Internal Layer Solutions in Quasilinear Integro-Differential Equations

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Abstract. The Dirichlet boundary value problem for a class of singularly perturbed quasilinear integro-differential equations is considered. The asymptotic expansion for a new class of solutions, which have internal layers, is constructed. Theorems on existence, local uniqueness and asymptotic stability of such internal layer solutions are proved.

1. Introduction

Mathematical problems concerning reaction-advection-diffusion problems describe many important practical applications in chemical kinetics, synergetics, astrophysics, biology, etc. In many important cases the solutions of these problems feature internal layers (see [1] and references therein). Recently there is an increasing interest to more complicated models, which include the effects of feedback or non-local interaction. These models are represented by integro-differential equations (see [2]).

In this work we consider the boundary value problem

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - A(u, x, \varepsilon)\frac{\partial u}{\partial x} - \int\limits_a^b g(u(x), u(s), x, s, \varepsilon)\, ds, \quad a < x < b, \tag{1}$$

$$u(a, t, \varepsilon) = u^a, \quad u(b, t, \varepsilon) = u^b \tag{2}$$

and investigate the existence and stability of equilibrium internal layer solutions. Here, $A(u, x, \varepsilon)$ and $g(u, v, x, s, \varepsilon)$ are sufficiently smooth functions (their actual degree of smoothness is specified below), u^a and u^b are prescribed numbers, and $\varepsilon > 0$ is a small parameter. Equilibrium solutions to (1), (2) are solutions to the boundary value problem

$$L[u] \equiv \varepsilon \frac{d^2 u}{dx^2} - A(u, x, \varepsilon)\frac{du}{dx} - \int\limits_a^b g(u(x), u(s), x, s, \varepsilon)\, ds = 0, \quad a < x < b, \tag{3}$$

$$u(a, \varepsilon) = u^a, \quad u(b, \varepsilon) = u^b. \tag{4}$$

The corresponding boundary value problem for the case $A \equiv 0$ was considered in [3]. Internal layer solutions for quasilinear differential equations were analyzed in [4], [5, Section 5] and later in [6]. Solutions with boundary layers for (3), (4) were investigated in [7].

Our results develop and extend methods proposed in [3] and [7] to a new more complicated class of problems.

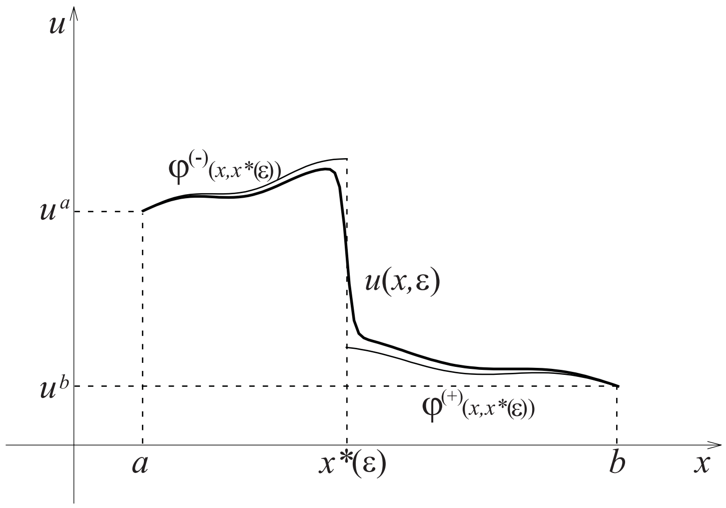

Fig. 1

The most important feature of problem (3), (4) is that the reduced equation

$$A(u,x,0)\frac{du}{dx}+\int\limits_a^b g(u(x),u(s),x,s,0)\,ds=0 \tag{5}$$

is a first-order integro-differential equation, and its continuous solutions can satisfy only one of the boundary conditions of the original problem, in general. Thus, in order to describe internal layer solutions we introduce a family of discontinuous solutions of problem (5), (4). Namely, we assume

