View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Dokumenten-Publikationsserver der Humboldt-Universität zu Berlin

# Parametric Representation of Feynman Amplitudes in Gauge Theories

Matthias Sars

# Parametric Representation of Feynman Amplitudes in Gauge Theories

Dissertation
zur Erlangung des akademischen Grades
doctor rerum naturalium
(Dr. rer. nat.)
im Fach Physik
Spezialisierung: Theoretische Physik
eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät
der Humboldt-Universität zu Berlin
von
Matthias Christiaan Bernhard Sars MSc

Präsident der Humboldt-Universität zu Berlin
Prof. Dr. Jan-Hendrik Olbertz
Dekan der Mathematisch-Naturwissenschaftlichen Fakultät
Prof. Dr. Elmar Kulke

Gutachter:
1. Prof. Dr. Dirk Kreimer
2. Dr. David Broadhurst
3. Dr. Walter D. van Suijlekom

Tag der mündlichen Prüfung: 1. September 2015

# Contents

# 1
# Introduction

Quantum field theory, or to be more precise, perturbative quantum field theory, provides the framework for theories or models in particle physics, such as the Standard Model of elementary particle physics. The Standard Model is our most complete description of nature on the small scale, although it has its problems.

Experimentally measurable quantities, such as scattering cross sections and decay rates, are obtained from the correlation functions. Feynman graphs and Feynman rules are the tools one uses to compute these functions. These computations involve integrals over momenta, and it is known that for scalar theories these can be rewritten systematically as integrals over positive parameters (Schwinger parameters), involving certain polynomials (the Symanzik polynomials). This will be discussed in chapter 2.

Many tools have been and are being developed to compute these parametric integrals and study the underlying mathematics.* Together with a program that generates Feynman graphs and finds the subdivergences†, one has in principle a powerful tool to do computations. However, a serious problem is that the expressions can get gigantic.

The goal of this thesis is to extend this parametric representation from scalar theories to gauge theories: quantum electrodynamics, scalar electrodynamics and Yang-Mills theories will be discussed here, in chapter 3, 4 and 5 respectively. This adds to to previous work for QED by Nakanishi, Cvitanović and Kinoshita.‡

Furthermore, the respective Ward identities in these theories are studied. These identities show that the gauge bosons, or photons in the case of (s)QED, are transversal, as expected from the classical theory.

*For example, see [5], [2], [13] and [14].
†such as [4]
‡[12], section 9-2 and [8] respectively. See also [1], section V.

# 2

# Scalar Theories

## 2.1 Feynman Graphs

We start by introducing the combinatorial tool we need for our computations:

**Definition 2.1.** A *Feynman graph*[*] $\Gamma$ is defined by:

- a finite set of *half-edges* $\Gamma^{\text{he}}$,
- a partition $\Gamma^{[0]}$ on $\Gamma^{\text{he}}$, which we call the set of *vertices*,
- and a set of *internal edges*[†] $\Gamma^{[1]}$, which consists of disjoint unordered pairs of half-edges.

The half-edges that do not show up in $\Gamma^{[1]}$ are called *external edges*[‡] and the set of external edges is denoted by $\Gamma^{\text{ext}}$:

$$\Gamma^{\text{ext}} := \Gamma^{\text{he}} \setminus \bigcup_{e \in \Gamma^{[1]}} e \,. \tag{2.1}$$

An edge $e \in \Gamma^{[1]}$ is called *incident* to a vertex $v \in \Gamma^{[0]}$ if $v \cap e \neq \emptyset$. Two vertices are said to be *adjacent* if there is an edge incident to both of them, and two edges are adjacent if they are incident to the same vertex.

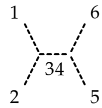
We use the words 'graphs', 'edges' and 'vertices' for a reason: we represent Feynman graphs indeed graphically:

**Example 2.2.** i. Let $\Gamma$ be given by

$$\Gamma^{\text{he}} = \{1,2,3,4,5,6\}\,, \qquad \Gamma^{[0]} = \big\{\{1,2,3\},\{4,5,6\}\big\}$$
$$\text{and} \qquad \Gamma^{[1]} = \big\{\{3,4\}\big\}\,.$$

[*] or *Feynman diagram*
[†] In physics literature the word *lines* is also used.
[‡] or *legs*

This graph looks like:

$\Gamma =$ 

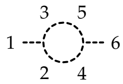
.

We have $\Gamma^{\text{ext}} = \{1, 2, 5, 6\}$.

ii. Let $\Gamma^{\text{he}}$ and $\Gamma^{[0]}$ be as above, but now take

$$\Gamma^{[1]} = \{\{2,4\}, \{3,5\}\}.$$

This one looks like:

$\Gamma =$ 

.

In this case: $\Gamma^{\text{ext}} = \{1, 6\}$.

iii. The *empty graph* $\emptyset$ ($\emptyset^{\text{he}} = \emptyset$) is a graph too.

The number of half-edges $\#v$ in a vertex $v$ is called the *valence* of $v$. If every vertex in a graph has the same valence $k$, we say that it is a *k-regular* graph. Both graphs in example 2.2.i and ii are 3-regular.

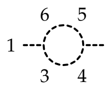
**Definition 2.3.** Let $\Gamma_1$ and $\Gamma_2$ be Feynman graphs. A *Feynman graph isomorphism* $\phi : \Gamma_1 \to \Gamma_2$ is given by a bijection $\phi : \Gamma_1^{\text{he}} \to \Gamma_2^{\text{he}}$ which respects the vertices, internal edges and external edges. By this we mean:

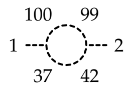
- if $v \in \Gamma_1^{[0]}$, then $\phi(v) \in \Gamma_2^{[0]}$,
- if $e \in \Gamma_1^{[1]}$, then $\phi(e) \in \Gamma_2^{[1]}$,
- and for every $h \in \Gamma_1^{\text{ext}}$: $\phi(h) = h$.

If such an isomorphism between $\Gamma_1$ and $\Gamma_2$ exists, we say that $\Gamma_1$ and $\Gamma_2$ are *equivalent* Feynman graphs: $\Gamma_1 \cong \Gamma_2$.

Note that the third condition above implies that $\Gamma_1$ and $\Gamma_2$ can only be equivalent if $\Gamma_1^{\text{ext}} = \Gamma_2^{\text{ext}}$.

**Example 2.4.** i. Let

$\Gamma_1 =$ 

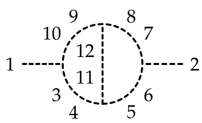
, $\Gamma_2 =$ 

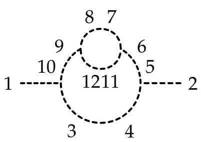

and $\phi : \Gamma_1 \to \Gamma_2$ given by

$$1 \mapsto 1, \quad 2 \mapsto 2, \quad 3 \mapsto 37, \quad 4 \mapsto 42, \quad 5 \mapsto 99, \quad 6 \mapsto 100.$$

$\phi$ is a isomorphism in the sense of definition 2.3 and hence $\Gamma_1 \cong \Gamma_2$.

ii. Let

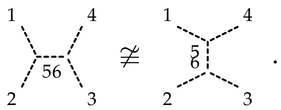
$\Gamma_1 =$ 

and $\Gamma_2 =$ 

.

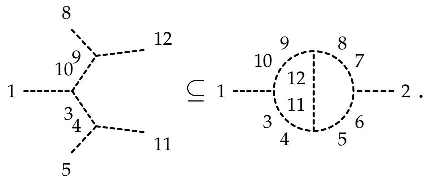
Bijections $\Gamma_1^{\text{he}} \to \Gamma_2^{\text{he}}$ exist, but none of them will meet the first two properties in above definition simultaneously. So $\Gamma_1 \ncong \Gamma_2$.

iii. Because of the third condition in above definition:




**Definition 2.5.** Let $\Gamma$ and $\gamma$ be Feynman graphs. We say that $\gamma$ is a *subgraph* of $\Gamma$ (notation: $\gamma \subseteq \Gamma$) if $\gamma^{[0]} \subseteq \Gamma^{[0]}$ and $\gamma^{[1]} \subseteq \Gamma^{[1]}$.

For example:

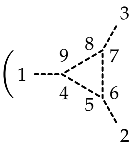


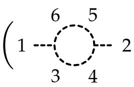

**Definition 2.6.** The *symmetry factor* of a Feynman graph $\Gamma$ is defined as

$$\mathrm{Sym}(\Gamma) := \#\,\mathrm{Aut}(\Gamma)\,, \tag{2.2}$$

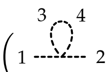
the order of the group of automorphisms on $\Gamma$ (i.e. isomorphisms $\Gamma \to \Gamma$).

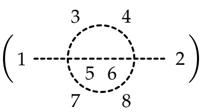
**Example 2.7.**

Sym (  ) $= \#\{\mathrm{id}\} = 1\,,$


Sym (  ) $= \#\{\mathrm{id}, (3\,6)(4\,5)\} = 2$


(using the cycle notation),

Sym (  ) $= \#\{\mathrm{id}, (3\,4)\} = 2\,,$


Sym (  ) $= \#\{\mathrm{id}, (3\,5)(4\,6), (3\,7)(4\,8), (5\,7)(6\,8),$


$(3\,5\,7)(4\,6\,8), (3\,7\,5)(4\,8\,6)\} = 6\,.$

**Definition 2.8.** i. A graph $\Gamma$ is *connected* if one can go from any vertex to any other one by hopping over only adjacent vertices. To put it differently: A graph is connected if it cannot be written as a disjoint union of several nonempty graphs.

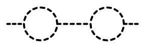
A graph that is not connected is called *disconnected*.

ii. The number of connected components of a graph $\Gamma$ is denoted by $c_\Gamma$.

iii. A graph $\Gamma$ is called *1-particle reducible* (*1PI*)[*] if for every $e \in \Gamma^{[1]}$: $\Gamma \setminus e$ is connected.

The graph

for example, is 1-particle irreducible; the graph



is not. Both are connected.

**Definition 2.9.** i. For a graph $\Gamma$ and an edge $e \in \Gamma^{[1]}$, we define the following operation: *cutting* the edge $e$ gives a new graph $\Gamma \setminus e$ given by

$$(\Gamma \setminus e)^{[0]} := \Gamma^{[0]} \tag{2.3}$$

and

$$(\Gamma \setminus e)^{[1]} := \Gamma^{[1]} \setminus \{e\}\,. \tag{2.4}$$

We use the following notation:

$$\Gamma \setminus \{e_1, \dots, e_n\} := \Gamma \setminus e_1 \setminus \cdots \setminus e_n\,.$$

ii. Let $e \in \Gamma^{[1]}$ be incident to the vertices $v_1$ and $v_2 \in \Gamma^{[1]}$, and assume $v_1 \neq v_2$. (Anticipating to definition 2.11.i: $e$ should not form a self-loop.) If we *contract* $e$, we get a new graph $\Gamma/e$ given by

$$(\Gamma/e)^{[0]} := \Gamma^{[0]} \setminus \{v_1, v_2\} \cup \{v_1 \cup v_2 \setminus e\} \tag{2.5}$$

and

$$(\Gamma/e)^{[1]} := \Gamma^{[1]} \setminus \{e\}\,. \tag{2.6}$$

For this operation, we also write

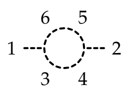
$$\Gamma/\{e_1, \dots, e_n\} := \Gamma/e_1/\cdots/e_n\,.$$

iii. For a subgraph $\gamma \subseteq \Gamma$ we define the *cograph* $\Gamma/\gamma$ by:

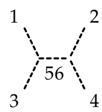
$$(\Gamma/\gamma)^{[0]} = \Gamma^{[0]} \setminus \gamma^{[0]} \cup \gamma^{\text{ext}} \qquad \text{and} \qquad (\Gamma/\gamma)^{[1]} = \Gamma^{[1]} \setminus \gamma^{[1]}\,.$$

**Example 2.10.**

i.

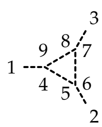



$\setminus \{3,4\} =$

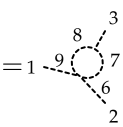



.

ii.



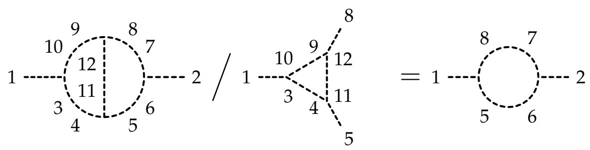

$/\{4,5\} =$



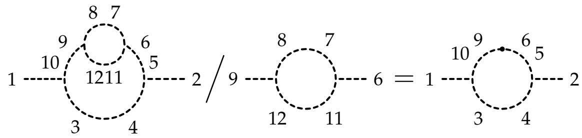

.

[*]Mathematicians would use the term 2-*connected*.

iii.




iv.




The dot indicates the 2-valent vertex $\{6, 9\}$.

**Definition 2.11.** i. A *loop** is a connected subgraph where every vertex contains two internal half-edges. We denote the set of loops of a graph $\Gamma$ by $\mathscr{L}_\Gamma$.

A loop with only one vertex is called a *self-loop*.†

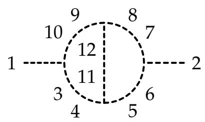
ii. A connected graph without loops is called a *tree* and a disjoint union of $n$ trees is a *forest*, or $n$*-forest*, if one wants to specify the number of connected components.

iii. The *loop order* $l_\Gamma$ of a connected graph $\Gamma$ is the number of edges one has to cut, such that the result is a tree.

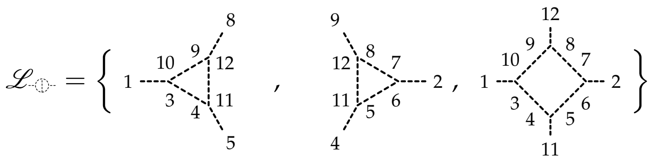
For a disconnected graph $\Gamma = \gamma_1 \cdots \gamma_{c_\Gamma}$, the loop order is

$$l_\Gamma = l_{\gamma_1} + \cdots + l_{\gamma_{c_\Gamma}} .$$

**Example 2.12.** The graph




has the following set of loops:



and loop order $l = 2$.

Note that if one cuts an edge $e \in \Gamma^{[1]}$, either the loop number of the graph decreases by 1, or one gets one more connected component:

$$l_{\Gamma\setminus e} - c_{\Gamma\setminus e} = l_\Gamma - c_\Gamma - 1 . \tag{2.7}$$

**Lemma 2.13** (Euler's formula)**.** For any graph

$$\#\Gamma^{[1]} - \#\Gamma^{[0]} = l_\Gamma - c_\Gamma . \tag{2.8}$$

*In mathematical literature, this is usually called a *cycle*.

†This is what mathematicians usually call a *loop*.

*Proof.* By induction in $\#\Gamma^{[1]}$:

- If $\Gamma$ has no internal edges, it is only a bunch of disconnected vertices. So $\#\Gamma^{[0]} = c_\Gamma$ and $l_\Gamma = 0$. So (2.8) holds.
- Let $e \in \Gamma^{[1]}$. Note that per definition $\#(\Gamma \setminus e)^{[1]} = \#\Gamma^{[1]} - 1$ (equation (2.4)). Assume (2.8) is true for $\Gamma \setminus e$. Then:

$$\begin{aligned}\#\Gamma^{[1]} - \#\Gamma^{[0]} &= \#(\Gamma \setminus e)^{[1]} - \#(\Gamma \setminus e)^{[0]} + 1 \\ &= l_{\Gamma\setminus e} - c_{\Gamma\setminus e} + 1 = l_\Gamma - c_\Gamma\,,\end{aligned}$$

where we used equations (2.3) and (2.7). □

**Lemma 2.14.** For $k$-regular Feynman graphs:

i.
$$\#\Gamma^{[0]} = \frac{\#\Gamma^{\text{ext}} + 2(l_\Gamma - c_\Gamma)}{k-2}\,, \tag{2.9}$$

ii.
$$\#\Gamma^{[1]} = \frac{\#\Gamma^{\text{ext}} + k(l_\Gamma - c_\Gamma)}{k-2}\,. \tag{2.10}$$

*Proof.* This follows from Euler's formula together with

$$k\#\Gamma^{[0]} = \#\Gamma^{\text{he}} = 2\#\Gamma^{[1]} + \#\Gamma^{\text{ext}}\,.$$ □

Although the graph

$$\Gamma = 1 \text{-----} 2\,,$$

does not fit in our definition 2.1, we will allow it. If we take $\Gamma^{\text{ext}} = \{1,2\}$ and $l_\Gamma = 0$, then from above lemma we have paradoxically $\#\Gamma^{[0]} = 0$ and $\#\Gamma^{[1]} = -1$.

Note that the 1-loop vacuum bubble,

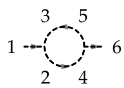
,

does not fit in our setup either.

**Definition 2.15.** An *orientation* on a Feynman graph $\Gamma$ is an assignment of a sign $\varepsilon_h \in \{1, -1\}$ to every half-edge $h \in \Gamma^{\text{he}}$, such that for all $\{h_1, h_2\} \in \Gamma^{[1]}$: $\varepsilon_{h_1} = -\varepsilon_{h_2}$.

If $e_h = 1$, we say $h$ is *ingoing* and if $e_h = -1$, we say it is *outgoing*.

We represent such an orientation by grey arrows. For example: the orientation on the graph




is given by $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = 1$ and $\varepsilon_2 = \varepsilon_5 = \varepsilon_6 = -1$.

In the rest of this thesis, instead of labelling the half-edges, we will give labels to the vertices and the edges.

## 2.2 Feynman Rules

In this chapter, we look at theories in $d$ space-time dimensions with a classical Lagrangian of the form

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \tfrac{1}{k!}\lambda \phi^k \,, \tag{2.11}$$

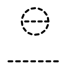
where $\phi$ is a real scalar field and $k \in \mathbb{N}$, $k \geq 3$.

For odd $k$, these theories are actually unphysical. The potential term is unbounded from below then, so there is no stable vacuum.

These theories are massless. For massive theories one includes a mass term $-\frac{1}{2}m^2\phi^2$. In this thesis, theories are assumed to be massless, because in the end we are interested in gauge theories. But occasionally a comment will be made on the massive case.

In the quantum theory we want to compute *correlation functions* or *Green's functions*, and to do so Feynman graphs and *Feynman rules* are used.

We exclude graphs with vacuum bubbles components (a *vacuum bubble* is a graph without any external edges), such as

 .

Furthermore, we exclude graphs with tadpole subgraphs (a *tadpole graph* is a graph with only one external edge), such as

 .

In $\phi^k$-theory, $k$-regular graphs are the graphs we need. The Feynman rules in this case are:

**Definition 2.16.** Let $\Gamma$ be a $\phi^k$-theory Feynman graph. Choose an orientation on $\Gamma$. Choose a set of $l_\Gamma$ loops $L \subseteq \mathscr{L}(\Gamma)$ and for each loop in $L$ a clockwise or anticlockwise orientation.* Assign a momentum vector $\xi_e$ to every edge $e \in \Gamma^{[1]}$ and a momentum vector $k_\ell$ to every loop $\ell \in L$. $\Gamma$'s *Feynman amplitude* is then:

$$\Phi(\Gamma) := \tfrac{1}{\pi^{d l_\Gamma/2}} \int \mathrm{d}\underline{k} \, \frac{1}{\prod\limits_{e \in \Gamma^{[1]}} p_e^2} \,, \tag{2.12}$$

where we use the short-hand notation

$$\int \mathrm{d}\underline{k} := \prod_{\ell \in L} \int \mathrm{d}^d k \tag{2.13}$$

and

$$p_e := \xi_e + \sum_{\substack{\ell \in L \\ \ell^{[1]} \ni e}} \varepsilon_{e\ell} k_\ell \,. \tag{2.14}$$

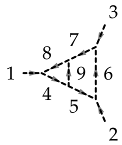
*By this we mean that for every vertex $v$ in the loop, the two internal half-edges $h_1, h_2 \in v$ are oriented opposite: $\varepsilon_{h_1} = -\varepsilon_{h_2}$.

The sign $\varepsilon_{e\ell} \in \{1, -1\}$ is 1 if $e$ is oriented the same way in $\Gamma$ and $\ell$, and $-1$ if it is oriented the opposite way.

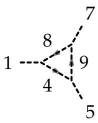
The reader might miss some factors $i$, $-i\lambda$ and $\frac{1}{(2\pi)^d}$; these will be included in definition 2.19. Also, the factor $\frac{1}{\pi^{dl_\Gamma/2}}$ which we included here will be compensated there. In example 2.21 and theorem 2.24 it will be clear why this is convenient.

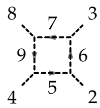
For massive theories we have $p_e^2 - m^2$ in the denominator instead of $p_e^2$.

**Example 2.17.** Consider the graph



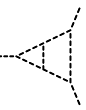

with $L = \{\ell_1, \ell_2\}$, where the loops are

$\ell_1 =$ 

and $\ell_2 =$ 

.

The Feynman amplitude is

$\Phi\Big($  $\Big) = \frac{1}{\pi^d} \iint \frac{\mathrm{d}^d k_{\ell_1} \mathrm{d}^d k_{\ell_2}}{p_4^2 p_5^2 p_6^2 p_7^2 p_8^2 p_9^2}$

$$= \frac{1}{\pi^d} \iint \frac{\mathrm{d}^d k_{\ell_1} \mathrm{d}^d k_{\ell_2}}{(\xi_4 + k_{\ell_1})^2 (\xi_5 + k_{\ell_2})^2 (\xi_6 + k_{\ell_2})^2 (\xi_7 + k_{\ell_2})^2 (\xi_8 + k_{\ell_1})^2 (\xi_9 + k_{\ell_1} - k_{\ell_2})^2} \,.$$

**Definition 2.18.** For a graph $\Gamma$, *momentum conservation* (abbreviation: m.c.) is given by the following system of equations:

$$\forall v \in \Gamma^{[0]} : \quad \sum_{h \in v} p_h = 0 \,, \tag{2.15}$$

or equivalently

$$\forall v \in \Gamma^{[0]} : \quad \sum_{h \in v} \xi_h = 0 \,. \tag{2.16}$$

(For an edge $e = \{h_1, h_2\} \in \Gamma^{[1]}$ we write $\xi_e = \xi_{h_1} = \xi_{h_1}$.) We also assign momenta $p_h = \xi_h$ to the external edges $h \in \Gamma^{\mathrm{ext}}$.

$\Phi(\Gamma)$ is a function of the internal $\xi_e$, and $\Phi(\Gamma)|_{\mathrm{m.c.}}$ is a function of the external momenta $p_e$, with the condition that overall momentum conservation holds:

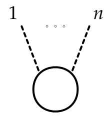
$$\sum_{h \in \Gamma^{\mathrm{ext}}} p_h = 0 \,. \tag{2.17}$$

*One-scale graphs* graphs are graphs for which the amplitude depends on only one momentum (with momentum conservation), such as all *propagator* *graphs* (i.e.: graphs with 2 external edges). For such graphs, we drop the index for the external momentum, and just write $p$.

In theorem 3.9 it will be clear why we do not impose momentum conservation from the beginning.

If for two graphs $\Gamma_1$ and $\Gamma_2$ $\Phi(\Gamma_1)|_{\text{m.c.}} = \Phi(\Gamma_2)|_{\text{m.c.}}$, we write $\Gamma_1 \sim \Gamma_2$. Note that $\Gamma_1 \cong \Gamma_2$ implies $\Gamma_1 \sim \Gamma_2$. In other words: $\Phi(\Gamma)|_{\text{m.c.}}$ does not depend on $\Gamma$'s internal labelling. Neither depends it on the orientation of its internal edges and the choice of the set $L$.

**Definition 2.19.** i. We represent a *full combinatorial Green's function* as follows:

$G =$ 

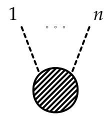
(2.18)

and define it as:

$$G := \sum_{\Gamma} \frac{1}{\mathrm{Sym}(\Gamma)} \frac{i^{\#\Gamma^{[1]}} (-i\lambda)^{\#\Gamma^{[0]}} \pi^{d l_\Gamma /2}}{(2\pi)^{d l_\Gamma}} \Gamma \tag{2.19}$$

where the sum runs over all Feynman graphs possible in the theory $\Gamma$ modulo equivalence in the given theory with the given external structure, in this case: $\Gamma^{\text{ext}} = \{1, \ldots, n\}$.

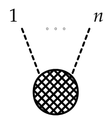
ii. We represent a *connected combinatorial Green's function* as

$G =$ 

(2.20)

and define it with the same formula (2.19), but with the sum restricted to only connected graphs.

iii. And we represent a *1PI combinatorial Green's function* as

$G =$ 

. (2.21)

Here the sum in (2.19) is restricted to only 1PI graphs.

In above definition we have the pre-factors we promised just after definition 2.16: for every edge we have a factor $i$, for every vertex a factor $-i\lambda$ and for every independent loop a factor $\frac{1}{(2\pi)^4}$. The factor $\frac{1}{\pi^{dl_\Gamma/2}}$ in equation (2.12) also gets compensated.

If $G$ is a connected or 1PI Green's function, using lemmata 2.13 and 2.14, we can rewrite it as:

$$G = -i\lambda^{\frac{n-2}{k-2}} \sum_l x^l G_{(l)} \tag{2.22}$$

where

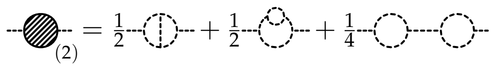
$$x = \frac{i\lambda^{\frac{2}{k-2}}}{2^d \pi^{d/2}}, \tag{2.23}$$

and

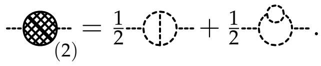
$$G_{(l)} := \sum_{\substack{\Gamma \\ l_\Gamma = l}} \frac{1}{\mathrm{Sym}(\Gamma)} \Gamma \tag{2.24}$$

is the *l-loop* combinatorial Green's function, or the combinatorial Green's function at *order l in perturbation theory*.

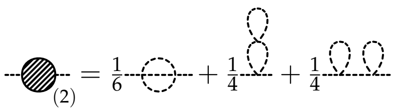
**Example 2.20.** i. In $\phi^3$ theory, the connected 2-loop propagator function is



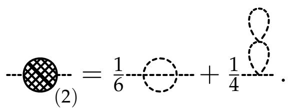
and the 1PI one is



ii. In $\phi^4$ theory they are



and



We use the word 'combinatorial' for $G$; the actual Green's function is given by applying the Feynman rules to $G$: $\Phi(G)|_{\text{m.c.}}$. ($G$ is a linear combination of graphs, so $\Phi$'s definition is extended linearly.)

### 2.2.1 Power Counting

A thing we have to worry about a lot is the convergence of the integral in equation (2.12). We will do this in section 2.4, but for now we can say a little bit about how much the amplitude of a graph diverges.*

For a graph $\Gamma$, the *superficial degree of divergence* $\omega_\Gamma$ is defined as follows: scale every momentum in $\Phi(\Gamma)$ by a factor $\alpha$, then

$$\Phi(\Gamma) \rightsquigarrow \alpha^{\omega_\Gamma} \Phi(\Gamma).$$

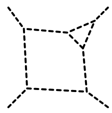
In $\phi^k$ theory it is

$$\omega_\Gamma = d l_\Gamma - 2\#\Gamma^{[1]}. \tag{2.25}$$

*See for example also [10], subsection 8-1-3.

We say that $\Gamma$ is superficially convergent if $\omega_\Gamma < 0$ and superficially divergent for $\omega_\Gamma \geq 0$. In particular: if $\omega_\Gamma = 0$, we say that $\Gamma$ is logarithmically divergent, if $\omega_\Gamma = 1$ we say it is linearly divergent (this will not occur in this chapter, but it will in the next ones) and for $\omega_\Gamma = 2$ it is quadratically divergent.

The word 'superficial' is used above, because $\omega_\Gamma$ does not say everything about convergence. It does not see *subdivergences*: divergent subgraphs. For example: in 6 dimensions,



is superficially convergent ($\omega$  $= -2$), while the triangle subgraph is logarithmically divergent, so the integral is undefined.

Using lemma 2.14, $\omega_\Gamma$ for $\phi^k$ theory can be expressed in the number of external edges and the loop order:

$$\omega_\Gamma = \frac{2(k - \#\Gamma^{\text{ext}})}{k-2} + \Big(d - \frac{2k}{k-2}\Big) l_\Gamma \,. \tag{2.26}$$

The divergences we talked about so far are *ultraviolet divergences*, called so because they arise from the contributions to the amplitude with large momenta. In massless theories, superficially convergent graphs turn out to have *infrared divergences*, caused by low-momentum contributions. In this thesis, we only deal with the ultraviolet ones.

## 2.3 Parametric Representation

In definition 2.16 we introduced the Feynman amplitude of a graph as an integral over loop momenta. In this section we will rewrite this as an integral over scalar parameters.

It all starts with the *Schwinger trick*:

$$\frac{1}{p_e^2} = \int_0^\infty \mathrm{d}A_e \, e^{-p_e^2 A_e} \,, \tag{2.27}$$

where $A_e$ is called the *Schwinger parameter*. If we introduce the *parametric integrand* to be

$$I(\Gamma) := \tfrac{1}{\pi^{d l_\Gamma/2}} \int \mathrm{d}\underline{k} \, e^{-\sum_{e \in \Gamma^{[1]}} p_e^2 A_e} \,, \tag{2.28}$$

the Feynman amplitude can be written as

$$\Phi(\Gamma) = \int \mathrm{d}\underline{A}_\Gamma \, I(\Gamma) \,, \tag{2.29}$$

where we use the following short-hand notation:

$$\int \mathrm{d}\underline{A}_\Gamma := \prod_{e \in \Gamma^{[1]}} \int_0^\infty \mathrm{d}A_e \,. \tag{2.30}$$

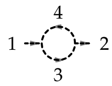
So the product of propagators in equation (2.12) turns into a sum in the exponent.

Note that the mass dimension of the Schwinger parameters is

$$[A_e] = \frac{1}{\text{mass}^2}\,.$$

The next step is to perform the integration over the loop momenta. Before discussing the general case, we look at a simple example:

**Example 2.21.** The parametric integrand of the graph




is

$$I\left(\text{-}\!\bigcirc\!\text{-}\right) = \tfrac{1}{\pi^{d/2}} \int \mathrm{d}^d k\, e^{-(\xi_3+k)^2 A_3 - (\xi_4+k)^2 A_4}\,.$$

Complete the square in the exponent

$$\begin{aligned} I\left(\text{-}\!\bigcirc\!\text{-}\right) &= \tfrac{1}{\pi^{d/2}} \int \mathrm{d}^d k\, e^{-k^2(A_3+A_4)+2k\cdot(\xi_3 A_3+\xi_4 A_4)+\xi_3^2 A_3+\xi_4^2 A_4} \\ &= \tfrac{1}{\pi^{d/2}} \int \mathrm{d}^d k\, e^{-\left(k+\frac{\xi_3 A_3+\xi_4 A_4}{A_3+A_4}\right)^2 (A_3+A_4) - \frac{(\xi_3-\xi_4)^2 A_3 A_4}{A_3+A_4}} \end{aligned}$$

and now it is just a Gaußian integral:

$$I\left(\text{-}\!\bigcirc\!\text{-}\right) = \frac{e^{-\frac{(\xi_3-\xi_4)^2 A_3 A_4}{A_3+A_4}}}{(A_3+A_4)^{d/2}}\,.$$

Here we see why we had the factor $\frac{1}{\pi^{dl_\Gamma/2}}$ in definition 2.16: it disappears here.

