen wir Existenz und Eindeutigkeit einer optimalen Strategie. Im zweiten Schritt beweisen wir eine Version des Maximumprinzips im singulären Fall, die eine notwendige und hinreichende Optimalitätsbedingung liefert. Daraus leiten wir eine weitere Charakterisierung mittels Kauf-, Verkaufs- und Nichthandelsregionen ab. Wir zeigen, dass Marktorders nur benutzt werden, wenn der Spread klein genug ist. Damit können wir präzise beschreiben, wann ein "Überqueren" des Spreads sinnvoll ist und beantworten damit eine fundamentale Frage des algorithmischen Handels. Wir schließen dieses Kapitel mit einer Fallstudie über Portfolioliquidierung ab.

Das dritte Kapitel thematisiert Marktmanipulation in illiquiden Märkten. Wenn Transaktionen einen Einfluß auf den Aktienpreis haben, dann können Optionsbesitzer damit den Wert ihres Portfolios beeinflussen. Wir analysieren optimale Strategien im Mehrspielerfall, indem wir strategische Interaktion in das Modell aus dem ersten Kapitel einführen. Wir betrachten mehrere Agenten, die europäische Derivate halten und den Preis des zugrundeliegenden Wertpapiers beeinflussen. Wir beschränken uns auf risikoneutrale und CARA-Investoren und zeigen die Existenz eines eindeutigen Gleichgewichts, das wir mittels eines gekoppelten Systems nichtlinearer PDEs charakterisieren. Für lineare Kostenfunktionen leiten wir die Lösungen explizit her. Abschliessend geben wir Bedingungen an, wie diese Art von Marktmanipulation verhindert werden kann.

Schlagwörter: Stochastische Kontrolltheorie, Maximumprinzip, BSDEs, Illiquide Märkte.

Contents

Introduction and Main Results

Standard financial market models assume that asset prices follow an exogenous stochastic process and that all transactions can be settled at the prevailing price without any impact on market dynamics. In recent years there has been an increasing interest in *illiquid* markets, where these assumptions are relaxed. In such markets, trading has an impact on prices and every trading strategy incurs liquidity costs. In this thesis we consider stochastic control problems in continuous time arising in the context of illiquid markets. In particular we solve the problems of curve following, portfolio liquidation and market manipulation under market impact.

Illiquid Markets

Kyle [1985] singles out three measures of liquidity: "*Tightness*, the cost of turning around a position over a short period of time, *depth*, the size of an order flow innovation required to change prices a given amount, and *resiliency*, the speed with which prices recover from a random, uninformative shock". Liquidity risk can also be defined as "the additional risk due to the timing and size of a trade" as in Çetin et al. [2004], but a clear-cut definition of liquidity is not available, to the best of our knowledge. A lack of liquidity might be due to asymmetric information (as in Kyle [1985] and Back [1992]), the presence of large investors (Frey and Stremme [1997] and Bank and Baum [2004]) or an imbalance in supply and demand (as in Çetin et al. [2004]). Liquidity risk affects the replication of derivatives (Jarrow [1994] or Çetin et al. [2009]), plays a role in algorithmic trading (see for instance Bertsimas and Lo [1998] and Almgren and Chriss [2001]) and may lead to market manipulation (Jarrow [1994] and Huberman and Stanzl [2005]).

Limit Order Markets

Almost all modern exchanges are organised as electronic limit order markets. These are designed as continuous double auctions, see O'Hara [1995] and Parlour and Seppi [2008] for a detailed discussion. In such markets, two types of orders are available. Limit orders are submitted for future execution and are stored in the limit order book. Each limit order indicates the intention to buy (or sell) a certain quantity of the asset for a certain price. Market orders are submitted for immediate execution, they are matched with other traders' outstanding limit orders and hence change the level of supply and demand. Investors can thus provide liquidity using limit orders or consume liquidity by means of a market order. As a result, the cash proceeds from a large order depend crucially on the order placement strategy. In general, limit orders yield a better price than market orders, but their execution is uncertain. Moreover, a sequence of small market orders might be realised at better prices than a single large market order. Most existing models however only allow for one type of orders, typically market orders. We refer the reader to Gökay et al. [2010] for a recent survey.

In contrast to standard financial models, in limit order markets there are *two* prices. Limit sell orders are available at prices higher than or equal to the best ask price, and limit buy orders are available at prices lower than or equal to the best bid price. The difference of these two prices is called bid ask spread. Prices are discrete and hence the best bid and best ask are multiples of a fixed tick size. A typical tick size is 0.01 cent (also denoted one basispoint). On the microscopic level, the order book can be described as a high-dimensional priority queueing[1] system. Such models have been proposed among others by Kruk [2003] and Osterrieder [2007]. Models of this kind are mainly designed to study the long run distribution of available liquidity and prices, but they are often too involved for the analysis of optimal trading strategies.

Market Impact

In limit order markets there is typically a limited amount of liquidity available on each price tick. Thus, a large market order moves the current best bid (or best ask) and widens the spread. As a result, trading has an impact on prices. There is a large body of empirical literature on price impact[2], let us only mention Kraus and Stoll [1972], Holthausen et al. [1987], Holthausen et al. [1990], Biais et al. [1995] and Almgren et al. [2005]. There is typically a distinction between instantaneous (recovers instantly), temporary (recovers gradually) and permanent (does not recover) price impact.

Instead of describing the interplay of supply and demand on a microscopic level, many mathematical papers on optimal order execution take a macroscopic view and model the market impact directly. Obizhaeva and Wang [2005], e.g., assume a continuous distribution of available liquidity with a constant order book height, whereas Alfonsi et al. [2010] allow for more general shape functions. In a second step, the resilience is modelled, i.e. how fast new limit orders arrive inside the bid ask spread after liquidity was consumed. Almgren and Chriss [2001] for instance assume instant recovery, whereas Obizhaeva and Wang [2005] consider a model with finite resilience. A further simplification concerns the class of admissible trading strategies ("controls"). Most papers only allow for one type of orders, typically market orders. Some authors, such as Almgren and Chriss [2001] and Schied and Schöneborn [2008], only consider absolutely continuous trading strategies, others allow for continuous and discrete trades, let us mention Obizhaeva and Wang [2005] and Predoiu et al. [2010]. A notable exception is Kratz [2011], who considers portfolio liquidation in a primary exchange *and* a dark pool in the multi-asset case. While his results for the single asset case are qualitatively similar to what will be derived in Chapter 1, the optimisation problem we consider here is more general and the solution technique is different. Unlike Kratz [2011] we distinguish between passive buy and sell orders (both of them being nonnegative) and as a result we face a *constrained*

[1] There are typically priority rules for limit order execution with respect to price and time.

[2] Here we mean the impact of market orders. The impact of limit orders is less well understood, see however Cebiroglu and Horst [2011].

optimisation problem in Chapter 2.

Many existing models for price impact are designed to study portfolio liquidation, which describes the task of selling a large number of shares of a given asset in a short period of time such that the expected liquidity costs are minimised. This is a fundamental problem of algorithmic trading and may serve as a building block for more involved algorithms. Moreover, this problem is closely related to the valuation of a large portfolio of assets under liquidity risk. In limit order markets, different valuation methods make sense: *Marked-to-market* (i.e. valuation under the best bid price), instant liquidation (which involves no volatility risk, but high liquidity costs) or valuation under the "optimal" liquidation strategy (which reflects a balance between revenue and risk).

One of the first papers in the mathematical finance literature concerned with portfolio liquidation is Bertsimas and Lo [1998]. They solve the portfolio liquidation problem for a basket of stocks and a risk-neutral investor in a discrete time model with permanent and instantaneous price impact. Almgren and Chriss [2001] extend this model to risk-averse investors by considering a mean variance optimisation criterion and still find closed form solutions. They allow for nonlinear instantaneous price impact and extend the model to continuous time in Almgren [2003]; an empirical calibration can be found in Almgren et al. [2005]. The permanent price impact is taken to be linear, it is shown in Huberman and Stanzl [2004] that this is necessary to prevent quasi-arbitrage. Schied and Schöneborn [2007b] give the solution first for exponential utility and then for general utility functions in Schied and Schöneborn [2008]. The common feature of all these models is that they allow for permanent and instantaneous, but no temporary price impact. It is assumed that the order book recovers instantly after a trade. Gatheral [2010] and Gatheral et al. [2010] also allow for temporary price impact, so that the transaction price at time t does not only depend on the order at t, but possibly on the trading strategy in $[0, t]$. They consider different decay functions and derive conditions to exclude liquidity-induced arbitrage. In their model, there is *one* price which is influenced by the investor's buy and sell trades. Thus, if the price reacts only slowly to past trades, a fast sequence of buy and sell trades might lead to quasi-arbitrage. This is of course not desirable, and we avoid this by considering two price processes in our model. In Chapter 2 we suppose that market buy orders increase the best ask price and market sell orders decrease the best bid price, then each trade incurs nonnegative liquidity costs and there is no liquidity-induced arbitrage.

The articles mentioned above restrict the analysis to absolutely continuous trading strategies. However, real world trading is discrete, and a large discrete trade may have a substantial impact on the best quotes. Obizhaeva and Wang [2005] take this into account and solve the portfolio liquidation problem in a block-shaped order book for singular control processes, so that both continuous and discrete market orders are allowed. It turns out that the optimal strategy for a risk-neutral investor is composed of initial and terminal discrete trades and a constant rate of continuous trading in between. Generalisations to arbitrary shape functions for the order book are given in Alfonsi et al. [2010] as well as Predoiu et al. [2010]. Fruth [2011] treats the case of stochastic order book height. These papers focus on portfolio liquidation and therefore only consider a one-sided order book. In addition, they only allow for market orders. In the present work, we solve a more general optimisation problem in a two-sided order book model and we also allow for passive orders.

Setup and Economic Contribution

In the first two chapters we consider the problem of *curve following*. We take the perspective of an investor trading in a limit order market who wants to keep his stock holdings close to a stochastic target function. He faces a tradeoff between the penalty for deviating and the costs of trading. This is a quite general framework which covers an array of interesting applications. We already discussed portfolio liquidation above, here the target function might be chosen to be identically equal to zero. A second application is hedging, in this case the target function represents a prespecified hedging portfolio. Let us also mention inventory management, where a trader (e.g. a market maker, a broker or an investment bank) receives random orders from customers. Incoming orders lead to jumps in stock holdings ("inventory") and the trader needs to rebalance his portfolio by trading in the open market. More generally, the curve follower may serve as a part of an algorithmic trading platform, where the target function is the output of some higher-level program. A typical example is trading at volume-weighted average prices (VWAP).

The optimisation problem outlined above is related to the well studied problem of tracking a stochastic process, also known as the monotone follower problem. Among others, Beneš et al. [1980], Karatzas and Shreve [1984] and Karatzas et al. [2000] solve the problem of tracking Brownian Motion with finite fuel, using methods of singular stochastic control. This is extended to more general stochastic processes and a dynamic fuel constraint in Bank [2005]. In the finance literature, Leland [2000] considers a situation where an investor aims to maintain fixed proportions of his wealth in a given selection of stocks, in a market where there are proportional transaction costs. Pliska and Suzuki [2004] reformulate the problem in a market with fixed and proportional transaction costs and compute explicit strategies using methods of impulse control. An extension to the multidimensional case can be found in Palczewski and Zabzyck [2005]. Most of the papers mentioned above use the dynamic programming approach and solve the associated Hamilton-Jacobi-Bellman equation. For the verification argument, one typically needs smoothness of the value function, which is not easy to prove in the present framework since the forward diffusion is not uniformly parabolic. Alternatively, the weaker concept of viscosity solutions can be used, but this gives less information on the control. In the first two chapters of this thesis, we shall instead prove suitable versions of the stochastic maximum principle, since it does not require regularity of the value function and provides a direct representation of the optimal control in terms of a forward backward stochastic differential equation (FBSDE). Analysing the FBSDE, we then derive necessary and sufficient conditions of optimality in terms of buy, sell and no-trade regions. This allows to describe the structure of the optimal trading strategy quite explicitly and, in special cases, even in closed form.

Our mathematical framework is flexible enough to allow for passive orders. Passive orders are understood as orders without price impact and with uncertain execution, such as limit orders or orders placed in a dark venue or crossing network. Dark pools are trading venues associated to a classical exchange. Liquidity available in the dark pool is not openly displayed and trades are only executed if matching liquidity is available. Dark pools can be used to reduce the liquidity costs due to market impact, see Hendershott and Mendelson [2000], Kratz and Schöneborn [2009], Kratz [2011] and the references therein for further details. In contrast to the model of Kratz [2011], in Chapter 2 we model passive buy and sell orders separately. This covers situations where the probability of execution is different for passive buy and sell orders. In our model, the target function which may be influenced by stochastic signals. This offers a large degree of flexibility in inputs and allows for a complicated target driven by different market phenomena such as the order book height or the bid ask spread.

It turns out that for general target functions it is necessary to assume that the stochastic signal is independent from passive order execution, since market and passive orders in this case have the *same* sign. We construct explicit examples which show that, if this assumption is relaxed, trading simultaneously on different sides of the market might be optimal. Specifically, the optimal strategy may be composed of market buy and passive sell orders, which is not a desirable feature. To the best of our knowledge, this problem has not been addressed in the literature on illiquid markets, since most papers either only allow for one type of orders or only consider portfolio liquidation where it is clear *a priori* that only sell orders and no buy orders are used. Unfortunately, if the signal represents bid ask spread, it is *not* independent from limit order execution. It follows that the signal cannot be interpreted as bid ask spread, in general. We will show however that in the important case of portfolio liquidation the undesirable feature described above does not occur, so our framework covers portfolio liquidation with stochastic spread.

The market model of Chapter 2 allows to answer the question of when to cancel a passive order and submit a market order instead ("cross the spread"). For small spread sizes, trading is relatively cheap and submitting a market order might be beneficial. For large spreads, market orders are expensive and it might be optimal to place only passive but no market orders and wait until the spread recovers. This decision is relevant for trading algorithms, and to our best knowledge no solution is yet available in the mathematical literature on limit order markets. This is mainly due to the fact that no existing paper explicitly models the bid ask spread *and* allows for both market and passive orders. For instance, Obizhaeva and Wang [2005] and Predoiu et al. [2010] do not allow for passive orders. They *do* model the spread, but in a way that submitting market orders is always beneficial and it is never optimal to stop trading. Fruth [2011] allows for stochastic order book height, then it is optimal to stop trading when the market is too thin. Again, the analysis is restricted to market orders. In the second chapter we construct a threshold for the bid ask spread and show that submitting a market order is optimal if and only if the spread is smaller than this threshold.

The optimisation problems and models discussed thus far are designed for a single player. In illiquid markets every trader potentially moves prices and interesting problems of strategic interaction arise. Typically investors want to *reduce* market impact, e.g.

when liquidating a large portfolio. However under certain circumstances it might even be beneficial to move prices. Specifically, investors holding options with cash delivery and illiquid underlying can drive up the stock price at maturity and thus increase their payoff. Kumar and Seppi [1992] call such trading strategies "punching the close". We model this situation in a multi-player framework in Chapter 3 by introducing strategic interaction into the model of the first chapter. We then characterise optimal trading strategies in form of a Nash equilibrium. Deliberately manipulating prices is not legal, however it is interesting from a mathematical perspective. Moreover, a better understanding of market manipulation may help to detect and prevent it. Different notions of market manipulation have been discussed in the literature, let us mention short squeezes, the use of private information or false rumours. We refer the reader to Kyle [1985], Back [1992], Jarrow [1994], Allen and Gale [1992], Pirrong [2001], Dutt and Harris [2005], Kyle and Viswanathan [2008]. Closest to our setup is the paper by Gallmeyer and Seppi [2000]. They consider a binomial model with three periods and finitely many risk-neutral agents holding call options on an illiquid underlying. Assuming a linear permanent price impact and linear transaction costs, and assuming that all agents are initially endowed with the same derivative they prove the existence of a Nash equilibrium trading strategy. We shall extend their results to a continuous time diffusion model, allowing for a more general liquidity cost term and different endowments.

We construct the solution for risk-neutral and risk-averse investors and characterise it in terms of a coupled system of partial differential equations. A detailed analysis of the solution allows to show how market manipulation can be reduced. In turns out that for zero sum games, i.e. for offsetting payoffs, the agents' aggregate manipulation strategy is zero. We further show that manipulation can be reduced by increasing the number of informed competitors. Similarly, splitting a product and selling it to several customers may be better than selling it to a single agent. We close the third chapter by showing that derivatives with physical delivery do not induce market manipulation in the sense of "punching the close".

Mathematical Results of Chapter 1

In Chapter 1 we start with the curve following problem in continuous time. We allow for passive and market orders, where the latter are restricted to absolutely continuous trading strategies as in Almgren and Chriss [2001]. This simplification allows us to concentrate on the tradeoff between accuracy and liquidity costs, the generalisation to singular market orders is given in Chapter 2. The main difficulties in the first chapter are due to the presence of jumps in the state variables and the fact that passive orders incur no liquidity costs, so the standard characterisation as pointwise maximisers of the Hamiltonian does not apply. As indicated above, our aim is to prove a suitable version of the stochastic maximum principle which provides a characterisation of the optimal control. As a first step, we derive a priori estimates on the control by comparing the problem to a simpler linear quadratic regulator problem whose solution is constructed explicitly via Riccati equations. Using our a priori estimates, we then apply a Komlós argument which provides the existence and uniqueness of an optimal trading strategy. In a second step, we define the adjoint equation, a backward SDE which involves the optimal trading strategy. We then show that the optimisation problem is convex and explicitly compute the Gâteaux derivative of the performance functional. Next, we apply arguments based on Cadenillas and Karatzas [1995] and Cadenillas [2002] to derive a necessary and sufficient condition for optimality. We note that the results therein cannot be applied directly to our setting, since they only allow for linear state dynamics (whereas the stochastic signal driving the target function in our model has more general dynamics). Our version of the stochastic maximum principle involves a coupled forward backward SDE, composed of a forward equation for the state process and a backward equation for the adjoint process, as well as a pointwise optimality condition on the control. Constructing the optimal solution is thus equivalent to solving a fully coupled FBSDE. The optimal market order is then given as the pointwise minimiser of the Hamiltonian, a function involving the state variable as well as the adjoint process. The optimal passive order however is characterised only *implicitly* by a condition on the solution of the backward equation.

To obtain more insight into the structure of the optimal trading strategy, in a third step we identify a (stochastic) threshold in terms of stock holdings. This threshold is defined in terms of the value function and we call it the *cost-adjusted target function*. We show that if stock holdings are above (below) this function, it is optimal to sell (resp. buy). It follows that trading is always directed towards this function, and not towards the original target function as might be expected. Intuitively, the cost-adjusted target function represents the expected future evolution of the target curve, weighted against expected trading costs. This function is key, it separates the buy from the sell region, and we discuss its dependence on the input parameters in detail. In the example section, we show that the FBSDE with a general signal and a general target function can be solved in closed form if the penalty functions are quadratic. As one application, we explicitly solve the portfolio liquidation problem allowing for the simultaneous use of market *and* passive orders. While the discrete time case is solved in Kratz and Schöneborn [2009], to the best of our knowledge our solution is the first[3] in continuous time. We close the first chapter with a technical section, where we show why it is necessary to assume that the signal process is independent of passive order execution. Specifically it turns out that if this assumption is weakened, it might be optimal to use market buy and passive sell orders at the same time and thus trade on both sides of the market simultaneously, a feature which is rather undesirable from the practitioner's point of view.

Mathematical Results of Chapter 2

The results of the first chapter are derived under the hypothesis of absolutely continuous market orders. Trading in real markets is discrete though. We therefore extend the model of Chapter 1 to singular market orders in the second chapter, so that continuous

[3]Simultaneously to our work, Kratz [2011] extended the results from Kratz and Schöneborn [2009] to continuous time using methods different to ours.

and discrete trading is allowed. The control now comprises absolutely continuous passive orders as well as singular market orders. The challenge is that there are now two sources of jumps, representing passive order execution and singular market orders. The singular nature of the market order complicates the analysis considerably, it cannot be characterised as the pointwise maximiser of the Hamiltonian as in the absolutely continuous case. In addition, there are now constraints on the control, since both passive buy and sell orders are nonnegative processes.

We start with the market model introduced in Obizhaeva and Wang [2005], and extend it by allowing for trading on both sides of the market and also include passive orders. As before, market orders have an impact on prices, they consume liquidity and increase the bid ask spread temporarily. The spread then narrows gradually and prices slowly recover to "normal levels". In contrast to the first chapter, the spread is now controlled, and trading costs at time t depend on the whole trading strategy in $[0, t]$. Our first mathematical result is an a priori estimate on the control. For the proof, we reduce the curve following problem to a simpler optimisation problem with quadratic cost terms and zero target function and then apply a scaling argument. The a priori estimate then provides the existence and uniqueness of an optimal strategy via a Komlós argument. We go on to prove a suitable version of the stochastic maximum principle, which yields a characterisation of the optimal control in terms of a coupled forward backward SDE which now involves singular terms. The proof combines arguments from Cadenillas and Haussmann [1994] with ideas developped in the first chapter. We note that the singular maximum principle derived in Cadenillas and Haussmann [1994] does not cover the present situation as it does not allow for jumps, state-dependent singular cost terms as well as general dynamics for the stochastic signal. The maximum principle given in Øksendal and Sulem [2010] includes jumps, but can also not be applied directly as it only allows for singular but no absolutely continuous controls (which are needed here for the passive order). Our maximum principle provides a characterisation of optimality which is quite implicit, and for this reason we prove a second characterisation in terms of buy, sell and no-trade regions. In contrast to Chapter 1, it now turns out that there is a nondegenerate no-trade region where the costs of trading are larger that the penalty for deviating. This region is defined in terms of a threshold for the bid ask spread. We show that spread crossing is optimal if the spread is smaller than or equal to the threshold. If it is larger, then no market orders should be used and trading stops. This result allows to characterise precisely when spread crossing is optimal for a large class of optimisation problems, a novel result in the mathematical literature on limit order markets. We also show that market orders are submitted in order to keep the controlled system inside (the closure of) the no-trade region, so that its trajectory is reflected at the boundary of the no-trade region. To make this precise, we show that the controlled system can be interpreted as the solution to a reflected BSDE. Due to the presence of jumps in the state process and due to the singular nature of the control, it is in general difficult to solve the coupled FBSDE explicitly. For quadratic penalty functions and zero target however we are able to construct the solution in closed form. As one application, we solve the portfolio liquidation problem with passive orders, which extends the result from Obizhaeva and Wang [2005] to a situation with both market and passive orders. The new feature is that the optimal strategy is then no longer deterministic, but is adapted to passive order execution. As a second illustration we provide a further example where it is optimal never to use market orders.

Mathematical Results of Chapter 3

In the third chapter we introduce strategic interaction into the model of Chapter 1. We consider a finite set of agents and assume that each of them is endowed with a fixed European option with cash settlement, for instance a Call option. We assume for simplicity that the endowment is fixed and the option does not trade. The underlying asset is illiquid, its price depends on the trading strategy of *all* the agents. As in Almgren [2003] we assume that the permanent price impact is linear. In addition, there is an instantaneous price impact which is modelled by a general liquidity cost function. Passive orders have no price impact, so we remove them and only allow for market orders.

Each agent has to balance the gain from driving the stock price into a favourable direction against the liquidity costs of trading, taking into account his competitors' strategies. We set this up as a stochastic differential game and look for solutions in the form of a Nash equilibrium. We consider the cases of risk-neutral investors and risk-averse agents with exponential utility functions separately. In both cases the preference functionals are translation invariant. It is then not necessary to keep track of each agent's trading costs, which simplifies the analysis. The first step is to establish a priori estimates on the controls, the proof is based on a linear growth condition on the payoffs. The methods used in the first two chapters are not applicable here, since the optimisation problem for each agent is not necessarily convex. Instead, we use the dynamic programming approach. The agents' value functions can be described by a coupled system of Hamilton-Jacobi-Bellman PDEs. In contrast to the preceding chapters, the forward diffusion is not degenerate and the HJB PDEs are uniformly parabolic. However, standard results of existence and uniqueness of a smooth solution to this coupled PDE do not apply since we work on an unbounded state space. Instead, we give a direct proof based on arguments from Taylor [1997] which exploits our a priori estimates on the controls.

For the special case of linear cost functions, we show that the coupled system of PDEs can be solved explicitly. We shall analyse these closed form solutions in detail in order to derive conditions on how market manipulation can be avoided. It turns out that the aggregate trading speed converges to zero if the number of informed competitors (without endowment) increases. We also show that in the case of physically settled Call options, the optimal trading strategy for each agent is zero, so manipulation is not beneficial.

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1. Curve Following in Illiquid Markets

1.1. Introduction

In modern financial institutions, due to external regulation as well as client preferences, there are often imposed trading targets which should be followed. These can take the form of a curve giving desired stock holdings over the course of some time horizon, one could think of a day. In an idealised setting one would simply trade so as to stay exactly on the target. Preventing this is the associated costs, thus one has to balance the two conflicting objectives of ensuring minimal deviation from the prespecified target and concurrently minimising trading costs. In the present chapter we address and solve the curve following problem using techniques of stochastic control. In particular, we prove existence and uniqueness of an optimal control and then give a characterisation via the stochastic maximum principle. Controls include both market and passive orders. Our optimal system is described by a fully coupled forward backward stochastic differential equation (FBSDE) and in special cases we provide closed form solutions. The main difficulties are due to the presence of jumps and the fact that passive orders incur no liquidity costs, so the usual optimality criterion via the maximisation of the Hamiltonian does not apply. Instead, we derive a characterisation via buy and sell regions.

A typical problem in the mathematical literature on price impact is that of how to optimally liquidate a given stock holding and we mention first the paper of Almgren [2003] in which he formulates a continuous time model for temporary and permanent market impact. He allows for absolutely continuous market orders and derives explicit solutions to the liquidation problem. This work has become very popular with practitioners as well as forming the basis for subsequent research articles including Almgren et al. [2005], Schied and Schöneborn [2008] and Almgren [2009]. Our model in the present chapter is build on Almgren's model and extends it to general cost functions as well as passive orders. In this limit order book model we consider the problem of curve following and construct the trading strategy which balances the penalty for deviating against the liquidity costs of trading. In the special case of tracking a Brownian Motion this is known as the "monotone follower problem" and has been discussed in Bayraktar and Egami [2008], Beneš et al. [1980] and Karatzas et al. [2000], among others. In the finance literature, Leland [2000] considers a situation where an investor aims to maintain fixed proportions of his wealth in a given selection of stocks, in a market where there are proportional transaction costs. His solution has a local time component as in Davis and Norman [1990]. Pliska and Suzuki [2004] reformulate the problem in a market with fixed and proportional transaction costs. Using techniques of impulse control, they compute explicit strategies and this time, due to the presence of fixed costs, there is no local time phenomenon. In addition, they calculate some sensitivities. Let us also mention

Palczewski and Zabzyck [2005] who extend the model of Pliska and Suzuki [2004] to the multidimensional case when the underlying prices are Markovian.

From an economic perspective, our first major contribution over the articles mentioned above is that we allow for the use of passive orders, which are understood as orders without price impact and with a random execution. We think of them as a reduced-form model for limit orders or dark pools. The second contribution is the introduction of a target which may depend on an array of stochastic signals. This offers a large degree of flexibility in inputs and allows for a target driven by different market phenomena. Relevant applications include tracking the output of an algorithmic trading program, portfolio liquidation, inventory management and hedging.

Let us now describe the mathematical results in more detail. The first step towards a solution is an a priori estimate on the control. For the proof, we reduce the curve following problem to a simpler problem with quadratic cost terms and without target function, for which a solution via Riccati equations can be contructed explicitly. Our a priori estimate then allows to prove existence and uniqueness of an optimal trading strategy via a Komlós argument. We go on to derive a suitable version of the stochastic maximum principle with jumps. The proof is based on ideas from Cadenillas and Karatzas [1995] and Cadenillas [2002]. However, their results cannot be directly applied to the present framework since they only allow for linear dynamics of the state variables, while in our case the SDE for the signal may be nonlinear. Our maximum principle provides a necessary and sufficient condition of optimality in terms of a FBSDE, which is composed of a forward equation for the state process, a backward equation for the adjoint process and a pointwise optimality condition on the control. Constructing the optimal trading strategy is then equivalent to solving a coupled FBSDE. The motivation for using such techniques is due to the fact that our model has a degenerate forward diffusion component and is therefore not uniformly parabolic. This means that standard arguments which may imply a smooth solution to the Hamilton-Jacobi-Bellman (HJB) equation do not apply. Secondly, our interest is not in the value function per se, but primarily in the optimal control, about which one gets more information with the present methods.

Our work also contributes to the stochastic control literature by showing that it is possible to describe very clearly the structure of the problem by analysing probabilistically the corresponding FBSDE rather than the HJB equation via viscosity solution techniques. Specifically, we provide a detailed analysis of the forward backward equation which yields a second characterisation of optimality in terms of buy and sell regions. We show that there is a threshold in terms of stock holdings above which it is optimal to sell and below which we buy. We call this function the *cost-adjusted target function*, it represents the expected future evolution of the target, weighted against expected trading costs. It turns out that stock holdings should be kept close to *this* function, and not to the original target function. For quadratic penalty and liquidity cost functions, we are able to solve the controlled system in (semi-)closed form. We also provide an explicit solution to the portfolio liquidation problem, which is the first[1] in continuous time al-

[1] Simultaneously to our work, a similar solution was derived in Kratz [2011] using different techniques.

lowing for *both* market and passive orders. In the final section some counterexamples are provided in closed form to illustrate that market and passive orders might have different signs if the signal is not independent from passive order execution. In economic terms, this corresponds to trading on different sides of the market simultaneously, which is not a desirable feature. To our best knowledge, this problem has not yet been addressed in the mathematical literature on limit order markets since most existing papers only allow for one type of orders.

The outline of this chapter is as follows, Section 1.2 derives the model as well as introducing the target functions, stochastic signal and control problem. Section 1.3 contains our main results, Sections 1.4, 1.5 and 1.6 discuss the proofs. We consider the quadratic case and some applications in Section 1.7 and close in Section 1.8 with some counterexamples. Parts of this chapter are published in Naujokat and Westray [2011].

1.2. The Control Problem

We consider a terminal time T together with a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(s) : s \in [0,T]\}, \mathbb{P})$ satisfying the usual conditions of right continuity and completeness.

Assumption 1.2.1. *The filtration is generated by the following three mutually independent processes,*

1. *A d-dimensional Brownian Motion W.*
2. *A one-dimensional Poisson process N with intensity λ.*
3. *A compound Poisson process M with compensator $m(d\theta)dt$, where $m(\mathbb{R}^k) < \infty$.*

We consider an investor whose stock holdings are governed by the following SDE,

$$dX^u(s) = u_1(s)N(ds) + u_2(s)ds, \tag{1.1}$$

for $s \in [t,T]$ and with $X^u(t) = x$. The control process u is an $\mathbb{R}^2$-valued process and chosen in the following set,

$$\begin{aligned}\mathcal{U}_t \triangleq \{&u \in L^2([t,T] \times \Omega) : \\ &u_1 \text{ predictably measurable and } u_2 \text{ progressively measurable}\}.\end{aligned}$$

The interpretation of u is as follows. The investor places a passive order of size u_1, when a jump of N occurs the order is executed and the portfolio adjusts accordingly. For ease of exposition we consider only full liquidation. The component u_2 represents the market order, interpreted here as a rate as in Almgren [2003]; more general market orders will be considered in Chapter 2. The investor can thus take and provide liquidity.

We use the notation $\|u\|_{L^2}$ to denote the $L^2([t,T] \times \Omega)$-norm of a control, where t will be understood from the context. To keep a distinction we use $\|\cdot\|_{\mathbb{R}^n}$ for the Euclidean norm of an n-dimensional vector, while $|\cdot|$ is reserved for real numbers. Inequalities with respect to random variables are assumed to hold a.s.

In addition to the controlled process X^u, there is an uncontrolled n-dimensional vector Z with dynamics given by

$$dZ(s) = \mu(s, Z(s))ds + \sigma(s, Z(s))dW(s) + \int_{\mathbb{R}^k} \gamma(s, Z(s-), \theta)\, \tilde{M}(ds, d\theta), \tag{1.2}$$

for $s \in [t, T]$ and with $Z(t) = z$. Observe that we write $\tilde{M}([0, s] \times A) \triangleq M([0, s] \times A) - m(A)s$ for the compensated Poisson martingale; similarly $\tilde{N} \triangleq N - \lambda s$. The functions μ and σ take values from $[t, T] \times \mathbb{R}^n$ and are valued in $\mathbb{R}^n$ and $\mathbb{R}^{n\times d}$ respectively, while γ takes values from $[t, T] \times \mathbb{R}^n \times \mathbb{R}^k$ and is valued in $\mathbb{R}^n$. The vector Z denotes a collection of n factors which may influence the costs of trading as well as the target curve to be followed, however it is not affected by the trading strategy of the investor.

Let us now introduce the performance functional,

$$\begin{aligned} J(t, x, z, u) \triangleq \mathbb{E}\Big[\int_t^T & g(u_2(s), Z(s)) + h(X^u(s) - \alpha(s, Z(s)))ds \\ & + f(X^u(T) - \alpha(T, Z(T))) \Big| X^u(t) = x, Z(t) = z \Big]. \end{aligned} \tag{1.3}$$

The function $\alpha : [t, T] \times \mathbb{R}^n \to \mathbb{R}$ is the target function and h and f penalise deviation from the target. The cost function g captures the liquidity costs of market orders and we now give a heuristic derivation. Trading takes place in a limit order market, which is characterised by a benchmark price D and a collection of other traders' outstanding limit orders. We assume that the process $(D(s))_{t\le s\le T}$ is a nonnegative martingale. At a given instant s, there are limit sell orders available at prices higher than $D(s)$ and limit buy orders at prices lower than $D(s)$. The investor's market buy order is matched with prevailing limit orders and executed at prices higher than $D(s)$. The more volume the trader demands, the higher the price paid per share, that is to say there is an increasing supply curve, as in Çetin et al. [2004]. Similarly, market sell orders are executed at prices lower than $D(s)$ and the price per share is decreasing in the volume sold. The investor may also use passive orders, these are placed and fully executed at $D(s)$. A passive order always achieves a better price, however its time of execution is uncertain.

Given a market order u_2, recall here interpreted as a rate, together with the stochastic signal Z, the above considerations lead us to define the asset price as

$$S(s, Z(s), u_2(s)) = D(s) + \tilde{g}(u_2(s), Z(s)), \tag{1.4}$$

where $\tilde{g}$ captures the *instantaneous* price impact of the market order per unit. We assume that $u_2 \mapsto \tilde{g}(u_2, z)$ is increasing and such that $\tilde{g}(0, z) = 0$. The cash flow over the interval $[t, T]$ is given by

$$\begin{aligned} \mathrm{CF}(u) &\triangleq \int_t^T u_2(s)S(s, Z(s), u_2(s))ds + \int_t^T u_1(s)D(s-)N(ds) \\ &= \int_t^T [u_2(s)D(s) + u_2(s)\tilde{g}(u_2(s), Z(s))]ds + \int_t^T u_1(s)D(s-)N(ds), \end{aligned}$$

where we assume all the necessary conditions for the above stochastic integrals to exist. The premium paid due to not being able to trade at the benchmark price, the cost of trading over the interval $[t, T]$, is then given by

$$\mathrm{CF}(u) - \int_t^T u_2(s)D(s)ds - \int_t^T u_1(s)D(s-)N(ds) = \int_t^T u_2(s)\tilde{g}(u_2(s), Z(s))ds.$$

Defining the liquidity cost function g as $g(u_2, z) \triangleq u_2\tilde{g}(u_2, z)$ gives precisely the term in (1.3).

Remark 1.2.2. There are two natural interpretations of the passive order. The first would be as an order placed in a dark venue, where the underlying level of liquidity is unobservable, see Hendershott and Mendelson [2000] and the references therein for further details. Let us also mention Kratz [2011] who discuss portfolio liquidation in the multi-asset case in the presence of a dark venue. For the special case of a single asset, they have portfolio dynamics similar to ours.
A second interpretation of the passive order is a stylised version of a limit order where placement is only at the benchmark price and there is no time priority and only full execution.

Remark 1.2.3.

- Let us compare the present setting with the literature. Without passive orders, our approach is close to Rogers and Singh [2010]. In their model, absolute liquidity costs are captured by a convex, nonnegative loss function. If we set $g(u_2, z) = \kappa u_2^2$ for some $\kappa > 0$, we recover the model of Almgren [2003]. However therein there is an additional permanent price impact, which is undesirable in the present case. In Chapter 3 we consider options with illiquid underlying where trading does have a permanent impact. In this case, market manipulation may be beneficial.
- In the present model we assume that trading only has an *instantaneous* price impact, i.e. the order book recovers instantly after a trade. A market with temporary price impact (i.e. finite resilience) will be discussed in Chapter 2. In that model, the bid ask spread depends on the trading strategy and recovers only gradually after a trade.

We now proceed to the main problem of interest. The value function associated to our optimisation problem is defined as

$$v(t, x, z) \triangleq \inf_{u \in \mathcal{U}_t} J(t, x, z, u).$$

In the sequel we slightly abuse notation and write $J(u) \triangleq J(t, x, z, u)$ if $(t, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$ is fixed. The curve following problem is then defined to be

Problem 1.2.4. *Find $\hat{u} \in \mathcal{U}_t$ such that $J(\hat{u}) = \min_{u \in \mathcal{U}_t} J(u)$.*

To ensure existence of an optimal control we need some assumptions on the input functions. We remark that here and throughout the constants may be different at each occurrence.

Assumption 1.2.5. *Each function $\psi = f(\cdot), g(\cdot, z), h(\cdot)$ satisfies:*

1. *The function ψ is strictly convex, nonnegative, $\mathcal{C}^1$ and normalised in the sense that $\psi(0) = 0$.*

2. *In addition, ψ has at least quadratic growth, i.e. there exists $\varepsilon > 0$ such that $|\psi(x)| \geq \varepsilon |x|^2$ for all $x \in \mathbb{R}$. In the case of g this is supposed to be uniform in z.*

3. *The functions μ, σ and γ are Lipschitz continuous, i.e. there exists a constant c such that for all $z, z' \in \mathbb{R}^n$ and $s \in [t, T]$,*

$$\|\mu(s,z) - \mu(s,z')\|^2_{\mathbb{R}^n} + \|\sigma(s,z) - \sigma(s,z')\|^2_{\mathbb{R}^{n\times d}} + \int_{\mathbb{R}^k} \|\gamma(s,z,\theta) - \gamma(s,z',\theta)\|^2_{\mathbb{R}^n} m(d\theta) \leq c\|z - z'\|^2_{\mathbb{R}^n}.$$

   In addition, they satisfy

$$\sup_{t \leq s \leq T} \left[\|\mu(s,0)\|^2_{\mathbb{R}^n} + \|\sigma(s,0)\|^2_{\mathbb{R}^{n\times d}} + \int_{\mathbb{R}^k} \|\gamma(s,0,\theta)\|^2_{\mathbb{R}^n} m(d\theta) \right] < \infty.$$

4. *The target function α has at most polynomial growth in the variable z uniformly in s, i.e. there exist constants $c_\alpha, \eta > 0$ such that for all $z \in \mathbb{R}^n$,*

$$\sup_{t \leq s \leq T} |\alpha(s,z)| \leq c_\alpha (1 + \|z\|^\eta_{\mathbb{R}^n}).$$

5. *The functions f and h have at most polynomial growth.*

Remark 1.2.6. Let us briefly comment on these assumptions. The nonnegativity assumption is motivated by the fact that trading is always costly together with it never being desirable to deviate from the target. Taking f and h normalised is no loss of generality, this may always be achieved by a linear shift of f, h and α.

The convexity and quadratic growth condition lead naturally to a convex coercive problem which then admits a unique solution. A typical candidate for the penalty function is $f(x) = h(x) = x^2$, which corresponds to minimising the squared error. We also note that our framework is flexible enough to cover nonsymmetric penalty functions, e.g. if falling behind the target curve is penalised stronger than going ahead.

Once existence and uniqueness of the optimal control has been established we shall need further assumptions for a characterisation of optimality via an FBSDE.

Assumption 1.2.7. *We require the existence of a constant c such that*

1. *The derivatives f' and h' have at most linear growth, i.e. for all $x \in \mathbb{R}$*

$$|f'(x)| + |h'(x)| \leq c(1 + |x|).$$

2. *The cost function g has polynomial style growth, i.e. for all $u_2 \in \mathbb{R}$*

$$|u_2 g_{u_2}(u_2, \cdot)| \leq c(1 + g(u_2, \cdot)).$$

3. *The cost function g satisfies a subadditivity condition, i.e. for all $u_2, w_2 \in \mathbb{R}$*

$$g(u_2 + w_2, \cdot) \leq c\,(1 + g(u_2, \cdot) + g(w_2, \cdot))\,.$$

4. *Constant deterministic controls have finite cost, in particular for all $u_2 \in \mathbb{R}$,*

$$\mathbb{E}\left[\int_t^T g(u_2, Z(s))ds\right] < \infty.$$

Remark 1.2.8. We need the linear growth on the derivatives of f and h to ensure that we can solve the adjoint BSDE. In particular this essentially limits us to quadratic penalty functions h and f. For the cost function g, one example satisfying the above assumptions would be to set

$$g(u_2, Z) = cu_2 \arctan(u_2) + u_2^2(Z + \varepsilon),$$

for some $\varepsilon > 0$, where Z is a nonnegative mean-reverting jump process. We think of Z as modelling the inverse order book height. The function $u_2 \arctan(u_2)$ represents a smooth approximation to $|u_2|$ and the constant $c > 0$ represents bid ask spread. This represents a model with fixed spread and stochastic order book height.

In the present setting, we are most interested in processes on $[t, T] \times \Omega$ and write that a given property $(\mathcal{P})$ holds "$ds \times d\mathbb{P}$ a.e. on B" for a measurable subset $B \subset [t, T] \times \Omega$ when $(\mathcal{P})$ holds for the restriction of the measure $ds \times d\mathbb{P}$ to B.

1.3. Main Results

Having formulated the problem and introduced the necessary assumptions, we can now give our main results of the present chapter.

Theorem 1.3.1. *The functional $u \mapsto J(u)$ is strictly convex for $u \in \mathcal{U}_t$. If Assumption 1.2.5 holds then for any initial triple $(t, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$ there is an optimal control, unique $ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$.*

We postpone the proof to Section 1.4. To characterise the optimal control $\hat{u}$ and the corresponding state process $\hat{X} \triangleq X^{\hat{u}}$ we define the following BSDE on $[t, T]$, the adjoint equation,

$$\begin{aligned} dP(s) = h'(\hat{X}(s) - \alpha(s, Z(s)))ds + Q(s)dW(s) + R_1(s)\tilde{N}(ds) \\ + \int_{\mathbb{R}^k} R_2(s, \theta)\tilde{M}(ds, d\theta), \end{aligned} \tag{1.5}$$

$$P(T) = -f'\big(\hat{X}(T) - \alpha(T, Z(T)\big).$$

Theorem 1.3.2. *Let Assumptions 1.2.5 and 1.2.7 hold. Then*

1. *The above BSDE has a unique solution for all starting triples $(t, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$.*

2. *A control $\hat{u}$ is optimal if and only if $ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$,*
   a) *$\hat{u}_2(s, \omega)$ is the pointwise minimiser of $u_2 \mapsto g(u_2, Z(s, \omega)) - P(s, \omega)u_2$.*
   b) *$P(s-, \omega) + R_1(s, \omega) = 0$.*

Remark 1.3.3. The second part of Theorem 1.3.2 is essentially a version of the stochastic maximum principle and we now describe how this relates to those in the literature. Our results are most similar to Cadenillas [2002], however in his setting one requires that (in our notation) the joint process (X, Z) have dynamics which are jointly affine as functions of (X, Z) and control u. This is not necessarily the case for only Lipschitz μ, σ and γ, so that we are outside the scope of the results therein.

The article Tang and Li [1994] considers the case where the dynamics of (X, Z) need not be affine, as in the present article, however they require that the control satisfies the following integrability condition

$$\sup_{t \le s \le T} \mathbb{E}\left[\|u(s)\|^8_{\mathbb{R}^2}\right] < \infty,$$

which excludes the L^2-framework considered here. Finally we mention Ji and Zhou [2006], where the authors allow for square integrable controls and non-affine dynamics but have no jumps, so that again their results do not cover the present situation.

The proof of the second item relies on the stochastic maximum principle and is dealt with in Section 1.5. The characterisation given in Theorem 1.3.2 is still rather implicit, we can describe $\hat{u}_1$ more precisely and for this require the following definition.

Definition 1.3.4. The *cost-adjusted target function* $\tilde{\alpha}$ is defined to be the pointwise minimiser (with respect to x) of the value function,

$$\tilde{\alpha}(t, z) \triangleq \arg\min_{x \in \mathbb{R}} v(t, x, z).$$

The fact that $\tilde{\alpha}$ is well defined is a consequence of the convexity of v as well as Lemma 1.4.2 where it is shown that v has at least quadratic growth in x. The next theorem shows that trading is directed towards the cost-adjusted target function, motivating its definition.

Theorem 1.3.5. *Let Assumptions 1.2.5 and 1.2.7 hold, then*

1. *The optimal passive order is given $ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$ by*

$$\hat{u}_1(s, \omega) = \tilde{\alpha}(s, Z(s-, \omega)) - \hat{X}(s-, \omega).$$

2. *Define the buy region via*

$$\mathcal{R}_{\text{buy}} \triangleq \{(s,x,z) \in [t,T] \times \mathbb{R} \times \mathbb{R}^n : x < \tilde{\alpha}(s,z)\},$$

as well as the set where the optimal state process is valued in $\mathcal{R}_{\text{buy}}$,

$$A_{\text{buy}} = \left\{(s,\omega) \in [t,T] \times \Omega : (s, \hat{X}(s-,\omega), Z(s-,\omega)) \in \mathcal{R}_{\text{buy}}\right\}.$$

Then we have that $\hat{u}_1, \hat{u}_2 > 0$, $ds \times d\mathbb{P}$ *a.e. on* A_{buy}. *The symmetric result holds for the sell region,* $\mathcal{R}_{\text{sell}}$, *defined similarly with* $>$ *replacing* $<$.

3. *For the corresponding boundary sets,*

$$\mathcal{R}_{\text{no trade}} \triangleq \{(s,x,z) \in [t,T] \times \mathbb{R} \times \mathbb{R}^n : x = \tilde{\alpha}(s,z)\},$$
$$A_{\text{no trade}} \triangleq \left\{(s,\omega) \in [t,T] \times \Omega : (s, \hat{X}(s-,\omega), Z(s-,\omega)) \in \mathcal{R}_{\text{no trade}}\right\},$$

we have $\hat{u}_1 = \hat{u}_2 = 0$, $ds \times d\mathbb{P}$ *a.e. on* $A_{\text{no trade}}$.

The proof of this result and a discussion of further properties of the cost-adjusted target function are given in Section 1.6.

1.4. Existence of a Solution

The aim of this section is to establish existence of an optimal control. This is done in several steps, first some a priori estimates on the growth of the value function are established. These are then used to show that it is sufficient to consider a subset of controls with a uniform L^2-norm bound. This then permits the use of a Komlós argument to construct the optimal control.

We begin with some estimates from the theory of SDEs.

Lemma 1.4.1. *Let* X^u *and* Z *have dynamics* (1.1) *and* (1.2) *respectively.*

1. *For every* $p \geq 2$ *there exists a constant* c_p *such that for every* $t \in [0,T]$ *we have*

$$\mathbb{E}\left[\sup_{t \leq s \leq T} \|Z(s)\|^p_{\mathbb{R}^n} \,\middle|\, Z(t) = z\right] \leq c_p \left(1 + \|z\|^p_{\mathbb{R}^n}\right).$$

2. *There exists a constant* c_x *such that for any* $u \in \mathcal{U}_t$ *we have*

$$\mathbb{E}\left[\sup_{t \leq s \leq T} |X^u(s)|^2 \,\middle|\, X^u(t) = x\right] \leq c_x \left(1 + \|u\|^2_{L^2}\right).$$

In particular, X^u *has square integrable supremum for all* $u \in \mathcal{U}_t$.

Proof. Item (1) is a well known estimate on the solution of an SDE with Lipschitz coefficients, see for example Barles et al. [1997] Proposition 1.1. Let us now prove item

(2). For $s \in [t,T]$ we have using (1.1) and Jensen's inequality

$$\begin{aligned}|X^u(s)|^2 =& \left| x + \int_t^s u_1(r)N(dr) + \int_t^s u_2(r)dr \right|^2 \\ \leq & 3\Big(x^2 + \int_t^s |u_1(r)|^2 N(dr) + \int_t^s |u_2(r)|^2 dr \Big) \\ \leq & c_x \Big(1 + \int_t^T |u_1(r)|^2 N(dr) + \int_t^T |u_2(r)|^2 dr \Big).\end{aligned}$$

We now use Lemma A.1.3 in the appendix and relabel the constant to get for each $s \in [t,T]$

$$\mathbb{E}\Big[\sup_{0 \leq s \leq T} |X^u(s)|^2 \,|X^u(t) = x \Big] \leq c_x \Big(1 + \int_t^T |u_1(r)|^2 dr + \int_t^T |u_2(r)|^2 dr \Big).$$

□

Since the cost and penalty functions have quadratic growth in x and the SDE for X^u is linear in the control u, it is sensible, at least intuitively, that controls with large L^2-norm cannot be optimal. Specifically, an a priori estimate on the control will be derived in Lemma 1.4.3. This estimate is necessary for the proof of our first main result, the existence of an optimal trading strategy. As a prerequisite for our estimate, we now establish a quadratic growth estimate on the value function. The proof relies on the quadratic growth assumptions on the penalty and liquidity cost functions, which allow to reduce the curve following problem to a simpler optimisation problem whose solution is known.

Lemma 1.4.2. *There exist constants $c_0, c_1, \eta > 0$ such that*

$$v(t,x,z) \geq c_0 x^2 - c_1(1 + \|z\|_{\mathbb{R}^n}^{\eta}),$$

for all $(t,x,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n$.