Condition I. *There exist two functions*

$$\varphi^{(-)}(x,y)\in C^{1,0}_{x,y}(\Omega^{(-)}),\quad \textit{where}\quad \Omega^{(-)}\equiv\{(x,y): a\le x\le y\le b\},$$
$$\varphi^{(+)}(x,y)\in C^{1,0}_{x,y}(\Omega^{(+)}),\quad \textit{where}\quad \Omega^{(+)}\equiv\{(x,y): a\le y\le x\le b\},$$

which for every fixed $y\in(a,b)$ satisfy the system of two coupled integro-differential equations

$$\begin{aligned}
&A(\varphi^{(-)}(x,y),x,0)\frac{d\varphi^{(-)}}{dx}+\int\limits_a^y g(\varphi^{(-)}(x,y),\varphi^{(-)}(s,y),x,s,0)\,ds+\\
&\qquad+\int\limits_y^b g(\varphi^{(-)}(x,y),\varphi^{(+)}(s,y),x,s,0)\,ds=0,\quad a<x<y,\\
&A(\varphi^{(+)}(x,y),x,0)\frac{d\varphi^{(+)}}{dx}+\int\limits_a^y g(\varphi^{(+)}(x,y),\varphi^{(-)}(s,y),x,s,0)\,ds+\\
&\qquad+\int\limits_y^b g(\varphi^{(+)}(x,y),\varphi^{(+)}(s,y),x,s,0)\,ds=0,\quad y<x<b
\end{aligned} \tag{6}$$

with boundary conditions

$$\varphi^{(-)}(a, y) = u^a, \qquad \varphi^{(+)}(b, y) = u^b. \tag{7}$$

We show that, under some additional assumptions, problem (3), (4) has internal layer solutions (see Fig. 1) which pointwise tend to one of the discontinuous solutions, represented in the Condition I when the small parameter ε tends to zero.

2. The asymptotic expansion of the internal layer solution

In accordance with the boundary layer function method to build asymptotics of the seeking solution we shall use the following ansatz

$$u(x, \varepsilon) = \begin{cases} \overline{u}^{(-)}(x, x^\star(\varepsilon), \varepsilon) + Q^{(-)}(\xi, \varepsilon) & \text{for} \quad a \le x < x^\star(\varepsilon), \\ \overline{u}^{(+)}(x, x^\star(\varepsilon), \varepsilon) + Q^{(+)}(\xi, \varepsilon) & \text{for} \quad x^\star(\varepsilon) < x \le b. \end{cases} \tag{8}$$

Here $\overline{u}^{(\pm)}(x, y, \varepsilon)$ is the regular part and $Q^{(\pm)}(\xi, \varepsilon)$ is the internal layer part. The last one serves to describe the quick transition layer in the small vicinity of the point $x^\star(\varepsilon) \in (a, b)$ and thus depends on the stretched variable $\xi = [x - x^\star(\varepsilon)]/\varepsilon$. Each term of the presented ansatz is treated as an integer power series with respect to the small parameter ε, namely

$$\overline{u}^{(\pm)}(x, y, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \overline{u}_k^{(\pm)}(x, y), \qquad Q^{(\pm)}(\xi, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k Q_k^{(\pm)}(\xi).$$

For simplicity we shall omit indices $(-)$ and $(+)$ in what follows if it is possible without misunderstanding.

We note that $x^\star(\varepsilon)$ is also unknown from the beginning. We seek it in the following form

$$x^\star(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k x_k \tag{9}$$

and define this point by equation

$$u(x^\star(\varepsilon), \varepsilon) = \frac{1}{2}\left[\varphi^{(-)}(x_0, x_0) + \varphi^{(+)}(x_0, x_0)\right]. \tag{10}$$

In order to find the terms of the asymptotic expansion (9) we use the condition of C^1-matching of the asymptotics at the point $x^\star(\varepsilon)$:

$$u(x^\star(\varepsilon) - 0, \varepsilon) = u(x^\star(\varepsilon) + 0, \varepsilon), \qquad \varepsilon \frac{du}{dx}(x^\star(\varepsilon) - 0, \varepsilon) = \varepsilon \frac{du}{dx}(x^\star(\varepsilon) + 0, \varepsilon),$$

or

$$\begin{gathered} Q_0^{(-)}(0) + \overline{u}_0^{(-)}(x_0, x_0) + \\ + \sum_{k=1}^{\infty} \varepsilon^k \left\{ Q_k^{(-)}(0) + \overline{u}_k^{(-)}(x_0, x_0) + x_k \left[\frac{\partial \overline{u}_0^{(-)}}{\partial x}(x_0, x_0) + \frac{\partial \overline{u}_0^{(-)}}{\partial y}(x_0, x_0) \right] + M_k^{(-)} \right\} = \\ = Q_0^{(+)}(0) + \overline{u}_0^{(+)}(x_0, x_0) + \\ + \sum_{k=1}^{\infty} \varepsilon^k \left\{ Q_k^{(+)}(0) + \overline{u}_k^{(+)}(x_0, x_0) + x_k \left[\frac{\partial \overline{u}_0^{(+)}}{\partial x}(x_0, x_0) + \frac{\partial \overline{u}_0^{(+)}}{\partial y}(x_0, x_0) \right] + M_k^{(+)} \right\} \end{gathered} \tag{11}$$

and

$$\frac{dQ_0^{(-)}}{d\xi}(0) + \sum_{k=1}^{\infty} \varepsilon^k \left\{ \frac{dQ_k^{(-)}}{d\xi}(0) + N_k^{(-)} \right\} = \frac{dQ_0^{(+)}}{d\xi}(0) + \sum_{k=1}^{\infty} \varepsilon^k \left\{ \frac{dQ_k^{(+)}}{d\xi}(0) + N_k^{(+)} \right\}. \quad (12)$$