Momentum conservation gives us the relation $\xi_3 - \xi_4 = p$. ($p$ is the external momentum. See the remark below equation (2.17).) So

$$I\left(\text{-}\!\bigcirc\!\text{-}\right)\Big|_{\text{m.c.}} = \frac{e^{-\frac{p^2 A_3 A_4}{A_3+A_4}}}{(A_3+A_4)^{d/2}}\,.$$

The amplitude of this graph is given by the following parametric integral:

$$\Phi\left(\text{-}\!\bigcirc\!\text{-}\right) = \int_{\mathbb{R}_+^2} \mathrm{d}A_3 \mathrm{d}A_4\, I\left(\text{-}\!\bigcirc\!\text{-}\right).$$

One remark has to be made: the Gaußian integration above is actually not defined in a Minkowski metric, since it is not positive definite. But with a Wick rotation it can be made positive, i.e. the space-time is made Euclidean. At the end of the computation one has to Wick rotate back.

For the general case, we need to define two polynomials in the Schwinger parameters:

**Definition 2.22.** For a connected graph $\Gamma$, define the set

$$\mathscr{C}_\Gamma^n := \{C \subseteq \Gamma^{[1]} \mid \Gamma \setminus C \text{ is an } n\text{-forest}\}\,. \tag{2.31}$$

i. $\Gamma$'s *first Symanzik polynomial* is defined as

$$\psi_\Gamma := \sum_{C \in \mathscr{C}^1_\Gamma} \prod_{e \in C} A_e \,, \tag{2.32}$$

ii. and its *second Symanzik polynomial* as

$$\varphi_\Gamma := \sum_{C \in \mathscr{C}^2_\Gamma} q_C^2 \prod_{e \in C} A_e \,, \tag{2.33}$$

where

$$q_C := \sum_{e \in C} \varepsilon_{Ce} \xi_e \,. \tag{2.34}$$

$\varepsilon_{Ce} \in \{1, 0, -1\}$ is defined as follows: $\Gamma \setminus C$ consists of two connected components: $\Gamma \setminus C = T_1 T_2$. Choose one of those, say $T_1$. Then

$$\varepsilon_{Ce} = \begin{cases} 1 & \text{if } e \text{ is oriented going into } T_1, \\ -1 & \text{if } e \text{ is oriented coming out of } T_1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that choosing $T_2$ instead of $T_1$ gives a minus sign, but since $q_C$ is squared, $\varphi_\Gamma$ does not depend on that choice.

At momentum conservation $q_C$ can be written as

$$q_C\big|_{\text{m.c.}} = -\sum_{h \in \Gamma^{\text{ext}} \cap T_1^{\text{ext}}} \varepsilon_h p_h = \sum_{h \in \Gamma^{\text{ext}} \cap T_2^{\text{ext}}} \varepsilon_h p_h \,. \tag{2.35}$$

For one-scale graphs we write

$$\varphi_\Gamma\big|_{\text{m.c.}} =: p^2 \varphi'_\Gamma \,. \tag{2.36}$$

Both $\psi_\Gamma$ and $\varphi_\Gamma$ are homogeneous polynomials of degrees

$$\deg \psi_\Gamma = l_\Gamma \tag{2.37}$$

and

$$\deg \varphi_\Gamma = l_\Gamma + 1 \,. \tag{2.38}$$

**Example 2.23.** i. The Symanzik polynomials for the graph in example 2.21 are

$$\psi = A_3 + A_4 \qquad \text{and} \qquad \varphi = q_{34}^2 A_3 A_4 \,,$$

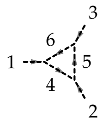
where

$$q_{34} = \xi_3 - \xi_4 \overset{\text{m.c.}}{=\!=} p \,.$$

Because it is one-scale we can write

$$\varphi' = A_3 A_4$$

ii. For the graph




the Symanzik polynomials are

$$\psi = A_4 + A_5 + A_6$$

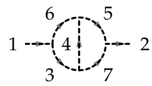
and

$$\varphi = q_{64}^2 A_6 A_4 + q_{45}^2 A_4 A_5 + q_{56}^2 A_5 A_6\,,$$

where

$$\begin{aligned} q_{64} &= \xi_6 - \xi_4 \overset{\text{m.c.}}{=\!=} p_1\,, \\ q_{45} &= \xi_4 - \xi_5 \overset{\text{m.c.}}{=\!=} p_2\,, \\ q_{56} &= \xi_5 - \xi_6 \overset{\text{m.c.}}{=\!=} p_3\,. \end{aligned}$$

iii. For




we have

$$\psi = (A_3 + A_6)(A_5 + A_7) + A_4(A_3 + A_5 + A_6 + A_7)$$

and

$$\begin{aligned} \varphi = {}& q_{36}^2 A_3 A_6 (A_4 + A_5 + A_7) + q_{57}^2 A_5 A_7 (A_3 + A_4 + A_6) \\ & + q_{345}^2 A_3 A_4 A_5 + q_{467}^2 A_4 A_6 A_7 \\ & + q_{347}^2 A_3 A_4 A_7 + q_{456}^2 A_4 A_5 A_6\,, \end{aligned}$$

where

$$\begin{aligned} q_{36} &= \xi_3 + \xi_6 \overset{\text{m.c.}}{=\!=} p\,, \\ q_{57} &= \xi_5 + \xi_7 \overset{\text{m.c.}}{=\!=} p\,, \\ q_{345} &= \xi_3 - \xi_4 + \xi_5 \overset{\text{m.c.}}{=\!=} p\,, \\ q_{467} &= \xi_4 + \xi_6 + \xi_7 \overset{\text{m.c.}}{=\!=} p\,, \\ q_{347} &= \xi_3 - \xi_4 - \xi_7 \overset{\text{m.c.}}{=\!=} 0\,, \\ q_{456} &= \xi_4 - \xi_5 + \xi_6 \overset{\text{m.c.}}{=\!=} 0\,. \end{aligned}$$

Because it is one-scale:

$$\begin{aligned} \varphi' = {}& A_3 A_6 (A_4 + A_5 + A_7) + A_5 A_7 (A_3 + A_4 + A_6) \\ & + (A_3 A_5 + A_6 A_7) A_4\,. \end{aligned}$$

The second Symanzik polynomial can also be written as:

$$\varphi_\Gamma = \sum_{C\in\mathscr{C}_\Gamma^{\prime 2}} q_C^2 \Big(\prod_{e\in C} A_e\Big)\psi_{\Gamma\setminus C}\,, \tag{2.39}$$

where $\mathscr{C}_\Gamma^{\prime 2}$ consists of the minimal $C \subseteq \Gamma^{[1]}$ (by 'minimal' we mean that for all $e \in C$: $\varepsilon_{Ce} \neq 0$) such that $\Gamma \setminus C$ has two connected components. Example 2.23.iii above is a good example of this.

**Theorem 2.24.** For a general Feynman graph, the parametric integrand with the loop momenta integreated out can be written as:

$$I(\Gamma) = \frac{e^{-\varphi_\Gamma/\psi_\Gamma}}{\psi_\Gamma^{d/2}}\,.^* \tag{2.40}$$

In the massive case, one gets mass terms in the exponential:

$$I(\Gamma) = \frac{e^{-\varphi_\Gamma/\psi_\Gamma - m^2\sum_{e\in\Gamma^{[1]}} A_e}}{\psi_\Gamma^{d/2}} \tag{2.41}$$

So, we have written the amplitude of a graph $\Gamma$ as an $\#\Gamma^{[1]}$-dimensional integral over positive parameters. The number of integrations can be reduced by one as follows:

**Proposition 2.25.** i.

$$\Phi(\Gamma) = \int \Omega_\Gamma\, \mathscr{I}(\Gamma)\,, \tag{2.42}$$

where

$$\mathscr{I}(\Gamma) := \int_0^\infty \mathrm{d}t\, t^{\#\Gamma^{[1]}-1} I(\Gamma)\big|_{\underline{A}=t\underline{a}} \tag{2.43}$$

and

$$\Omega_\Gamma := \mathrm{d}\underline{a}_\Gamma\, \delta\Big(1 - \sum_{e\in\Gamma^{[1]}} \lambda_e a_e\Big)\,. \tag{2.44}$$

All $\lambda_e \geq 0$ and are such that there is at least one $\lambda_e \neq 0$.

This also holds in other theories than $\phi^k$.

ii. In $\phi^k$ theory $\mathscr{I}(\Gamma)$ is

$$\mathscr{I}(\Gamma) = \frac{\varphi_\Gamma^{\omega_\Gamma/2}}{\psi_\Gamma^{(\omega_\Gamma+d)/2}}\Gamma(-\tfrac{1}{2}\omega_\Gamma)\,. \tag{2.45}$$

($\Gamma$ stands for the Euler $\Gamma$-function.)

*Proof.* i. First note that the number 1 can be written as

$$\int_0^\infty \mathrm{d}t\, \delta\Big(t - \sum_{e\in\Gamma^{[1]}} \lambda_e A_e\Big) = 1\,,$$

*For a proof, we refer to [10], subsection 6-2-3 together with [3], and to [14], subsection 2.1.1.

because of the restrictions we have put on the $\lambda_e$. Plug this into equation (2.29):

$$\Phi(\Gamma) = \int_0^\infty \mathrm{d}t \int \mathrm{d}\underline{A}_\Gamma \, \delta\Big(t - \sum_{e\in\Gamma^{[1]}} \lambda_e A_e\Big) I(\Gamma)\,.$$

Substitute $\underline{A}_\Gamma = t\underline{a}_\Gamma$ (by this we mean $A_e = ta_e$ for every $e \in \Gamma^{[1]}$):

$$\Phi(\Gamma) = \int_0^\infty \mathrm{d}t \int \mathrm{d}\underline{a}_\Gamma \, t^{\#\Gamma^{[1]}-1} \delta\Big(1 - \sum_{e\in\Gamma^{[1]}} \lambda_e a_e\Big) I(\Gamma)\big|_{\underline{A}_\Gamma = t\underline{a}_\Gamma}\,.$$

Note that the form of the integrand is not used, which means that it also holds for other theories.

ii. If we use the expression for $I(\Gamma)$ (theorem 2.24), we get

$$\begin{aligned}\mathscr{I}(\Gamma) &= \frac{1}{\psi_\Gamma^{d/2}} \int_0^\infty \mathrm{d}t \, t^{\#\Gamma^{[1]}-dl/2-1} e^{-t\varphi_\Gamma/\psi_\Gamma} \\ &= \frac{1}{\psi_\Gamma^{d/2}} \int_0^\infty \mathrm{d}t \, t^{-\omega_\Gamma/2-1} e^{-t\varphi_\Gamma/\psi_\Gamma},.\end{aligned}$$

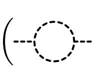
Recall (2.37) and (2.38). (We did not explicitly write that $\psi_\Gamma$ and $\varphi_\Gamma$ are polynomials in the parameters $a_e$ instead of $A_e$.) In the second step equation (2.25) is used. Doing the integral by using the definition of the Γ-function gives the result.

For this integration, we have to assume an Euclidean space-time, such that $\phi_\Gamma \geq 0$. See the remark about Wick rotation after example 2.21 □

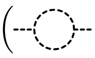
**Remark 2.26.** Because of the Γ-function, $\mathscr{I}(\Gamma)$ diverges if $\omega_\Gamma \geq 0$ and converges if $\omega_\Gamma < 0$. This is precisely the ultraviolet divergence we described in subsection 2.2.1. Actually, it is also convergent for odd $\omega_\Gamma > 0$, but we will not see such a case. Sub- and infrared divergences arise if we do the $\Omega_\Gamma$-integration.

One is free to choose the $\lambda_e$ in equation (2.44); a different choice is just a change of integration variables. A choice where one $\lambda_e = 1$ and the other ones are 0 is usually the best for doing the computations.

**Example 2.27.** We continue with example 2.21 / 2.23.i, for which $\omega = d-4$. With proposition 2.25.ii we have

$\mathscr{I}\Big($  $\Big) \overset{\text{m.c.}}{=\!=} \frac{(p^2 a_3 a_4)^{d/2-2}}{(a_3+a_4)^{d-2}} \Gamma(2-\tfrac{1}{2}d),.$

This diverges (ultraviolet) for $d \in \{4, 6, 8, \ldots\}$.

$\Phi\Big($  $\Big) = (p^2)^{d/2-2} \int_0^\infty \mathrm{d}a_3 \frac{a_3^{d/2-2}}{(a_3+1)^{d-2}} \Gamma(2-\tfrac{1}{2}d)$

$$= 8\sqrt{\pi} 2^{-d} (p^2)^{d/2-2} \frac{\Gamma(\frac{1}{2}d-1)}{\Gamma(\frac{1}{2}d-\frac{1}{2})} \Gamma(2-\tfrac{1}{2}d)\,.$$

Here we see another divergence: $\Gamma(\frac{1}{2}d - 1)$ diverges for $d \in \{0, 2\}$. This is the infrared divergence.

## 2.4 Renormalization

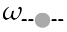
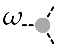
### 2.4.1 $\phi^3$ Theory in 6 Dimensions

So, we have these divergent integrals. In the following we will show how we deal with it in the case of $\phi^3$ theory in 6 space-time dimensions, although this theory is not physical.

With equation (2.26), one can see that the superficial degree of divergence is

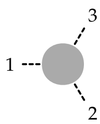
$$\omega_\Gamma = 6 - 2\#\Gamma^{\text{ext}} . \tag{2.46}$$

Note that it does not depend on the loop order, only on the external structure. The only divergent graphs are propagator (quadratically divergent) and vertex graphs (logarithmically divergent):

$\omega$  $= 2$ and $\omega$  $= 0$ .

First, we look at graphs without subdivergences.* Loosely said, we make sense of these divergent integrals by subtracting another divergence. To keep things defined, we do this subtraction on the level of the integrand.

**Definition 2.28.** Let $\Gamma$ be a vertex graph:

$\Gamma =$  ,


and assume that it has no subdivergences. We introduce a momentum scale $\mu$ and define the *renormalized integrand* as:

$$I^{\text{ren}}(\Gamma) := I(\Gamma) - I^\circ(\Gamma) , \tag{2.47}$$

where the superscript $^\circ$ means evaluation at a point in the space of external momenta $p_1$, $p_2$ and $p_3$ given by $p_1^2 = p_2^2 = p_3^2 = \mu^2$. Momentum conservation is assumed, so $p_1 \cdot p_2 = p_1 \cdot p_3 = p_2 \cdot p_3 = -\frac{1}{2}\mu^2$. The renormalized integrand fulfills the renormalization condition

$$I^{\text{ren}}(\Gamma)\big|_{p_1^2 = p_2^2 = p_3^2 = \mu^2} = 0 . \tag{2.48}$$

Doing one integration, as in proposition 2.25, gives:

$$\mathscr{I}^{\text{ren}}(\Gamma) = \frac{1}{\psi^3} \int_0^\infty \frac{\mathrm{d}t}{t} \left( e^{-t\varphi_\Gamma/\psi_\Gamma} - e^{-t\varphi_\Gamma^\circ/\psi_\Gamma} \right) . \tag{2.49}$$

With the identity

$$\int_c^\infty \frac{\mathrm{d}t}{t} e^{-t\varphi_\Gamma/\psi_\Gamma} = -\ln c - \gamma_{\text{E}} - \ln \frac{\varphi_\Gamma}{\psi_\Gamma} + \mathscr{O}(c) \tag{2.50}$$

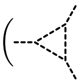
*In Hopf-algebraic language one says *primitive* graphs.

(as $c \to 0$), can be written as

$$\mathscr{I}^{\mathrm{ren}}(\Gamma) = -\frac{1}{\psi_\Gamma^3} \ln \frac{\varphi_\Gamma}{\varphi_\Gamma^\circ} \tag{2.51}$$

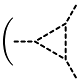
The number $\gamma_{\mathrm{E}} \approx 0.577$ is the Euler-Mascheroni constant.

**Example 2.29.** Take the graph from example 2.23.ii. For this one:



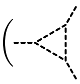
$$\mathscr{I}^{\mathrm{ren}}\Big(\quad\Big) = -\frac{1}{(a_4+a_5+a_6)^3} \ln \frac{p_1^2 a_6 a_4 + p_2^2 a_4 a_5 + p_3^2 a_5 a_6}{\mu^2(a_6 a_4 + a_4 a_5 + a_5 a_6)} \,.$$

If one takes $p_1^2 = p_2^2 = p_3^2 = p^2$, to make life easier, it is



$$\mathscr{I}^{\mathrm{ren}}\Big(\quad\Big) = -\frac{1}{(a_4+a_5+a_6)^3} \ln \frac{p^2}{\mu^2} \,.$$

The amplitude is then:



$$\Phi^{\mathrm{ren}}\Big(\quad\Big) = -\int_{\mathbb{R}_+^2} \frac{\mathrm{d}a_4 \mathrm{d}a_5}{(a_4+a_5+1)^3} \ln \frac{p^2}{\mu^2} = -\tfrac{1}{2} \ln \frac{p^2}{\mu^2} \,.$$

**Definition 2.30.** For propagator graphs, the following renormalization conditions are assumed:

$$I^{\mathrm{ren}}(\Gamma)\big|_{p^2=0} = 0 \tag{2.52}$$

and

$$\frac{I^{\mathrm{ren}}(\Gamma)}{p^2}\Big|_{p^2=\mu^2} = 0 \,. \tag{2.53}$$

So for a propagator graph $\Gamma$ without subdivergences, we define:

$$\begin{aligned} I^{\mathrm{ren}}(\Gamma) &:= I(\Gamma) - I(\Gamma)\big|_{p^2=0} - \frac{p^2}{\mu^2}\big(I(\Gamma)\big|_{p^2=\mu^2} - I(\Gamma)\big|_{p^2=0}\big) \\ &= \frac{1}{\psi_\Gamma^3}\Big(e^{-p^2\varphi'_\Gamma/\psi_\Gamma} - 1 - \frac{p^2}{\mu^2}(e^{-\mu^2\varphi'_\Gamma/\psi_\Gamma} - 1)\Big) \,. \end{aligned} \tag{2.54}$$

(Recall equation (2.36).)

$\mathscr{I}^{\mathrm{ren}}(\Gamma)$ is:

$$\mathscr{I}^{\mathrm{ren}}(\Gamma) = \frac{1}{\psi_\Gamma^3} \int_0^\infty \frac{\mathrm{d}t}{t^2}\Big(e^{-tp^2\varphi'_\Gamma/\psi_\Gamma} - 1 - \frac{p^2}{\mu^2}(e^{-t\mu^2\varphi'_\Gamma/\psi_\Gamma} - 1)\Big) \tag{2.55}$$

A partial integration and equation (2.50) give:

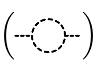
$$\begin{aligned} \int_c^\infty \frac{\mathrm{d}t}{t^2}(e^{-tp^2\varphi'_\Gamma/\psi_\Gamma} - 1) &= -\frac{p^2\varphi'_\Gamma}{\psi_\Gamma} \int_c^\infty \frac{\mathrm{d}t}{t} e^{-tp^2\varphi'_\Gamma/\psi_\Gamma} + \frac{1}{c}(e^{-cp^2\varphi'_\Gamma/\psi_\Gamma} - 1) \\ &= \frac{p^2\varphi'_\Gamma}{\psi_\Gamma}\Big(\gamma_{\mathrm{E}} + \ln \frac{p^2\varphi'_\Gamma}{\psi_\Gamma} + \ln c - 1\Big) + \mathscr{O}(c) \,, \end{aligned} \tag{2.56}$$

and so:

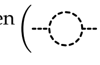
$$\mathscr{I}^{\mathrm{ren}}(\Gamma) = \frac{\varphi_\Gamma}{\psi_\Gamma^4} \ln \frac{p^2}{\mu^2} \tag{2.57}$$

Note that the boundary terms from the partial integration cancel.

**Example 2.31.** Actually, there is only one primitive propagator graph in $\phi^3$-theory: the 1-loop graph in example 2.27. For this one:

$$\mathscr{I}^{\mathrm{ren}}\Big($$  $$\Big) = \frac{p^2 a_3 a_4}{(a_3+a_4)^4} \ln \frac{p^2}{\mu^2}\,,$$

and so the amplitude is

$$\Phi^{\mathrm{ren}}\Big($$  $$\Big) = p^2 \int_0^\infty \mathrm{d}a_3 \frac{a_3}{(a_3+1)^4} \ln \frac{p^2}{\mu^2} = \tfrac{1}{6} p^2 \ln \frac{p^2}{\mu^2}\,.$$

For the renormalization of subdivergences, we need the following definition:

**Definition 2.32.** A *forest (of subdivergences)* $f$ of a graph $\Gamma$ is a set of divergent, connected subgraphs of $\Gamma$ such that for every $\gamma_1, \gamma_2 \in f$: either $\gamma_1 \subseteq \gamma_2$, or $\gamma_2 \subseteq \gamma_1$, or $\gamma_1 \cap \gamma_2 = \emptyset$.
The set of all forests of $\Gamma$ is denoted by $\mathscr{F}(\Gamma)$.

In definition 2.11.iii the word 'forest' was used already. Forests of subdivergences have an interpretation as forest graphs.

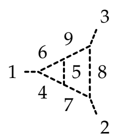
**Definition 2.33.** Let $\Gamma$ be a graph with only logarithmic subdivergences. To make life slightly easier, propagator subdivergences are excluded. Then the renormalized integrand is given by the *forest formula*:*

$$I^{\mathrm{ren}}(\Gamma) = \sum_{f \in \mathscr{F}(\Gamma)} (-)^{\# f} I^\circ(f) I(\Gamma/f)\,. \tag{2.58}$$

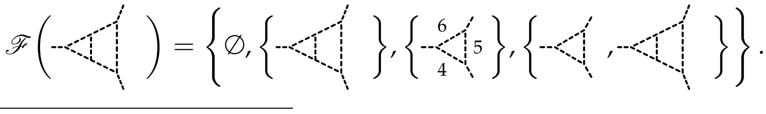
The integrand of a forest is the following product of integrands of cographs:

$$I^\circ(f) = \prod_{\gamma \in f} I^\circ\Big(\gamma \Big/ \bigcup_{\substack{\gamma' \subsetneq \gamma \\ \gamma' \in f}} \gamma'\Big)\,. \tag{2.59}$$

**Example 2.34.** The graph

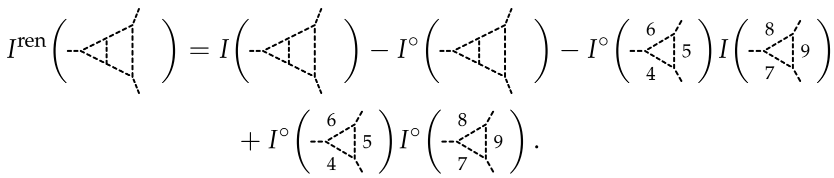



has the following forests:




*See [10], subsection 8-2-3 and [6], equation (40). In the latter, propagator divergences in the parametric context are discussed as well.

The renormalized integrand is




For an overall divergent graph $\Gamma$, the forest formula can be split in two sums, one with the forests that do not contain $\Gamma$ itself, and one with the forests that do. So:

$$I^{\text{ren}}(\Gamma) = \sum_{f \in \mathscr{F}'(\Gamma)} (-)^{\#f} \big( I^\circ(f) I(\Gamma/f) - I^\circ(f) I^\circ(\Gamma/f) \big) \,, \tag{2.60}$$

where

$$\mathscr{F}'(\Gamma) = \{ f \in \mathscr{F}(\Gamma) \,|\, f \not\ni \Gamma \} \,. \tag{2.61}$$

Let us denote the renormalized integrand of a graph $\Gamma$, where the subdivergences are ignored by $I^{\overline{\text{ren}}}(\Gamma)$. Then

$$I^{\text{ren}}(\Gamma) = \sum_{f \in \mathscr{F}'(\Gamma)} (-)^{\#f} I^\circ(f) I^{\overline{\text{ren}}}(\Gamma/f) \,. \tag{2.62}$$

### 2.4.2 Other Theories

Three classes of theories are distinguished:

- *Superrenormalizable theories*: theories with only a finite number of superficially divergent graphs.
- *Renormalizable theories*: theories with infinitely many superficially divergent graphs, but with a finite number of divergent Green's functions. The degree of divergence does not depend on the order in perturbation theory.
- *Unrenormalizable theories*: theories where every Green's function is divergent from some point three in perturbation theory.

Looking at equation (2.26), we see that the renormalizable $\phi^k$-theories are the ones for which $d = \frac{2k}{k-2}$, in order to let the $l_\Gamma$ dependency disappear. The three only ones are:

- 6-dimensional $\phi^3$ theory ($\omega_\Gamma$ is given in equation (2.46)),
- 4-dimensional $\phi^4$ theory, where

$$\omega_\Gamma = 4 - \#\Gamma^{\text{ext}} \,, \tag{2.63}$$

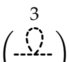
- and 3-dimensional $\phi^6$ theory, where

$$\omega_\Gamma = 3 - \tfrac{1}{2}\#\Gamma^{\text{ext}} \,. \tag{2.64}$$

Note that for these theories propagator graphs are always quadratically divergent and vertex graphs (i.e. $k$-point graphs) are always logarithmically divergent. Furthermore, the propagator and vertex graphs are the only superficially divergent ones in these theories. (4-regular 3-point graphs and 6-regular 3-, 4- and 5-point graphs do not exist and we disregard vacuum and tadpole graphs.)

We conclude this chapter with remark on self-loops in $\phi^4$ theory:

**Remark 2.35.** The integrand of a self-loop graph in 4-dimensional $\phi^4$ theory is:

$I\Big($  $\Big) = \dfrac{1}{A_3^2}\,.$

Because it does not depend on the momentum, the renormalized integrand vanishes:

$I^{\mathrm{ren}}\Big($  $\Big) = 0\,.$

Together with the forest formula, this implies that every graph with self-loops, and also more general graphs like

have a vanishing integrand after renormalization.

# 3 Quantum Electrodynamics

## 3.1 Feynman Rules

### 3.1.1 Lagrangian

First of all: from now on, everything will be in 4-dimensional space-time.

In *quantum electrodynamics* (*QED*) we have two fields: a spinor field $\psi$ for the fermions and a vector field $A$, called the *gauge field* for the photons. The Lagrangian is

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\not{D}\psi\,. \tag{3.1}$$

Here,

$$D_\mu = \partial_\mu + ieA_\mu \tag{3.2}$$

is the *covariant derivative* and

$$F_{\mu\nu} = -\frac{i}{e}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.3}$$

is the *field tensor*.

Furthermore, we need the *Clifford algebra*, which is generated by $4 \times 4$ matrices $\gamma^\mu$ that fulfill the *Clifford relation*:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,. \tag{3.4}$$

The Feynman slash notation is a short-hand notation for the *Clifford representation* of a Lorentz vector $a$:

$$\not{a} := \gamma_\mu a^\mu\,. \tag{3.5}$$

This Lagrangian describes massless QED. For massive fermions, one adds a term $-m\overline{\psi}\psi$.

An important property of this Lagrangian is *gauge invariance*, $\mathrm{U}(1)$ gauge invariance to be precise. This means that for a $\mathrm{U}(1)$-valued function $U$ on the space-time, the Lagrangian is invariant under the *gauge transformation*

$$\psi \mapsto U\psi\,, \qquad D_\mu \mapsto UD_\mu U^{-1}\,. \tag{3.6}$$

If you like the Lie algebra formalism better than the Lie group formalism, let $i\alpha$ be a $\mathfrak{u}(1) = i\mathbb{R}$-valued function and write

$$U = e^{i\alpha} \,. \tag{3.7}$$

Then the gauge transformation can be written as

$$\psi \mapsto e^{i\alpha}\psi \,, \qquad A_\mu \mapsto A_\mu - \frac{1}{e}\partial_\mu \alpha \,. \tag{3.8}$$

### 3.1.2 Feynman Graphs

For QED, we need to enrich the notion of Feynman graphs from section 2.1 a bit: half-edges occur in three types instead of one. We have photon half-edges and incoming and outgoing fermion half-edges, which we represent graphically as

respectively.

Edges come in two types: photon edges consists of 2 photon-half-edges and fermion edges consist of an incoming and an outgoing fermion edge. Naturally, they look like

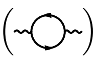
respectively.

There is one vertex type with a photon and an incoming and an outgoing fermion:

 .

We denote the set of photon edges by $\Gamma^{[1]}_{\sim}$, the set of fermion edges by $\Gamma^{[1]}_{\rightarrow}$, the set of external ingoing fermion half-edges by $\Gamma^{\mathrm{ext}}_{\leftarrow}$ etcetera.

Feynman graph isomorphisms need an extra condition with respect to in definition 2.3.i: an isomorphism also has to respect half-edge type. This has for example the implication that

Sym (  ) = 1

instead of $\frac{1}{2}$.

Note that this implies that in QED every symmetry factor is simply 1, because the vertex has no symmetries.

There is an analogon of lemma 2.14 for QED:

**Lemma 3.1.** For a QED graph $\Gamma$:

i. $$\#\Gamma^{[0]} = \#\Gamma^{\mathrm{ext}} + 2(l_\Gamma - c_\Gamma) \,, \tag{3.9}$$

ii. $$\#\Gamma^{[1]} = \#\Gamma^{\mathrm{ext}} + 3(l_\Gamma - c_\Gamma) \,, \tag{3.10}$$

iii. $$\#\Gamma^{[1]}_{\sim} = \#\Gamma^{\mathrm{ext}}_{\leftarrow} + l_\Gamma - c_\Gamma \,, \tag{3.11}$$

iv. $$\#\Gamma^{[1]}_{\rightarrow} = \#\Gamma^{\mathrm{ext}}_{\sim} + \#\Gamma^{\mathrm{ext}}_{\leftarrow} + 2(l_\Gamma - c_\Gamma) \,. \tag{3.12}$$

*Proof.* Taking $k = 3$ in lemma 2.14 gives i and ii. For iii and iv, use

$$2\Gamma^{[1]}_{\sim} + \#\Gamma^{\mathrm{ext}}_{\sim} = \#\Gamma^{\mathrm{he}}_{\sim} = \#\Gamma^{\mathrm{he}}_{\leftarrow} = \#\Gamma^{[1]}_{\rightarrow} + \#\Gamma^{\mathrm{ext}}_{\leftarrow} \,.$$ □

### 3.1.3 Feynman Rules

To write down the Feynman amplitude of a QED graph $\Gamma$, assign to every internal and external photon half-edge $h \in \Gamma^{\mathrm{he}}_{\sim}$ a Lorentz index $\mu_h$ and to every fermion edge $e \in \Gamma^{[1]}_{\rightarrow}$ a Lorentz index $\mu_e$. Actually, the fermion half-edges also carry a spinor indices, but these will not be written explicitly in this thesis. The Feynman amplitude is

$$\Phi(\Gamma) := \frac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}_L \frac{N(\Gamma)}{\prod\limits_{e \in \Gamma^{[1]}} p_e^2} . \tag{3.13}$$

The numerator $N(\Gamma)$ is a product of the following:

- for every photon edge $e = \{h_1, h_2\} \in \Gamma^{[1]}_{\sim}$ a factor

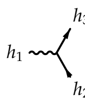
$$g_{\mu_{h_1}\mu_{h_2}} - (1-\alpha)\frac{p_{e\mu_{h_1}} p_{e\mu_{h_2}}}{p_e^2} \tag{3.14}$$

  ($\alpha$ is the gauge parameter),

- for every fermion edge $e \in \Gamma^{[1]}_{\rightarrow}$ a factor

$$\gamma_{\mu_e} p_e^{\mu_e} = \not{p}_e , \tag{3.15}$$

- and for every vertex




$$\in \Gamma^{[0]} \qquad \text{a factor} \qquad \gamma^{\mu_{h_1}} . \tag{3.16}$$

We have to be careful with the order of the $\gamma$-matrices, since they do not commute. We write the numerator as

$$\begin{aligned} N(\Gamma) = \gamma(\Gamma) \Bigg( & \prod_{\{h_1,h_2\}=e \in \Gamma^{[1]}_{\sim}} \Big( g_{\mu_{h_1}\mu_{h_2}} - (1-\alpha)\frac{p_{e\mu_{h_1}} p_{e\mu_{h_2}}}{p_e^2} \Big) \Bigg) \\ & \times \Big( \prod_{e \in \Gamma^{[1]}_{\rightarrow}} p_e^{\mu_e} \Big) , \end{aligned} \tag{3.17}$$

where all the $\gamma$-matrices are collected in $\gamma(\Gamma)$.