Proof. To ease notation we write the expectation in (1.3) as $\mathbb{E}_{t,x,z}[\cdot]$. Using the quadratic growth of f, g, h yields

$$\begin{aligned}v(t,x,z) \geq \varepsilon \inf_{u \in \mathcal{U}_t} \mathbb{E}_{t,x,z}\Big[\int_t^T & |u_2(s)|^2 + \big(X^u(s) - \alpha(s, Z(s))\big)^2 ds \\ & + \big(X^u(T) - \alpha(T, Z(T))\big)^2 \Big].\end{aligned}$$

Next an application of the inequality $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ provides

$$v(t,x,z) \geq \frac{\varepsilon}{2} \inf_{u \in \mathcal{U}_t} \mathbb{E}_{t,x,z}\left[\int_t^T |u_2(s)|^2 + |X^u(s)|^2 ds + |X^u(T)|^2 \right]$$

$$- \varepsilon \mathbb{E}_{t,x,z} \left[\int_t^T |\alpha(s, Z(s))|^2 \, ds + |\alpha(T, Z(T))|^2 \right].$$

The polynomial growth of α coupled with Lemma 1.4.1 allows us to write

$$v(t, x, z) \geq \frac{\varepsilon}{2} \inf_{u \in \mathcal{U}_t} \mathbb{E}_{t,x,z} \left[\int_t^T |u_2(s)|^2 + |X^u(s)|^2 \, ds + |X^u(T)|^2 \right] - c_1 \left(1 + \|z\|_{\mathbb{R}^n}^\eta\right).$$

It now remains only to estimate the infimum. This term may be interpreted as a stochastic control problem with quadratic penalty and cost functions and zero target. It is known that such a control problem admits an analytic solution via Riccati equations. In particular we have that

$$v(t, x, z) \geq \frac{\varepsilon}{2} a(t) x^2 - c_1 (1 + \|z\|_{\mathbb{R}^n}^\eta),$$

for a function a given by the solution of the differential equation

$$a'(s) = a^2(s) - 1 + \lambda a(s), \; s \in [t, T], \quad a(T) = 1.$$

Solving explicitly for a one finds that it is monotone and that $a(t) > 0$. If we set

$$c_0 \triangleq \frac{\varepsilon}{2} \min\{a(t), a(T)\} > 0,$$

this completes the proof. □

Using the preceding estimate on the value function, we now show that it is enough to consider trading strategies which satisfy a uniform L^2-norm bound. We note here that from Assumption 1.2.5 together with Lemma 1.4.1 we deduce that $J(0) < \infty$, which is needed in the proof of the following result.

Lemma 1.4.3. *There is a constant $c_{\max}$ such that $\|u\|_{L^2}^2 \geq c_{\max}$ implies that u cannot be optimal.*

Proof. For a control $u \in \mathcal{U}_t$ we want to show that we may bound $J(u)$ from below in terms of $\|u\|_{L^2}^2$. For the market order u_2 we have

$$J(u) \geq \mathbb{E} \left[\int_t^T g(u_2(s), Z(s)) ds \right] \geq \varepsilon \mathbb{E} \left[\int_t^T |u_2(s)|^2 ds \right], \tag{1.6}$$

where we have used Assumption 1.2.5 (2).

The estimate in terms of the passive order u_1 is slightly more involved. Let τ_1 denote the first jump time of the Poisson process N after t, an exponentially distributed random variable with parameter λ, and set $\tau \triangleq \tau_1 \wedge T$. The functions f, g and h are nonnegative, combining this with the definition of v as an infimum we derive

$$J(u) = \mathbb{E}_{t,x,z} \left[\int_t^\tau g(u_2(s), Z(s)) + h\left(X^u(s) - \alpha(s, Z(s))\right) ds \right]$$

$$\begin{aligned}&+\mathbb{E}_{t,x,z}\left[J(\tau, X^u(\tau), Z(\tau), u)\right]\\ \geq &\mathbb{E}_{t,x,z}\left[v(\tau, X^u(\tau), Z(\tau))\right],\end{aligned}$$

where J in the above is evaluated at controls on the stochastic interval $[\tau, T]$. Noting the nonnegativity of v this implies the lower bound

$$J(u) \geq \mathbb{E}_{t,x,z}\left[\mathbb{1}_{\{\tau_1<T\}}v(\tau_1, X^u(\tau_1), Z(\tau_1))\right].$$

Applying first the growth estimates from Lemma 1.4.2, then combining the inequality

$$\mathbb{1}_{\{\tau_1<T\}}\|Z(\tau_1)\|^\eta_{\mathbb{R}^n} \leq \sup_{t\leq s\leq T}\|Z(s)\|^\eta_{\mathbb{R}^n},$$

with Lemma 1.4.1 provides the existence of a constant $c_{1,z}$ such that

$$J(u) \geq c_{1,z} + c_0\mathbb{E}_{t,x,z}\left[\mathbb{1}_{\{\tau_1<T\}}|X^u(\tau_1)|^2\right],$$

where $c_0 > 0$ is as in Lemma 1.4.2. We may write $X^u(\tau_1) = X^u(\tau_1-) + u_1(\tau_1)$ and observe that on the set $\{\tau_1 < T\}$ we have the relation

$$X^u(\tau_1-) = x + \int_t^{\tau_1} u_2(s)ds.$$

Using the inequality $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$ twice, together with the Jensen inequality, we get

$$J(u) \geq c_{1,x,z} + c_0\mathbb{E}\left[\mathbb{1}_{\{\tau_1<T\}}|u_1(\tau_1)|^2\right] - c_2\mathbb{E}\left[\int_t^{\tau_1}|u_2(s)|^2ds\right],$$

for some constant $c_2 > 0$, where we drop the subscript $\{t, x, z\}$. In light of inequality (1.6) we derive

$$\left(1+\frac{c_2}{\varepsilon}\right)J(u) \geq c_{1,x,z} + c_0\mathbb{E}\left[\mathbb{1}_{\{\tau_1<T\}}|u_1(\tau_1)|^2\right].$$

An application of the law of total expectation and relabelling the constants provides the estimate

$$J(u) \geq c_{1,x,z} + c_0\int_t^T \lambda\mathbb{E}\left[|u_1(s)|^2\right]e^{-\lambda(s-t)}ds.$$

We apply the uniform bound $e^{-\lambda(s-t)} \geq e^{-\lambda(T-t)}$ for $s \in [t, T]$ in the above, then combine with (1.6) to see that

$$J(u) \geq c_{1,x,z} + c_0\|u\|^2_{L^2}.$$

In particular if

$$\|u\|_{L^2}^2 \geq c_{\max} \triangleq \frac{J(0) - c_{1,x,z}}{c_0} + 1,$$

then we see that $J(u) > J(0)$ and the control u is clearly not optimal. □

We remark that a generalisation of the growth estimate on the value function from Lemma 1.4.2 to the case of singular controls will be derived in Lemma 2.3.4. Similarly, an a priori estimate which also covers the singular control case is given in Lemma 2.3.5.

Before completing the proof of Theorem 1.3.1, we recall a definition and refer the reader to Protter [2004] for further details.

Definition 1.4.4. A sequence of processes $(Y^n)_{n\in\mathbb{N}}$ defined on $[t,T]\times\Omega$ and valued in $\mathbb{R}$ converges to a process $Y : [t,T]\times\Omega \mapsto \mathbb{R}$ *uniformly on compacts in probability (UCP)* if, for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{t\leq s\leq T} |Y_s^n - Y_s| > \varepsilon \right) = 0.$$

We may now complete the proof of our first main result, the existence and uniqueness of an optimal trading strategy. The proof combines our a priori estimate on the control with a Komlós argument.

Proof of Theorem 1.3.1. The strict convexity of J is a direct consequence of the strict convexity of f, g and h. From the previous lemma it follows that (using the notation therein) if we set

$$\mathcal{U}_t^{c_{\max}} \triangleq \left\{ u \in \mathcal{U}_t : \|u\|_{L^2}^2 \leq c_{\max} \right\},$$

then

$$\inf_{u\in\mathcal{U}_t} J(u) = \inf_{u\in\mathcal{U}_t^{c_{\max}}} J(u).$$

We take a sequence of minimising processes $(u^n)_{n\in\mathbb{N}} \subset \mathcal{U}_t^{c_{\max}}$. Due to the uniform bound on the L^2-norms we may proceed as in Beneš et al. [2004] Theorem 2.1 to find a subsequence (also indexed by n) together with a process $\hat{u} : [t,T]\times\Omega \to \mathbb{R}^2$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} u^j = \hat{u}$$

$ds \times d\mathbb{P}$ a.e. on $[t,T]\times\Omega$. To be precise, we note here that the superscripts index the sequence and the subscripts the components of the process. Due to Karatzas and Shreve [1991] Proposition 1.2 we may assume that $\hat{u}_2$ is progressively measurable, whereas the predictability of $\hat{u}_1$ follows as in Applebaum [2009] Lemma 4.1.3. In particular we deduce first the appropriate measurability of $\hat{u} \in \mathcal{U}_t$ and then from Fatou's lemma that

$\hat{u} \in \mathcal{U}_t^{c_{\max}}$. Before proving the optimality of $\hat{u}$ we first show some convergence results. For $n \in \mathbb{N}$ we set

$$\bar{u}^n \triangleq \frac{1}{n}\sum_{j=1}^{n} u^j \text{ and } \bar{X}^n \triangleq \frac{1}{n}\sum_{j=1}^{n} X^{u^j}.$$

We have the following estimate,

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\le s\le T}|\bar{X}^n(s)-\bar{X}^m(s)|\right]\\ \le&\mathbb{E}\left[\int_t^T|\bar{u}_2^n(s)-\bar{u}_2^m(s)|ds\right]+\mathbb{E}\left[\int_t^T|\bar{u}_1^n(s)-\bar{u}_1^m(s)|N(ds)\right].\end{aligned}$$

Via the de-la-Vallée-Poussin Theorem, a consequence of the uniform bound on the L^2-norms is that $(\bar{u}^n)_{n\in\mathbb{N}}$ also converges in $L^1([t,T]\times\Omega)$ to $\hat{u}$. It now follows that $(\bar{X}^n)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{D}$, the space of càdlàg processes equipped with the UCP topology. Hence there exists a process $\hat{X}$ such that $\bar{X}^n$ converges to $\hat{X}$. Here by convergence we mean that

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{t\le s\le T}|\bar{X}^n(s)-\hat{X}(s)|\Big]=0.$$

In particular the above argument implies that $\hat{X}=X^{\hat{u}}$ up to indistinguishability and is well defined. For the optimality, applying Fatou's lemma together with the convexity of f, g and h gives

$$J(\hat{u})\le\liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}J(u^j)=\inf_{u\in\mathcal{U}_t}J(u).$$

Turning to uniqueness, suppose $\hat{u}$ and $\bar{u}$ are optimal controls. The strict convexity of $u_2\mapsto g(u_2,z)$ implies that $\hat{u}_2=\bar{u}_2$ $ds\times d\mathbb{P}$ a.e. on $[t,T]\times\Omega$. We also have $X^{\hat{u}}=X^{\bar{u}}$ $ds\times d\mathbb{P}$ a.e. since otherwise $J\left(\frac{\hat{u}+\bar{u}}{2}\right)<J(\hat{u})=J(\bar{u})$ due to the strict convexity of $x\mapsto h(x)$. An application of Lemma A.1.1 now provides $\bar{u}=\hat{u}$ $ds\times d\mathbb{P}$ a.e. on $[t,T]\times\Omega$. □

1.5. The Stochastic Maximum Principle

In this section we are concerned with the proof of Theorem 1.3.2. In particular we show the existence of a solution to the adjoint equation as well as providing a characterisation of the optimal control. To avoid difficulties related to controls for which the performance functional is not finite we define a subclass of controls given by

$$\mathcal{U}_{\text{adm}}\triangleq\{u\in\mathcal{U}_t: J(u)<\infty\}.$$

It is clear that $\hat{u} \in \mathcal{U}_{\mathrm{adm}}$, that minimising J over $\mathcal{U}_{\mathrm{adm}}$ is equivalent to minimising over $\mathcal{U}_t$ and that for any $u \in \mathcal{U}_{\mathrm{adm}}$ we have

$$\mathbb{E}\left[\int_t^T g(u_2(s), Z(s))ds\right] < \infty.$$

We shall use these properties throughout.

We begin by recalling the adjoint BSDE on $[t, T]$ given in equation (1.5),

$$\begin{aligned} dP(s) =& h'\left(\hat{X}(s) - \alpha(s, Z(s))\right) ds + Q(s)dW(s) + R_1(s)\tilde{N}(ds) \\ &+ \int_{\mathbb{R}^k} R_2(s,\theta)\tilde{M}(ds, d\theta), \\ P(T) =& - f'\left(\hat{X}(T) - \alpha(T, Z(T))\right). \end{aligned}$$

Remark 1.5.1. As before Q and R are $\mathbb{R}^d$ and $\mathbb{R}^2$-valued respectively. Strictly speaking the adjoint equation should be for a vector $(P_1, \ldots, P_{n+1})$, where P_1 satisfies the BSDE above. However when one writes down the full system one sees that, due to the fact that the control does not enter the signal Z, P_1 and $(P_2, \ldots, P_{n+1})$ are decoupled. Moreover, we shall see that the optimality criterion only involves P_1, thus it is sufficient to omit $(P_2, \ldots, P_{n+1})$ and to consider only the above BSDE.

Due to the simple structure of the adjoint equation, the proof of existence and uniqueness of a solution is straightforward, we only give it here for completeness.

Lemma 1.5.2. *There is a solution (P, Q, R) to the adjoint BSDE such that*

$$\begin{aligned} \mathbb{E}\Big[\sup_{t\le s\le T} |P(s)|^2 + \int_t^T \|Q(s)\|^2_{\mathbb{R}^d} ds \\ + \int_t^T |R_1(s)|^2 ds + \int_t^T \int_{\mathbb{R}^k} |R_2(s,\theta)|^2 m(d\theta) ds\Big] < \infty. \end{aligned}$$

It is unique amongst triples (P, Q, R) satisfying the above integrability criterion.

Proof. The functions f', h' and α have, respectively, linear and polynomial growth. Applying Lemma 1.4.1 we see that the process

$$\Phi(s) \triangleq -\mathbb{E}\left[\int_t^T h'(\hat{X}(r) - \alpha(r, Z(r)))dr + f'(\hat{X}(T) - \alpha(T, Z(T)))\Big|\mathcal{F}_s\right]$$

is a square integrable martingale on $[t, T]$. Since the filtration is generated by the Brownian motion and the (compound) Poisson processes by Tang and Li [1994] Lemma 2.3 we deduce the existence of processes Q, R_1 and R_2 such that

$$\mathbb{E}\left[\int_t^T \|Q(r)\|^2_{\mathbb{R}^d} dr + \int_t^T |R_1(r)|^2 dr + \int_t^T \int_{\mathbb{R}^k} |R_2(r,\theta)|^2 m(d\theta) dr\right] < \infty,$$

such that for $s \in [t,T]$ we have

$$\Phi(s) = \Phi(t) + \int_t^s Q(r)dW(r) + \int_t^s R_1(r)\tilde{N}(dr) + \int_t^s R_2(r,\theta)\tilde{M}(dr,d\theta).$$

A calculation now shows that

$$\begin{aligned} P(s) \triangleq & \Phi(s) + \int_t^s h'(\hat{X}(r) - \alpha(r, Z(r)))dr \\ = & -\mathbb{E}\left[\int_s^T h'(\hat{X}(r) - \alpha(r, Z(r)))dr + f'(\hat{X}(T) - \alpha(T, Z(T)))\middle|\mathcal{F}_s\right] \end{aligned}$$

is the required solution. Uniqueness follows from the uniqueness in the martingale representation theorem. It remains to show that P has integrable supremum. For this, we first apply Doob's inequality to the martingale Φ, then use the linear growth of f', h' and Lemma 1.4.1 to get

$$\begin{aligned} \mathbb{E}\Big[\sup_{t\le s\le T}|\Phi(s)|^2\Big] \le & c_1\mathbb{E}[|\Phi(T)|^2] \\ \le & c_1\mathbb{E}\Big[\int_t^T \big|h'(\hat{X}(r) - \alpha(r, Z(r)))\big|^2 dr + \big|f'(\hat{X}(T) - \alpha(T, Z(T)))\big|^2\Big] \\ \le & c_1\Big(1 + \mathbb{E}\Big[\sup_{t\le s\le T}|\hat{X}(s)|^2\Big] + \|z\|^\eta_{\mathbb{R}^n}\Big) < \infty. \end{aligned}$$

Now from the definition of P it follows that

$$\mathbb{E}\Big[\sup_{t\le s\le T}|P(s)|^2\Big] \le c_2\Big(\mathbb{E}\Big[\sup_{t\le s\le T}|\Phi(s)|^2\Big] + \mathbb{E}\Big[\int_t^T \big|h'(\hat{X}(r) - \alpha(r, Z(r)))\big|^2 dr\Big]\Big) < \infty.$$

This completes the proof. □

The stochastic maximum principle exploits the convexity of the performance functional J together with the fact that the minimum of J can be characterised using the subgradient inequality, which allows us to give an explicit condition for the optimality of a control $\hat{u}$. Given controls $u, w \in \mathcal{U}_{\text{adm}}$, the *Gâteaux derivative* of J is defined as

$$\langle J'(w), u\rangle = \lim_{\rho\to 0}\frac{J(w+\rho u) - J(w)}{\rho}.$$

The following lemma provides the explicit formula for the Gâteaux derivative of the performance functional. We give a proof based upon Cadenillas and Karatzas [1995] Lemma 1.1, however we avoid the measurable selection argument therein. In this chapter the starting point (t,x,z) is fixed and we therefore write $\mathbb{E}[\cdot]$ for $\mathbb{E}_{t,x,z}[\cdot]$.

Lemma 1.5.3. *For $u, w \in \mathcal{U}_{\text{adm}}$ the Gâteaux derivative of J is given by*

$$\begin{aligned}\langle J'(w), u\rangle =&\mathbb{E}\left[\int_t^T (X^u(s)-x)h'(X^w(s)-\alpha(s,Z(s))) + u_2(s)g_{u_2}(w_2(s),Z(s))\,ds\right]\\&+\mathbb{E}\left[(X^u(T)-x)f'\left(X^w(T)-\alpha(T,Z(T))\right)\right].\end{aligned}$$

Proof. We first note that for $s \in [t,T]$, $\rho \in [0,1]$ and $u, w \in \mathcal{U}_{\text{adm}}$ we have

$$\begin{aligned}X^{w+\rho u}(s) =&x + \int_t^s (w_2(r)+\rho u_2(r))dr + \int_t^s (w_1(r)+\rho u_1(r))N(dr)\\=&X^w(s)+\rho(X^u(s)-x).\end{aligned}$$

Using the fact that the signal is unaffected by the control together with the mean value theorem we may compute, for $\rho \in [0,1]$,

$$\begin{aligned}\langle J'(w), u\rangle &= \lim_{\rho\to 0}\frac{J(w+\rho u)-J(w)}{\rho}\\&= \lim_{\rho\to 0}\Bigg\{\mathbb{E}\left[\int_t^T\int_0^1 (X^u(s)-x)h'(X^w(s)+\zeta\rho(X^u(s)-x)-\alpha(s,Z(s)))d\zeta ds\right]\\&\quad+\mathbb{E}\left[\int_t^T\int_0^1 u_2(s)g_{u_2}(w_2(s)+\zeta\rho u_2(s),Z(s))\,d\zeta ds\right]\\&\quad+\mathbb{E}\left[\int_0^1 (X^u(T)-x)f'(X^w(T)+\zeta\rho(X^u(T)-x)-\alpha(T,Z(T)))d\zeta\right]\Bigg\}.\end{aligned}$$

Due to the convexity of the functions $f, g(\cdot,z)$ and h, exactly as in Cadenillas and Karatzas [1995] Lemma 1.1, the integrands are all decreasing as ρ decreases so that all limits are well defined as $\rho\to 0$. The statement of the lemma will follow from the monotone convergence theorem once we show that

$$\begin{aligned}&\mathbb{E}\left[\int_t^T (X^u(s)-x)h'(X^w(s)+(X^u(s)-x)-\alpha(s,Z(s)))ds\right]\\+&\mathbb{E}\left[\int_t^T u_2(s)g_{u_2}(w_2(s)+u_2(s),Z(s))\,ds\right]\\+&\mathbb{E}\left[(X^u(T)-x)f'\left(X^w(T)+(X^u(T)-x)-\alpha(T,Z(T))\right)\right]<\infty.\end{aligned} \tag{1.7}$$

Using the linear growth of h' together with the Young inequality one can find a constant c such that for $s \in [t,T]$

$$\begin{aligned}&(X^u(s)-x)h'(X^w(s)+X^u(s)-\alpha(s,Z(s)))\\\le& c(1+|X^u(s)|^2+|X^w(s)|^2+|\alpha(s,Z(s))|^2).\end{aligned}$$

The right hand side is integrable over $[t,T]\times\Omega$ thanks to the growth estimates from Lemma 1.4.1 and the polynomial growth of α. An identical argument may be applied to the term involving f'. Thus to complete the proof we need an estimate for the second term in (1.7). The subgradient inequality together with the nonnegativity of g implies that for $s \in [t, T]$ the following holds,

$$
\begin{aligned}
&u_2(s) g_{u_2}\left(w_2(s) + u_2(s), Z(s)\right) \\
\leq &g\left(u_2(s), Z(s)\right) + \left(w_2(s) + u_2(s)\right) g_{u_2}\left(w_2(s) + u_2(s), Z(s)\right).
\end{aligned}
$$

Applying now Assumption 1.2.7 (2) and (3) and observing that $u, w \in \mathcal{U}_{\text{adm}}$ completes the proof. □

Having constructed a formula for the Gâteaux derivative we may now turn to characterising the optimal control. As a prerequisite for some algebraic manipulations of the Gâteaux derivative, let us compute $P \cdot X$. From the integration by parts formula we derive, for a control $u \in \mathcal{U}_{\text{adm}}$ and $s \in [t, T]$,

$$
\begin{aligned}
&P(s)X^u(s) - P(t)X^u(t) - \int_t^s \psi(r, u(r))dr \\
=&\int_t^s X^u(r-)Q(r)dW(r) + \int_t^s \left[\left(P(r-) + R_1(r)\right)u_1(r) + X^u(r-)R_1(r)\right]\tilde{N}(dr) \\
&+ \int_t^s \int_{\mathbb{R}^k} X^u(r-)R_2(r,\theta)\tilde{M}(dr, d\theta),
\end{aligned}
$$

where we have used that $[N, M] = 0$, a consequence of the independence of M and N together with Applebaum [2009] Proposition 1.3.12, as well as setting

$$
\psi(r, u(r)) \triangleq P(r)u_2(r) + \lambda u_1(r)[P(r-) + R_1(r)] + X^u(r)h'\left(\hat{X}(r) - \alpha(r, Z(r))\right).
$$

We rewrite the above as

$$
Y^u(s) = P(t)X^u(t) + L^u(s), \tag{1.8}
$$

where the "local martingale part" L^u is defined for $s \in [t, T]$ by

$$
\begin{aligned}
L^u(s) \triangleq &\int_t^s X^u(r-)Q(r)dW(r) + \int_t^s \int_{\mathbb{R}^k} X^u(r-)R_2(r,\theta)\tilde{M}(dr, d\theta) \\
&+ \int_t^s \left[\left(P(r-) + R_1(r)\right)u_1(r) + X^u(r-)R_1(r)\right]\tilde{N}(dr).
\end{aligned}
$$

and the "non-martingale part" Y^u is defined by

$$
Y^u(s) \triangleq P(s)X^u(s) - \int_t^s \psi(r, u(r))dr.
$$

We now verify that L is a true martingale.

Lemma 1.5.4. *For all $u \in \mathcal{U}_{\text{adm}}$ the process L^u is a martingale starting in* 0.

Proof. For the continuous local martingale part, the result follows from the Burkholder-

Davis-Gundy and Hölder inequalities noting that X^u has square integrable supremum and $Q \in L^2([t,T] \times \Omega)$. For each of the remaining terms we may apply Lemma A.1.3, where the appropriate integrability follows from the fact that X^u and P have square integrable suprema and Q, R_1, R_2 as well as u are square integrable. □

We can now turn to the proof of our second main result, the stochastic maximum principle. It is based on ideas given in Cadenillas [2002] and extended to the case where the dynamics are not affine in (X, Z), as discussed in Remark 1.3.3. We note however that our theorem is not a straightforward extension thereof. We must verify firstly that Cadenillas [2002] Assumptions 3.1 and 4.1 hold in this new setting and secondly that one can still derive an apriori L^2-estimate on the control for our choice of cost functions and dynamics. This motivates the present detailed derivation of the maximum principle.

Proof of Theorem 1.3.2. We are minimising a convex functional over $\mathcal{U}_{\text{adm}}$, so by Ekeland and Témam [1999] Proposition 2.2.1 a necessary and sufficient condition for optimality of $\hat{u}$ is that

$$\langle J'(\hat{u}), u - \hat{u}\rangle \geq 0 \text{ for all } u \in \mathcal{U}_{\text{adm}}. \tag{1.9}$$

Due to Lemma 1.5.4 we know that L^u is a martingale starting in 0 for all $u \in \mathcal{U}_{\text{adm}}$. In particular from equation (1.8) we have that $\mathbb{E}[Y^u(T) - Y^{\hat{u}}(T)] = 0$. The definition of Y^u together with the terminal condition in (1.5) allows us to write this as

$$\begin{aligned}
0 =& \mathbb{E}\left[P(T)\left[X^u(T) - \hat{X}(T)\right] - \int_t^T \left[\psi(s, u(s)) - \psi(s, \hat{u}(s))\right] ds\right] \\
=& \mathbb{E}\Big[- f'\big(\hat{X}(T) - \alpha(T, Z(T))\big)\left[X^u(T) - \hat{X}(T)\right] \\
& \quad - \int_t^T \left[\psi(s, u(s)) - \psi(s, \hat{u}(s))\right] ds\Big].
\end{aligned}$$

Using the known form of the Gâteaux derivative we derive

$$\begin{aligned}
\langle J'(\hat{u}), u - \hat{u}\rangle &= \langle J'(\hat{u}), u - \hat{u}\rangle + \mathbb{E}\left[Y^u(T) - Y^{\hat{u}}(T)\right] \\
&= \mathbb{E}\left[\int_t^T \left[u_2(s) - \hat{u}_2(s)\right]\left[g_{u_2}\left(\hat{u}_2(s), Z(s)\right) - P(s)\right] ds\right] \\
&\quad - \lambda\mathbb{E}\left[\int_t^T \left[u_1(s) - \hat{u}_1(s)\right]\left(P(s-) + R_1(s)\right) ds\right],
\end{aligned} \tag{1.10}$$

for every control $u \in \mathcal{U}_{\text{adm}}$.

A consequence of (1.9) and (1.10) is that $\hat{u}$ is optimal if and only if we have $ds \times d\mathbb{P}$ a.e. on $[t,T] \times \Omega$,

1. $[u_2(s) - \hat{u}_2(s)]\left[g_{u_2}\left(\hat{u}_2(s), Z(s)\right) - P(s)\right] \geq 0 \quad \forall u = (u_1, u_2) \in \mathcal{U}_{\text{adm}}$,
2. $P(s-) + R_1(s) = 0$.

It remains to show that the first item is equivalent to $\hat{u}_2$ being the pointwise minimiser of the function

$$u_2 \mapsto g(u_2, Z(s,\omega)) - P(s,\omega)u_2. \tag{1.11}$$

By Assumption 1.2.7 constant deterministic market orders $\hat{u}_2(s,\omega) \equiv u_2 \in \mathbb{R}$ incur finite liquidity costs, in particular they are admissible and item (1) is equivalent to

$$[u_2 - \hat{u}_2(s)]\,[g_{u_2}(\hat{u}_2(s), Z(s)) - P(s)] \geq 0 \quad \forall u_2 \in \mathbb{R}.$$

This is the subgradient condition for the strictly convex function defined in (1.11), so $\hat{u}_2$ is indeed the pointwise minimiser and the proof is complete. □

Corollary 1.5.5. *The optimal market order satisfies $ds \times d\mathbb{P}$ a.e.*

1. $\hat{u}_2(s) = \left[g_{u_2}(\cdot, Z(s))\right]^{-1}(P(s))$.
2. *If $P(s) > 0$ then $\hat{u}_2(s) > 0$. Similarly, if $P(s) < 0$ then $\hat{u}_2(s) < 0$ and if $P(s) = 0$ then $\hat{u}_2(s) = 0$.*

Proof. By assumption 1.2.5 we have that $g(\cdot, z)$ is strictly convex and thus $g_{u_2}(\cdot, z)$ is strictly increasing for each fixed $z \in \mathbb{R}^n$. This function is normalised since $g(\cdot, z)$ admits its minimum at zero. Moreover, the range of $g_{u_2}(\cdot, z)$ is $\mathbb{R}$ which is due to Assumption 1.2.5(2). It is now clear that this function is a bijection from $\mathbb{R}$ into $\mathbb{R}$, its inverse exists and is also a bijection. Item (1) is now a direct consequence of Theorem 1.3.2(2). Item (2) is true since $g_{u_2}(\cdot, z)$ is strictly increasing and normalised. □

1.6. The Cost-Adjusted Target Function

In Section 1.5 we have established a necessary and sufficient condition for the optimality of a given control. Unfortunately in the case of the passive order it is far from explicit. In the present section we prove Theorem 1.3.5, which provides a more explicit characterisation of optimality in terms of buy and sell regions. These regions are defined in terms of the cost-adjusted target function as given in Definition 1.3.4. As a first step, we show that the passive order is of the form $\hat{u}_1 = \tilde{\alpha} - \hat{X}$, so it is a sell (buy) order if stock holdings are above (resp. below) the cost-adjusted target function. For the proof of this result, we need the following estimate.

Proposition 1.6.1. *There exist constants $c_{\tilde{\alpha}}$ and η such that for each $z \in \mathbb{R}^n$,*

$$\sup_{t \leq s \leq T} |\tilde{\alpha}(s,z)| \leq c_{\tilde{\alpha}} \left(1 + \|z\|^{\eta}_{\mathbb{R}^n}\right).$$

Proof. Choosing the zero control $u \equiv 0$ and using the polynomial growth of the functions f, h and α as well as Lemma 1.4.1 we see that there exist c_1 and η_1 such that

$$v(t,0,z) \leq J(t,0,z,0) \leq c_1 \left(1 + \|z\|^{\eta_1}_{\mathbb{R}^n}\right).$$

If we now apply Lemma 1.4.2 we find further constants $c_2 > 0$, c_3 and η_2 such that

$$v(t, \tilde{\alpha}(t, z), z) \geq c_2 \left|\tilde{\alpha}(t, z)\right|^2 - c_3 \left(1 + \|z\|_{\mathbb{R}^n}^{\eta_2}\right).$$

Since $\tilde{\alpha}$ is the pointwise minimiser of v with respect to x, combining the above inequalities and relabelling constants provides the result. □

The following proposition shows that the optimal passive order is directed towards the cost-adjusted target function.

Proposition 1.6.2. *The optimal passive order $\hat{u}_1$ is given $ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$ by*

$$\hat{u}_1(s, \omega) = \tilde{\alpha}(s, Z(s-, \omega)) - \hat{X}(s-, \omega).$$

Proof. We consider the process $\tilde{X}$ which is defined in terms of the optimal market order $\hat{u}_2$ by the following SDE on $[t, T]$,

$$d\tilde{X}(s) = \hat{u}_2(s)ds + \left(\tilde{\alpha}(s, Z(s-)) - \tilde{X}(s-)\right) N(ds), \quad \tilde{X}(t) = x,$$

and want to show that the control $\tilde{u}$ defined for $s \in [t, T]$ by

$$\tilde{u}(s) \triangleq \begin{pmatrix} \tilde{\alpha}(s, Z(s-)) - \tilde{X}(s-) \\ \hat{u}_2(s) \end{pmatrix},$$

is admissible. The predictability and progressive measurability are straightforward and thus we need only check the L^2-nature of the control, which is a consequence of the following estimate,

$$\sup_{t \leq s \leq T} |\tilde{X}(s)|^2 \leq c \left(1 + \int_t^T |\hat{u}_2(s)|^2 ds + \sup_{t \leq s \leq T} |\tilde{\alpha}(s, Z(s))|^2\right),$$

together with Proposition 1.6.1 and Lemma 1.4.1.

Now let us prove that such a strategy is in fact optimal. We let τ_1 be the first jump time of N after t and $\tau \triangleq \tau_1 \wedge T$. By the dynamic programming principle we have

$$\begin{aligned}
&\mathbb{E}_{t,x,z}\left[\int_t^\tau g(\hat{u}_2(s), Z(s)) + h(\tilde{X}(s) - \alpha(s, Z(s)))ds + v\left(\tau, \tilde{X}(\tau), Z(\tau)\right)\right] \\
\geq &\mathbb{E}_{t,x,z}\left[\int_t^\tau g(\hat{u}_2(s), Z(s)) + h(\hat{X}(s) - \alpha(s, Z(s)))ds + v\left(\tau, \hat{X}(\tau), Z(\tau)\right)\right].
\end{aligned}$$

On the stochastic time interval $[t, \tau)$ we have that the optimal trajectories $\hat{X}$ and $\tilde{X}$ coincide, and on the set $\{\tau_1 > T\}$ we have equality in the above. If we define the set $A \triangleq \{\tau_1 \leq T\}$ then the above inequality leads to

$$\begin{aligned}
\mathbb{E}_{t,x,z}\left[v\left(\tau_1, \tilde{X}(\tau_1), Z(\tau_1)\right) \mathbb{1}_A\right] &\geq \mathbb{E}_{t,x,z}\left[v\left(\tau_1, \hat{X}(\tau_1), Z(\tau_1)\right) \mathbb{1}_A\right] \\
&= \mathbb{E}_{t,x,z}\left[v\left(\tau_1, \hat{X}(\tau_1-) + \hat{u}_1(\tau_1), Z(\tau_1)\right) \mathbb{1}_A\right].
\end{aligned}$$

Independence of N and M together with Applebaum [2009] Proposition 1.3.12 implies $Z(\tau_1-) = Z(\tau_1)$ so that by the construction of the process $\tilde{X}$ we get

$$\mathbb{E}_{t,x,z}\Big[v\left(\tau_1, \tilde{X}(\tau_1), Z(\tau_1)\right)\mathbb{1}_A\Big] = \mathbb{E}_{t,x,z}\Big[v\Big(\tau_1, \tilde{\alpha}(\tau_1, Z(\tau_1)), Z(\tau_1)\Big)\mathbb{1}_A\Big].$$

We combine this with the fact that τ_1 is exponentially distributed to derive

$$\begin{aligned} 0 \leq & \mathbb{E}_{t,x,z}\left[\left(v\Big(\tau_1, \tilde{\alpha}(\tau_1, Z(\tau_1)), Z(\tau_1)\Big) - v\left(\tau_1, \hat{X}(\tau_1-) + \hat{u}_1(\tau_1), Z(\tau_1)\right)\right)\mathbb{1}_A\right] \\ = & \mathbb{E}_{t,x,z}\left[\int_t^T \lambda e^{-\lambda(r-t)}\Big(v(r, \tilde{\alpha}(r, Z(r)), Z(r)) - v(r, \hat{X}(r-) + \hat{u}_1(r), Z(r))\Big)\mathbb{1}_A dr\right]. \end{aligned}$$

Since $\tilde{\alpha}$ is the pointwise minimiser of the value function with respect to x, the integrand on the right hand side is nonpositive, and even strictly negative on the set

$$\big\{(r, \omega)\big|\hat{u}_1(r, \omega) \neq \tilde{u}_1(r, \omega)\big\}.$$

This proves that $\hat{u}_1 = \tilde{u}_1$ $ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$ and completes the proof. □

The preceding proposition shows that passive sell (buy) orders are used if and only if stock holdings are above (resp. below) the cost-adjusted target function. The aim is now to establish the corresponding result for the optimal market order. The proof relies on a careful analysis of the FBSDE and some technical results which are stated in Lemma 1.6.3 as well as Propositions 1.6.4 and 1.6.5. We denote by

$$(\hat{X}^{t,x,z}, Z^{t,z}, P^{t,x,z})$$

the solution to the coupled FBSDE given by equations (1.1), (1.2) and (1.5), started at $(t, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$. We now show that $\hat{X}^{t,x,z}$ is monotone in x.

Lemma 1.6.3. *If $x < y$ then $\hat{X}^{t,x,z}(s) \leq \hat{X}^{t,y,z}(s)$ for each $s \in [t, T]$.*

Proof. As above we denote by τ_1 the first jump time of N after time t. If there is a jump in $[t, T]$ then by Proposition 1.6.2

$$\hat{X}^{t,x,z}(\tau_1) = \hat{X}^{t,y,z}(\tau_1) = \tilde{\alpha}(\tau_1, Z(\tau_1-)).$$

Due to the uniqueness of the solution to the FBSDE for any initial data we derive the flow property, exactly as in Pardoux and Tang [1999] Theorem 5.1,

$$\hat{X}^{s, \hat{X}^{t,x,z}(s), Z^{t,z}(s)}(r) = \hat{X}^{t,x,z}(r), \quad t \leq s \leq r \leq T. \tag{1.12}$$

This implies that $\hat{X}^{t,x,z}$ and $\hat{X}^{t,y,z}$ coincide on $[\tau_1, T]$. Before a jump of N, $\hat{X}^{t,x,z}$ and $\hat{X}^{t,y,z}$ evolve continuously and we define the stopping time

$$\tau_2 \triangleq \inf\left\{s \geq t : \hat{X}^{t,x,z}(s) = \hat{X}^{t,y,z}(s)\right\} \wedge \tau_1 \wedge T.$$

By continuity $\hat{X}^{t,x,z} < \hat{X}^{t,y,z}$ in $[s, \tau_2)$ and $\hat{X}^{t,x,z} = \hat{X}^{t,y,z}$ in $[\tau_2, \tau_1 \wedge T]$ again thanks to the flow property. □

The following result is classical in the study of fully coupled FBSDEs, see for instance Bender and Zhang [2008] Corollary 6.2. Since we have jumps we provide a proof.

Proposition 1.6.4. *There exists a deterministic measurable function $\varphi : [t,T] \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ such that for $s \in [t,T]$ we have*

$$P^{t,x,z}(s) = \varphi(s, \hat{X}^{t,x,z}(s), Z^{t,z}(s)).$$

Proof. When $t = 0$ since $P^{0,x,z}$ is adapted and the filtration is generated by the (compound) Poisson processes and the Brownian motion we have that $P^{0,x,z}(0)$ is constant so that the map $(x,z) \mapsto P^{0,x,z}(0)$ is well defined. Using a time shift argument exactly as in El Karoui et al. [1997b] Proposition 4.2 one can show that $P^{t,x,z}(t)$ is deterministic so that the map

$$\varphi(t,x,z) = P^{t,x,z}(t)$$

is well defined. Using the flow property (1.12) we see that for $s \in [t,T]$

$$P^{t,x,z}(s) = P^{s,\hat{X}^{t,x,z}(s),Z^{t,z}(s)}(s) = \varphi(s, \hat{X}^{t,x,z}(s), Z^{t,z}(s)),$$

as required. □

We now show that φ (or equivalently P), viewed as a function of x, is strictly decreasing and normalised at $\tilde{\alpha}$. Combining this with the representation $\hat{u}_2 = g_{u_2}^{-1}(P)$ from Corollary 1.5.5 will then lead to buy and sell regions which are separated by $\tilde{\alpha}$.

Proposition 1.6.5. *For all $s \in [t,T]$ and $z \in \mathbb{R}^n$ the map $x \mapsto \varphi(s,x,z)$ is strictly decreasing. Moreover we have $\varphi(s, \tilde{\alpha}(s,z), z) = 0$.*

Proof. Using Proposition 1.6.4 and the definition of the adjoint equation (1.5) we have the representation

$$\begin{aligned} P^{t,x,z}(t) = \varphi(t,x,z) = &- \mathbb{E}_{t,x,z}\left[\int_t^T h'\left(\hat{X}^{t,x,z}(s) - \alpha(s, Z^{t,z}(s))\right) ds\right] \\ &- \mathbb{E}_{t,x,z}\left[f'\left(\hat{X}^{t,x,z}(T) - \alpha(T, Z^{t,z}(T))\right)\right]. \end{aligned} \tag{1.13}$$

Suppose $x < y$, then from Lemma 1.6.3 together with the càdlàg property of the paths of $\hat{X}^{t,x,z}$ and the fact that h', f' are normalised and strictly increasing, it follows that $\varphi(s,x,z) \geq \varphi(s,y,z)$.

We observe that $\varphi(t,x,z) = \varphi(t,y,z)$ would imply $\hat{X}^{t,x,z}(s) = \hat{X}^{t,y,z}(s)$ $ds \times d\mathbb{P}$ a.e. on $[t,T] \times \Omega$ so that $\hat{X}^{t,x,z}$ and $\hat{X}^{t,y,z}$ would be indistinguishable by Lemma A.1.1, which contradicts $\hat{X}^{t,x,z}(t) = x < y = \hat{X}^{t,y,z}(t)$.

To prove the second claim, let τ_1 denote the first jump time of N after t and define $\tau \triangleq \tau_1 \wedge T$. Due to the independence of N and M, we see from Applebaum [2009]

Proposition 1.3.12 that they do not jump at the same time. In particular, using that τ_1 is exponentially distributed with parameter λ, we may write

$$\begin{aligned}\mathbb{E}\left[P(\tau)^2\right] &= \mathbb{E}\left[\left(P(\tau-)+R_1(\tau)\right)^2\right]\\ &= \mathbb{E}\left[\int_t^T \lambda e^{-\lambda(s-t)}\left(P(s-)+R_1(s)\right)^2 ds\right] = 0.\end{aligned}$$

The final equality follows since $P(s-)+R_1(s)=0$, $ds\times d\mathbb{P}$ a.e. on $[t,T]\times\Omega$ by Theorem 1.3.2 and we have dropped the superscripts as we now consider a fixed starting point $(t,x,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^n$. Using Proposition 1.6.4 we may write this as

$$\begin{aligned}0 =&\mathbb{E}\left[P(\tau)^2\right] = \mathbb{E}\left[\varphi\left(\tau,\hat{X}(\tau),Z(\tau)\right)^2\right]\\ =&\mathbb{E}\left[\int_t^T \lambda e^{-\lambda(s-t)}\varphi\left(s,\hat{X}(s),Z(s)\right)^2 ds\right].\end{aligned}$$

A consequence of Proposition 1.6.2 is that

$$\hat{X}(\tau) = \hat{X}(\tau-)+\hat{u}_1(\tau) = \tilde{\alpha}(\tau,Z(\tau-)) = \tilde{\alpha}(\tau,Z(\tau)),$$

so we have

$$0 = \mathbb{E}\left[\int_t^T \lambda e^{-\lambda(s-t)}\varphi\left(s,\tilde{\alpha}(s,Z(s)),Z(s)\right)^2 ds\right],$$

and thus $\varphi\left(s,\tilde{\alpha}(s,Z(s)),Z(s)\right)=0$ $ds\times d\mathbb{P}$ a.e. on $[t,T]\times\Omega$. Since the process P (and hence φ) is càdlàg, an argument as in Lemma A.1.1 now shows $\varphi\left(s,\tilde{\alpha}(s,Z(s)),Z(s)\right)=0$ for all $s\in[t,T]$. □

We are now in a position to prove our third main result, Theorem 1.3.5, which provides a necessary and sufficient condition of optimality in terms of buy and sell regions. The proof is now basically a consequence of Propositions 1.6.2 and 1.6.5.

Proof of Theorem 1.3.5. The first assertion is the content of Proposition 1.6.2. To prove the second part, first recall the buy region

$$\mathcal{R}_{\text{buy}} \triangleq \left\{(s,x,z)\in[t,T]\times\mathbb{R}\times\mathbb{R}^n : x<\tilde{\alpha}(s,z)\right\}.$$

Using Proposition 1.6.5 we conclude that for (s,ω) such that $(s,\hat{X}(s-,\omega),Z(s-,\omega))$ is in the buy region we have $P(s,\omega)>0$, recall that $P(s-,\omega)$ and $P(s,\omega)$ are equal $ds\times d\mathbb{P}$ a.e. on $[t,T]\times\Omega$. An application of Corollary 1.5.5(2) shows that in this case $\hat{u}_2(s)>0$ $ds\times d\mathbb{P}$ a.e. The fact that $\hat{u}_1>0$ $ds\times d\mathbb{P}$ a.e. for such (s,ω) is a consequence of the definition of the buy region together with Proposition 1.6.2 . The proof for the sell and no-trade regions is symmetric. □

One particular consequence of Theorem 1.3.5 is the following: If stock holdings are above the cost-adjusted target function, then it is optimal to use market sell orders; if they are below, it is optimal to use buy orders. Only if stock holdings and the cost-adjusted target function agree, no market orders are used. In this sense, the no-trade region consists of only one point and is degenerate. Equivalently, we have

$$\hat{u}_2(s) = 0 \text{ if and only if } P(s) = 0. \tag{1.14}$$

Technically, this is due to the representation

$$\hat{u}_2(s) = \left[g_{u_2}(\cdot, Z(s))\right]^{-1}(P(s))$$

from Corollary 1.5.5 coupled with the strict convexity and smoothness of the cost function $u_2 \mapsto g(u_2, \cdot)$. These assumptions also imply that the marginal costs of trading are zero, i.e. for each $z \in \mathbb{R}^n$ we have

$$\lim_{u_2 \to 0} g_{u_2}(u_2, z) = g_{u_2}(0, z) = 0. \tag{1.15}$$

The following counterexample shows that if the marginal costs of trading are nonzero, the considerations above do not hold. In this example, there is a fixed positive bid ask spread which makes market orders less attractive. We remark that in Chapter 2 we will consider a model with temporary price impact and resilience. In that case, we also have a positive spread and it will turn out that the no-trade regions is then not degenerate (as in the present case).

Example 1.6.6. *Let us illustrate that Theorem 1.3.5 and in particular relation* (1.14) *are no longer true if the assumption of g being $\mathcal{C}^1$ is dropped. We consider the liquidity cost function defined by*

$$g(u_2, z) = c|u_2| + \varepsilon u_2^2$$

for a constant $c > 0$ representing bid ask spread as in Remark 1.2.8. The function $u_2 \mapsto g(u_2, \cdot)$ is not $\mathcal{C}^1$ and the marginal costs of trading are given by

$$\lim_{u_2 \to 0,\, u_2 \neq 0} \left|g_{u_2}(u_2, z)\right| = c > 0.$$

In this case one can still prove a maximum principle and show that the optimal market order is the pointwise minimiser of $u_2 \mapsto g(u_2, Z(s)) - P(s)u_2$, which is now given by

$$\hat{u}_2(s) = \frac{1}{2\varepsilon} \operatorname{sign}(P(s))\left(|P(s)| - c\right)^+.$$

It follows that if $|P(s)| \leq c$ then $\hat{u}_2(s) = 0$ and (1.14) *does not hold.*

We have now established a necessary and sufficient condition of optimality in terms of buy and sell regions, which are defined via the cost-adjusted target function. In order to gain further insight into the structure of these regions, the remainder of this section is devoted to some qualitative properties of the function $\tilde{\alpha}$. Specifically, we show that the map $\alpha \mapsto \tilde{\alpha}$ is translation invariant and preserves orderings and boundedness. Moreover, in the case when α is a deterministic function $\tilde{\alpha}$ coincides with α if and only if α is constant. We first demonstrate that the map $\alpha \mapsto \tilde{\alpha}$ is monotone. This property is natural, a larger target function cannot correspond to a smaller cost-adjusted target function.

Proposition 1.6.7. *If $\alpha(s,z) \geq \beta(s,z)$ for all $(s,z) \in [t,T] \times \mathbb{R}^n$ then we have*

$$\tilde{\alpha}(s,z) \geq \tilde{\beta}(s,z) \text{ for all } (s,z) \in [t,T] \times \mathbb{R}^n.$$

Proof. We prove the claim by contradiction and assume there exists $(t_0, z_0) \in [t,T] \times \mathbb{R}^n$ with $\tilde{\alpha}(t_0,z_0) < \tilde{\beta}(t_0,z_0)$ so that one may choose x_0 with

$$\tilde{\alpha}(t_0,z_0) < x_0 < \tilde{\beta}(t_0,z_0).$$

We denote by $(\hat{X}^\alpha, P^\alpha)$ and $(\hat{X}^\beta, P^\beta)$ the optimal pairs for the problem started at (t_0, x_0, z_0) with targets α and β, respectively.

Observe first that by Proposition 1.6.5 we have $P^\alpha(t_0) < 0 < P^\beta(t_0)$. Define the stopping time

$$\tau_{\alpha,\beta} \triangleq \inf\left\{ s \in [t_0,T] \ : \ P^\alpha(s) \geq P^\beta(s) \right\}$$

as the first time that P^α is larger than or equal to P^β, with the convention $\inf \emptyset = \infty$. To deduce a contradiction we must first establish several properties of the stopping time $\tau_{\alpha,\beta}$.

If τ_1 denotes, as in Lemma 1.4.3, the first jump time after t_0 of the Poisson process N then from Propositions 1.6.2 and 1.6.5 we have $P^\alpha(\tau_1) = P^\beta(\tau_1) = 0$. Thus we conclude

$$\tau_{\alpha,\beta} \leq \tau_1. \tag{1.16}$$

The waiting time until the first jump of N or M is exponentially distributed and since P^α, P^β evolve continuously in the absence of jumps we have

$$\tau_{\alpha,\beta} > t_0. \tag{1.17}$$

We now want to compare the processes $\hat{X}^\alpha$ and $\hat{X}^\beta$ up to the stopping time $\tau_{\alpha,\beta}$. The function $g(\cdot, z)$ is assumed to be smooth and strictly convex for fixed $z \in \mathbb{R}^n$, in addition it has uniform quadratic growth in u_2. This implies that $g_{u_2}(\cdot, z)$ is invertible with a well defined strictly increasing inverse, for all $z \in \mathbb{R}^n$. Thus we deduce that $ds \times d\mathbb{P}$ a.e. on $(t_0, \tau_{\alpha,\beta} \wedge T) \times \Omega$

$$\hat{u}_2^\alpha(s) = [g_{u_2}(\cdot, Z(s))]^{-1}(P^\alpha(s)) \leq [g_{u_2}(\cdot, Z(s))]^{-1}(P^\beta(s)) = \hat{u}_2^\beta(s).$$

Using (1.16) we have that for $s \in [t_0, \tau_{\alpha,\beta})$

$$\hat{X}^{\alpha}(s) = x_0 + \int_{t_0}^{s} \hat{u}_2^{\alpha}(r)dr \leq x_0 + \int_{t_0}^{s} \hat{u}_2^{\beta}(r)dr = \hat{X}^{\beta}(s), \tag{1.18}$$

which shows that

$$\hat{X}^{\alpha}(s) - \alpha(s, Z(s)) \leq \hat{X}^{\beta}(s) - \beta(s, Z(s)). \tag{1.19}$$

Finally, consider the set $\{\tau_{\alpha,\beta} > T\}$. On this set we have $P^{\alpha}(T) < P^{\beta}(T)$, thus using the terminal condition of the BSDE (1.5) we see

$$f'(\hat{X}^{\alpha}(T) - \alpha(T, Z(T))) > f'(\hat{X}^{\beta}(T) - \beta(T, Z(T))).$$

However comparing this with (1.19) and noting $\tau_{\alpha,\beta} > T$ we get a contradiction as f' is strictly increasing. Thus we also have

$$\tau_{\alpha,\beta} \leq T. \tag{1.20}$$

Let us now derive a contradiction. We write using (1.19)

$$\begin{aligned}
\mathbb{E}_{t_0,x_0,z_0}\left[P^{\alpha}(\tau_{\alpha,\beta}) - P^{\alpha}(t_0)\right] &= \mathbb{E}_{t_0,x_0,z_0}\left[\int_{t_0}^{\tau_{\alpha,\beta}} h'(\hat{X}^{\alpha}(s) - \alpha(s, Z(s)))ds\right] \\
&\leq \mathbb{E}_{t_0,x_0,z_0}\left[\int_{t_0}^{\tau_{\alpha,\beta}} h'(\hat{X}^{\beta}(s) - \beta(s, Z(s)))ds\right] \\
&= \mathbb{E}_{t_0,x_0,z_0}\left[P^{\beta}(\tau_{\alpha,\beta}) - P^{\beta}(t_0)\right].
\end{aligned}$$

This implies

$$\mathbb{E}_{t_0,x_0,z_0}\left[P^{\alpha}(\tau_{\alpha,\beta}) - P^{\beta}(\tau_{\alpha,\beta})\right] \leq P^{\alpha}(t_0) - P^{\beta}(t_0) < 0,$$

but $P^{\alpha}(\tau_{\alpha,\beta}) \geq P^{\beta}(\tau_{\alpha,\beta})$ by definition of $\tau_{\alpha,\beta}$, which is the desired contradiction. □

Next, we show that the map $\alpha \mapsto \tilde{\alpha}$ is translation invariant.