Here $M_k^{(\pm)}$ and $N_k^{(\pm)}$ are certain numbers recurrently expressed in terms of the preceding orders of the asymptotics, in particular

$$M_1^{(\pm)} = 0, \qquad N_1^{(\pm)} = \frac{\partial \overline{u}_0^{(\pm)}}{\partial x}(x_0, x_0).$$

To formulate problems that determine the terms appearing in this series, it is necessary to represent Eq. (3) in the form of a sum of regular and boundary layer parts. To do this, we represent the last two terms of the Eq. (3) as the sum

$$A(u(x,\varepsilon), x, \varepsilon)\frac{du}{dx} + \int_a^b g(u(x,\varepsilon), u(s,\varepsilon), x, s, \varepsilon)\, ds = \sum_{k=1}^{2} \overline{L}_k(x, y, \varepsilon) + \sum_{k=1}^{3} QL_k(\xi, \varepsilon). \quad (13)$$

Here and in what follows, the following notation is used:

$$\overline{L}_1(x, y, \varepsilon) \equiv A(\overline{u}(x, y, \varepsilon), x, \varepsilon)\frac{\partial \overline{u}}{\partial x} + \int_a^b g(\overline{u}(x, y, \varepsilon), \overline{u}(s, y, \varepsilon), x, s, \varepsilon)\, ds,$$

$$\overline{L}_2(x, \varepsilon) \equiv \int_a^b \left[g(\overline{u}(x, y, \varepsilon), u(s, \varepsilon), x, s, \varepsilon) - g(\overline{u}(x, y, \varepsilon), \overline{u}(s, x^\star(\varepsilon), \varepsilon), x, s, \varepsilon) \right]\, ds,$$

$$QL_1(\xi, \varepsilon) \equiv A(u(x, x^\star(\varepsilon), \varepsilon), x, \varepsilon)\frac{du}{dx} - A(\overline{u}(x, x^\star(\varepsilon), \varepsilon), x, \varepsilon)\frac{d\overline{u}}{dx}$$

$$QL_2(\xi, \varepsilon) \equiv \int_a^b \left[g(u(x, \varepsilon), \overline{u}(s, x^\star(\varepsilon), \varepsilon), x, s, \varepsilon) - g(\overline{u}(x, x^\star(\varepsilon), \varepsilon), \overline{u}(s, x^\star(\varepsilon), \varepsilon), x, s, \varepsilon) \right]\, ds$$

and

$$QL_3(\xi, \varepsilon) \equiv \int_a^b \left[g(u(x,\varepsilon), u(s,\varepsilon), x, s, \varepsilon) - g(\overline{u}(x, x^\star(\varepsilon), \varepsilon), u(s, \varepsilon), x, s, \varepsilon) - \right.$$
$$\left. - g(u(x, \varepsilon), \overline{u}(s, x^\star(\varepsilon), \varepsilon), x, s, \varepsilon) + g(\overline{u}(x, x^\star(\varepsilon), \varepsilon), \overline{u}(s, x^\star(\varepsilon), \varepsilon), x, s, \varepsilon) \right]\, ds.$$

Let us separately transform each of the above terms. Represent the first term $\overline{L}_1$ in the form

$$\overline{L}_1(x, y, \varepsilon) = A(\overline{u}_0(x, y), x, 0)\frac{\partial \overline{u}_0}{\partial x} + \int_a^b g(\overline{u}_0(x, y), \overline{u}_0(s, y), x, s, 0)\, ds +$$
$$+ \sum_{k=1}^{\infty} \varepsilon^k \left[A(\overline{u}_0(x, y), x, 0)\frac{\partial \overline{u}_k}{\partial x} + P(x, y)\overline{u}_k + \right.$$
$$\left. + \int_a^b g_v(\overline{u}_0(x, y), \overline{u}_0(s, y), x, s, 0)\overline{u}_k(s, y)\, ds + D_k(x, y) \right],$$

where

$$P(x,y) = A_u(\overline{u}_0(x,y),x,0)\frac{\partial \overline{u}_0}{\partial x} + \int\limits_a^b g_u(\overline{u}_0(x,y),\overline{u}_0(s,y),x,s,0)\,ds,$$

and $D_k(x,y)$ are recurrently expressed in terms of the preceding orders of the asymptotics, in particular,

$$D_1(x,y) = A_\varepsilon(\overline{u}_0(x,y),x,0)\frac{\partial \overline{u}_0}{\partial x} + \int\limits_a^b g_\varepsilon(\overline{u}_0(x,y),\overline{u}_0(s,y),x,s,0)\,ds.$$