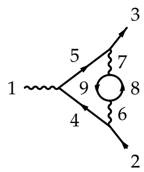
Note that $\Phi(\Gamma)$ has 'open' Lorentz indices for the external photons. The other Lorentz indices are contracted.

For the Feynman gauge, i.e. $\alpha = 1$, the numerator can be simplified with some abuse of notation. For this, instead of assigning Lorentz indices to the photon half-edges, we assign them to the internal and external photon edges. We drop the $g_{\mu_{h_1}\mu_{h_2}}$ and do not care about upper or lower indices, but still use Einstein's summation convention for repeated indices. The numerator is then simply

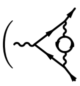
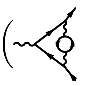
$$N(\Gamma) = \gamma(\Gamma) \prod_{e \in \Gamma^{[1]}_{\rightarrow}} p_e^{\mu_e} . \tag{3.18}$$

The Feynman gauge is assumed unless indicated otherwise. We will briefly come back to other covariant gauges in remark 3.14.

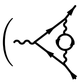
**Example 3.2.** For the graph




we have

$N\Big($  $\Big) = \gamma\Big($  $\Big) p_4^{\mu_4} p_5^{\mu_5} p_8^{\mu_8} p_9^{\mu_9} \,,$

with

$\gamma\Big($  $\Big) = \gamma^{\mu_7}\gamma^{\mu_5}\gamma^{\mu_1}\gamma^{\mu_4}\gamma^{\mu_6}\,\mathrm{Tr}(\gamma^{\mu_6}\gamma^{\mu_9}\gamma^{\mu_7}\gamma^{\mu_8})\,.$

If external fermions are in a physical state, a spinor $u_e$ has to be included if it is ingoing and $\overline{u}_e$ if it is outoing. These spinors fulfill the *Dirac equation* in momentum space:

$$\not{p}_e u_e = 0 \tag{3.19}$$

and

$$\overline{u}_e \not{p}_e = 0\,. \tag{3.20}$$

(Remember that that our fermions are massless.) For anti-fermions, it is customary to write $\overline{v}_e$ and $v_e$.

For physical external photons, one has to include a polatization vector $\varepsilon_e^{\mu_e}$, which is transversal:

$$p_e \cdot \varepsilon_e = 0\,. \tag{3.21}$$

Furthermore, physical photons have lightlike momentum:

$$p_e^2 = 0\,. \tag{3.22}$$

We represent physical external particles graphically by a dot:

Analogous to definition 2.19, we define Green's functions as

$$G := \sum_\Gamma (-)^{\#\mathscr{L}_\Gamma^{\rightarrow}} \frac{1}{\mathrm{Sym}(\Gamma)} \frac{(-i)^{\#\Gamma^{[1]}_{\sim}} i^{\#\Gamma^{[1]}_{\rightarrow}} (ie)^{\#\Gamma^{[0]}} \pi^{2l_\Gamma}}{(2\pi)^{4l_\Gamma}} \Gamma\,. \tag{3.23}$$

Note the sign in front: every fermion loop in $\Gamma$ gives a minus sign. ($\mathscr{L}_\Gamma^{\rightarrow}$ denotes the set of fermion loops in $\Gamma$.) This is a consequence of Fermi statistics.

Using lemma 3.1, it can be written as

$$G = (-)^{\#\Gamma^{\mathrm{ext}}_{\sim} + \#\Gamma^{\mathrm{ext}}_{\rightarrow}} ie^{\Gamma^{\mathrm{ext}}-2} \sum_{l=0}^{\infty} x^l G_{(l)}\,, \tag{3.24}$$

with

$$x = -\frac{ie^2}{16\pi^2} \tag{3.25}$$

and

$$G_{(l)} = \sum_{\substack{\Gamma \\ l_\Gamma = l}} (-)^{\#\mathscr{L}_\Gamma} \frac{1}{\mathrm{Sym}(\Gamma)} \Gamma\,. \tag{3.26}$$

### 3.1.4 Power Counting

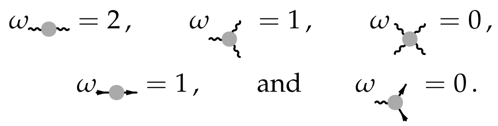
Looking at equations (3.13) and (3.18), we see that the superficial degree of divergence is

$$\omega_\Gamma = 4l_\Gamma - 2\#\Gamma^{[1]}_{\sim} - \#\Gamma^{[1]}_{-} = 2l_\Gamma + \#\Gamma^{[1]}_{-} - 2\#\Gamma^{[1]}\,. \tag{3.27}$$

With the use of lemma 3.1 it can be written as

$$\omega_\Gamma = 4 - \#\Gamma^{\mathrm{ext}}_{\sim} - 3\#\Gamma^{\mathrm{ext}}_{-}\,. \tag{3.28}$$

This means we have the following superficial divergences:

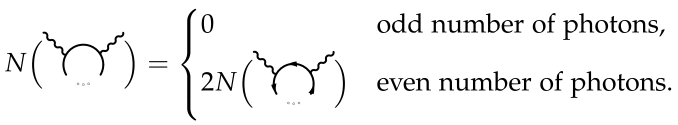


(3.29)

We will get back on this at the beginning of section 3.4.

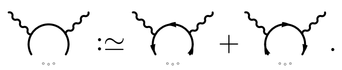
The following result will be useful there:

**Lemma 3.3** (Furry's theorem)**.**



(3.30)

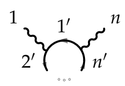
By this unoriented fermion loop we mean the sum over both orientations:



(3.31)

The relation $\simeq$ means that the left- and the right-hand side have exactly the same Feynman rules.

*Proof.*




$$= \mathrm{Tr}(\gamma^{\mu_n}\gamma^{\mu_{n'}} \cdots \gamma^{\mu_1}\gamma^{\mu_{1'}})\, p_{n'}^{\mu_{n'}} \cdots p_{1'}^{\mu_{1'}}$$

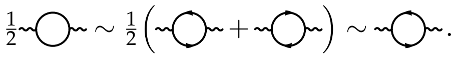
$$+ (-)^n\, \mathrm{Tr}(\gamma^{\mu_{1'}}\gamma^{\mu_1} \cdots \gamma^{\mu_{n'}}\gamma^{\mu_n})\, p_{n'}^{\mu_{n'}} \cdots p_{1'}^{\mu_{1'}}\,.$$

The $n$ minus signs appear because in the clockwise orientation, the momenta are oriented opposite to the fermion arrow. The $\gamma$-matrices have the following property:*

$$\mathrm{Tr}(\gamma^{\mu_n}\gamma^{\mu_{n'}}\cdots\gamma^{\mu_1}\gamma^{\mu_{1'}}) = \mathrm{Tr}(\gamma^{\mu_{1'}}\gamma^{\mu_1}\cdots\gamma^{\mu_{n'}}\gamma^{\mu_n}),$$

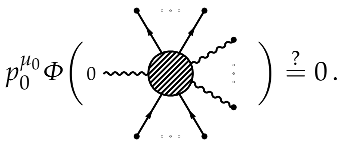
so the statement is proven. □

Note that unoriented fermion loops have symmetry factor and that they respect them, for example:



## 3.2 Ward Identities

In classical electrodynamics we know that electromagnetic waves are transverse. The *Ward identities* confirm that in the quantized theory longitudinal photons are indeed unphysical:



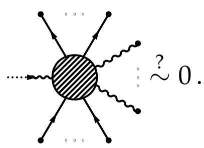
(We omit writing 'm.c.' in this section, but momentum conservation is assumed everywhere.)

If we introduce a new notation for external edges (a longitudinal photon):

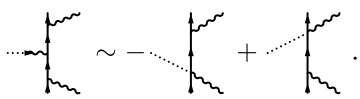
with the Feynman rule that one has to include a factor

$$p_e^{\mu_e} \tag{3.32}$$

for such an external edge $e$, the Ward identities can be written as



**Lemma 3.4.**

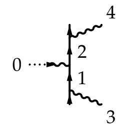


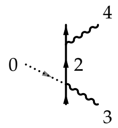
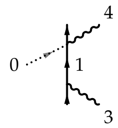
(3.33)

*[15], equation (A.28)

The dotted line is just there to keep it consistent with momentum conservation; it does not alter the Feynman rules.

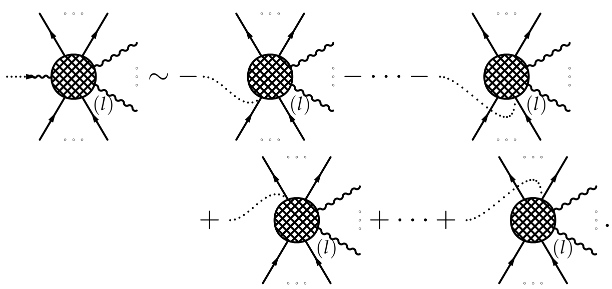
*Proof.* With momentum conservation, $p_0 = -p_1 + p_2$, one has

$$\Phi\Bigg($$  $$\Bigg) = p_0^{\mu_0} \frac{\gamma^{\mu_4} \not{p}_2 \gamma^{\mu_0} \not{p}_1 \gamma^{\mu_3}}{p_1^2 p_2^2} = -\frac{\gamma^{\mu_4} \not{p}_2 \gamma^{\mu_3}}{p_2^2} + \frac{\gamma^{\mu_4} \not{p}_1 \gamma^{\mu_3}}{p_1^2}$$

$$= \Phi\Bigg( -$$  $$+$$  $$\Bigg).$$ □

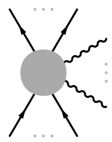
Before we go to the Ward identities, we first give the *Ward-Takahashi identities*, which relate of-shell 1PI functions to each other:*

**Theorem 3.5** (Ward-Takahashi identities)**.**

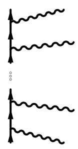


(3.34)

*Proof.* Consider a 1PI graph $\Gamma$ of the form

$$\Gamma =$$ 

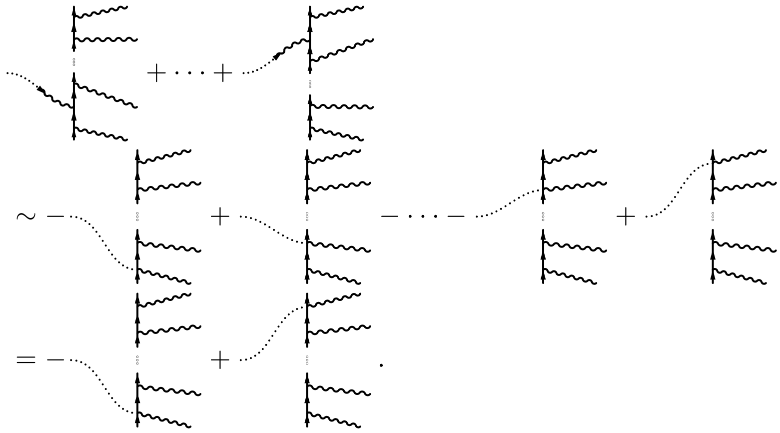
and take a fermion line that is going through it:

 .

The next step is to sum over the fermion edges in the line and insert a longi-

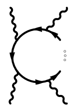
*See also for example [15], section 7.4

tudinal photon into each of these edges. With lemma 3.4 we get:

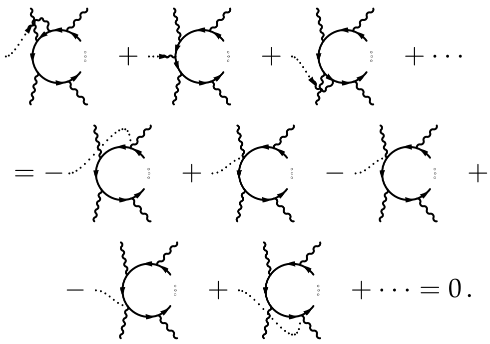


The terms in the middle line cancel in pairs, except for the two outer ones.

Now take fermion loop in $\Gamma$:

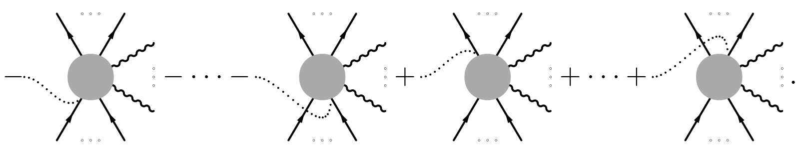


and do the same thing:



Here we see that the whole thing cancels pairwise.

So, if we insert a longitudinal photon in every internal fermion edge in $\Gamma$, we get for every open fermion line two contributions:

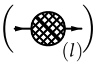


Note that the graph remains 1PI after inserting a photon into an internal fermion edges.

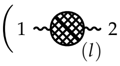
We do not have to worry about symmetry factors. In subsection 3.1.2 we remarked that in QED we do not have symmetry factors oter than 1. (We do not use the notation of lemma 3.3.)

Summing over all such graphs completes the proof. □

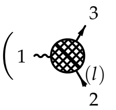
**Corollary 3.6.** Write the 1PI fermion propagator function as

$$\Sigma_{(l)}(p) := \Phi\Big($$  $$\Big)\,, \tag{3.35}$$

the photon propagator function as

$$\Pi^{\mu_1\mu_2}_{(l)}(p) := \Phi\Big($$  $$\Big) \tag{3.36}$$

and the vertex function as

$$\Gamma^{\mu_1}_{(l)}(p_2, p_3) := \Phi\Big($$  $$\Big)\,. \tag{3.37}$$

Then:

i.
$$p^\mu \Pi^{\mu\nu}_{(l)}(p) = 0\,, \tag{3.38}$$

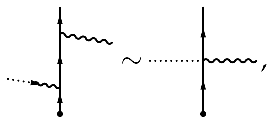
ii.
$$p_1^\mu \Gamma^\mu_{(l)}(p_2, p_2 + p_1) = \Sigma_{(l)}(p_2) - \Sigma_{(l)}(p_2 + p_1)\,, \tag{3.39}$$

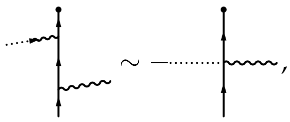
iii.
$$\Gamma^\mu_{(l)}(p, p) = -\frac{\mathrm{d}\Sigma_{(l)}(p)}{\mathrm{d}p_\mu}\,. \tag{3.40}$$

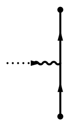
*Proof.* The identities i and ii follow directly from theorem 3.5. Identity iii follows from ii by differentiating to $p_1$ and setting it to 0. □

For the Ward identities, we first need something similar to lemma 3.4, but with physical external fermions:

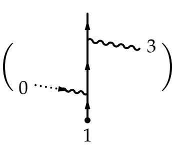
**Lemma 3.7.**

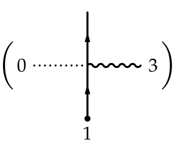
i.
 (3.41)

ii.
 (3.42)

iii.
 $$\sim 0\,. \tag{3.43}$$

*Proof.* i. With the Dirac equation (3.19):

$$\Phi\Big($$  $$\Big) = \frac{\gamma^{\mu_3} \not{p}_2 \not{p}_0 u_1}{p_2^2} = \frac{\gamma^{\mu_3} \not{p}_2 (-\not{p}_1 + \not{p}_2) u_1}{p_2^2} = \gamma^{\mu_3} u_1$$

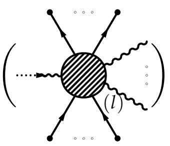
$$= \Phi\Big($$  $$\Big)\,.$$

ii. This is proven analogously using (3.20).

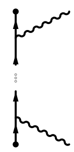
iii. And for this one, use both (3.19) and (3.20). ☐

**Theorem 3.8** (Ward identities).

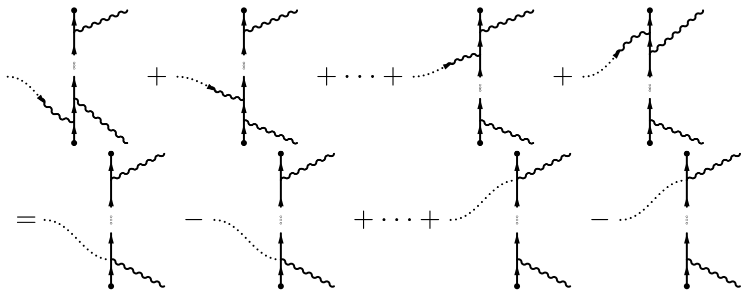
$$\Phi\left( \quad \right) = 0 \tag{3.44}$$




*Proof.* The proof is the same as in theorem 3.5, except if one takes a fermion line going through the graph,



we do not only insert the photon in the internal fermion edges, but also in the external ones. With lemmata 3.4 and 3.7, one sees that




The rest of the proof is the same. ☐

## 3.3 Parametric Representation

In analogy with equation (2.28), we define the parametric integrand in QED as

$$I(\Gamma) := \tfrac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}\, N(\Gamma) e^{-\sum_{e\in\Gamma^{[1]}} p_e^2 A_e}\,, \tag{3.45}$$

such that

$$\Phi(\Gamma) = \int \mathrm{d}\underline{A}_\Gamma\, I(\Gamma)\,. \tag{3.46}$$

The numerator $N(\Gamma)$ contains loop momenta, so theorem 2.24 cannot be applied here directly. In the following we will use a little trick using a suitable differential operator acting on the parametric integrand in scalar theory.

**Theorem 3.9.** Define the differential operator

$$\widehat{p}_e^{\mu_e} := -\frac{1}{2A_e}\frac{\partial}{\partial \xi_{e\mu_e}} \tag{3.47}$$

and let $\widehat{N}(\Gamma)$ be the differential operator obtained by replacing every momentum $p_e$ ($e \in \Gamma^{[1]}_{\text{-}}$) in $N(\Gamma)$ by $\widehat{p}_e$:

$$\widehat{N}(\Gamma) := N(\Gamma)\big|_{\forall e \in \Gamma^{[1]}: p_e \rightsquigarrow \widehat{p}_e} = \gamma(\Gamma) \prod_{e \in \Gamma^{[1]}_{\text{-}}} \widehat{p}_e^{\mu_e}\,, \tag{3.48}$$

Then, the parametric integrand in QED can be written as

$$I(\Gamma) = \widehat{N}(\Gamma) \frac{e^{-\varphi_\Gamma/\psi_\Gamma}}{\psi_\Gamma^2}\,. \tag{3.49}$$

*Proof.* First note that

$$\widehat{p}_e^\mu e^{-\sum_{e' \in \Gamma^{[1]}} p_{e'}^2 A_{e'}} = p_e^\mu e^{-\sum_{e' \in \Gamma^{[1]}} p_{e'}^2 A_{e'}}\,. \tag{3.50}$$

This is the reason we assigned an independent $\xi_e$ to each edge in definition 2.16, instead of using momentum conservation right away.

The integrand can be written as

$$I(\Gamma) = \widehat{N}(\Gamma) \tfrac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}\, e^{-\sum_{e \in \Gamma^{[1]}} p_e^2 A_e}\,.$$

Since every $p_e$ appears in $N(\Gamma)$ at most once, we do not have to take the Leibniz rule (the product rule) into account.

The object the differential operator $\widehat{N}(\Gamma)$ acts on is exactly the integrand in scalar theory (equation (2.28)), so we can apply theorem 2.24. □

**Remark 3.10.** i. Before we go to some examples, let us introduce some useful notations. The first one is:

$$\widetilde{p}_e^\mu := -\widehat{p}_e^\mu \varphi_\Gamma\,. \tag{3.51}$$

It is homogeneous of degree

$$\deg \widetilde{p}_e = l_\Gamma \tag{3.52}$$

in the Schwinger parameters. For one-scale graphs we can write

$$\widetilde{p}_e^\mu\big|_{\text{m.c.}} =: p^\mu \alpha_e\,. \tag{3.53}$$

$\varphi_\Gamma$ is quadratic in the momenta. This means that $\widehat{p}_e^\mu \widetilde{p}_f^\nu$ is always proportional to $g^{\mu\nu}$, so we write

$$\widehat{p}_{e_1}^\mu \widetilde{p}_{e_2}^\nu =: g^{\mu\nu} \beta_{e_1 e_2}\,, \tag{3.54}$$

where $\beta_{e_1e_2}$ is of degree

$$\deg \beta_{e_1 e_2} = l_\Gamma - 1\,. \tag{3.55}$$

Furthermore,

$$\widehat{p}_{e_1}^{\mu_1} \widehat{p}_{e_2}^{\mu_2} \widehat{p}_{e_3}^{\mu_3} \varphi_\Gamma = 0\,.$$

ii. Applying the differential operators and using the Leibniz rule, we see that the integrand can be written as

$$I(\Gamma) = \sum_{i=0}^{\lfloor \#\Gamma_{\rightarrow}^{[1]}/2 \rfloor} \frac{B_i(\Gamma)}{\psi^{\#\Gamma_{\rightarrow}^{[1]}-i+2}} e^{-\varphi_\Gamma/\psi_\Gamma} \,. \tag{3.56}$$

The index $i$ counts the number of times the Leibniz rule is applied. $B_i(\Gamma)$ is:

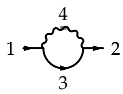
$$\begin{aligned} B_i(\Gamma) := \gamma(\Gamma) \tfrac{1}{2^i i!(k-2i)!} & \sum_{\text{perm. of } \Gamma_{\rightarrow}^{[1]}} g^{\mu_{e_1}\mu_{e_2}} \cdots g^{\mu_{e_{2i-1}}\mu_{e_{2i}}} \\ & \times \beta_{e_1 e_2} \cdots \beta_{e_{2i-1}e_{2i}} \widetilde{p}_{e_{2i+1}}^{\mu_{e_{2i+1}}} \cdots \widetilde{p}_{e_k}^{\mu_{e_k}} \,, \end{aligned} \tag{3.57}$$

where we labelled $\Gamma_{\rightarrow}^{[1]} = \{e_1, \ldots, e_k\}$. The combinatorial factor compensates double counting. $B_i(\Gamma)$ is of degree

$$\deg B_i(\Gamma) = l_\Gamma(\#\Gamma_{\rightarrow}^{[1]} - i) - i \tag{3.58}$$

in the Schwinger parameters.

**Example 3.11.** i. Take the graph




.

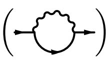
In example 2.23.i the Symanzik polynomials were given, but with this orientation

$$q_{34} = \xi_3 + \xi_4 \overset{\text{m.c.}}{=\!=} p \,.$$

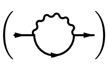
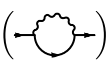
So:

$$\widetilde{p}_3^{\mu} = q_{34}^{\mu} A_4 \overset{\text{m.c.}}{=\!=} p^{\mu} A_4 \,.$$

The $\gamma$-structure is

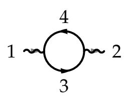
$\gamma\Big($  $\Big) = \gamma^{\mu_4}\gamma^{\mu_3}\gamma^{\mu_4} = -2\gamma^{\mu_3} \,.$

This gives us the parametric integrand

$I\Big($  $\Big) = \widehat{N}\Big($  $\Big) \dfrac{e^{-\varphi_\Gamma/\psi_\Gamma}}{\psi_\Gamma^2} = -2\gamma^{\mu_3}\, \widehat{p}_3^{\mu_3} \dfrac{e^{-\varphi_\Gamma/\psi_\Gamma}}{\psi_\Gamma^2}$

$$= -2\gamma^{\mu_3} \frac{\widetilde{p}_3^{\mu_3}}{\psi_\Gamma^3} e^{-\varphi_\Gamma/\psi_\Gamma} \overset{\text{m.c.}}{=\!=} -2\not{p} \frac{A_4}{(A_3+A_4)^3} e^{-\frac{p^2 A_3 A_4}{A_3+A_4}} \,.$$

ii. For the graph



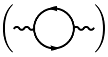
we have:

$$\widetilde{p}_3^{\mu} = q_{34}^{\mu} A_4 \overset{\text{m.c.}}{=\!=\!=} p^{\mu} A_4 \,,$$
$$\widetilde{p}_4^{\mu} = -q_{34}^{\mu} A_3 \overset{\text{m.c.}}{=\!=\!=} -p^{\mu} A_3 \,,$$

and

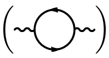
$$g^{\mu\nu}\beta_{34} = \widehat{p}_3^{\mu}\widetilde{p}_4^{\nu} = \tfrac{1}{2} g^{\mu\nu} \,.$$

The $\gamma$-structure is:

$\gamma\Big($  $\Big)$

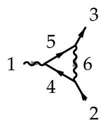
$$= \mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_4}\gamma^{\mu_2}\gamma^{\mu_3})$$
$$= 4(g^{\mu_1\mu_4}g^{\mu_2\mu_3} - g^{\mu_1\mu_2}g^{\mu_4\mu_3} + g^{\mu_1\mu_3}g^{\mu_4\mu_2}) \,.$$

Putting this together, we get for the integrand:

$I\Big($  $\Big)$

$$\begin{aligned}
&= \mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_4}\gamma^{\mu_2}\gamma^{\mu_3})\, \widehat{p}_3^{\mu_3}\widehat{p}_4^{\mu_4} \frac{e^{-\varphi_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2} \\
&= \mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_4}\gamma^{\mu_2}\gamma^{\mu_3})\Big(\frac{\widetilde{p}_3^{\mu_3}\widetilde{p}_4^{\mu_4}}{\psi_{\circ}^4} + \frac{g^{\mu_3\mu_4}\beta_{34}}{\psi_{\circ}^3}\Big) e^{-\varphi_{\circ}/\psi_{\circ}} \\
&= \mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_4}\gamma^{\mu_2}\gamma^{\mu_3})\Big( - \frac{p^{\mu_3}p^{\mu_4}A_3A_4}{(A_3+A_4)^4} \\
&\qquad + \frac{g^{\mu_3\mu_4}}{2(A_3+A_4)^3}\Big) e^{-\frac{p^2 A_3 A_4}{A_3+A_4}} \\
&= 4\Big((-2p^{\mu_1}p^{\mu_2} + g^{\mu_1\mu_2}p^2)\frac{A_3A_4}{(A_3+A_4)^4} \\
&\qquad - g^{\mu_1\mu_2}\frac{1}{(A_3+A_4)^3}\Big) e^{-\frac{p^2 A_3 A_4}{A_3+A_4}} \,.
\end{aligned}$$

iii. The Symanzik polynomials of the graph




are given in example 2.23.ii, but with this orientation

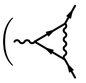
$$q_{45} = -\xi_4 + \xi_5 \overset{\text{m.c.}}{=\!=\!=} p_1 \,,$$
$$q_{46} = \xi_4 + \xi_6 \overset{\text{m.c.}}{=\!=\!=} p_2 \,,$$
$$q_{56} = \xi_5 + \xi_6 \overset{\text{m.c.}}{=\!=\!=} p_3 \,.$$

Then

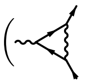
$$\widetilde{p}_4^{\mu} = -q_{45}^{\mu}A_5 + q_{46}^{\mu}A_6 \overset{\text{m.c.}}{=\!=\!=} -p_1^{\mu}A_5 + p_2^{\mu}A_6 \,,$$
$$\widetilde{p}_5^{\mu} = q_{45}^{\mu}A_4 + q_{56}^{\mu}A_6 \overset{\text{m.c.}}{=\!=\!=} p_1^{\mu}A_4 + p_3^{\mu}A_6 \,,$$

$$g^{\mu\nu}\beta_{45} = \tfrac{1}{2} g^{\mu\nu} \,.$$

We have

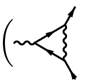
$\gamma\Big($  $\Big) = \gamma^{\mu_6}\gamma^{\mu_5}\gamma^{\mu_1}\gamma^{\mu_4}\gamma^{\mu_6} = -2\gamma^{\mu_4}\gamma^{\mu_1}\gamma^{\mu_5}\,.$

and the integrand is

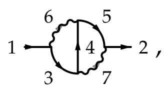
$I\Big($  $\Big)$

$$= -2\gamma^{\mu_4}\gamma^{\mu_1}\gamma^{\mu_5}\,\widehat{p}_4^{\mu_4}\widehat{p}_5^{\mu_5}\,\frac{e^{-\varphi_{\triangleleft}/\psi_{\triangleleft}}}{\psi_{\triangleleft}^2}$$
$$= -2\gamma^{\mu_4}\gamma^{\mu_1}\gamma^{\mu_5}\Big(\frac{\widetilde{p}_4^{\mu_4}\widetilde{p}_5^{\mu_5}}{\psi_{\triangleleft}^4} + \frac{g^{\mu_4\mu_5}\beta_{45}}{\psi_{\triangleleft}^3}\Big)e^{-\varphi_{\triangleleft}/\psi_{\triangleleft}}$$
$$= 2\Big(-\frac{\widetilde{\not p}_4\gamma^{\mu_1}\widetilde{\not p}_5}{\psi_{\triangleleft}^4} + \gamma^{\mu_1}\frac{1}{\psi_{\triangleleft}^3}\Big)e^{-\varphi_{\triangleleft}/\psi_{\triangleleft}}\,.$$

If we take the photon momentum $p_1 = 0$ and the fermion momenta $p_2 = p_3 = p$, it simplifies to

$I\Big($  $\Big)\Big|_{\substack{p_1=0\\ p_2=p_3=p}} = 2\Big(-\not p\gamma^{\mu_1}\not p\,\frac{A_6^2}{\psi_{\triangleleft}^4} + \gamma^{\mu_1}\frac{1}{\psi_{\triangleleft}^3}\Big)e^{-\frac{p^2(A_4+A_5)A_6}{\psi_{\triangleleft}}}\,.$

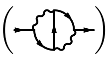
iv. And finally a slightly more complicated 2-loop example:

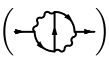



for which the Symanzik polynomials were given in example 2.23.iii. For this one, one has:

$$\widetilde{p}_3^{\mu} = q_{36}^{\mu}A_6(A_4+A_5+A_7) + q_{345}^{\mu}A_4A_5 + q_{347}^{\mu}A_4A_7$$
$$\stackrel{\text{m.c}}{=\!=} p^{\mu}\big(A_6(A_4+A_5+A_7) + A_4A_5\big) = p^{\mu}\alpha_3\,,$$
$$\widetilde{p}_4^{\mu} = -q_{345}^{\mu}A_3A_5 + q_{467}^{\mu}A_6A_7 - q_{347}^{\mu}A_3A_7 + q_{456}^{\mu}A_5A_6$$
$$\stackrel{\text{m.c}}{=\!=} p^{\mu}\big(-A_3A_5 + A_6A_7\big) = p^{\mu}\alpha_4\,,$$
$$\widetilde{p}_5^{\mu} = q_{57}^{\mu}A_7(A_3+A_4+A_6) + q_{345}^{\mu}A_3A_4 - q_{456}^{\mu}A_4A_6$$
$$\stackrel{\text{m.c}}{=\!=} p^{\mu}\big(A_7(A_3+A_4+A_6) + A_3A_4\big) = p^{\mu}\alpha_5\,,$$

$$g^{\mu\nu}\beta_{34} = \tfrac{1}{2}g^{\mu\nu}(A_5+A_7)\,,$$
$$g^{\mu\nu}\beta_{35} = -\tfrac{1}{2}g^{\mu\nu}A_4\,,$$
$$g^{\mu\nu}\beta_{45} = \tfrac{1}{2}g^{\mu\nu}(A_3+A_6)\,.$$

$\gamma\Big($  $\Big) = \gamma^{\mu_7}\gamma^{\mu_5}\gamma^{\mu_6}\gamma^{\mu_4}\gamma^{\mu_7}\gamma^{\mu_3}\gamma^{\mu_6} = -8g^{\mu_5\mu_3}\gamma^{\mu_4}\,.$

$$I\Big($$  $$\Big) = -8g^{\mu_5\mu_3}\gamma^{\mu_4}\,\widehat{p}_3^{\mu_3}\,\widehat{p}_4^{\mu_4}\,\widehat{p}_5^{\mu_5}\,\frac{e^{-\varphi_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2}$$

$$= -8g^{\mu_5\mu_3}\gamma^{\mu_4}\Big(\frac{\widetilde{p}_3^{\mu_3}\,\widetilde{p}_4^{\mu_4}\,\widetilde{p}_5^{\mu_5}}{\psi_{\circ}^5} + \frac{g^{\mu_3\mu_4}\beta_{34}\widetilde{p}_5^{\mu_5} + g^{\mu_3\mu_5}\beta_{35}\widetilde{p}_4^{\mu_4} + g^{\mu_4\mu_5}\beta_{45}\widetilde{p}_3^{\mu_3}}{\psi_{\circ}^4}\Big) \times e^{-\varphi_{\circ}/\psi_{\circ}}$$

$$\overset{\text{m.c.}}{=\!=} -8\not{p}\Big(p^2\frac{\alpha_3\alpha_4\alpha_5}{\psi_{\circ}^5} + \frac{\beta_{34}\alpha_5 + 4\beta_{35}\alpha_4 + \beta_{45}\alpha_3}{\psi_{\circ}^4}\Big) \times e^{-p^2\varphi'_{\circ}/\psi_{\circ}}\,.$$

**Remark 3.12.** Applying $\widehat{p}_e^{\mu}$ on equation (2.39), gives us:

$$\widetilde{p}_e^{\mu} = \sum_{\substack{C\in\mathscr{C}'^2_\Gamma\\ C\ni e}} \varepsilon_{Ce}q_C^{\mu}\Big(\prod_{e'\in C\setminus\{e\}} A_{e'}\Big)\psi_{\Gamma\setminus C}\,. \tag{3.59}$$

By applying another $\widehat{p}$, one can see that

$$\beta_{ee'} = -\tfrac{1}{2}\sum_{\substack{C\in\mathscr{C}'^2_\Gamma\\ C\ni e,e'}} \varepsilon_{Ce}\varepsilon_{Ce'}\Big(\prod_{e''\in C\setminus\{e,e'\}} A_{e''}\Big)\psi_{\Gamma\setminus C}\,, \tag{3.60}$$

for $e \neq e'$. For the case $e = e'$, one has

$$\beta_{ee} = -\frac{1}{2A_e}\sum_{\substack{C\in\mathscr{C}'^2_\Gamma\\ C\ni e}} \Big(\prod_{e'\in C\setminus\{e\}} A_{e'}\Big)\psi_{\Gamma\setminus C}\,. \tag{3.61}$$

The case $e = e'$ does not occur in QED in the Feynman gauge, but it does in other gauges (see the remark 3.14) and sQED and non-Abelian gauge theories (see the next two chapters). (Note that because of the $\frac{1}{A_e}$, $\beta_{ee}$ this is not a homogeneous polynomial, but a homogeneous rational function.)