Proposition 1.6.8. *For any constant c, if $\beta = \alpha + c$ then $\tilde{\beta} = \tilde{\alpha} + c$.*

Proof. Let us denote by v^{α} and v^{β} the value functions corresponding to the targets α and β, respectively. We use that $u \mapsto X^u$ is affine to deduce

$$\begin{aligned}
v^{\alpha}(t, x, z) = \inf_{u \in \mathcal{U}_t} \mathbb{E}_{t,x,z}\Bigg[\int_t^T & g(u_2, Z(s)) + h(X^u(s) + c - \alpha(s, Z(s)) - c)ds \\
&+ f(X^u(T) + c - \alpha(T, Z(T)) - c)\Bigg] \\
= v^{\beta}(t, x + c, z).&
\end{aligned}$$

Where the final line follows from using translation properties of the expectation together with the definition of v^β. Since the cost-adjusted target is defined to be

$$\tilde{\alpha}(t,z) = \arg\min_{x\in\mathbb{R}} v^\alpha(t,x,z),$$

the result follows. $\square$

In the case that α is independent of z, we can say more about the structure of the cost-adjusted target. Specifically, if the target function is constant (e.g. in the case of portfolio liquidation) then it agrees with the cost-adjusted target function. In the more interesting case of nonconstant target, these two functions are *not* the same.

Proposition 1.6.9. *Let α be independent of z and continuously differentiable in t. Then $\tilde{\alpha} \equiv \alpha$ if and only if α is constant.*

Proof. Suppose $\alpha \equiv c$, a constant. If $x = c$ then the control $u \equiv 0$ yields $J(t,x,z,u) = 0$. Since J is nonnegative we see that $v(t,c,z) = 0$ and that $v(t,x,z) > 0$ for $x \neq c$. This implies that $\tilde{\alpha}(t,z) = \arg\min_x v(t,x,z) = c$. This proves the "if"-part.

We prove the opposite implication by contradiction. Suppose that $\tilde{\alpha} \equiv \alpha$ and α is *not* constant, by continuity there is a global minimum and maximum on $[0,T]$. At least one of them is attained at some $t_0 \in (0,T]$, and we only consider the case that α attains a maximum at t_0 (the case of a minimum is symmetric). Now there is $\delta > 0$ such that α is strictly increasing on $[t_0 - \delta, t_0]$. For the remainder of the proof we assume $s \in [t_0 - \delta, t_0)$.

We denote by $\hat{X}$ the process $\hat{X}^{t_0-\delta,\alpha(t_0-\delta),z}$ started at the point $(t_0 - \delta, \alpha(t_0 - \delta), z)$ and P the corresponding solution to the backward equation. The crucial observation is that stock holdings are never above the target function, i.e.

$$\hat{X}(s) \leq \alpha(s), \tag{1.21}$$

for each $s \in [t_0 - \delta, t_0)$. Indeed, if τ denotes a jump time of the Poisson process N, then by Proposition 1.6.2 and the assumption $\tilde{\alpha} \equiv \alpha$ we have

$$\hat{X}(\tau) = \tilde{\alpha}(\tau, Z(\tau-)) = \alpha(\tau).$$

In particular $\hat{X}$ does not jump above α on the time interval $[t_0 - \delta, t_0)$.

Furthermore, if there is no jump and if we have $\hat{X}(s) = \alpha(s) = \tilde{\alpha}(s)$ then by Theorem 1.3.5 (3) we have $ds \times d\mathbb{P}$ a.e. on $[t_0 - \delta, t_0] \times \Omega$

$$\hat{u}_2(s,\omega) = 0 < \alpha'(s),$$

as α is smooth and strictly increasing on $[t_0 - \delta, t_0)$ by assumption. In other words, if stock holdings are *on* the cost-adjusted target function, no market orders are used and trading stops. The implication is that $\hat{X}$ does not cross α from below and (1.21) holds.

As $\hat{X}$ is never above α (and thus never above $\tilde{\alpha}$) the monotonicity property of P given in Proposition 1.6.5 implies

$$P(s) \geq 0, \tag{1.22}$$

for $s \in [t_0 - \delta, t_0)$. However from the definition of P we have

$$\begin{aligned}&\mathbb{E}_{t_0-\delta,\alpha(t_0-\delta),z}\left[P(s) - P(t_0 - \delta)\right]\\ =&\mathbb{E}_{t_0-\delta,\alpha(t_0-\delta),z}\left[\int_{t_0-\delta}^{s} h'(\hat{X}(r) - \alpha(r))dr\right] \leq 0.\end{aligned}$$

The last inequality follows from noting that $\hat{X}(r) - \alpha(r) \leq 0$ and that h' is increasing and normalised.

Rearranging the above inequality we see that

$$\mathbb{E}_{t_0-\delta,\alpha(t_0-\delta),z}\left[P(s)\right] \leq P(t_0 - \delta). \tag{1.23}$$

In addition we have $P(t_0 - \delta) = 0$, since $\hat{X}$ starts *on* the cost-adjusted target function. Combining (1.22) and (1.23) we now see that $P(s) = 0$ on the whole time interval $[t_0 - \delta, t_0)$.

An application of Corollary 1.5.5 now shows that we have $\hat{u}_2(s) = 0$ $ds \times d\mathbb{P}$ a.e. on $[t_0 - \delta, t_0)$, i.e. no market orders are used. Moreover, from $P(s) = 0$ and Proposition 1.6.5 it follows that

$$\hat{X}(s) = \tilde{\alpha}(s) = \alpha(s), \tag{1.24}$$

a.e. on $[t_0 - \delta, t_0)$ so that $\hat{u}_1(s) = \tilde{\alpha}(s) - \hat{X}(s) = 0$ a.e. and passive orders are also not used. This implies that $\hat{X}$ has paths which are almost surely constant on the interval $[t_0 - \delta, t_0)$.

However by assumption α is strictly increasing on this interval, so (1.24) provides the necessary contradiction. □

As a corollary we show that the cost-adjusted target function can be bounded above (below) by the maximum (resp. minimum) of the target function. This is natural, the cost-adjusted target should not exceed the maximum of the target function.

Corollary 1.6.10. *Let $(s, z) \in [t, T] \times \mathbb{R}^n$. We have the following estimate,*

$$\inf_{t \leq r \leq T} \inf_{y \in \mathbb{R}^n} \alpha(r, y) \leq \tilde{\alpha}(s, z) \leq \sup_{t \leq r \leq T} \sup_{y \in \mathbb{R}^n} \alpha(r, y).$$

Proof. We only prove the first inequality and define

$$c \triangleq \inf_{t \leq r \leq T} \inf_{y \in \mathbb{R}^n} \alpha(r, y).$$

If $c = -\infty$, there is nothing to prove. Since the function α has polynomial growth, we may assume $c \in \mathbb{R}$. From Proposition 1.6.7 we have $\tilde{\alpha}(s, z) \geq \tilde{c}$ for all $(s, z) \in [t, T] \times \mathbb{R}^n$ and $\tilde{c} = c$ from Proposition 1.6.9. □

We now move on to consider some special cases.

1.7. Examples

In this section, we shall show that Theorem 1.3.2 may be used to derive a closed form solution for the optimal control when the penalty and cost functions are quadratic. In this case our problem becomes one of quadratic linear regulator type, which have been well studied in the literature, see Yong and Zhou [1999] Chapter 6 for an overview. The novelty in the present applications is the interpretation of the jumps in terms of passive order execution.

1.7.1. Curve Following with Signal

The following proposition gives an example under which we can find the optimal control (semi)explicitly. Here we have quadratic penalty and liquidity cost functions and a general signal which feeds into the target function.

Proposition 1.7.1. *Let $t = 0$ and $g(u_2, z) = \kappa u_2^2$ for a constant $\kappa > 0$, $h(x) = f(x) = x^2$ and α be any continuous function satisfying Assumption 1.2.5. Suppose that the signal is defined by*

$$dZ(s) = \mu(s, Z(s))ds + \sigma(s, Z(s))dW(s), \quad Z(0) = z.$$

where μ and σ are bounded continuous real-valued functions with $\sigma \geq \delta > 0$ for some constant δ. Suppose in addition that μ, σ, α are Hölder continuous for some exponent less than 1, uniformly in t. Then the optimal control is given $ds \times d\mathbb{P}$ a.e by

$$\begin{cases} \hat{u}_1(s, \hat{X}(s-), Z(s)) & = -\dfrac{b(s, Z(s))}{a(s)} - \hat{X}(s-), \\ \hat{u}_2(s, \hat{X}(s), Z(s)) & = -\dfrac{a(s)}{2\kappa}\left(-\dfrac{b(s, Z(s))}{a(s)} - \hat{X}(s)\right), \end{cases} \tag{1.25}$$

where the functions a and b satisfy linear PDEs and can be given in (semi)explicit form.

Proof. We first note that one can check by direct computation that the functions

$$g(u_2, \cdot) = \kappa u_2^2 \text{ and } f(x) = h(x) = x^2$$

satisfy Assumptions 1.2.5 and 1.2.7. We know from Theorem 1.3.2 that $ds \times d\mathbb{P}$ a.e. on $[0, T] \times \Omega$, the optimal market order is given by the pointwise minimiser of

$$u_2 \mapsto \kappa u_2^2 - P(s)u_2.$$

Thus we compute that $u_2(s) = (2\kappa)^{-1}P(s)$. This leads to the following coupled FBSDE,

$$\begin{cases} d\hat{X}(s) &= (2\kappa)^{-1}P(s)ds + \hat{u}_1(s)N(ds), \\ dP(s) &= 2\big(\hat{X}(s) - \alpha(s, Z(s))\big)ds + Q(s)dW(s) + R_1(s)\tilde{N}(ds), \\ \hat{X}(0) &= x,\ P(T) = -2\big(\hat{X}(T) - \alpha(T, Z(T))\big). \end{cases} \tag{1.26}$$

We have omitted the process Z in the above as it may be solved independently of $(P, \hat{X})$ and then regarded as an input.

It follows from Theorem 1.3.2 that any solution of the above for which $P(s-)+R_1(s) = 0$, $ds \times d\mathbb{P}$ a.e. on $[0, T] \times \Omega$ provides the optimal control. Moreover, since the coupled FBSDE is in one to one correspondence with the optimal control (again by Theorem 1.3.2) there is at most one solution. We make the ansatz

$$P(s) = a\big(s, Z(s)\big)\hat{X}(s) + b\big(s, Z(s)\big), \tag{1.27}$$

for deterministic functions a and b to be determined.

A consequence of the above ansatz is that the jumps of P are equal to a times the jumps of $\hat{X}$. In particular we know that $\hat{u}_1$ should ensure

$$P(s-) + R_1(s) = a\big(s, Z(s)\big)[\hat{X}(s-) + \hat{u}_1(s)] + b\big(s, Z(s)\big) = 0. \tag{1.28}$$

This gives the following form for the passive order,

$$\hat{u}_1\big(s, \hat{X}(s-), Z(s)\big) = -\frac{b\big(s, Z(s)\big)}{a\big(s, Z(s)\big)} - \hat{X}(s-).$$

Inserting this into (1.26) leads to the FBSDE which we hope to solve by the above ansatz.

We proceed similarly to the four step scheme of Ma et al. [1994], applying Itô's formula to the expression for P from equation (1.27) and using the definition of the optimal controls we derive that

$$\begin{aligned} dP =& \hat{X}\left(a_s + \mu a_z + \frac{1}{2}\sigma^2 a_{zz}\right)ds + \sigma\left(\hat{X}a_z + b_z\right)dW(s) - \left(\frac{b}{a} + \hat{X}\right)\tilde{N}(ds) \\ &+ \left(\frac{1}{2\kappa}a\left[a\hat{X} + b\right] - \lambda b - \lambda a\hat{X}\right)ds \\ &+ \left(b_s + \mu b_z + \frac{1}{2}\sigma^2 b_{zz}\right)ds, \end{aligned}$$

where we suppress the arguments (s, z) of a, b, μ, σ and $\hat{X}$ for brevity. These dynamics must coincide with those of equation (1.26) so that matching the coefficients we derive that the functions a and b should solve the following partial differential equations on

$[0, T] \times \mathbb{R}$,

$$a_s + \mu a_z + \frac{1}{2}\sigma^2 a_{zz} - \lambda a + \frac{1}{2\kappa}a^2 - 2 = 0, \quad a(T, z) = -2, \tag{1.29}$$

$$b_s + \mu b_z + \frac{1}{2}\sigma^2 b_{zz} - \lambda b + \frac{1}{2\kappa}ab + 2\alpha = 0, \quad b(T, z) = 2\alpha(T, z), \tag{1.30}$$

where we suppress the (s, z) for notational simplicity. The equation (1.29) for a can be solved independently of z as a standard Riccati equation,

$$a(s) = \kappa \left(\lambda + \zeta - \frac{2\zeta}{1 - c_a e^{\zeta s}} \right), \tag{1.31}$$

where we define $\zeta \triangleq \sqrt{\lambda^2 + 4/\kappa}$ and choose c_a such that the boundary condition at T is satisfied. A calculation shows that $a(s) < 0$ for all $s \in [0, T]$, so the terms in (1.25) are well defined. The PDE (1.30) for b then becomes a heat equation with a source term and bounded Hölder continuous coefficients for which Friedman [1964] Theorem 1.7.12 gives the existence of a solution as well as a semi-explicit formula in terms of Green's functions. Using the Feynman-Kac formula we can give a more explicit probabilistic solution which better displays its relation with the original problem. Namely we have

$$b(s, z) = 2\mathbb{E}_{s,z} \left[\int_s^T \exp\left(\int_s^r \rho(w)dw \right) \alpha(r, Z(r))dr + \exp\left(\int_s^T \rho(w)dw \right) \alpha(T, Z(T)) \right],$$

where we use the notation $\mathbb{E}_{s,z}[\cdot]$ as in Lemma 1.4.2 and the function $\rho(s) \triangleq \frac{a(s)}{2\kappa} - \lambda$ for $s \in [0, T]$. □

Remark 1.7.2. In the case of quadratic cost functions there is a clear economic interpretation of the controls. The function b encodes the expected future motion, appropriately discounted by λ and $\frac{a}{2\kappa}$, of the target function with respect to the distribution of the signal Z. This then feeds into the cost-adjusted target function via the ratio $\frac{b}{a}$. We see that this is forward looking, evolves in time and reacts to changes in Z. A passive order is placed and continuously adjusted so that when a jump occurs the stock holdings move to the cost-adjusted target function. Simultaneously market orders are used with a rate $\frac{-a(s)}{2\kappa}$ proportional to the amount held in the passive order.

Such a control structure is intuitive, for fixed (s, z) trading slows down as one approaches the cost-adjusted target $\tilde{\alpha}$ and speeds up as one moves away. Put another way, when the agent has stock holdings near the function $\tilde{\alpha}$ he reduces the trading in market orders, preferring to wait for passive order execution. The parameter κ describes how expensive trading is, when it is large the agent uses less market orders and relies more on passive orders. This again coincides with trading strategies seen in markets with low liquidity.

We point out a final point on the nature of the solution. We have chosen the functions $f(x) = h(x) = \eta x^2$, with the value $\eta = 1$. For general η one can simply scale the value function by η^{-1} to reduce to the current setting, with parameter $\kappa' = \frac{\kappa}{\eta}$. Now we can

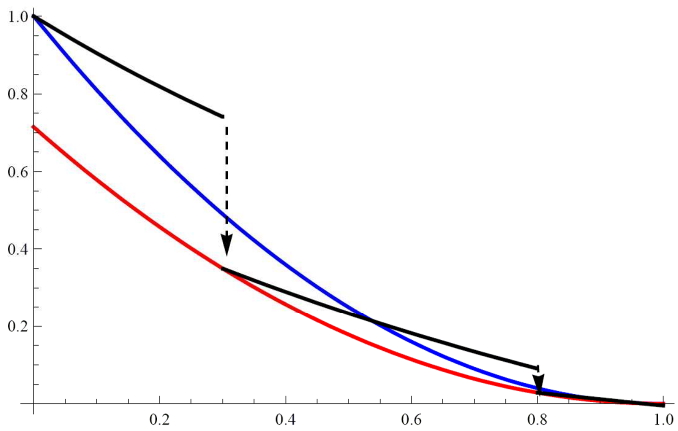


Figure 1.1.: Target function (blue), cost-adjusted target (red) and a typical trajectory of stock holdings (black).

interpret η as urgency parameter, a key feature present in all modern trading algorithms, this controls how close one should adhere to the target and thus decides the allocation between market and passive orders.

Figure 1.1 illustrates the relation of target, cost-adjusted target and a typical trajectory of stock holdings.

1.7.2. Portfolio Liquidation

We consider an investor who wants to sell x stock shares over the interval $[0, T]$, however trading incurs liquidity costs. The question then becomes what the optimal strategy should be. The construction of such a trading program has received much attention recently, see for instance Almgren and Chriss [2001], Obizhaeva and Wang [2005] as well as Schied and Schöneborn [2008]. With a little extra work one can embed this in the present setup. When we assume, as in Almgren [2003], that market orders incur quadratic trading costs and there is a quadratic penalty on stock holdings we are led to the following formulation,

$$\begin{aligned} v(0,x) &\triangleq \inf_{u \in \mathcal{U}_0, X^u(T)=0} J(0,x,u), \\ J(0,x,u) &\triangleq \mathbb{E}_{0,x}\left[\int_0^T \kappa |u_2(s)|^2 + |X^u(s)|^2 ds\right], \end{aligned}$$

where $\kappa > 0$ and we have omitted a signal for ease of exposition. Observe the new feature in the present optimisation problem is that we now have a binding constraint, we are interested in only those controls for which $X^u(T) = 0$.

Proposition 1.7.3. *The optimal control is given $ds \times d\mathbb{P}$ a.e. by*

$$\begin{cases} \hat{u}_1(s, \hat{X}(s-)) &= -\hat{X}(s-), \\ \hat{u}_2(s, \hat{X}(s)) &= \dfrac{a(s)}{2\kappa}\hat{X}(s), \end{cases}$$

where a is given by

$$a(s) = \kappa\left(\lambda + \zeta - \frac{2\zeta}{1 - e^{\zeta(s-T)}}\right), \tag{1.32}$$

with $\zeta \triangleq \sqrt{\lambda^2 + 4/\kappa}$.

Proof. Let $n \in \mathbb{N}$ and define the sequence of approximate optimisation problems without binding constraint, but with an additional penalty term at time T, by

$$\begin{aligned} v^n(0,x) &\triangleq \inf_{u \in \mathcal{U}_0} J^n(0,x,u), \\ J^n(0,x,u) &\triangleq \mathbb{E}_{0,x}\left[\int_0^T \kappa|u_2(s)|^2 + |X^u(s)|^2 ds + n|X^u(T)|^2\right]. \end{aligned}$$

Applying the methods of the previous subsection we see that the optimal control corresponding to v^n is given $ds \times d\mathbb{P}$ a.e. by

$$\begin{cases} \hat{u}_1^n(s, \hat{X}^n(s-)) &= -\hat{X}^n(s-), \\ \hat{u}_2^n(s, \hat{X}^n(s)) &= \dfrac{a^n(s)}{2\kappa}\hat{X}^n(s), \end{cases}$$

where the a^n are defined by

$$a^n(s) \triangleq \kappa\left(\lambda + \zeta - \frac{2\zeta}{1 - c_n e^{\zeta(s-T)}}\right),$$

with the constant c_n chosen so that $a^n(T) = -2n$. A calculation shows that the functions a^n converge towards a and hence the optimal controls $\hat{u}^n$ converge to $\hat{u}$, for a and $\hat{u}$ given in the statement of the proposition. We may rewrite the performance functional associated to the portfolio liquidation problem as

$$J(0,x,u) = \mathbb{E}_{0,x}\left[\int_0^T \kappa|u_2(s)|^2 + |X^u(s)|^2 ds + \delta_{\{\mathbb{R}\setminus\{0\}\}}(X^u(T))\right],$$

where $\delta_{\{\mathbb{R}\setminus\{0\}\}}$ is the indicator function in the sense of convex analysis. This leads to the following inequality,

$$J^n(0,x,u) \leq J(0,x,u) \text{ for all } x \in \mathbb{R}, u \in \mathcal{U}_0. \tag{1.33}$$

Next we show that the stock holdings associated to $\hat{u} = \lim_{n\to\infty} u^n$ satisfy the liquidation contraint $\hat{X}(T) = 0$. Its dynamics are given by

$$d\hat{X}(s) = \frac{a(s)}{2\kappa}\hat{X}(s)ds - \hat{X}(s-)N(ds), \quad \hat{X}(0) = x.$$

If N jumps in $[0, T]$ we clearly end up with zero stock holdings, thus we are reduced to showing that

$$\hat{X}(T) = x \exp\left(\frac{1}{2\kappa}\int_0^T a(s)ds\right)\mathbb{1}_{T\leq\tau_1} = 0,$$

where τ_1 is again the first jump time of N. Using the explicit formula for a from (1.32), this equality can be verified.

Before we prove the optimality of $\hat{u}$, we need to show that the integrand of the performance functional J^n converges. Indeed, we have $ds \times d\mathbb{P}$ a.e.

$$\lim_{n\to\infty}\{\kappa|\hat{u}_2^n(s)|^2 + |X^{\hat{u}^n}(s)|^2\} = \kappa|\hat{u}_2(s)|^2 + |\hat{X}(s)|^2.$$

A computation shows that the stock holdings associated to $\hat{u}^n$ are given by

$$\begin{aligned} X^{\hat{u}^n}(s) =& x\exp\left(\int_0^s \hat{u}_2(r)dr\right)\mathbb{1}_{\{s<\tau_1\}} = x\exp\left(\int_0^s \frac{1}{2\kappa}a^n(r)dr\right)\mathbb{1}_{\{s<\tau_1\}} \\ =& x\exp\left(\frac{1}{2}(\lambda-\xi)s\right)\frac{1-c_n\exp(\xi(s-T))}{1-c_n\exp(-\xi T)}\mathbb{1}_{\{s<\tau_1\}}, \end{aligned}$$

and a further computation shows that the terminal penalty satisfies

$$\lim_{n\to\infty}\{n|X^{\hat{u}^n}(T)|^2\} = 0.$$

Now an application of Fatou's Lemma together with (1.33) yields for any $u \in \mathcal{U}_0$

$$\begin{aligned} J(0, x, \hat{u}) =& \mathbb{E}_{0,x}\left[\int_0^T \kappa|\hat{u}_2(s)|^2 + |\hat{X}(s)|^2 ds\right] \\ \leq& \liminf_{n\to\infty} \mathbb{E}_{0,x}\left[\int_0^T \kappa|\hat{u}_2^n(s)|^2 + |X^{\hat{u}^n}(s)|^2 ds + n|X^{\hat{u}^n}(T)|^2\right] \\ =& \liminf_{n\to\infty} J^n(0, x, \hat{u}^n) \leq \liminf_{n\to\infty} J^n(0, x, u) \leq J(0, x, u). \end{aligned}$$

This shows that $\hat{u}$ is optimal and completes the proof. □

Remark 1.7.4. The optimal strategy is to simultaneously place all outstanding shares as a passive order and sell using market orders with the rate $\frac{1}{2\kappa}|a(s)|$, proportional to current stock holdings. A similar result is obtained in Kratz [2011].

Remark 1.7.5. Our main results, Theorems 1.3.2 and 1.3.5, can be used to obtain a solution to the portfolio liquidation problem under quite weak conditions, e.g. for general cost and penalty functions and a general stochastic signal Z. The process Z could represent e.g. the order book height, thereby extending Almgren [2003] to a setting with stochastic liquidity parameters as well as passive orders. The example discussed in Proposition 1.7.3 provides an *explicit* solution and therefore needs strong assumptions (constant parameters, no spread, quadratic costs, quadratic penalty).

1.8. Bid-Ask Spread and the Independence of the Jump Processes

One choice for the stochastic signal Z would be bid ask spread. However, in our model a jump of N represents a liquidity event which executes the investor's passive order. In real markets a liquidity event which executes passive orders might also temporarily widen the bid ask spread on one side of the book. This is in contrast with our requirement that Z and N be independent. In the present section we shall show that when one relaxes this assumption an interesting feature occurs, it might be optimal to place passive sell and market buy orders at the same time. From a practitioner's point of view, trading on different sides of the market simultaneously is not desirable and we discuss this further in the present section. To the best of our knowledge, this problem has not been addressed in the literature on illiquid markets, since most papers either only allow for one type of orders or only consider portfolio liquidation where it is clear *a priori* that only sell orders and no buy orders are used.

We now drop the assumption of Z and N being independent and suppose that the dynamics of Z introduced in (1.2) are replaced by

$$\begin{aligned} dZ(s) &= \mu(s, Z(s))ds + \sigma(s, Z(s))dW(s) \\ &\quad + \int_{\mathbb{R}^k} \gamma\left(s, Z(s-), \theta\right) \tilde{M}(ds, d\theta) + \delta\left(s, Z(s-)\right) N(ds), \end{aligned}$$

for some function $\delta : [t, T] \times \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz in z, uniformly in s. Using an identical proof to that of Theorem 1.3.1 one can show existence and uniqueness of a solution as well as the characterisation of Theorem 1.3.2(2). In this section we present examples to show that Theorem 1.3.5(2) is no longer valid, in particular it can be optimal to place both market buy (sell) and passive sell (buy) orders simultaneously.

When one interprets the process $\hat{u}_1$ as an order placed in a dark pool, the above behaviour amounts to selling in the dark venue and buying in the visible venue (or vice versa). This phenomenon arises because when the passive order is executed, the signal jumps as well and the passive order "foresees" this, whereas the market order does not so that they may have different signs.

However when $\hat{u}_1$ is interpreted as a limit order this behaviour is equivalent to the investor placing liquidity on the buy (sell) side and then consuming it themselves. This is rather counterintuitive and is not economically rational or realistic, thus we conclude from our examples that in the general setting it is necessary to retain the assumption of independence between N and Z. Hence for arbitrary target functions Z may not be

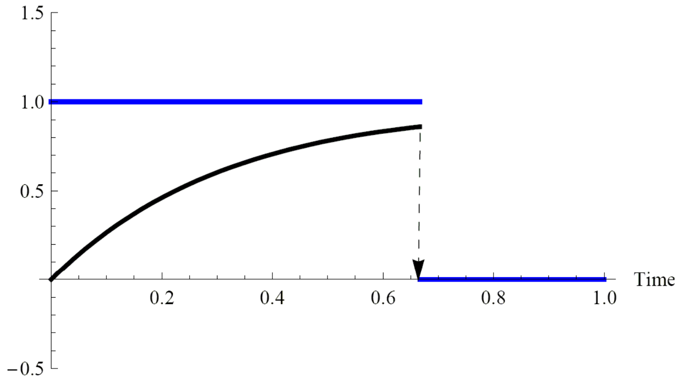


Figure 1.2.: Target function (blue) and stock holdings (black) for Example 1.8.1. Before a jump, market buy orders are used to reduce deviation. At the same time, a passive sell order is placed such that stock holdings and signal both jump to zero in the case of execution.

interpretable as spread. However by imposing specific conditions on α we show that, even when independence does not hold, the optimal control does not exhibit the undesirable behaviour described above. Thus in certain circumstances an interpretation of Z as spread is compatible with the notion of $\hat{u}_1$ as a limit order.

Our first example shows that if the signal enters the target but not the cost function then one may simultaneously buy and sell with positive probability.

Example 1.8.1. *Suppose the target function is given by $\alpha(t,z) = z$ and the stochastic signal satisfies*

$$dZ(s) = -Z(s-)N(ds), \quad Z(0-) = z > 0.$$

The signal is piecewise constant, before a jump of the process N it takes the value z, afterwards it takes the value 0 (and stays there). We assume as above that N is a Poisson process with intensity $\lambda > 0$. Let the cost and penalty functions be given by $g(u_2, z) = \kappa u_2^2$ for some $\kappa > 0$ and $h(y) = y^2$. The investor wants to minimise the deviation of stock holdings from the target function, where his stock holdings satisfy as above

$$dX^u(s) = u_2(s)ds + u_1(s)N(ds), \quad X^u(0-) = x \in (0, z).$$

The performance functional is now defined as

$$J(t,x,z,u) \triangleq \mathbb{E}_{t,x,z}\left[\int_t^T \kappa u_2(s)^2 + (X^u(s) - Z(s))^2\, ds\right].$$

Applying the same ideas as in the proof of Proposition 1.7.1, we first note that equation (1.28) *turns into*

$$P(s-) + R_1(s) = a(s)[\hat{X}(s-) + \hat{u}_1(s)] + b(s,0) = 0,$$

where the coefficient b is evaluated at 0 since the signal jumps to 0 in case of a jump. In particular the new feature for the passive order is that the signal Z is not evaluated before, *but* after *the jump of N. Intuitively, a jump of N executes the passive order and lets the signal jump to zero. The optimal control can then be computed as above as*

$$\begin{cases} \hat{u}_1(s, \hat{X}(s-), Z(s-)) & = -\dfrac{b(s,0)}{a(s)} - \hat{X}(s-), \\ \hat{u}_2(s, \hat{X}(s-), Z(s-)) & = -\dfrac{a(s)}{2\kappa}\left(-\dfrac{b(s,Z(s-))}{a(s)} - \hat{X}(s-)\right). \end{cases} \tag{1.34}$$

The economic interpretation of the optimal controls is identical with that of Remark 1.7.2 save for where the cost-adjusted target function is evaluated after the jump time of N. As above, the coefficients a and b are given by the following differential equations

$$\begin{cases} a_s - \lambda a + \dfrac{1}{2\kappa}a^2 - 2 & = 0, \quad a(T) = 0, \\ b_s - \lambda b + \dfrac{1}{2\kappa}ab + 2z & = 0, \quad b(T,z) = 0. \end{cases}$$

The explicit solutions are given by

$$\begin{aligned} a(s) =& \kappa\left(\lambda + \zeta - \frac{2\zeta}{1 - c_a e^{\zeta s}}\right), \\ b(s,z) =& -a(s)z = -\kappa z\left(\lambda + \zeta - \frac{2\zeta}{1 - c_a e^{\zeta s}}\right), \end{aligned}$$

where the integration constant c_a is chosen such that the terminal condition $a(T) = 0$ is met and we set $\zeta \triangleq \sqrt{\lambda^2 + 4/\kappa}$. It can be verified that a is strictly negative on $[0,T)$. It now follows that the cost-adjusted target function is given by, for $s \in [0,T)$,

$$\tilde{\alpha}(s,z) = -\frac{b(s,z)}{a(s)} = z. \tag{1.35}$$

In particular, the cost-adjusted target function agrees with the target function. Combining (1.34) *and* (1.35) *we see that the optimal control is*

$$\begin{cases} \hat{u}_1(s, \hat{X}(s-), Z(s-)) & = -\hat{X}(s-), \\ \hat{u}_2(s, \hat{X}(s-), Z(s-)) & = -\dfrac{a(s)}{2\kappa}\left(Z(s-) - \hat{X}(s-)\right). \end{cases}$$

Recall that $\hat{X}(0-) = x < z = Z(0-)$ and that $a < 0$. Before *a jump of N, the dynamics* *of* X^u *are now given by*

$$d\hat{X}(s) = -\frac{a(s)}{2\kappa}\left(z - \hat{X}(s-)\right) ds,$$

which is a mean reverting process with mean z*. It grows monotonically towards* z *without reaching it. The fact that* $\hat{X} < z$ *a.e. before the first jump of* N *implies that* $\hat{u}_2 > 0$*, i.e. market buy orders are used. If a jump of* N *occurs, the cost-adjusted target function jumps to zero. At the same time, the passive order is executed and all stocks are sold. To sum up, we have* $ds \times d\mathbb{P}$ *a.e. before a jump*

$$\hat{u}_1 < 0 \text{ and } \hat{u}_2 > 0.$$

Figure 1.2 illustrates the dynamics of the optimal trajectories.

The preceding example illustrates that it might be optimal to use market buy and passive sell orders at the same time. In this example, the signal only influences the target function, but not the liquidity costs. One might conjecture that such counterintuitive behaviour can be excluded in a model where the signal affects the liquidity costs, but not the target function. Unfortunately, this is wrong, as illustrated by the following example. In addition it provides an explicit solution to a curve following problem with a regime shift in liquidity.

Example 1.8.2. *This is a stylised model of liquidity breakdowns. There are times with high liquidity (low liquidity costs) and with sparse liquidity (high liquidity costs). This regime shift might be triggered by a news event or a very large trade.*

Consider an investor who wants to keep his stock holdings close to a deterministic function α*. The optimisation problem is*

$$\inf_{u\in\mathcal{U}_0} \mathbb{E}_{0,x,z}\left[\int_0^T Z(s)u_2(s)^2 + (\alpha(s) - X^u(s))^2\, ds\right].$$

Here Z *is a liquidity parameter, we might think of the inverse order book height. The higher* Z*, the more expensive market orders are. We assume that* Z *can only take two values. In the first stage (before a jump of the Poisson process* N*),* Z *equals* $\kappa_1 > 0$*. In the second stage (after the first jump of* N*),* Z *takes the value* $\kappa_2 > 0$ *and remains there until maturity. This assumption is made for simplicity. It allows to solve the optimisation problem in two stages, corresponding to the two possible values of* Z*. The dynamics of* Z *are then given by*

$$dZ(s) = (\kappa_2 - Z(s-))\, N(ds), \quad Z(0-) = \kappa_1.$$

As above, we model the investor's stock holdings by

$$dX^u(s) = u_2(s)ds + u_1(s)N(ds), \quad X^u(0) = x.$$

We will solve the above optimisation problem in two stages, before and after a jump *of the Poisson process N. In the second stage, the signal is constant, $Z \equiv \kappa_2$. The performance functional and the value function are defined by*

$$\bar{J}(t,x,u) \triangleq \mathbb{E}_{t,x}\left[\int_t^T \kappa_2 u_2(s)^2 + (\alpha(s) - X^u(s))^2\, ds\right],$$
$$\bar{v}(t,x) \triangleq \inf_{u\in\mathcal{U}_t} \bar{J}(t,x,u).$$

The HJB equation associated to this optimisation problem is

$$0 = \inf_{u\in\mathbb{R}^2}\left\{\bar{v}_s + u_2\bar{v}_x + \kappa_2 u_2^2 + (\alpha - x)^2 + \lambda\left[\bar{v}(t,x+u_1) - \bar{v}(t,x)\right]\right\},$$

with terminal condition $\bar{v}(T,x) = 0$. We try the following quadratic ansatz for $\bar{v}$:

$$\bar{v}(t,x) = \frac{1}{2}\bar{a}(t)x^2 + \bar{b}(t)x + \bar{c}(t),$$

for coefficients $\bar{a}, \bar{b}, \bar{c} : [0,T] \to \mathbb{R}$. The optimal control as well as the cost-adjusted target function in the second stage then satisfy

$$\begin{cases} \hat{u}_1(s, \hat{X}(s-)) & = \tilde{\alpha}_2(s) - \hat{X}(s-), \\ \hat{u}_2(s, \hat{X}(s-)) & = \dfrac{\bar{a}(s)}{2\kappa_2}\Big(\tilde{\alpha}_2(s) - \hat{X}(s-)\Big), \\ \tilde{\alpha}_2(s) & = -\dfrac{\bar{b}(s)}{\bar{a}(s)}, \end{cases}$$

where the coefficients $\bar{a}, \bar{b}, \bar{c}$ are given as the solution to the following system of Riccati equations

$$\begin{cases} \bar{a}_s - \dfrac{1}{2\kappa_2}\bar{a}^2 + 2 - \lambda\bar{a} & = 0, \quad \bar{a}(T) = 0, \\ \bar{b}_s - \dfrac{1}{2\kappa_2}\bar{a}\bar{b} - 2\alpha - \lambda\bar{b} & = 0, \quad \bar{b}(T) = 0, \\ \bar{c}_s - \dfrac{1}{4\kappa_2}\bar{b}^2 + \alpha^2 - \lambda\frac{\bar{b}^2}{2\bar{a}} & = 0, \quad \bar{c}(T) = 0. \end{cases}$$

The closed form solution to this system of ODEs is given by

$$\bar{a}(s) = \kappa_2\left(-\lambda - \zeta + \frac{2\zeta}{1 - c_{\bar{a}}e^{\zeta s}}\right),$$
$$\bar{b}(s) = -\int_s^T 2\alpha(r)\exp\left(-\frac{1}{2\kappa_2}\int_s^r \bar{a}(w)dw - \lambda(r-s)\right)dr,$$
$$\bar{c}(s) = \int_s^T \Big(-\frac{1}{4\kappa_2}\bar{b}^2(r) + \alpha^2(r) - \lambda\frac{\bar{b}^2(r)}{2\bar{a}(r)}\Big)dr,$$

where the integration constant $c_{\bar{a}}$ *is chosen such that the terminal conditions* $\bar{a}(T) = 0$ *is satisfied and we defined* $\zeta \triangleq \sqrt{\lambda^2 + 4/\kappa_2}$. *Now that we have solved the optimisation problem in the second stage, let us consider the first stage (before the first jump of* N*). We then have* $Z \equiv \kappa_1$. *Let* τ_1 *denote the first jump time of* N *and* $\tau \triangleq \tau_1 \wedge T$. *The optimisation problem under consideration is now*

$$\inf_{u\in\mathcal{U}_0} \mathbb{E}_{0,x}\left[\int_0^\tau \left[\kappa_1 u_2(s)^2 + (\alpha(s) - X^u(s))^2\right] ds + \bar{v}\left(\tau, X^u(\tau-) + \hat{u}_1(\tau)\right)\right].$$

By definition, τ_1 *is exponentially distributed with density* $\phi(s) \triangleq \lambda e^{-\lambda s}$. *We proceed as in Pham [2009] Section 3.6.2 and rewrite the optimisation problem as*

$$\inf_{u\in\mathcal{U}_0} \mathbb{E}_{0,x}\Bigg[\int_0^T \lambda e^{-\lambda s}\left(\int_0^s \left[\kappa_1 u_2(r)^2 + (\alpha(r) - X^u(r))^2\right] dr + \bar{v}\left(s, X^u(s-) + \hat{u}_1(s)\right)\right) ds$$
$$+ \int_T^\infty \lambda e^{-\lambda s}\left(\int_0^T \left[\kappa_1 u_2(r)^2 + (\alpha(r) - X^u(r))^2\right] dr + \bar{v}\left(T, X^u(T)\right)\right) ds\Bigg].$$

Changing the order of integration and combining with $\bar{v}(T,x) = 0$ *and*

$$\int_r^\infty \lambda e^{-\lambda s} ds = e^{-\lambda r} \triangleq \beta(r)$$

leads to

$$\inf_{u\in\mathcal{U}_0} \mathbb{E}_{0,x}\Bigg[\int_0^T \Big(e^{-\lambda r}\left[\kappa_1 u_2(r)^2 + (\alpha(r) - X^u(r))^2\right]$$
$$+ \lambda e^{-\lambda r}\bar{v}\left(r, X^u(r-) + \hat{u}_1(r)\right)\Big) dr\Bigg]$$
$$= \inf_{u\in\mathcal{U}_0} \mathbb{E}_{0,x}\Bigg[\int_0^T \Big(\beta(r)\left[\kappa_1 u_2(r)^2 + (\alpha(r) - X^u(r))^2\right] + \phi(r)\bar{v}\left(r, X^u(r-) + \hat{u}_1(r)\right)\Big) dr\Bigg].$$

The HJB equation associated to this optimisation problem is

$$0 = \inf_{u\in\mathbb{R}^2}\Big\{v_s + u_2 v_x + \beta\kappa_1 u_2^2 + \beta(\alpha - x)^2 + \phi(s)\bar{v}(s, x + u_1)\Big\},$$

with terminal condition $v(T,x) = 0$. *The pointwise minimisers are*

$$\hat{u}_1 = \arg\min_{x\in\mathbb{R}} \bar{v}(s,x) - x = -\frac{\bar{b}(s)}{\bar{a}(s)} - x,$$
$$\hat{u}_2 = -\frac{1}{2\beta(s)\kappa_1} v_x(s,x),$$

and we remark that for $s \in [0,T)$ $\bar{a}(s)$ *is strictly positive, so that the above expressions* *are well defined. The HJB equation in the first stage turns into*

$$0 = v_s - \frac{1}{4\beta(s)\kappa_1} v_x^2 + \beta(s)(\alpha - x)^2 + \phi(s)\left(-\frac{\bar{b}(s)^2}{2\bar{a}(s)} + \bar{c}(s)\right).$$

As above, we try a quadratic ansatz for v:

$$v(t,x) = \frac{1}{2}a(t)x^2 + b(t)x + c(t),$$

for coefficients $a, b, c : [0,T] \to \mathbb{R}$. The optimal control as well as the cost-adjusted target function in the first stage then satisfy

$$\begin{cases} \hat{u}_1(s, \hat{X}(s-)) & = \tilde{\alpha}_2(s) - \hat{X}(s-) \\ \hat{u}_2(s, \hat{X}(s-)) & = \dfrac{a(s)}{2\beta(s)\kappa_1}\left(\tilde{\alpha}_1(s) - \hat{X}(s-)\right), \\ \tilde{\alpha}_1(s) & = -\dfrac{b(s)}{a(s)}, \end{cases} \tag{1.36}$$

where the coefficients a, b, c are given by

$$\begin{cases} a_s - \dfrac{1}{2\beta(s)\kappa_1}a^2 + 2\beta(s) & = 0, \quad a(T) = 0, \\ b_s - \dfrac{1}{2\beta(s)\kappa_1}ab - 2\beta(s)\alpha(s) & = 0, \quad b(T) = 0, \\ c_s - \dfrac{1}{4\beta(s)\kappa_1}b^2 + \beta(s)\alpha(s)^2 + \phi(s)\left(-\frac{\bar{b}^2(s)}{2\bar{a}(s)} + \bar{c}(s)\right) & = 0, \quad c(T) = 0. \end{cases}$$

This system admits the following solution

$$\begin{aligned} a(s) &= \beta(s)\kappa_1\left(-\lambda - \zeta + \frac{2\zeta}{1 - c_a e^{\zeta s}}\right), \\ b(s) &= -\int_s^T 2\alpha(r)2\beta(r)\exp\left(-\frac{1}{2\kappa_1}\int_s^r \frac{a(w)}{\beta(w)}dw\right)dr, \\ c(s) &= \int_s^T \left(-\frac{1}{4\beta(r)\kappa_1}b(r)^2 + \beta(r)\alpha(r)^2 + \lambda\beta(r)\left(-\frac{\bar{b}(r)^2}{2\bar{a}(r)} + \bar{c}(r)\right)\right)dr, \end{aligned}$$

where the integration constant c_a is chosen such that the terminal condition $a(T) = 0$ is met and we set $\zeta \triangleq \sqrt{\lambda^2 + 4/\kappa_1}$.

To conclude, we see from (1.36) *that in the first stage the optimal market order is directed towards the cost-adjusted target function of the* first *stage, but the optimal passive order is directed towards the cost-adjusted target function of the* second *stage. The reason for this asymmetric behaviour is that the passive order takes into account the jump of N, i.e. it "foresees" the regime shift from the first to the second stage. A consequence is that (in general) there are regions where market buy and passive sell orders are used,*

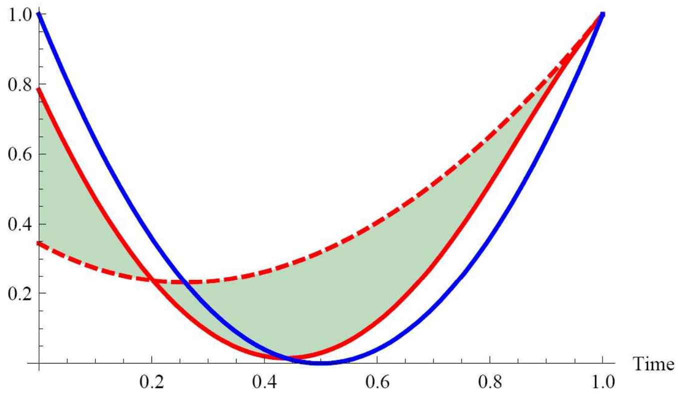


Figure 1.3.: Cost-adjusted target functions for Example 1.8.2 before (red) and after (dashed red) a jump of the signal. The blue curve is the target function. Before a jump, market orders are used to keep the stock holdings close to the red curve. Simultaneously, passive orders are directed towards the dashed red curve. In the shaded region, market sell and passive buy orders are used, or vice versa.

and vice versa.

To give a concrete example we simulated these functions for the target function $\alpha(s) = (2s-1)^2$, *intensity* $\lambda = 10$, *maturity* $T = 1$ *and liquidity parameters* $\kappa_1 = 0.01$ *and* $\kappa_2 = 1$, *see Figure 1.3.*

Examples 1.8.1 and 1.8.2 show that counterintuitive trading might occur if the signal enters the cost, but not the penalty function, and vice versa. The reader might conjecture that a sufficient condition to exclude this is that the target function is deterministic and strictly decreasing and stock holdings start above the target function. Heuristically, the trajectory of stock holdings should then also be decreasing, so that only sell orders are used. The following counterexample shows that this conjecture is wrong.

Example 1.8.3. *We remain in the framework of Example 1.8.2 and choose the target function* $\alpha(s) = \tanh(100(0.5 - s)) + 1$. *Then* α *is deterministic and strictly decreasing. Furthermore, we choose the maturity* $T = 1$, *intensity* $\lambda = 10$ *and liquidity cost parameters* $\kappa_1 = 10^{-2}$ *(in the first stage) and* $\kappa_2 = 10^{-4}$ *(in the second stage). The stock holdings are assumed to be above the target function initially, i.e.* $\hat{X}(0) = \alpha(0) + 0.1$.

Figure 1.4 shows a plot of the target function (blue) as well as the cost-adjusted target functions in the first (red) and second (dashed red) stage. The optimal trajectory of stock holdings in the absence of jumps is given in black. We see that the stock holdings enters the shaded region where $\tilde{\alpha}_1(s) < \hat{X}(s) < \tilde{\alpha}_2(s)$. *In this region, market sell and passive buy orders are used.*

The reason is the following: In the first stage, rapid trading is (relatively) expensive and so it is optimal to reduce stock holdings slowly and start selling early. In the second stage, trading is (relatively) cheap, so it is optimal to stay very close to the target func-

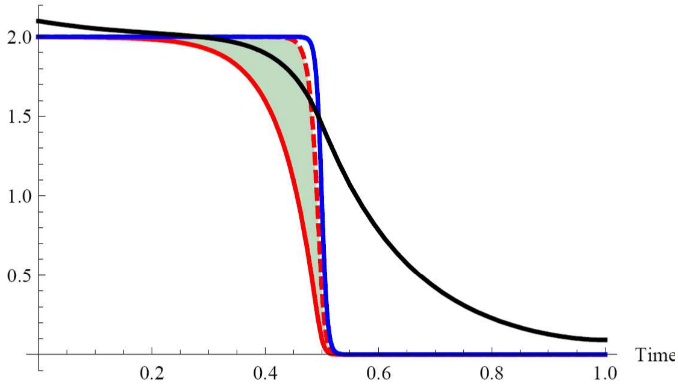


Figure 1.4.: Cost-adjusted target functions for Example 1.8.3 before (red) and after (dashed red) a jump of the signal. The blue curve is the target function. The black curve represents the optimal stock holdings if no jump occurs. We see that this function enters the shaded region, where market sell and passive buy orders are optimal.

tion. If stock holdings are in the shaded area and a jump happens, it is optimal to use a passive buy *order which take stock holdings up to the dashed red curve again and then closely follow the blue curve.*

In conclusion, having a strictly decreasing target function and stock holdings which start above the target is not sufficient to exclude counterintuitive optimal controls.

The preceding counterexamples show that, in general, the signal cannot be interpreted as bid ask spread. We conclude this section with a *positive* result and show that in the case of a constant target function, passive and market orders always have the same sign, i.e. the counterintuitive results described above do not occur. This extends Proposition 1.6.9 to the case where the signal is *not* independent from passive order execution. The implication is that for constant target functions, e.g. for the important case of portfolio liquidation, we may indeed interpret Z as bid ask spread. Specifically, the cost function discussed in Remark 1.2.8 may be generalised to

$$g(u_2, Z) = Z_1 u_2 \arctan(u_2) + u_2^2(Z_2 + \varepsilon),$$

where Z_1 and Z_2 are mean-reverting positive jump processes representing spread and the inverse order book height, respectively. Such a cost function extends the model from Almgren [2003] to stochastic liquidity parameters.

Lemma 1.8.4. *Let $\alpha(s,z) = c$ for all $(s,z) \in [t,T] \times \mathbb{R}^n$ and some $c \in \mathbb{R}$. Then $u_1(s)$ and $u_2(s)$ have the same sign $ds \times d\mathbb{P}$ a.e.*

Proof. We only consider the case $c = 0$ and we first assume $X^u(0) > 0$. We will use throughout that each of the functions $\psi \triangleq g(\cdot, z), h$ and f is strictly convex and attains its minimum at zero. In particular $\psi'(x) \leq 0$ if $x \leq 0$.