The term $\overline{L}_2$ is transformed in a somewhat different way. Before expressing it as a series in the small parameter ε, we change the integration variable by the formula $s = x^\star(\varepsilon) + \varepsilon\tau$. As a result, we obtain

$$\begin{aligned}
\overline{L}_2(x,y,\varepsilon) = \varepsilon \int\limits_{(a-x^\star(\varepsilon))/\varepsilon}^{(b-x^\star(\varepsilon))/\varepsilon} &[g(\overline{u}(x,y,\varepsilon),\overline{u}(x^\star(\varepsilon)+\varepsilon\tau,x^\star(\varepsilon),\varepsilon)+Q(\tau),x,x^\star(\varepsilon)+\varepsilon\tau,\varepsilon)- \\
&-g(\overline{u}(x,y,\varepsilon),\overline{u}(x^\star(\varepsilon)+\varepsilon\tau,x^\star(\varepsilon),\varepsilon),x,x^\star(\varepsilon)+\varepsilon\tau,\varepsilon)]\;d\tau = \\
&= \varepsilon\int\limits_{-\infty}^{+\infty}[\ldots]\,d\tau - \varepsilon\int\limits_{-\infty}^{(a-x^\star(\varepsilon))/\varepsilon}[\ldots]\,d\tau - \varepsilon\int\limits_{(b-x^\star(\varepsilon))/\varepsilon}^{+\infty}[\ldots]\,d\tau = \\
= \varepsilon\int\limits_{-\infty}^{+\infty}[g(\overline{u}_0(x,y),\overline{u}_0(x_0,x_0)&+Q_0(\tau),x,x_0,0) - g(\overline{u}_0(x,y),\overline{u}_0(x_0,x_0),x,x_0,0)]\;d\tau + \\
&+\sum_{k=1}^{\infty}\varepsilon^{k+1}T_{k+1}(x,y) - \Phi_1(x,y,\varepsilon) - \Phi_2(x,y,\varepsilon).
\end{aligned}$$

Here, $T_k(x,y)$ are certain functions recurrently expressed in terms of the preceding orders of the asymptotics and

$$\Phi_1(x,y,\varepsilon) \equiv \varepsilon\int\limits_{-\infty}^{(a-x^\star(\varepsilon))/\varepsilon}[\ldots]\,d\tau,\quad \Phi_2(x,y,\varepsilon) \equiv \varepsilon\int\limits_{(b-x^\star(\varepsilon))/\varepsilon}^{+\infty}[\ldots]\,d\tau.$$

It is shown below that all the boundary functions $Q_k(\xi)$ have an exponential estimate at infinity (see formulae (33) below). Under this condition, the integrand in the definition of $\Phi_1(x,y,\varepsilon)$ also satisfies this estimate, i.e., there exist constants $C > 0$ and $\nu > 0$ such that the inequality $|[\ldots]| \le Ce^{\nu\tau}$ is fulfilled for all $\tau \le 0$. Then, the estimate $|\Phi_1(x,y,\varepsilon)| \le (\varepsilon C/\nu)e^{\nu(a-x^\star(\varepsilon))/\varepsilon}$ is valid for the function Φ_1, therefore, the relationship $\Phi_1(x,y,\varepsilon) = o(\varepsilon^n)$ is fulfilled for all $n \ge 0$ as $\varepsilon \to +0$. Analogously, $\Phi_2(x,y,\varepsilon) = o(\varepsilon^n)$ for all $n \ge 0$ as $\varepsilon \to +0$. Thus it is possible to neglect terms $\Phi_1(x,y,\varepsilon)$ and $\Phi_2(x,y,\varepsilon)$ in comparison with any other term with a power dependence on ε.