**Remark 3.13.** Recall proposition 2.25.i. Using equation (3.56), we can see that in QED

$$\mathscr{I}(\Gamma) = \sum_{i=0}^{\lfloor \#\Gamma_{\leftharpoondown}^{[1]}/2\rfloor} \frac{B_i(\Gamma)}{\psi^{\#\Gamma_{\leftharpoondown}^{[1]}-i+2}}\int_0^\infty \mathrm{d}t\, t^{\#\Gamma^{[1]}-2l_\Gamma-i-1}e^{-t\varphi_\Gamma/\psi_\Gamma}\,. \tag{3.62}$$

With equations (3.58) and (3.27), we have:

$$\begin{aligned}\mathscr{I}(\Gamma) &= \sum_{i=0}^{\lfloor \#\Gamma_{\leftharpoondown}^{[1]}/2\rfloor} \frac{B_i(\Gamma)}{\psi^{\#\Gamma_{\leftharpoondown}^{[1]}-i+2}}\int_0^\infty \mathrm{d}t\, t^{(-\omega_\Gamma+\#\Gamma_{\leftharpoondown}^{[1]})/2-i-1}e^{-t\varphi_\Gamma/\psi_\Gamma}\\ &= \sum_{i=0}^{\lfloor \#\Gamma_{\leftharpoondown}^{[1]}/2\rfloor} \frac{B_i(\Gamma)\varphi_\Gamma^{(\omega_\Gamma-\#\Gamma_{\leftharpoondown}^{[1]})/2+i}}{\psi^{(\omega_\Gamma+\#\Gamma_{\leftharpoondown}^{[1]})/2+2}}\Gamma\big(\tfrac{1}{2}(-\omega_\Gamma+\#\Gamma_{\leftharpoondown}^{[1]})-i\big)\,.\end{aligned} \tag{3.63}$$

For an even number of internal fermions, the most divergent term of $\mathscr{I}(\Gamma)$ is at $i = \frac{1}{2}\#\Gamma^{[1]}_{\to}$; there we have $\Gamma(-\frac{1}{2}\omega_\Gamma)$, just like in remark 2.26. For odd $\#\Gamma^{[1]}_{\to}$, the most divergent term is at $i = \frac{1}{2}(\#\Gamma^{[1]}_{\to} - 1)$. Then we have $\Gamma(-\frac{1}{2}\omega_\Gamma + 1)$. So in this case the integral is a little bit less divergent than we would expect.

**Remark 3.14.** With a little bit more effort, we can make a parametric integrand for other gauges than the Feynman gauge. Recall (3.14). Instead of the replacement $p_e \rightsquigarrow \widehat{p}_e$, we replace

$$\frac{p_{e\mu_{h_1}} p_{e\mu_{h_2}}}{p_e^2} \rightsquigarrow A_e \widehat{p}_{e\mu_{h_1}} \widehat{p}_{e\mu_{h_2}} + \tfrac{1}{2} g_{\mu_{h_1}\mu_{h_2}}$$

to obtain $\widehat{N}(\Gamma)$:

$$\begin{aligned} \widehat{N}(\Gamma) = \gamma(\Gamma)\Big( \prod_{\{h_1,h_2\}=e\in\Gamma^{[1]}_{\sim}} \big(\tfrac{1}{2}(1+\alpha) g_{\mu_{h_1}\mu_{h_2}} - (1-\alpha) A_e \widehat{p}_{e\mu_{h_1}} \widehat{p}_{e\mu_{h_2}}\big)\Big) \\ \times \Big( \prod_{e\in\Gamma^{[1]}_{\to}} \widehat{p}_e^{\mu_e} \Big) . \end{aligned} \tag{3.64}$$

*Proof.*

$$\begin{aligned} \int_0^\infty \mathrm{d}A_e (A_e \widehat{p}_{e\mu_{h_1}} \widehat{p}_{e\mu_{h_2}} + \tfrac{1}{2} g_{\mu_{h_1}\mu_{h_2}}) e^{-p_e^2 A_e} &= p_{e\mu_{h_1}} p_{e\mu_{h_2}} \int_0^\infty \mathrm{d}A_e\, A_e e^{-p_e^2 A_e} \\ &= \frac{p_{e\mu_{h_1}} p_{e\mu_{h_2}}}{(p_e^2)^2} . \end{aligned}$$

We used that

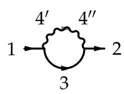
$$\widehat{p}_e^\mu p_e^\nu = -\frac{1}{2A_e} g^{\mu\nu} , \tag{3.65}$$

so the term from the Leibniz rule vanishes against $\frac{1}{2} g^{\mu_{h_1}\mu_{h_2}}$, and

$$\int_0^\infty \mathrm{d}A_e\, A_e e^{-p_e^2 A_e} = \frac{1}{(p_e^2)^2} . \tag{3.66}$$

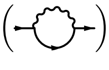
□

**Example 3.15.** Let us go back to example 3.11.i. We label the two half-edges of the photon edge 4 with $4'$ and $4''$:

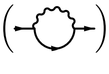


.

The Feynman rules give the numerator

$N\Big($  $\Big) = \gamma^{\mu_{4''}} \gamma^{\mu_3} \gamma^{\mu_{4'}} \Big( g^{\mu_{4'}\mu_{4''}} - (1-\alpha) \dfrac{p_4^{\mu_{4'}} p_4^{\mu_{4''}}}{p_4^2} \Big) p_3^{\mu_3} .$

The corresponding differential operator is then

$\widehat{N}\Big($  $\Big) = \gamma^{\mu_{4''}} \gamma^{\mu_3} \gamma^{\mu_{4'}} \Big( \tfrac{1}{2}(1+\alpha) g^{\mu_{4'}\mu_{4''}} - (1-\alpha) A_4 \widehat{p}_4^{\mu_{4'}} \widehat{p}_4^{\mu_{4''}} \Big) \widehat{p}_3^{\mu_3} .$

Whith this, we get for the interand

$$
\begin{aligned}
I\Big(\!-\!\bigcirc\!-\!\Big) &= \widehat{N}\Big(\!-\!\bigcirc\!-\!\Big)\frac{e^{-\varphi_{\bigcirc}/\psi_{\bigcirc}}}{\psi^2_{\bigcirc}} \\
&\overset{\text{m.c.}}{=\!=} \gamma^{\mu_{4''}}\gamma^{\mu_3}\gamma^{\mu_{4'}}\Big(\tfrac{1}{2}(1+\alpha)g^{\mu_{4'}\mu_{4''}}p^{\mu_3}\frac{A_4}{(A_3+A_4)^3} \\
&\qquad -(1-\alpha)p^{\mu_{4'}}p^{\mu_{4''}}p^{\mu_3}\frac{A_3^2A_4^2}{(A_3+A_4)^5} \\
&\qquad +\tfrac{1}{2}(1-\alpha)(g^{\mu_{4'}\mu_{4''}}p^{\mu_3}+g^{\mu_{4'}\mu_3}p^{\mu_{4''}}+g^{\mu_{4''}\mu_3}p^{\mu_{4'}})\frac{A_3A_4}{(A_3+A_4)^4}\Big) \\
&\quad \times e^{-\frac{p^2A_3A_4}{A_3+A_4}} \\
&= \not{p}\Big(\frac{(2-4\alpha)A_3A_4-(1+\alpha)A_4^2}{(A_3+A_4)^4} \\
&\quad -p^2(1-\alpha)\frac{A_3^2A_4^2}{(A_3+A_4)^5}\Big)e^{-\frac{p^2A_3A_4}{A_3+A_4}}\,.
\end{aligned}
$$

For $\alpha = 1$ we indeed get back the result of example 3.11.i.

### 3.3.1 A Ward-Takahashi Identity Revisited

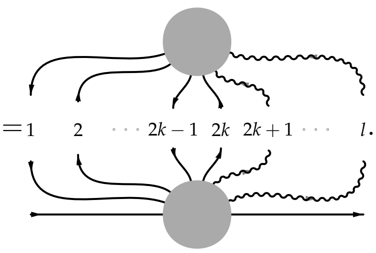
In this subsection we give an alternative proof of the Ward identity in corollary 3.6.iii using the parametric representation.

**Lemma 3.16.** Let $\Gamma$ be a fermion propagator graph. Then:

$$\frac{\mathrm{d}I(\Gamma)|_{\text{m.c.}}}{\mathrm{d}p_\mu} = \sum_{e\in\Gamma^{[1]}_{\to}}\frac{\partial I(\Gamma)}{\partial\xi_{e\mu}}\Big|_{\text{m.c.}}\,. \tag{3.67}$$

*Proof.* Let $C \in \mathscr{C}^{\prime 2}_{\Gamma}$ and label $C = \{1,\dots,l\}$. Two things can happen:

- Assume that $C$ is such that $\Gamma\backslash C$ is of the form

$\Gamma\backslash C =$ 


Then

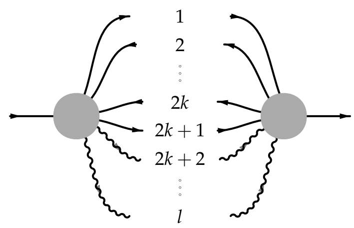
$$q_C = \xi_1 - \xi_2 + \cdots + \xi_{2k+1} - \xi_{2k} + \xi_{2k+1} + \cdots + \xi_l \overset{\text{m.c.}}{=\!=} 0\,,$$

and

$$\sum_{e\in\Gamma^{[1]}_{\to}}\frac{\partial q_C^\nu}{\partial\xi_{e\mu}} = \sum_{e=1}^{2k}\frac{\partial q_C^\nu}{\partial\xi_{e\mu}} = g^{\mu\nu} - g^{\mu\nu} + \cdots + g^{\mu\nu} - g^{\mu\nu} = 0\,.$$

So
$$\frac{\mathrm{d}q_C^\nu|_{\text{m.c.}}}{\mathrm{d}p_\mu} = 0 = \sum_{e\in\Gamma^{[1]}_{\to}} \frac{\partial q_C^\nu}{\partial \xi_{e\mu}} \, .$$

- Assume that $C$ is such that $\Gamma\backslash C$ is of the form

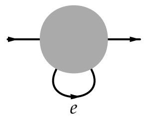
$$\Gamma\backslash C =$$



Then
$$q_C = \xi_1 - \xi_2 + \cdots - \xi_{2k} + \xi_{2k+1} + \xi_{2k+2} + \cdots + \xi_l \overset{\text{m.c.}}{=\!=} p \, ,$$
and
$$\sum_{e\in\Gamma^{[1]}_{\to}} \frac{\partial q_C^\nu}{\partial \xi_{e\mu}} = \sum_{e=1}^{2k+1} \frac{\partial q_C^\nu}{\partial \xi_{e\mu}} = g^{\mu\nu} - g^{\mu\nu} + \cdots - g^{\mu\nu} + g^{\mu\nu} = g^{\mu\nu} \, .$$
So
$$\frac{\mathrm{d}q_C^\nu|_{\text{m.c.}}}{\mathrm{d}p_\mu} = g^{\mu\nu} = \sum_{e\in\Gamma^{[1]}_{\to}} \frac{\partial q_C^\nu}{\partial \xi_{e\mu}} \, .$$

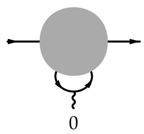
So for any $C \in \mathscr{C}'^2_\Gamma$:
$$\frac{\mathrm{d}q_C^\nu|_{\text{m.c.}}}{\mathrm{d}p_\mu} = \sum_{e\in\Gamma^{[1]}_{\to}} \frac{\partial q_C^\nu}{\partial \xi_{e\mu}} \, .$$
From this
$$\frac{\mathrm{d}\varphi_\Gamma|_{\text{m.c.}}}{\mathrm{d}p_\mu} = \sum_{e\in\Gamma^{[1]}_{\to}} \frac{\partial \varphi_\Gamma}{\partial \xi_{e\mu}}\Big|_{\text{m.c.}} \, ,$$
and
$$\frac{\mathrm{d}\tilde{p}_e^\nu|_{\text{m.c.}}}{\mathrm{d}p_\mu} = \sum_{e\in\Gamma^{[1]}_{\to}} \frac{\partial \tilde{p}_e^\nu}{\partial \xi_{e\mu}} \, .$$
follow. □

**Lemma 3.17.** Let $\Gamma$ be a fermion propagator graph. Then:
$$\frac{\partial \Phi(\Gamma)|_{\text{m.c.}}}{\partial p_{\mu_0}} = - \sum_{e\in\Gamma^{[1]}_{\to}} \Phi(\Gamma_{(e)})\big|_{\text{m.c.}} \, , \tag{3.68}$$

where $\Gamma_{(e)}$ is the graph one gets by inserting an external photon edge (labelled 0) in fermion edge $e \in \Gamma^{[1]}_{\rightarrow}$: for a $\Gamma$ of the form

$\Gamma =$  ,

$\Gamma_{(e)}$ looks like

$\Gamma_{(e)} =$  .

The momentum of this photon is $p_0 = 0$; so momentum is conserved.

*Proof.* Integrating lemma 3.16 over all Schwinger parameters yields

$$\frac{\partial \Phi(\Gamma)|_{\text{m.c.}}}{\partial p_{\mu_0}} = \sum_{e \in \Gamma^{[1]}_{\rightarrow}} \frac{\partial \Phi(\Gamma)}{\partial \xi_{e\mu_0}}\Big|_{\text{m.c.}} .$$

From the Clifford relation (3.4) follows

$$\frac{\partial}{\partial \xi_{e\mu_0}} \frac{\not{p}}{p_e^2} = \frac{p_e^2 \gamma^{\mu_0} - 2 p_e^{\mu_0} \not{p}_e}{(p_e^2)^2} = -\frac{\not{p}_e \gamma^{\mu_0} \not{p}_e}{(p_e^2)^2} ,$$

so

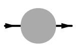
$$\frac{\partial \Phi(\Gamma)}{\partial \xi_{e\mu_0}} = -\Phi(\Gamma_{(e)}) .$$ □

Corollary 3.6 follows from this by summing over all 1PI fermion propagator graphs at loop order $l$.

## 3.4 Renormalization

The superficially divergent graphs are given in equation (3.29). From Furry's theorem (lemma 3.3) we know that the 3-photon Green's function vanishes. Furthermore, because of the Ward identity (theorem 3.5), the 4-photon function is finite, despite the superficial degree of divergence being 0.* This is why we can regard the fermion and photon propagator graphs and the vertex graphs to be the only divergent ones.

**Definition 3.18.** Let $\Gamma$ be a fermion propagator graph:

$\Gamma =$  .

*See [15], around equation (10.9).

The integrand $I(\Gamma)$ is proportional to $\not{p}$ (see equation (3.72)):

$$I(\Gamma) =: \not{p} I'(\Gamma) \tag{3.69}$$

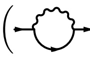
Let

$$I^{\circ}(\Gamma) := I'(\Gamma)\big|_{p^2=\mu^2} . \tag{3.70}$$

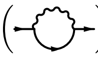
Then, the overall divergence of $\Gamma$ is renormalized as follows:

$$I^{\overline{\text{ren}}}(\Gamma) = I - \not{p} I^{\circ}(\Gamma) . \tag{3.71}$$

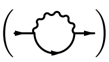
**Example 3.19.** In example 3.15, the integrand for the 1-loop fermion propagator graph was computed for a general covariant gauge. The renormalized integrand is:

$$I^{\text{ren}}\Big($$  $$\Big) = \not{p}\frac{(2-4\alpha)A_3A_4 - (1+\alpha)A_4^2}{(A_3+A_4)^4}\Big(e^{-\frac{p^2A_3A_4}{A_3+A_4}} - e^{-\frac{\mu^2A_3A_4}{A_3+A_4}}\Big)$$
$$- \not{p}(1-\alpha)\frac{A_3^2A_4^2}{(A_3+A_4)^5}\Big(p^2e^{-\frac{p^2A_3A_4}{A_3+A_4}} - \mu^2e^{-\frac{\mu^2A_3A_4}{A_3+A_4}}\Big) .$$

Integrating $t$ gives (equation (2.43)):

$$\mathscr{I}^{\text{ren}}\Big($$  $$\Big) = \not{p}\frac{(2-4\alpha)a_3a_4 - (1+\alpha)a_4^2}{(a_3+a_4)^4}\int_0^\infty \frac{\mathrm{d}t}{t}\Big(e^{-t\frac{p^2a_3a_4}{a_3+a_4}} - e^{-t\frac{\mu^2a_3a_4}{a_3+a_4}}\Big)$$
$$- \not{p}(1-\alpha)\frac{a_3^2a_4^2}{(a_3+a_4)^5}\int_0^\infty \mathrm{d}t\Big(p^2e^{-t\frac{p^2a_3a_4}{a_3+a_4}} - \mu^2e^{-t\frac{\mu^2a_3a_4}{a_3+a_4}}\Big)$$
$$= -\not{p}\frac{(2-4\alpha)a_3a_4 - (1+\alpha)a_4^2}{(a_3+a_4)^4}\ln\frac{p^2}{\mu^2} .$$

The amplitude of this graph, and hence the 1-loop Green's function, is then

$$\Sigma_{(1)}(p) = \Phi^{\text{ren}}\Big($$  $$\Big) = -\not{p}\int_0^\infty \mathrm{d}a_3\frac{(2-4\alpha)a_3 - (1+\alpha)}{(a_3+1)^4}\ln\frac{p^2}{\mu^2} = \alpha\not{p}\ln\frac{p^2}{\mu^2} .$$

**Remark 3.20.** From lemma 3.1.iv, it follows that for fermion propagator graphs $\Gamma$ $\#\Gamma^{[1]}_{-} = 2l_\Gamma - 1$. Now go back to equation (3.56): $i$ runs from 0 to $l_\Gamma - 1$. $\Gamma$ is 1-scale, and there are $2l_\Gamma - 2i - 1$ powers of $p$ in $B_i(\Gamma)$, so $B_i(\Gamma)$ is of the form:

$$B_i(\Gamma) =: \not{p}(p^2)^{l_\Gamma - i - 1}B'_i(\Gamma) , \tag{3.72}$$

where $B'_i(\Gamma)$ contains no momenta. So:

$$I^{\overline{\text{ren}}}(\Gamma) = I - \not{p}I^{\circ}(\Gamma)$$
$$= \not{p}\sum_{i=0}^{l_\Gamma-1}\frac{B'_i(\Gamma)}{\psi^{2l_\Gamma-i+1}}\big((p^2)^{l_\Gamma-i-1}e^{-p^2\varphi'_\Gamma/\psi_\Gamma} - (\mu^2)^{l_\Gamma-i-1}e^{-\mu^2\varphi'_\Gamma/\psi_\Gamma}\big) .$$

With equation (2.43), one has:

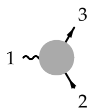
$$\begin{aligned}\mathscr{I}^{\overline{\text{ren}}}(\Gamma) = \not{p} \sum_{i=0}^{l_\Gamma - 1} \frac{B_i'(\Gamma)}{\psi^{2l_\Gamma - i + 1}} \int_0^\infty \mathrm{d}t\, t^{l_\Gamma - i - 2} \big( (p^2)^{l_\Gamma - i - 1} e^{-tp^2 \varphi_\Gamma' / \psi_\Gamma} \\ - (\mu^2)^{l_\Gamma - i - 1} e^{-t\mu^2 \varphi_\Gamma' / \psi_\Gamma} \big) \\ = -\not{p} \frac{B'_{l_\Gamma - 1}(\Gamma)}{\psi^{l_\Gamma + 2}} \ln \frac{p^2}{\mu^2}\,,\end{aligned} \tag{3.73}$$

where we used

$$t^{\#\Gamma^{[1]} - 1 + \deg B_i(\Gamma) - (2l_\Gamma - i + 1)\deg \psi_\Gamma} = t^{l_\Gamma - i - 2}\,.$$

Note that it simplifies to only one remaining term; the terms with $i < l_\Gamma - 1$ all vanish.

**Definition 3.21.** Let $\Gamma$ be a vertex graph:

$\Gamma =$  .



At $p_1 = 0$ and $p_2 = p_3 = p$, the integrand is of the form

$$I(\Gamma) = \gamma^\mu I'(\Gamma) + \not{p} p^\mu I''(\Gamma)\,. \tag{3.74}$$

We subtract for the overall divergence as follows:

$$I^{\overline{\text{ren}}}(\Gamma) = I(\Gamma) - \gamma^{\mu_1} I^\circ(\Gamma)\,, \tag{3.75}$$

where

$$I^\circ(\Gamma) = I'(\Gamma)\big|_{p^2 = \mu^2}\,. \tag{3.76}$$

This is motivated as follows: Recall the definitions (3.35) and (3.37). These are of the form

$$\Gamma^\mu_{(l)}(p,p) = \gamma^\mu \Gamma'(p^2) + \not{p} p^\mu \Gamma''(p^2) \qquad \text{and} \qquad \Sigma_{(l)}(p) = \not{p}\Sigma'_{(l)}(p^2)\,. \tag{3.77}$$

Then, the Ward-Takahashi identity (3.40) can be written as

$$\begin{cases} \Gamma'(p^2) = -\Sigma'_{(l)}(p^2)\,, \\ \Gamma''(p^2) = -2 \dfrac{\mathrm{d}\Sigma'_{(l)}(p^2)}{\mathrm{d}p^2}\,. \end{cases} \tag{3.78}$$

With the renormalization scheme given in definitions 3.18 and 3.21, one has

$$\Sigma^{\overline{\text{ren}}\prime}_{(l)}(p)^2 = \Sigma'_{(l)}(p^2) - \Sigma'_{(l)}(\mu^2) = -\Gamma'(p^2) + \Gamma'(\mu^2) = -\Gamma^{\overline{\text{ren}}\prime}(p^2) \tag{3.79}$$

and

$$-2\frac{\mathrm{d}\Sigma^{\overline{\text{ren}}\prime}_{(l)}(p^2)}{\mathrm{d}p^2} = -2\frac{\mathrm{d}\Sigma'_{(l)}(p^2)}{\mathrm{d}p^2} = \Gamma''(p^2) = \Gamma^{\overline{\text{ren}}\prime\prime}(p^2)\,. \tag{3.80}$$

So this scheme is compatible with the Ward identities.*

**Example 3.22.** Continue with example 3.11.iii: with the Clifford relation (3.4), the integrand with $p_1 = 0$ can be written as

$$I\Big(\vartriangleleft\Big)\Big|_{\substack{p_1=0\\ p_2=p_3=p}} = 2\Big(p^2\gamma^{\mu_1}\frac{A_6^2}{\psi_{\vartriangleleft}^4} - 2\not{p}\,p^{\mu_1}\frac{A_6^2}{\psi_{\vartriangleleft}^4} + \gamma^{\mu_1}\frac{1}{\psi_{\vartriangleleft}^3}\Big)e^{-\frac{p^2(A_4+A_5)A_6}{\psi_{\vartriangleleft}}}\,,$$

so the counter-term is

$$I^{\circ}\Big(\vartriangleleft\Big) = 2\Big(\mu^2\frac{A_6^2}{\psi_{\vartriangleleft}^4} + \frac{1}{\psi_{\vartriangleleft}^3}\Big)e^{-\frac{\mu^2(A_4+A_5)A_6}{\psi_{\vartriangleleft}}}\,,$$

and the renormalized integrand is

$$\begin{aligned}I^{\mathrm{ren}}\Big(\vartriangleleft\Big) &= -2\frac{1}{\psi_{\vartriangleleft}^4}\big(\widetilde{\not{p}}_4\gamma^{\mu_1}\widetilde{\not{p}}_5 e^{-\varphi_{\vartriangleleft}/\psi_{\vartriangleleft}} + \mu^2 A_6^2 e^{-\frac{\mu^2(A_4+A_5)A_6}{\psi_{\vartriangleleft}}}\big)\\ &\quad + 2\gamma^{\mu_1}\frac{1}{\psi_{\vartriangleleft}^3}\big(e^{-\varphi_{\vartriangleleft}/\psi_{\vartriangleleft}} - e^{-\frac{\mu^2(A_4+A_5)A_6}{\psi_{\vartriangleleft}}}\big)\,.\end{aligned}$$

With equation (2.43) the $t$-integration can be done:

$$\begin{aligned}\mathscr{I}^{\mathrm{ren}}\Big(\vartriangleleft\Big) &= -2\frac{1}{\psi_{\vartriangleleft}^4}\int_0^\infty \mathrm{d}t\big(\widetilde{\not{p}}_4\gamma^{\mu_1}\widetilde{\not{p}}_5 e^{-t\varphi_{\vartriangleleft}/\psi_{\vartriangleleft}} + \gamma^{\mu_1}\mu^2 a_6^2 e^{-t\frac{\mu^2(a_4+a_5)a_6}{\psi_{\vartriangleleft}}}\big)\\ &\quad + 2\gamma^{\mu_1}\frac{1}{\psi_{\vartriangleleft}^3}\int_0^\infty \frac{\mathrm{d}t}{t}\big(e^{-t\varphi_{\vartriangleleft}/\psi_{\vartriangleleft}} - e^{-t\frac{\mu^2(a_4+a_5)a_6}{\psi_{\vartriangleleft}}}\big)\\ &= -2\Big(\frac{\widetilde{\not{p}}_4\gamma^{\mu_1}\widetilde{\not{p}}_5}{\varphi_{\vartriangleleft}} + \gamma^{\mu_1}\frac{a_6}{a_4+a_5}\Big)\frac{1}{\psi_{\vartriangleleft}^3}\\ &\quad - 2\gamma^{\mu_1}\frac{1}{\psi_{\vartriangleleft}^3}\ln\frac{\varphi_{\vartriangleleft}}{\mu^2(a_4+a_5)a_6}\,.\end{aligned}$$

To make life easier, we make the graph 1-scale by taking $p_1 = 0$ and $p_2 = p_3 = p$. Then, $\mathscr{I}$ simplifies to

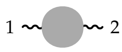
$$\begin{aligned}\mathscr{I}^{\mathrm{ren}}\Big(\vartriangleleft\Big) &= -2\Big(\frac{\not{p}\gamma^{\mu_1}\not{p}}{p^2} + \gamma^{\mu_1}\Big)\frac{a_6}{\psi_{\vartriangleleft}^3\,(a_4+a_5)} - 2\gamma^{\mu_1}\frac{1}{\psi_{\vartriangleleft}^3}\ln\frac{p^2}{\mu^2}\,.\\ &= -4\frac{\not{p}p^{\mu}}{p^2}\frac{a_6}{\psi_{\vartriangleleft}^3\,(a_4+a_5)} - 2\gamma^{\mu_1}\frac{1}{\psi_{\vartriangleleft}^3}\ln\frac{p^2}{\mu^2}\,a\end{aligned}$$

so the amplitude, and hence the 1-loop Green's function, is

$$\Gamma_{(1)}^{\mu_1}(p) = \Phi^{\mathrm{ren}}\Big(\vartriangleleft\Big) = -2\frac{\not{p}p^{\mu}}{p^2} - \gamma^{\mu_1}\ln\frac{p^2}{\mu^2}\,.$$

*In [16] it is discussd that it also works with subdivergences.