A first consequence is that $J(t, 0, z, 0) = 0$ for each $(t, z) \in [0, T] \times \mathbb{R}^n$ and $J(t, 0, z, u) > 0$ for each $u \in \mathcal{U}_t$ which is not identically zero. This implies $v(t, 0, z) = 0$ and $v(t, x, z) > 0$ for each $x \neq 0$ since v is strictly convex and nonnegative. It now follows that

$$\tilde{\alpha}(t, z) = \underset{x \in \mathbb{R}}{\arg\min}\, v(t, x, z) = 0.$$

An argument as in the proof of Proposition 1.6.2 shows that the optimal passive order satisfies $ds \times d\mathbb{P}$ a.e.

$$\hat{u}_1(s) = -\hat{X}(s).$$

A consequence of $0 = J(t, 0, z, 0) < J(t, 0, z, u)$ is that if the optimal trajectory of stock holdings $\hat{X}$ hits zero, it stays there until maturity. Let τ_1 denote the first jump time of the Poisson process N. At the jump time, we have $\hat{X}(\tau) = \hat{X}(\tau-) + \hat{u}_1(\tau) = 0$. $\hat{X}$ starts above zero and evolves continuously before τ_1. It is now clear that $\hat{X} \geq 0$ $ds \times d\mathbb{P}$ a.e. Combining this with the explicit representation of the adjoint process (1.13) we get $P \leq 0$ $ds \times d\mathbb{P}$ a.e. An argument as in the proof of Theorem 1.3.2 yields that a.e.

$$\hat{u}_2(s) = \left[g_{u_2}(\cdot, Z(s))\right]^{-1}(P(s)) \leq 0.$$

In conclusion, both the optimal passive and market order are nonnegative. A similar argument shows that they are nonpositive if $X^u(0) < 0$ and zero if $X^u(0) = 0$. $\square$

2. When to Cross the Spread: Curve Following with Singular Control

2.1. Introduction

In the previous chapter we solved the problem of curve following in an order book model with instantaneous price impact and absolutely continuous market orders. In particular it was assumed that trades have no lasting impact on future prices. However in limit order markets the best bid and best ask prices typically recover only slowly after large discrete trades. In the present chapter we therefore consider a two-sided limit order market model with temporary market impact and resilience, where the price impact of trading decays only gradually. Trading strategies now include continuous and discrete trades, so that we are in the framework of singular stochastic control. The additional difficulty is that there are now two sources of jumps, representing passive order execution and singular market orders. The singular nature of the market order complicates the analysis considerably, the optimal market order cannot be characterised as the pointwise maximiser of the Hamiltonian as in the absolutely continuous case. Moreover, we now face an optimisation problem with *constraints* , since passive buy and sell orders are modelled separately and both are nonnegative.

Methods of singular control have been applied in different fields including the monotone follower problem as in Beneš et al. [1980], the consumption-investment problem with proportional transaction costs in Davis and Norman [1990] and finite fuel problems as in Karatzas et al. [2000]. Most of them rely on the dynamic programming approach or a direct martingale optimality principle. Here we shall prove a version of the stochastic maximum principle. In contrast to dynamic programming, this does not require regularity of the value function and provides information on the optimal control directly. Maximum principles for singular stochastic control problems can be found in Cadenillas and Haussmann [1994], Øksendal and Sulem [2001] and Bahlali and Mezerdi [2005], among others. Unfortunately these results cannot directly be applied to the optimisation problem under consideration, since it involves jumps and state dependent singular cost terms. The recent paper Øksendal and Sulem [2010] provides necessary and sufficient maximum principles for jump diffusions with partial information. Despite being fairly general, their setup does not cover the particular model we consider; instead we give a direct proof based on Cadenillas and Haussmann [1994] and the ideas developped in Chapter 1.

As in the previous chapter we consider an investor who wants to minimise the deviation of his stock holdings from a prespecified target function, where the latter is driven by a vector of uncontrolled stochastic signals. The applications we have in mind are index tracking, portfolio liquidation, hedging and inventory management. Our problem can be seen as an extension of the monotone follower problem to an order book framework with market and passive orders. In our model, the investor faces a tradeoff between the penalty for deviation and the liquidity costs of trading in the order book, i.e. the costs of crossing the spread and buying into the order book. The investor's market orders widen the spread temporarily; the gap then attracts new limit orders from other market participants and the spread recovers. The key decision the trader has to take is the following: If the spread is small, trading is cheap and a market order might be beneficial. For large spreads however it might be better to stop trading and wait until the spread recovers. When to cross the spread is a fundamental question of algorithmic trading in limit order markets. An equivalent question would be when to convert a limit into a market order. To the best of our knowledge, the problem of when to cross the bid ask spread has not been addressed in the mathematical finance literature on limit order markets. Obizhaeva and Wang [2005] and Predoiu et al. [2010], for instance, consider portfolio liquidation for a one-sided order book with initial spread zero; in this case it is optimal *never* to stop trading.

Our order book model is inspired by Obizhaeva and Wang [2005], a model which has recently been generalised to arbitrary shape functions by Alfonsi et al. [2010] as well as Predoiu et al. [2010] and stochastic order book height in Fruth [2011]. While the mentioned articles focus on portfolio liquidation, we consider here the more general problem of curve following and therefore need a two-sided order book model. In addition, we allow for passive orders. These are orders with random execution which do not induce liquidity costs, such as limit orders or orders placed in a dark venue.

Our first mathematical result is an a priori estimate on the control. For the proof, we reduce the curve following problem to an optimisation problem with quadratic penalty and without target function and then use a scaling argument. This result provides the existence and uniqueness of an optimal control via a Komlós argument. Next we prove a suitable version of the stochastic maximum principle and characterise the optimal trading strategy in terms of a coupled forward backward stochastic differential equation (FBSDE). The proof builds on results from Cadenillas and Haussmann [1994] and extends them to the present case where we have jumps, state-dependent singular cost terms and general dynamics for the stochastic signal. Next we give a second characterisation of optimality in terms of buy, sell and no-trade regions. It turns out that there is always a region where the costs of trading are larger than the penalty for deviating, so that it is optimal to stop trading when the controlled system is inside this region. This is in contrast to the previous chapter, where only absolutely continuous trading strategies are allowed and a smoothness condition on the cost function is imposed. It was shown in Chapter 1 that under these conditions the no-trade region is degenerate, so that the investor always trades. In the present model the no-trade region is defined in terms of a threshold for the bid ask spread. We show that spread crossing is optimal if the spread is smaller than or equal to the threshold. If it is larger, then no market orders should be used. The threshold is given explicitly in terms of the FBSDE and as a result, we can precisely characterise when spread crossing is optimal for a large class of optimisation problems. We will see that market orders are applied such that the controlled system remains inside (the closure of) the no-trade region at all times, and that its trajectory is reflected at the boundary. To make this precise, we show that the adjoint process together with the optimal control provides the solution to a reflected BSDE.

In general it is difficult to solve the coupled forward backward SDE (or the corresponding Hamilton-Jacobi-Bellman quasi variational inequality, henceforth HJB QVI) explicitly. This is due to the Poisson jumps (leading to nonlocal terms) and the singular nature of the control. For quadratic penalty function and zero target function though the solution can be given in closed form. This corresponds to the portfolio liquidation problem in limit order markets and extends the result of Obizhaeva and Wang [2005] to trading strategies with passive orders. The new feature is that the optimal strategy is not deterministic, but adapted to passive order execution, and the trading rate is not constant but increasing in time.

The remainder of this chapter is organised as follows: We describe the market environment and the control problem in Section 2.2 and show in Section 2.3 that a unique optimal control exists. We then provide two characterisations of optimality, first via the stochastic maximum principle in Section 2.4 and then via buy, sell and no-trade regions in Section 2.5. The link to reflected BSDEs is presented in Section 2.6 and we conclude with some examples in Section 2.7.

2.2. The Control Problem

Let $(\Omega, \mathcal{F}, \{\mathcal{F}(s) : s \in [0,T]\}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions of right continuity and completeness and $T > 0$ be the terminal time.

Assumption 2.2.1. *The filtration is generated by the following mutually independent processes,*

1. *A d-dimensional Brownian Motion W, $d \geq 1$.*
2. *Two one-dimensional Poisson processes N_i with intensities λ_i for $i = 1, 2$.*
3. *A compound Poisson process M on $[0,T] \times \mathbb{R}^k$ with compensator $m(d\theta)dt$ where $m(\mathbb{R}^k) < \infty$.*

Trading takes place in a two sided limit order market. Limit order markets are a special type of illiquid markets; we give a detailed discussion of their properties in the introductory chapter of this thesis. We postulate the existence of three price processes: the benchmark price (a nonnegative martingale), the best ask price (which is above the benchmark) and the best bid price (which is below the benchmark). On the buy side of the order book liquidity is available for prices higher than the best ask price, and we assume a block shaped distribution of available liquidity with constant height $\frac{1}{\kappa_1} > 0$. This assumption is also made in Obizhaeva and Wang [2005]; it is key for the current approach as it leads to linear dynamics for the bid ask spread. Similarly, liquidity is available on the sell side for prices lower than the best bid. We assume a block shaped distribution of liquidity available on the sell side with constant height $\frac{1}{\kappa_2} > 0$. The investor's trades have a temporary impact[1] on the best bid and ask prices, this will be made more precise below. The benchmark price is hypothetical and cannot be observed directly in the market. As in Chapter 1, it represents the "fair" price of the underlying or a reference price in the absence of liquidity costs. We assume that the benchmark price is uncontrolled. A permanent price impact as in Almgren and Chriss [2001] can easily be incorporated here, but we do not include it and instead focus on the tradeoff between accuracy and liquidity costs. A stylised snapshot of the order book and a typical trajectory of the price processes are plotted in Figure 2.1.

The investor can apply market buy (sell) orders to consume liquidity on the buy (sell) side of the order book. His cumulated market buy (sell) orders are denoted by η_1 (η_2, respectively). These are nondecreasing càdlàg processes, and hence we allow for continuous as well as discrete trades and denote by

$$\Delta\eta_i(s) \triangleq \eta_i(s) - \eta_i(s-) \geq 0$$

for $s \in [0, T]$ and $i = 1, 2$ the jumps of η_i. Such control processes are more general than absolutely continuous trading strategies and they seem better suited to describe real world trading strategies. Real world trading is purely discrete and continuous strategies are merely an analytical device, they represent the limit of a sequence of small discrete trades. The additional difficulty is then to cope with the jumps in the control. The Hamilton-Jacobi-Bellman equation associated to a singular stochastic control problem is a quasi variational inequality and not only a partial differential equation, cf. Øksendal and Sulem [2001] Chapter 5.

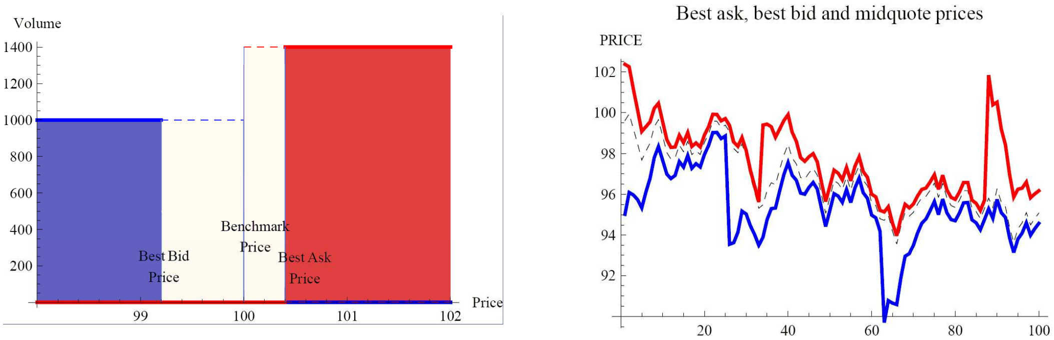
In addition, just as in the previous chapter, the investor can use passive buy (sell) orders u_1 (u_2, respectively). We assume that they are placed and fully executed at the benchmark price. Thus a passive order always achieves a better price than the corresponding market order, however its execution is uncertain. We think of them as orders placed in a dark venue or as a stylised form of limit orders. Market and passive orders represent taking and providing liquidity.

The class of admissible controls is now defined for $t \in [0, T]$ as

$$\begin{aligned}\mathcal{U}_t \triangleq \Bigg\{ (\eta, u) : [t, T] \times \Omega \to \mathbb{R}^2_+ \times \mathbb{R}^2_+ \;\Bigg|\; &\eta_i(t-) = 0,\, \mathbb{E}\Big[\eta_i(T)^2 + \int_t^T u_i(r)^2 dr\Big] < \infty, \\ &\eta_i \text{ is nondecreasing, càdlàg and progressively measurable and} \\ &u_i \text{ is predictably measurable, for } i = 1, 2 \Bigg\}.\end{aligned}$$

Each control consists of the four components η_1, η_2, u_1, u_2, each of them being nonnegative. In particular, we face an optimisation problem with *constraints*. We note that $\eta_1(s)$ (resp. $\eta_2(s)$) denotes the market buy (resp. sell) orders *accumulated* in $[t, s]$. In contrast, $u_1(s)$ (resp. $u_2(s)$) represents the volume placed as a passive buy (resp. sell)

[1] A fundamental property of illiquid markets is that trades move prices. There is a large body of empirical literature on the price impact of trading, we refer the reader to Kraus and Stoll [1972], Holthausen et al. [1987], Holthausen et al. [1990], Biais et al. [1995] and Almgren et al. [2005].



(a) (b)

Figure 2.1.: (a) This stylised snapshot of the order book shows the best bid, benchmark and best ask price as well as liquidity that is available (dark) and consumed (light). (b) Here we see a typical evolution of the price processes over time. The best ask (red) is above the benchmark price (dashed black), which is above the best bid price (blue). Market buy (resp. sell) orders lead to jumps in the best ask (resp. bid) price. In the absence of trading, the best ask and best bid converge to the benchmark.

order at time $s \in [t, T]$. We remark that from $\eta_i(T) < \infty$ a.s. for $i = 1, 2$ it follows that $\sum_{r \in [t,T]} \Delta\eta_i(r) < \infty$ a.s. and we have the following decomposition of the singular term into a continuous and a pure jump part for $s \in [t, T]$:

$$\eta_i(s) = \left(\eta_i(s) - \sum_{r \in [t,s]} \Delta\eta_i(r)\right) + \sum_{r \in [t,s]} \Delta\eta_i(r).$$

Having defined the admissible controls, let us now specify the price dynamics. As pointed out in the introductory chapter of this thesis, a model for a limit order market should satisfy the following properties:

- The best ask price is larger than or equal to the best bid price.
- The investor's market buy orders have a temporary impact on the best ask price, but not on the best bid.
- The investor's market sell orders have a temporary impact on the best bid price, but not on the best ask.
- The impact of a trade on the price decays over time (resilience).
- In the absence of trading, both processes are (or converge to) martingales.
- The investor's passive orders do not move prices. More precisely, the price impact of passive orders as compared to the impact of market orders is negligible.

Specifying price dynamics with these properties is not straightforward. We find it more convenient to work with the buy and sell spreads instead. Specifically, we denote by X_1 the distance of the best ask price to the benchmark price and call this process the buy spread. As in Obizhaeva and Wang [2005] and Alfonsi et al. [2010] we assume exponential recovery of the buy spread with resilience parameter $\rho_1 > 0$. The dynamics of the *buy spread* are then given for $s \in [t, T]$ by

$$X_1(s) - X_1(t-) = -\int_t^s \rho_1 X_1(r)dr + \int_{[t,s]} \kappa_1 d\eta_1(r), \quad X_1(t-) = x_1 \geq 0.$$

As a convention, we write $\int_{[t,s]}$ for integrals with respect to the singular processes η_i for $i = 1, 2$ to indicate that possible jumps at times s and t are included. Similarly, the *sell spread* X_2 is defined as the distance of the best bid price to the benchmark price and it satisfies

$$X_2(s) - X_2(t-) = -\int_t^s \rho_2 X_2(r)dr + \int_{[t,s]} \kappa_2 d\eta_2(r), \quad X_2(t-) = x_2 \geq 0.$$

An immediate consequence is that the spreads X_1 and X_2 are nonnegative, mean reverting and the price system thus defined satisfies the properties given above.

Remark 2.2.2.

- In the literature the bid ask spread is typically defined as the distance of the best ask from the best bid price; in our notation this process is given by $X_1 + X_2$.
- In the seminal paper Kyle [1985] three measures of liquidity are defined, all of which are captured in the model we propose. *Depth*, "the size of an order flow innovation required to change prices a given amount", is given by the parameters κ_1 and κ_2 which denote the inverse order book height. *Resiliency*, "the speed with which prices recover from a random, uninformative shock", is captured by the resilience parameters ρ_1 and ρ_2. Finally, *tightness*, "the cost of turning around a position over a short period of time", can be measured in terms of the bid ask spread $X_1 + X_2$.
- We remark that the spread dynamics specified above are linear in the control and the state variable. This is necessary to compute the Gâteaux derivative of the cost functional, which then allows to derive a necessary and sufficient condition for optimality. General shape functions for the order book lead to nonlinear dynamics, which are not compatible with the current approach.
- If the passive orders are interpreted as limit orders placed on the benchmark price, they *do* change the best quotes. Strictly speaking, in this case X_1 and X_2 represent the buy and sell spreads *modulo the investor's own limit order*.
- As in the previous chapter, the dynamics of the benchmark price is not important, aside from the fact that this process is a nonnegative martingale. Then it does not contribute to the expected trading costs and the liquidity costs only depend on the buy and sell spread.

- We assume that, in the absence of trading, the spreads converge back to zero. More generally, they might converge to any number $z \in \mathbb{R}_+$. This is equivalent to introducing an additional fixed spread of size z as in Obizhaeva and Wang [2005]. However, our assumption is no loss of generality, since an additional fixed spread can be incorporated in the present model by replacing X_i by $X_i + z$ in the performance functional defined in (2.1) below, for $i = 1, 2$.

We shall need a third state process X_3 representing the investor's stock holdings. They are the sum of the market and passive buy orders less the market and passive sell orders, and thus for $s \in [t, T]$ given by

$$\begin{aligned} X_3(s) - X_3(t-) &= \int_{[t,s]} d\eta_1(r) - \int_{[t,s]} d\eta_2(r) + \int_t^s u_1(r)N_1(dr) - \int_t^s u_2(r)N_2(dr), \\ X_3(t-) &= x_3 \in \mathbb{R}. \end{aligned}$$

A jump of the Poisson process N_i represents a liquidity event which executes the passive order u_i, for $i = 1, 2$. For simplicity we consider full execution only, this assumption is also made in Kratz [2011] and in the first chapter of this thesis. We define the vector $X \triangleq (X_i)_{i=1,2,3}$ and write $X = X^{\eta,u}$ if we want to emphasise the dependence of the state process on the control. Note that there are two sources of jumps, the Poisson processes and the discrete market orders. More precisely, the jump of the state process at time $s \in [t, T]$ is given by

$$\begin{aligned} \Delta X(s) &= \begin{pmatrix} X_1(s) - X_1(s-) \\ X_2(s) - X_2(s-) \\ X_3(s) - X_3(s-) \end{pmatrix} = \Delta_N X(s) + \Delta_\eta X(s) \\ &\triangleq \begin{pmatrix} 0 \\ 0 \\ u_1(s)\Delta N_1(s) - u_2(s)\Delta N_2(s) \end{pmatrix} + \begin{pmatrix} \kappa_1 \Delta\eta_1(s) \\ \kappa_2 \Delta\eta_2(s) \\ \Delta\eta_1(s) - \Delta\eta_2(s) \end{pmatrix}. \end{aligned}$$

The formulation of the curve following problem is close to the previous chapter. We briefly describe the relevant changes here. The trader wants to minimise the deviation of his stock holdings to a prespecified target function $\alpha : [t, T] \times \mathbb{R}^n \to \mathbb{R}$. This function depends on a vector of uncontrolled stochastic signals Z with dynamics given for $s \in [t, T]$ by

$$\begin{aligned} Z(s) - Z(t-) = &\int_t^s \mu(r, Z(r))dr + \int_t^s \sigma(r, Z(r-))dW(r) \\ &+ \int_t^s \int_{\mathbb{R}^k} \gamma(r, Z(r-), \theta)\tilde{M}(dr, d\theta), \quad Z(t-) = z \in \mathbb{R}^n. \end{aligned}$$

We think of Z as a stochastic factor which drives the target function, it might represent a stock price index, the price of some underlying or some other kind of risk factor. As above we denote the compensated Poisson martingale by $\tilde{M}([0, s] \times A) \triangleq M([0, s] \times A) - m(A)s$; similarly $\tilde{N}_i \triangleq N_i - \lambda_i s$ for $i = 1, 2$.

Having defined the state processes and their respective dynamics, let us now specify the optimisation criterion. The *performance functional* is defined for $(t, x, z) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^n$ and a control $(\eta, u) \in \mathcal{U}_t$ as

$$
\begin{aligned}
&J(t, x, z, \eta, u) \\
&\triangleq \mathbb{E}_{t,x,z}\Bigg[\int_{[t,T]} \left[X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\right] d\eta_1(r) + \int_{[t,T]} \left[X_2(r-) + \frac{\kappa_2}{2}\Delta\eta_2(r)\right] d\eta_2(r) \\
&\qquad + \int_t^T h\left(X_3(r) - \alpha(r, Z(r))\right) dr + f\left(X_3(T) - \alpha(T, Z(T))\right)\Bigg]. \qquad (2.1)
\end{aligned}
$$

There are four cost terms representing the conflicting interests of liquidity costs and accuracy and we now explain them briefly. The first two terms on the right hand side of (2.1) capture trading costs of market buy (sell) orders, i.e. the costs of crossing the spread and buying (selling) into the order book. Specifically, an infinitesimal market buy order $d\eta_1(r)$ is executed at the best ask price, so that the costs of crossing the spread are given by $X_1(r-)d\eta_1(r)$. A discrete buy order $\Delta\eta_1(r)$ eats into the block shaped order book and shifts the spread from $X_1(r-)$ to $X_1(r-) + \kappa_1\Delta\eta_1(r)$. Its liquidity costs are given by

$$
\left(X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\right)\Delta\eta_1(r).
$$

In particular, the jump part in (2.1) is understood as, for $i = 1, 2$

$$
\begin{aligned}
\mathbb{E}\left[\int_{[t,T]} \Delta\eta_i(r) d\eta_i(r)\right] &= \mathbb{E}\left[\sum_{r\in[t,T]} (\Delta\eta_i(r))^2\right] \\
&\leq \mathbb{E}\left[\left(\sum_{r\in[t,T]} \Delta\eta_i(r)\right)^2\right] \leq \mathbb{E}\left[\eta_i(T)^2\right] < \infty. \qquad (2.2)
\end{aligned}
$$

The last two terms on the right hand side of (2.1) penalise deviation from the target function, the term involving h is referred to as running costs, while the term involving f represents terminal costs. As a shorthand, we sometimes write $J(\eta, u) \triangleq J(t, x, z, \eta, u)$ if $(t, x, z) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^n$ is fixed. The optimisation problem under consideration is

Problem 2.2.3. *Minimise* $J(\eta, u)$ *over* $(\eta, u) \in \mathcal{U}_t$.

For $(t, x, z) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^n$ the *value function* is defined as

$$
v(t, x, z) = \inf_{(\eta,u)\in\mathcal{U}_t} J(t, x, z, \eta, u).
$$

Remark 2.2.4. Problem 2.2.3 is a singular stochastic control problem. Maximum principles for singular control are derived for instance in Cadenillas and Haussmann [1994], Øksendal and Sulem [2001] and Bahlali and Mezerdi [2005]. However, the above problem is not covered by their results for several reasons. Firstly, it involves jumps. Secondly, the singular cost terms $\int_{[t,T]} \left[X_i(r-) + \frac{\kappa_i}{2}\Delta\eta_i(r)\right] d\eta_i(r)$ for $i = 1, 2$ depend on the state variable and on the jumps of the control, which is not the case in the "usual" formulation. The standard setup only allows for cost terms of the form $\int_t^T k(s,\omega)d\eta(s)$. In financial applications, e.g. in Davis and Norman [1990], the integrand k is often interpreted as proportional transaction costs. In the illiquid market model we propose the buy and sell spreads play the role of the transaction cost parameters, and they are themselves controlled. A third difficulty in the present model is that the absolutely continuous control u (the passive order) does not incur trading costs, so the "standard" characterisation as the pointwise maximiser of the Hamiltonian does not apply.

The recent article Øksendal and Sulem [2010] provides necessary and sufficient maximum principles for the singular control of jump diffusions, where the singular cost term may depend on the state variable. However, they do not allow for terms like $\int_{[t,T]} \Delta\eta_i(r)d\eta_i(r)$, they do not incorporate absolutely continuous controls (which are needed in the present framework for the passive order u) and their sufficient condition is based on a convexity condition on the Hamiltonian which is not satisfied in our specific case. Instead we give a direct proof based on Cadenillas and Haussmann [1994] and ideas used in the previous chapter.

To ensure existence and uniqueness of an optimal control, we impose the following assumptions. Here and throughout, we write c for a generic constant, which might be different at each occurrence.

Assumption 2.2.5. *1. The penalty functions $f, h : \mathbb{R} \to \mathbb{R}$ are strictly convex, continuously differentiable, normalised and nonnegative.*

2. *In addition, f and h have at least quadratic growth, i.e. there exists $\varepsilon > 0$ such that $|f(x)|, |h(x)| \geq \varepsilon |x|^2$ for all $x \in \mathbb{R}$.*

3. *The functions μ, σ and γ are Lipschitz continuous, i.e. there exists a constant c such that for all $z, z' \in \mathbb{R}^n$ and $s \in [t,T]$,*

$$\begin{aligned}&\|\mu(s,z)-\mu(s,z')\|^2_{\mathbb{R}^n} + \|\sigma(s,z)-\sigma(s,z')\|^2_{\mathbb{R}^{n\times d}}\\ &\quad + \int_{\mathbb{R}^k} \|\gamma(s,z,\theta)-\gamma(s,z',\theta)\|^2_{\mathbb{R}^n} m(d\theta) \leq c\|z-z'\|^2_{\mathbb{R}^n}.\end{aligned}$$

   In addition, they satisfy

$$\sup_{t\leq s\leq T}\left[\|\mu(s,0)\|^2_{\mathbb{R}^n} + \|\sigma(s,0)\|^2_{\mathbb{R}^{n\times d}} + \int_{\mathbb{R}^k}\|\gamma(s,0,\theta)\|^2_{\mathbb{R}^n} m(d\theta)\right] < \infty.$$

4. *The target function α has at most polynomial growth in the variable z uniformly in s, i.e. there exist constants $c_\alpha, q > 0$ such that for all $z \in \mathbb{R}^n$,*

$$\sup_{t\leq s\leq T} |\alpha(s,z)| \leq c_\alpha(1+\|z\|^q_{\mathbb{R}^n}).$$

5. *The penalty functions f and h have at most polynomial growth.*

Remark 2.2.6. Let us briefly comment on these assumptions. Taking f and h non-negative is reasonable for penalty functions. Normalisation is no loss of generality, this may always be achieved by a linear shift of f, h and α. Quadratic growth of f and h is only needed in Lemma 2.3.4 for an a priori L^2-norm bound on the control, which is then used for a Komlós argument. The convexity condition leads naturally to a convex coercive problem which then admits a unique solution. A typical candidate for the penalty function is $f(x) = h(x) = x^2$, which corresponds to minimising the squared error. We require that the signal SDE admits a unique strong solution which has moments of all orders, the Lipschitz assumptions on μ, σ and γ are sufficient to guarantee this, however we remark that any condition guaranteeing existence, uniqueness and all moments would be appropriate here.

Once the existence of an optimal control is established, we need one further assumption. It guarantees the existence and uniqueness of the adjoint process.

Assumption 2.2.7. *The derivatives f' and h' have at most linear growth, i.e. for all $x \in \mathbb{R}$ we have $|f'(x)| + |h'(x)| \leq c(1 + |x|)$.*

2.3. Existence of a Solution

The aim of the present section is to show that the performance functional is strictly convex and that it is enough to consider controls with a uniform L^2-norm bound. Combining these results with a Komlós argument, we then prove that there is a unique optimal control.

We begin with some growth estimates for the state processes. Henceforth we impose Assumption 2.2.5.

Lemma 2.3.1. *1. For every $p \geq 2$ there exists a constant c_p such that for every $(t, x, z) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^n$ we have*

$$\mathbb{E}_{t,x,z}\left[\sup_{t \leq s \leq T} \|Z(s)\|^p_{\mathbb{R}^n}\right] \leq c_p \left(1 + \|z\|^p_{\mathbb{R}^n}\right).$$

2. There exists a constant c_x such that for any $(\eta, u) \in \mathcal{U}_t$ we have

$$\mathbb{E}_{t,x,z}\left[\sup_{t \leq s \leq T} \|X^{\eta,u}(s)\|^2_{\mathbb{R}^3}\right] \leq c_x \left(1 + \mathbb{E}_{t,x,z}\left[\|\eta(T)\|^2_{\mathbb{R}^2}\right] + \mathbb{E}\left[\int_t^T \|u(r)\|^2_{\mathbb{R}^2}\, dr\right]\right).$$

In particular, $X^{\eta,u}$ has square integrable supremum for all $(\eta, u) \in \mathcal{U}_t$.

Proof. Item (1) is a well known estimate on the solution of an SDE with Lipschitz coefficients, see for example Barles et al. [1997] Proposition 1.1. We now prove item (2). For $s \in [t, T]$ we have

$$|X_1(s)|^2 = \left|x_1 - \int_t^s \rho_1 X_1(r) dr + \kappa_1 \int_{[t,s]} d\eta_1(r)\right|^2$$

$$\leq 3\left[|x_1|^2 + \rho_1^2\left|\int_t^s X_1(r)dr\right|^2 + \kappa_1^2|\eta_1(s)|^2\right]$$
$$\leq 3\left[|x_1|^2 + \rho_1^2\int_t^s \sup_{z\in[t,r]}|X_1(z)|^2 dr + \kappa_1^2|\eta_1(T)|^2\right],$$

where we have used Jensen's inequality and the fact that η_1 is nondecreasing in the last line. An application of Gronwall's Lemma now provides the existence of a constant $c > 0$ such that

$$\mathbb{E}_{t,x,z}\left[\sup_{t\leq s\leq T}|X_1(s)|^2\right] \leq c\Big(1 + |x_1|^2 + \mathbb{E}_{t,x,z}\left[|\eta_1(T)|^2\right]\Big).$$

Similar estimates hold for X_2 and X_3. □

A first consequence of the above lemma is that the zero control incurs finite costs.

Corollary 2.3.2. *The zero control incurs finite costs, i.e. for each $(t, x, z) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^n$ we have*

$$J(t, x, z, 0, 0) < \infty.$$

Proof. The polynomial growth of f, h and α implies the existence of constants $c_1, c_2 > 0$ such that

$$J(t, x, z, 0, 0) = \mathbb{E}_{t,x,z}\Big[\int_t^T h(x_3 - \alpha(r, Z(r)))dr + f(x_3 - \alpha(T, Z(T)))\Big]$$
$$\leq c_1\Big(1 + \mathbb{E}_{t,x,z}\Big[\int_t^T \|Z(r)\|_{\mathbb{R}^n}^{c_2} dr + \|Z(T)\|_{\mathbb{R}^n}^{c_2}\Big]\Big).$$

The terms in the last line are finite due to Lemma 2.3.1(1). □

We now show that the performance functional is strictly convex in the control, so that methods of convex analysis can be applied.

Proposition 2.3.3. *The performance functional $(\eta, u) \mapsto J(t, x, z, \eta, u)$ is strictly convex, for every $(t, x, z) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^n$.*

Proof. From the definition of X_i for $i = 1, 2$ we have $d\eta_i(s) = \frac{dX_i(s)+\rho_i X_i(s)ds}{\kappa_i}$. We use this to rewrite the performance functional as

$$\begin{aligned} &J(t, x, z, \eta, u) \\ =&\mathbb{E}_{t,x,z}\Big[\frac{X_1(T)^2 - x_1^2}{2\kappa_1} + \frac{X_2(T)^2 - x_2^2}{2\kappa_2} + \int_t^T \frac{\rho_1}{\kappa_1}X_1(r)^2 dr + \int_t^T \frac{\rho_2}{\kappa_2}X_2(r)^2 dr \\ &+ \int_t^T h\left(X_3(r) - \alpha(r, Z(r))\right)dr + f\left(X_3(T) - \alpha(T, Z(T))\right)\Big]. \end{aligned} \tag{2.3}$$

The right hand side is strictly convex in X. Due to the fact that $(\eta, u) \mapsto X^{\eta,u}$ is affine, it follows that $(\eta, u) \mapsto J(t, x, z, \eta, u)$ is strictly convex. □

The aim in this section is to prove existence and uniqueness of an optimal control. For the proof of this result, we need two auxiliary lemmata. We first show a quadratic growth estimate on the value function in Lemma 2.3.4. This extends Lemma 1.4.2 to the singular control case. As above, the idea is to reduce the curve following problem to a simpler linear-quadratic regulator problem. In contrast to Lemma 1.4.2, a solution to this LQ problem via Riccati equations is not available and we use a scaling argument instead.

Lemma 2.3.4. *For each $(t, x, z) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^n$ there are constants $c_{1,t}, c_2, c_3 > 0$ such that*

$$v(t, x, z) \geq c_{1,t} x_3^2 - c_2 \left(1 + \|z\|_{\mathbb{R}^n}^{c_3}\right).$$

Proof. The idea is to use the growth conditions on the penalty functions to reduce the optimisation problem to simpler linear-quadratic problem, which can then be estimated in terms of x_3^2. Using the quadratic growth of f and h yields

$$\begin{aligned} &v(t, x, z) \\ &\geq \inf_{(\eta,u)\in\mathcal{U}_t} \mathbb{E}_{t,x,z}\Bigg[\int_{[t,T]} \left[X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\right] d\eta_1(r) + \int_{[t,T]} \left[X_2(r-) + \frac{\kappa_2}{2}\Delta\eta_2(r)\right] d\eta_2(r) \\ &\qquad + \int_t^T \varepsilon \left(X_3(r) - \alpha(r, Z(r))\right)^2 dr + \varepsilon \left(X_3(T) - \alpha(T, Z(T))\right)^2\Bigg]. \end{aligned}$$

Next an application of the inequality $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ leads to

$$\begin{aligned} &v(t, x, z) \\ &\geq \inf_{(\eta,u)\in\mathcal{U}_t} \mathbb{E}_{t,x,z}\Bigg[\int_{[t,T]} \left[X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\right] d\eta_1(r) + \int_{[t,T]} \left[X_2(r-) + \frac{\kappa_2}{2}\Delta\eta_2(r)\right] d\eta_2(r) \\ &\quad + \int_t^T \frac{\varepsilon}{2} |X_3(r)|^2 dr + \frac{\varepsilon}{2} |X_3(T)|^2\Bigg] - \varepsilon\mathbb{E}_{t,x,z}\left[\int_t^T |\alpha(r, Z(r))|^2 dr + |\alpha(T, Z(T))|^2\right]. \end{aligned}$$

The polynomial growth of α coupled with Lemma 2.3.1 provides the existence of constants $c_2, c_3 > 0$ such that

$$\begin{aligned} &v(t, x, z) \qquad\qquad (2.4)\\ &\geq \inf_{(\eta,u)\in\mathcal{U}_t} \mathbb{E}_{t,x,z}\Bigg[\int_{[t,T]} \left[X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\right] d\eta_1(r) + \int_{[t,T]} \left[X_2(r-) + \frac{\kappa_2}{2}\Delta\eta_2(r)\right] d\eta_2(r) \\ &\qquad + \int_t^T \frac{\varepsilon}{2} |X_3(r)|^2 dr + \frac{\varepsilon}{2} |X_3(T)|^2\Bigg] - c_2 \left(1 + \|z\|_{\mathbb{R}^n}^{c_3}\right). \end{aligned}$$

This provides an estimate of the original value function in terms of an easier optimisation problem with a quadratic penalty function and zero target function. Economically, this may be interpreted as a portfolio liquidation problem. To continue the estimate, we define the following "value" function,

$$
\begin{aligned}
&v_1(t,x)\\
\triangleq &\inf_{(\eta,u)\in\mathcal{U}_t} \mathbb{E}_{t,x}\bigg[\int_{[t,T]}\Big[X_1(r-)+\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\,d\eta_1(r)+\int_{[t,T]}\Big[X_2(r-)+\frac{\kappa_2}{2}\Delta\eta_2(r)\Big]\,d\eta_2(r)\\
&\qquad+\int_t^T\frac{\varepsilon}{2}\,|X_3(r)|^2\,dr+\frac{\varepsilon}{2}\,|X_3(T)|^2\bigg].
\end{aligned}
$$

We remark that the value function v_1 is monotone in X_i for $i=1,2$ and X_i is monotone in the starting value x_i, so that v_1 does not increase if we replace the initial spread $X_i(t-)=x_i\geq 0$ by zero for $i=1,2$ and only consider starting values $x=(0,0,x_3)^*$, i.e.

$$
v_1(t,x)\geq v_1(t,(0,0,x_3)^*)\triangleq v_2(t,x_3).
$$

Let us denote by J_2 the performance functional associated to the value function v_2. Due to $x_i=0$ for $i=1,2$ the mappings $(\eta,u)\mapsto X_i^{\eta,u}$ are linear and the mapping $(x_3,\eta,u)\mapsto X_3^{\eta,u}-x_3$ is also linear. We use this to show that J_2 scales quadratically. We write for $(t,x_3)\in[0,T]\times\mathbb{R}$ and a scaling factor $\beta>0$

$$
\begin{aligned}
&J_2(t,\beta x_3,\beta\eta,\beta u)\\
=&\mathbb{E}_{t,(0,0,\beta x_3)^*}\bigg[\int_{[t,T]}\Big[X_1^{\beta\eta,\beta u}(r-)+\frac{\kappa_1}{2}\beta\Delta\eta_1(r)\Big]\,\beta d\eta_1(r)\\
&+\int_{[t,T]}\Big[X_2^{\beta\eta,\beta u}(r-)+\frac{\kappa_2}{2}\beta\Delta\eta_2(r)\Big]\,\beta d\eta_2(r)\\
&+\int_t^T\frac{\varepsilon}{2}\Big|X_3^{\beta\eta,\beta u}(r)\Big|^2\,dr+\frac{\varepsilon}{2}\Big|X_3^{\beta\eta,\beta u}(T)\Big|^2\bigg]\\
=&\mathbb{E}_{t,(0,0,\beta x_3)^*}\bigg[\int_{[t,T]}\Big[X_1^{\beta\eta,\beta u}(r-)+\frac{\kappa_1}{2}\beta\Delta\eta_1(r)\Big]\,\beta d\eta_1(r)\\
&+\int_{[t,T]}\Big[X_2^{\beta\eta,\beta u}(r-)+\frac{\kappa_2}{2}\beta\Delta\eta_2(r)\Big]\,\beta d\eta_2(r)\\
&+\int_t^T\frac{\varepsilon}{2}\bigg|\beta\Big(x_3+\int_{[t,r]}d\eta_1(z)-\int_{[t,r]}d\eta_2(z)+\int_t^r u_1(z)N_1(dz)-\int_t^r u_2(z)N_2(dz)\Big)\bigg|^2dr\\
&+\frac{\varepsilon}{2}\bigg|\beta x_3+\beta\Big(\int_{[t,T]}d\eta_1(z)-\int_{[t,T]}d\eta_2(z)+\int_t^T u_1(z)N_1(dz)-\int_t^T u_2(z)N_2(dz)\Big)\bigg|^2\bigg]\\
=&\beta^2\mathbb{E}_{t,(0,0,x_3)^*}\bigg[\int_{[t,T]}\Big[X_1^{\eta,u}(r-)+\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\,d\eta_1(r)\\
&\qquad+\int_{[t,T]}\Big[X_2^{\eta,u}(r-)+\frac{\kappa_2}{2}\Delta\eta_2(r)\Big]\,d\eta_2(r)+\int_t^T\frac{\varepsilon}{2}\,|X_3^{\eta,u}(r)|^2\,dr+\frac{\varepsilon}{2}\,|X_3^{\eta,u}(T)|^2\bigg]\\
=&\beta^2 J_2(t,x_3,\eta,u).
\end{aligned}
$$

Next we claim that also v_2 scales quadratically. Indeed, let $(u^n, \eta^n) \subset \mathcal{U}_t$ be a minimising sequence for $v_2(t, x_3)$ and let $\beta > 0$ be a scaling factor. We use that J_2 scales quadratically to write

$$v_2(t, \beta x_3) = \lim_{n\to\infty} J_2(t, \beta x_3, \eta^n, u^n) = \beta^2 \lim_{n\to\infty} J_2(t, x_3, \frac{\eta^n}{\beta}, \frac{u^n}{\beta}) \geq \beta^2 v_2(t, x_3). \tag{2.5}$$

We now use (2.5) with the scaling factor $\frac{1}{\beta}$ to get the reverse inequality:

$$\beta^2 v_2(t, x_3) = \beta^2 v_2(t, \frac{1}{\beta}(\beta x_3)) \geq \frac{1}{\beta}^2 \beta^2 v_2(t, \beta x_3) = v_2(t, \beta x_3). \tag{2.6}$$

Combining (2.5) and (2.6) we see that

$$v_2(t, \beta x_3) = \beta^2 v_2(t, x_3).$$

One can check that if $x_3 = 0$ then $v_2(t, 0) = 0$. Choosing now $\beta = |x_3|$ for $x_3 \neq 0$ we get

$$v_2(t, x_3) = \begin{cases} x_3^2 v_2(t, 1), & x_3 > 0 \\ 0, & x_3 = 0 \\ x_3^2 v_2(t, -1), & x_3 < 0, \end{cases}$$

and defining $c_{1,t} \triangleq \min\{v_2(t, 1), v_2(t, -1)\}$ leads to

$$v_2(t, x_3) \geq c_{1,t} x_3^2.$$

Plugging this result into (2.4) provides the following estimate

$$v(t, x, z) \geq c_{1,t} x_3^2 - c_2 \left(1 + \|z\|_{\mathbb{R}^n}^{c_3}\right).$$

To prove the assertion of the lemma, it remains to show that the constant $c_{1,t}$ is strictly positive and finite for each $t \in [0, T]$. The proof of this result is relegated to Lemma A.2.1 in the appendix. □

We are now ready to prove an a priori estimate on the control, which will be needed in the Komlós argument below. This result extends Lemma 1.4.3 to the singular control case.

Lemma 2.3.5. *There is a constant K such that any control with*

$$\mathbb{E}_{t,x,z} \left[\|\eta(T)\|_{\mathbb{R}^2}^2 + \int_t^T \|u(r)\|_{\mathbb{R}^2}^2 \, dr \right] > K$$

cannot be optimal.

Proof. We first consider the market order η. The dynamics of X_i for $i = 1, 2$ imply that for $s \in [t,T]$ we have

$$X_i(s) = e^{-\rho_i(s-t)}x_i + \kappa_i \int_{[t,s]} e^{-\rho_i(r-t)} d\eta_i(r), \tag{2.7}$$

and thus $X_i(T) \geq \kappa_i e^{-\rho_i T}\eta_i(T)$. Combining this with (2.3) yields

$$J(\eta,u) \geq \mathbb{E}_{t,x,z}\left[\frac{X_i(T)^2}{2\kappa_i} - \frac{x_1^2}{2\kappa_1} - \frac{x_2^2}{2\kappa_2}\right] \geq K_1\mathbb{E}_{t,x,z}[\eta_i(T)^2] - K_{2,x}. \tag{2.8}$$

for constants $K_1, K_{2,x} > 0$. It follows that if $\mathbb{E}_{t,x,z}[\eta_i(T)^2] > \frac{J(0,0)+K_{2,x}+1}{K_1}$ then η cannot be optimal. We have $J(0,0) < \infty$ due to Corollary 2.3.2.

The estimate in terms of the passive order u is slightly more involved. Let τ_i denote the first jump time of the Poisson process N_i after t for $i = 1,2$, an exponentially distributed random variable with parameter λ_i, and set $\tau \triangleq \tau_1 \wedge \tau_2 \wedge T$. At the jump time τ the state process jumps from $X(\tau-)$ to

$$X(\tau-) + \Delta_N X(\tau) \triangleq X(\tau-) + \begin{pmatrix} 0 \\ 0 \\ u_1(\tau)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}} \end{pmatrix}.$$

We use the definition of the cost functional and the fact that the cost terms are nonnegative to get

$$\begin{aligned} J(\eta,u) =& \mathbb{E}_{t,x,z}\Bigg[\int_{[t,\tau)}\left[X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\right]d\eta_1(r) + \int_{[t,\tau)}\left[X_2(r-) + \frac{\kappa_2}{2}\Delta\eta_2(r)\right]d\eta_2(r) \\ &+ \int_t^\tau h\left(X_3(r) - \alpha(r,Z(r))\right)dr + J\left(\tau, X(\tau-) + \Delta_N X(\tau), Z(\tau), \eta, u\right)\Bigg] \\ \geq& \mathbb{E}_{t,x,z}\left[J(\tau, X(\tau-) + \Delta_N X(\tau), Z(\tau), \eta, u)\right] \\ \geq& \mathbb{E}_{t,x,z}\left[v(\tau, X(\tau-) + \Delta_N X(\tau), Z(\tau))\right], \end{aligned} \tag{2.9}$$

where J in the above is evaluated at controls on the stochastic interval[2] $[\tau, T]$. Combining this with Lemma 2.3.4 we get

$$J(\eta,u) \geq \mathbb{E}_{t,x,z}\left[c_{1,t}|X_3(\tau-) + \Delta_N X_3(\tau)|^2 - c_2(1 + \|Z(\tau)\|_{\mathbb{R}^n}^{c_3})\right].$$

[2]More precisely, we split the interval $[t,T]$ into the subintervals $[t,\tau]$ and $(\tau,T]$. By definition of the cost functional, the singular order on the second subinterval $(\tau,T]$ includes a possible jump at the left endpoint τ, so this jump must be excluded from the first subinterval $[t,\tau]$. For this reason, the state process directly after the Poisson jump in (2.9) is given by $X(\tau-) + \Delta_N X(\tau)$ and *not* by $X(\tau-) + \Delta_N X(\tau) + \Delta_\eta X(\tau) = X(\tau)$.

In view of Lemma 2.3.1 we have

$$\mathbb{E}_{t,x,z}\left[\|Z(\tau)\|_{\mathbb{R}^n}^{c_3}\right] \leq \mathbb{E}_{t,x,z}\left[\sup_{s\in[t,T]} \|Z(s)\|_{\mathbb{R}^n}^{c_3}\right] \leq c\left(1+\|z\|_{\mathbb{R}^n}^{c_3}\right),$$

and thus there is a constant $c_{2,z} \geq 0$ such that

$$J(\eta,u) \geq -c_{2,z} + c_{1,t}\mathbb{E}\left[|X_3(\tau-)+\Delta_N X_3(\tau)|^2\right]. \tag{2.10}$$

By definition, the stock holdings directly after a jump of the Poisson process are given by

$$X_3(\tau-)+\Delta_N X_3(\tau) = x_3+\eta_1(\tau-)-\eta_2(\tau-)+u_1(\tau)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}},$$

and an application of the inequality $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$ leads to

$$\begin{aligned}
&|X_3(\tau-)+\Delta_N X_3(\tau)|^2\\
\geq&\frac{1}{2}\left(u_1(\tau_1)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}}\right)^2 - \left(x_3+\eta_1(\tau-)-\eta_2(\tau-)\right)^2\\
\geq&\frac{1}{2}\left(u_1(\tau_1)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}}\right)^2 - 3\left(|x_3|^2+|\eta_1(\tau-)|^2+|\eta_2(\tau-)|^2\right)\\
\geq&\frac{1}{2}\left(u_1(\tau_1)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}}\right)^2 - 3\left(|x_3|^2+|\eta_1(T)|^2+|\eta_2(T)|^2\right).
\end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11) we get

$$\begin{aligned}
&\frac{1}{2}c_{1,t}\mathbb{E}_{t,x,z}\left[\left(u_1(\tau_1)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}}\right)^2\right]\\
\leq& J(\eta,u)+c_{2,z}+3c_{1,t}\left(|x_3|^2+\mathbb{E}_{t,x,z}\left[|\eta_1(T)|^2+|\eta_2(T)|^2\right]\right).
\end{aligned}$$

Due to equation (2.8) we have for $i=1,2$

$$\mathbb{E}_{t,x,z}[|\eta_i(T)|^2] \leq \frac{K_{2,x}}{K_1} + \frac{1}{K_1}J(\eta,u),$$

so combining the last two displays and relabelling constants provides

$$\mathbb{E}_{t,x,z}\left[\left(u_1(\tau_1)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}}\right)^2\right] \leq c_{1,t,x,z}+c_{2,t}J(\eta,u). \tag{2.12}$$

We shall now compute the term on the left hand side of inequality (2.12). The jump times τ_i are independent and exponentially distributed with parameter λ_i, for $i=1,2$. We thus have

$$\mathbb{E}_{t,x,z}\left[\left(u_1(\tau_1)\mathbb{1}_{\{\tau_1<\tau_2\wedge T\}} - u_2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\wedge T\}}\right)^2\right]$$

$$
\begin{aligned}
&= \int_t^\infty \int_t^\infty \lambda_1 e^{-\lambda_1(r_1-t)} \lambda_2 e^{-\lambda_2(r_2-t)} \left(u_1(r_1) \mathbb{1}_{\{r_1 < r_2 \wedge T\}} - u_2(r_2) \mathbb{1}_{\{r_2 < r_1 \wedge T\}} \right)^2 dr_1 dr_2 \\
&\geq \int_T^\infty \int_t^T \lambda_1 e^{-\lambda_1(r_1-t)} \lambda_2 e^{-\lambda_2(r_2-t)} |u_1(r_1)|^2 dr_1 dr_2,
\end{aligned}
$$

where we have used the nonnegativity of the integrand in the last line and restricted integration to $(r_1, r_2) \in [t, T] \times [T, \infty)$. We now compute

$$
\begin{aligned}
&\mathbb{E}_{t,x,z} \left[\left(u_1(\tau_1) \mathbb{1}_{\{\tau_1 < \tau_2 \wedge T\}} - u_2(\tau_2) \mathbb{1}_{\{\tau_2 < \tau_1 \wedge T\}} \right)^2 \right] \\
\geq &\int_T^\infty \lambda_2 e^{-\lambda_2(r_2-t)} dr_2 \int_t^T \lambda_1 e^{-\lambda_1(r_1-t)} \mathbb{E}_{t,x,z} \left[|u_1(r_1)|^2 \right] dr_1 \\
= &e^{-\lambda_2(T-t)} \int_t^T \lambda_1 e^{-\lambda_1(r_1-t)} \mathbb{E}_{t,x,z} \left[|u_1(r_1)|^2 \right] dr_1 \\
\geq &e^{-\lambda_2(T-t)} \lambda_1 e^{-\lambda_1(T-t)} \int_t^T \mathbb{E}_{t,x,z} \left[|u_1(r_1)|^2 \right] dr_1.
\end{aligned}
$$

Combining this with equation (2.12) and relabelling constants we get

$$
\mathbb{E}_{t,x,z} \left[\int_t^T |u_1(r)|^2 dr \right] \leq c_{1,t,x,z} + c_{2,t} J(\eta, u).
$$

In particular if

$$
\mathbb{E}_{t,x,z} \left[\int_t^T |u_1(r)|^2 dr \right] \geq c_{1,t,x,z} + c_{2,t} J(0, 0) + 1,
$$

then we see that $J(\eta, u) > J(0, 0)$ and the control (η, u) is clearly not optimal. A similar estimate holds for the passive sell order u_2. □

Theorem 2.3.6. *There is a unique optimal control $(\hat{\eta}, \hat{u}) \in \mathcal{U}_t$ for Problem 2.2.3.*

Proof. Let $(\eta^n, u^n)_{n \in \mathbb{N}} \subset \mathcal{U}_t$ be a minimising sequence, i.e.

$$
\lim_{n \to \infty} J(\eta^n, u^n) = \inf_{(\eta, u) \in \mathcal{U}_t} J(\eta, u).
$$

Recall that the singular control η^n is a nondecreasing càdlàg process, whereas u^n is absolutely continuous. Identifying u^n with the nondecreasing càdlàg process $\int_t^\cdot u^n(r) dr$, we can also interpret u^n as a singular control. Due to the uniform L^2-norm bound from Lemma 2.3.5 we can then apply the Komlós theorem for singular stochastic control given in Kabanov [1999] Lemma 3.5. It provides the existence of a subsequence (also indexed by n) and adapted processes $\hat{\eta} : [t, T] \times \Omega \to \mathbb{R}_+^2$ such that

$$
\bar{\eta}^n \triangleq \frac{1}{n} \sum_{i=1}^n \eta^i
$$

converges weakly to $\hat{\eta}$ in the sense that for almost all $\omega \in \Omega$ the measures $\bar{\eta}^n(\omega)$ on $[t,T]$ converge weakly to $\hat{\eta}(\omega)$. Similarly there is an adapted process $\xi : [t,T] \times \Omega \to \mathbb{R}^2_+$ such that

$$\bar{u}^n \triangleq \frac{1}{n}\sum_{i=1}^{n} u^i$$

converges weakly to ξ. However it is not yet clear that the limit ξ is absolutely continuous with respect to Lebesgue measure, so it is not an admissible passive order. Therefore we now fix $\hat{\eta}$ and consider the mapping $u \mapsto J(\hat{\eta}, u)$. The sequence of controls $(u^n)_{n\in\mathbb{N}}$ is still a minimising sequence, i.e.

$$\lim_{n\to\infty} J(\hat{\eta}, u^n) = \inf_u J(\hat{\eta}, u).$$

A Komlós argument as in the proof of Theorem 1.3.1 now provides the existence of a further subsequence (also indexed by n) and a predictable process $\hat{u} = \hat{u}$ which takes values from $[t,T] \times \Omega$ and is valued in $\mathbb{R}^2_+$ such that

$$\bar{u}^n \triangleq \frac{1}{n}\sum_{i=1}^{n} u^i$$

converges to $\hat{u}$ $ds \times d\mathbb{P}$ a.e. on $[t,T] \times \Omega$.