The terms QL_1 and QL_2 are transformed using the conventional scheme. Thus, after transformation of QL_1, we obtain

$$QL_1(\xi,\varepsilon) = \tfrac{1}{\varepsilon} A(\overline{u}_0(x_0,x_0) + Q_0, x_0, 0)\frac{dQ_0}{d\xi} +$$
$$+ \sum_{k=1}^{\infty} \varepsilon^{k-1} \left\{ \frac{d}{d\xi} \left[A(\overline{u}_0(x_0,x_0) + Q_0, x_0, 0)[Q_k + \overline{u}_k(x_0,x_0)] \right] + \right.$$
$$\left. + x_k \left[\tilde{A}_u(\xi) \left(\frac{\partial \overline{u}_0}{\partial x}(x_0,x_0) + \frac{\partial \overline{u}_0}{\partial y}(x_0,x_0) \right) + \tilde{A}_x(\xi) \right] \frac{dQ_0}{d\xi} + R_k(\xi) \right\},$$

where $R_k(\xi)$ are the terms recurrently expressed in terms of the preceding orders of the asymptotics; in particular,

$$R_1(\xi) = \left\{ \xi \left[\tilde{A}_u(\xi) \frac{\partial \overline{u}_0}{\partial x}(x_0,x_0) + \tilde{A}_x(\xi) \right] + \tilde{A}_\varepsilon(\xi) \right\} \frac{dQ_0}{d\xi} + \left[\tilde{A}(\xi) - \overline{A} \right] \frac{\partial \overline{u}_0}{\partial x}(x_0,x_0),$$

where $\tilde{A}(\xi) \equiv A(\overline{u}_0(x_0,x_0) + Q_0(\xi), x_0, 0)$ and $\overline{A} \equiv A(\overline{u}_0(x_0,x_0), x_0, 0)$ (here and in what follows it is assumed that, in expressions containing partial derivatives of the function A, we first calculate a partial derivative and then substitute the values marked with a tilde for the arguments of the resulting function).

With a little manipulation, the term QL_2 is reduced to the form

$$QL_2(\xi,\varepsilon) = \int_a^b [g(\overline{u}_0(x_0,x_0) + Q_0(\xi), \overline{u}_0(s,x_0), x_0, s, 0) - g(\overline{u}_0(x_0,x_0), \overline{u}_0(s,x_0), x_0, s, 0)]\ ds +$$
$$+ \sum_{k=1}^{\infty} \varepsilon^k S_k(\xi).$$

Finally, when transforming the term QL_3, the same sequence of operations as in the case with the term $\overline{L}_2$ is required. As a result, we obtain

$$QL_3(\xi,\varepsilon) = \varepsilon \int_{-\infty}^{+\infty} [g(\overline{u}_0(x_0,x_0) + Q_0(\xi), \overline{u}_0(x_0,x_0) + Q_0(\tau), x_0, x_0, 0) -$$
$$- g(\overline{u}_0(x_0,x_0), \overline{u}_0(x_0,x_0) + Q_0(\tau), x_0, x_0, 0) -$$
$$- g(\overline{u}_0(x_0,x_0) + Q_0(\xi), \overline{u}_0(x_0,x_0), x_0, x_0, 0) +$$
$$+ g(\overline{u}_0(x_0,x_0), \overline{u}_0(x_0,x_0), x_0, x_0, 0)]\ d\tau + \sum_{k=1}^{\infty} \varepsilon^{k+1} \Theta_{k+1}(\xi) - \Psi(\xi,\varepsilon).$$

Here $\Theta_k(\tau)$ are certain functions recurrently expressed in terms of the preceding orders of the asymptotics and the function $\Psi(\xi,\varepsilon)$ has the same origin as functions $\Phi_1(x,\varepsilon)$ and $\Phi_2(x,\varepsilon)$ in the case of the expansion for $\overline{L}_2$. By analogy with the above discussion, it can be shown that, in the subsequent reasoning, $\Psi(\xi,\varepsilon) = o(\varepsilon^n)$ for all $n \geq 0$, therefore, the function Ψ can be neglected in comparison with any term with a power dependence on ε.

Using the above scheme, we can also rewrite the second derivative in Eq. (3) in the form

$$\varepsilon \frac{d^2 u}{dx^2} = \sum_{k=0}^{\infty} \varepsilon^{k+1} \frac{\partial^2 \overline{u}_k}{\partial x^2} + \sum_{k=0}^{\infty} \varepsilon^{k-1} \frac{d^2 Q_k}{d\xi^2}.$$

Equating the sum of coefficients of the equal powers of ε to zero, we easily obtain equations for determining all terms of the asymptotic.

It is obvious that the equation for the zeroth order regular function $\overline{u}_0(x,y)$ coincides with the reduced equation (5). Thus, let us put $\overline{u}_0^{(\pm)}(x,y) = \varphi^{(\pm)}(x,y)$. Note, that on this step value x_0 is unknown.