**Definition 3.23.** For a photon propagator graph

$\Gamma =$  $,$

the integrand is of the form

$$I(\Gamma) = p^{\mu_1} p^{\mu_2} I'(\Gamma) + p^2 g^{\mu_1\mu_2} I'(\Gamma)\,. \tag{3.81}$$

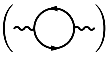
Up to subdivergences, we define the renormalized integrand as

$$\begin{aligned} I^{\overline{\text{ren}}}(\Gamma) = p^{\mu_1} p^{\mu_2} \big( J'(\Gamma) - J'(\Gamma)\big|_{p^2=\mu^2} \big) \\ + p^2 g^{\mu_1\mu_2} \big( J''(\Gamma) - J''(\Gamma)\big|_{p^2=\mu^2} \big)\,, \end{aligned} \tag{3.82}$$

where

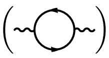
$$J(\Gamma) := J(\Gamma) - J(\Gamma)\big|_{p^2=0}\,. \tag{3.83}$$

**Example 3.24.** Continue with example 3.11.ii: The renormalized integrand is

$I^{\text{ren}}\Big($  $\Big)$

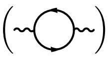
$$= 4\frac{(-2p^{\mu_1}p^{\mu_2} + g^{\mu_1\mu_2}p^2)A_3A_4}{(A_3+A_4)^4}\big(e^{-\frac{p^2A_3A_4}{A_3+A_4}} - e^{-\frac{\mu^2A_3A_4}{A_3+A_4}}\big) - 4\frac{p^2 g^{\mu_1\mu_2}}{(A_3+A_4)^3}\Big(\frac{1}{p^2}\big(e^{-\frac{p^2A_3A_4}{A_3+A_4}} - 1\big) - \frac{1}{\mu^2}\big(e^{-\frac{\mu^2A_3A_4}{A_3+A_4}} - 1\big)\Big)\,.$$

Do the $t$-integration:

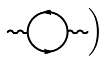
$\mathscr{I}^{\text{ren}}\Big($  $\Big)$

$$\begin{aligned} &= 4\frac{(-2p^{\mu_1}p^{\mu_2} + g^{\mu_1\mu_2}p^2)a_3a_4}{(a_3+a_4)^4}\int_0^\infty \frac{\mathrm{d}t}{t}\big(e^{-t\frac{p^2a_3a_4}{a_3+a_4}} - e^{-t\frac{\mu^2a_3a_4}{a_3+a_4}}\big) \\ &\quad - 4\frac{p^2g^{\mu_1\mu_2}}{(a_3+a_4)^3}\int_0^\infty \frac{\mathrm{d}t}{t^2}\Big(\frac{1}{p^2}\big(e^{-t\frac{p^2a_3a_4}{a_3+a_4}} - 1\big) \\ &\quad - \frac{1}{\mu^2}\big(e^{-t\frac{\mu^2a_3a_4}{a_3+a_4}} - 1\big)\Big) \\ &= 8(p^{\mu_1}p^{\mu_2} - g^{\mu_1\mu_2}p^2)\frac{a_3a_4}{(a_3+a_4)^4}\ln\frac{p^2}{\mu^2}\,. \end{aligned}$$

Here we can see already that the amplitude of this graph is transversal. The amplitude is:

$\Phi^{\text{ren}}\Big($  $\Big) = \frac{4}{3}(p^{\mu_3}p^{\mu_4} - p^2g^{\mu_3\mu_4})\ln\frac{p^2}{\mu^2}\,.$

For the 1-loop Green's function, we have to include a minus sign for the fermion loop (equation 3.26):

$\Pi^{\mu_1\mu_2}_{(1)}(p) = \Phi^{\text{ren}}\Big( -$  $\Big) = \frac{4}{3}(-p^{\mu_3}p^{\mu_4} + p^2g^{\mu_3\mu_4})\ln\frac{p^2}{\mu^2}\,.$

**Remark 3.25.** For a photon propagator graph $\Gamma$, there is a similar simplification as we have seen for fermion propagators in remark 3.20. From lemma 3.1.iv follows that $\#\Gamma_-^{[1]} = 2l_\Gamma$, so $i$ in equation (3.56) runs from 0 to $l_\Gamma$. There are $2l_\Gamma - 2i$ powers of $p$ in $B_i(\Gamma)$, so because of Lorentz covariance, $B_i(\Gamma)$ has to be of the form:

$$B_i(\Gamma) =: p^{\mu_1} p^{\mu_2} (p^2)^{l_\Gamma - i - 1} B_i'(\Gamma) + g^{\mu_1\mu_2} (p^2)^{l_\Gamma - i} B_i''(\Gamma) \tag{3.84}$$

where $B_i'(\Gamma)$ and $B_i''(\Gamma)$ contain no momenta. Note that $B_{l_\Gamma}'(\Gamma) = 0$. The integrand is now:

$$\begin{aligned} I(\Gamma) = &\sum_{i=0}^{l_\Gamma - 1} (p^2)^{l_\Gamma - i - 1} \frac{p^{\mu_1} p^{\mu_2} B_i'(\Gamma) + p^2 g^{\mu_1\mu_2} B_i''(\Gamma)}{\psi^{2l_\Gamma - i + 2}} e^{-p^2 \varphi_\Gamma' / \psi_\Gamma} \\ &+ g^{\mu_1\mu_2} \frac{B_{l_\Gamma}''(\Gamma)}{\psi^{l_\Gamma + 2}} e^{-p^2 \varphi_\Gamma' / \psi_\Gamma} . \end{aligned}$$

Subtraction for the overall divergence gives:

$$\begin{aligned} I^{\overline{\text{ren}}}(\Gamma) = &\sum_{i=0}^{l_\Gamma - 1} \frac{p^{\mu_1} p^{\mu_2} B_i'(\Gamma) + p^2 g^{\mu_1\mu_2} B_i''(\Gamma)}{\psi^{2l_\Gamma - i + 2}} \\ &\times \left( (p^2)^{l_\Gamma - i - 1} e^{-p^2 \varphi_\Gamma' / \psi_\Gamma} - (\mu^2)^{l_\Gamma - i - 1} e^{-\mu^2 \varphi_\Gamma' / \psi_\Gamma} \right) \\ &+ g^{\mu_1\mu_2} \frac{B_{l_\Gamma}''(\Gamma)}{\psi^{l_\Gamma + 2}} \left( e^{-p^2 \varphi_\Gamma' / \psi_\Gamma} - 1 - \frac{p^2}{\mu^2} \left( e^{-\mu^2 \varphi_\Gamma' / \psi_\Gamma} - 1 \right) \right) . \end{aligned}$$

Performing the $t$-integration, with

$$t^{\#\Gamma^{[1]} - 1 + \deg B_i(\Gamma) - (2l_\Gamma - i + 2) \deg \psi_\Gamma} = t^{l_\Gamma - i - 2} ,$$

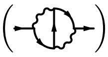
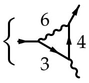
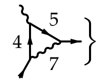
one obtains:

$$\begin{aligned} \mathscr{I}^{\overline{\text{ren}}}(\Gamma) = &\sum_{i=0}^{l_\Gamma - 1} \frac{p^{\mu_1} p^{\mu_2} B_i'(\Gamma) + p^2 g^{\mu_1\mu_2} B_i''(\Gamma)}{\psi^{2l_\Gamma - i + 2}} \int_0^\infty \mathrm{d}t \, t^{l_\Gamma - i - 2} \\ &\times \left( (p^2)^{l_\Gamma - i - 1} e^{-tp^2 \varphi_\Gamma' / \psi_\Gamma} - (\mu^2)^{l_\Gamma - i - 1} e^{-t\mu^2 \varphi_\Gamma' / \psi_\Gamma} \right) \\ &+ g^{\mu_1\mu_2} \frac{B_{l_\Gamma}''(\Gamma)}{\psi^{l_\Gamma + 2}} \int_0^\infty \frac{\mathrm{d}t}{t^2} \\ &\times \left( e^{-tp^2 \varphi_\Gamma' / \psi_\Gamma} - 1 - \frac{p^2}{\mu^2} \left( e^{-t\mu^2 \varphi_\Gamma' / \psi_\Gamma} - 1 \right) \right) \\ = &\frac{-p^{\mu_1} p^{\mu_2} B_{l_\Gamma - 1}'(\Gamma) + p^2 g^{\mu_1\mu_2} \left( - B_{l_\Gamma - 1}''(\Gamma) + B_{l_\Gamma}''(\Gamma) \varphi_\Gamma' \right)}{\psi^{l_\Gamma + 3}} \\ &\times \ln \frac{p^2}{\mu^2} . \end{aligned} \tag{3.85}$$

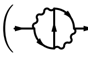
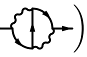
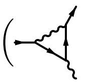
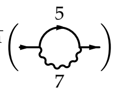
Only three terms are left.

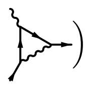
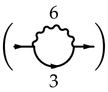
To conclude this chapter, we give an example with subdivergences:

**Example 3.26.** Continue with example 3.11.iv. For the renormalization we use the forest formula 2.62. The forests for our graph (only the ones that do not contain the graph itself) are

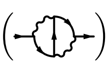
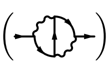
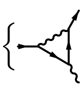
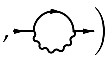
$\mathscr{F}'\Big($  $\Big) = \Big\{\emptyset, \Big\{$  $\Big\}, \Big\{$  $\Big\}\Big\},$

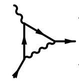
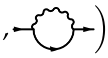
so with 2.62 we have for the renormalized integrand

$I^{\mathrm{ren}}\Big($  $\Big) = I^{\overline{\mathrm{ren}}}\Big($  $\Big) - I^{\circ}\Big($  $\Big) I^{\overline{\mathrm{ren}}}\Big($  $\Big)$

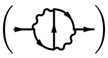
$- I^{\circ}\Big($  $\Big) I^{\overline{\mathrm{ren}}}\Big($  $\Big).$

Do the $t$-integration:

$\mathscr{I}^{\mathrm{ren}}\Big($  $\Big) = \mathscr{I}^{\overline{\mathrm{ren}}}\Big($  $\Big) - M\Big(\Big\{$  $\Big\},$  $\Big)$

$- M\Big(\Big\{$  $\Big\},$  $\Big)$

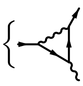
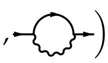
The first term is (see remark 3.20):

$\mathscr{I}^{\overline{\mathrm{ren}}}\Big($  $\Big) = 8\frac{\beta_{34}\alpha_5 + 4\beta_{35}\alpha_4 + \beta_{45}\alpha_3}{\psi^4_{\bigcirc}} \ln\frac{p^2}{\mu^2}.$

For the second and third term, we used the notation

$$M(f, \Gamma/f) = \int_0^\infty \mathrm{d}t\, t^{\#\Gamma^{[1]}-1} I^{\circ}(f) I^{\overline{\mathrm{ren}}}(\Gamma/f)\big|_{\underline{A}=t\underline{a}}.$$

The second term turns out to be

$M\Big(\Big\{$  $\Big\},$  $\Big)$

$$= -4\not{p}\frac{a_6^2 a_7}{\psi^3_{\triangleleft}\psi^2_{\bigcirc}}\Big(\frac{1}{\phi'_{\triangleleft}\psi_{\bigcirc} + \frac{p^2}{\mu^2}\phi'_{\bigcirc}\psi_{\triangleleft}} - \frac{1}{\phi'_{\triangleleft}\psi_{\bigcirc} + \phi'_{\bigcirc}\psi_{\triangleleft}}\Big)$$

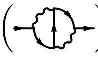
$$+ 4\not{p}\frac{a_7}{\psi^3_{\triangleleft}\psi^3_{\bigcirc}} \ln\frac{\phi'_{\triangleleft}\psi_{\bigcirc} + \frac{p^2}{\mu^2}\phi'_{\bigcirc}\psi_{\triangleleft}}{\phi'_{\triangleleft}\psi_{\bigcirc} + \phi'_{\bigcirc}\psi_{\triangleleft}},$$

whith

$$\psi_{\bigcirc} = a_5 + a_7, \qquad \psi_{\triangleleft} = a_3 + a_4 + a_6,$$

$$\phi'_{\bigcirc} = a_5 a_7, \qquad \text{and} \qquad \phi'_{\triangleleft} = (a_3 + a_4)a_6.$$

The third one is something similar. This can be integrated to:*

$\Phi^{\mathrm{ren}}\Big($  $\Big) = -\not{p}\left( \ln^2 \frac{p^2}{\mu^2} + \ln \frac{p^2}{\mu^2} \right).$

*Erik Panzer's Maple program HyperInt is used for this; see [13] and [14], chapter 4.

# 4

# Scalar Quantum Electrodynamics

## 4.1 Feynman Rules

### 4.1.1 Lagrangian

In this chapter we study *scalar quantum electrodynamics* (*sQED*),* which is a theory similar to QED, but with a complex scalar field $\phi$ instead of the spinor field. The Lagrangian is

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)(D^\mu\phi^*) - \tfrac{1}{4}\lambda(\phi^*\phi)^2\,. \qquad (4.1)$$

Just like QED, this is $\mathrm{U}(1)$ gauge invariant.

### 4.1.2 Feynman Graphs

For the Feynman graphs, we have photon half-edges (as in QED) and incoming and outgoing scalar half-edges, which we represent graphically as

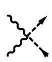
 ,  and 

respectively. In chapter 2 we had real scalar fields; now they are complex. That is why we have an arrow here.

As in QED, these half-edges combine to two types of edges:

 and  .

But unlike QED we have several types of vertices:

 ,  and  .

*See [10], subsection 6-1-4.

### 4.1.3 Feynman Rules

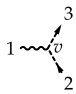
We take the Feynman gauge again, which means that we can use the same abuse of notation as in subsection 3.1.3. Assign to every internal and external photon edge $e \in \Gamma^{[1]}_{\sim} \cup \Gamma^{\text{ext}}_{\sim}$ a Lorentz index $\mu_e$.

As in the previous chapter, the Feynman amplitude is

$$\Phi(\Gamma) = \frac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}_L \frac{N(\Gamma)}{\prod_{e \in \Gamma^{[1]}} p_e^2} . \tag{4.2}$$

Here, the numerator $N(\Gamma)$ is a product of:

- for every vertex

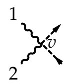


$\in \Gamma^{[1]}_{\prec}$

a factor

$$(p_2 + p_3)^{\mu_1} =: V_v , \tag{4.3}$$

- and for every vertex



$\in \Gamma^{[1]}_{\times}$

a factor

$$-2g^{\mu_1 \mu_2} =: V_v^{\times} , \tag{4.4}$$

so the numerator $N(\Gamma)$ looks like

$$N(\Gamma) = \Big( \prod_{v \in \Gamma^{[1]}_{\times}} V_v^{\times} \Big) \Big( \prod_{v \in \Gamma^{[0]}_{\prec}} V_v \Big) . \tag{4.5}$$

The Green's functions are

$$G = \sum_\Gamma \frac{1}{\operatorname{Sym} \Gamma} \frac{i^{\#\Gamma^{[1]}_{-}} (-i)^{\#\Gamma^{[1]}_{\sim}} (-ie)^{\#\Gamma^{[0]}_{\prec}} (-ie^2)^{\#\Gamma^{[0]}_{\times}} (-i\lambda)^{\#\Gamma^{[0]}_{\times}} \pi^{2l_\Gamma}}{(2\pi)^{4l_\Gamma}} \Gamma . \tag{4.6}$$

Take $\lambda = -e^2$. Then the connected and 1PI functions can be written as

$$G = (-)^{\#\Gamma^{\text{ext}}_{-}} i e^{\#\Gamma^{\text{ext}} - 2} \sum_{l=0}^{\infty} x^l G_{(l)} , \tag{4.7}$$

where

$$x := -\frac{ie^2}{16\pi^2} . \tag{4.8}$$

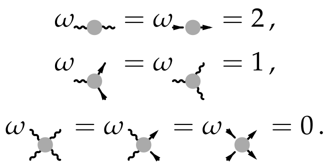
The superficial degree of divergence in sQED is

$$\omega_\Gamma = 4l_\Gamma + \#\Gamma^{[0]}_{\prec} - 2(\#\Gamma^{[1]}_{-} + \#\Gamma^{[1]}_{\sim}) . \tag{4.9}$$

This turns out to be the same as in $\phi^4$ theory:

$$\omega_\Gamma = 4 - \#\Gamma^{\text{ext}} , \tag{4.10}$$

so we have the following superficial divergences:



(4.11)

It is not difficult to see that Furry's theorem (lemma 3.3) also holds here. The Feynman rule for 3-valent vertex gives a minus sign when the arrow is flipped. So the numerators for both orientations cancel.

Note that the 4-scalar function is divergent. In order to renormalize sensibly, we therefore need the 4-scalar vertex.

### 4.1.4 The 2-Scalar-2-Photon Vertex

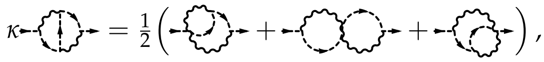
**Definition 4.1.** i. For a scalar edge $e \in \Gamma^{[1]}_{\dashrightarrow}$, we define the operator

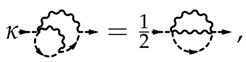
$$\kappa_e \Gamma := \begin{cases} \frac{1}{2}\,\Gamma \backslash e & \text{if } e \text{ is incident to two 3-valent vertices:} \\ & \quad \text{[diagram]} \subseteq \Gamma\,, \\ 0 & \text{otherwise,} \end{cases} \tag{4.12}$$

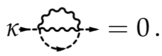
ii. and we define

$$\kappa\Gamma := \sum_{e \in \Gamma^{[1]}_{\dashrightarrow}} \kappa_e \Gamma\,. \tag{4.13}$$

**Example 4.2.**

i. 

ii. 

iii. 

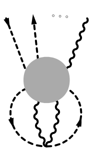
**Lemma 4.3.** Let $G$ be a connected Green's function. Then

$$\tfrac{1}{k+1}\kappa G\big|_{k\times} = G\big|_{k+1\times}\,. \tag{4.14}$$

$G|_{k\times}$ is $G$, restricted to the graphs with exactly $k$ 2-photon-2-scalar vertices.

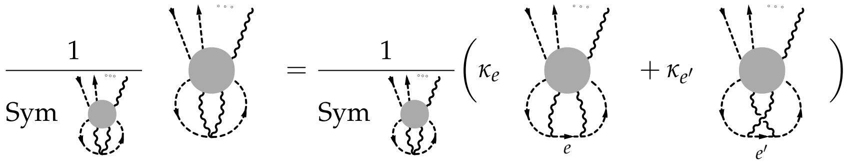
*Proof.* It is clear that the left and the right hand side contain the same graphs. The point of this proof is to show that the coefficients for these graphs are equal.

To do this, we start with a graph $\Gamma$ with $\#\Gamma^{[0]}_{\times} = k+1$ and let $v \in \Gamma^{[0]}_{\times}$. We represent $\Gamma$ as:

$\Gamma =$ 

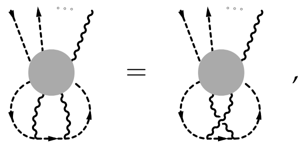
$v$

Using definition 4.1, we can write:

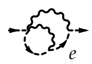



Here we see why $\kappa$ is defined with a factor $\frac{1}{2}$: It compensates for the two ways of making a 2-scalar-2-scalar vertex.

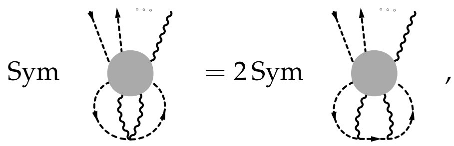
The following two cases can occur:

- 

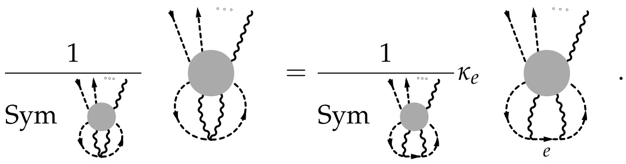
  for example if

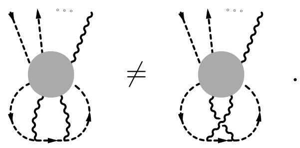
  $\Gamma =$ 


  Then

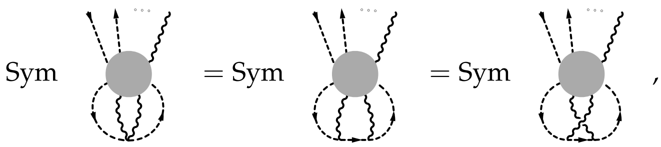
  


  and so

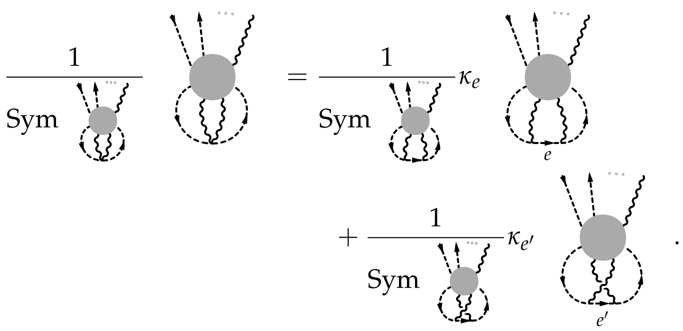
  


- 

  Then

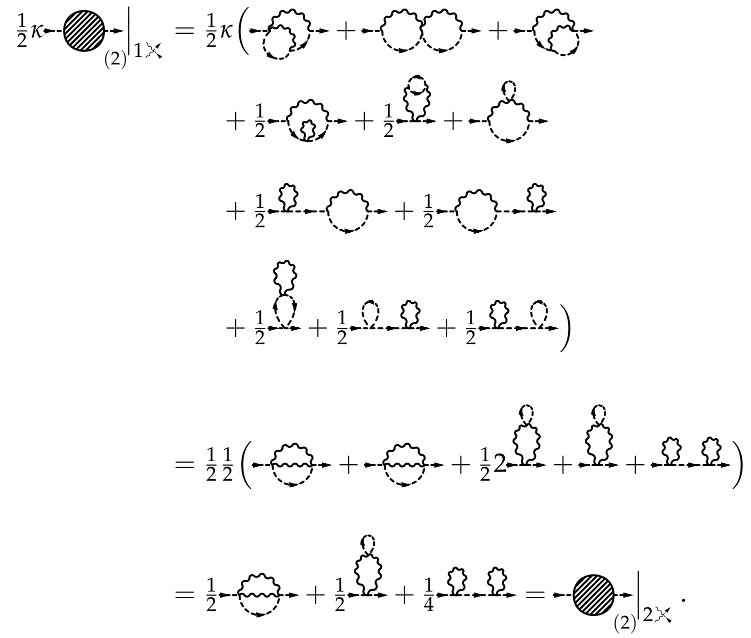

  and so:

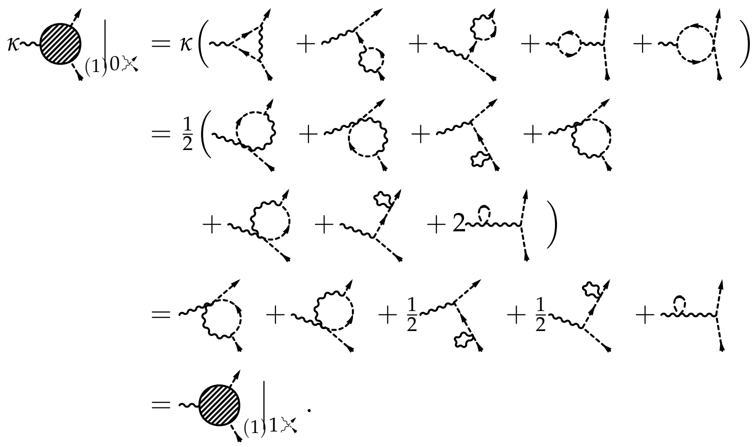
Symmetrizing over all vertices in $\Gamma$ (that gives the factor $\frac{1}{k+1}$) and then summing over all graphs $\Gamma$ with $\#\Gamma^{[0]}_{\times}$ (as always, with a given external structure and modulo equivalence) proves the lemma. □

**Example 4.4.**

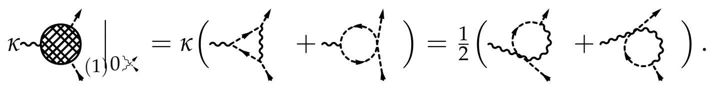
i.



ii.



Lemma 4.3 does not work for 1PI Green's functions. For example:



We miss the graphs that add the factor $\frac{1}{2}$ up to 1.

**Lemma 4.5.** For connected Green's functions $G$:

$$e^{\kappa} G\big|_{0\times} = G\,. \tag{4.15}$$

*Proof.* Using lemma 4.3 and induction in $k$, one can see that

$$\tfrac{1}{k!}\kappa^k G\big|_{0\times} = G\big|_{k\times}\,. \tag{4.16}$$

Summing over all $k$ proves the lemma:

$$e^{\kappa} G\big|_{0\times} = \sum_{k\geq 0} \tfrac{1}{k!}\kappa^k G\big|_{0\times} = \sum_{k\geq 0} G\big|_{k\times} = G\,. \tag{4.17}$$

□

**Remark 4.6.** The exponent

$$e^{\kappa}\Gamma := \sum_{k\geq 0} \tfrac{1}{k!}\kappa^k \Gamma \tag{4.18}$$

is defined as an infinite sum, but actually it is just a finite one. Let $m_\Gamma$ be the number such that $\kappa^{m_\Gamma}\Gamma \neq 0$ and $\kappa^{m_\Gamma+1}\Gamma = 0$, for example

$m$  $= 2\,.$

We can write

$$e^{\kappa}\Gamma := \sum_{k\geq 0}^{m_\Gamma} \tfrac{1}{k!}\kappa^k \Gamma\,. \tag{4.19}$$

The exponent can also be written as

$$e^{\kappa}\Gamma = \sum_{k\geq 0} \sum_{\{e_1,\ldots,e_k\}\subseteq \Gamma^{[1]}} \kappa_{e_1}\cdots\kappa_{e_k}\Gamma\,. \tag{4.20}$$

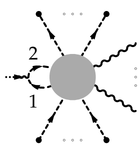
The factors $\frac{1}{k!}$ were just there to compensate for double counting.

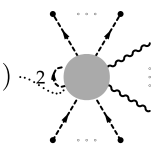
## 4.2 Ward Identities

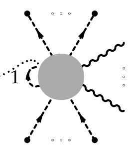
First, a lemma analogous to lemmata 3.4 and 3.7:

**Lemma 4.7.**

i.

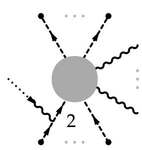
$(1+\kappa_1+\kappa_2)$ 

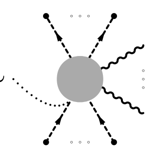
$\sim -(1+\kappa_2)$ 

$+ (1+\kappa_1)$ 

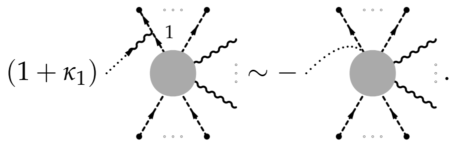
, (4.21)

ii.

$(1+\kappa_2)$ 

$\sim$  , (4.22)

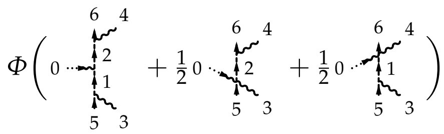
iii.



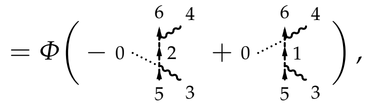
(4.23)

*Proof.* i. Actually, a lot of cases have to be distinguished: Both edges 1 and 2 can be incident to a 1-photon-2-scalar vertex, a 2-photon-2-scalar vertex or a 4-scalar vertex.

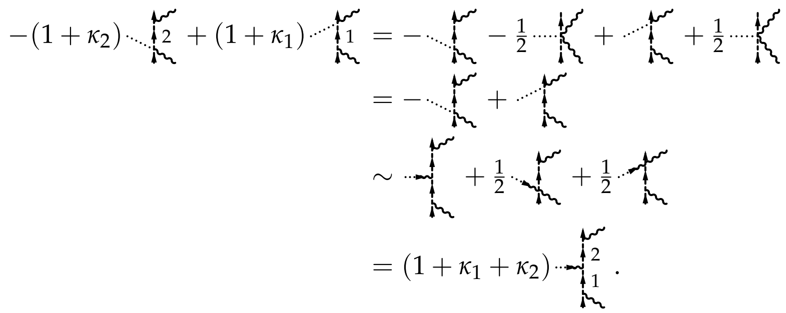
- If both edges 1 and 2 are incident to 1-photon-2-scalar vertices, one has (using momentum conservation, $p_0 = -p_1 + p_2$):



$$
\begin{aligned}
&= p_0^{\mu_0} \frac{(p_5 + p_1)^{\mu_3}(p_1 + p_2)^{\mu_0}(p_2 + p_6)^{\mu_4}}{p_1^2 p_2^2} \\
&\quad - \frac{p_0^{\mu_3}(p_2 + p_6)^{\mu_4}}{p_2^2} - \frac{(p_5 + p_1)^{\mu_3} p_0^{\mu_4}}{p_1^2} \\
&= -\frac{(p_5 + p_1)^{\mu_3}(p_2 + p_6)^{\mu_4}}{p_2^2} + \frac{(p_5 + p_1)^{\mu_3}(p_2 + p_6)^{\mu_4}}{p_1^2} \\
&\quad - \frac{(-p_1 + p_2)^{\mu_3}(p_2 + p_6)^{\mu_4}}{p_2^2} + \frac{(p_5 + p_1)^{\mu_3}(p_1 - p_2)^{\mu_4}}{p_1^2} \\
&= -\frac{(p_5 + p_2)^{\mu_3}(p_2 + p_6)^{\mu_4}}{p_2^2} + \frac{(p_5 + p_1)^{\mu_3}(p_1 + p_6)^{\mu_4}}{p_1^2}
\end{aligned}
$$

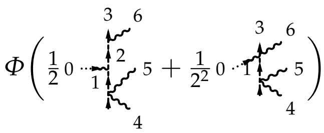


so

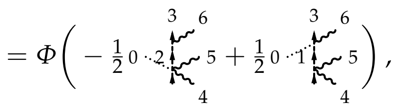


- If edge 1 is incident to a 2-photon-2-scalar vertex and edge 2 to a

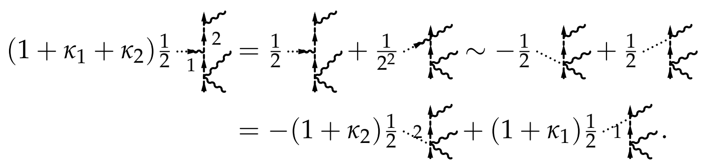
1-photon-2-scalar vertex:



$$= -p_0^{\mu_0}\frac{g^{\mu_3\mu_4}(p_1+p_2)^{\mu_0}(p_2+p_3)^{\mu_6}}{p_1^2p_2^2} + p_0^{\mu_0}\frac{g^{\mu_3\mu_4}g^{\mu_0\mu_6}}{p_1^2}$$
$$= \frac{g^{\mu_3\mu_4}(p_2+p_3)^{\mu_6}}{p_2^2} - \frac{g^{\mu_3\mu_4}(p_2+p_3)^{\mu_6}}{p_1^2} + \frac{g^{\mu_3\mu_4}(-p_1+p_2)^{\mu_6}}{p_1^2}$$
$$= \frac{g^{\mu_3\mu_4}(p_2+p_3)^{\mu_6}}{p_2^2} - \frac{g^{\mu_3\mu_4}(p_1+p_3)^{\mu_6}}{p_1^2}$$

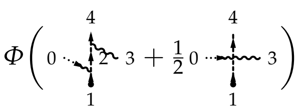


so

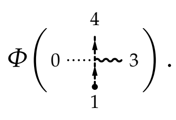


All other cases are proven similarly.

ii. Here too some different cases have to be distinguished. The edge 2 can be incident to three differend types of vertices, for instance the 1-photon-2-scalar vertex:



$$= p_0^{\mu_0}\frac{(p_1+p_2)^{\mu_0}(p_2+p_4)^{\mu_3}}{p_2^2} - p_0^{\mu_3}$$
$$= -\frac{p_1^2(p_2+p_4)^{\mu_3}}{p_2^2} + (p_2+p_4)^{\mu_3} - (-p_1+p_2)^{\mu_3}$$

$= (p_1+p_4)^{\mu_3} =$ 

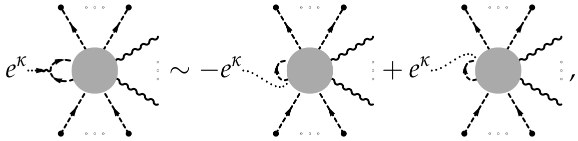
The external edge 1 representes a physical photon, wich has a null momentum: $p_1^2 = 0$.

iii. This is proven analogously to ii. □

From this lemma follows:

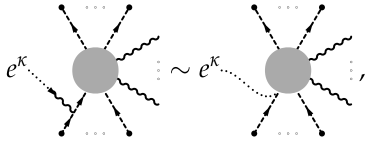
**Corollary 4.8.** The following blobs represent a graph without any 2-boson-2-scalar vertices.

i.