We now show that $(\hat{\eta}, \hat{u})$ is an optimal control. The weak convergence coupled with equation (2.7) implies that for $s \in [t,T]$ such that $\Delta\hat{\eta}(s) = 0$ we have a.s.

$$\begin{aligned}\lim_{n\to\infty} X_1^{\bar{\eta}^n,\bar{u}^n}(s) &= \lim_{n\to\infty}\left[e^{-\rho_1(s-t)}x_1 + \kappa_1\int_{[t,s]} e^{-\rho_1(r-t)}d\bar{\eta}_1^n(r)\right]\\ &= e^{-\rho_1(s-t)}x_1 + \kappa_1\int_{[t,s]} e^{-\rho_1(r-t)}d\hat{\eta}_1(r) = X_1^{\hat{\eta},\hat{u}}(s),\end{aligned}$$

and similarly for X_2 and X_3. Combining Fatou's lemma with the convexity of J gives

$$J(\hat{\eta},\hat{u}) \le \liminf_{n\to\infty} J(\bar{\eta}^n, \bar{u}^n) \le \liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} J(\eta^i, u^i) = \inf_{(\eta,u)\in\mathcal{U}_t} J(\eta,u),$$

which shows that $(\hat{\eta}, \hat{u})$ minimises J over $\mathcal{U}_t$. Uniqueness is due to the strict convexity of $(\eta, u) \mapsto J(\eta, u)$ and can be shown as in the proof of Theorem 1.3.1. ☐

Throughout, we denote by $(\hat{\eta}, \hat{u})$ the optimal control and by $\hat{X} = X^{\hat{\eta},\hat{u}}$ the optimal state trajectory.

2.4. The Stochastic Maximum Principle

In the preceding section we showed that Problem 2.2.3 admits a unique solution under Assumption 2.2.5. We shall now prove a version of the stochastic maximum principle which yields a characterisation of the optimal control in terms of the adjoint equation. In the sequel, we impose Assumption 2.2.7 and we write $\mathbb{E}$ instead of $\mathbb{E}_{t,x,z}$.

The *adjoint equation* is defined as the following BSDE on $[t,T]$,

$$\begin{aligned}
\begin{pmatrix} P_1(s)-P_1(t-) \\ P_2(s)-P_2(t-) \\ P_3(s)-P_3(t-) \end{pmatrix} &= \int_t^s \begin{pmatrix} \rho_1 P_1(r) \\ \rho_2 P_2(r) \\ h'(\hat{X}_3(r)-\alpha(r,Z(r))) \end{pmatrix} dr + \int_t^s \begin{pmatrix} Q_1(r) \\ Q_2(r) \\ Q_3(r) \end{pmatrix} dW(r) \\
&\quad + \int_t^s \begin{pmatrix} R_{1,1}(r) \\ R_{1,2}(r) \\ R_{1,3}(r) \end{pmatrix} \tilde{N}_1(dr) + \int_t^s \begin{pmatrix} R_{2,1}(r) \\ R_{2,2}(r) \\ R_{2,3}(r) \end{pmatrix} \tilde{N}_2(dr) \\
&\quad + \int_t^s \int_{\mathbb{R}^k} \begin{pmatrix} R_{3,1}(r,\theta) \\ R_{3,2}(r,\theta) \\ R_{3,3}(r,\theta) \end{pmatrix} \tilde{M}(dr,d\theta) \\
&\quad + \int_{[t,s]} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} d\hat{\eta}_1(r) + \int_{[t,s]} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} d\hat{\eta}_2(r), \qquad (2.13) \\
\begin{pmatrix} P_1(T) \\ P_2(T) \\ P_3(T) \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ -f'(\hat{X}_3(T)-\alpha(T,Z(T))) \end{pmatrix}.
\end{aligned}$$

Remark 2.4.1. Note that the optimal control $\hat{\eta}$ now enters the adjoint equation, which is not the case in the "usual" formulation of singular control problems, see e.g. Cadenillas and Haussmann [1994]. We will show in Section 2.6 that the solution to the BSDE defined above provides the solution to a *reflected* BSDE, where the bid ask spread plays the role of the reflecting barrier.

The adjoint process is then a triple of processes (P,Q,R) defined for $j=1,2,3$ on $[t,T]$ by

$$P(s) \triangleq \begin{pmatrix} P_1(s) \\ P_2(s) \\ P_3(s) \end{pmatrix}, Q(s) \triangleq \begin{pmatrix} Q_1(s) \\ Q_2(s) \\ Q_3(s) \end{pmatrix} \text{ and } R(s) \triangleq \begin{pmatrix} R_{1,1}(s) & R_{1,2}(s) & R_{1,3}(s) \\ R_{2,1}(s) & R_{2,2}(s) & R_{2,3}(s) \\ R_{3,1}(s) & R_{3,2}(s) & R_{3,3}(s) \end{pmatrix},$$

which satisfy for $i=1,2,3$

$$\begin{aligned}
P_i &: [t,T]\times\Omega \to \mathbb{R}, \quad Q_i : [t,T]\times\Omega \to \mathbb{R}^d, \\
R_{1,i} &: [t,T]\times\Omega \to \mathbb{R}, \quad R_{2,i} : [t,T]\times\Omega \to \mathbb{R}, \quad R_{3,i} : [t,T]\times\mathbb{R}^k\times\Omega \to \mathbb{R}
\end{aligned}$$

and which also satisfy the dynamics (2.13) where P is adapted and Q,R are predictable.

Proposition 2.4.2. *The BSDE* (2.13) *admits a unique solution which satisfies for* $i=$

1, 2, 3

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\le s\le T}|P_i(s)|^2\right]+\mathbb{E}\left[\int_t^T\|Q_i(r)\|^2_{\mathbb{R}^d}dr\right]+\mathbb{E}\left[\int_t^T|R_{1,i}(r)|^2dr\right]\\
+&\mathbb{E}\left[\int_t^T|R_{2,i}(r)|^2dr\right]+\mathbb{E}\left[\int_t^T\int_{\mathbb{R}^k}|R_{3,i}(r,\theta)|^2m(d\theta)dr\right]<\infty.
\end{aligned}
$$

It is unique among triples (P,Q,R) satisfying the above integrability criterion.

Proof. We claim that the explicit solution to (2.13) is given by

$$
\begin{cases}
P_1(s)= & \mathbb{E}\left[-\int_{(s,T]}e^{-\rho_1(r-s)}d\hat{\eta}_1(r)\,|\,\mathcal{F}_s\right],\\
P_2(s)= & \mathbb{E}\left[-\int_{(s,T]}e^{-\rho_2(r-s)}d\hat{\eta}_2(r)\,|\,\mathcal{F}_s\right],\\
P_3(s)= & \mathbb{E}\left[-\int_s^T h'\left(\hat{X}_3(r)-\alpha(r,Z(r))\right)dr-f'(\hat{X}_3(T)-\alpha(T,Z(T)))\,|\,\mathcal{F}_s\right].
\end{cases}
\tag{2.14}
$$

To show that this is true, we first note that due to the linear growth assumptions on h' and f' and the growth estimates from Lemma 2.3.1, the functions f and h satisfy

$$
\mathbb{E}\left[\int_t^T\left|h'(\hat{X}_3(r)-\alpha(r,Z(r)))\right|^2dr\right]+\mathbb{E}\left[\left|f'(\hat{X}_3(T)-\alpha(T,Z(T)))\right|^2\right]<\infty.
$$

Moreover, we have by assumption for $i=1,2$

$$
\mathbb{E}\left[\left(\int_{[t,T]}d\hat{\eta}_i(r)\right)^2\right]=\mathbb{E}\left[\hat{\eta}_i(T)^2\right]<\infty.
$$

We only construct the solution for P_1, the representations for P_2 and P_3 follow by similar arguments. The proof proceeds in three steps. To start with, we consider the following "standard" BSDE without singular terms on $[t,T]$,

$$
\begin{aligned}
\bar{P}_1(s)-\bar{P}_1(t-)&=\int_t^s\bar{Q}_1(r)dW(r)+\int_t^s\bar{R}_{1,1}(r)\tilde{N}_1(dr)+\int_t^s\bar{R}_{2,1}(r)\tilde{N}_2(dr)\\
&\quad+\int_t^s\int_{\mathbb{R}^k}\bar{R}_{3,1}(r,\theta)\tilde{M}(dr,d\theta),\\
\bar{P}_1(T)&=-\int_{[t,T]}e^{\rho_1(T-r)}d\hat{\eta}_1(r).
\end{aligned}
$$

We claim that the solution to this BSDE is given by

$$
\bar{P}_1(s)=\mathbb{E}\left[-\int_{[t,T]}e^{\rho_1(T-r)}d\hat{\eta}_1(r)\middle|\mathcal{F}(s)\right]\triangleq\Phi(s).
$$

Indeed, Φ is a square integrable martingale and by the martingale representation theorem

Tang and Li [1994] Lemma 2.3 there exist unique predictable processes

$$\bar{Q}_1 : [t,T] \times \Omega \to \mathbb{R}^d, \quad \bar{R}_{1,1} : [t,T] \times \Omega \to \mathbb{R},$$
$$\bar{R}_{2,1} : [t,T] \times \Omega \to \mathbb{R}, \quad \bar{R}_{3,1} : [t,T] \times \mathbb{R}^k \times \Omega \to \mathbb{R}$$

satisfying

$$\mathbb{E}\left[\int_t^T \|\bar{Q}_1(r)\|^2_{\mathbb{R}^d} dr\right] + \mathbb{E}\left[\int_t^T |\bar{R}_{1,1}(r)|^2 dr\right]$$
$$+\mathbb{E}\left[\int_t^T |\bar{R}_{2,1}(r)|^2 dr\right] + \mathbb{E}\left[\int_t^T \int_{\mathbb{R}^k} |\bar{R}_{3,1}(r,\theta)|^2 m(d\theta) dr\right] < \infty,$$

such that for $s \in [t,T]$ we have

$$\begin{aligned}\Phi(s) - \Phi(t-) = &\int_t^s \bar{Q}_1(r) dW(r) + \int_t^s \bar{R}_{1,1}(r)\tilde{N}_1(dr) + \int_t^s \bar{R}_{2,1}(r)\tilde{N}_2(dr) \\ &+ \int_t^s \int_{\mathbb{R}^k} \bar{R}_{3,1}(r,\theta)\tilde{M}(dr,d\theta), \\ \Phi(T) = &- \int_{[t,T]} e^{\rho_1(T-r)} d\hat{\eta}_1(r).\end{aligned}$$

In a second step, we define the process

$$\tilde{P}_1(s) \triangleq \bar{P}_1(s) + \int_{[t,s]} e^{\rho_1(T-r)} d\hat{\eta}_1(r) = \mathbb{E}\left[- \int_{(s,T]} e^{\rho_1(T-r)} d\hat{\eta}_1(r) \mid \mathcal{F}(s)\right].$$

This process satisfies

$$\begin{aligned}\tilde{P}_1(s) - \tilde{P}_1(t-) = &\int_t^s \bar{Q}_1(r) dW(r) + \int_t^s \bar{R}_{1,1}(r)\tilde{N}_1(dr) + \int_t^s \bar{R}_{2,1}(r)\tilde{N}_2(dr) \\ &+ \int_t^s \int_{\mathbb{R}^k} \bar{R}_{3,1}(r,\theta)\tilde{M}(dr,d\theta) + \int_{[t,s]} e^{\rho_1(T-r)} d\hat{\eta}_1(r), \\ \tilde{P}_1(T) = &0.\end{aligned}$$

In a third step, we set

$$\begin{aligned}P_1(s) \triangleq &e^{-\rho_1(T-s)}\tilde{P}_1(s) = \mathbb{E}\Big[- \int_{(s,T]} e^{-\rho_1(r-s)} d\hat{\eta}_1(r) \,|\, \mathcal{F}_s\Big], \\ Q_1(s) \triangleq &e^{-\rho_1(T-s)}\bar{Q}_1(s), \quad R_{1,1}(s) \triangleq e^{-\rho_1(T-s)}\bar{R}_{1,1}(s), \\ R_{2,1}(s) \triangleq &e^{-\rho_1(T-s)}\bar{R}_{2,1}(s), \quad R_{3,1}(s) \triangleq e^{-\rho_1(T-s)}\bar{R}_{3,1}(s)\end{aligned}$$

and apply the product rule to see that these processes satisfy

$$P_1(s) - P_1(t-) = \int_t^s \rho_1 P_1(r) dr + \int_t^s Q_1(r) dW(r) + \int_t^s R_{1,1}(r)\tilde{N}_1(dr)$$

$$
\begin{aligned}
&+\int_t^s R_{2,1}(r)\tilde{N}_2(dr)+\int_t^s\int_{\mathbb{R}^k} R_{3,1}(r,\theta)\tilde{M}(dr,d\theta)\\
&+\int_{[t,s]} d\hat{\eta}_1(r),\\
P_1(T)=&0.
\end{aligned}
$$

This proves that the process P_1 defined above is the solution of (the first component of) the adjoint equation (2.13). It remains to show that P_1 has integrable supremum. We first apply Doob's inequality to the martingale Φ,

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{t\le s\le T}|\Phi(s)|^2\Big] \le& c\mathbb{E}\big[\Phi(T)^2\big] = c\mathbb{E}\Big[\Big(\int_{[t,T]} e^{\rho_1(T-r)}d\hat{\eta}_1(r)\Big)^2\Big]\\
\le& ce^{2\rho_1 T}\mathbb{E}\big[\hat{\eta}_1(T)^2\big] < \infty.
\end{aligned}
$$

From the definition of P_1 we have for $s\in[t,T]$

$$
P_1(s)=e^{-\rho_1(T-s)}\tilde{P}_1(s)=e^{-\rho_1(T-s)}\left(\Phi(s)+\int_{[t,s]} e^{\rho_1(T-r)}d\hat{\eta}_1(r)\right)\le \Phi(s)+e^{\rho_1 T}\hat{\eta}_1(T).
$$

Combining the above two displays leads to

$$
\mathbb{E}\Big[\sup_{t\le s\le T}|P_1(s)|^2\Big]\le c\Big(\mathbb{E}\Big[\sup_{t\le s\le T}|\Phi(s)|^2\Big]+e^{\rho_1 T}\hat{E}\big[\eta_1(T)^2\big]\Big)<\infty.
$$

Similar arguments lead to the corresponding representations of P_2 and P_3. Uniqueness follows from the corresponding uniqueness in the martingale representation theorem. □

The characterisation of the optimal control we shall derive exploits an optimality condition in terms of the Gâteaux derivative of J. Given controls $(\eta,u),(\bar{\eta},\bar{u})\in\mathcal{U}_t$, it is defined as

$$
\langle J'(\bar{\eta},\bar{u}),(\eta,u)\rangle=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[J\left(\bar{\eta}+\varepsilon\eta,\bar{u}+\varepsilon u\right)-J(\bar{\eta},\bar{u})\right].
$$

In our particular case, the Gâteaux derivative can be computed explicitly. This is the content of the following lemma.

Lemma 2.4.3. *The performance functional $J:(\eta,u)\mapsto J(\eta,u)$ is Gâteaux differentiable. Its derivative is given by, for controls $(\eta,u),(\bar{\eta},\bar{u})\in\mathcal{U}_t$,*

$$
\begin{aligned}
&\langle J'(\bar{\eta},\bar{u}),(\eta,u)\rangle\\
=&\mathbb{E}\Big[\int_{[t,T]}\left[X_1^{\eta,u}(r-)-e^{-\rho_1(r-t)}x_1+\frac{\kappa_1}{2}\Delta\eta_1(r)\right]d\bar{\eta}_1(r)\\
&+\int_{[t,T]}\left[X_2^{\eta,u}(r-)-e^{-\rho_2(r-t)}x_2+\frac{\kappa_2}{2}\Delta\eta_2(r)\right]d\bar{\eta}_2(r)
\end{aligned}
$$

$$+\int_{[t,T]}\left[X_1^{\bar\eta,\bar u}(r-)+\frac{\kappa_1}{2}\Delta\bar\eta_1(r)\right]d\eta_1(r)+\int_{[t,T]}\left[X_2^{\bar\eta,\bar u}(r-)+\frac{\kappa_2}{2}\Delta\bar\eta_2(r)\right]d\eta_2(r)$$
$$+\int_t^T X_3^{\eta,u}(r)h'\left(X_3^{\bar\eta,\bar u}(r)-\alpha(r,Z(r))\right)dr+X_3^{\eta,u}(T)f'\left(X_3^{\bar\eta,\bar u}(T)-\alpha(T,Z(T))\right)\Bigg].$$

Proof. The terms involving h and f can be treated exactly as in Lemma 1.5.3, so it is enough to compute the Gâteaux derivative of

$$J_1(\eta,u)\triangleq\mathbb{E}\left[\int_{[t,T]}\left[X_1(r-)+\frac{\kappa_1}{2}\Delta\eta_1(r)\right]d\eta_1(r)\right].$$

From equation (2.7) it follows that the map $(\eta,u)\mapsto X_1^{\eta,u}$ is affine, so for $s\in[t,T]$, $\varepsilon\in[0,1]$ and $(\eta,u),(\bar\eta,\bar u)\in\mathcal{U}_t$ we have

$$\begin{aligned}X_1^{\bar\eta+\varepsilon\eta,\bar u+\varepsilon u}(s)=&X_1^{\bar\eta,\bar u}(s)+\varepsilon\kappa_1\int_{[t,s]}e^{-\rho_1(r-t)}d\eta_1(r)\\=&X_1^{\bar\eta,\bar u}(s)+\varepsilon\big(X_1^{\eta,u}(s)-e^{-\rho_1(s-t)}x_1\big).\end{aligned}$$

We can now compute

$$\begin{aligned}&\langle J_1'(\bar\eta,\bar u),(\eta,u)\rangle\\
=&\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\left[J_1\left(\bar\eta+\varepsilon\eta,\bar u+\varepsilon u\right)-J_1(\bar\eta,\bar u)\right]\\
=&\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\mathbb{E}\Big[\int_{[t,T]}\left[X_1^{\bar\eta+\varepsilon\eta,\bar u+\varepsilon u}(r-)+\frac{\kappa_1}{2}\Delta\bar\eta_1(r)+\varepsilon\frac{\kappa_1}{2}\Delta\eta_1(r)\right]d\left(\bar\eta_1(r)+\varepsilon\eta_1(r)\right)\\
&\quad-\int_{[t,T]}\left[X_1^{\bar\eta,\bar u}(r-)+\frac{\kappa_1}{2}\Delta\bar\eta_1(r)\right]d\bar\eta_1(r)\Big]\\
=&\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\mathbb{E}\Big[\int_{[t,T]}\Big[X_1^{\bar\eta,\bar u}(r-)+\varepsilon\big(X_1^{\eta,u}(r-)-e^{-\rho_1(r-t)}x_1\big)\\
&\quad+\frac{\kappa_1}{2}\Delta\bar\eta_1(r)+\varepsilon\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]d\bar\eta_1(r)-\int_{[t,T]}\left[X_1^{\bar\eta,\bar u}(r-)+\frac{\kappa_1}{2}\Delta\bar\eta_1(r)\right]d\bar\eta_1(r)\\
&\quad+\varepsilon\int_{[t,T]}\left[X_1^{\bar\eta,\bar u}(r-)+\varepsilon\big(X_1^{\eta,u}(r-)-e^{-\rho_1(r-t)}x_1\big)+\frac{\kappa_1}{2}\Delta\bar\eta_1(r)+\varepsilon\frac{\kappa_1}{2}\Delta\eta_1(r)\right]d\eta_1(r)\Big]\\
=&\mathbb{E}\Big[\int_{[t,T]}\left[X_1^{\eta,u}(r-)-e^{-\rho_1(r-t)}x_1+\frac{\kappa_1}{2}\Delta\eta_1(r)\right]d\bar\eta_1(r)\\
&\quad+\int_{[t,T]}\left[X_1^{\bar\eta,\bar u}(r-)+\frac{\kappa_1}{2}\Delta\bar\eta_1(r)\right]d\eta_1(r)\Big].\end{aligned}$$

This completes the proof. □

Our version of the maximum principle is based on an optimality condition on the Gâteaux derivative. As a prerequisite for some algebraic manipulations of the Gâteaux derivative, let us now compute $d(P\cdot X)$ for a fixed control $(\eta,u)\in\mathcal{U}_t$. Using integration by parts, we have for $s \in [t, T]$

$$
\begin{aligned}
&P(s)X(s) - P(t-)X(t-) \\
&= \int_t^s X_3(r-)h'\big(\hat{X}_3(r) - \alpha(r, Z(r))\big)dr \\
&\quad + \int_t^s \big[\lambda_1 u_1(r)\big(P_3(r) + R_{1,3}(r)\big) - \lambda_2 u_2(r)\big(P_3(r) + R_{2,3}(r)\big)\big]dr \\
&\quad + \int_t^s \big[X_1(r-)Q_1(r) + X_2(r-)Q_2(r) + X_3(r-)Q_3(r)\big]dW(r) \\
&\quad + \int_t^s \Big[X_1(r-)R_{1,1}(r) + X_2(r-)R_{1,2}(r) + X_3(r-)R_{1,3}(r) \\
&\qquad\quad + u_1(r)\big(P_3(r-) + R_{1,3}(r)\big)\Big]\tilde{N}_1(dr) \\
&\quad + \int_t^s \Big[X_1(r-)R_{2,1}(r) + X_2(r-)R_{2,2}(r) + X_3(r-)R_{2,3}(r) \\
&\qquad\quad - u_2(r)\big(P_3(r-) + R_{2,3}(r)\big)\Big]\tilde{N}_2(dr) \\
&\quad + \int_t^s \int_{\mathbb{R}^k} \big[X_1(r-)R_{3,1}(r,\theta) + X_2(r-)R_{3,2}(r,\theta) + X_3(r-)R_{3,3}(r,\theta)\big]\tilde{M}(dr, d\theta) \\
&\quad + \int_{[t,s]} \big[\kappa_1 P_1(r) + P_3(r)\big]d\eta_1(r) + \int_{[t,s]} \big[\kappa_2 P_2(r) - P_3(r)\big]d\eta_2(r) \\
&\quad + \int_{[t,s]} X_1(r-)d\hat{\eta}_1(r) + \int_{[t,s]} X_2(r-)d\hat{\eta}_2(r).
\end{aligned}
$$

This can be written as

$$
Y^{\eta,u}(s) = P(t-)X(t-) + L^{\eta,u}(s), \tag{2.15}
$$

where we define the "local martingale part" $L^{\eta,u}$ for $s \in [t, T]$ by

$$
\begin{aligned}
L^{\eta,u}(s) \triangleq &\int_t^s \big[X_1(r-)Q_1(r) + X_2(r-)Q_2(r) + X_3(r-)Q_3(r)\big]dW(r) \\
&+ \int_t^s \Big[X_1(r-)R_{1,1}(r) + X_2(r-)R_{1,2}(r) + X_3(r-)R_{1,3}(r) \\
&\qquad + u_1(r)\big(P_3(r-) + R_{1,3}(r)\big)\Big]\tilde{N}_1(dr) \\
&+ \int_t^s \Big[X_1(r-)R_{2,1}(r) + X_2(r-)R_{2,2}(r) + X_3(r-)R_{2,3}(r) \\
&\qquad - u_2(r)\big(P_3(r-) + R_{2,3}(r)\big)\Big]\tilde{N}_2(dr) \\
&+ \int_t^s \int_{\mathbb{R}^k} \big[X_1(r-)R_{3,1}(r,\theta) + X_2(r-)R_{3,2}(r,\theta) + X_3(r-)R_{3,3}(r,\theta)\big]\tilde{M}(dr, d\theta),
\end{aligned}
$$

and the "non-martingale part" $Y^{\eta,u}$ for $s \in [t,T]$ by

$$\begin{aligned}
Y^{\eta,u}(s) \triangleq & P(s)X(s) - \int_t^s X_3(r)h'\left(\hat{X}_3(r) - \alpha(r, Z(r))\right) dr \\
& - \int_t^s \left[\lambda_1 u_1(r)(P_3(r) + R_{1,3}(r)) - \lambda_2 u_2(r)(P_3(r) + R_{2,3}(r))\right] dr \\
& - \int_{[t,s]} [\kappa_1 P_1(r) + P_3(r)] d\eta_1(r) - \int_{[t,s]} [\kappa_2 P_2(r) - P_3(r)] d\eta_2(r) \\
& - \int_{[t,s]} X_1(r-) d\hat{\eta}_1(r) - \int_{[t,s]} X_2(r-) d\hat{\eta}_2(r).
\end{aligned}$$

Let us now check that L is indeed a martingale.

Lemma 2.4.4. *For each $(\eta, u) \in \mathcal{U}_t$, the process $L^{\eta,u}$ is a martingale starting in* 0*.*

Proof. We first consider the process $\int_t^\cdot X_1(r-)Q_1(r)dW(r)$. To prove that it is a true martingale it is enough to check that

$$\mathbb{E}\left[\sup_{s\in[t,T]} \left|\int_t^s X_1(r-)Q_1(r)dW(r)\right|\right] < \infty.$$

An application of the Burkholder-Davis-Gundy and Hölder inequalities yields

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[t,T]} \left|\int_t^s X_1(r-)Q_1(r)dW(r)\right|\right] \leq & c\mathbb{E}\left[\left(\int_t^T \|X_1(r-)Q_1(r)\|^2_{\mathbb{R}^d} dr\right)^{\frac{1}{2}}\right] \\
\leq & c\mathbb{E}\left[\sup_{r\in[t,T]} |X_1(r)|^2\right]^{\frac{1}{2}} \mathbb{E}\left[\int_t^T \|Q_1(r)\|^2_{\mathbb{R}^d} dr\right]^{\frac{1}{2}}.
\end{aligned}$$

The last expression is finite due to Lemma 2.3.1 and Proposition 2.4.2. Now consider the process $\int_t^\cdot \int_{\mathbb{R}^k} X_1(r-)R_1(r,\theta)\tilde{M}(dr, d\theta)$. A Hölder argument as above shows that

$$\mathbb{E}\left[\int_t^T \int_{\mathbb{R}^k} |X_1(r)R_1(r,\theta)|\, m(d\theta)dr\right] < \infty.$$

The martingale property now follows from Lemma A.1.3. The remaining terms of $L^{\eta,u}$ can be treated similarly. □

We are now in a position to formulate our second main result, the stochastic maximum principle in integral form.

Theorem 2.4.5. *A control $(\hat{\eta}, \hat{u}) \in \mathcal{U}_t$ is optimal if and only if for each $(\eta, u) \in \mathcal{U}_t$ we* *have*

$$\begin{cases} \mathbb{E}\Big[\int_{[t,T]} \Big[\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r)\Big] \, d\left(\eta_1(r) - \hat{\eta}_1(r)\right) \Big] \geq 0, \\ \mathbb{E}\Big[\int_{[t,T]} \Big[\hat{X}_2(r) - \kappa_2 P_2(r) + P_3(r)\Big] \, d\left(\eta_2(r) - \hat{\eta}_2(r)\right) \Big] \geq 0, \\ \mathbb{E}\Big[\int_t^T \left[u_1(r) - \hat{u}_1(r)\right] \left[R_{1,3}(r) + P_3(r)\right] dr \Big] \leq 0, \\ \mathbb{E}\Big[\int_t^T \left[u_2(r) - \hat{u}_2(r)\right] \left[R_{2,3}(r) + P_3(r)\right] dr \Big] \geq 0. \end{cases} \tag{2.16}$$

Proof. We proceed as in Cadenillas and Haussmann [1994] Theorem 4.1. We are minimising the convex functional J over $\mathcal{U}_t$, so by Ekeland and Témam [1999] Proposition 2.2.1 a necessary and sufficient condition for optimality of $(\hat{\eta}, \hat{u})$ is that

$$\langle J'(\hat{\eta}, \hat{u}), (\eta - \hat{\eta}, u - \hat{u})\rangle \geq 0 \text{ for each } (\eta, u) \in \mathcal{U}_t.$$

Due to Lemma 2.4.4 we know that $L^{\eta,u}$ is a martingale starting in zero for each $(\eta, u) \in \mathcal{U}_t$. In particular from equation (2.15) we have that $\mathbb{E}[Y^{\eta,u}(T) - Y^{\hat{\eta},\hat{u}}(T)] = 0$. The definition of $Y^{\eta,u}$ together with the terminal condition (2.13) for the adjoint equation allows us to write this as

$$\begin{aligned} 0 = \mathbb{E}\Big[& f'\left(\hat{X}_3(T) - \alpha(T, Z(T))\right) \left[X_3(T) - \hat{X}_3(T)\right] \\ & + \int_t^T h'\left(\hat{X}_3(r) - \alpha(r, Z(r))\right) \left[X_3(r) - \hat{X}_3(r)\right] dr \\ & + \int_{[t,T]} \left[P_3(r) + \kappa_1 P_1(r)\right] d(\eta_1(r) - \hat{\eta}_1(r)) \\ & + \int_{[t,T]} \left[-P_3(r) + \kappa_2 P_2(r)\right] d(\eta_2(r) - \hat{\eta}_2(r)) \\ & + \int_{[t,T]} \left[X_1(r-) - \hat{X}_1(r-)\right] d\hat{\eta}_1(r) + \int_{[t,T]} \left[X_2(r-) - \hat{X}_2(r-)\right] d\hat{\eta}_2(r) \\ & + \lambda_1 \int_t^T \left[u_1(r) - \hat{u}_1(r)\right] \left[P_3(r) + R_{1,3}(r)\right] dr \\ & - \lambda_2 \int_t^T \left[u_2(r) - \hat{u}_2(r)\right] \left[P_3(r) + R_{2,3}(r)\right] dr \Big]. \end{aligned}$$

Combining this with the explicit formula for the Gâteaux derivative given in Proposition 2.4.3 yields

$$\begin{aligned} & \langle J'(\hat{\eta}, \hat{u}), (\eta - \hat{\eta}, u - \hat{u})\rangle \\ =& \mathbb{E}\Big[\int_{[t,T]} \frac{\kappa_1}{2} \left[\Delta\eta_1(r) - \Delta\hat{\eta}_1(r)\right] d\hat{\eta}_1(r) + \int_{[t,T]} \frac{\kappa_2}{2} \left[\Delta\eta_2(r) - \Delta\hat{\eta}_2(r)\right] d\hat{\eta}_2(r) \\ & + \int_{[t,T]} \left[\hat{X}_1(r-) + \frac{\kappa_1}{2}\Delta\hat{\eta}_1(r) - P_3(r) - \kappa_1 P_1(r)\right] d\left(\eta_1(r) - \hat{\eta}_1(r)\right) \end{aligned}$$

$$
\begin{aligned}
&+ \int_{[t,T]} \left[\hat{X}_2(r-) + \frac{\kappa_2}{2}\Delta\hat{\eta}_2(r) + P_3(r) - \kappa_2 P_2(r)\right] d\left(\eta_2(r) - \hat{\eta}_2(r)\right) \\
&- \lambda_1 \int_t^T \left[u_1(r) - \hat{u}_1(r)\right]\left[P_3(r) + R_{1,3}(r)\right] dr \\
&+ \lambda_2 \int_t^T \left[u_2(r) - \hat{u}_2(r)\right]\left[P_3(r) + R_{2,3}(r)\right] dr\Bigg].
\end{aligned}
$$

Note that for $i = 1, 2$ we have, using the notation from equation (2.2),

$$
\begin{aligned}
&\mathbb{E}\Bigg[\int_{[t,T]} \frac{\kappa_i}{2}\left[\Delta\eta_i(r) - \Delta\hat{\eta}_i(r)\right] d\hat{\eta}_i(r) + \int_{[t,T]} \left[\hat{X}_i(r-) + \frac{\kappa_i}{2}\Delta\hat{\eta}_i(r)\right] d(\eta_i(r) - \hat{\eta}_i(r))\Bigg] \\
=&\mathbb{E}\Bigg[\int_{[t,T]} \hat{X}_i(r-) d(\eta_i(r) - \hat{\eta}_i(r)) \\
&\quad + \frac{\kappa_i}{2} \sum_{r\in[t,T]} \left[\Delta\eta_i(r) - \Delta\hat{\eta}_i(r)\right]\Delta\hat{\eta}_i(r) + \Delta\hat{\eta}_i(r)\left[\Delta\eta_i(r) - \Delta\hat{\eta}_i(r)\right]\Bigg] \\
=&\mathbb{E}\Bigg[\int_{[t,T]} \left[\hat{X}_i(r-) + \kappa_i\Delta\hat{\eta}_i(r)\right] d(\eta_i(r) - \hat{\eta}_i(r))\Bigg] \\
=&\mathbb{E}\Bigg[\int_{[t,T]} \hat{X}_i(r) d(\eta_i(r) - \hat{\eta}_i(r))\Bigg].
\end{aligned}
$$

Combining the above two displays leads to

$$
\begin{aligned}
\langle J'(\hat{\eta}, \hat{u}), (\eta - \hat{\eta}, u - \hat{u})\rangle = \mathbb{E}\Bigg[&\int_{[t,T]} \left[\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r)\right] d\left(\eta_1(r) - \hat{\eta}_1(r)\right) \\
&+ \int_{[t,T]} \left[\hat{X}_2(r) - \kappa_2 P_2(r) + P_3(r)\right] d\left(\eta_2(r) - \hat{\eta}_2(r)\right) \\
&-\lambda_1 \int_t^T \left[u_1(r) - \hat{u}_1(r)\right]\left[P_3(r) + R_{1,3}(r)\right] dr \\
&+\lambda_2 \int_t^T \left[u_2(r) - \hat{u}_2(r)\right]\left[P_3(r) + R_{2,3}(r)\right] dr\Bigg].
\end{aligned}
$$

We conclude that $(\hat{\eta}, \hat{u})$ is optimal if and only if for all $(\eta, u) \in \mathcal{U}_t$ we have

$$
\begin{aligned}
&\mathbb{E}\Bigg[\int_{[t,T]} \left[\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r)\right] d\left(\eta_1(r) - \hat{\eta}_1(r)\right)\Bigg] \geq 0, \\
&\mathbb{E}\Bigg[\int_{[t,T]} \left[\hat{X}_2(r) - \kappa_2 P_2(r) + P_3(r)\right] d\left(\eta_2(r) - \hat{\eta}_2(r)\right)\Bigg] \geq 0, \\
&\mathbb{E}\Bigg[\int_t^T \left[u_1(r) - \hat{u}_1(r)\right]\left[R_{1,3}(r) + P_3(r)\right] dr\Bigg] \leq 0, \\
&\mathbb{E}\Bigg[\int_t^T \left[u_2(r) - \hat{u}_2(r)\right]\left[R_{2,3}(r) + P_3(r)\right] dr\Bigg] \geq 0.
\end{aligned}
$$

□

2.5. Buy, Sell and No-Trade Regions

In the preceding section we derived a characterisation of optimality in terms of all admissible controls. This condition is not always easy to verify. Therefore, we derive a further characterisation in the present section, this time in terms of buy, sell and no-trade regions. As a byproduct, this result shows that spread crossing is optimal if and only if the spread is smaller than some threshold.

We start with the main result of this section, which provides a necessary and sufficient condition of optimality in terms of the trajectory of the controlled system

$$\left(s, \hat{X}(s), P(s)\right)_{s\in[t,T]}.$$

The proof builds on arguments from Cadenillas and Haussmann [1994] Theorem 4.2 and extends them to the present framework where we have jumps and state-dependent singular cost terms.

Theorem 2.5.1. *A control* $(\hat{\eta}, \hat{u}) \in \mathcal{U}_t$ *is optimal if and only if it satisfies*

$$\begin{cases} \mathbb{P}\Big(\hat{X}_1(s) - \kappa_1 P_1(s) - P_3(s) \geq 0 \;\forall s \in [t,T]\Big) & = 1, \\ \mathbb{P}\Big(\hat{X}_2(s) - \kappa_2 P_2(s) + P_3(s) \geq 0 \;\forall s \in [t,T]\Big) & = 1, \end{cases} \tag{2.17}$$

as well as

$$\begin{cases} \mathbb{P}\Big(\int_{[t,T]} \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)>0\}} d\hat{\eta}_1(r) = 0\Big) & = 1, \\ \mathbb{P}\Big(\int_{[t,T]} \mathbb{1}_{\{\hat{X}_2(r)-\kappa_2 P_2(r)+P_3(r)>0\}} d\hat{\eta}_2(r) = 0\Big) & = 1, \end{cases} \tag{2.18}$$

and $ds \times d\mathbb{P}$ *a.e. on* $[t,T]\times\Omega$

$$\begin{cases} R_{1,3} + P_3 \leq 0 \text{ and } (R_{1,3} + P_3)\,\hat{u}_1 = 0, \\ R_{2,3} + P_3 \geq 0 \text{ and } (R_{2,3} + P_3)\,\hat{u}_2 = 0. \end{cases} \tag{2.19}$$

Proof. First, let $(\hat{\eta}, \hat{u})$ be optimal and define the stopping time

$$\nu(\omega) \triangleq \inf\left\{s \in [t,T] \,:\, \hat{X}_1(s) - \kappa_1 P_1(s) - P_3(s) < 0\right\},$$

with the convention $\inf\emptyset \triangleq \infty$. Consider the control defined by $u = \hat{u}, \eta_2 = \hat{\eta}_2$ and

$$\eta_1(s,\omega) \triangleq \hat{\eta}_1(s,\omega) + \mathbb{1}_{[\nu(\omega),T]}(s).$$

Then η_1 is equal to $\hat{\eta}_1$ except for an additional jump of size one at time ν. It also is càdlàg and increasing on $[t, T]$. An application of Theorem 2.4.5 yields

$$\begin{aligned} 0 \leq & \mathbb{E}\Big[\int_{[t,T]} \Big[\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r)\Big]\, d\left(\eta_1(r) - \hat{\eta}_1(r)\right)\Big] \\ = & \mathbb{E}\Big[\Big(\hat{X}_1(\nu) - \kappa_1 P_1(\nu) - P_3(\nu)\Big)\, \mathbb{1}_{\{\nu \leq T\}}\Big] \leq 0, \end{aligned}$$

which implies that $\mathbb{P}(\nu = \infty) = 1$. This proves the first line of (2.17), the second line follows by similar arguments. Now consider the control defined by $u = \hat{u}, \eta_2 = \hat{\eta}_2$ and

$$\begin{cases} \eta_1(t-) & = 0, \\ d\eta_1(s,\omega) & \triangleq \mathbb{1}_{\{\hat{X}_1(s,\omega) - \kappa_1 P_1(s,\omega) - P_3(s,\omega) \leq 0\}} d\hat{\eta}_1(s,\omega). \end{cases}$$

Then η_1 is càdlàg and increasing on $[t, T]$, since $\hat{\eta}_1$ is. Due to Theorem 2.4.5 we have

$$\begin{aligned} 0 \leq & \mathbb{E}\Big[\int_{[t,T]} \Big[\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r)\Big]\, d\left(\eta_1(r) - \hat{\eta}_1(r)\right)\Big] \\ = & \mathbb{E}\Big[\int_{[t,T]} \Big[\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r)\Big]\, \mathbb{1}_{\{\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r) > 0\}} d\left(-\hat{\eta}_1(r)\right)\Big] \leq 0, \end{aligned}$$

and in particular

$$0 = \mathbb{E}\Big[\int_{[t,T]} \mathbb{1}_{\{\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r) > 0\}} d\hat{\eta}_1(r)\Big],$$

which proves the first part of (2.18), the second part follows by similar arguments. It remains to prove (2.19). Again by Theorem 2.4.5 we have for every control $(\eta, u) \in \mathcal{U}_t$

$$0 \geq E\left[\int_t^T (u_1(r) - \hat{u}_1(r))(R_{1,3}(r) + P_3(r))dr\right].$$

Choosing the control $(\hat{\eta}, u)$ with $u_2 = \hat{u}_2$ and

$$u_1(r,\omega) = \hat{u}_1(r,\omega) + \mathbb{1}_{\{R_{1,3}(r-,\omega) + P_3(r-,\omega) > 0\}}$$

we first note that u_1 is predictable and we get

$$0 \geq E\left[\int_t^T \mathbb{1}_{\{R_{1,3}(r) + P_3(r) > 0\}}(R_{1,3}(r) + P_3(r))dr\right] \geq 0,$$

which shows that $R_{1,3} + P_3 \leq 0$ $ds \times d\mathbb{P}$ a.e. Recall that we also have $\hat{u}_1 \geq 0$ by definition. We now want to show at least one of the processes $R_{1,3} + P_3$ and $\hat{u}_1$ is zero. To this end, consider the control $(\hat{\eta}, u)$ whose passive orders are defined by $u_1 = \frac{1}{2}\hat{u}_1$ and $u_2 = \hat{u}_2$.

We then get

$$\begin{aligned} 0 \geq & E\left[\int_t^T (u_1(r) - \hat{u}_1(r))(R_{1,3}(r) + P_3(r))dr\right] \\ = & E\left[\int_t^T -\frac{1}{2}\hat{u}_1(r)(R_{1,3}(r) + P_3(r))dr\right] \geq 0, \end{aligned}$$

and it follows that $ds \times d\mathbb{P}$ a.e. we have $u_1(R_{1,3} + P_3) = 0$. The argument for the second line in (2.18) is similar. This proves the "only if" part of the assertion.

In order to prove the "if" part, let conditions (2.17), (2.18) and (2.19) be satisfied. We then have for each $(\eta, u) \in \mathcal{U}_t$

$$\begin{aligned} & \mathbb{E}\Big[\int_{[t,T]} \Big[\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r)\Big]\, d\left(\eta_1(r) - \hat{\eta}_1(r)\right)\Big] \\ = & \mathbb{E}\Big[\int_{[t,T]} \Big[\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r)\Big]\, d\eta_1(r)\Big] && (2.20) \\ & + \mathbb{E}\Big[\int_{[t,T]} \Big[\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r)\Big]\, \mathbb{1}_{\{\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r) > 0\}} d\left(-\hat{\eta}_1(r)\right)\Big] && (2.21) \\ & + \mathbb{E}\Big[\int_{[t,T]} \Big[\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r)\Big]\, \mathbb{1}_{\{\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r) \leq 0\}} d\left(-\hat{\eta}_1(r)\right)\Big]. && (2.22) \end{aligned}$$

The integrand of (2.20) is nonnegative due to condition (2.17), so (2.20) is nonnegative. The term (2.21) is zero due to condition (2.18). The term (2.22) has a nonpositive integrand and a decreasing integrator and is therefore also nonnegative. In conclusion, we have

$$\mathbb{E}\Big[\int_{[t,T]} \Big[\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r)\Big]\, d\left(\eta_1(r) - \hat{\eta}_1(r)\right)\Big] \geq 0,$$

and by a similar argument

$$\mathbb{E}\Big[\int_{[t,T]} \Big[\hat{X}_2(r) - \kappa_2 P_2(r) + P_3(r)\Big]\, d\left(\eta_2(r) - \hat{\eta}_2(r)\right)\Big] \geq 0.$$

Still for arbitrary $(\eta, u) \in \mathcal{U}_t$ we have using (2.19) and $u_1 \geq 0$

$$\mathbb{E}\Big[\int_t^T [u_1(r) - \hat{u}_1(r)]\, [R_{1,3}(r) + P_3(r)]\, dr\Big] = \mathbb{E}\Big[\int_t^T u_1(r)\, [R_{1,3}(r) + P_3(r)]\, dr\Big] \leq 0.$$

By a similar argument

$$\mathbb{E}\Big[\int_t^T [u_2(r) - \hat{u}_2(r)]\, [R_{2,3}(r) - P_3(r)]\, dr\Big] \leq 0.$$

An application of Theorem 2.4.5 now shows that $(\hat{\eta}, \hat{u})$ is indeed optimal. $\square$

The preceding theorem gives an optimality condition in terms of the controlled system $(P, \hat{X})$. We now show how Theorem 2.5.1 can be used to describe the optimal market order quite explicitly in terms of buy, sell and no-trade regions.

Definition 2.5.2. We define the buy, sell and no-trade regions (with respect to market orders) by

$$\begin{aligned}\mathcal{R}_{buy} &\triangleq \Big\{(s,x,p) \in [t,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \,|\, x_1 - \kappa_1 p_1 - p_3 < 0\Big\}, \\ \mathcal{R}_{sell} &\triangleq \Big\{(s,x,p) \in [t,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \,|\, x_2 - \kappa_2 p_2 + p_3 < 0\Big\}, \\ \mathcal{R}_{nt} &\triangleq \Big\{(s,x,p) \in [t,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \,|\, x_1 - \kappa_1 p_1 - p_3 > 0 \text{ and } x_2 - \kappa_2 p_2 + p_3 > 0\Big\}.\end{aligned}$$

Moreover, we define the boundaries of the buy and sell regions by

$$\begin{aligned}\partial R_{buy} &\triangleq \Big\{(s,x,p) \in [t,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \,|\, x_1 - \kappa_1 p_1 - p_3 = 0\Big\}, \\ \partial R_{sell} &\triangleq \Big\{(s,x,p) \in [t,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \,|\, x_2 - \kappa_2 p_2 + p_3 = 0\Big\}.\end{aligned}$$

Let us emphasise that each of the three regions defined above is open. We remark that the time variable s is included into the definition of the buy, sell and no-trade regions such that statements like "the trajectory of the process $(s, \hat{X}(s), P(s))$ under the optimal control is inside the no-trade region" make sense. Specifically, we now show that the optimal control remains inside the closure of the no-trade region at all times, i.e. it is either inside the no-trade region or on the boundary of the buy or sell region. Moreover, as long as the controlled system is inside the no-trade region, market orders are not used, i.e. $\hat{\eta}_i$ does not increase for $i = 1, 2$.

Proposition 2.5.3. *1. If $(s, \hat{X}(s), P(s))$ is in the no-trade region, it is optimal not to use market orders, i.e. for $i = 1, 2$*

$$\mathbb{E}\left[\int_{[t,T]} \mathbb{1}_{\{(r,\hat{X}(r),P(r)) \in \mathcal{R}_{nt}\}} d\hat{\eta}_i(r)\right] = 0.$$

2. The optimal trajectory remains a.s. inside the closure of the no-trade region,

$$\mathbb{P}\Big((s, \hat{X}(s), P(s)) \in \overline{\mathcal{R}_{nt}} \;\forall s \in [t,T]\Big) = 1.$$

In particular, it spends no time inside the buy and sell regions.

Proof. Item (1) is a direct consequence of (2.18), while (2) follows from (2.17). □

Example 2.5.4. *The particular case of portfolio liquidation is solved in Subsection 2.7.1. In this case, the optimal strategy is composed of discrete sell orders at times $t = 0, T$ and a constant rate of sell orders in $(0, T)$. Specifically, these are chosen such that the process $(s, \hat{X}(s), P(s))$ remains on the boundary between the sell region and the no-trade region for all $s \in [0, T]$. This provides an example where the controlled system is on the* *boundary of the sell region at all times. Another extreme case is studied in Subsection 2.7.3. There, we discuss an optimisation problem where the controlled system is always inside the no-trade region and never hits the boundary, so that it is optimal not to use market orders at all.*

The above proposition shows that the controlled system remains inside the closure of the no-trade region and market orders are not used inside the no-trade region. This suggests that markets orders are only used on the boundary, and we shall now make this more precise. To this end, we first note that for $i = 1, 2$ the nondecrasing process $\hat{\eta}_i$ induces a measure on $[t, T] \times \Omega$ by the following map

$$[t, s] \times A \mapsto \mathbb{E} \int_{[t,s]} \mathbb{1}_A d\hat{\eta}_i(r).$$

Proposition 2.5.5. *1. We have*

$$\mathbb{P}\Big(\hat{\eta}_1(s) = \int_{[t,s]} \mathbb{1}_{\{(r,\hat{X}(r),P(r))\in\partial\mathcal{R}_{buy}\}} d\hat{\eta}_1(r)\, \forall s \in [t, T] \Big) = 1.$$

In particular, the support of the measure induced by $\hat{\eta}_1$ is a subset of

$$\Big(r, \hat{X}(r), P(r)\Big) \in \partial\mathcal{R}_{buy},$$

i.e. market buy orders are only used if the controlled system is on the boundary of the buy region.