Extracting leading terms from expansions (11) and (13), we obtain the problem for determination of the boundary functions $Q_0^{(\pm)}(\xi)$, namely

$$\frac{d^2 Q_0^{(\pm)}}{d\xi^2} = A(\overline{u}_0^{(\pm)}(x_0,x_0) + Q_0^{(\pm)}, x_0, 0)\frac{dQ_0^{(\pm)}}{d\xi}, \quad \xi \in \mathbb{R}^{\pm}, \tag{14}$$

$$Q_0^{(\pm)}(0) + \overline{u}_0^{(\pm)}(x_0,x_0) = u^\star(x_0) \equiv [\varphi^{(-)}(x_0,x_0) + \varphi^{(+)}(x_0,x_0)]/2, \tag{15}$$

$$Q_0^{(-)}(-\infty) = Q_0^{(+)}(+\infty) = 0. \tag{16}$$

Let us introduce the so called stability condition.

Condition II. *There exists $a_0 > 0$ such that*

$$\begin{aligned} &A(\varphi^{(-)}(x,y),x,0) \geq a_0 \quad \text{for all} \quad (x,y) \in \Omega^{(-)}, \\ &A(\varphi^{(+)}(x,y),x,0) \leq -a_0 \quad \text{for all} \quad (x,y) \in \Omega^{(+)}. \end{aligned} \tag{17}$$

As shown in the lemma below this condition guarantees that the Q_0-functions have exponential decay at infinity.

Lemma 1. *Let $A(u,x,0) \in C(\mathbb{R} \times [a,b])$ and Condition II is satisfied. Then for every solution to the problem (14)-(16) there exist constants $C_i > 0$ and $\nu_i > 0$ $(i = 1,2,3)$ such that $\left|Q_0^{(\pm)}(\xi)\right| \leq C_1 e^{-\nu_1|\xi|}$ and $C_2 e^{-\nu_2|\xi|} \leq \left|\frac{dQ_0^{(\pm)}}{d\xi}\right| \leq C_3 e^{-\nu_3|\xi|}$ for all $\xi \in \mathbb{R}$.*

Proof: Suppose that existing solution $Q_0^{(-)}(\xi)$ is putted in Eq. (14) for $\xi \leq 0$. Multiplying obtained identity by $\exp\left\{-\int\limits_0^\xi A(\overline{u}_0^{(-)}(x_0,x_0) + Q_0^{(-)}(\zeta), x_0, 0)\, d\zeta\right\}$ we have

$$\frac{d}{d\xi}\left[\frac{dQ_0^{(-)}}{d\xi}\exp\left\{-\int\limits_0^\xi A(\overline{u}_0^{(-)}(x_0,x_0) + Q_0^{(-)}(\zeta), x_0, 0)\, d\zeta\right\}\right] = 0.$$

Therefore,

$$\frac{dQ_0^{(-)}}{d\xi} = C\exp\left\{\int\limits_0^\xi A(\overline{u}_0^{(-)}(x_0,x_0) + Q_0^{(-)}(\zeta), x_0, 0)\, d\zeta\right\}$$

for some constant C. We have $C \neq 0$ because Condition II implies that $\overline{u}_0^{(\pm)}(x_0,x_0) \neq [\varphi^{(-)}(x_0,x_0) + \varphi^{(+)}(x_0,x_0)]/2$, and, hence, $Q_0^{(-)}(\xi) = 0$ can not satisfy both boundary conditions (15) and (16).

Boundary condition (16) and Condition II imply that there exist $\xi_0 > 0$ such that

$$\frac{a_0}{2} \leq A(\overline{u}_0^{(-)}(x_0, x_0) + Q_0^{(-)}(\zeta), x_0, 0) \leq \frac{3a_0}{2} \quad \text{for all} \quad \zeta \leq -\xi_0. \tag{18}$$

Thus, taking constant $C_2 > 0$ sufficiently small and constant $C_3 > 0$ sufficiently large we obtain

$$C_2 \exp\left(\frac{3a_0\xi}{2}\right) \leq \left|\frac{dQ_0^{(-)}}{d\xi}\right| \leq C_3 \exp\left(\frac{a_0\xi}{2}\right) \quad \text{for all} \quad \xi \leq 0.$$

Now let us integrate Eq. (14) by ξ in interval between ξ_1 and ξ_2 ($\xi_1, \xi_2 \leq 0$). As a result we obtain

$$\frac{dQ_0^{(-)}}{d\xi}(\xi_2) - \frac{dQ_0^{(-)}}{d\xi}(\xi_1) = \int\limits_{\overline{u}_0^{(-)}(x_0,x_0)+Q_0^{(-)}(\xi_1)}^{\overline{u}_0^{(-)}(x_0,x_0)+Q_0^{(-)}(\xi_2)} A(u, x_0, 0)du. \tag{19}$$