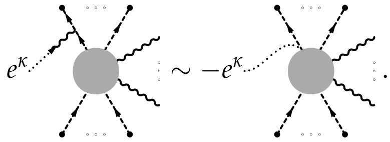

(4.24)

ii.



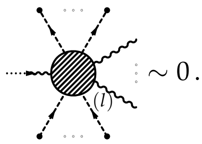

(4.25)

iii.



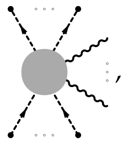

(4.26)

**Theorem 4.9** (Ward identities)**.**

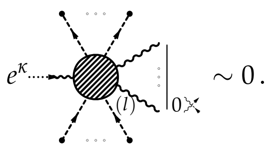



(4.27)

*Proof.* Start by taking take a graph of the form



that has no 2-boson-2-scalar vertices. As in the proof of theorem 3.8, insert a longitudinal photon in every internal and external scalar edge. Next, apply the operator $e^{\kappa}$ and sum over all $l$-loop connected graphs whithout 2-boson-2-scalar vertices (with the given external structure, modulo equivalence and weighted by the symmetry factors). This gives, using corollary 4.8:




With lemma 4.5 one can see that the theorem is true. □

## 4.3 Parametric Representation

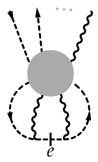
### 4.3.1 Marking Edges

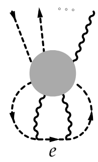
The operator $\kappa_e$ forgets information about the topology of a graph. In this section this information is useful, so therefore we introduce a related operator

$\chi_e$ that keeps the topology: instead of contracting the edge $e$, it puts a little mark on it:

**Definition 4.10.** i. For a scalar edge $e \in \Gamma^{[1]}_{-}$, we define

$$\chi_e \Gamma := \begin{cases} \text{[diagram]} & \text{if } \Gamma = \text{[diagram]} \,, \\ 0 & \text{otherwise,} \end{cases} \tag{4.28}$$

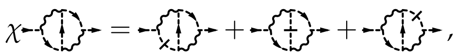


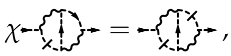


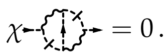
ii. and we define

$$\chi \Gamma := \sum_{e \in \Gamma^{[1]}_{-}} \chi_e \Gamma \,. \tag{4.29}$$

**Example 4.11.** Analogously to example 4.2, we have:

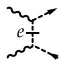
i. 

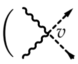
ii. 

iii. 

This marked edge is just a different notation for the 2-scalar-2-photon vertex, and as such this edge type does not represent a propagator. The Feynman rule for this new edge type is:

$$W_e := \Phi\Big(\text{[diagram]}\Big) = \tfrac{1}{2}\Phi\Big(\text{[diagram]}\Big) = \tfrac{1}{2} V_v^{\times} = -g^{\mu_1\mu_2} \,. \tag{4.30}$$





We replace the 2-scalar-2-photon vertex by this marked edge. The denominator is now

$$N(\Gamma) = \Big( \prod_{e \in \Gamma^{[1]}_{+}} W_e \Big) \Big( \prod_{v \in \Gamma^{[0]}} V_v \Big) \tag{4.31}$$

Because the marked edges are not propagators, they have to be excluded from the denominator:

$$\Phi(\Gamma) = \frac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}_L \frac{N(\Gamma)}{\prod_{e \in \Gamma^{[1]} \setminus \Gamma^{[1]}_{+}} p_e^2} \,. \tag{4.32}$$

Furthermore, they are only allowed as internal edges.

## 4.3.2 Parametric Representation

Just like in the previous chapters we define the parametric integral as:

$$I(\Gamma) := \frac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}_L \, N(\Gamma) e^{-\sum_{e \in \Gamma^{[1]} \setminus \Gamma^{[1]}_{+}} p_e^2 A_e} \,, \tag{4.33}$$

but now, we omit the marked edges from the exponent. For the amplitude, we do not integrate over them:

$$\Phi(\Gamma) = \int \mathrm{d}\underline{A}_{\Gamma/\Gamma^{[1]}_{+}} I(\Gamma) \tag{4.34}$$

**Theorem 4.12.** First some things have to be defined: As in theorem 3.9, $\widehat{N}(\Gamma)$ is the differential operator obtained by replacing the momenta $p_e$ by differential operators $\widehat{p}_e$ (equation (3.47)) in $N(\Gamma)$. So in sQED

$$\widehat{N}(\Gamma) = \Big( \prod_{e \in \Gamma^{[1]}_{+}} W_e \Big) \Big( \prod_{v \in \Gamma^{[0]}_{\curlywedge}} \widehat{V}_v \Big) . \tag{4.35}$$

Let $\overline{\varphi}_\Gamma$ be $\varphi_\Gamma$ plus contributions for the external scalar edges:

$$\overline{\varphi}_\Gamma := \varphi_\Gamma + \sum_{h \in \Gamma^{\mathrm{ext}}_{-}} \xi_h^2 A_h \psi . \tag{4.36}$$

Define:

$$U(\Gamma) := \widehat{N}(\Gamma) \frac{e^{-\overline{\varphi}_\Gamma / \psi_\Gamma}}{\psi_\Gamma^2} \Big|_{\underline{A}_{\Gamma^{\mathrm{ext}}}=0} , \tag{4.37}$$

where $\underline{A}_{\Gamma^{\mathrm{ext}}} = 0$ is a short-hand notation for $\forall h \in \Gamma^{\mathrm{ext}} : A_h = 0$.

Having defined this, one has

$$U(\Gamma) = \sum_{i \geq 0} \tfrac{1}{i!} \sum_{e_1, \ldots e_i \in \Gamma^{[1]}_{-}} \frac{1}{2^i A_{e_1} \cdots A_{e_i}} u(\chi_{e_1} \cdots \chi_{e_i} \Gamma) , \tag{4.38}$$

where the $u(\Gamma)$ have the property

$$u(\Gamma)\big|_{\underline{A}_{\Gamma^{[1]}_{+}}=0} = I(\Gamma) . \tag{4.39}$$

*Proof.* Using theorem 2.24 reversely, we have

$$U(\Gamma) = \tfrac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}_L \Big( \prod_{e \in \Gamma^{[1]}_{+}} W_e \Big) \Big( \prod_{v \in \Gamma^{[0]}_{\curlywedge}} \widehat{V}_v \Big) e^{-\sum_{e \in \Gamma^{[1]} \cup \Gamma^{\mathrm{ext}}_{-}} p_e^2 A_e} \Big|_{\underline{A}_{\Gamma^{\mathrm{ext}}}=0} .$$

Note that in the sum in the exponent the external scalar edges are also included. For $e \in \Gamma^{[1]} \cup \Gamma^{\mathrm{ext}}_{-}$,

$$\widehat{p}_e^\mu e^{-\sum_{e \in \Gamma^{[1]} \cup \Gamma^{\mathrm{ext}}_{-}} p_e^2 A_e} \Big|_{\underline{A}_{\Gamma^{\mathrm{ext}}}=0} = p_e^\mu e^{-\sum_{e \in \Gamma^{[1]}} p_e^2 A_e} ,$$

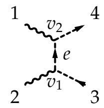
so for $v \in \Gamma^{[0]}_{\curlywedge}$:

$$\widehat{V}_v e^{-\sum_{e \in \Gamma^{[1]} \cup \Gamma^{\mathrm{ext}}_{-}} p_e^2 A_e} \Big|_{\underline{A}_{\Gamma^{\mathrm{ext}}}=0} = V_v e^{-\sum_{e \in \Gamma^{[1]}} p_e^2 A_e} .$$

This equation also holds for vertices $v$ with an external scalar edge incident to it. That is the reason why $\overline{\varphi}_\Gamma$ is used rather than just $\varphi_\Gamma$. The Schwinger parameters of these external edges are set to 0, after applying the differential operator.

Unlike QED, we have to take the Leibniz rule into account. If $v_1, v_2 \in \Gamma^{[0]}_{\rightsquigarrow}$ are not adjacent, $\widehat{V}_{v_1} V_{v_2} = 0$. If they are adjacent, then with equation (3.65) one sees that:

- If there is one scalar edge, $e$, incident to both $v_1$ and $v_2$,

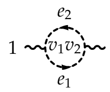



$$\subseteq \Gamma\,,$$

then

$$\widehat{V}_{v_1} V_{v_2} = (\widehat{p}_3 + \widehat{p}_e)^{\mu_2} (p_e + p_4)^{\mu_1} = -\frac{1}{2A_e} g^{\mu_2 \mu_1} = \frac{W_e}{2A_e}\,.$$

- If there are two scalar edges, $e_1$ and $e_2$, incident to both $v_1$ and $v_2$,




$$\subseteq \Gamma\,,$$

then

$$\begin{aligned}\widehat{V}_{v_1} V_{v_2} &= (\widehat{p}_{e_2} + \widehat{p}_{e_1})^{\mu_1} (p_{e_1} + p_{e_2})^{\mu_1} = -\frac{1}{2A_{e_1}} g^{\mu_1 \mu_1} - \frac{1}{2A_{e_2}} g^{\mu_1 \mu_1} \\ &= \frac{W_{e_1}}{2A_{e_1}} + \frac{W_{e_2}}{2A_{e_2}}\,.\end{aligned}$$

So

$$\begin{aligned}U(\Gamma) = \tfrac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}_L \Big( \prod_{e \in \Gamma^{[1]}_{\dashrightarrow +}} W_e \Big) \Big( &\prod_{v \in \Gamma^{[0]}_{\rightsquigarrow}} V_v + \sum_{e \in \Gamma^{[1]}_{\dashrightarrow}} \frac{W_e}{2A_e} \prod_{\substack{v \in \Gamma^{[1]}_{\rightsquigarrow} \\ e \text{ not inc. to } v}} V_v \\ &+ \tfrac{1}{2} \sum_{\substack{e_1, e_2 \in \Gamma^{[1]}_{\dashrightarrow} \\ \text{not adj.}}} \frac{W_{e_1} W_{e_2}}{2^2 A_{e_1} A_{e_2}} \prod_{\substack{v \in \Gamma^{[1]}_{\rightsquigarrow} \\ e_1, e_2 \text{ not inc. to } v}} V_v \\ &+ \tfrac{1}{3!} \cdots \Big) e^{-\sum_{e \in \Gamma^{[1]}} p_e^2 A_e}\,.\end{aligned}$$

(The factors $\frac{1}{2}$, $\frac{1}{3!}$ etc. are just there to compensate for double counting.) If we introduce

$$u(\Gamma) := \tfrac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}_L \, N(\Gamma) e^{-\sum_{e \in \Gamma^{[1]}} p_e^2 A_e}\,, \tag{4.40}$$

it can we written as

$$U(\Gamma) = u(\Gamma) + \sum_{e \in \Gamma^{[1]}_{\dashrightarrow}} \frac{1}{2A_e} u(\chi_e \Gamma) + \tfrac{1}{2} \sum_{e_1, e_2 \in \Gamma^{[1]}_{\dashrightarrow}} \frac{1}{2^2 A_{e_1} A_{e_2}} u(\chi_{e_1} \chi_{e_2} \Gamma) + \tfrac{1}{3!} \cdots\,.$$

And indeed, $u(\Gamma)$ has the property

$$u(\Gamma)\big|_{\underline{A}_{\Gamma_{+}^{[1]}}=0} = \tfrac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}_L \, N(\Gamma) e^{-\sum_{e\in\Gamma^{[1]}\setminus\Gamma_{+}^{[1]}} p_e^2 A_e} = I(\Gamma)\,. \qquad \square$$

For the following, we alter definition (3.51) a bit:

$$\widetilde{p}_e^\mu := -\widehat{p}_e^\mu \overline{\varphi}_\Gamma\,. \tag{4.41}$$

For internal edges $e$ nothing changes actually; for external edges $e$:

$$\widetilde{p}_e^\mu = \xi_e^\mu \psi_\Gamma = p_e^\mu \psi_\Gamma\,. \tag{4.42}$$

Furthermore, it is convenient to define

$$\widetilde{V}_v := -\widehat{V}_v \overline{\varphi}_\Gamma \tag{4.43}$$

and

$$\widetilde{W}_{v_1 v_2} := \widehat{V}_{v_1} \widetilde{V}_{v_2}\,. \tag{4.44}$$

$\widetilde{W}_{v_1 v_2}$ is proportional to $g^{\mu_{e_1}\mu_{e_2}}$, if $e_1$ and $e_2$ are the photon edges incident to $v_1$ and $v_2$ respectively. And for 1-scale graphs, $\widetilde{V}_{v_1}$ is proportional to $p^{\mu_{e_1}}$.

**Remark 4.13.** Analogously to remark 3.10.ii, we have

$$U(\Gamma) = \sum_{i=0}^{\lfloor \#\Gamma_{\rightsquigarrow}^{[0]}/2 \rfloor} \frac{B_i(\Gamma)}{\psi_\Gamma^{\#\Gamma_{\rightsquigarrow}^{[0]}-i+2}} e^{-\varphi_\Gamma/\psi_\Gamma}\,, \tag{4.45}$$

where

$$B_i(\Gamma) := \Big(\prod_{e\in\Gamma_{+}^{[1]}} W_e\Big) \tfrac{1}{2^i i!(k-2i)!} \sum_{\text{perm. of } \Gamma_{\rightsquigarrow}^{[0]}} \widetilde{W}_{v_1 v_2} \cdots \widetilde{W}_{v_{2i+1} v_{2i}} \widetilde{V}_{v_{2i+1}} \cdots \widetilde{V}_{v_k} \tag{4.46}$$

and we labelled $\Gamma_{\rightsquigarrow}^{[0]} = \{v_1, \ldots, v_k\}$.

So $U(\Gamma)$ can be computed. The question is now how to get the $u(\Gamma)$ from this, because if one has those, it is not difficult to get the parametric integrands $I(\Gamma)$.

**Theorem 4.14.** i. $u(\Gamma)$ can be computed recursively:

$$u(\Gamma) = U(\Gamma) - \sum_{i\geq 1} \tfrac{1}{i!} \sum_{e_1,\ldots,e_i\in\Gamma^{[1]}} \frac{1}{2^i A_{e_1}\cdots A_{e_i}} u(\chi_{e_1}\cdots\chi_{e_i}\Gamma)\,, \tag{4.47}$$

ii. or directly from the $U$s:

$$u(\Gamma) = \sum_{i\geq 0} (-)^i \tfrac{1}{i!} \sum_{e_1,\ldots,e_i\in\Gamma^{[1]}} \frac{1}{2^i A_{e_1}\cdots A_{e_i}} U(\chi_{e_1}\cdots\chi_{e_i}\Gamma)\,. \tag{4.48}$$

*Proof.* i. This is equation (4.38).

ii. Proof by strong induction in $m_\Gamma$ (this is defined in remark 4.6):

- For $m_\Gamma = 0$: we have $u(\Gamma) = U(\Gamma)$
- Assume that (4.48) holds for all graphs of the form $\chi_{e_1}\cdots\chi_{e_i}\Gamma \neq 0$, where $e_1,\dots,e_j \in \Gamma^{[1]}_{-}$:

$$u(\chi_{e_1}\cdots\chi_{e_i}\Gamma) = \sum_{j\geq 0}(-)^j\tfrac{1}{j!}\sum_{e'_1,\dots,e'_j\in\Gamma^{[1]}}\frac{1}{2^j A_{e'_1}\cdots A_{e'_j}} \times U(\chi_{e'_1}\cdots\chi_{e'_j}\chi_{e_1}\cdots\chi_{e_i}\Gamma)\,.$$

Note that $m_{\chi_{e'_1}\cdots\chi_{e'_i}\Gamma} = m_\Gamma - i$. Use theorem i:

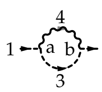
$$\begin{aligned}
u(\Gamma) &= U(\Gamma) - \sum_{i\geq 1}\sum_{j\geq 0}(-)^j\tfrac{1}{i!j!}\sum_{e_1,\dots,e_{i+j}\in\Gamma^{[1]}}\frac{1}{2^{i+j}A_{e_1}\cdots A_{e_{i+j}}}\\
&\quad\times U(\chi_{e_1}\cdots\chi_{e_{i+j}}\Gamma)\\
&= U(\Gamma) - \sum_{k\geq 1}\sum_{j=0}^{k-1}(-)^j\tfrac{1}{(k-j)!j!}\sum_{e_1,\dots,e_k\in\Gamma^{[1]}}\frac{1}{2^k A_{e_1}\cdots A_{e_k}}\\
&\quad\times U(\chi_{e_1}\cdots\chi_{e_k}\Gamma)\\
&= U(\Gamma) + \sum_{k\geq 1}(-)^k\tfrac{1}{k!}\sum_{e_1,\dots,e_k\in\Gamma^{[1]}}\frac{1}{2^k A_{e_1}\cdots A_{e_k}}U(\chi_{e_1}\cdots\chi_{e_k}\Gamma)\,.
\end{aligned}$$

$k = i + j$ is substituted and the trick

$$\sum_{j=0}^{k}(-)^j\tfrac{1}{(k-j)!j!} = \tfrac{1}{k!}\sum_{j=0}^{k}(-)^j\tbinom{k}{j} = (1-1)^k = 0$$

is used. □

**Example 4.15.** i. Take the graph




.

The Symanzik polynomials and $\widetilde{p}_3$ were given in example 3.11.i. With

$$\widehat{V}_\mathrm{a} = (\widehat{p}_1 + \widehat{p}_3)^{\mu_4} \qquad\text{and}\qquad \widehat{V}_\mathrm{b} = (\widehat{p}_3 + \widehat{p}_2)^{\mu_4}$$

we have

$$\begin{aligned}
\widetilde{V}_\mathrm{a} &= (\widetilde{p}_1 + \widetilde{p}_3)^{\mu_4} \stackrel{\text{m.c.}}{=\!=} p^{\mu_4}(\psi_{\circ} + A_4) = p^{\mu_4}(A_3 + 2A_4)\,,\\
\widetilde{V}_\mathrm{b} &= (\widetilde{p}_3 + \widetilde{p}_2)^{\mu_4} \stackrel{\text{m.c.}}{=\!=} p^{\mu_4}(A_4 + \psi_{\circ}) = p^{\mu_4}(A_3 + 2A_4)\,,
\end{aligned}$$

and

$$\widetilde{W}_\mathrm{ab} = \widehat{p}_3^{\mu_4}\widehat{p}_3^{\mu_4} = -\frac{A_4}{2A_3}g^{\mu_4\mu_4} = -\frac{2A_4}{A_3}\,.$$

Applying the differential operator gives

$$U\Big(\bigcirc\Big) = \widehat{V}_{\mathrm{a}}\widehat{V}_{\mathrm{b}}\frac{e^{-\overline{\varphi}_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2}\Big|_{A_1=A_2=0} = \Big(\frac{\widetilde{V}_{\mathrm{a}}\widetilde{V}_{\mathrm{b}}}{\psi_{\circ}^4} + \frac{\widetilde{W}_{\mathrm{ab}}}{\psi_{\circ}^3}\Big)e^{-\varphi_{\circ}/\psi_{\circ}}$$
$$\overset{\text{m.c.}}{=\!=} \Big(p^2\frac{(A_3+2A_4)^2}{\psi_{\circ}^4} - \frac{2A_4}{A_3\psi_{\circ}^3}\Big)e^{-p^2\varphi'_{\circ}/\psi_{\circ}}\,.$$

We also need

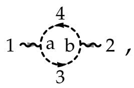
$$U\Big(\bigcirc\!\!\!\!+\Big) \overset{\text{m.c.}}{=\!=} W_3\frac{e^{-p^2\varphi'_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2} = -g^{\mu_4\mu_4}\frac{e^{-p^2\varphi'_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2}$$
$$= -4\frac{e^{-p^2\varphi'_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2}\,.$$

Using theorem 4.14, one obtains the integrand

$$I\Big(\bigcirc\Big) = u\Big(\bigcirc\Big) = U\Big(\bigcirc\Big) - \frac{1}{2A_3}U\Big(\bigcirc\!\!\!\!+\Big)$$
$$= \Big(p^2\frac{(A_3+2A_4)^2}{\psi_{\circ}^4} + \frac{2}{\psi_{\circ}^3}\Big)e^{-p^2\varphi'_{\circ}/\psi_{\circ}}\,.$$

Note that the pole $\frac{1}{A_3}$ disappears.

ii. For the graph



,

the Symanzik polynomials and $\widetilde{p}_3$ and $\widetilde{p}_4$ were given in example 3.11.ii. With

$$\widehat{V}_{\mathrm{a}} = (\widehat{p}_4 + \widehat{p}_3)^{\mu_1} \qquad \text{and} \qquad \widehat{V}_{\mathrm{b}} = (\widehat{p}_3 + \widehat{p}_4)^{\mu_2}$$

one has

$$\widetilde{V}_{\mathrm{a}} = (\widetilde{p}_4 + \widetilde{p}_3)^{\mu_1} \overset{\text{m.c.}}{=\!=} p^{\mu_1}(-A_3 + A_4)\,,$$
$$\widetilde{V}_{\mathrm{b}} = (\widetilde{p}_3 + \widetilde{p}_4)^{\mu_2} \overset{\text{m.c.}}{=\!=} p^{\mu_2}(-A_3 + A_4)\,,$$

and

$$\widetilde{W}_{\mathrm{ab}} = (\widehat{p}_4 + \widehat{p}_3)^{\mu_1}(\widehat{p}_3 + \widehat{p}_4)^{\mu_2} = g^{\mu_1\mu_2}\Big(1 - \frac{A_4}{2A_3} - \frac{A_3}{2A_4}\Big)\,.$$

Applying the differential operator gives us

$$U\Big(\bigcirc\Big) = \widehat{V}_{\mathrm{a}}\widehat{V}_{\mathrm{b}}\frac{e^{-\varphi_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2} = \Big(\frac{\widetilde{V}_{\mathrm{a}}\widetilde{V}_{\mathrm{b}}}{\psi_{\circ}^4} + \frac{\widetilde{W}_{\mathrm{ab}}}{\psi_{\circ}^3}\Big)e^{-\varphi_{\circ}/\psi_{\circ}}$$
$$\overset{\text{m.c.}}{=\!=} \Big(p^{\mu_1}p^{\mu_2}\frac{(A_3 - A_4)^2}{\psi_{\circ}^4}$$
$$+ g^{\mu_1\mu_2}\Big(1 - \frac{A_4}{2A_3} - \frac{A_3}{2A_4}\Big)\frac{1}{\psi_{\circ}^3}\Big)e^{-p^2\varphi'_{\circ}/\psi_{\circ}}\,.$$

We also need

$$U\left(\bigcirc\right) \overset{\text{m.c.}}{=\!=\!=} W_3 \frac{e^{-p^2\varphi'_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2} = -g^{\mu_1\mu_2} \frac{e^{-p^2\varphi'_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2}$$

and likewise

$$U\left(\bigcirc\right) = -g^{\mu_1\mu_2} \frac{e^{-p^2\varphi'_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2}\,.$$

Using theorem 4.14, we get integrand

$$\begin{aligned} I\left(\bigcirc\right) &= U\left(\bigcirc\right) - \frac{1}{2A_3}U\left(\bigcirc\right) - \frac{1}{2A_4}U\left(\bigcirc\right) \\ &= \left(p^{\mu_1}p^{\mu_2}\frac{(A_3-A_4)^2}{\psi_{\circ}^4} + 2g^{\mu_1\mu_2}\frac{1}{\psi_{\circ}^3}\right)e^{-p^2\varphi'_{\circ}/\psi_{\circ}}\,, \end{aligned}$$

which does not have the poles $\frac{1}{A_3}$ and $\frac{1}{A_4}$. Renormalizing as in definition 3.23 gives

$$\mathscr{I}^{\text{ren}}\left(\bigcirc\right) = \left(-p^{\mu_1}p^{\mu_2}(a_3-a_4)^2 + 2p^2g^{\mu_1\mu_2}\varphi'_{\circ}\right)\frac{1}{\psi_{\circ}^3}\ln\frac{p^2}{\mu^2}\,.$$

This gives the amplitude:

$$\Phi^{\text{ren}}\left(\bigcirc\right) = \tfrac{1}{3}(-p^{\mu_1}p^{\mu_2} + p^2g^{\mu_1\mu_2})\ln\frac{p^2}{\mu^2}\,.$$

It is transversal, as one would expect.

By the way, with the same argument as in remark 2.35, we see that

$$\tfrac{1}{2}I^{\text{ren}}\left(\bigcirc\right) = I^{\text{ren}}\left(\bigcirc\right) = 0\,.$$

In remark 3.14 it is explained how in QED a parametric integrand can be constructed for other covariant gauges than the Feynman gauge. Exactly the same thing can be done for scalar QED.

# 5 Non-Abelian Gauge Theories

## 5.1 Feynman Rules

### 5.1.1 Lagrangian

In the previous two chapters we had an Abelian gauge group: U(1). In this chapter we look at *non-Abelian gauge theories* or *Yang-Mills theories*,* which have a non-Abelian gauge group $G$.

The gauge group is a Lie group, and we denote the generators of the Lie algebra $\mathfrak{g}$ corresponding to $G$ by $t^a$. Since the Lie algebra is closed under the bracket, we introduce the *structure constants* $f^{abc}$:

$$[t^a, t^b] =: i f^{abc} t^c \,. \tag{5.1}$$

(Einstein's summation convention is used.) They are antisymmetric in every index, because the Lie bracket is antisymmetric. In terms of the structure constants, the *Jacobi identity* reads:

$$f^{a_0 a_1 b} f^{a_2 a_3 b} + f^{a_0 a_3 b} f^{a_1 a_2 b} + f^{a_0 a_2 b} f^{a_3 a_1 b} = 0 \,. \tag{5.2}$$

The Yang-Mills Lagrangian is

$$\mathscr{L} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \bar{c}^a \partial^\mu D^{\mathrm{ad}\,ab}_\mu c^b \,. \tag{5.3}$$

This needs some explanation. The first term is the generalization of the first term in equation (3.1). The covariant derivative is now

$$D_\mu = \partial_\mu - i g A^a_\mu t^a \tag{5.4}$$

and the field tensor $F^a_{\mu\nu}$ is given by

$$F^a_{\mu\nu} t^a = \frac{i}{g} [D_\mu, D_\nu] \,, \tag{5.5}$$

*See [15], chapters 15 and 16 and [10], sections 12-1 and 12-2.

so

$$F^a_{\mu\nu} = \partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g f^{abc} A^b_\mu A^c_\nu \,. \tag{5.6}$$

In the second term of the Lagrangian we have the *Faddeev-Popov ghost* field $c$. This is a Graßmannian field: it has spin 0, but fulfills anti-commutation relations. Under gauge transformations it transforms in the adjoint representation, therefore one has the covariant derivative in the adjoint representation $((t^c)^{ab} = f^{abc})$:

$$D^{\text{ad}ab}_\mu = \delta^{ac}\partial_\mu - g f^{abc} A^c_\mu \,. \tag{5.7}$$

This Lagrangian is gauge invariant; the two terms are even gauge invariant separately. The reason ghosts are introduced is to make the 'measure' of the Feynman path integral, and hence the path integral itself, gauge invariant. Because of their anti-commutativity, the ghost fields form a kind of a determinant, the *Faddeev-Popov determinant*, which acts as a Jacobian if one changes the gauge.*

Ghosts violate spin-statistics: they anti-commute and have integer spin. This means that they cannot be physical, so they will not occur in a physical initial of final state of a scattering process.

To keep notations a bit simpler, we only focus on the pure gauge theory; we do not consider couplings to fermion or scalar fields.

### 5.1.2 Feynman Graphs

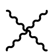
There are the following half-edges:

,  and  :

the gauge boson, and the ingoing and outgoing ghost respectively, which combine to the edges

 and .

There is a 3-boson, 4-boson and ghost vertex:

 ,  and  .

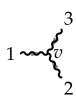
### 5.1.3 Feynman Rules

As always, the Feynman amplitude of a graph $\Gamma$ is given by:

$$\Phi(\Gamma) = \int \mathrm{d}\underline{k} \frac{N(\Gamma)}{\prod\limits_{e \in \Gamma^{[1]}} p_e^2} \,. \tag{5.8}$$

In the Feynman gauge, the numerator $N(\Gamma)$ is now given as follows: As in QED and sQED, assign to each internal and external boson edge $e \in \Gamma^{[1]}_{\sim} \cup \Gamma^{\text{ext}}_{\sim}$ a Lorentz index $\mu_e$, but now also assign to every internal and external edge $e \in \Gamma^{[1]} \cup \Gamma^{\text{ext}}$ a 'color' index $a_e$. Then to obtain $N(\Gamma)$, include

- for every 3-boson vertex



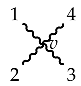
$\in \Gamma^{[1]}_{\prec}$

*See [15], section 16.2 and [17], section 15.5-6.

a factor

$$
\begin{aligned}
f^{a_1 a_2 a_3} \big( & g^{\mu_2 \mu_3} (p_2 - p_3)^{\mu_1} \\
& + g^{\mu_3 \mu_1} (p_3 - p_1)^{\mu_2} \\
& + g^{\mu_1 \mu_2} (p_1 - p_2)^{\mu_3} \big) =: V_v \,,
\end{aligned} \tag{5.9}
$$

- for every 4-boson vertex

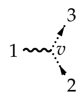


$$\in \Gamma^{[1]}_{\times}$$

a factor

$$
\begin{aligned}
& f^{a_1 a_2 b} f^{a_3 a_4 b} \big( g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3} \big) \\
& \quad + f^{a_1 a_3 b} f^{a_2 a_4 b} \big( g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_3 \mu_2} \big) \\
& \quad + f^{a_1 a_4 b} f^{a_2 a_3 b} \big( g^{\mu_1 \mu_2} g^{\mu_4 \mu_3} - g^{\mu_1 \mu_3} g^{\mu_4 \mu_2} \big) =: V_v^{\times} \,,
\end{aligned} \tag{5.10}
$$

- and for every ghost vertex



$$\in \Gamma^{[1]}_{\prec\!\cdot}$$

a factor

$$f^{a_1 a_2 a_3} p_3^{\mu_1} =: V_v^{\prec\!\cdot} \,. \tag{5.11}$$

So the numerator is

$$N(\Gamma) = \Big( \prod_{v \in \Gamma^{[0]}_{\times}} V_v^{\times} \Big) \Big( \prod_{v \in \Gamma^{[0]}_{\prec}} V_v \Big) \Big( \prod_{v \in \Gamma^{[0]}_{\prec\cdot}} V_v^{\prec\cdot} \Big) \,. \tag{5.12}$$

The Green's functions are given by

$$G = \sum_{\Gamma} (-)^{\# \mathscr{L}_{\Gamma}^{\cdot\cdot}} \frac{1}{\operatorname{Sym} \Gamma} \frac{g^{\# \Gamma^{[0]}_{\prec}} (-i g^2)^{\# \Gamma^{[0]}_{\times}} (-g)^{\# \Gamma^{[0]}_{\prec\cdot}} (-i)^{\# \Gamma^{[1]}_{\sim}} i^{\# \Gamma^{[1]}_{\cdot\cdot}}}{2^{4 l_{\Gamma}} \pi^{2 l_{\Gamma}}} \Gamma \,. \tag{5.13}$$

Because the ghost fields anti-commute, there is a minus-sign for every ghost loop. Connected and 1PI functions can be written as

$$G = i^{-\# \Gamma^{\text{ext}}_{\sim} - 1} g^{\# \Gamma^{\text{ext}} - 2} \sum_{l=0}^{\infty} x^l G_{(l)} \,, \tag{5.14}$$

where

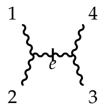
$$x := \frac{i g^2}{16 \pi^2} \tag{5.15}$$

and

$$G_{(l)} := \sum_{\Gamma} (-)^{\# \mathscr{L}_{\Gamma}^{\cdot\cdot}} \frac{1}{\operatorname{Sym} \Gamma} \Gamma \,. \tag{5.16}$$

### 5.1.4 Marking Edges

As in subsection 4.3.1, we replace the 4-valent vertex by an edge with a little mark. Here we give it the following Feynman rule: for every marked edge $e \in \Gamma^{[1]}_{\text{-+-}}$, for which the adjacent edges are labeled as

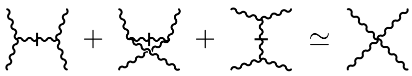



, (5.17)

include a factor

$$f^{a_1 a_2 b} f^{a_3 a_4 b} (g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3}) := W_e \,. \tag{5.18}$$

This is one of the three terms of (5.10), so:




. (5.19)

The amplitude is now

$$\Phi(\Gamma) = \int \mathrm{d}\underline{k} \frac{N(\Gamma)}{\prod\limits_{e \in \Gamma^{[1]} \setminus \Gamma^{[1]}_{\text{-+-}}} p_e^2} \tag{5.20}$$

with the numerator

$$N(\Gamma) = \Big( \prod_{e \in \Gamma^{[1]}_{\text{-+-}}} W_e \Big) \Big( \prod_{v \in \Gamma^{[0]}_{\text{-<}}} V_v \Big) \Big( \prod_{v \in \Gamma^{[0]}_{\text{-<}}} V_v^{\text{-<}} \Big) \,. \tag{5.21}$$

As already said in subsection 4.3.1, it is important to note that these marked edges are not propagators, and that they are only allowed as internal edges.