2. Similarly, we have

$$\mathbb{P}\Big(\hat{\eta}_2(s) = \int_{[t,s]} \mathbb{1}_{\{(r,\hat{X}(r),P(r))\in\partial\mathcal{R}_{sell}\}} d\hat{\eta}_2(r)\, \forall s \in [t, T] \Big) = 1.$$

In particular, the support of the measure induced by $\hat{\eta}_2$ is a subset of

$$\Big(r, \hat{X}(r), P(r)\Big) \in \partial\mathcal{R}_{sell},$$

i.e. market sell orders are only used if the controlled system is on the boundary of the sell region.

Proof. We only show the first assertion. For $s \in [t, T]$ we have using $\hat{\eta}_1(t-) = 0$

$$\begin{aligned}\hat{\eta}_1(s) &= \int_{[t,s]} d\hat{\eta}_1(r)\\ &= \int_{[t,s]} \big[\mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)<0\}} + \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)>0\}}\\ &\qquad + \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)=0\}}\big] d\hat{\eta}_1(r).\end{aligned}$$

We shall show that terms in the second line vanish a.s. By Proposition 2.5.3 (2) we have

$$\mathbb{P}\Big((r, \hat{X}(r), P(r)) \notin \mathcal{R}_{buy} \ \forall r \in [t,T]\Big) = 1$$

i.e. the optimal trajectory spends no time in the buy region, so that a.s. for each $s \in [t,T]$

$$\int_{[t,s]} \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)<0\}} d\hat{\eta}_1(r) = \int_{[t,s]} \mathbb{1}_{\{(r,\hat{X}(r),P(r))\in\mathcal{R}_{buy}\}} d\hat{\eta}_1(r) = 0.$$

Due to equation (2.18) we have a.s. for each $s \in [t,T]$

$$0 = \int_{[t,T]} \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)>0\}} d\hat{\eta}_1(r) \geq \int_{[t,s]} \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)>0\}} d\hat{\eta}_1(r) \geq 0,$$

so that a.s. for each $s \in [t,T]$

$$\int_{[t,s]} \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)>0\}} d\hat{\eta}_1(r) = 0.$$

This shows that a.s. for each $s \in [t,T]$ we have

$$\hat{\eta}_1(s) = \int_{[t,s]} \mathbb{1}_{\{\hat{X}_1(r)-\kappa_1 P_1(r)-P_3(r)=0\}} d\hat{\eta}_1(r) = \int_{[t,s]} \mathbb{1}_{\{(r,\hat{X}(r),P(r))\in\partial\mathcal{R}_{buy}\}} d\hat{\eta}_1(r).$$

□

In view of the preceding propositions we have now achieved our main goal, namely to show when spread crossing is optimal. Specifically, there is a threshold $\kappa_1 P_1 + P_3$ for the buy spread. If the buy spread is larger than this threshold, i.e. the controlled system is inside the no-trade region, then the costs of market buy orders are large as compared to the penalty for deviating from the target, and no market orders are used. Note that the threshold can be negative, in this case buying is not optimal at all, irrespective of the spread size. Market orders are only used to prevent a downward crossing of the threshold and as a result the buy spread is never smaller than the threshold. In this sense, the trajectory of the controlled system is reflected at the boundary of the no-trade region. This will be made more precise in Subsection 2.6 where the link to reflected BSDEs is discussed. A similar interpretation holds for the sell spread, where the threshold is given by $\kappa_2 P_2 - P_3$.

Proposition 2.5.5 shows that the support of the measure induced by $\hat{\eta}_1$ (respectively $\hat{\eta}_2$) is concentrated on the boundary of the buy (respectively sell) region. We now show that the supports of these measures are disjoint. The economic interpretation is that market buy and market sell orders are never used simultaneously. This makes sense, since trading on both sides of the market at the same time is counterintuitive.

Proposition 2.5.6. *The supports of the measures induced by $\hat{\eta}_1$ and $\hat{\eta}_2$, respectively, are disjoint. In particular, market buy and market sell orders are not used simultaneously.*

Proof. Let us denote by $supp(\hat{\eta}_i)$ the support of the measure induced by $\hat{\eta}_i$, for $i = 1, 2$. We use Proposition 2.5.5 to write

$$\begin{aligned} 0 \leq & \mathbb{E} \int_{[t,T]} \mathbb{1}_{supp(\hat{\eta}_1)} d\hat{\eta}_2(r) \\ = & \mathbb{E} \int_{[t,T]} \mathbb{1}_{supp(\hat{\eta}_1)} \mathbb{1}_{supp(\hat{\eta}_2)} d\hat{\eta}_2(r) \\ \leq & \mathbb{E} \int_{[t,T]} \mathbb{1}_{\{(r,\hat{X}(r),P(r)) \in \partial\mathcal{R}_{buy}\}} \mathbb{1}_{\{(r,\hat{X}(r),P(r)) \in \partial\mathcal{R}_{sell}\}} d\hat{\eta}_2(r). \end{aligned} \tag{2.23}$$

We define the stopping time ν as the first time the controlled system is on the boundary of the buy and on the boundary of the sell region,

$$\nu \triangleq \inf \left\{ r \in [t, T] \,\middle|\, (r, \hat{X}(r), P(r)) \in \partial\mathcal{R}_{buy} \cap \partial\mathcal{R}_{sell} \right\},$$

with the convention $\inf \emptyset = \infty$. By definition of the boundaries, on the set $\nu < \infty$ we then have

$$\hat{X}_1(\nu) - \kappa_1 P_1(\nu) = P_3(\nu) = \kappa_2 P_2(\nu) - \hat{X}_2(\nu).$$

Combining this with $\hat{X}_i \geq 0$ a.s. and $P_i \leq 0$ a.s. for $i = 1, 2$ we see that the left hand side is nonnegative and the right hand side is nonpositive, so both terms are zero and in particular

$$P_1(\nu) = P_2(\nu) = 0.$$

The representation (2.14) now implies

$$0 = \mathbb{E} \left[\int_{[\nu,T]} d\hat{\eta}_2(r) \right].$$

We combine this with inequality (2.23) to get

$$0 \leq \mathbb{E} \int_{[t,T]} \mathbb{1}_{supp(\hat{\eta}_1)} d\hat{\eta}_2(r) \leq \mathbb{E} \int_{[\nu,T]} \mathbb{1}_{supp(\hat{\eta}_1)} d\hat{\eta}_2(r) \leq \mathbb{E} \int_{[\nu,T]} d\hat{\eta}_2(r) = 0.$$

This proves that $supp(\hat{\eta}_1) \cap supp(\hat{\eta}_2) = \emptyset$. $\square$

2.6. Link to Reflected BSDEs

In this section we use the results from the preceding section to show that the adjoint process together with the optimal control is the solution to a reflected BSDE, where the obstacle is the spread. The following definition is taken from Øksendal and Sulem [2010].

Definition 2.6.1. Let $F : [t,T]\times\mathbb{R}\times\Omega \to \mathbb{R}$ be a measurable function, $L : [t,T]\times\Omega \to \mathbb{R}$ be an adapted càdlàg process and $G \in L^2$. We say that $(\tilde{P}, \tilde{Q}, \tilde{R}, K)$ is a solution to the *reflected BSDE* with driver F, reflecting barrier L and terminal condition G on the time interval $[t,T]$ if the following holds:

1. $\tilde{P}$ is adapted, $\tilde{Q}$ and $\tilde{R} \triangleq \begin{pmatrix}\tilde{R}_1\\ \tilde{R}_2\\ \tilde{R}_3\end{pmatrix}$ are predictable and they satisfy

$$\begin{aligned}\tilde{P} &: [t,T]\times\Omega \to \mathbb{R}, \quad \tilde{Q} : [t,T]\times\Omega \to \mathbb{R}^d,\\ \tilde{R}_1 &: [t,T]\times\Omega \to \mathbb{R}, \quad \tilde{R}_2 : [t,T]\times\Omega \to \mathbb{R}, \quad \tilde{R}_3 : [t,T]\times\mathbb{R}^k\times\Omega \to \mathbb{R}.\end{aligned}$$

2. K is nondecreasing and càdlàg with $K(t-) = 0$.

3. For all $s \in [t,T]$ we have

$$\begin{aligned}\tilde{P}(s) - \tilde{P}(t-) =& \int_t^s F(r, \tilde{P}(r))dr + \int_t^s \tilde{Q}(r)dW(r)\\ &+ \int_t^s \tilde{R}(r,\theta)\tilde{N}(d\theta, dr) - \int_{[t,s]} dK(r),\\ \tilde{P}(T) =& G.\end{aligned}$$

4. We have a.s. for all $s \in [t,T]$ that $\tilde{P}(s) \geq L(s)$.

5. We have a.s. that $\int_{[t,T]}(\tilde{P}(r) - L(r))dK(r) = 0$.

The interpretation is as follows: By item (4), the process $\tilde{P}$ is never below the barrier L. (5) means that the process K increases only if $\tilde{P}$ is at the barrier and is flat otherwise. Let us now define the following linear combinations of the adjoint processes:

$$\begin{aligned}\begin{pmatrix}\bar{P}_1\\ \bar{P}_2\end{pmatrix} &\triangleq \begin{pmatrix}-\kappa_1 P_1 - P_3\\ -\kappa_2 P_2 + P_3\end{pmatrix},\\ \begin{pmatrix}\bar{Q}_1\\ \bar{Q}_2\end{pmatrix} &\triangleq \begin{pmatrix}-\kappa_1 Q_1 - Q_3\\ -\kappa_2 Q_2 + Q_3\end{pmatrix},\\ \begin{pmatrix}\bar{R}_{1,1} & \bar{R}_{1,2}\\ \bar{R}_{2,1} & \bar{R}_{2,2}\\ \bar{R}_{3,1} & \bar{R}_{3,2}\end{pmatrix} &\triangleq \begin{pmatrix}-\kappa_1 R_{1,1} - R_{1,3} & -\kappa_2 R_{1,2} + R_{1,3}\\ -\kappa_1 R_{2,1} - R_{2,3} & -\kappa_2 R_{2,2} + R_{2,3}\\ -\kappa_1 R_{3,1} - R_{3,3} & -\kappa_2 R_{3,2} + R_{3,3}\end{pmatrix}.\end{aligned}$$

Proposition 2.6.2. *The process*

$$\left(\bar{P}_1, \bar{Q}_1, \begin{pmatrix}\bar{R}_{1,1}\\ \bar{R}_{2,1}\\ \bar{R}_{3,1}\end{pmatrix}, \kappa_1\hat{\eta}_1\right)$$

is a solution to the reflected BSDE with driver

$$-\kappa_1\rho_1 P_1(r) - h'(\hat{X}_3(r) - \alpha(r, Z(r))),$$

reflecting barrier $-\hat{X}_1$ *and terminal condition* $f'(\hat{X}_3(T) - \alpha(T, Z(T)))$. *Similarly, the process*

$$\left(\bar{P}_2, \bar{Q}_2, \begin{pmatrix}\bar{R}_{1,2}\\ \bar{R}_{2,2}\\ \bar{R}_{3,2}\end{pmatrix}, \kappa_2\hat{\eta}_2\right)$$

is a solution to the reflected BSDE with driver

$$-\kappa_2\rho_2 P_2(r) + h'(\hat{X}_3(r) - \alpha(r, Z(r))),$$

reflecting barrier $-\hat{X}_2$ *and terminal condition* $-f'(\hat{X}_3(T) - \alpha(T, Z(T)))$.

Proof. We only check the first assertion. The first two items of Definition 2.6.1 are clear. Item (3) follows from the dynamics of the adjoint process by direct computation. Specifically, we have for $s \in [t, T]$

$$\begin{aligned}
\bar{P}_1(s) - \bar{P}_1(t-) =& -\kappa_1(P_1(s) - P_1(t-)) - (P_3(s) - P_3(t-))\\
=& \int_t^s -\kappa_1\rho_1 P_1(r) - h'\left(\hat{X}_3(r) - \alpha(r, Z(r))\right) dr\\
&+ \int_t^s -\kappa_1 Q_1(r) - Q_3(r) dW(r)\\
&+ \int_t^s -\kappa_1 R_{1,1}(r) - R_{1,3}(r)\tilde{N}_1(dr) + \int_t^s -\kappa_1 R_{2,1}(r) - R_{2,3}(r)\tilde{N}_2(dr)\\
&+ \int_t^s \int_{\mathbb{R}^k} -\kappa_1 R_{3,1}(r,\theta) - R_{3,3}(r,\theta)\tilde{M}(dr, d\theta) - \int_{[t,s]} d(\kappa_1\hat{\eta}_1(r)),\\
\bar{P}_1(T) =& -\kappa_1 P_1(T) - P_3(T) = h'\left(\hat{X}_3(T) - \alpha(T, Z(T))\right).
\end{aligned}$$

Item (4) follows from equation (2.17) in Theorem 2.5.1. In order to verify item (5) we apply Proposition 2.5.5 to get

$$\begin{aligned}
&\int_{[t,T]} (\bar{P}_1(r) + \hat{X}_1(r)) d(\kappa_1\hat{\eta}_1(r)) \qquad (2.24)\\
=&\kappa_1 \int_{[t,T]} (-\kappa_1 P_1(r) - P_3(r) + \hat{X}_1(r)) \mathbb{1}_{\{\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r) = 0\}} d\hat{\eta}_1(r) = 0.
\end{aligned}$$

The second assertion follows from similar arguments. □

As our main focus is on a solution to the curve following problem and not on reflected BSDEs, we shall not pursue this further and instead refer the interested reader to Øksendal and Sulem [2010], El Karoui et al. [1997a] as well as Cvitanic and Ma [2001].

2.7. Examples

In this example section we shall apply the general results on curve following to specific subproblems. We first consider portfolio liquidation, where an investor wants to unwind a large position of stock shares in a short period of time, with as little adverse price impact as possible. Models and solutions have been proposed among others by Almgren and Chriss [2001] and Schied and Schöneborn [2008]. Our framework is inspired by Obizhaeva and Wang [2005]. In Subsection 2.7.1 we show how our characterisation of optimality can be used to recover their[3] solution. We extend the model to passive orders in Subsection 2.7.2. In the concluding Subsection 2.7.3 we analyse an optimisation problem where it is optimal never to trade.

2.7.1. Portfolio Liquidation with Singular Market Orders

The investor starts with stock holdings $X_3(0-) = x_3 > 0$ and wants to sell them such that

$$X_3(T) = 0. \tag{2.25}$$

In the present subsection, we remove passive orders and only consider controls from the following set

$$\mathcal{U}_0^\eta \triangleq \Big\{ \eta : [0,T] \to \mathbb{R}_+^2 \,\Big|\, \eta_i(0-) = 0,\ \eta_i(T)^2 < \infty, \\ \eta_i \text{ is nondecreasing and càdlàg for } i = 1,2 \Big\}.$$

The solution with passive orders will be given in Subsection 2.7.2. Heuristically, it should be optimal to use only market sell and no buy orders, however we allow for both types of orders and then prove that buying is not optimal. The constraint (2.25) ensures that the portfolio is liquidated by maturity. Thus we do not need to penalise deviation and may choose $h = f = \alpha = 0$. The portfolio liquidation problem is

Problem 2.7.1. *Minimise*

$$J(\eta) \triangleq \int_{[0,T]} \left[X_1(r-) + \frac{\kappa_1}{2} \Delta\eta_1(r) \right] d\eta_1(r) + \int_{[0,T]} \left[X_2(r-) + \frac{\kappa_2}{2} \Delta\eta_2(r) \right] d\eta_2(r)$$

over controls $\eta \in \mathcal{U}_0^\eta$ *such that* $\hat{X}_3(T) = 0$.

Note that this is a deterministic problem, so that P, X and η are deterministic functions. We give the explicit solution in the following proposition. While the result is known from Obizhaeva and Wang [2005] Proposition 3, who use an ad hoc martingale optimality principle, we show here how the result can be derived from the more gen-

[3] To be precise, they consider the equivalent task of *acquiring* a large number of stock shares.

eral maximum principle given in Theorem 2.4.5. As in Obizhaeva and Wang [2005] we assume that the initial spreads are zero, i.e. $x_i = X_i(0-) = 0$ for $i = 1, 2$.

As in the proof of Proposition 1.7.3, we do not solve the above constrained problem directly. Instead we introduce a sequence of auxiliary control problems without constraints, but with a penalty for stock holdings at maturity. For $n \in \mathbb{N}$ we define the relaxed optimisation problem with additional terminal costs $nX_3(T)^2$ by

Problem 2.7.2. *Minimise*

$$J^n(\eta) \triangleq \int_{[0,T]} \left[X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\right] d\eta_1(r) + \int_{[0,T]} \left[X_2(r-) + \frac{\kappa_2}{2}\Delta\eta_2(r)\right] d\eta_2(r) + nX_3(T)^2$$

over controls $\eta \in \mathcal{U}_0^\eta$.

We first solve the auxiliary control problem.

Proposition 2.7.3. *The solution to Problem 2.7.2 is given by discrete market sell orders at times* $s = 0$ *and* $s = T$ *of size*

$$\Delta\hat{\eta}_2^n(0) = \Delta\hat{\eta}_2^n(T) = \frac{x_3}{\rho_2 T + 2 + \frac{\kappa_2}{n}}$$

and a constant rate of market sell orders in $(0, T)$ *given by*

$$d\hat{\eta}_2^n(s) = \rho_2 \Delta\hat{\eta}_2^n(0)ds = \frac{\rho_2 x_3}{\rho_2 T + 2 + \frac{\kappa_2}{n}} ds.$$

Market buy orders are not used, i.e. $\hat{\eta}_1^n \equiv 0$.

Proof. We first note that Assumptions 2.2.5 and 2.2.7 are satisfied in the present framework, so that the results from the previous sections can be applied. For fixed $n \in \mathbb{N}$, the state dynamics are given for $s \in [0, T]$ by

$$\begin{cases} X_1^{\hat{\eta}^n}(s) = & 0, \\ X_2^{\hat{\eta}^n}(s) = & -\int_0^s \rho_2 X_2^{\hat{\eta}^n}(r)dr + \int_{[0,s]} \kappa_2 d\hat{\eta}_2^n(r), \\ X_3^{\hat{\eta}^n}(s) = & x_3 - \int_{[0,s]} d\hat{\eta}_2^n(r), \end{cases} \tag{2.26}$$

and the adjoint equation is now for $0 \leq t \leq s \leq T$

$$\begin{cases} P_1(s) - P_1(t-) = & \int_t^s \rho_1 P_1(r)dr + \int_{[t,s]} d\hat{\eta}_1^n(r), \quad P_1(T) = 0, \\ P_2(s) - P_2(t-) = & \int_t^s \rho_2 P_2(r)dr + \int_{[t,s]} d\hat{\eta}_2^n(r), \quad P_2(T) = 0, \\ P_3(s) - P_3(t-) = & 0, \quad P_3(T) = -2n\hat{X}_3(T). \end{cases} \tag{2.27}$$

It can be verified by direct computation that if $\hat{\eta}^n$ is chosen as in the assertion of the proposition, then the state dynamics satisfy

$$\begin{aligned}
\hat{X}_1(s) =&0,\\
\hat{X}_2(s) =&\kappa_2\Delta\hat{\eta}_2^n(0)\mathbb{1}_{\{s<T\}} + 2\kappa_2\Delta\hat{\eta}_2^n(0)\mathbb{1}_{\{s=T\}},\\
\hat{X}_3(s) =&\frac{\rho_2(T-s)+1+\frac{\kappa_2}{n}}{\rho_2 T+2+\frac{\kappa_2}{n}}x_3\mathbb{1}_{\{s<T\}} + \frac{\frac{\kappa_2}{n}}{\rho_2 T+2+\frac{\kappa_2}{n}}x_3\mathbb{1}_{\{s=T\}}.
\end{aligned}$$

We claim that the solution to the adjoint equation is then given on $[0,T]$ by

$$\begin{aligned}
P_1(s) =&0,\\
P_2(s) =&-\Delta\hat{\eta}_2^n(0)\mathbb{1}_{\{s<T\}},\\
P_3(s) =&-2n\hat{X}_3(T).
\end{aligned}$$

Indeed, P_1 and P_3 are constant on $[0,T]$ and P_2 is constant on $[0,T)$ and jumps to zero at T. We note that for $s\in[0,T)$ we have

$$\int_t^s \rho_2 P_2(r)dr + \int_{[t,s]} d\hat{\eta}_2^n(r) = \int_t^s \big[-\rho_2\Delta\hat{\eta}_2^n(0) + \rho_2\Delta\hat{\eta}_2^n(0)\big]dr = 0,$$

so that (2.27) is satisfied. A further computation shows that for all $s\in[0,T]$ we then have

$$\begin{cases}\hat{X}_1(s)-\kappa_1 P_1(s) - P_3(s) = & -P_3(s) = 2n\hat{X}_3(T)\geq 0,\\ \hat{X}_2(s)-\kappa_2 P_2(s)+P_3(s) = & 0.\end{cases}$$

In particular, conditions (2.17) and (2.18) of Theorem 2.5.1 are satisfied, and thus $\hat{\eta}^n$ is the unique solution of Problem 2.7.2. □

Now that the solution to the unconstrained optimisation problem is known, we take the limit $n\to\infty$ to get the solution to the problem with terminal constraint.

Proposition 2.7.4. *The solution to Problem 2.7.1 is given by discrete market sell orders at times $s=0$ and $s=T$ of size*

$$\Delta\hat{\eta}_2(0) = \Delta\hat{\eta}_2(T) = \frac{x_3}{\rho_2 T+2}$$

and a constant rate of market sell orders in $(0,T)$ given by

$$d\hat{\eta}_2(s) = \rho_2\Delta\hat{\eta}_2(0)ds = \frac{\rho_2 x_3}{\rho_2 T+2}ds.$$

Market buy orders are not used, i.e. $\hat{\eta}_1\equiv 0$.

Proof. We rewrite the performance functional in the following way:

$$J(\eta) =\mathbb{E}\Big[\int_{[0,T]}\Big[X_1(r-)+\frac{\kappa_1}{2}\Delta\eta_1(r)\Big]d\eta_1(r)$$

$$+ \int_{[0,T]} \left[X_2(r-) + \frac{\kappa_2}{2} \Delta\eta_2(r) \right] d\eta_2(r) + \delta_{\{\mathbb{R}\setminus\{0\}\}}(X_3(T)) \Big].$$

where $\delta_{\{\mathbb{R}\setminus\{0\}\}}$ is the indicator function in the sense of convex analysis. We then have for each $\eta \in \mathcal{U}_0^\eta$

$$J^n(\eta) \leq J(\eta). \tag{2.28}$$

Moreover, one can check by direct calculation that the strategy $\hat{\eta}$ satisfies the liquidation constraint (2.25), i.e. we have $\hat{X}_3(T) = 0$ and thus $\hat{\eta}$ is admissible. Before we prove the optimality, let us establish some convergence results. We first note that the optimal strategies converge in the sense that $\lim_{n\to\infty} \hat{\eta}^n(s) = \hat{\eta}(s)$ for all $s \in [0, T]$ a.s. We now show that the associated trading costs also converge. Indeed, using the known form of $X_3^{\hat{\eta}^n}(T)$ from (2.26) implies that the terminal costs satisfy

$$\begin{aligned} \lim_{n\to\infty} \{nX_3^{\hat{\eta}^n}(T)^2\} &= \lim_{n\to\infty} \left\{ n \left[x3 - \frac{\rho_2 T + 2}{\rho_2 T + 2 + \frac{\kappa_2}{n}} x3 \right]^2 \right\} \\ &= \lim_{n\to\infty} \left\{ \frac{1}{n} \left[\frac{\kappa_2}{\rho_2 T + 2 + \frac{\kappa_2}{n}} x3 \right]^2 \right\} = 0. \end{aligned}$$

The integrand of the singular cost term defined in Problem 2.7.1 converges pointwise in the sense

$$\begin{aligned} &\lim_{n\to\infty} \left\{ \left[X_2^{\hat{\eta}^n}(r-) + \frac{\kappa_2}{2} \Delta\hat{\eta}_2^n(r) \right] d\hat{\eta}_2^n(r) \right\} \\ =&\lim_{n\to\infty} \left\{ X_2^{\hat{\eta}^n}(r-)\rho_2 \Delta\hat{\eta}_2^n(0)dr + \frac{\kappa_2}{2} \Delta\hat{\eta}_2^n(0)^2 + \frac{\kappa_2}{2} \Delta\hat{\eta}_2^n(T)^2 \right\} \\ =&\hat{X}_2(r-)\rho_2 \Delta\hat{\eta}_2(0)dr + \frac{\kappa_2}{2} \Delta\hat{\eta}_2(0)^2 + \frac{\kappa_2}{2} \Delta\hat{\eta}_2(T)^2 \\ =&\left[X_2^{\hat{\eta}}(r-) + \frac{\kappa_2}{2} \Delta\hat{\eta}_2(r) \right] d\hat{\eta}_2(r). \end{aligned}$$

We now apply Fatou's Lemma together with (2.28) to get for each $\eta \in \mathcal{U}_0$

$$\begin{aligned} J(\hat{\eta}) =& \mathbb{E}\left[\int_{[0,T]} \left[\hat{X}_2(r-) + \frac{\kappa_2}{2} \Delta\hat{\eta}_2(r) \right] d\hat{\eta}_2(r) \right] \\ \leq& \liminf_{n\in\mathbb{N}} \mathbb{E}\left[\int_{[0,T]} \left[X_2^{\hat{\eta}^n}(r-) + \frac{\kappa_2}{2} \Delta\hat{\eta}_2^n(r) \right] d\hat{\eta}_2^n(r) + nX_3^{\hat{\eta}^n}(T)^2 \right] \\ =& \liminf_{n\in\mathbb{N}} J^n(\hat{\eta}^n) \leq \liminf_{n\in\mathbb{N}} J^n(\eta) \leq J(\eta). \end{aligned}$$

This proves that $\hat{\eta}$ is indeed the solution to Problem 2.7.1. □

Remark 2.7.5.

- The optimal liquidation strategy has the following structure: An initial discrete trade is chosen such that $\left(0, \hat{X}(0), P(0)\right)$ is on the boundary of the sell region. Afterwards, a constant rate of sell orders and a final discrete trade are used such that $(s, \hat{X}(s), P(s))$ is on the boundary of the sell region for all $s \in (0, T]$.

- While Obizhaeva and Wang [2005] work in a one sided model and only consider market sell orders, we consider a larger class of controls and allow for both market buy and sell orders. It is a *consequence* of Proposition 2.7.4 that market buy orders are never used.
- The solution given above only holds for initial spread zero. If we start with a larger spread, it might be optimal not to use market orders for a certain period of time and wait for the spread to grow back.

2.7.2. Portfolio Liquidation with Singular and Passive Orders

In this section, we extend the model described in Subsection 2.7.1 such that it also allows for passive orders. The portfolio liquidation problem with passive orders is

Problem 2.7.6. *Minimise*

$$J(\eta, u) \triangleq \mathbb{E}\Big[\int_{[0,T]} \Big[X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\, d\eta_1(r) + \int_{[0,T]} \Big[X_2(r-) + \frac{\kappa_2}{2}\Delta\eta_2(r)\Big]\, d\eta_2(r)\Big]$$

over controls $(\eta, u) \in \mathcal{U}_0$ *such that* $\hat{X}_3(T) = 0$.

Note that in contrast to Problem 2.7.1 this is now a stochastic problem, because passive order execution is random. We shall see that the optimal control is no longer deterministic, but adapted to the jumps of the Poisson process N_2. Again we introduce a sequence of auxiliary control problems without constraints, but with a penalty for stock holdings at maturity. For $n \in \mathbb{N}$ we define

Problem 2.7.7. *Minimise*

$$\begin{aligned} J^n(\eta, u) \triangleq \mathbb{E}\Big[&\int_{[0,T]} \Big[X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\Big]\, d\eta_1(r) \\ &+ \int_{[0,T]} \Big[X_2(r-) + \frac{\kappa_2}{2}\Delta\eta_2(r)\Big]\, d\eta_2(r) + nX_3(T)^2\Big] \end{aligned}$$

over controls $(\eta, u) \in \mathcal{U}_0$.

Again we first solve the auxiliary control problem.

Proposition 2.7.8. *The solution to Problem 2.7.7 is given* $ds \times d\mathbb{P}$ *a.e. on* $[0, T] \times \Omega$ *by a passive sell order of size*

$$\hat{u}_2^n(s) = \hat{X}_3(s-),$$

an initial discrete market sell order of size

$$\Delta\hat{\eta}_2^n(0) = \frac{2n\lambda_2\rho_2}{2ne^{\lambda_2 T}(\lambda_2 + \rho_2)^2 + e^{\lambda_2 T}\lambda_2\kappa_2(\lambda_2 + 2\rho_2) - 2n\rho_2^2} x_3,$$

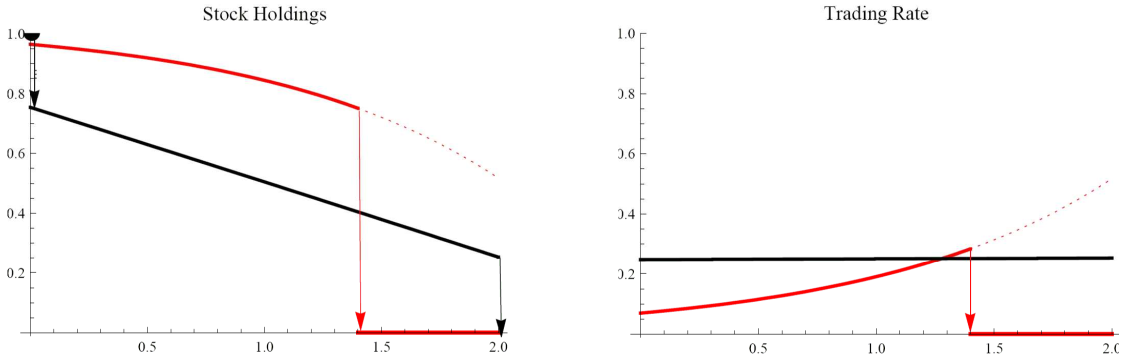


Figure 2.2.: Stock holdings and trading rate with (red, $\lambda_2 = 1$) and without (black, $\lambda_2 = 0$) passive orders. If there are no passive orders, there are equally sized initial and terminal discrete trades and a constant trading rate in between. If passive orders are allowed, the initial trade is smaller and the trading rate is increasing in time. If the passive order is executed, the stock holdings jump to zero. The parameters in this simulation are $T = 2, x_3 = 1, \rho_2 = 1$ and $\kappa_2 = 0.01$.

a terminal discrete market sell order of size

$$\Delta\hat{\eta}_2^n(T) = \frac{\lambda_2 + \rho_2}{\rho_2} e^{\lambda_2 T} \Delta\hat{\eta}_2^n(0) \mathbb{1}_{\{T<\tau_2\}}$$

and the following rate of market sell orders in $(0, T)$*,*

$$d\hat{\eta}_2^n(s) = (\lambda_2 + \rho_2) e^{\lambda_2 s} \Delta\hat{\eta}_2^n(0) \mathbb{1}_{\{s<\tau_2\}} ds,$$

where τ_2 *denotes the first jump time of the Poisson process* N_2*. Market and passive buy orders are not used, i.e. a.s.* $\hat{\eta}_1^n(s) = 0$ *for each* $s \in [0, T]$ *and* $\hat{u}_1^n = 0$ $ds \times d\mathbb{P}$ *a.e. on* $[0, T] \times \Omega$*.*

Proof. The proof proceeds as follows: Taking the candidate optimal control $(\hat{\eta}^n, \hat{u}^n)$ as given, we first compute the associated state process and then the adjoint equation. This provides a solution to the forward backward system and it then only remains to check the optimality conditions from Theorem 2.5.1.

The state trajectory associated to the control $(\hat{\eta}^n, \hat{u}^n)$ is given on $[0, T]$ by

$$\hat{X}_1(s) = x_1 e^{-\rho_1 s},$$

$$\hat{X}_2(s) = \begin{cases} \kappa_2 e^{\lambda_2 s} \Delta\hat{\eta}_2^n(0), & \text{if } s \leq \tau_2 \text{ and } s < T, \\ \hat{X}_2(\tau_2) e^{-\rho_2(s-\tau_2)}, & \text{if } \tau_2 < s, \\ \frac{\kappa_2}{\rho_2}(\lambda_2 + 2\rho_2) e^{\lambda_2 T} \Delta\hat{\eta}_2^n(0), & \text{if } s = T < \tau_2, \end{cases}$$

$$\hat{X}_3(s) = \begin{cases} x_3 - \frac{\lambda_2+\rho_2}{\lambda_2}(e^{\lambda_2 s} - 1)\Delta\hat{\eta}_2^n(0), & \text{if } s < \tau_2 \text{ and } s < T, \\ \frac{1}{2n}\frac{\kappa_2}{\rho_2}(\lambda_2 + 2\rho_2) e^{\lambda_2 T} \Delta\hat{\eta}_2^n(0), & \text{if } s = T < \tau_2, \\ 0, & \text{if } s \geq \tau_2. \end{cases} \quad (2.29)$$

Note that the stock holdings $\hat{X}_3$ are strictly positive on $[0, \tau_2)$ and jump to zero at τ_2, i.e. if N_2 jumps and the passive order is executed. At this instant, the investor stops trading. Afterwards, the sell spread $\hat{X}_2$ recovers exponentially due to resilience. We will now use the representation (2.14) to construct the adjoint process. First note that $\hat{\eta}_1^n \equiv 0$ implies $P_1 = 0$ $ds \times d\mathbb{P}$ a.e. We now compute P_3. For $s \in [0, T]$ we have using (2.14)

$$P_3(s) = -\,\mathbb{E}_{s,x}\big[2n\hat{X}_3(T)\big].$$

We know from (2.29) that $\hat{X}_3 = 0$ on the stochastic interval $[\tau_2, T]$, so that

$$P_3(s)\mathbb{1}_{\{s \geq \tau_2\}} = 0.$$

We also have $P_3(T) = -2n\hat{X}_3(T)$. It remains to consider $s \in [0, \tau_2 \wedge T)$ and for such s we compute using the exponential density of τ_2

$$\begin{aligned} P_3(s) &= -\,\mathbb{E}_{s,x}\left[2n\hat{X}_3(T)\right] \\ &= -\,\mathbb{E}_{s,x}\left[2n\frac{1}{2n}\frac{\kappa_2}{\rho_2}(\lambda_2 + 2\rho_2)e^{\lambda_2 T}\Delta\hat{\eta}_2^n(0)\mathbb{1}_{\{T<\tau_2\}}\right] \\ &= -\,\frac{\kappa_2}{\rho_2}(\lambda_2 + 2\rho_2)e^{\lambda_2 T}\Delta\hat{\eta}_2^n(0)\int_T^\infty \lambda_2 e^{-\lambda_2(z-s)}dz \\ &= -\,\frac{\kappa_2}{\rho_2}(\lambda_2 + 2\rho_2)e^{\lambda_2 T}\Delta\hat{\eta}_2^n(0)e^{-\lambda_2(T-s)} \\ &= -\,\frac{\kappa_2}{\rho_2}(\lambda_2 + 2\rho_2)e^{\lambda_2 s}\Delta\hat{\eta}_2^n(0). \end{aligned}$$

We now turn to P_2. A calculation based on the known form of $\hat{\eta}_2^n$, the representation (2.14) and the density of τ_2 shows that

$$\begin{aligned} P_2(s) =& \mathbb{E}_{s,x}\left[-\int_{(s,T]} e^{-\rho_2(r-s)}d\hat{\eta}_2^n(r)\right] \\ =& -\int_s^T \lambda_2 e^{-\lambda_2(z-s)}\int_s^z e^{\rho_2 s}e^{(\lambda_2-\rho_2)r}(\lambda_2+\rho_2)\Delta\hat{\eta}_2^n(0)drdz \\ & -\int_T^\infty \lambda_2 e^{-\lambda_2(z-s)}\Big\{\int_s^T e^{\rho_2 s}e^{(\lambda_2-\rho_2)r}(\lambda_2+\rho_2)\Delta\hat{\eta}_2^n(0)dr \\ & \qquad + e^{\rho_2 s}e^{(\lambda_2-\rho_2)T}\frac{1}{\rho_2}(\lambda_2+\rho_2)\Delta\hat{\eta}_2^n(0)\Big\}dz \\ =& -\frac{\lambda_2+\rho_2}{\rho_2}e^{\lambda_2 s}\Delta\hat{\eta}_2^n(0). \end{aligned}$$

To sum up, the adjoint process is given explicitly as

$$P_1(s) = 0,$$

$$P_2(s) = \begin{cases} -\frac{\lambda_2+\rho_2}{\rho_2} e^{\lambda_2 s} \Delta\hat{\eta}_2^n(0), & \text{if } s < \tau_2 \text{ and } s < T, \\ 0, & \text{else,} \end{cases}$$

$$P_3(s) = \begin{cases} -\frac{\kappa_2}{\rho_2}(\lambda_2 + 2\rho_2) e^{\lambda_2 s} \Delta\hat{\eta}_2^n(0), & \text{if } s < \tau_2 \text{ and } s < T, \\ -2n\hat{X}_3(T), & \text{if } s = T < \tau_2 \\ 0, & \text{else.} \end{cases}$$

In particular, P_i is zero on the stochastic interval $[\tau_2, T]$ for $i = 2, 3$.

Having constructed a solution to the forward backward system, we will now use Theorem 2.5.1 to show that the control $(\hat{u}^n, \hat{\xi}^n)$ is indeed optimal. Using the known form of $\hat{X}_i$ and P_i for $i = 1, 2, 3$, we check the optimality conditions and compute that a.s.

$$\begin{cases} \hat{X}_1(s) - P_3(s) - \kappa_1 P_1(s) = -P_3(s) & \geq 0,\ s \in [0, T] \\ \hat{X}_2(s) + P_3(s) - \kappa_2 P_2(s) & = 0,\ s \in [0, \tau_2 \wedge T], \\ \hat{X}_2(s) + P_3(s) - \kappa_2 P_2(s) = \hat{X}_2(s) & \geq 0,\ s \in (\tau_2 \wedge T, T], \end{cases}$$

so that condition (2.17) is satisfied. In order to check (2.18), we first note that $\hat{\eta}_1^n(r) = 0$ for each $r \in [0, T]$ a.s. so that

$$\mathbb{P}\Big(\int_{[0,T]} \mathbb{1}_{\{\hat{X}_1(r) - \kappa_1 P_1(r) - P_3(r) > 0\}} d\hat{\eta}_1^n(r) = 0 \Big) = 1.$$

In addition, we have $\hat{X}_2 - \kappa_2 P_2 + P_3 = 0$ on $[0, \tau_2 \wedge T]$ and $\hat{\eta}_2^n$ is constant on $[\tau_2 \wedge T, T]$ so that

$$\begin{aligned} &\int_{[0,T]} \mathbb{1}_{\{\hat{X}_2(r) - \kappa_2 P_2(r) + P_3(r) > 0\}} d\hat{\eta}_2^n(r) \\ =& \int_{[0,\tau_2 \wedge T]} \mathbb{1}_{\{\hat{X}_2(r) - \kappa_2 P_2(r) + P_3(r) > 0\}} d\hat{\eta}_2^n(r) + \int_{(\tau_2 \wedge T, T]} \mathbb{1}_{\{\hat{X}_2(r) - \kappa_2 P_2(r) + P_3(r) > 0\}} d\hat{\eta}_2^n(r) \\ =& 0. \end{aligned}$$

Finally, let us check condition (2.19). A consequence of $P_1 = 0$ is that $R_{1,3} = 0$ $ds \times d\mathbb{P}$ a.e. and we have

$$R_{1,3}(s) + P_3(s-) = P_3(s-) \leq 0 \text{ and } \hat{u}_1(s) = 0.$$

If the Poisson process N_2 jumps, then P_3 jumps to zero, so we have $ds \times d\mathbb{P}$ a.e. on $[0, T] \times \Omega$

$$R_{2,3}(s) + P_3(s-) = 0.$$

An application of Theorem 2.5.1 now yields that $(\hat{u}^n, \hat{\eta}^n)$ is optimal. □

We now proceed to the portfolio liquidation problem with passive orders and terminal constraint.

Proposition 2.7.9. *The solution to Problem 2.7.6 is given $ds \times d\mathbb{P}$ a.e. on $[0,T] \times \Omega$ by a passive sell order of size*

$$\hat{u}_2(s) = \hat{X}_3(s-),$$

an initial discrete market sell order of size

$$\Delta\hat{\eta}_2(0) = \frac{\lambda_2 \rho_2}{e^{\lambda_2 T}(\lambda_2 + \rho_2)^2 - \rho_2^2} x_3,$$

a terminal discrete market sell order of size

$$\Delta\hat{\eta}_2(T) = \frac{\lambda_2 + \rho_2}{\rho_2} e^{\lambda_2 T} \Delta\hat{\eta}_2(0) \mathbb{1}_{\{T<\tau_2\}} = \frac{\lambda_2(\lambda_2 + \rho_2)e^{\lambda_2 T}}{e^{\lambda_2 T}(\lambda_2 + \rho_2)^2 - \rho_2^2} x_3 \mathbb{1}_{\{T<\tau_2\}},$$

and the following rate of market sell orders in $(0,T)$,

$$d\hat{\eta}_2(s) = (\lambda_2 + \rho_2) e^{\lambda_2 s} \Delta\hat{\eta}_2(0) \mathbb{1}_{\{s<\tau_2\}} ds = \frac{\lambda_2 \rho_2 (\lambda_2 + \rho_2)}{e^{\lambda_2 T}(\lambda_2 + \rho_2)^2 - \rho_2^2} e^{\lambda_2 s} x_3 \mathbb{1}_{\{s<\tau_2\}} ds,$$

where τ_2 denotes the first jump time of the Poisson process N_2. Market and passive buy orders are not used, i.e. a.s. $\hat{\eta}_1(s) = 0$ for each $s \in [0,T]$ and $\hat{u}_1 = 0$ $ds \times d\mathbb{P}$ a.e. on $[0,T] \times \Omega$.

Proof. The argument is the same as in the proof of Proposition 2.7.4. □

We conclude with some remarks on the structure of the optimal control.

Remark 2.7.10.

- It is optimal to offer all outstanding shares as a passive order, and simultaneously trade using market orders.
- Let us compare the solutions with and without passive orders. Proposition 2.7.4 shows that in the latter, there are equally sized initial and terminal discrete trades and a constant trading rate in between. If passive orders are allowed, it follows from Proposition 2.7.9 that the initial discrete trade is small and the investor starts with a small trading rate, which increases as maturity approaches. The interpretation is that he is reluctant to use market orders and rather waits for passive order execution. See Figure 2.2 for an illustration.
- The sell region is in this case

  $$\mathcal{R}_{sell} = \left\{ (s,x,p) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \middle| \, x_2 + p_3 - \kappa_2 p_2 < 0 \right\}.$$

  The initial discrete trade is chosen such that the controlled system jumps to the boundary of the sell region. Then a rate of market sell orders is chosen such that the state process remains on this boundary until the passive order is executed.
- The optimal strategy does not depend on the inverse order book height κ_2 and is linear in the initial portfolio size $x_3 = X_3(0-)$.

- The solution to the portfolio liquidation problem with passive orders given in Proposition 2.7.9 is similar to the one obtained in Kratz and Schöneborn [2009] Proposition 4.2; what they call dark pool can be interpreted as a passive order in our setup. Note however that they work in discrete time in a model without spread and resilience. Our solution is also similar to the one obtained in Proposition 1.7.3, where the portfolio liquidation problem is solved in continuous time using passive and market, but no discrete orders and without resilience.

Remark 2.7.11. As the jump intensity λ_2 tends to zero, the solution given in Proposition 2.7.9 for the model with passive orders converges to the solution given in Proposition 2.7.4 for the model without passive orders. Specifically we have for $s \in (0, T)$

$$\begin{aligned}
\lim_{\lambda_2 \to 0} \Delta\hat{\eta}_2(0) &= \lim_{\lambda_2 \to 0} \frac{\lambda_2 \rho_2}{e^{\lambda_2 T}(\lambda_2 + \rho_2)^2 - \rho_2^2} x_3 = \frac{x_3}{\rho_2 T + 2}, \\
\lim_{\lambda_2 \to 0} \Delta\hat{\eta}_2(T) &= \lim_{\lambda_2 \to 0} \frac{\lambda_2(\lambda_2 + \rho_2)e^{\lambda_2 T}}{e^{\lambda_2 T}(\lambda_2 + \rho_2)^2 - \rho_2^2} x_3 = \frac{x_3}{\rho_2 T + 2}, \\
\lim_{\lambda_2 \to 0} d\hat{\eta}_2(s) &= \lim_{\lambda_2 \to 0} \frac{\lambda_2 \rho_2(\lambda_2 + \rho_2)}{e^{\lambda_2 T}(\lambda_2 + \rho_2)^2 - \rho_2^2} e^{\lambda_2 s} x_3 ds = \frac{\rho_2 x_3}{\rho_2 T + 2} ds.
\end{aligned}$$

This shows that Proposition 2.7.4 is a special case of Proposition 2.7.9.

2.7.3. An Example Where It Is Optimal Never To Trade

We know from Proposition 2.5.3 that the controlled system never leaves (the closure of) the no-trade region. In the preceding subsections we solved the problem of curve following with and without passive orders and found that in this case the controlled system remains on the boundary between the sell and the no-trade region at all times. We shall now present another example where the controlled system always remains *inside* the no-trade region, and thus it is optimal never to use market orders.

In the present subsection we remove passive orders and only consider deterministic market orders from the set

$$\mathcal{U}_0^\eta \triangleq \Big\{\eta : [0,T] \to \mathbb{R}_+^2 \,\Big|\, \eta_i(0-) = 0,\ \eta_i(T)^2 < \infty,$$
$$\eta_i \text{ is nondecreasing and càdlàg for } i = 1, 2\Big\}.$$

Let α be a deterministic bounded target function and let f, h be penalty functions satisfying Assumptions 2.2.5 and 2.2.7. We consider the following optimisation problem

Problem 2.7.12. *Minimise*

$$\begin{aligned}
J(\eta) \triangleq & \int_{[0,T]} \left[X_1(r-) + \frac{\kappa_1}{2}\Delta\eta_1(r)\right] d\eta_1(r) + \int_{[0,T]} \left[X_2(r-) + \frac{\kappa_2}{2}\Delta\eta_2(r)\right] d\eta_2(r) \\
& + \int_0^T h(X_3(r) - \alpha(r))dr + f(X_3(T) - \alpha(T)).
\end{aligned}$$

over controls $\eta \in \mathcal{U}_0^\eta$.

This is again a deterministic optimisation problem since passive order execution as well as the signal process Z do not play a role here. We want to construct the problem such that it is optimal never to trade. To this end, we choose initial values for the buy and sell spreads which are so high that market orders are never beneficial. Specifically, we assume for $i = 1, 2$

$$x_i = X_i(0-) > e^{\rho_i T}\Big\{ \sup_{r\in[0,T]} \big|h'(x_3 - \alpha(r))\big|T + \big|f'(x_3 - \alpha(T))\big|\Big\}. \tag{2.30}$$

Proposition 2.7.13. *The solution to Problem 2.7.12 is given by* $\hat{\eta}_i(s) = 0$ *for* $i = 1, 2$ *and all* $s \in [0, T]$.

Proof. A consequence of (2.7) is that the state dynamics associated to the control $\hat{\eta} \equiv 0$ are given for $s \in [0, T]$ by

$$\begin{aligned}\hat{X}_1(s) =& e^{-\rho_1 s} x_1,\\ \hat{X}_2(s) =& e^{-\rho_2 s} x_2,\\ \hat{X}_3(s) =& x_3.\end{aligned}$$

The adjoint processes can now be represented using (2.14) by

$$\begin{aligned}P_1(s) =& 0,\\ P_2(s) =& 0,\\ P_3(s) =& -\int_s^T h'(x_3 - \alpha(r))dr - f'(x_3 - \alpha(T)).\end{aligned}$$

To confirm that $\hat{\eta} \equiv 0$ is indeed optimal, by Theorem 2.5.1 it is enough to check that for each $s \in [0, T]$ we have

$$\begin{cases}\hat{X}_1(s) - \kappa_1 P_1(s) - P_3(s) \geq 0,\\ \hat{X}_2(s) - \kappa_2 P_2(s) + P_3(s) \geq 0.\end{cases} \tag{2.31}$$

We only check the first inequality, the second follows from similar arguments. We have for $s \in [0, T]$ using (2.30)

$$\begin{aligned}\kappa_1 P_1(s) + P_3(s) =& P_3(s)\\ =& -\int_s^T h'(x_3 - \alpha(r))dr - f'(x_3 - \alpha(T))\\ \leq& \sup_{r\in[0,T]} \big|h'(x_3 - \alpha(r))\big|T + \big|f'(x_3 - \alpha(T))\big|\\ <& x_1 e^{-\rho_1 T} \leq x_1 e^{-\rho_1 s} = \hat{X}_1(s).\end{aligned}$$

This proves (2.31) even with strict inequality and completes the proof. □

Remark 2.7.14. The proof of the preceding proposition shows that for each $s \in [0, T]$ we have the strict inequalities

$$\begin{cases} \hat{X}_1(s) - \kappa_1 P_1(s) - P_3(s) > 0, \\ \hat{X}_2(s) - \kappa_2 P_2(s) + P_3(s) > 0, \end{cases}$$

so that the controlled system is always inside the no-trade region and never hits the boundary.

3. On Market Manipulation in Illiquid Markets

3.1. Introduction

A key feature of illiquid markets is that large transactions move prices. This is a clear disadvantage for traders that need to liquidate large portfolios or keep their stock holdings close to a prespecified target as discussed in the previous chapters. Typically, market impact should be avoided, but there are situations where investors may benefit from moving prices. Specifically, a trader that holds a large number of options may have an incentive to utilise his impact on the dynamics of the underlying and to move the option value in a favorable direction if the increase in the option value outweighs the trading costs in the underlying. Gallmeyer and Seppi [2000] provide some empirical evidence that in illiquid markets option traders are in fact able to increase a derivative's value by moving the price of the underlying. [Pirrong, 2001, p.222f] writes that "a trader with a large long position in a cash-settled contract can drive up its settlement value by buying excessive quantities [of the underlying]". Kumar and Seppi [1992] call such trading behavior "punching the close". In this chapter we provide a continuous time framework to model the interaction between several investors which have an incentive to punch the close. We set this up as a stochastic differential game and establish existence and uniqueness of Markov equilibria for risk neutral investors and those with exponential utility and constant absolute risk aversion (CARA). For certain cases we have explicit solutions which allow to discuss some ideas how manipulation in the sense of "punching the close" could potentially be reduced.