Expressions (16) and (19) imply $\frac{dQ_0^{(-)}}{d\xi}(-\infty) = 0$. Thus, tending $\xi_1 \to -\infty$ in (19) and recalling ξ_2 as ξ we obtain

$$\frac{dQ_0^{(-)}}{d\xi} = \int\limits_{\overline{u}_0^{(-)}(x_0,x_0)}^{\overline{u}_0^{(-)}(x_0,x_0)+Q_0^{(-)}(\xi)} A(u, x_0, 0)du. \tag{20}$$

Using mean value theorem and inequality (18) we found that

$$\frac{a_0}{2}\left|Q_0^{(-)}(\xi)\right| \leq \left|\int\limits_{\overline{u}_0^{(-)}(x_0,x_0)}^{\overline{u}_0^{(-)}(x_0,x_0)+Q_0^{(-)}(\xi)} A(u, x_0, 0)du\right| = \left|\frac{dQ_0^{(-)}}{d\xi}\right| \leq C_3 \exp\left(\frac{a_0\xi}{2}\right)$$

for all $\xi \leq -\xi_0$. Therefore we have

$$\left|Q_0^{(-)}(\xi)\right| \leq C_1 \exp\left(\frac{a_0\xi}{2}\right) \quad \text{for all} \quad \xi \leq 0,$$

where C_1 is appropriate number. The same estimates for function $Q_0^{(+)}(\xi)$ and its derivative for $\xi \geq 0$ are obtained analogously. ▲

The problem (14)-(16) is not fully defined until value x_0 is unknown. As we show in the below lemma this value can be calculated from the zeroth order C^1-matching condition (see (12))

$$\frac{dQ_0^{(-)}}{d\xi}(0) = \frac{dQ_0^{(+)}}{d\xi}(0). \tag{21}$$

Lemma 2. *Let $A(u,x,0) \in C(\mathbb{R} \times [a,b])$. Then problem (14)-(16) with additional condition (21) have solution only if $J(x_0) = 0$, where*

$$J(x) \equiv \int\limits_{\varphi^{(-)}(x,x)}^{\varphi^{(+)}(x,x)} A(u,x,0)du. \tag{22}$$

Proof: Using the same arguments as in Lemma 1 we can write by analogy with formula (20) integral expression for $\frac{dQ_0^{(+)}}{d\xi}$. Putting expressions for $\frac{dQ_0^{(-)}}{d\xi}$ and $\frac{dQ_0^{(+)}}{d\xi}$ from $(20)_{(-)}$ and $(20)_{(+)}$ in match condition (21) and simplifying it with the help of condition (15) we shall have $J(x_0) = 0$. ▲

Suppose below the next conditions

Condition III. *Let equation $J(x) = 0$ has root x_0 such that $J'_x(x_0) \neq 0$.*

Condition IV. *Let $\int\limits_{\varphi^{(-)}(x_0,x_0)}^{v} A(u,x_0,0)du \neq 0$ for all v from interval between $\varphi^{(-)}(x_0,x_0)$ and $\varphi^{(+)}(x_0,x_0)$.*

If Conditions I-IV are satisfied then problem (14)-(16) with x_0 from equation $J(x_0) = 0$ has unique solution. Integrating Eq. $(20)_{(-)}$ and $(20)_{(+)}$ we can represent this solution in implicit form

$$\int\limits_{u^\star(x_0)}^{\varphi^{(\pm)}(x_0,x_0)+Q_0^{(\pm)}} \left[\int\limits_{\varphi^{(\pm)}(x_0,x_0)}^{z} A(u,x_0,0)\, du \right]^{-1} dz = \xi.$$

We note here that Condition IV guarantees that functions $Q_0^{(-)}(\xi)$ and $Q_0^{(+)}(\xi)$ are monotone in intervals $(-\infty, 0]$ and $[0, +\infty)$, respectively. Moreover,

$$\operatorname{sign}\left[\frac{dQ_0^{(\pm)}}{d\xi}\right] = \operatorname{sign}\left[\varphi^{(+)}(x_0,x_0) - \varphi^{(-)}(x_0,x_0)\right] \quad \text{for all} \quad \xi \in \mathbb{R}^{\pm}. \tag{23}$$