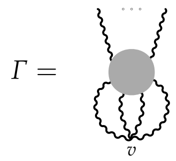
**Lemma 5.1.** A connected graph $\Gamma$ without any marked edges (but possibly with 4-valent vertices) can as follows be written in terms of graphs with marked edges and no 4-valent vertices:

$$\frac{1}{\operatorname{Sym} \Gamma} \Gamma \simeq \sum_{\substack{\Gamma' \\ \#\Gamma'^{[0]}_{\times} = 0 \\ \Gamma' / \Gamma'^{[1]}_{\text{-+-}} = \Gamma}} \frac{1}{\operatorname{Sym} \Gamma'} \Gamma' \,. \tag{5.22}$$

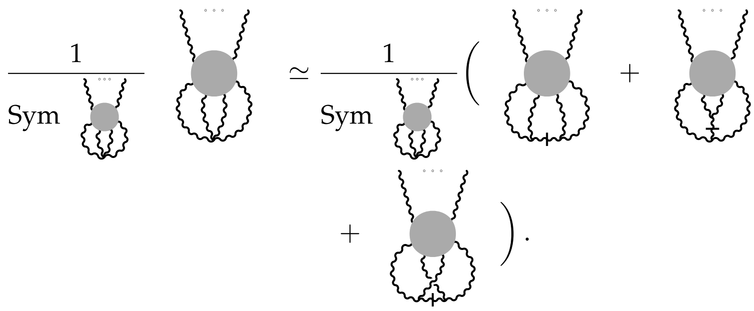
The sum runs over all connected graphs $\Gamma'$ modulo equivalence with the same external structure as $\Gamma$.

*Proof.* It is clear that on the right hand side we have the right graphs to make $\Gamma$ using (5.19); the point of the following proof is to show that the symmetry factors are correct. The proof is quite similar to the proof of lemma 4.3.

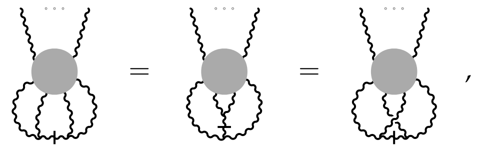
We start by taking a $v \in \Gamma^{[0]}_{\times}$. We represent $\Gamma$ as:



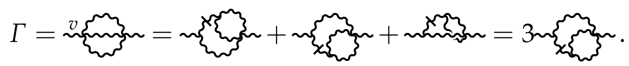
and apply equation (5.19):



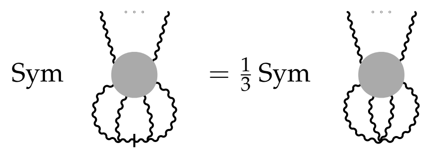
The following three cases can occur:

- 

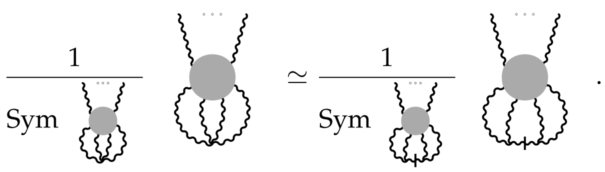
  for example for

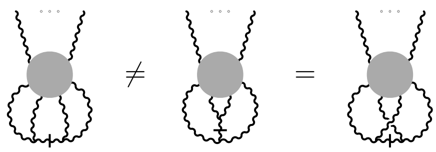
  

  Then

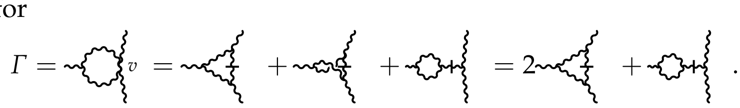
  

  and so

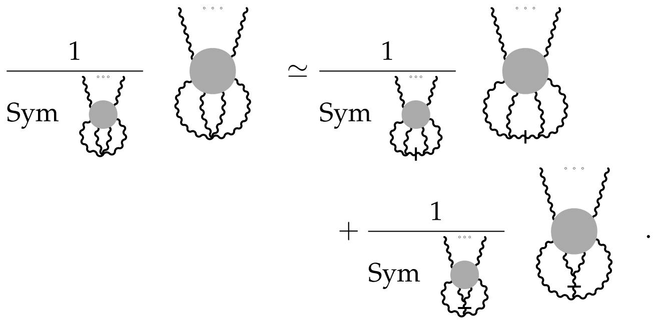
  

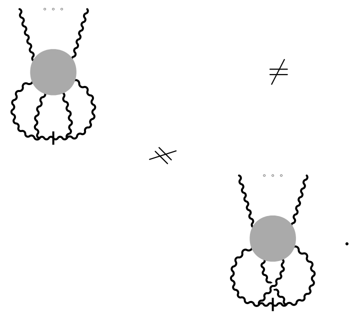
- 

  (or another combination of two inequalities and one equality), for example for

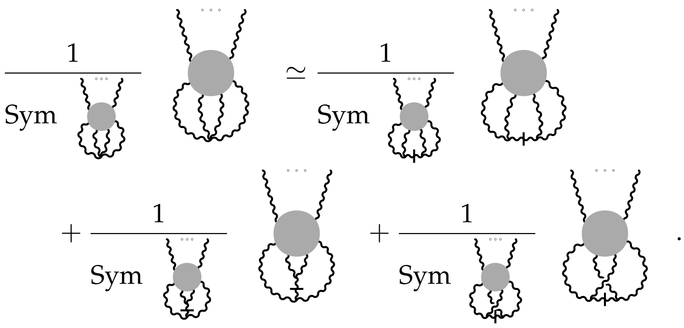
  

Then



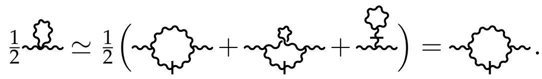
- 

Then

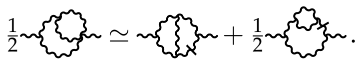


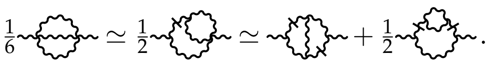
This can be repeated until all 4-valent vertices are converted into marked edges. □

**Example 5.2.**

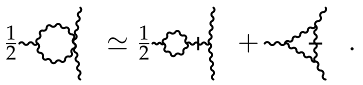
i. 

The graph with the tadpole does not contribute, because it has a vanishing color factor.

ii. 

iii. 

Even for a 1PI graph $\Gamma$, we need the sum in equation (5.22) to run over connected graphs $\Gamma'$. For example:



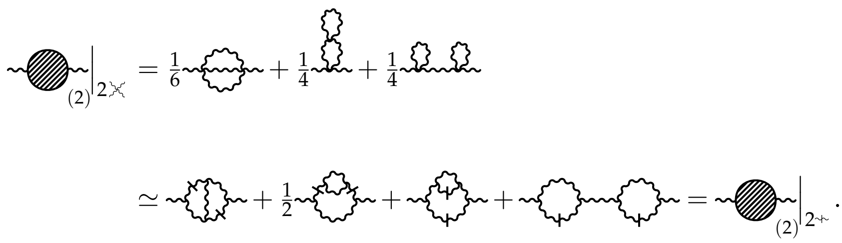
From lemma 5.1 follows:

**Corollary 5.3.** Using the 4-boson vertex or using the marked edge is completely equivalent. In other words: for a connected Green's function $G$,

$$G\big|_{k\times} \simeq G\big|_{k\nsim}, \tag{5.23}$$

where $G|_{k\times}$ is $G$ restricted to graphs with exactly $k$ 4-valent vertices (and no marked edges) and $G|_{k\nsim}$ is $G$ restricted to graphs with exactly $k$ marked edges (and no 4-valent vertices).

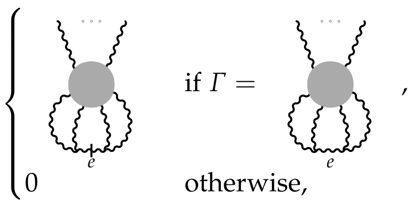
**Example 5.4.** With example 5.2.i and ii, we can write:



As in definition 4.10, we define operators $\chi_e$ and $\chi$ that mark edges:

**Definition 5.5.** i. For a graph $\Gamma$ and an edge $e \in \Gamma^{[1]}_{\sim}$:

$\chi_e\Gamma :=$



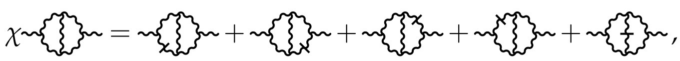

(5.24)

ii. and

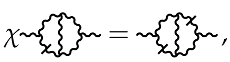
$$\chi\Gamma := \sum_{e\in\Gamma^{[1]}_{\sim}} \chi_e\Gamma\,. \tag{5.25}$$

**Example 5.6.**

i.



ii.

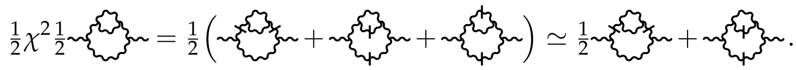


iii.

This operator can be used to express connected Green's functions in fully 3-valent Green's functions:

$$G\big|_{k\,\text{-}} \simeq \tfrac{1}{k!}\chi^k G\big|_{0\,\text{-}}\,. \tag{5.26}$$

For example:



Summing (5.26) over all $k$ gives:

$$G \simeq e^{\chi} G\big|_{0\,\text{-}}\,. \tag{5.27}$$

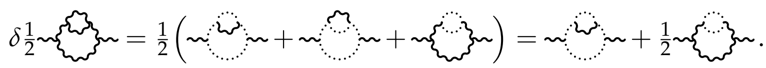
The same thing can be done with ghost loops. For this we define:

**Definition 5.7.**

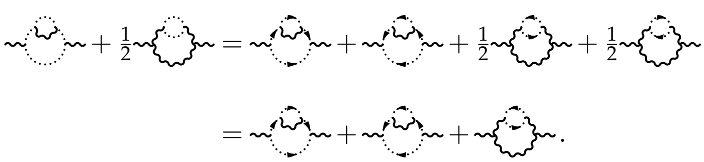
i.
$$\delta_\ell \Gamma := \begin{cases} \Gamma\big|_{\text{every internal edge in } \ell \text{ replaced by } \cdots} & \text{if } \ell^{[0]} = \ell^{[0]}_{\prec}\,, \\ 0 & \text{otherwise,} \end{cases} \tag{5.28}$$

ii. and
$$\delta\Gamma = \sum_{\ell \in \mathscr{L}_\Gamma} \delta_\ell \Gamma\,. \tag{5.29}$$

**Example 5.8.**



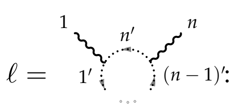
As in lemma 3.3, a ghost loop without arrows is a short-hand notation for both orientations:

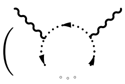
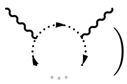


The numerator for such a graph can be written as

$$N(\Gamma) = \Big(\prod_{e \in \Gamma^{[1]}_{\sim}} W_e\Big)\Big(\prod_{v \in \Gamma^{[0]}_{\prec}} V_v\Big)\Big(\prod_{\ell \in \mathscr{L}^{\cdots}_\Gamma} C_\ell\Big)\,, \tag{5.30}$$

where for an unoriented ghost loop



$C_\ell := N\Big($  $+$  $\Big)$

$$= f^{a_1 a_{n'} a_{1'}} \cdots f^{a_n a_{(n-1)'} a_{n'}} p_{1'}^{\mu_1} \cdots p_{n'}^{\mu_n} \tag{5.31}$$
$$+ f^{a_1 a_{1'} a_{n'}} \cdots f^{a_n a_{n'} a_{(n-1)'}} (-p_{n'})^{\mu_1} (-p_{1'})^{\mu_2} \cdots (-p_{(n-1)'})^{\mu_n}$$
$$= f^{a_1 a_{n'} a_{1'}} \cdots f^{a_n a_{(n-1)'} a_{n'}} \big(p_{1'}^{\mu_1} \cdots p_{n'}^{\mu_n} + p_{n'}^{\mu_1} p_{1'}^{\mu_2} \cdots p_{(n-1)'}^{\mu_n}\big)\,.$$

Then:

$$G \simeq e^{-\delta} G\big|_{0\circ}\,. \tag{5.32}$$

$G|_{0\circ}$ is the Green's function $G$ without the graphs with ghost loops. The minus sign in fron of the $\delta$ is the Fermi minus for the ghost loops.

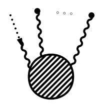
With these two operators, Green's functions can be expressed in fully 3-valent, ghost-less Green's functions.

$$G \simeq e^{\chi} e^{-\delta} G\big|_{\substack{0\times \\ 0\circ}}\,. \tag{5.33}$$

## 5.2 Ward Identities

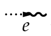
Like the Ward-Takahashi identities give relations between off-shell functions in QED, there are more complicated relations for Yang-Mills theories, the *Slavnov-Taylor identities*.

In this section, we go straight to the Ward identities:

 $\overset{?}{\sim} 0\,.$ (5.34)

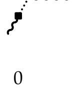
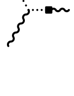
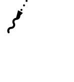
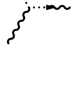
(Recall the diagrammatic notation from equation (3.32).) The proof given in this section is similar to Gerard 't Hooft's in [9], section 4. See also Predrag Cvitanović' treatise in [7], chapter 7.

It is convenient to extend the Feynman rule (3.32) also for internal edges: for every edge

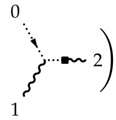
 , include a factor $\begin{cases} p_e^{\mu_e} & \text{if } e \text{ is external,} \\ \dfrac{p_e^{\mu_e}}{p_e^2} & \text{if } e \text{ is internal.} \end{cases}$ (5.35)

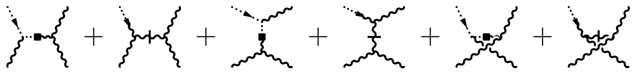
Before we prove the Ward identities, it's useful to prove some identities using this new notation:

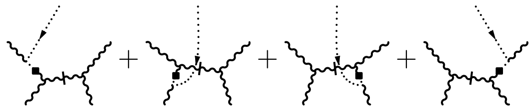
**Lemma 5.9.**

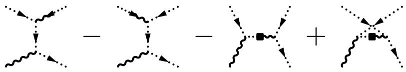
i.  $\sim$  $+$  $+$  $+$  , (5.36)

where we introduced:

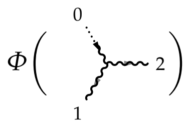
$\Phi\Big($  $\Big) := f^{a_0 a_1 a_2} p_2^2 g^{\mu_1 \mu_2}\,.$ (5.37)

ii.  $\sim 0\,.$ (5.38)

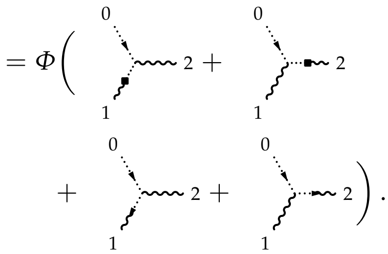
iii.  $\sim 0\,.$ (5.39)

iv.  $\sim 0\,.$ (5.40)

*Proof.*

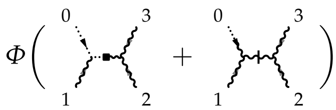
i. 

$$= f^{a_0a_1a_2}(p_1+p_2)^{\mu_0}\big(g^{\mu_1\mu_2}(-p_1+p_2)^{\mu_0} - g^{\mu_2\mu_0}(2p_2+p_1)^{\mu_1} + g^{\mu_0\mu_1}(2p_1+p_2)^{\mu_2}\big)$$
$$= f^{a_0a_1a_2}(-p_1^2 g^{\mu_1\mu_2} + p_2^2 g^{\mu_1\mu_2} + p_1^{\mu_1}p_1^{\mu_2} - p_2^{\mu_1}p_2^{\mu_2})$$



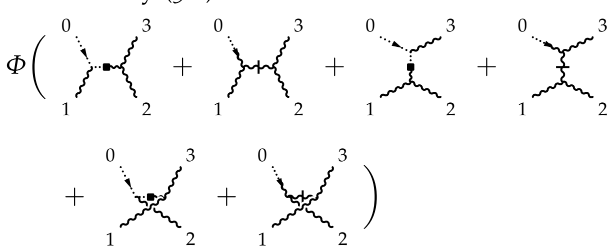
The first two terms are precisely our newly defined (5.37), in the last two we recognize the ghost vertex (5.11) and equation (5.35).

ii. For the first two terms we have:



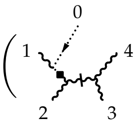
$$= f^{a_0a_1b}f^{a_2a_3b}\big(g^{\mu_3\mu_1}(2p_3+p_2)^{\mu_2} - g^{\mu_1\mu_2}(2p_2+p_3)^{\mu_3} + g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1}\big)$$
$$+ f^{a_1a_2b}f^{a_3a_4b}\big(-g^{\mu_1\mu_3}(p_1+p_2+p_3)^{\mu_2} + g^{\mu_1\mu_2}(p_1+p_2+p_3)^{\mu_3}\big)$$
$$= f^{a_0a_1b}f^{a_2a_3b}\big(g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1} + g^{\mu_3\mu_1}(p_3-p_1)^{\mu_2} + g^{\mu_1\mu_2}(p_1-p_2)^{\mu_3}\big)\,.$$

Up to the color factor, this is cyclic in the indices 1, 2 and 3. So with the Jacobi identity (5.2) we have:

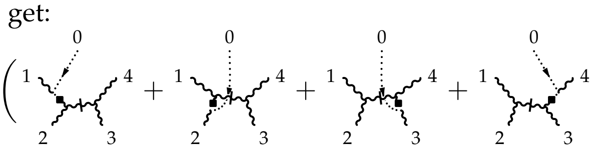


$$= (f^{a_0a_1b}f^{a_2a_3b} + f^{a_0a_3b}f^{a_1a_2b} + f^{a_0a_2b}f^{a_3a_1b})$$
$$\times\big(g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1} + g^{\mu_3\mu_1}(p_3-p_1)^{\mu_2} + g^{\mu_1\mu_2}(p_1-p_2)^{\mu_3}\big) = 0\,.$$

iii.

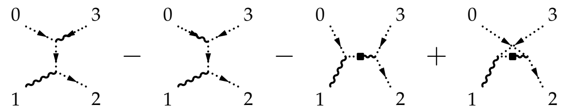
$$\Phi\Bigg($$  $$\Bigg) = f^{a_0 a_1 b} f^{b a_2 c} f^{a_3 a_4 c} \big( g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3} \big) \,.$$

Using the antisymmetry of the structure constants and Jacobi identity we get:

$$\Phi\Bigg($$  $$\Bigg)$$

$$\begin{aligned}
&= f^{a_0 a_1 b} f^{b a_2 c} f^{a_3 a_4 c} \big( g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3} \big) \\
&\quad + f^{a_0 a_2 b} f^{b a_1 c} f^{a_3 a_4 c} \big( g^{\mu_2 \mu_3} g^{\mu_1 \mu_4} - g^{\mu_2 \mu_4} g^{\mu_1 \mu_3} \big) \\
&\quad + f^{a_0 a_3 b} f^{b a_4 c} f^{a_1 a_2 c} \big( g^{\mu_3 \mu_1} g^{\mu_4 \mu_2} - g^{\mu_3 \mu_2} g^{\mu_4 \mu_1} \big) \\
&\quad + f^{a_0 a_4 b} f^{b a_3 c} f^{a_1 a_2 c} \big( g^{\mu_4 \mu_1} g^{\mu_3 \mu_2} - g^{\mu_4 \mu_2} g^{\mu_3 \mu_1} \big) \\
&= \big( ( f^{a_0 a_1 b} f^{b a_2 c} - f^{a_0 a_2 b} f^{b a_1 c} ) f^{a_3 a_4 c} + ( f^{a_0 a_3 b} f^{b a_4 c} - f^{a_0 a_4 b} f^{b a_3 c} ) f^{a_1 a_2 c} \big) \\
&\quad \times \big( g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3} \big) \\
&= - \big( f^{a_0 c b} f^{b a_1 a_2} f^{a_3 a_4 c} + f^{a_0 c b} f^{b a_3 a_4} f^{a_1 a_2 c} \big) \big( g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3} \big) \\
&= - \big( f^{a_0 c b} f^{b a_1 a_2} f^{a_3 a_4 c} + f^{a_0 b c} f^{c a_3 a_4} f^{a_1 a_2 b} \big) \big( g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3} \big) = 0 \,.
\end{aligned}$$

iv.

$$\Phi\Bigg($$  $$\Bigg)$$

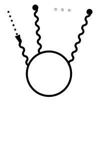
$$\begin{aligned}
&= f^{a_0 b a_3} f^{b a_2 a_1} \frac{p_3 \cdot (p_0 + p_3) p_2^{\mu_1}}{(p_0 + p_3)^2} - f^{a_3 b a_0} f^{b a_2 a_1} \frac{p_0 \cdot (p_0 + p_3) p_2^{\mu_1}}{(p_0 + p_3)^2} \\
&\quad - f^{a_0 a_1 b} f^{a_3 a_2 b} p_2^{\mu_1} + f^{a_0 a_2 b} f^{a_3 a_1 b} p_2^{\mu_1} \\
&= \big( f^{a_0 b a_3} f^{b a_2 a_1} - f^{a_0 a_1 b} f^{a_3 a_2 b} + f^{a_0 a_2 b} f^{a_3 a_1 b} \big) p_2^{\mu_1} = 0 \,.
\end{aligned}$$

Here we used antisymmetry and the Jacobi identity again. □

From the last two terms in lemma 5.9.i we see that ghosts are more or less longitudinal gauge bosons. The general idea is that they cancel, because ghost loops provide a Fermi −-sign. We will make this precise in the following.

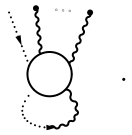
Before we continue to the case of connected functions, we prove the Ward identities for the full functions, i.e. including disconnected graphs.

**Theorem 5.10** (Ward identities (full Green's functions))**.**

 $$\sim 0 \,. \tag{5.41}$$

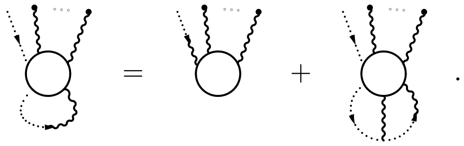
*Proof.* In lemma 5.9.i we see some kind of recursivity; the longitudinal degrees of freedom 'travel' though the graph (if we neglect the contributions drawn with the squared).

We take the following full Green's function:

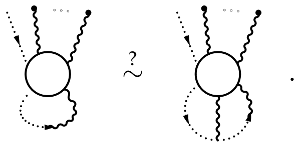


(5.42)

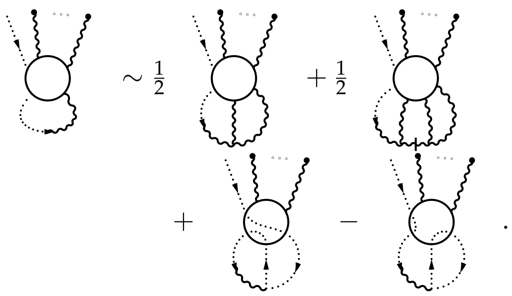
The outgoing ghost on the bottom is connected to the external one at the top; either directly or via one or more interactions with a gauge boson:



The first term of the right-hand side is interesting; this is the object we want to show to be 0, which means that we have to show that

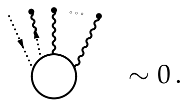


Let us do the same thing with the boson on the bottom in (5.42). It can be incident to a 3-valent vertex, a 4-valent one, or a ghost:

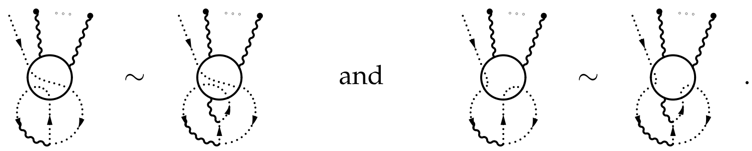


(5.43)

In the last two terms we have to distinguish two cases: the longitudinal line ends in itself, or in a ghost loop. For the latter case we have to include a Fermi $-$-sign for that ghost loop. We did not include the possibility that it is an external boson since these contributions vanish because of transversality (equation (3.21)) anyway:

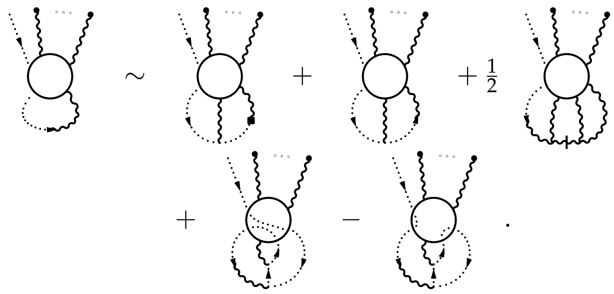


The last two terms in equation (5.43) can be written as

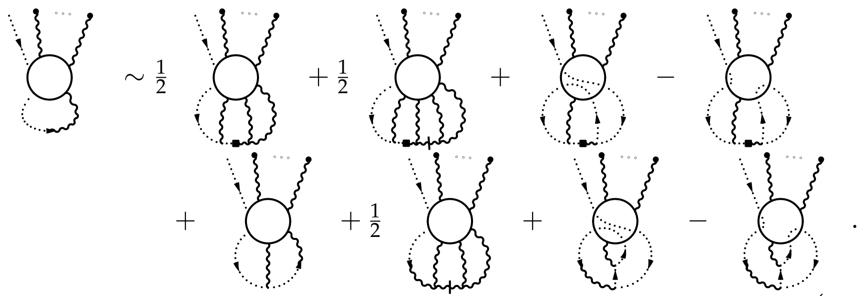


because tadpoles have vanishing color factors.

Apply this and lemma 5.9.i on the first term of the right-hand side of (5.43):

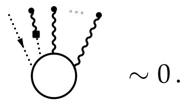


We can do the same thing to the first term as we did in (5.43):

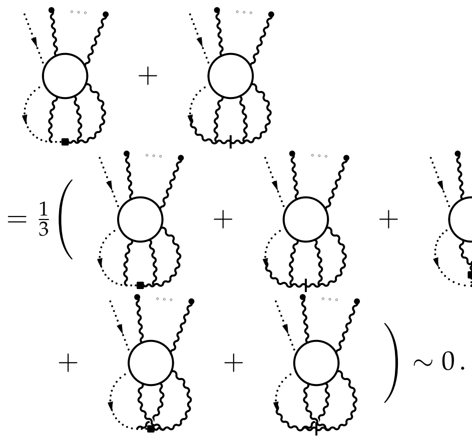


(5.44)

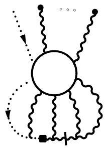
Note that because of the mass-shell condition (equation (3.22)) we have



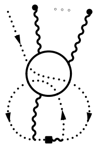
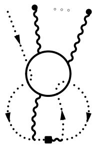
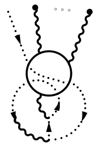
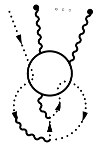
The first and the sixth term in (5.44) cancel (lemma 5.9.ii):



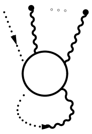
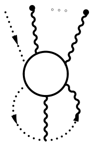
With a similar symmetrization argument and using lemma 5.9.iii it can be seen that the second term of (5.44) is zero:

 $\sim 0\,.$

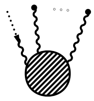
The third, fourth, seventh and eighth term cancel because of lemma 5.9.iv:

 $-$  $+$  $-$  $\sim 0\,.$

We are left with:

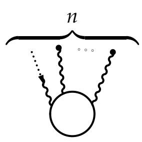
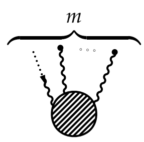
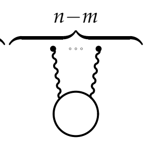
 $\sim$  .  □

**Theorem 5.11** (Ward identities (connected Green's functions))**.**

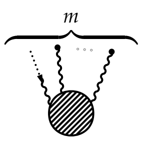
 $\sim 0\,.$ (5.45)

*Proof.* We use complete induction in over the number of external legs:

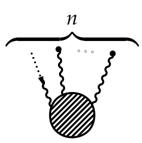
- The statement is trivially true for tadpole functions, because the color factor of tadpole graphs always vanish.
- First note that

  $0 \sim$  $= \sum_{m=1}^{n} \sum_{\text{perm.}}$   .

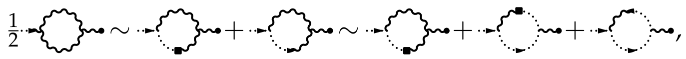
  Assume we have already proven that

  $\forall m < n :$  $\sim 0\,.$

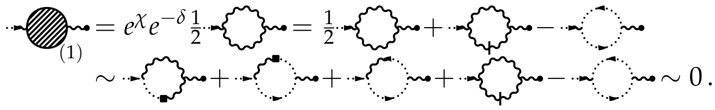
  Recall that we exclude vacuum graphs, so then

   $\sim 0\,.$  □

**Example 5.12.** i. Apply lemma 5.9.i twice on the following little graph:

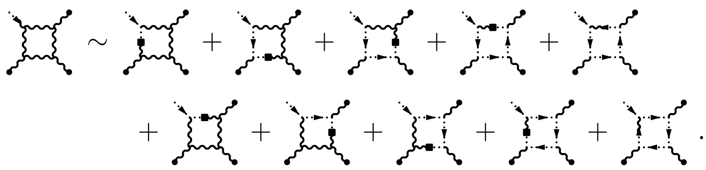


and use this to show that the 1-loop 2-point function is indeed transversal:



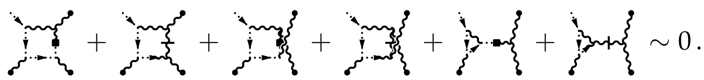
The first and fourth term in the third line cancel because of lemma 5.9.ii, the other three because of lemma 5.9.iv.

ii. Apply lemma 5.9.i repetitively to the following 4-point graph:

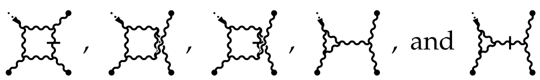


We will not prove the transversality of the 1-loop 4-point function, but show using two examples that for each of these terms, there are contributions from other 4-point graphs to which they cancel.