Our work builds on previous research in at least three different fields. The first is the mathematical modeling of illiquid financial markets. The role of liquidity as a source of financial risk has been extensively investigated in both the mathematical finance and financial economics literature over the last couple of years. Much of the literature focusses on either optimal hedging and portfolio liquidation strategies for a single large investor under market impact (Çetin et al. [2004], Alfonsi et al. [2010], Rogers and Singh [2010]), predatory trading (Brunnermeier and Pedersen [2005], Carlin et al. [2007], Schied and Schöneborn [2007a]) or the role of derivative securities including the problem of market manipulation using options (Jarrow [1994], Kumar and Seppi [1992]). It has been shown by Jarrow [1994], for instance, that by introducing derivatives into an otherwise complete and arbitrage-free market, certain manipulation strategies for a large trader may appear, such as market corners and front runs. Schönbucher and Wilmott [2000] discuss an illiquid market model where a large trader can influence the stock price with vanishing costs and risk. They argue that the risk of manipulation on the part of the large trader makes the small traders unwilling to trade derivatives any more. In particular, they predict that the option market breaks down. Our analysis indicates that markets do not necessarily break down when stock price manipulation is costly as it is in our model. Kraft and Kühn [2009] analyse the behaviour of an investor in a Black Scholes type market, where trading has a linear permanent impact on the stock's drift. They construct the hedging strategy and the indifference price of a European payoff for a CARA investor, and show that the optimal strategy is a combination of hedging and manipulation. In order to exploit her market impact, the investor over- or underhedges the option, depending on his endowment and the sign of the impact term.

The second line of research our work is connected to are stochastic differential games, see e.g. Fleming and Soner [1993] chapter XI and Hamadène and Lepeltier [1995] for zero sum games, Friedman [1972] and Buckdahn et al. [2004] for nonzero sum games or Nisio [1988] and Buckdahn and Li [2008] for viscosity solutions of Hamilton-Jacobi-Bellman (henceforth HJB) equations. The strategic interaction between large investors and its implications for market microstructure are discussed in Kyle [1985], Foster and Viswanathan [1996], Back et al. [2000], and Chau and Vayanos [2008], for instance. Brunnermeier and Pedersen [2005], Carlin et al. [2007] and Schied and Schöneborn [2007a] consider predatory trading, where liquidity providers try to benefit from the liquidity demand that comes from some "large" investor. Vanden [2005] considers a pricing game in continuous time where the option issuer controls the volatility of the underlying but does not incur liquidity costs. He derives a Nash equilibrium in the two player, risk neutral case and shows that "seemingly harmless derivatives, such as ordinary bull spreads, offer incentives for manipulation that are identical to those offered by digital options" (p.1892). Closest to our setup is the paper by Gallmeyer and Seppi [2000]. They consider a binomial model with three periods and finitely many risk neutral agents holding call options on an illiquid underlying. Assuming a linear permanent price impact and linear transaction costs, and assuming that all agents are initially endowed with the same derivative they prove the existence of a Nash equilibrium trading strategy and indicate how market manipulation can be reduced.

A third line of research we build on is market manipulation. Different notions of market manipulations have been discussed in the literature including short squeezes, the use of private information or false rumours, cf. Kyle [1985], Back [1992], Jarrow [1994], Allen and Gale [1992], Pirrong [2001], Dutt and Harris [2005], Kyle and Viswanathan [2008]. Most of these articles are set up in discrete time. We suggest a general mathematical framework in continuous time within which to value derivative securities in illiquid markets under strategic interactions. Specifically, we consider a stochastic differential game between a finite number of large investors ("players") holding European claims written on an illiquid stock. Their goal is to maximise the expected portfolio value at maturity, composed of trading costs and the option payoff, which depends on the trading strategies of all the other players through their impact on the dynamics of the underlying. Following Almgren and Chriss [2001] we assume that the players have a permanent impact on stock prices and that all trades are settled at the prevailing market price plus a liquidity premium. The liquidity premium can be viewed as an instantaneous price impact that affects transaction prices but not the value of the players' inventory. This form of market impact modeling is analytically more tractable than that of Obizhaeva and Wang [2005] which also allows for temporary price impacts and resilience effects. It has also been adopted by, e.g. Carlin et al. [2007] and Schied and Schöneborn [2007a] and some practitioners from the financial industry, as pointed out by Schied and Schöneborn [2008].

Our framework is flexible enough to allow for rather general liquidity costs including the linear cost function of Almgren and Chriss [2001] and some form of bid ask spread, cf. Example 3.2.3. We show that when the market participants are risk neutral or have CARA utility functions the pricing game has a unique Nash equilibrium. We solve the problem of equilibrium pricing using techniques from the theory of stochastic optimal control and stochastic differential games. We show that the family of the players' value functions can be characterised as the solution to a coupled system of non-linear PDEs. Coupled systems of non-linear PDEs arise naturally in differential stochastic games. Since general existence and uniqueness of solution results for systems of non-linear PDEs on unbounded state spaces are hard to prove much of the literature on stochastic differential games is confined to bounded state spaces; see e.g. the seminal paper of Friedman [1972]. We prove an a priori estimate for Nash equilibria. More precisely we prove that under rather mild conditions any equilibrium trading strategy is uniformly bounded. This allows us to prove that the PDE system that describes the equilibrium dynamics has a unique classical solution. The equilibrium problem can be solved in closed form for a specific market environment, namely the linear cost structure and risk neutral agents.

It is important to know which measures may reduce market manipulation. For instance, Dutt and Harris [2005] propose position limits; Pirrong [2001] suggests efficient contract designs. We use the explicit solution for risk neutral investors to show when "punching the close" is not beneficial. For instance, no manipulation occurs in zero sum games, i.e. in a game between an option writer and an option issuer. In our model manipulation decreases with the number of informed liquidity providers and with the number of competitors, if the product is split between them. Furthermore, we find that the bid ask spread is important determinant of market manipulation. It turns out that the higher the spread, the less beneficial market manipulation: high spread crossing costs make trading more costly and hence discourage frequent re-balancing of portfolio positions.

The remainder of this chapter is organised as follows: We present the market model as well as the optimisation problem and some a priori estimates in Section 3.2. The solution for risk neutral and CARA investors are given in Sections 3.3 and 3.4, respectively. We use the explicit solution for the risk neutral case in Section 3.5 to show how market manipulation can be reduced. Parts of this chapter are published in Horst and Naujokat [2011].

3.2. The Model

We adopt the market impact model of Schied and Schöneborn [2007a] with a finite set of *agents*, or *players*, trading a single stock whose price process depends on the agents' trading strategies. It is a multiplayer extension of the model introduced in Chapter 1, with an additional permanent price impact as in Almgren and Chriss [2001]. All trades are settled at prevailing market prices plus a liquidity premium which depends on the *change* in the players' portfolios. In order to be able to capture changes in portfolio positions in an analytically tractable way, we assume that the stock holdings of player $j \in \{1, ..., N\}$ are governed by the following SDE,

$$dX^j(s) = u^j(s)ds, \quad X^j(0) = 0,$$

where the trading speed $u^j = \dot{X}^j$ is chosen from the following set of admissible controls, for $t \in [0, T]$:

$$\mathcal{U}_t \triangleq \{u : [t, T] \times \Omega \to \mathbb{R} \text{ progressively measurable}\}.$$

There is a an array of large investor models which assume that stock holdings are absolutely continuous and that the price dynamics depend on the *change* of the investors' positions, e.g. Almgren et al. [2005], Almgren and Lorenz [2007], Schied and Schöneborn (2007, 2008), Carlin et al. [2007] and Rogers and Singh [2010]. In all these papers the assumption of absolute continuity is made merely for analytical convenience. We also remark that the dynamics specified above are similar to the setup in Chapter 1, but we do not consider passive orders here since their market impact is typically negligible.

3.2.1. Price dynamics and the liquidity premium

Our focus is on optimal manipulation strategies (in the sense of "punching the close") for derivatives with short maturities under strategic market interactions. For short trading periods we deem it appropriate to model the *fundamental stock price*, i.e. the value of the stock in the absence of any market impact, as a Brownian Motion with volatility $\sigma > 0$. Market impact is accounted for by assuming that the investors' accumulated stock holdings $\sum_{i=1}^{N} X^i$ have a linear permanent impact on the stock process P so that for $s \in [0, T]$

$$P(s) = P(0) + \sigma W(s) + \lambda \sum_{i=1}^{N} X^i(s) \tag{3.1}$$

with an impact parameter $\lambda > 0$. The linear permanent impact is consistent with the work of Huberman and Stanzl [2004] who argue that linearity of the permanent price impact is important to exclude quasi-arbitrage. We remark that in Chapters 1 and 2 our focus was on the tradeoff between the liquidity costs of trading and the penalty for deviating from a target function. In that case the liquidity costs were determined by the trading strategy and the bid ask spread. It was therefore not necessary to model the stock price explicitly and we only considered instantaneous and temporary (but no permanent) price impact. However in the present chapter we investigate how to increase the payoff of a given option with illiquid underlying. We *do* model the price of the underlying explicitly and we now also include a permanent price impact.

A trade at time $s \in [0, T]$ is settled at a *transaction price* $\tilde{P}(s)$ that includes an additional instantaneous price impact, or *liquidity premium.* Specifically,

$$\tilde{P}(s) = P(s) + g\left(\sum_{i=1}^{N} u^i(s)\right) \tag{3.2}$$

with a *cost function* g that depends on the instantaneous change $\sum_{i=1}^{N} u^i$ in the agents' position in a possibly non-linear manner, just as in the first chapter. The liquidity premium accounts for limited available liquidity, transaction costs, fees or spread crossing costs, cf. Example 3.2.3.

Remark 3.2.1.

- In the single player case discussed in Chapter 1 we considered a liquidity cost function $g(u, Z)$ which was driven by a stochastic factor Z. In the present case with several agents we remove Z for tractability.
- In our model the liquidity costs are the same for all traders and depend only on the aggregate demand throughout the entire set of agents. This captures situations where the agents trade through a market maker or clearing house that reduces the trading costs by collecting all orders and matching incoming demand and supply prior to settling the outstanding balance $\sum_{i=1}^{N} u^i(s)$ at market prices.

We assume that g is normalised, $g(0) = 0$ and smooth. The following additional mild assumptions on g will guarantee that the equilibrium pricing problem has a solution for risk neutral and CARA investors.

Assumption 3.2.2.

- *The derivative g' is bounded away from zero, that is $g' > \epsilon > 0$.*
- *The mapping $z \mapsto g(z) + zg'(z)$ is strictly increasing.*

The first assumption is a technical condition needed in the proof of Proposition 3.2.9. It appears not too restrictive for a cost function. Since the liquidity costs associated with a net change in the overall position z is given by $zg(z)$, the second assumption states that the agents face increasing marginal costs of trading.

Example 3.2.3. *Among the cost functions which satisfy Assumption 3.2.2 are the linear cost function $g(z) = \kappa z$ with $\kappa > 0$ and cost functions of the form*

$$g(z) = \kappa z + c\frac{2}{\pi}\arctan(Cz) \quad \textit{with} \quad c, C > 0.$$

The former is the cost function associated with a block-shaped limit order book. The latter can be viewed as a smooth approximation of the map $z \mapsto \kappa z + c \cdot \operatorname{sign}(z)$ which is the cost function associated with a block-shaped limit order book and bid ask spread $c > 0$.

3.2.2. The Optimisation Problem

Each agent is initially endowed with a contingent claim $H^j = H^j(P(T))$, whose payoff depends on the stock price at maturity. Our focus is on optimal trading strategies in the stock, given an initial endowment. As in Gallmeyer and Seppi [2000] and Kraft and Kühn [2009], we assume that the agents do not trade the option in $[0, T]$. A consistent model for trading an illiquid option with illiquid underlying in a multiplayer framework in continuous time is not available, to the best of our knowledge. Our work might be considered a step in this direction. We assume that the functions H^j are smooth and bounded with bounded derivatives H^j_p. This is needed in the a priori estimates as well as in the proof of existence of a smooth solution to the HJB equation.

Remark 3.2.4. We only consider options with cash settlement. This assumption is key. While cash settlement is susceptible to market manipulation, we show in Proposition 3.5.5 below that when deals are settled physically, i.e. when the option issuer delivers the underlying, market manipulation is not beneficial: Any price increase is outweighed by the liquidity costs of subsequent liquidation. We notice that this only applies to "punching the close". There are other types of market manipulation, such as corners and short squeezes, which might be beneficial when deals are settled physically, but which are not captured by our model, cf. Jarrow [1994] or Kyle and Viswanathan [2008].

We shall now give a heuristic derivation of the optimisation problem. Consider a single risk neutral investor who builds up a position in stock holdings $X(T)$ using the trading strategy u in $[0, T]$ and afterwards liquidates his stock holdings using a constant rate of liquidation η, so that at time $T' \triangleq T + \left|\frac{X(T)}{\eta}\right|$ the portfolio is liquidated. In view of (3.2), the proceeds from such a round trip strategy are

$$
\begin{aligned}
&\int_0^T -u(s)\tilde{P}(s)ds + \int_T^{T'} \eta\tilde{P}(s)ds \\
=&\int_0^{T'} \sigma W(s)dX(s) - \lambda \int_0^{T'} X(s)dX(s) - \int_0^T u(s)g(u(s))ds - X(T)g(\eta).
\end{aligned}
$$

Using integration by parts and $X(0) = X(T') = 0$ we see that the first term in the second line has zero expectation[1] and the second term also vanishes. The last term describes the liquidity costs of the constant liquidation rate η and goes to zero if η goes to zero since $g(0) = 0$. In this sense, infinitely slow liquidation incurs no costs. It follows that the round trip strategy described above incurs expected liquidity costs of $-\int_0^T u(s)g(u(s))ds$. Taking into account the option payoff, the optimisation problem for a single risk neutral investor becomes

$$
\sup_{u\in\mathcal{U}_0} \mathbb{E}\left[-\int_0^T u(s)g\left(u(s)\right)ds + H(P(T))\right]. \tag{3.3}
$$

[1] We will prove an a priori estimate in Proposition 3.2.9 and then only consider bounded strategies, so that the stochastic integral $\int_0^T X(t)dW(t)$ is indeed a martingale.

This reflects the tradeoff between liquidity costs (the costs of "punching the close") and an increased option payoff. In our model, the only purpose of trading is an increased option payoff and not, for instance, hedging. For a study on the interplay of hedging and manipulation we refer the reader to Kraft and Kühn [2009]. Unfortunately, the heuristic derivation given above has no direct counterpart in the multiplayer case. As one prerequisite one would need the optimal liquidation strategies (and corresponding liquidation value) of several agents in a market with general liquidity structure. Defining a notion of liquidation value under strategic interaction is still an open question (Carlin et al. [2007] and Schied and Schöneborn [2007a] derived solutions in special cases) and it is not the focus of the present work. Our focus is on the tradeoff between increased option payoff and liquidity costs in a multiplayer framework. Specifically, we assume that the preferences of player j at time $t \in [0, T]$ are described by a preference functional Ψ_t^j (conditional expected value or conditional entropic risk measure) and that his goal at time $t = 0$ is to maximise the utility from the option payoff minus the cost of trading (given the other players' strategies). We hence consider the following optimisation problem:

Problem 3.2.5. *Given the strategies $u^i \in \mathcal{U}_0$ for all the players $i \neq j$ the optimisation problem of player $j \leq N$ is*

$$\sup_{u^j \in \mathcal{U}_0} \Psi_0^j \left(- \int_0^T u^j(s) g \left(\sum_{i=1}^N u^i(s) \right) ds + H^j(P(T)) \right).$$

Remark 3.2.6. We remark that in the preceding chapters the dynamics of the state were degenerate, so the dynamic programming approach was not directly applicable and we based our characterisation of optimality on a suitable version of the stochastic maximum principle instead. In the present case the state variable P is given by a nondegenerate diffusion. As a result, we shall see that the HJB equation is uniformly parabolic and we proceed via the dynamic programming approach. In contrast, the methods from the previous chapters do not apply here since the above optimisation problem is not necessarily convex in the control.

The case where all investors are risk neutral, $\Psi_t^j(Z) = \mathbb{E}[Z|\mathcal{F}_t]$, is studied in Section 3.3. The case of conditional expected exponential utility maximising investors is studied in Section 3.4. In that case we may choose $\Psi_t^j(Z) = -\frac{1}{\alpha^j} \log \mathbb{E}\left[\exp(-\alpha^j Z)|\mathcal{F}_t\right]$ where $\alpha^j > 0$ denotes the risk aversion of player j. Both preference functionals are *translation invariant*[2]. This means that $\Psi_t^j(Z + Y) = \Psi_t^j(Z) + Y$ for any random variable Y that is measurable with respect to the information available at time $t \in [0, T]$. As a result, the trading costs incurred up to time t do not affect the optimal trading strategy at later times. This property is key and will allow us to establish the existence of Nash equilibria in our financial market model.

Definition 3.2.7. We say that a vector of strategies $\left(u^1, ..., u^N\right)$ is a *Nash equilibrium* if for each agent $j \leq N$ his trading strategy u^j is a best response against the behavior of

[2] Translation invariant preferences have recently attracted much attention in the mathematical finance literature in the context of optimal risk sharing and equilibrium pricing in dynamically incomplete markets. We refer to Cheridito et al. [2009] for further details.

all the other players, i.e. if u^j solves Problem 3.2.5, given the other players' aggregate trading $u^{-j} \triangleq \sum_{i \neq j} u^i$.

Remark 3.2.8. Our results hinge on two key assumptions: the restriction to absolutely continuous trading strategies and the focus on the tradeoff between trading costs and market manipulation. Both restrictions may be considered undesirable. On the other hand, if singular controls are considered, the dynamic programming approach would lead to a system of quasi-variational inequalities, which is beyond the scope of our work. Also, it is not obvious how the maximum principle approach we derived in Chapter 2 can be extended to a multiplayer framework. Instead we work in a rather simple framework in the spirit of Almgren and Chriss [2001] and our model should be viewed as a first benchmark to more sophisticated models. Despite its many simplifications, it allows for explicit solutions and thus yields some insight into the qualitative behaviour of optimal manipulation strategies as well as "rules of thumb" for traders or regulators. Moreover, the closed-form solutions will be used in Section 3.5 to indicate how manipulation can be reduced.

3.2.3. A Priori Estimates

In the sequel we show that Problem 3.2.5 admits a unique solution for risk neutral and CARA investors. The proof uses the following a priori estimates for the optimal trading strategies. It states that, if an equilibrium exists, then each player's trading speed is bounded. The reason is that the derivatives H^j_p of the payoff functions H^j are assumed to be bounded, so each investor benefits at most linearly from fast trading. However, trading costs grow more than linearly, and thus very fast trading is not beneficial. Note that this result does not depend on the preference functional.

Proposition 3.2.9. *Let $\left(u^1, ..., u^N\right)$ be a Nash equilibrium for Problem 3.2.5. Then each strategy u^j satisfies $ds \times d\mathbb{P}$ a.e.*

$$\left|u^j(s)\right| \leq N \frac{\lambda}{\epsilon} \left(\max_{i \leq N} \left\| H^i_p \right\|_\infty + 1 \right),$$

where ϵ is taken from Assumption 3.2.2.

Proof. Let $j \leq N$, $h \triangleq \max_i \left\| H^i_p \right\|_\infty$ and

$$A \triangleq \left\{ (s, \omega) \in [0, T] \times \Omega : \sum_{i=1}^N u^i(s, \omega) \geq 0 \right\}$$

be the set where the aggregate trading speed is nonnegative. Let us fix the sum of the competitors' strategies u^{-j}. On the set A the best response $u^j(s)$ is bounded from above by $K \triangleq \frac{\lambda}{\varepsilon}(h+1)$. Otherwise the truncated strategy $\bar{u}^j(s) \triangleq u^j(s) \wedge K \, \mathbb{1}_A + u^j(s) \, \mathbb{1}_{A^c}$ would outperform $u^j(s)$. To see this, let us compare the payoffs associated with u^j and $\bar{u}^j$. We denote by $P^{\bar{u}^j}(T)$ and $P^{u^j}(T)$ the stock price under the strategies $\bar{u}^j$ and u^j, respectively. The payoff associated with $\bar{u}^j$ minus the payoff associated with u^j can be estimated from below as

$$\begin{aligned}
&-\int_0^T \bar{u}^j(s) g\left(\bar{u}^j(s) + u^{-j}(s)\right) ds + H^j(P^{\bar{u}^j}(T)) \\
&+\int_0^T u^j(s) g\left(u^j(s) + u^{-j}(s)\right) ds - H^j(P^{u^j}(T)) \\
\geq &\int_0^T \bar{u}^j(s)\left(g\left(u^j(s) + u^{-j}(s)\right) - g\left(\bar{u}^j(s) + u^{-j}(s)\right)\right) ds \\
&+\int_0^T \left(u^j(s) - \bar{u}^j(s)\right) g\left(u^j(s) + u^{-j}(s)\right) ds - \lambda(X^j(T) - Y^j(T)) \left\|H_p\right\|_\infty .
\end{aligned}$$

Note that $u^j(s) + u^{-j}(s) \geq 0$ on A and thus $g\left(u^j(s) + u^{-j}(s)\right) \geq 0$ due to Assumption 3.2.2. Furthermore, $g\left(u^j(s) + u^{-j}(s)\right) - g\left(\bar{u}^j(s) + u^{-j}(s)\right) \geq \epsilon\left(u^j(s) - \bar{u}^j(s)\right)$, again by Assumption 3.2.2. The difference in the payoffs is therefore larger than

$$\begin{aligned}
&\int_0^T \bar{u}^j(s)\epsilon\left(u^j(s) - \bar{u}^j(s)\right) ds - \lambda h \int_0^T \left(u^j(s) - \bar{u}^j(s)\right) ds \\
=&\int_{u^j(s) > \bar{u}^j(s)} \left(\epsilon \bar{u}^j(s) - \lambda h\right)\left(u^j(s) - \bar{u}^j(s)\right) ds
\end{aligned}$$

On the set $\{u^j(s) > \bar{u}^j(s)\}$ we have $\bar{u}^j(s) = K = \frac{\lambda}{\varepsilon}(h+1)$ and the above expression is strictly positive, a contradiction. This shows that $u^j(s)$ is bounded above by K on the set A for each $j \leq N$. Still on the set A, we get the following lower bound:

$$u^j(s) = \sum_{i=1}^N u^i(s) + \sum_{i \neq j} -u^i(s) \geq 0 - (N-1)K. \tag{3.4}$$

A symmetric argument on the set $B \triangleq \left\{(s,\omega) \in [0,T] \times \Omega : \sum_{i=1}^N u^i(s,\omega) \leq 0\right\}$ completes the proof. $\square$

3.3. Solution for Risk Neutral Investors

In this section we use dynamic programming to show that Problem 3.2.5 admits a unique solution (in a certain class) for risk neutral agents. Here the preference functional is $\Psi_t^j(Z) = \mathbb{E}[Z|\mathcal{F}_t]$ for each $j \leq N$. We also show that the solution can be given in closed form for the special case of a linear cost function.

The idea is to consider the value function associated to Problem 3.2.5 for player j, where his competitors' strategies are fixed, and to characterise it as the solution of the HJB PDE. Solving the resulting coupled system of PDEs for all players simultaneously then provides an equilibrium point of the stochastic differential game, cf. Friedman [1972]. To begin with, we fix the strategies $(u^i)_{i \neq j}$ and define the *value function* for player $j \le N$ as

$$V^j(t,p) = \sup_{u^j \in \mathcal{U}_t} \mathbb{E}_{t,p}\left[-\int_t^T u^j(s) g\left(\sum_{i=1}^N u^i(s)\right) ds + H^j(P(T))\right],$$

subject to the state dynamics

$$dP(s) = \sigma dW(s) + \lambda \sum_{i=1}^N u^i(s) ds, \quad P(t) = p.$$

Here we use the notation $\mathbb{E}_{t,p}[\cdot] \triangleq \mathbb{E}[\cdot | P_t = p]$. Given time $t \in [0,T]$ and stock price $p \in \mathbb{R}$ the value function represents the conditional expected portfolio value at maturity that player j can achieve by trading optimally, given the other players' strategies. The associated HJB equation is, cf. Fleming and Soner [1993] Theorem IV.3.1,

$$0 = v_t^j + \frac{1}{2}\sigma^2 v_{pp}^j + \sup_{c^j \in \mathbb{R}} \left[\lambda \left(c^j + u^{-j}\right) v_p^j - c^j g\left(c^j + u^{-j}\right)\right], \tag{3.5}$$

with terminal condition $v^j(T,p) = H^j(p)$, where v_t and v_p denote time and spatial derivatives, respectively. The HJB equation is formulated in terms of the *candidate* value functions $v^1, ..., v^N$ instead of the actual value functions $V^1, ..., V^N$. We first need to show existence and uniqueness of a smooth solution to (3.5) before we can identify v^i with V^i. Given the aggregate trading strategy u^{-j} of all the other agents, a candidate for the maximiser $c^j = u^j$ in (3.5) should satisfy

$$0 = \lambda v_p^j - g\left(c^j + u^{-j}\right) - c^j g'\left(c^j + u^{-j}\right). \tag{3.6}$$

We have one equation of this type for each player $j \le N$. Summing them up and defining the *aggregate trading speed* as

$$u^{ag} \triangleq \sum_{i=1}^N u^i$$

yields the following condition

$$\begin{aligned} 0 &= \lambda \sum_{i=1}^N v_p^i - Ng\left(\sum_{i=1}^N u^i(s)\right) - \left(\sum_{i=1}^N u^i(s)\right) g'\left(\sum_{i=1}^N u^i(s)\right) \\ &= \lambda \sum_{i=1}^N v_p^i - Ng\left(u^{ag}(s)\right) - u^{ag}(s) g'\left(u^{ag}(s)\right). \end{aligned} \tag{3.7}$$

In view of Assumption 3.2.2 the map $z \mapsto Ng(z) + zg'(z)$ is strictly increasing. Hence condition (3.7) admits a unique solution u^{ag} which depends on $\sum_{i=1}^N v_p^i$. Plugging u^{ag}

back into (3.6) allows to compute the candidate optimal control for player $j \leq N$ as

$$c^j = u^j = \frac{\lambda v_p^j - g(u^{ag})}{g'(u^{ag})}. \tag{3.8}$$

This expression is well defined since $g' > 0$ again by Assumption 3.2.2. Plugging this candidate optimal control into the HJB equation, we see that the system of HJB PDEs now takes the form

$$0 = v_t^j + \frac{1}{2}\sigma^2 v_{pp}^j + \lambda \left(u^{ag} - \frac{g(u^{ag})}{g'(u^{ag})} \right) v_p^j + \frac{g(u^{ag})^2}{g'(u^{ag})} \tag{3.9}$$

with terminal condition $v^j(T,p) = H^j(p)$ for $j \leq N$. Note that the coupling stems from the aggregate trading speed u^{ag} via condition (3.7).

Remark 3.3.1. Looking back, we have turned the individual HJB equations (3.5) into the system of coupled PDEs (3.9). Systems of this form appear naturally in the theory of differential games, but we did not find a reference which covers this particular case. Theorem 1 of Friedman [1972] for instance is valid only on a bounded state space. We shall use our a priori estimates of Proposition 3.2.9 in order to prove existence of a unique solution to (3.9).

The following theorem, whose proof is given in Appendix A.3.1, shows that the system of PDEs (3.9) has a unique classical solution if $H^j \in \mathcal{C}_b^2$, i.e. H^j is twice continuously differentiable and its derivatives up to order 2 are bounded, for each j. Similarly, $\mathcal{C}^{1,2}$ is the space of functions which are continuously differentiable in time and twice continuously differentiable in space.

Theorem 3.3.2. *Let $H \in \mathcal{C}_b^2$. Then the Cauchy problem (3.9) admits a unique classical solution in $\mathcal{C}^{1,2}$, which is the vector of value functions.*

Remark 3.3.3. An alternative way of solving the system (3.9) is the following: If we sum up the N equations, we get a Cauchy problem for the aggregate value function $v \triangleq \sum_{i=1}^N v^i$, namely

$$0 = v_t + \frac{1}{2}\sigma^2 v_{pp} + u^{ag}\left[\lambda v_p - g\left(u^{ag}\right)\right] \tag{3.10}$$

with terminal condition $v(T,p) = \sum_{i=1}^N H^i(p)$. Existence and uniqueness of a solution to this one-dimensional problem can be shown using Theorem V.8.1 in Ladyzenskaja et al. [1968]. Once the solution is known, we can plug it back into (3.9) and get N decoupled equations. This technique is applied in the following section where we construct an explicit solution for linear cost functions.

It is hard to find a closed form solution for the coupled PDE (3.9). However, for the particular choice $g(z) = \kappa z$ with a liquidity parameter $\kappa > 0$ the solution to (3.9) can be given explicitly. Here and throughout, we denote by

$$f_{\mu,\sigma^2}(z) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right)$$

the normal density with mean μ and variance σ^2.

Proposition 3.3.4. *Let $g(z) = \kappa z$. Then the solution of (3.9) can be given in closed form as the solution to a nonhomogeneous heat equation.*

Proof. The optimal trading speed from (3.8) and the aggregate trading speed from (3.7) are

$$u^j = \frac{\lambda}{\kappa}\left(v_p^j - \frac{1}{N+1}\sum_{i=1}^N v_p^i\right) \tag{3.11}$$

$$u^{ag} = \sum_{i=1}^N u^i = \frac{\lambda}{\kappa(N+1)}\sum_{i=1}^N v_p^i = \frac{\lambda}{\kappa(N+1)} v_p. \tag{3.12}$$

Equation (3.9) for player j's value function now becomes

$$0 = v_t^j + \frac{1}{2}\sigma^2 v_{pp}^j + \kappa(u^{ag})^2.$$

Combining this with (3.12) and summing up for $j = 1, ..., N$ yields the following PDE for the aggregate value function $v = \sum_{i=1}^N v^i$:

$$0 = v_t + \frac{1}{2}\sigma^2 v_{pp} + \frac{\lambda^2 N}{\kappa(N+1)^2} v_p^2 \tag{3.13}$$

with terminal condition $v(T,p) = \sum_{i=1}^N H^i(p)$. This PDE is a variant of *Burgers' equation*, cf. Rosencrans [1972]. It allows for an explicit solution, which we cite in Lemma A.3.3 in the appendix. With this solution at hand, we can solve for each single investor's value function. We plug the solution v back into the equations (3.11) and (3.12) for the trading speeds, and those into the PDE (3.9). This yields

$$0 = v_t^j + \frac{1}{2}\sigma^2 v_{pp}^j + \frac{\lambda^2}{\kappa(N+1)^2} v_p^2$$

with terminal condition $v^j(T,p) = H^j(p)$. This is now a PDE in the unknown function v^j with known function v_p. We see that it is a nonhomogeneous heat equation with solution given by

$$v^j(T-t,p) = \int_{\mathbb{R}} H^j(z) f_{p,\sigma^2 t}(z)dz + \frac{\lambda^2}{\kappa(N+1)^2}\int_0^t \int_{\mathbb{R}} v_p^2(s,z) f_{p,\sigma^2(t-s)}(z)dzds$$

where v is known from Lemma A.3.3 (in particular it is bounded and integrable). □

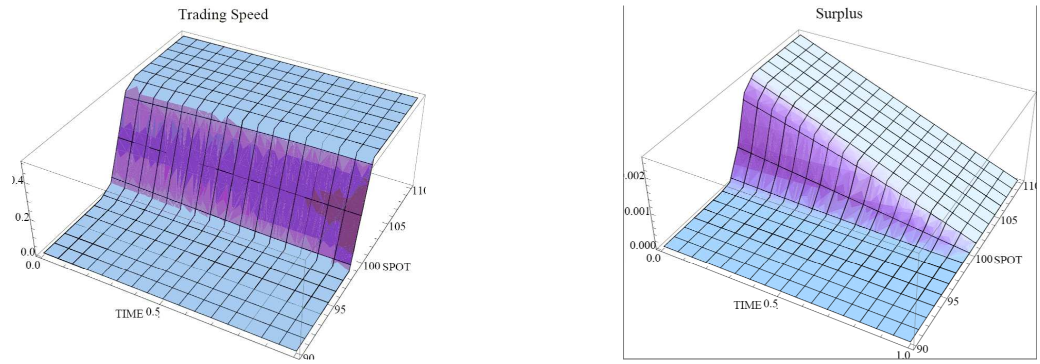


Figure 3.1.: Trading speed and surplus for one risk neutral investor holding a European Call option.

Let us conclude this section with some numerical illustrations. For risk neutral players and a linear cost structure, we reduced the system of PDEs to the one-dimensional PDE (3.13) for the aggregate value function. This can be interpreted as the value function of the representative agent. Such reduction to a representative agent is not always possible for more general utility functions. In the sequel we illustrate the optimal trading speed $u(s,p)$ and surplus of a representative agent as functions of time and spot prices for a European call option $H(P(T)) = (P(T) - K)^+$ and digital option $H(P(T)) = \mathbb{1}_{\{P(T) \geq K\}}$, respectively.[3] By surplus, we mean the difference between the representative agent's optimal expected portfolio value $v(t,p)$ and the conditional expected payoff $\mathbb{E}_{t,p}[H(P(T))]$ in the absence of any market impact. It represents the expected net benefit due to price manipulation.

We choose a linear cost function, strike $K = 100$, maturity $T = 1$, volatility $\sigma = 1$ and liquidity parameters $\lambda = \kappa = 0.01$. We see from Figure 3.1 that for the case of a call option both the optimal trading speed and the surplus increases with the spot; the latter also increases with the time to maturity. Furthermore, the increase in the trading speed is maximal when the option is at the money. For digital options (figure 2) the trading speed is highest for at the money options close to maturity as the trader tries to push the spot above the strike. If the spot is far away from the strike, the trading speed is very small as it is unlikely that the trader can push the spot above the strike before expiry. For both option types a high spread renders manipulation unattractive. Figures 3 and 4 show the optimal trading speed and the surplus at time $t = 0$ for the Call and Digital option for a representative agent. We used the cost function

$$g(z) = \kappa z + c \cdot \operatorname{sign}(z) \quad \text{for different spreads } c \in \{0, 0.001, 0.002, 0.003, 0.004\} \tag{3.14}$$

with the remaining parameters as above. We see that the higher the spread, the smaller

[3] Note that the cost function in (3.14) is not smooth, and the Call and Digital options are not smooth and bounded, so Theorem 3.3.2 does not apply directly. There are two ways to overcome this difficulty: We could either approximate g and H by smooth and bounded functions. Or we could interpret v not as a classical, but only as a viscosity solution of (3.5), cf. Fleming and Soner [1993] chapter V.

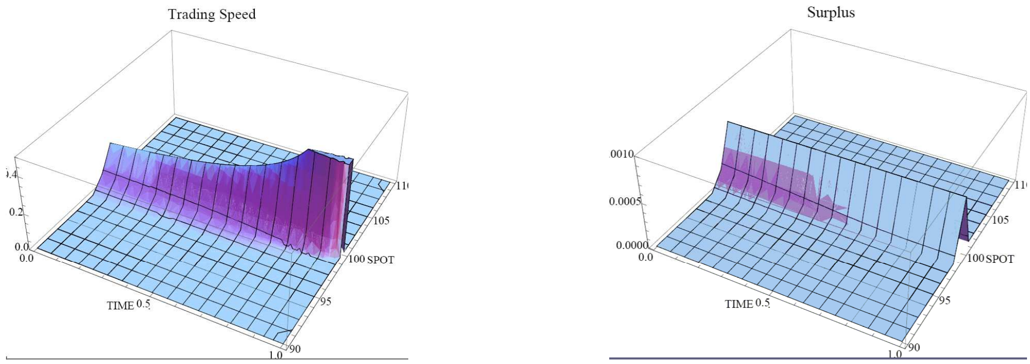


Figure 3.2.: Trading speed and surplus for one risk neutral investor holding a Digital option.

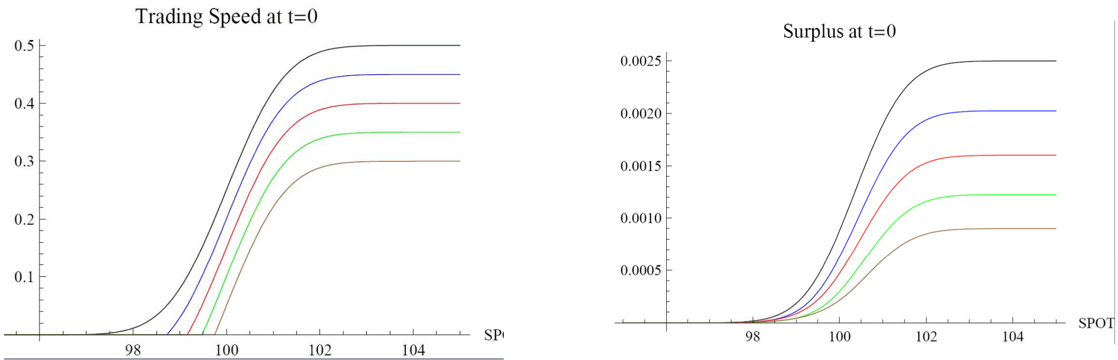


Figure 3.3.: Trading speed and surplus for a risk neutral investor holding a European Call option for different spread sizes $s = 0$ (black), 0.001 (blue), 0.002 (red), 0.003 (green), 0.004 (brown). The higher the spread, the smaller the trading speed and the surplus.

the trading speed and the surplus. This is intuitive as frequent trading, in particular when the option is at the money, incurs high spread crossing costs. The same is true for fixed transaction costs which also discourage frequent trading.

3.4. Solution for CARA Investors

In the preceding section we considered risk neutral investors. We shall now extend the analysis of Problem 3.2.5 to the class of entropic preference functionals with risk aversion coefficient $\alpha^j > 0$, given by

$$\Psi_t^j(Z) = -\frac{1}{\alpha^j} \log \mathbb{E}\left[\exp(-\alpha^j Z)|\mathcal{F}_t\right].$$

As pointed out by [Cheridito et al., 2009, p.9], these mappings induce the same preferences as conditional expected exponential utility functions. Due to the translation invariance of Ψ_t^j the trading costs $R(t) \triangleq \int_0^t u^j(s) g\left(\sum_{i=1}^N u^i(s)\right) ds$ that player j incurred in $[0,t]$ do not affect the player's optimal strategy in the time interval $[t,T]$ (they

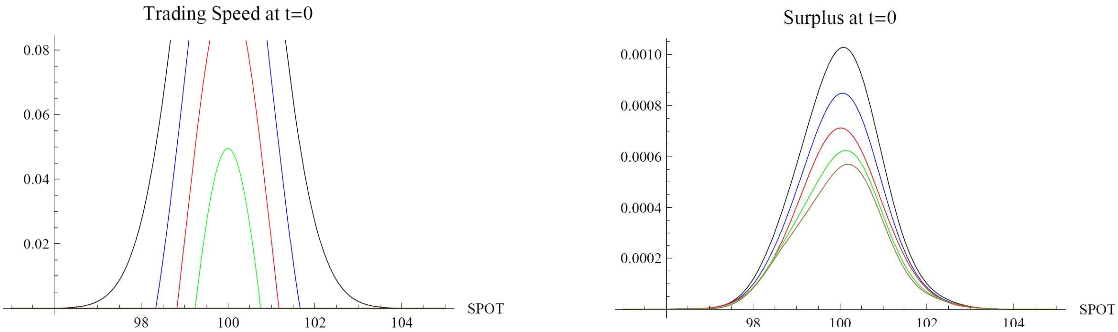


Figure 3.4.: Trading speed and surplus for a risk neutral investor holding a Digital option for different spread sizes $s = 0$ (black), 0.001 (blue), 0.002 (red), 0.003 (green), 0.004 (brown). The higher the spread, the smaller the trading speed and the surplus.

affect only the utility). As a result, we may consider the cost adjusted preference functional $\Psi_t^j + R(t)$. So, given the strategies $(u^i)_{i\neq j}$ of the other players, the value function for player $j \leq N$ is

$$V^j(t,p)$$
$$\triangleq \sup_{u^j\in\mathcal{U}_t}\left\{-\frac{1}{\alpha^j}\log\mathbb{E}_{t,p}\left[\exp\left(-\alpha^j\left(-\int_t^T u^j(s)g\left(\sum_{i=1}^N u^i(s)\right)ds + H^j(P(T))\right)\right)\right]\right\}.$$

As a result, the HJB equation[4] for player j is now given by

$$0 = v_t^j + \frac{1}{2}\sigma^2 v_{pp}^j - \frac{1}{2}\sigma^2\alpha^j\left(v_p^j\right)^2 + \sup_{c^j\in\mathbb{R}}\left[\lambda\left(c^j + u^{-j}\right)v_p^j - c^j g\left(c^j + u^{-j}\right)\right] \tag{3.15}$$

with terminal condition $v^j(T,p) = H^j(p)$. Note that this equation equals the HJB equation (3.5) in the risk neutral setting, up to the quadratic term $-\frac{1}{2}\sigma^2\alpha^j\left(v_p^j\right)^2$. Applying the same arguments as in Section 3.3, the candidate optimal trading speeds are for $j \leq N$

$$c^j = u^j = -\frac{1}{g'(u^{ag})}\left[-\lambda v_p^j + g(u^{ag})\right]$$

where the aggregate trading speed u^{ag} is the unique solution to

$$0 = \lambda\sum_{i=1}^N v_p^i - Ng\left(\sum_{i=1}^N u^i(s)\right) - \left(\sum_{i=1}^N u^i(s)\right)g'\left(\sum_{i=1}^N u^i(s)\right). \tag{3.16}$$

[4]This PDE can be derived by considering the exponential utility function first and then applying a logarithmic transformation. In this approach it is necessary to introduce new state variables which keep track of each agent's trading costs. Due to translation invariance, these variables factor out and can be dropped again.

If we plug u^{ag} and u^j back into (3.15), we get

$$0 = v_t^j + \frac{1}{2}\sigma^2 v_{pp}^j - \frac{1}{2}\sigma^2\alpha^j \left(v_p^j\right)^2 + \lambda\left(u^{ag} - \frac{g(u^{ag})}{g'(u^{ag})}\right) v_p^j + \frac{g(u^{ag})^2}{g'(u^{ag})}. \tag{3.17}$$

We can show existence and uniqueness of a solution.

Theorem 3.4.1. *Let $H^j \in \mathcal{C}_b^2$ for each $j \leq N$. The Cauchy problem (3.15) admits a unique solution, which is the vector of value functions.*

Proof. See Appendix A.3.1. □

For the one player case with linear cost structure, we have an explicit solution:

Corollary 3.4.2. *Let $N = 1$ and $g(z) = \kappa z$. Then the Cauchy problem (3.15) admits a unique solution, which can be given in closed form.*

Proof. The maximiser in (3.15) is now

$$c = u = \frac{\lambda}{2\kappa} v_p$$

and the Cauchy problem (3.17) turns into

$$0 = v_t + \frac{1}{2}\sigma^2 v_{pp} + \left(\frac{\lambda^2}{4\kappa} - \frac{1}{2}\sigma^2\alpha\right) v_p^2$$

with terminal condition $v(T, p) = H(p)$. This is Burgers' equation. Its explicit solution is given in Lemma A.3.3 in the appendix. □

Let us conclude this section with numerical illustrations for the two player case. Figure 3.5 shows the aggregate optimal trading speed and the surpluses $v^j(0,p) - \Psi_0^j\left(H^j(P(T))\right)$ for time $t = 0$ and different spot prices $p \in [90, 100]$ for the European Call option $H(P(T)) = (P(T) - K)^+$. We assume that Player 1 (blue) is the option writer and Player 2 (red) the option issuer. We chose the strike $K = 100$, maturity $T = 1$, volatility $\sigma = 1$ and liquidity parameters $\lambda = 0.1$, $\kappa = 0.01$ and risk aversion parameters $\alpha^1 = 0.01, \alpha^2 = 0.01$ (solid), respectively, $\alpha^1 = 0.1, \alpha^2 = 0.001$ (dashed). Since Player 1 has a long position in the option, he has an incentive to buy the underlying; for the same reason Players 2 has an incentive to sell it (Panel (b)). Our simulations suggest that the option issuer is slightly more active than the option writer, in particular near the strike. Furthermore, we see from Panel (d) that the issuer benefits more from reducing his loss than the writer benefits from increasing his gains; this effect is due to the concavity of the utility function. If the option issuer is less risk averse than the option writer, he trades and benefits slightly more (dashed).

Figure 3.6 shows the same plots for the Digital option. Now the option writer trades faster and benefits more if the option is in the money, while the issuer trades faster and gains more if the option is out of the money (Panels (c) and (d)).

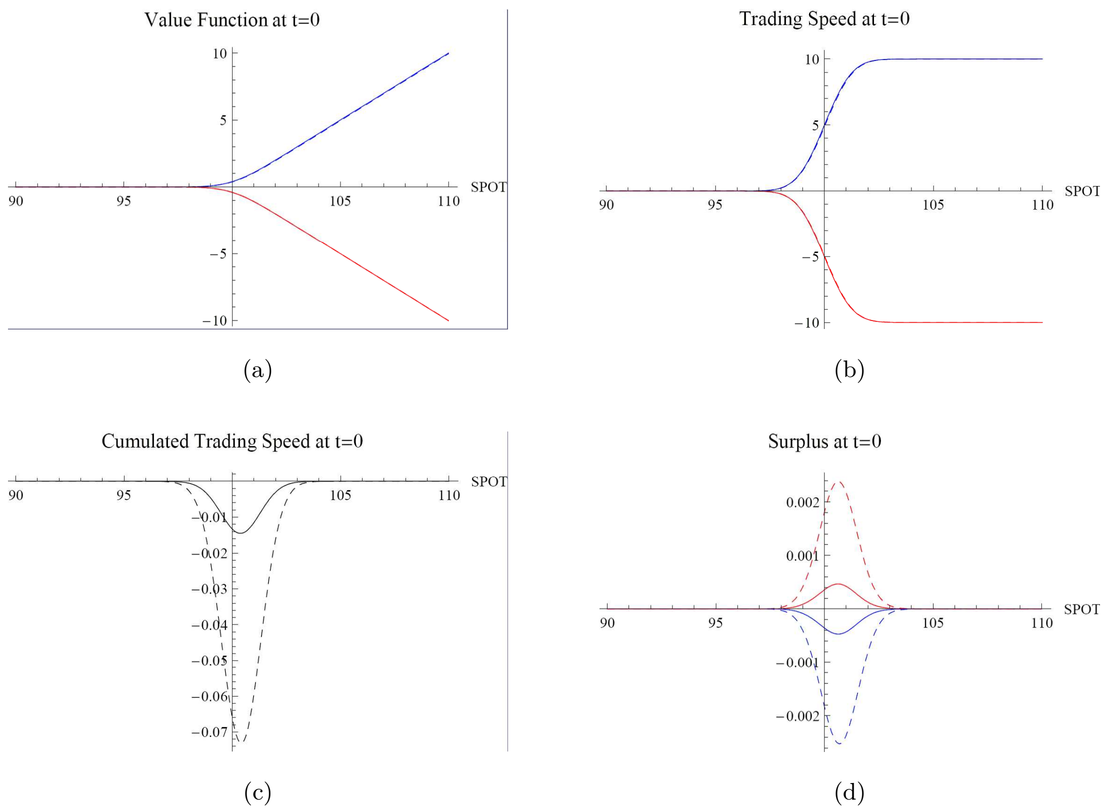


Figure 3.5.: Value function, trading speed, aggregate trading speed and surplus for the writer (blue) and issuer (red) of a European Call option when both agents are risk averse. The solid (dashed) curves display the case where issuer is about as (less) risk averse than the option writer.

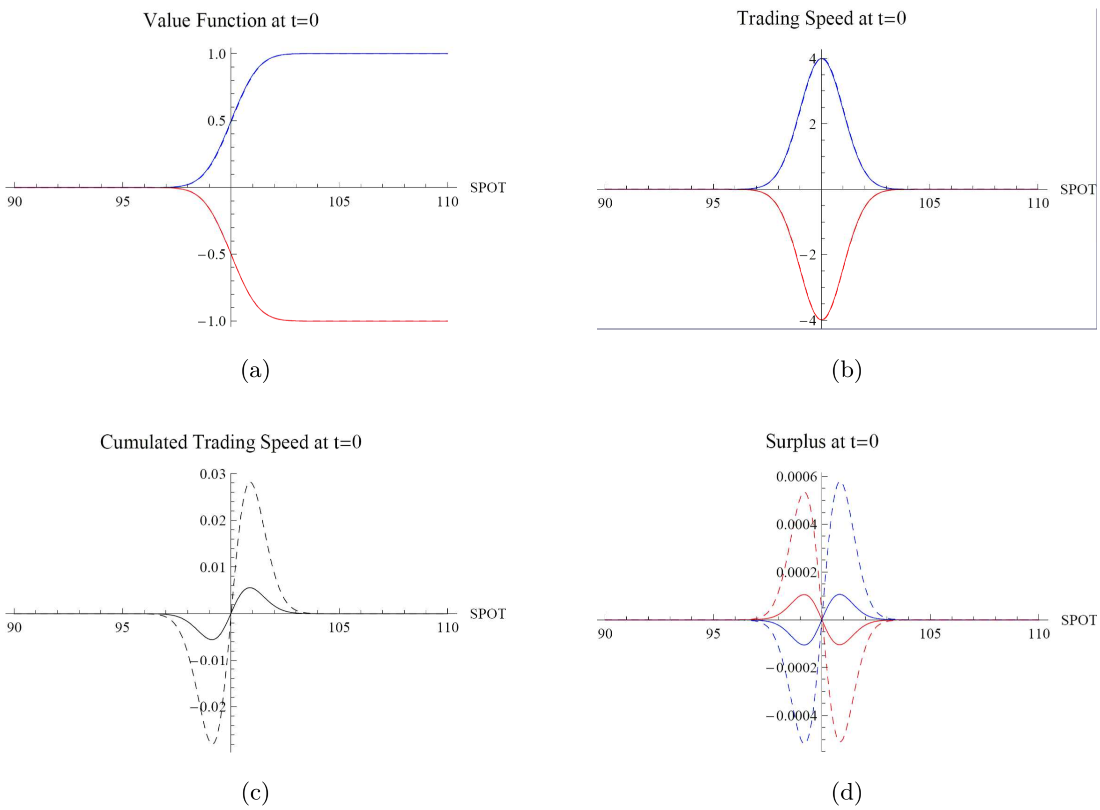


Figure 3.6.: Value function, trading speed, aggregate trading speed and surplus for the writer (blue) and issuer (red) of a European Digital option when both agents are risk averse. The solid (dashed) curves display the case where issuer is about as (less) risk averse than the option writer.