For example, the third term gets cancelled as follows, using lemma 5.9.ii:

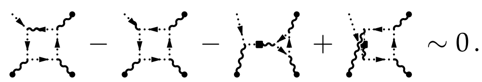


These are contributions from

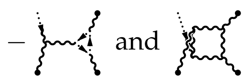


respectively.

And the fifth one gets cancelled in this way (lemma 5.9.iv):



The last two terms are contributions from



respectively.

## 5.3 Parametric Representation

Our approach for a parametric representation for amplitudes in non-Abelian gauge theories is very similar to our method for scalar quantum electrodynamics in subsection 4.3.2.

The parametric integrand is again

$$I(\Gamma) := \frac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}\, N(\Gamma) e^{-\sum_{e\in\Gamma^{[1]}\setminus\Gamma^{[1]}_{\leftarrow}} p_e^2 A_e}, \tag{5.46}$$

such that the Feynman amplitude is

$$\Phi(\Gamma) = \int \mathrm{d}\underline{A}_{\Gamma/\Gamma^{[1]}_{\leftarrow}} \Gamma\,. \tag{5.47}$$

**Theorem 5.13.** As before, $\widehat{N}(\Gamma)$ is the differential operator obtained by replacing the momenta $p_e$ by differential operators $\widehat{p}_e$ (equation (3.47)) in $N(\Gamma)$. So in non-Abelian gauge theory

$$\widehat{N}(\Gamma) = \Big(\prod_{e\in\Gamma^{[1]}_{\leftarrow}} W_e\Big)\Big(\prod_{v\in\Gamma^{[0]}_{\prec}} \widehat{V}_v\Big)\Big(\prod_{\ell\in\mathscr{L}_\Gamma} \widehat{C}_\ell\Big)\,. \tag{5.48}$$

The polynomial $\overline{\varphi}_\Gamma$ is $\varphi_\Gamma$ plus contributions for the external edges:

$$\overline{\varphi}_\Gamma := \varphi_\Gamma + \sum_{h\in\Gamma^{\text{ext}}} \xi_h^2 A_h \psi\,. \tag{5.49}$$

Define:

$$U(\Gamma) := \widehat{N}(\Gamma) \frac{e^{-\overline{\varphi}_\Gamma/\psi_\Gamma}}{\psi_\Gamma^2}\Big|_{\underline{A}_{\Gamma^{\text{ext}}}=0}\,. \tag{5.50}$$

Having defined this, one has

$$U(\Gamma) = \sum_{i\geq 0} \tfrac{1}{i!} \sum_{e_1,\ldots e_i\in\Gamma^{[1]}} \frac{1}{A_{e_1}\cdots A_{e_i}} u(\chi_{e_1}\cdots\chi_{e_i}\Gamma)\,, \tag{5.51}$$

where the $u(\Gamma)$ have the property

$$u(\Gamma)\big|_{\underline{A}_{\Gamma^{[1]}_{\leftarrow}}=0} = I(\Gamma)\,. \tag{5.52}$$

*Proof.* Using theorem 2.24, we have

$$U(\Gamma) = \tfrac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}_L \Big(\prod_{e\in\Gamma^{[1]}_{\leftarrow}} W_e\Big)\Big(\prod_{v\in\Gamma^{[0]}_{\prec}} \widehat{V}_v\Big)\Big(\prod_{\ell\in\mathscr{L}_\Gamma} \widehat{C}_\ell\Big) e^{-\sum_{e\in\Gamma^{[1]}\cup\Gamma^{\text{ext}}} p_e^2 A_e}\Big|_{\underline{A}_{\Gamma^{\text{ext}}}=0}\,.$$

For $e \in \Gamma^{[1]} \cup \Gamma^{\text{ext}}$,

$$\widehat{p}_e^\mu e^{-\sum_{e\in\Gamma^{[1]}\cup\Gamma^{\text{ext}}} p_e^2 A_e} = p_e^\mu e^{-\sum_{e\in\Gamma^{[1]}\cup\Gamma^{\text{ext}}} p_e^2 A_e}\,,$$

so for $v \in \Gamma^{[0]}_{\prec}$:

$$\widehat{V}_v e^{-\sum_{e\in\Gamma^{[1]}\cup\Gamma^{\text{ext}}} p_e^2 A_e} = V_v e^{-\sum_{e\in\Gamma^{[1]}\cup\Gamma^{\text{ext}}} p_e^2 A_e}$$

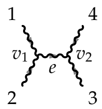
and for ghost loops $\ell$:

$$\widehat{C}_\ell e^{-\sum_{e\in\Gamma^{[1]}\cup\Gamma^{\text{ext}}} p_e^2 A_e} = C_\ell e^{-\sum_{e\in\Gamma^{[1]}\cup\Gamma^{\text{ext}}} p_e^2 A_e}\,.$$

There is no momentum appearing more than once in $C_\ell$; that is why there is no Leibniz rule involved.

For the product over the 3-boson vertices, we do have to consider the Leibniz rule. If $v_1, v_2 \in \Gamma^{[0]}_{\prec}$ are not adjacent, $\widehat{V}_{v_1} V_{v_2} = 0$. If they are adjacent, then with equation (3.65) one sees that:

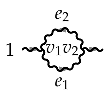
- If there is one edge, $e$, incident to both $v_1$ and $v_2$,




$$\subseteq \Gamma\,,$$

$$\widehat{V}_{v_1} V_{v_2} = \frac{1}{A_e} f^{a_1 a_2 a_e} f^{a_3 a_4 a_e} \left(g^{\mu_1\mu_3} g^{\mu_2\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3}\right) = \frac{W_e}{A_e}\,.$$

- If there are two edges, $e_1$ and $e_2$, incident to both $v_1$ and $v_2$,




$$\subseteq \Gamma\,,$$

$$\begin{aligned}\widehat{V}_{v_1} V_{v_2} &= \frac{1}{A_{e_1}} f^{a_1 a_{e_2} a_{e_1}} f^{a_2 a_{e_2} a_{e_1}} \left(g^{\mu_1\mu_2} g^{\mu_{e_2}\mu_{e_2}} - g^{\mu_1\mu_{e_2}} g^{\mu_{e_2}\mu_2}\right)\\ &\quad + \frac{1}{A_{e_2}} f^{a_1 a_{e_1} a_{e_2}} f^{a_2 a_{e_1} a_{e_2}} \left(g^{\mu_1\mu_2} g^{\mu_{e_1}\mu_{e_1}} - g^{\mu_1\mu_{e_1}} g^{\mu_{e_1}\mu_2}\right)\\ &= \frac{W_{e_1}}{A_{e_1}} + \frac{W_{e_2}}{A_{e_2}}\,.\end{aligned}$$

So

$$\begin{aligned}U(\Gamma) = \tfrac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}_L \Big(\prod_{e\in\Gamma^{[1]}_{\sim}} W_e\Big)\Big(&\prod_{v\in\Gamma^{[0]}_{\prec}} V_v + \sum_{e\in\Gamma^{[1]}_{\sim}} \frac{W_e}{A_e} \prod_{\substack{v\in\Gamma^{[1]}_{\prec}\\ e \text{ not inc. to } v}} V_v\\ &+ \tfrac{1}{2} \sum_{\substack{e_1,e_2\in\Gamma^{[1]}_{\sim}\\ \text{not adj.}}} \frac{W_{e_1} W_{e_2}}{A_{e_1} A_{e_2}} \prod_{\substack{v\in\Gamma^{[1]}_{\prec}\\ e_1,e_2 \text{ not inc. to } v}} V_v\\ &+ \tfrac{1}{3!}\cdots\Big)\Big(\prod_{\ell\in\mathscr{L}_\Gamma} C_\ell\Big) e^{-\sum_{e\in\Gamma^{[1]}} p_e^2 A_e}\,.\end{aligned}$$

With

$$u(\Gamma) := \tfrac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}_L\, N(\Gamma) e^{-\sum_{e\in\Gamma^{[1]}} p_e^2 A_e}\,,$$

it can we written as

$$U(\Gamma) = u(\Gamma) + \sum_{e\in\Gamma^{[1]}_{\sim}} \frac{1}{A_e} u(\chi_e \Gamma) + \tfrac{1}{2} \sum_{e_1,e_2\in\Gamma^{[1]}_{\sim}} \frac{1}{A_{e_1}A_{e_2}} u(\chi_{e_1}\chi_{e_2}\Gamma) + \tfrac{1}{3!}\cdots .$$

And indeed,

$$u(\Gamma)\big|_{\underline{A}_{\Gamma^{[1]}_{\nsim}}=0} = \tfrac{1}{\pi^{2l_\Gamma}} \int \mathrm{d}\underline{k}_L \, N(\Gamma) e^{-\sum_{e\in\Gamma^{[1]}\setminus\Gamma^{[1]}_{\nsim}} p_e^2 A_e} = I(\Gamma) . \qquad \square$$

Note that we do not have the factor $\frac{1}{2}$ we have in sQED (theorem 4.12). If one wants, one can include fermions without problems.;

**Theorem 5.14.**

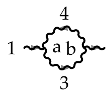
i.
$$u(\Gamma) = U(\Gamma) - \sum_{i\geq 1} \tfrac{1}{i!} \sum_{e_1,\ldots,e_i\in\Gamma^{[1]}} \frac{1}{A_{e_1}\cdots A_{e_i}} u(\chi_{e_1}\cdots\chi_{e_i}\Gamma) , \tag{5.53}$$

ii.
$$u(\Gamma) = \sum_{i\geq 0} (-)^i \tfrac{1}{i!} \sum_{e_1,\ldots,e_i\in\Gamma^{[1]}} \frac{1}{A_{e_1}\cdots A_{e_i}} U(\chi_{e_1}\cdots\chi_{e_i}\Gamma) . \tag{5.54}$$

*Proof.* See the proof of theorem 4.14. $\square$

Recall equations (4.43) and (4.44) from previous chapter. We use these notations in the following example too.

**Example 5.15.** Take the graph



.

See example 3.11.ii for the Symanzik polynomials and $\widetilde{p}_3$ and $\widetilde{p}_4$.

For this graph, we have:

$$\widehat{V}_{\mathrm{a}} = f^{a_1a_3a_4}\big(g^{\mu_3\mu_4}(-\widehat{p}_3 - \widehat{p}_4)^{\mu_1} + g^{\mu_4\mu_1}(\widehat{p}_4 - \widehat{p}_1)^{\mu_3} + g^{\mu_1\mu_3}(\widehat{p}_1 + \widehat{p}_3)^{\mu_4}\big) ,$$
$$\widehat{V}_{\mathrm{b}} = f^{a_2a_4a_3}\big(g^{\mu_4\mu_3}(-\widehat{p}_4 - \widehat{p}_3)^{\mu_2} + g^{\mu_3\mu_2}(\widehat{p}_3 + \widehat{p}_2)^{\mu_4} + g^{\mu_2\mu_4}(-\widehat{p}_2 + \widehat{p}_4)^{\mu_3}\big) ,$$

which give,

$$\begin{aligned}\widetilde{V}_{\mathrm{a}} &\overset{\text{m.c.}}{=\!=} f^{a_1a_3a_4}\big(g^{\mu_3\mu_4}p^{\mu_1}(-A_4 + A_3) + g^{\mu_4\mu_1}p^{\mu_3}(-2A_3 - A_4) \\ &\quad + g^{\mu_1\mu_3}p^{\mu_4}(A_3 + 2A_4)\big) , \\ \widetilde{V}_{\mathrm{b}} &\overset{\text{m.c.}}{=\!=} f^{a_2a_4a_3}\big(g^{\mu_4\mu_3}p^{\mu_2}(A_3 - A_4) + g^{\mu_3\mu_2}p^{\mu_4}(A_3 + 2A_4) \\ &\quad + g^{\mu_2\mu_4}p^{\mu_3}(-2A_3 - A_4)\big) .\end{aligned}$$

Their product is

$$\begin{aligned}\widetilde{V}_{\mathrm{a}}\widetilde{V}_{\mathrm{b}} &\overset{\text{m.c.}}{=\!=} C_2^{\mathrm{ad}}\delta^{a_1a_2}\big(p^{\mu_1}p^{\mu_2}(2A_3^2 + 2A_4^2 + 14A_3A_4) \\ &\quad - p^2 g^{\mu_1\mu_2}(5A_3^2 + 5A_4^2 + 8A_3A_4)\big) ,\end{aligned}$$

where $C_2^{\mathrm{ad}}$ is the quadratic Casimir operator of the adjoint representation of $\mathfrak{g}$, which is defined as:*

$$f^{a_1a_3a_4}f^{a_2a_3a_4} =: C_2^{\mathrm{ad}}\delta^{a_1a_2}\,. \tag{5.55}$$

It also appears in

$$\widetilde{W}_{\mathrm{ab}} = 3C_2^{\mathrm{ad}}\delta^{a_1a_2}g^{\mu_1\mu_2}\Big(-1+\frac{A_4}{A_3}+\frac{A_3}{A_4}\Big)\,.$$

Apply the differential operator:

$$\begin{aligned}
U\big(\circ\big) &= \widehat{V}_{\mathrm{a}}\widehat{V}_{\mathrm{b}}\frac{e^{-\overline{\varphi}_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2}\Big|_{A_1=A_2=0} = \Big(\frac{\widetilde{V}_{\mathrm{a}}\widetilde{V}_{\mathrm{b}}}{\psi_{\circ}^4}+\frac{\widetilde{W}_{\mathrm{ab}}}{\psi_{\circ}^3}\Big)e^{-\varphi_{\circ}/\psi_{\circ}}\\
&\overset{\text{m.c.}}{=\!=} C_2^{\mathrm{ad}}\delta^{a_1a_2}\bigg(\big(p^{\mu_1}p^{\mu_2}(2A_3^2+2A_4^2+14A_3A_4)\\
&\qquad - p^2g^{\mu_1\mu_2}(5A_3^2+5A_4^2+8A_3A_4)\big)\frac{1}{\psi_{\circ}^4}\\
&\qquad + 3g^{\mu_1\mu_2}\Big(-1+\frac{A_4}{A_3}+\frac{A_3}{A_4}\Big)\frac{1}{\psi_{\circ}^3}\bigg)e^{-p^2\varphi'_{\circ}/\psi_{\circ}}\,.
\end{aligned}$$

We also need

$$U\big(\circ\big) = W_3\frac{e^{-\varphi_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2} \overset{\text{m.c.}}{=\!=} -3C_2^{\mathrm{ad}}\delta^{a_1a_2}g^{\mu_1\mu_2}\frac{e^{-p^2\varphi'_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2}$$

and

$$U\big(\circ\big) \overset{\text{m.c.}}{=\!=} -3C_2^{\mathrm{ad}}\delta^{a_1a_2}g^{\mu_1\mu_2}\frac{e^{-p^2\varphi'_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2}\,.$$

So now, the integrand is

$$\begin{aligned}
I\big(\circ\big) &= U\big(\circ\big) - \frac{1}{A_3}U\big(\circ\big) - \frac{1}{A_4}U\big(\circ\big)\\
&= C_2^{\mathrm{ad}}\delta^{a_1a_2}\Big(\big(p^{\mu_1}p^{\mu_2}(2A_3^2+2A_4^2+14A_3A_4)\\
&\qquad - p^2g^{\mu_1\mu_2}(5A_3^2+5A_4^2+8A_3A_4)\big)\frac{1}{\psi_{\circ}^4} - 9\frac{g^{\mu_1\mu_2}}{\psi_{\circ}^3}\Big)e^{-\varphi_{\circ}/\psi_{\circ}}\,.
\end{aligned}$$

The integrand of the ghost loop graph can be computed as:

$$\begin{aligned}
I\big(\circ\big) &= U\big(\circ\big)\\
&= -C_2^{\mathrm{ad}}\delta^{a_1a_2}(\widehat{p}_3^{\mu_1}\widehat{p}_4^{\mu_2}+\widehat{p}_4^{\mu_1}\widehat{p}_3^{\mu_2})\frac{e^{-\overline{\varphi}_{\circ}/\psi_{\circ}}}{\psi_{\circ}^2}\Big|_{A_1=A_2=0}\\
&= -C_2^{\mathrm{ad}}\delta^{a_1a_2}\Big(\frac{\widetilde{p}_3^{\mu_1}\widetilde{p}_4^{\mu_2}+\widetilde{p}_4^{\mu_1}\widetilde{p}_3^{\mu_2}}{\psi_{\circ}^4}+\frac{2g^{\mu_1\mu_2}\beta_{34}}{\psi_{\circ}^3}\Big)e^{-\varphi_{\circ}/\psi_{\circ}}\\
&\overset{\text{m.c.}}{=\!=} C_2^{\mathrm{ad}}\delta^{a_1a_2}\Big(2\frac{p^{\mu_1}p^{\mu_2}A_3A_4}{\psi_{\circ}^4}-\frac{g^{\mu_1\mu_2}}{\psi_{\circ}^3}\Big)e^{-p^2\varphi'_{\circ}/\psi_{\circ}}\,.
\end{aligned}$$

*See [15], equation (15.93).

The two computed integrals combine to

$$\begin{aligned} I\Big(e^{-\delta}\rightsquigarrow\!\bigcirc\!\rightsquigarrow\Big) &= I\Big(\rightsquigarrow\!\bigcirc\!\rightsquigarrow - \rightsquigarrow\!\circ\!\rightsquigarrow\Big) \\ &= C_2^{\mathrm{ad}}\delta^{a_1a_2}\Big(\big(p^{\mu_1}p^{\mu_2}(2A_3^2+2A_4^2+12A_3A_4) \\ &\qquad - p^2g^{\mu_1\mu_2}(5A_3^2+5A_4^2+8A_3A_4)\big)\frac{1}{\psi^4_{\circ}} \\ &\qquad + 8\frac{g^{\mu_1\mu_2}}{\psi^3_{\circ}}\Big)e^{-\varphi_{\circ}/\psi_{\circ}}\,. \end{aligned}$$

Renormalize it as in definition 3.23:

$$\begin{aligned} \mathscr{I}^{\mathrm{ren}}\Big(e^{-\delta}\rightsquigarrow\!\bigcirc\!\rightsquigarrow\Big) &= C_2^{\mathrm{ad}}\delta^{a_1a_2}\big(-p^{\mu_1}p^{\mu_2}(2a_3^2+2a_4^2+12a_3a_4) \\ &\qquad + p^2g^{\mu_1\mu_2}(5a_3^2+5a_4^2)\big)\frac{1}{\psi^4_{\circ}}\ln\frac{p^2}{\mu^2}\,, \end{aligned}$$

and this integrates to

$$\Phi^{\mathrm{ren}}\Big(e^{-\delta}\rightsquigarrow\!\bigcirc\!\rightsquigarrow\Big) = \tfrac{10}{3}C_2^{\mathrm{ad}}\delta^{a_1a_2}(-p^{\mu_1}p^{\mu_2}+p^2g^{\mu_1\mu_2})\ln\frac{p^2}{\mu^2}\,.$$

As expected, it is transversal.

To get the Green's function, the only thing one has to do is to include a symmetry factor $\frac{1}{2}$:

$$\Phi^{\mathrm{ren}}\Big(\rightsquigarrow\!\bullet_{(1)}\!\rightsquigarrow\Big) = \Phi^{\mathrm{ren}}\Big(e^{\chi}e^{-\delta}\tfrac{1}{2}\rightsquigarrow\!\bigcirc\!\rightsquigarrow\Big) = \tfrac{5}{3}C_2^{\mathrm{ad}}\delta^{a_1a_2}(-p^{\mu_1}p^{\mu_2}+p^2g^{\mu_1\mu_2})\ln\frac{p^2}{\mu^2}\,.$$

The $\chi$ does not do much here actually, because it results in a self-loop, for wich the renormalized amplitude vanishes. (See remark 2.35.)

### 5.3.1 The Corolla Polynomial

In [11], the operator $\widehat{N}(e^{-\delta}\Gamma)$ (where $\Gamma$ has only 3-boson vertices) is introduced using the so-called *corolla polynomial*. This is a polynomial in the half-edge variables $a_h$.

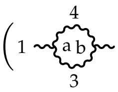
For a graph $\Gamma$ that has no 4-valent vertices, but possibly internal unoriented ghost loops, one first defines the polynomial:

$$\mathscr{C}(\Gamma;\underline{a}) := \Big(\prod_{v\in\Gamma^{[0]}_{\prec}}\sum_{h\in v}a_h\Big)\Big(\prod_{v\in\Gamma^{[1]}_{\sim}}a_{h_v^{\sim}}\Big)$$

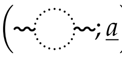
where $h_v^{\sim}\in v$ is the boson half-edge in the vertex $v$. Then, the corolla polynomial for a graph with only 3-boson vertices is defined as

$$\overline{\mathscr{C}}(\Gamma;\underline{a}) := \mathscr{C}(e^{-\delta}\Gamma;\underline{a})\,.$$

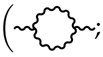
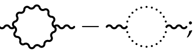
**Example 5.16.** We label the half-edges by the label of the vertex and the edge they belong to.

$\mathscr{C}\Big($  $;\underline{a}\Big) := (a_{\mathrm{a}1} + a_{\mathrm{a}3} + a_{\mathrm{a}4})(a_{\mathrm{b}1} + a_{\mathrm{b}3} + a_{\mathrm{b}4})$

and

$\mathscr{C}\Big($  $;\underline{a}\Big) = a_{\mathrm{a}1}a_{\mathrm{b}2}\,,$

so the corolla polynomial is

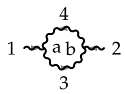
$\overline{\mathscr{C}}\Big($  $;\underline{a}\Big) = \mathscr{C}\Big($  $;\underline{a}\Big)$

$$= (a_{\mathrm{a}1} + a_{\mathrm{a}3} + a_{\mathrm{a}4})(a_{\mathrm{b}1} + a_{\mathrm{b}3} + a_{\mathrm{b}4}) - a_{\mathrm{a}1}a_{\mathrm{b}2}\,.$$

Next, we define for an half-edge $h \in \Gamma^{\mathrm{he}}$ the differential operator

$$D_h = f^{a_e a_{e_1} a_{e_2}} g^{\mu_{e_1}\mu_{e_2}} (\varepsilon_{h_1}\widehat{p}_{e_1} - \varepsilon_{h_1}\widehat{p}_{e_2})^{\mu_e}$$

where $\{h, h_1, h_2\} \in \Gamma^{[0]}$ is the vertex containing $h$ and $e, e_1, e_2 \in \Gamma^{[1]}$ are the edges incident to that vertex: $e \ni h$, $e_1 \ni h_1$ and $e_2 \ni h_2$. For example: for the graph

 ,

$$D_{\mathrm{a}1} = f^{a_1 a_3 a_4} g^{\mu_3\mu_4}(-\widehat{p}_3 - \widehat{p}_4)^{\mu_1} \qquad \text{and} \qquad D_{\mathrm{b}2} = f^{a_2 a_3 a_4} g^{\mu_3\mu_4}(\widehat{p}_3 + \widehat{p}_4)^{\mu_2}\,.$$

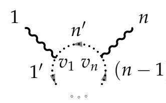
With this definition, one has, for $v \in \Gamma^{[0]}_{\prec}$,

$$\sum_{h\in v} D_h = \widehat{V}_v\,,$$

so if $\Gamma$ has no ghost loops

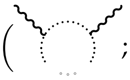
$$\mathscr{C}(\Gamma; \underline{D}) = \prod_{v\in\Gamma^{[0]}} \widehat{V}_v = \widehat{N}(\Gamma)\,.$$

Now take a ghost loop:

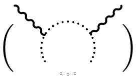
$\mathscr{C}\Big($  $;\underline{D}\Big) = D_{v_1 1}\cdots D_{v_n n}$

$$= f^{a_1 a_{n'} a_{1'}} \cdots f^{a_n a_{(n-1)'} a_{n'}} g^{\mu_{n'}\mu_{1'}} \cdots g^{\mu_{(n-1)'}\mu_{n'}} (\widehat{p}_{n'} + \widehat{p}_{1'})^{\mu_1} \cdots (\widehat{p}_{(n-1)'} + \widehat{p}_{n'})^{\mu_n}$$
$$= 4 f^{a_1 a_{n'} a_{1'}} \cdots f^{a_n a_{(n-1)'} a_{n'}} (\widehat{p}_{n'} + \widehat{p}_{1'})^{\mu_1} \cdots (\widehat{p}_{(n-1)'} + \widehat{p}_{n'})^{\mu_n}\,.$$

The string of metric tensors gives a factor 4. Working out the brackets gives $2^n$ terms. The two terms where every $\widehat{p}_e$ (or equivalently: every $\frac{1}{A_e}$) shows up exactly once give the ghost contributions. So if we get rid of the other $2^n - 2$ terms we get:

$\mathscr{C}\Big($  $;\underline{D}\Big)\Big|_{\frac{1}{A_{1'}^2},\ldots,\frac{1}{A_{n'}^2}\rightsquigarrow 0}$

$$= 4 f^{a_1 a_{n'} a_{1'}} \cdots f^{a_n a_{(n-1)'} a_{n'}} \left(\widehat{p}_{n'}^{\mu_1} \widehat{p}_{1'}^{\mu_2} \cdots \widehat{p}_{(n-1)'}^{\mu_n} + \widehat{p}_{1'}^{\mu_1} \cdots \widehat{p}_{n'}^{\mu_n}\right) = 4\widehat{N}\Big($$



$\Big).$

(See equation (5.31).)

So in general:

$$\widehat{\mathscr{C}}(\Gamma) := \mathscr{C}(\Gamma;\underline{D})\Big|_{\forall e \in \Gamma^{[1]}_{\cdots}:\frac{1}{A_e^2}\rightsquigarrow 0} = 4^{\#\mathscr{L}_\Gamma^{\cdots}}\, \widehat{N}(\Gamma)\,.$$

In the same way the corolla polynomial was defined for a graph $\Gamma$ with $\Gamma^{[0]} = \Gamma^{[0]}_{\prec}$, the following differential operator is defined:

$$\widehat{\widehat{\mathscr{C}}}(\Gamma) := \widehat{\mathscr{C}}(e^{-\delta/4}\Gamma)\,.$$

The factor $\frac{1}{4}$ in the exponent compensates the factor 4 that arises for every ghost loop, so:

$$\widehat{\widehat{\mathscr{C}}}(\Gamma) = \widehat{N}(e^{-\delta}\Gamma)\,.$$

# Summary

In this thesis a systematic method is given for writing the amplitudes in (scalar) quantum electrodynamics and non-Abelian gauge theories in Schwinger parametric form. This is done by turning the numerator of the Feynman rules in momentum space into a differential operator. It acts then on the parametric integrand of the scalar theory. For QED it is the most straightforward, because the Leibniz rule is not involved here. In the case of sQED and non-Abelian gauge theories, the contributions from the Leibniz rule are satisfyingly related to 4-valent vertices. Another feature of this method is that in the used renormalization scheme, the subtractions for 1-scale graphs cause significant simplifications.

Furthermore, the Ward identities for mentioned three theories are studied.

# Zusammenfassung

In dieser Arbeit wird eine systematische Methode gegeben um die Amplituden in (skalarer) Quantenelektrodynamik und nicht-Abelsche Eichtheorien in Schwinger-parametrische Form zu schreiben. Dies wird erreicht in dem der Zähler der Feynmanregeln im Impulsraum in einem Differentialoperator umgewandelt wird. Dieser Differentialoperator wirkt dann auf den parametrichen Integranden der skalaren Theorie. Für die QED ist das am einfachsten, weil die Leibnizregel hier nicht nötig ist. Im Fall der sQED und den nicht-Abelsche Eichtheorien stehen die Beiträge der Leibnizregel in Verbindung mit 4-valente Vertices. Eine andere Eigenschaft dieser Methode ist, dass mit dem hier benutzten Renormierungsschema die Subtraktionen für 1-scale Graphen signifikante Vereinfachungen verursachen.

Weiterhin wurden die Ward-Identitäte für die genannten drei Theorien studiert.

# Bibliography


[1] M. C. Bergère & J. B. Zuber, *Renormalization of Feynman amplitudes and parametric integral representation*, *Communications in Mathematical Physics* **35** (1974) 113–140

[2] Christian Bogner & Francis Brown, *Feynman integrals and iterated integrals on moduli spaces of curves of genus zero*, arXiv:1408.1862 [hep-th] (2014)

[3] Christian Bogner & Stefan Weinzierl, *Feynman graph polynomials*, *International Journal of Modern Physics* **A25** (2010) 2585–2618, arXiv:1002.3458 [hep-ph]

[4] Michael Borinsky, *Feynman graph generation and calculations in the Hopf algebra of Feynman graphs*, *Computer Physics Communications* **185** (2014) 3317-3330, arXiv:1402.2613 [hep-th]

[5] Francis Brown, *The Massless Higher-Loop Two-Point Function*, *Communications in Mathematical Physics* **287** (2009) 925–958

[6] Francis Brown & Dirk Kreimer, *Angles, Scales and Parametric Renormalization*, *Letters in Mathematical Physics* **103** (2013) 933–1007, arXiv:1112.1180 [hep-th]

[7] Predrag Cvitanović, *Field Theory*, Nordita Lecture Notes (1983), http://chaosbook.org/FieldTheory/

[8] Predrag Cvitanović & T. Kinoshita, *Feynman-Dyson rules in parametric space*, *Physical Review* **D10** (1974) 3478–3991

[9] G. 't Hooft, *Renormalization of Massless Yang-Mills Fields*, *Nuclear Physics* **B33** (1971) 173–199

[10] Claude Itzykson & Jean-Bernard Zuber, *Quantum Field Theory* (1980), Dover Publications

[11] Dirk Kreimer, Matthias Sars & Walter D. van Suijlekom, *Quantization of gauge fields, graph polynomials and graph homology*, *Annals of Physics* **336** (2013) 180–222, arXiv:1112.1180 [hep-th]

[12] Noboru Nakanishi, *Graph Theory and Feynman Integrals* (1971), Gordon and Breach

[13] Erik Panzer, *Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals* (2014), arXiv:1403.3385

[14] Erik Panzer, *Feynman integrals and hyperlogarithms*, PhD thesis, Humboldt-Universität zu Berlin (2014)

[15] Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory* (1995), Westview Press

[16] Walter D. van Suijlekom, *The Hopf Algebra of Feynman Graphs in Quantum Electrodynamics*, *Letters in Mathematical Physics* **77** (2006) 265–281, arXiv:hep-th/0602126

[17] Steven Weinberg, *The Quantum Theory of Fields, Volume II: Modern Applications* (1996), Cambridge University Press

Dear Dirk, thank you for welcoming me in your group, for your ideas and support, and thanks to Susanne and you for the wonderful dinners.

Dear Walter, thank you for getting me in touch with Dirk, for your help, and for welcoming me in Nijmegen for a month.

Dear colleagues of the Kreimer Gang: Claire, Erik, Henry, Konrad, Lutz, Marcus, Marko and Michael, and everyone else in the group, thanks for the good time and fruitful discussions.

Dear mam en pap, all my siblings, thanks for being there and for all your moral support.

Dear Janneke, thank you for everything during your last and my first year in Berlin. Thanks to all friends, far away and here Berlin. I feel privileged that I got to know many great people here. Dear Sako, Lora and Age, thank you for being there.

*Leve de fysica!*

December 2014