3.5. How to Reduce Manipulation

In this section, we use the results for risk neutral agents derived in Section 3.3 to illustrate how an option issuer may prevent[5] other market participants from trading against him by using their impact on the dynamics of the underlying. Some of our observations were already made in Kumar and Seppi [1992] for Futures in a two period model and in Gallmeyer and Seppi [2000] for Call options in a three period binomial model. Note that the results of this section only hold for risk neutral investors.

As a first step, we show that market manipulation is not beneficial if traders have no permanent impact on the price of the underlying.

Proposition 3.5.1. *If* $\lambda = 0$ *then* $u^j \equiv 0$ *for each* $j \leq N$.

Proof. First note that $u^{ag} = \sum_{i=1}^N u^i = 0$ is the unique solution to (3.7). Now (3.8) implies that $u^j \equiv 0$ for each $j \leq N$. □

Let us now consider the more interesting case of $\lambda > 0$. We show next that in the case of offsetting payoffs, the aggregate trading speed is zero. Put differently, in a zero sum game of risk neutral investors willing to move the market in their favor, their combined effect cancels. We note that this is no longer true for general utility functions, as illustrated in figure 3.5 for the CARA case.

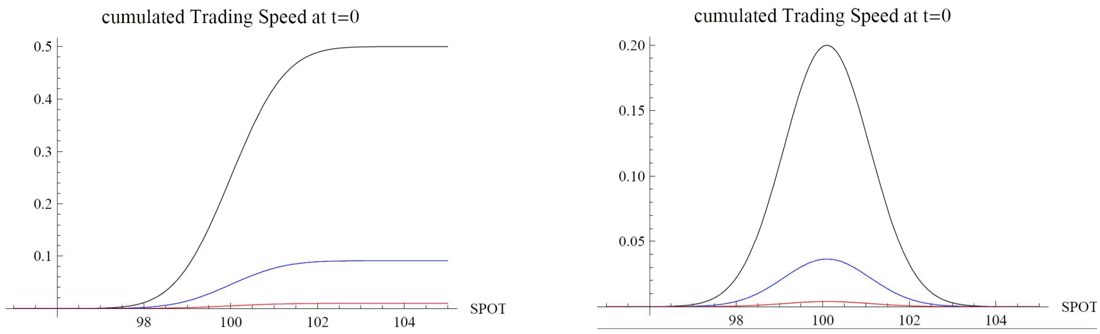
Proposition 3.5.2. *If* $\sum_{i=1}^N H^i = 0$ *then* $\sum_{i=1}^N u^i \equiv 0$.

Proof. Consider the PDE (3.10) for the aggregate value function with terminal condition zero and the characterisation (3.7) of the aggregate trading speed. $u^{ag} = \sum_{i=1}^N u^i \equiv 0$ and $v = \sum_{i=1}^N v^i \equiv 0$ is the unique solution to this coupled system. □

In reality, some (or all) of the investors might not want to manipulate, e.g. for legal reasons[6]. This is why we now look at the following asymmetric situation: The option issuer, Player 0, does not trade the underlying; his competitor, Player 1, owns the payoff $H^1 \neq 0$ and intends to move the stock price to his favor. In addition, there are $N - 1$ informed investors without option endowment in the market. They are "predators" that may supply liquidity and thus reduce the first player's market impact, cf. Carlin et al. [2007] and Schied and Schöneborn [2007a]. The following result states that the aggregate trading speed is decreasing in the number of players. More liquidity suppliers lead to more competition for profit and less (cumulated) market manipulation. If the number of players goes to infinity, manipulation vanishes. Note that Propositions 3.5.4 and 3.5.3 are only valid for the linear cost function, as the proofs hinge on the closed form solution obtained in Proposition 3.3.4, and for nondecreasing payoff functions.

[5] Let us emphasise again that our results only apply to the practice of "punching the close", i.e. manipulating the stock price in order to increase a given option payoff. There are other types of market manipulation not covered by our setup, such as market corners, short squeezes, the use of private information or false rumours. We refer the interested reader to Jarrow [1994] and Kyle and Viswanathan [2008].

[6] A discussion of legal issues is beyond the scope of our work, but see the discussion in Kyle and Viswanathan [2008].



(a) Call

(b) Digital

Figure 3.7.: Aggregate trading speed u^{ag} at time $t = 0$ for $N = 1$ (black), 10 (blue), 100 (red) players each holding $1/N$ shares of a Call (left) and Digital (right) option with strike $K = 100$. The more agents, the less aggregate manipulation.

Proposition 3.5.3. *Let $g(z) = \kappa z$. Let $H^1 \in \mathcal{C}_b^2$ be nondecreasing and $H^i = 0$ for $i = 2, ..., N$. Then for $s \in [0, T]$ the aggregate trading speed $\sum_{i=1}^N u^i(s)$ is decreasing in N and*

$$\lim_{N\to\infty} \sum_{i=1}^{N} u^i(s) = 0.$$

Proof. See Appendix A.3.2. □

Let us modify the preceding setting a little. Again, Player 0 issues a product H and does not intend to manipulate the underlying, while his competitors do. More precisely, assume that player 0 splits the product H into pieces and sells them to N risk neutral competitors, such that each of them gets $\frac{1}{N}H$. We find that their aggregate trading speed $\sum_{i=1}^N u^i$ is decreasing in the number of competitors N. Consequently, the option issuer should sell his product to as many investors as possible in order not to be susceptible to manipulation. We illustrate this result in figure 3.7, which shows the aggregate trading speed at time $t = 0$ of N players each holding $1/N$ option shares.

Proposition 3.5.4. *Let $g(z) = \kappa z$. Let $H \in \mathcal{C}_b^2$ be nondecreasing and $H^i = \frac{1}{N}H$ for $i = 1, ..., N$. Then for $s \in [0, T]$ the aggregate trading speed $\sum_{i=1}^N u^i(s)$ is decreasing in N and*

$$\lim_{N\to\infty} \sum_{i=1}^{N} u^i(s) = 0.$$

Proof. See Appendix A.3.2. □

The preceding results indicate how an option issuer can prevent his competitors from manipulation. One strategy is public announcement of the transaction: the more informed liquidity suppliers are on the market, the smaller the impact on the underlying. A second strategy is splitting the product into pieces - the more option writers, the less manipulation. Let us conclude this section with a surprisingly simple way to avoid manipulation: using options with physical settlement. In contrast to cash settlement the option holder does not receive (pay) the current price of the underlying, but receives (delivers) stock shares. In the case of Call options, for instance, let us denote by c^j the number of Calls player j decides to execute at maturity, he then holds $X^j(T) + c^j$ stock shares whose liquidation value under infinitely slow liquidation in $[T, \infty)$ is now defined as

$$\left(X^j(T) + c^j\right)\left(P(T) - \frac{1}{2}\lambda\left(X^j(T) + c^j\right)\right).$$

The following proposition shows that in a framework of several risk neutral players holding physically settled Calls, Puts and Forwards, it is optimal not to manipulate the underlying.

Proposition 3.5.5. *Consider N risk neutral agents holding European Call, Put or Forward options with physical settlement. Then $u^j \equiv 0$ for each $j \leq N$ is a Nash equilibrium.*

Proof. We only prove the assertion for Call options. The case of Puts and Forwards (or combinations thereof) follows by the same arguments. Suppose that agent $j \leq N$ is endowed with $C^j \geq 0$ Call options with physical settlement and strike K^j. At maturity, the agent decides how many options he exercises. The agent's strategy is now a pair (u^j, c^j), where $u^j \in \mathcal{U}_0$ denotes his trading speed in the underlying and $c^j \in [0, C^j]$ the number of Call options exercised. At maturity, the agent receives c^j stock shares for the price $c^j K^j$. Suppose that $u^i \equiv 0$ for each $i \neq j$, i.e. none of player j's competitors trades. His optimisation problem is then

$$\sup_{u^j, c^j} \mathbb{E}\left[\int_0^T -u^j(s)\tilde{P}(s)ds - c^j K^j + \left(X^j(T) + c^j\right)\left(P(T) - \frac{1}{2}\lambda\left(X^j(T) + c^j\right)\right)\right].$$

Here the first term represents the expected trading costs in $[0, T]$ and the second term is the cost of exercising the options. The last term describes the liquidation value of $X^j(T) + c^j$ stock shares under infinitely slow liquidation in $[T, \infty)$. Using the stock price dynamics (3.1), (3.2) and $X^j(0) = 0$, it can be shown that this equals

$$\sup_{u^j, c^j} \mathbb{E}\left[\int_0^T -u^j(s)g(u^j(s))ds - c^j K^j + c^j\left(P(0) + \sigma W(T) - \frac{1}{2}\lambda c^j\right)\right].$$

The cost term $\int_0^T u^j(s)g(u^j(s))ds$ is nonnegative and the remaining terms do not depend on u^j, so the optimal trading strategy in the stock is $u^j \equiv 0$. This shows that $u^j \equiv 0$ for each $j \leq N$ is a Nash equilibrium. □

At first glance, Proposition 3.5.5 might contradict [Pirrong, 2001, p.221]. He states that "replacement of delivery settlement of futures contracts with cash settlement is frequently proposed to reduce the frequency of market manipulation". While his notion of market manipulation refers to market corners and short squeezes (see also Garbade and Silber [1983]), Proposition 3.5.5 shows that this is not always true for manipulation strategies in the sense of "punching the close". It is not beneficial to drive up the stock price at maturity if the option is settled physically and the investor needs to liquidate the stocks he receives at maturity. Any price increase is outweighed by subsequent liquidation and has no positive effect, but it is costly. This confirms a claim made in [Kumar and Seppi, 1992, p.1497], who argue that whether "futures contracts with a 'physical delivery' option [are] also susceptible to liquidity-driven manipulation [...] depends on whether 'offsetting' trades can be used to unwind a futures position with little price impact".

A. Appendix

A.1. Auxiliary Results for Chapter 1

We collect here two lemmata which are needed in Chapter 1.

Lemma A.1.1. *Given controls $u, \bar{u} \in \mathcal{U}_t$ such that $X^u = X^{\bar{u}}$ $ds \times d\mathbb{P}$ a.e. on $[t,T] \times \Omega$*

1. *The processes X^u and $X^{\bar{u}}$ are indistinguishable, i.e*

$$\sup_{s \in [t,T]} \left| X^u(s) - X^{\bar{u}}(s) \right| = 0.$$

2. *The controls are identical, i.e*

$$u = \bar{u} \quad ds \times d\mathbb{P} \text{ a.e. on } [t,T] \times \Omega.$$

Proof. We write $X \triangleq X^u$ and $\bar{X} \triangleq X^{\bar{u}}$. From the $ds \times d\mathbb{P}$ equality we get

$$\mathbb{E}\left[\int_t^T \left(X(s) - \bar{X}(s) \right)^2 ds \right] = 0 \tag{A.1}$$

which immediately implies

$$\int_t^T \left(X(s) - \bar{X}(s) \right)^2 ds = 0.$$

Since X and $\bar{X}$ are càdlàg semimartingales with finitely many jumps on $[t,T]$ we conclude that

$$\sup_{s \in [t,T)} \left| X(s) - \bar{X}(s) \right| = 0,$$

i.e equality on $[t,T)$. Since N is a Poisson process we have $\mathbb{P}(\Delta N(T) > 0) = 0$ so we can extend the equality to $[0,T]$ which establishes the first claim. A consequence of item (1) is that the quadratic variation is zero and thus we have

$$0 = \int_t^T d[X - \bar{X}, X - \bar{X}](s) = \int_t^T \left(u_1(s) - \bar{u}_1(s) \right)^2 N(ds)$$

Taking expectation and using the L^2-property of the controls u and $\bar{u}$, we get

$$0 = \mathbb{E}\left[\int_t^T (u_1(s) - \bar{u}_1(s))^2 N(ds)\right] = \lambda \mathbb{E}\left[\int_t^T (u_1(s) - \bar{u}_1(s))^2 ds\right]$$

which implies $u_1 = \bar{u}_1$, $ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$. Combining this with equation (A.1) and repeating the argument above leads to the following relation for the second components,

$$\int_t^T \left(\int_t^s u_2(r) - \bar{u}_2(r) dr\right)^2 ds = 0$$

which then implies that

$$\sup_{t \leq s \leq T} \left| \int_t^s u_2(r) - \bar{u}_2(r) dr \right| = 0.$$

Thus the total variation of the process $\int_t^\cdot u_2(s) - \bar{u}(s) ds$ is zero, which leads to

$$\int_t^T |u_2(s) - \bar{u}_2(s)| \, ds = 0$$

and so $u_2 = \bar{u}_2$, $ds \times d\mathbb{P}$ a.e. on $[t, T] \times \Omega$, which completes the proof. □

Remark A.1.2. Observe that item (1) remains valid for any càdlàg process with finitely many jumps at exponentially distributed random times, in particular P, as in the second part of Proposition 1.6.5

The next lemma provides a sufficient condition for the stochastic integral with respect to a compensated Poisson random measure to be a true martingale. We provide a proof for completeness.

Lemma A.1.3. *Let L be a Poisson random measure with compensator l and H be a predictable L-integrable process such that*

$$\mathbb{E}\left[\int_t^T \int_{\mathbb{R}^k} |H(s, \theta)| l(ds, d\theta)\right] < \infty.$$

Then the process

$$\int_t^\cdot \int_{\mathbb{R}^k} H(s, \theta) \tilde{L}(ds, d\theta)$$

is a true martingale, where $\tilde{L}$ denotes the compensated Poisson random measure.

Proof. We proceed by an approximation argument. Let $n \in \mathbb{N}$ and define the truncated strategy $H^n(r, \theta) \triangleq H(r, \theta) \wedge n \vee (-n)$ and consider the process

$$M^n(s) \triangleq \int_t^s \int_{\mathbb{R}^k} H^n(r, \theta) \tilde{L}(dr, d\theta), \quad s \in [t, T].$$

The process M^n is a martingale thanks to Protter [2004] Theorem II.2.29. We now have the following estimate,

$$\mathbb{E}\left[\sup_{t\le s\le T}|M^n(s)-M^m(s)|\right]\le 2\mathbb{E}\left[\int_t^T\int_{\mathbb{R}^k}|H^n(s,\theta)-H^m(s,\theta)|\,l(ds,d\theta)\right].$$

Letting m and n go to infinity and using the assumptions of the lemma, it follows then that $(M^n)_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{H}^1$, the Banach space of uniformly integrable martingales on $[t,T]$ equipped with the norm

$$\|M\|_{\mathcal{H}^1}\triangleq\mathbb{E}\left[\sup_{t\le s\le T}|M(s)|\right].$$

This sequence therefore has a limit M, which is also a martingale. On the other hand we deduce from the Dominated Convergence Theorem Protter [2004] Theorem IV.32 that

$$\lim_{n\to\infty}\sup_{t\le s\le T}\left|M^n(s)-\int_t^s\int_{\mathbb{R}^k}H(r,\theta)\tilde{L}(dr,d\theta)\right|=0,$$

i.e. $(M^n)_{n\in\mathbb{N}}$ converges in UCP to

$$\int_t^{\cdot}\int_{\mathbb{R}^k}H(s,\theta)\tilde{L}(ds,d\theta).$$

We thus conclude that this process is indistinguishable from M and hence a true martingale. □

Remark A.1.4. At first sight this lemma may appear obvious, however one must be careful as in general it is not true that the stochastic integral with respect to a compensated Poisson random measure is even a local martingale, see the example of Emery [1980]. This motivates the need for the approximation in the above.

A.2. Auxiliary Results for Chapter 2

The following lemma is needed in the proof of Lemma 2.3.4. Specifically, we need to show that for fixed t the constant

$$c_{1,t}\triangleq\min\{v_2(t,1),v_2(t,-1)\}$$

is strictly positive and finite. We only show that $v_2(t,1)\in(0,\infty)$ for each $t\in[0,T]$, the proof for $v_2(t,-1)$ is similar. Recall that v_2 and J_2 are given by

$$v_2(t,x_3)=\inf_{(\eta,u)\in\mathcal{U}_t}J_2(t,x_3,\eta,u),$$
$$J_2(t,x_3,\eta,u)=\mathbb{E}_{t,(0,0,x_3)^*}\left[\int_{[t,T]}\left[X_1(r-)+\frac{\kappa_1}{2}\Delta\eta_1(r)\right]d\eta_1(r)\right.$$

$$+\int_{[t,T]}\left[X_2(r-)+\frac{\kappa_2}{2}\Delta\eta_2(r)\right]d\eta_2(r)+\int_t^T\frac{\varepsilon}{2}\left|X_3(r)\right|^2dr+\frac{\varepsilon}{2}\left|X_3(T)\right|^2\Bigg].$$

Lemma A.2.1. *For each $t \in [0,T]$ we have that $v_2(t,1) > 0$ and $v_2(t,1) < \infty$.*

Proof. The fact that $v_2(t,1)$ is finite for each $t \in [0,T]$ is a consequence of

$$v_2(t,1) \leq J_2(t,1,0,0) = \frac{\varepsilon}{2}(T-t+1) < \infty.$$

To show that $v_2(t,1)$ is strictly positive is a bit more involved. We first prove this for $t = T$. In this case, the control consists only of possible discrete trades $\Delta\eta_i(T)$ for $i = 1,2$ and the value function v_2 is given by

$$v_2(T,1) = \inf_{\Delta\eta_1(T),\Delta\eta_2(T)\in\mathbb{R}_+}\left[\frac{\kappa_1}{2}\Delta\eta_1(T)^2+\frac{\kappa_2}{2}\Delta\eta_2(T)^2+\frac{\varepsilon}{2}\left|1+\Delta\eta_1(T)-\Delta\eta_2(T)\right|^2\right].$$

The term on the right hand side is strictly positive, the minimisers $\left(\Delta\hat{\eta}_1(T),\Delta\hat{\eta}_2(T)\right)$ can be computed explicitly and one can check that $v_2(T,1) > 0$.

Next we consider $t \in [0,T)$. Just as in (2.7), the dynamics of X_i for $i = 1,2$ and $s \in [t,T]$ imply that

$$X_i(s) = \kappa_i\int_{[t,s]} e^{-\rho_i(r-t)}d\eta_i(r) \geq \kappa_i e^{-\rho_i T}\eta_i(s), \tag{A.2}$$

where we have used $x_i = X_i(t-) = 0$. As above we denote by $\tau \triangleq \tau_1 \wedge \tau_2 \wedge T$ the first jump time of the Poisson processes N_1 or N_2 in $[t,T]$. We then have for $s \in [t,\tau)$ using $x_3 = X_3(t-) = 1$

$$X_3(s) = 1 + \eta_1(s) - \eta_2(s), \tag{A.3}$$

i.e. before a limit buy or sell order is executed, the stock holdings are given by the initial position plus market buy less market sell orders. Just as in the proof of Proposition 2.3.3 we use the relation $d\eta_i(s) = \frac{dX_i(s)+\rho_i X_i(s)ds}{\kappa_i}$ to write the performance functional J_2 for arbitrary control $(u,\eta) \in \mathcal{U}_t$ as

$$\begin{aligned}
J_2(t,1,\eta,u) =&\mathbb{E}_{t,(0,0,1)^*}\Bigg[\frac{X_1(T)^2-x_1^2}{2\kappa_1}+\frac{X_2(T)^2-x_2^2}{2\kappa_2}\\
&+\int_t^T\frac{\rho_1}{\kappa_1}X_1(r)^2dr+\int_t^T\frac{\rho_2}{\kappa_2}X_2(r)^2dr+\int_t^T\frac{\varepsilon}{2}\left|X_3(r)\right|^2dr+\frac{\varepsilon}{2}\left|X_3(T)\right|^2\Bigg]\\
\geq& c_1\mathbb{E}_{t,(0,0,1)^*}\left[\int_t^T X_1(r)^2+X_2(r)^2+X_3(r)^2dr\right]\\
\geq& c_1\mathbb{E}_{t,(0,0,1)^*}\left[\int_t^\tau X_1(r)^2+X_2(r)^2+X_3(r)^2dr\right],
\end{aligned}$$

where we have used $x_i = 0$ for $i = 1, 2$ and defined the constant c_1 by

$$c_1 \triangleq \min\left\{\frac{\rho_1}{\kappa_1}, \frac{\rho_2}{\kappa_2}, \frac{\varepsilon}{2}\right\} > 0.$$

We combine this with (A.2) and (A.3) to get

$$\begin{aligned}
&J_2(t,1,\eta,u)\\
\geq& c_1 \mathbb{E}_{t,(0,0,1)^*}\left[\int_t^\tau \left(\kappa_1 e^{-\rho_1 T}\eta_1(r)\right)^2 + \left(\kappa_2 e^{-\rho_2 T}\eta_2(r)\right)^2 + (1+\eta_1(r)-\eta_2(r))^2\, dr\right]\\
\geq& c_2 \mathbb{E}_{t,(0,0,1)^*}\left[\int_t^\tau \left[\eta_1(r)^2 + \eta_2(r)^2 + (1+\eta_1(r)-\eta_2(r))^2\right] dr\right],
\end{aligned}$$

where we have defined the constant c_2 by

$$c_2 \triangleq c_1 \min\left\{(\kappa_1 e^{-\rho_1 T})^2, (\kappa_2 e^{-\rho_2 T})^2, 1\right\} > 0.$$

The integrand $\eta_1(r)^2 + \eta_2(r)^2 + (1+\eta_1(r)-\eta_2(r))^2$ in the above is strictly positive. Even more is true: One can check by direct computation that there is a constant $c_3 > 0$ such that for any pair of reals $(a_1, a_2)^* \in \mathbb{R}_+^2$ we have

$$a_1^2 + a_2^2 + (1 + a_1 - a_2)^2 \geq c_3 > 0.$$

We now continue the estimate of J_2,

$$\begin{aligned}
J_2(t,1,\eta,u) \geq& c_2 \mathbb{E}_{t,(0,0,1)^*}\left[\int_t^\tau \eta_1(r)^2 + \eta_2(r)^2 + (1+\eta_1(r)-\eta_2(r))^2\, dr\right]\\
\geq& c_2 \mathbb{E}_{t,(0,0,1)^*}\left[\int_t^\tau c_3 dr\right] = c_2 c_3 \mathbb{E}_{t,(0,0,1)^*}[\tau - t].
\end{aligned}$$

Recall that $\tau \triangleq \tau_1 \wedge \tau_2 \wedge T$ where τ_1 and τ_2 are independent, exponentially distributed random variables on $[t, T]$, so that $\mathbb{E}_{t,(0,0,1)^*}[\tau - t] > 0$. The above estimate for J_2 holds for any control $(u, \eta) \in \mathcal{U}_t$. As a consequence, we have

$$v_2(t,1) = \inf_{(\eta,u)\in\mathcal{U}_t} J_2(t,1,\eta,u) \geq c_2 c_3 \mathbb{E}_{t,(0,0,1)^*}[\tau - t] > 0,$$

which completes the proof. □

A.3. Auxiliary Results for Chapter 3

A.3.1. An Existence Result

In this subsection, we prove Theorems 3.3.2 and 3.4.1 where the PDE (3.9) in the risk-neutral setting is a special case of the system (3.15) for risk-averse agents, with $\alpha^j = 0$ for each j. In order to establish our existence and uniqueness of equilibrium result, we adopt the proof of Proposition 15.1.1 in Taylor [1997] to our framework. After time inversion from t to $T-t$ both systems of PDEs are of the form

$$v_t = Lv + F(v_p) \tag{A.4}$$

for $v \triangleq \left(v^1, ..., v^N\right)$, where L is the Laplace-operator

$$L = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial p^2}$$

and $F = \left(F^1, ..., F^N\right)$ is of the form

$$F^j(v_p) = -\frac{1}{2}\sigma^2 \alpha^j \left(v_p^j\right)^2 + \lambda \left(u^{ag} - \frac{g(u^{ag})}{g'(u^{ag})}\right) v_p^j + \frac{g(u^{ag})^2}{g'(u^{ag})}.$$

Here $u^{ag} = u^{ag}(v_p)$ is given implicitly by (3.7). The initial condition is

$$v(0,p) = H(p) = \left(H^1, ..., H^N\right). \tag{A.5}$$

We rewrite (A.4) in terms of an integral equation as

$$v(t) = e^{tL} + \int_0^t e^{(t-s)L} F(v_p(s))ds \triangleq \Gamma v(t). \tag{A.6}$$

and seek a fixed point of the operator Γ on the following set of functions:

$$\mathbb{X} = \mathcal{C}_b^1(\mathbb{R}, \mathbb{R}^N) \triangleq \left\{ v \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^N) \middle| v, v_p \text{ bounded} \right\}$$

equipped with the norm

$$\|v\|_{\mathbb{X}} \triangleq \|v\|_\infty + \|v_p\|_\infty.$$

We set $\mathbb{Y} \triangleq \mathcal{C}_b$. Note that $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces and the semi-group e^{tL} associated with the Laplace operator is strongly continuous on $\mathbb{X}$, sends $\mathbb{Y}$ on $\mathbb{X}$ and satisfies

$$\left\|e^{tL}\right\|_{\mathcal{L}(\mathbb{Y},\mathbb{X})} \leq Ct^{-\gamma}$$

for some $C > 0$, $\gamma < 1$ and $t \leq 1$. Furthermore, the nonlinearity F is locally Lipschitz and belongs to $\mathcal{C}^\infty$. Indeed, if we apply the implicit function theorem to u^{ag} given by (3.7), we see that the map $a \mapsto u^{ag}(a)$ is $\mathcal{C}^\infty$ with first derivative

$$\frac{\partial}{\partial v_p} u^{ag}(v_p) = \frac{\lambda}{(N+1)g'(u^{ag}(v_p)) + u^{ag}(v_p)g''(u^{ag}(v_p))}$$

where the denominator is positive due to Assumption 3.2.2. The cost function g is $\mathcal{C}^\infty$ by assumption. In particular, the assumptions of Proposition 15.1.1 in Taylor [1997] are satisfied.

Before we proceed, we need the following lemma. It states that the value function satisfies $\|V^j\|_{\mathbb{X}} \leq K$ for each $j \leq N$ and some constant K, so it suffices to construct a solution in the following set:

$$\mathbb{X}_K \triangleq \left\{ v \in \mathbb{X} \,\middle|\, \|v\|_{\mathbb{X}} \leq K \right\}.$$

Lemma A.3.1. *There is a constant K such that $\|V^j\|_{\mathbb{X}} \leq K$ for each $j \leq N$.*

Proof. We proof the assertion for risk-neutral agents, the CARA case follows by the same arguments. Our a priori estimates of Proposition 3.2.9 yield that the trading strategy u^j is bounded for each $j \leq N$, and hence the aggregate trading strategy u^{ag} is bounded as well. By definition, the value function $V^j(t,p)$ is then also bounded. Finally, equation (3.8) implies that v_p^j is bounded. □

We are now ready to prove existence and uniqueness of a solution to (A.6). In a nutshell, the argument is the following: Using Proposition 15.1.1 in Taylor [1997], we construct a solution to (A.4)-(A.5) for a small time horizon $[0,\tau]$, with $\tau > 0$ specified below. The vector v is the vector of value functions by Theorem IV.3.1 in Fleming and Soner [1993], so by Lemma A.3.1 the constructed solution is in $\mathbb{X}_K$. We apply this argument recursively to extend the solution to $[0,T]$.

Proposition A.3.2. *There is $\tau > 0$ such that for each $n \in \mathbb{N}_0$, the PDE (A.6) with initial condition (A.5) admits a unique classical, bounded solution in $\mathbb{X}_K$ on the time horizon $[0, n\tau \wedge T]$. This solution is the value function.*

Proof. 1. For $n = 0$, there is nothing to prove. Pick $n \in \mathbb{N}$ such that $n\tau < T$. By induction, we can assume that there is a solution $v^{(n)} \in \mathbb{X}_K$ on the time horizon $[0, n\tau]$. In particular, the initial condition for the next recursion step $h^{(n)} \triangleq v^{(n)}(n\tau)$ is in $\mathbb{X}_K$.

2. Fix $\delta > 0$. We construct a short time solution on the following set of functions:

$$Z^{(n+1)} \triangleq \Big\{ v \in \mathcal{C}\left([n\tau,(n+1)\tau], \mathbb{X}\right) \,\Big|\, v(n\tau) = h^{(n)} \text{ and } \left\| v(t) - h^{(n)} \right\|_{\mathbb{X}} \leq \delta \;\forall t \in [n\tau,(n+1)\tau] \Big\}.$$

We first show that $\Gamma : Z^{(n+1)} \to Z^{(n+1)}$ is a contraction, if $\tau > 0$ is chosen small enough. For this, let τ_1 be small enough such that for $t \leq \tau_1$ and any $v \in \mathbb{X}_K$ we have

$$\left\| e^{tL} v - v \right\|_{\mathbb{X}} \leq \frac{1}{2}\delta.$$

Here we used that e^{tL} is a continuous semigroup and $\|v\|_{\mathbb{X}} \leq K$. In particular, for $v = h^{(n)}$:

$$\left\| e^{tL} h^{(n)} - h^{(n)} \right\|_{\mathbb{X}} \leq \frac{1}{2}\delta.$$

For $v \in Z^{(n+1)}$, the derivative v_p is uniformly bounded in the sense $\|v_p\|_\infty \leq \left\| h^{(n)} \right\|_{\mathbb{X}} + \delta \leq K + \delta$. Hence, we only evaluate F on compact sets. By assumption, F is locally Lipschitz. In particular, F is Lipschitz on compact sets. In other words, there is a constant K_1 such that for any $v, w \in Z^{(n+1)}$ we have

$$\|F(v_p) - F(w_p)\|_{\mathbb{Y}} \leq K_1 \|v - w\|_{\mathbb{X}}$$

This implies, for $w = h^{(n)}$

$$\|F(v_p)\|_{\mathbb{Y}} \leq \left\| F(h_p^{(n)}) \right\|_{\mathbb{Y}} + K_1 \left\| v - h^{(n)} \right\|_{\mathbb{X}} \leq K + K_1\delta \triangleq K_2.$$

This, together with the boundedness assumption on e^{tL}, yields

$$\left\| \int_{n\tau}^{t} e^{(t-y)L} F(v_p(y)) dy \right\|_{\mathbb{X}} \leq t \left\| e^{tL} \right\| \sup_{n\tau \leq y \leq t} \|F(v_p(y))\|_{\mathbb{Y}} \leq t^{1-\gamma} C K_2.$$

This quantity is $\leq \frac{1}{2}\delta$ if $t \leq \tau_2 \triangleq \left(\frac{\delta}{2CK_2} \right)^{\frac{1}{1-\gamma}}$.

Finally, it follows that for $v \in Z^{(n+1)}$ we have

$$\left\| \Gamma v - h^{(n)} \right\|_{\mathbb{X}} \leq \left\| e^{tL} h^{(n)} - h^{(n)} \right\|_{\mathbb{X}} + \left\| \int_{n\tau}^{t} e^{(t-y)L} F(v_p(y)) dy \right\|_{\mathbb{X}} \leq \frac{1}{2}\delta + \frac{1}{2}\delta = \delta.$$

This shows that Γ maps $Z^{(n+1)}$ into itself.

It remains to show that Γ is a contraction. Let $v, w \in Z^{(n+1)}$. Then

$$\begin{aligned} \|\Gamma v(t) - \Gamma w(t)\|_{\mathbb{X}} &= \left\| \int_{n\tau}^{t} e^{(t-y)L} \left[F(v_p(y)) - F(w_p(y)) \right] dy \right\|_{\mathbb{X}} \\ &\leq t \left\| e^{tL} \right\| \sup_{n\tau \leq y \leq t} \|F(v_p(y)) - F(w_p(y))\|_{\mathbb{Y}} \\ &\leq t^{1-\gamma} C K_2 \sup_{n\tau \leq y \leq t} \|v(y) - w(y)\|_{\mathbb{X}} \end{aligned}$$

The quantity $t^{1-\gamma} C K_2$ is $\leq \frac{1}{2}$ if $t \leq \tau_3 \triangleq \left(\frac{1}{2CK_2} \right)^{\frac{1}{1-\gamma}}$. This proofs that Γ is a contraction in $Z^{(n+1)}$, if τ is small in the sense

$$0 < \tau \triangleq \min\{\tau_1, \tau_2, \tau_3\}.$$

Note that the time step τ does not depend on n. It is the same in every recursion step.

3. It follows that Γ has a unique fix point v in $Z^{(n+1)}$. In other words, we constructed a function $v \in \mathcal{C}([n\tau,(n+1)\tau],\mathbb{X}) = \mathcal{C}^{0,1}[n\tau,(n+1)\tau]$ which solves the PDE (A.6) with initial condition $v(s) = h^{(n)} = v^{(n)}(n\tau)$ on the time interval $[n\tau,(n+1)\tau]$.

   This solution is actually in $\mathcal{C}^{1,2}\left((n\tau,(n+1)\tau]\times\mathbb{R},\mathbb{R}^N\right)$, due to Proposition 15.1.2 in Taylor [1997]. Furthermore, v is bounded by construction. Indeed, $\|v\|_\infty \leq \left\|h^{(n)}\right\|_{\mathbb{X}} + \delta \leq K + \delta$. We define the new solution as

$$v^{(n+1)} \triangleq v^{(n)}\mathbb{1}_{\{0\leq t\leq n\tau\}} + v\mathbb{1}_{\{n\tau<t\leq(n+1)\tau\}}.$$

   By construction, $v^{(n+1)}$ solves (A.6) on the time horizon $[0,(n+1)\tau]$ and is bounded and in $\mathcal{C}^{1,2}$. Hence, we can apply the Verification Theorem IV.3.1 from Fleming and Soner [1993], which yields that $v^{(n+1)}$ is the vector of value functions (up to time reversal). Due to Lemma A.3.1 we have $v^{(n+1)} \in \mathbb{X}_K$. In particular, $\left\|v^{(n+1)}((n+1)\tau)\right\|_{\mathbb{X}} \leq K$, which is necessary for the next recursion step.

This completes the proof. □

A.3.2. Proof of Propositions 3.5.3 and 3.5.4

The argument is the same for both propositions. Fix $N \in \mathbb{N}$. The aggregate trading speed for N players is given from equation (3.12) as

$$u^{ag} = \sum_{i=1}^{N} u^i = \frac{\lambda}{\kappa}\frac{1}{N+1}v_p,$$

where the aggregate value function $v = \sum_{i=1}^N v_i$ from (3.13) solves Burgers' equation

$$0 = v_t + \frac{1}{2}\sigma^2 v_{pp} + \frac{\lambda^2}{\kappa}\frac{N}{(N+1)^2}v_p^2 \tag{A.7}$$

with terminal condition $v(T,p) = \sum_{i=1}^N H^i(p) = H^1(p) \triangleq H(p)$. On the other hand, the aggregate trading speed for $N+1$ players is

$$\bar{u}^{ag} = \sum_{i=1}^{N+1} \bar{u}^i = \frac{\lambda}{\kappa}\frac{1}{N+2}w_p,$$

where the aggregate value function $w = \sum_{i=1}^{N+1} w_i$ solves

$$0 = w_t + \frac{1}{2}\sigma^2 w_{pp} + \frac{\lambda^2}{\kappa}\frac{N+1}{(N+2)^2}w_p^2$$

with terminal condition $w(T,p) = H(p)$. We have to show that $u^{ag} \geq \bar{u}^{ag}$. To this end, let us define $\tilde{w} \triangleq \frac{N+1}{(N+2)^2}\frac{(N+1)^2}{N}w$. It is enough to show that $v_p \geq \tilde{w}_p$, since then

$\frac{1}{N+1}v_p \geq \frac{1}{N+1}\tilde{w}_p$ and, by definition, $\frac{1}{N+1}\tilde{w}_p \geq \frac{1}{N+2}w_p$. This implies $u^{ag} \geq \bar{u}^{ag}$.

To show $v_p \geq \tilde{w}_p$, first note that $\tilde{w}$ is chosen such that it satisfies the same PDE (A.7) as v, namely

$$0 = \tilde{w}_t + \frac{1}{2}\sigma^2 \tilde{w}_{pp} + \frac{\lambda^2}{\kappa}\frac{N}{(N+1)^2}\tilde{w}_p^2 \tag{A.8}$$

with a smaller terminal condition: $\tilde{w}(T,p) = \frac{N+1}{(N+2)^2}\frac{(N+1)^2}{N}H(p) \triangleq (1-\delta)H(p)$. The solutions to (A.7) and (A.8) are given in Lemma A.3.3 as

$$v(t,p) = c_1 \log \int_{\mathbb{R}} \exp\left(c_2 H(c_3 z)\right) f_{c_4 p, T-t}(z) dz$$

and

$$\tilde{w}(t,p) = c_1 \log \int_{\mathbb{R}} \exp\left(c_2 (1-\delta) H(c_3 z)\right) f_{c_4 p, T-t}(z) dz$$

with constants $c_1, c_2, c_3, c_4 \in \mathbb{R}$ and $\delta \in (0,1)$. To verify $v_p \geq \tilde{w}_p$, it is enough to show

$$\frac{\partial}{\partial p} \log \int_{\mathbb{R}} \exp\left(G\right) f_{p,1}(z) dz \geq \frac{\partial}{\partial p} \log \int_{\mathbb{R}} \exp\left((1-\delta)G\right) f_{p,1}(z) dz$$

for an increasing function $G \in \mathcal{C}_b^2$. This is equivalent to

$$\frac{\int_{\mathbb{R}} (z-p) e^G f_{p,1}(z) dz}{\int_{\mathbb{R}} e^G f_{p,1}(z) dz} \geq \frac{\int_{\mathbb{R}} (z-p) e^{(1-\delta)G} f_{p,1}(z) dz}{\int_{\mathbb{R}} e^{(1-\delta)G} f_{p,1}(z) dz}$$

or

$$\int_{\mathbb{R}} z e^{\delta G} \frac{e^{(1-\delta)G} f_{p,1}(z) dz}{\int_{\mathbb{R}} e^{(1-\delta)G} f_{p,1}(z) dz} \geq \int_{\mathbb{R}} z \frac{e^{(1-\delta)G} f_{p,1}(z) dz}{\int_{\mathbb{R}} e^{(1-\delta)G} f_{p,1}(z) dz} \int_{\mathbb{R}} e^{\delta G} \frac{e^{(1-\delta)G} f_{p,1}(z) dz}{\int_{\mathbb{R}} e^{(1-\delta)G} f_{p,1}(z) dz}$$

or

$$\text{cov}_{\mathbb{Q}}\left(id, e^{\delta G}\right) \geq 0$$

under the measure $\mathbb{Q}$ with $d\mathbb{Q} \triangleq \frac{e^{(1-\delta)G} f_{p,1}(z) dz}{\int_{\mathbb{R}} e^{(1-\delta)G} f_{p,1}(z) dz}$. The covariance of two increasing functions is surely nonnegative. This finally proofs the assertion $u^{ag} \geq \bar{u}^{ag}$.

It remains to show $\lim_{N\to\infty} \sum_{i=1}^N u^i(t) = 0$. We have

$$\begin{aligned} u^{ag}(t,p) &= \sum_{i=1}^N u^i(t) = \frac{\lambda}{\kappa}\frac{1}{N+1} v_p(t,p) \\ &= \frac{\partial}{\partial p}\frac{\lambda}{\kappa}\frac{1}{N+1}\frac{\sigma^2\kappa(N+1)^2}{2\lambda^2 N} \log \int_{\mathbb{R}} \exp\left(\frac{2\lambda^2 N}{\sigma^2\kappa(N+1)^2} H(\sigma z)\right) f_{\frac{p}{\sigma}, T-t}(z) dz \end{aligned}$$

$$= \frac{\partial}{\partial p}\frac{\lambda}{\kappa}\frac{1}{N+1}\frac{\sigma^2\kappa(N+1)^2}{2\lambda^2 N}\log\int_{\mathbb{R}}\exp\left(\frac{2\lambda^2 N}{\sigma^2\kappa(N+1)^2}H\left(\sigma z+\frac{p}{\sigma}\right)\right)f_{0,T-t}(z)dz$$
$$= \frac{\lambda}{\kappa}\frac{1}{N+1}\frac{1}{\sigma}\frac{\int_{\mathbb{R}} H_p\left(\sigma z+\frac{p}{\sigma}\right)\exp\left(\frac{2\lambda^2 N}{\sigma^2\kappa(N+1)^2}H\left(\sigma z+\frac{p}{\sigma}\right)\right)f_{0,T-t}(z)dz}{\int_{\mathbb{R}}\exp\left(\frac{2\lambda^2 N}{\sigma^2\kappa(N+1)^2}H\left(\sigma z+\frac{p}{\sigma}\right)\right)f_{0,T-t}(z)dz},$$

where we used Lemma A.3.3 in the second line. This expression is nonnegative, since $H_p \geq 0$. Furthermore, we have $\|H_p\|_\infty < \infty$ by assumption. It follows that

$$0 \leq \sum_{i=1}^{N} u^i(t) \leq \frac{\lambda}{\kappa}\frac{1}{N+1}\frac{1}{\sigma}\|H_p\|_\infty \overset{N\to\infty}{\longrightarrow} 0.$$

This completes the proof.

A.3.3. Burgers' Equation

In the proofs of Proposition 3.3.4 and Corollary 3.4.2 we need the solution to a variant of *Burgers' equation.* Recall our notation

$$f_{\mu,\sigma^2}(z) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right).$$

Lemma A.3.3. *Let $A \in \mathbb{R}_{>0}, B \in \mathbb{R}\setminus\{0\}$ and $G:\mathbb{R}\to\mathbb{R}$ be smooth and bounded. The PDE*

$$0 = 2v_t + Av_{pp} + Bv_p^2$$

with terminal condition $v(T,p) = G(p)$ is solved by

$$v(t,p) = \frac{A}{B}\log\left[\int_{\mathbb{R}}\exp\left(\frac{B}{A}G\left(\sqrt{A}z\right)\right)f_{\frac{p}{\sqrt{A}},T-t}(z)dz\right]. \tag{A.9}$$

Proof. We use the linear transformation $v(t,p) \triangleq \frac{A}{B}w(t,\frac{p}{\sqrt{A}})$ and note that

$$v_t = \frac{A}{B}w_t, \quad v_p = \frac{\sqrt{A}}{B}w_p, \quad v_{pp} = \frac{1}{B}w_{pp}.$$

The PDE under consideration is then equivalent to (after cancelling the factor $\frac{A}{B}$)

$$0 = 2w_t + w_{pp} + w_p^2,$$

with terminal condition $w(T,p) = \frac{B}{A}G\left(\sqrt{A}p\right)$. Next we apply the transformation $w(t,p) \triangleq \log h(t,p)$, which turns the above PDE into

$$0 = h_t + \frac{1}{2}h_{pp},$$

with terminal condition $h(T,p) = \exp\left(\frac{B}{A} G\left(\sqrt{A}p\right)\right)$. The solution to this heat equation is

$$h(t,p) = \int_{\mathbb{R}} \exp\left(\frac{B}{A} G\left(\sqrt{A}z\right)\right) f_{p,T-t}(z)dz.$$

This function is well defined since G is assumed to be bounded. Now it becomes clear that $v(t,p) = \frac{A}{B} \log h(t, \frac{p}{\sqrt{A}})$ is given by (A.9). See also Rosencrans [1972]. $\square$

Concluding Remarks

This thesis is concerned with stochastic control problems in limit order markets, in particular curve following, portfolio liquidation and market manipulation. We want to conclude our work by discussing possible extensions and the limitations of our approach.

In Chapter 1 we solve the problem of curve following with passive orders and absolutely continuous market orders. Existence and uniqueness of an optimal control is established, the optimal trading strategy is first characterised via a FBSDE and then in terms of buy and sell regions.

- We consider only one passive order, which is executed at the benchmark price. More generally, limit orders might be placed on different price ticks in the order book. While our approach is flexible enough to cover this situation, the notation becomes more cumbersome.

  Let us illustrate this with limit buy orders on the first and second price tick next to the benchmark price. We denote by $\delta > 0$ the tick size and by $u_{1,1}$ ($u_{1,2}$) the limit buy order placed on the first (resp. second) price tick. Let N_1 (N_2) denote independent Poisson processes; they represent liquidity events which trigger the execution of the first (resp. the first and the second) limit buy order. Recall that if the limit buy order on the k-th tick is executed, then each of the limit buy orders placed on the ticks $1, 2, ..., k-1$ is also executed. The investor's stock holdings given by equation (1.1) now turn into

  $$dX^u(s) = u_{1,1}(s)N_1(ds) + (u_{1,1} + u_{1,2})N_2(ds) + u_2(s)ds.$$

  Similarly, limit buy and sell orders on an arbitrary number of price ticks can be modelled. We note that a limit buy order on the k-th tick is executed at the benchmark price less $k\delta$, so it yields a relative gain of $k\delta$.

- We only consider full execution. In reality, passive orders might be partially executed. This can be accounted for by replacing the dynamics of stock holdings (1.1) by

  $$dX^u(s) = \int_0^\infty \min\{u_1(s), \theta\} M(d\theta, ds) + u_2(s)ds,$$

  for a compound Poisson process M. The interpretation is that a jump of size θ represents another agent's market order which executes our passive order u_1, so the investor receives the minimum of u_1 and θ. Unfortunately, the current approach does not capture this more general setup since the state variable is no longer linear in the control.

In Chapter 2 we extend the results of the first chapter to market orders with discrete components. The spread is now also controlled and price impact recovers only gradually. We show existence and uniqueness of an optimal control and prove a version of the maximum principle for singular control which provides a characterisation of optimality in terms of a FBSDE with singular terms. A second characterisation via buy, sell and no-trade regions is derived.

- The above remarks concerning limit orders on different ticks and partial execution also apply to the more general model we consider in the second chapter.
- It would be desirable to allow for stochastic jumps in the dynamics of the buy and sell spread to account for the random flow of other traders' market and limit orders. However, in this case the representation (2.7) does no longer hold. This representation is needed in the proofs of Lemma 2.3.5, Lemma 2.4.3 as well as Theorem 2.3.6.
- We assume that there is a fixed and constant distribution of prevailing limit orders. It is not clear whether our approach can be extended to more general shape functions for the order book, since they lead to nonlinear dynamics for the state variables. Specifically, if $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ denotes the shape function of the order book, a buy order of size $\Delta\eta_1(t)$ moves the buy spread from $X_1(t-)$ to $X_1(t)$, which is determined by

  $$\int_{X_1(t-)}^{X_1(t)} \phi(y)dy = \Delta\eta_1(t).$$

  We see that the dynamics of X_1 are linear in the control if and only if ϕ is constant, i.e. the order book is flat. Linear dynamics are key for our version of the maximum principle.
- In our model, market orders have discrete and absolutely continuous components. In real markets it is typically not possible to trade continuously; such strategies can only be approximated by a sequence of small discrete trades. We can force the optimal strategy to be *purely discrete* by introducing fixed transaction costs and then using methods of impulse control. The liquidity costs of a discrete buy order $\Delta\eta_1(t)$ in such a model can be expressed as

  $$\text{costs}(\Delta\eta_1(t)) = \begin{cases} 0, & \Delta\eta_1(t) = 0, \\ g(\Delta\eta_1(t)) + c, & \text{else,} \end{cases}$$

  for a constant $c > 0$ and a function $g : \mathbb{R}_+ \to \mathbb{R}_+$. However, this cost function is not convex. As a result, the optimisation problem for this set of controls is no longer convex and the current approach fails.

In Chapter 3 we consider a set of agents holding options with illiquid underlying. We construct the optimal manipulation strategies for risk neutral and CARA investors in form of a Nash equilibrium and show how manipulation can be avoided.

- In Problem 3.2.5 we only consider the tradeoff between liquidity costs and an increased option value. We ignored that the liquidation value at time T of the stock shares acquired in $[0, T]$ depends on the liquidation strategies of *all* the agents. Defining a liquidation value under strategic interaction is still an open question, to the best of our knowledge.
- Only European options are considered, the extension to path-dependent or American payoffs is left for future research.

Symbols and Notation

$\cdot^*$	Transposition of a vector or a matrix
$\mathbb{R}$	Set of real numbers
$\lvert\cdot\rvert, \lVert\cdot\rVert_{\mathbb{R}^d}$,	Euclidean norm in $\mathbb{R}$ and $\mathbb{R}^d$
$\mathbb{N}$	Set of natural numbers
$\mathbb{P}$	Probability measure
Ω	Probability space
ω	Event
$X, Y, Z, ...$	Stochastic processes
c, C, K	Constants (typically large)
ε, δ	Constants (typically small)
T	Time horizon, maturity
W	Brownian Motion
N	Poisson process with intensity λ
M	Compound Poisson process with intensity m
$\tilde{N}, \tilde{M}$	Compensated (compound) Poisson processes
ΔN	Jump of the Poisson process N
τ	Stopping time, first jump time of the Poisson process N
u	Control (typically absolutely continuous)
u^j	Control of player j (in Chapter 3)
u^{ag}	Aggregate trading speed of all players (in Chapter 3)
η	Singular control, nondecreasing process (in Chapter 2)
$\Delta\eta$	Jump of the process η
$\hat{u}, \hat{\eta}$	Optimal controls
$\mathcal{U}_t$	Set of admissible controls
$X, X^u, X^{\eta,u}$	State process, controlled by u or (η, u) resp. (in Chapters 1 and 2)
Z	Signal process (in Chapters 1 and 2)
μ, σ, γ	Coefficients in the dynamics of the state process Z
(P, Q, R)	Adjoint process, solution to a BSDE (in Chapters 1 and 2)
P	State process, price of the underlying (in Chapter 3)
J	Performance functional
v	Value function
V^j	Value function of player j (in Chapter 3)
$\mathcal{H}$	Hamiltonian
L^2	Set of square integrable stochastic processes (resp. random variables)
$\lVert\cdot\rVert_{L^2}$	L^2-norm
f, h	Penalty functions (in Chapters 1 and 2)
α	Target function (in Chapters 1 and 2)

$\tilde{\alpha}$	Cost-adjusted target function (in Chapter 1)
g	Liquidity cost function (in Chapters 1 and 3)
$\mathcal{R}_{buy}, \mathcal{R}_{sell}, \mathcal{R}_{nt}$	Buy, sell and no-trade regions
Ψ_t^j	Preference functional of player j (in Chapter 3)
H^j	Contingent claim, endowment of player j (in Chapter 3)
N	Number of players (in Chapter 3)
$f', f_x, \frac{\partial}{\partial x} f$	Derivative of the function f with respect to x
$\mathcal{C}_b^2$	Set of functions which are twice continuously differentiable and bounded, along with its derivatives
$\mathbb{X}, \mathbb{Y}$	Sets of functions defined in Appendix A.3.1
$\mathbb{E}_{t,x,z}[\cdot]$	Shorthand for $\mathbb{E}[\cdot \vert X(t) = x, Z(t) = z]$

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 4. Oktober 2011 Felix Naujokat