We now turn to the analysis of problems for determining higher order terms in the asymptotic expansion. The function $\overline{u}_1(x,y)$ is determined by the equation

$$\begin{gathered} A(\overline{u}_0(x,y),x,0)\frac{\partial \overline{u}_1}{\partial x} + P(x,y)\overline{u}_1(x,y) + \\ + \int\limits_a^b g_v(\overline{u}_0(x,y),\overline{u}_0(s,y),x,s,0)\overline{u}_1(s,y)\, ds + D_1(x,y) + T_1(x,y) = \frac{\partial^2 \overline{u}_0}{\partial x^2} \end{gathered} \tag{24}$$

with the homogenous boundary conditions

$$\overline{u}_1(a,y) = 0, \quad \overline{u}_1(b,y) = 0. \tag{25}$$

By virtue of Condition II, both sides of Eq (24) can be divided by $A(\overline{u}_0(x,y),x,0)$. Thus, by analogy with (6), (7), one can rewrite problem (24), (25) in the form of two coupled linear integro-differential equations

$$\begin{aligned}
&\frac{d\overline{u}_1^{(-)}}{dx} + p^{(-)}(x,y)\overline{u}_1^{(-)} + \int\limits_a^y K^{(--)}(x,s,y)\overline{u}_1^{(-)}(s,y)\,ds + \\
&\qquad + \int\limits_y^b K^{(-+)}(x,s,y)\overline{u}_1^{(+)}(s,y)\,ds = f_1^{(-)}(x,y), \quad a < x < y, \\
&\frac{d\overline{u}_1^{(+)}}{dx} + p^{(+)}(x,y)\overline{u}_1^{(+)} + \int\limits_a^y K^{(+-)}(x,s,y)\overline{u}_1^{(-)}(s,y)\,ds + \\
&\qquad + \int\limits_y^b K^{(++)}(x,s,y)\overline{u}_1^{(+)}(s,y)\,ds = f_1^{(+)}(x,y), \quad y < x < b
\end{aligned} \tag{26}$$

with boundary conditions

$$\overline{u}_1^{(-)}(a,y) = 0, \qquad \overline{u}_1^{(+)}(b,y) = 0. \tag{27}$$

Here

$$p^{(i)}(x,y) = \frac{P^{(i)}(x,y)}{A(\overline{u}_0^{(i)}(x,y),x,0)}, \qquad K^{(ij)}(x,s,y) = \frac{g_v(\overline{u}_0^{(i)}(x,y),\overline{u}_0^{(j)}(s,y),x,s,0)}{A(\overline{u}_0^{(i)}(x,y),x,0)},$$

$$f_1^{(i)}(x,y) = \frac{\dfrac{\partial^2\overline{u}_0^{(i)}}{\partial x^2} - D_1^{(i)}(x,y) - T_1^{(i)}(x,y)}{A(\overline{u}_0^{(i)}(x,y),x,0)}, \quad i,j = \{-,+\}.$$

Problem (26), (27) is investigated in details in Appendix.

Suppose that the following condition is satisfied.

Condition V. *The system of coupled integral equations*

$$\begin{aligned}
v^{(-)}(x) &= \int\limits_a^{x_0} N^{(--)}(x,s,x_0)v^{(-)}(s)\,ds + \int\limits_{x_0}^b N^{(-+)}(x,s,x_0)v^{(+)}(s)\,ds, \quad a \le x \le y, \\
v^{(+)}(x) &= \int\limits_a^{x_0} N^{(+-)}(x,s,x_0)v^{(-)}(s)\,ds + \int\limits_{x_0}^b N^{(++)}(x,s,x_0)v^{(+)}(s)\,ds, \quad x \le y \le b,
\end{aligned}$$

where

$$\begin{aligned}
N^{(--)}(x,s,y) &= -\int\limits_a^x K^{(--)}(z,s,y)\exp\left\{-\int\limits_z^x p^{(-)}(\xi,y)d\xi\right\}dz, \\
N^{(-+)}(x,s,y) &= -\int\limits_a^x K^{(-+)}(z,s,y)\exp\left\{-\int\limits_z^x p^{(-)}(\xi,y)d\xi\right\}dz, \\
N^{(+-)}(x,s,y) &= \int\limits_x^b K^{(+-)}(z,s,y)\exp\left\{-\int\limits_z^x p^{(+)}(\xi,y)d\xi\right\}dz, \\
N^{(++)}(x,s,y) &= \int\limits_x^b K^{(++)}(z,s,y)\exp\left\{-\int\limits_z^x p^{(+)}(\xi,y)d\xi\right\}dz,
\end{aligned} \tag{28}$$

does not have any nontrivial solution.

If Condition V is fulfilled then Lemmas A_1, A_2, A_5 (see Appendix) imply that there exist $\Delta > 0$ such that for all $y \in [x_0 - \Delta, x_0 + \Delta]$ problem (26), (27) has unique solution $\overline{u}_1^{(\pm)}(x, y)$ with discontinuity at the point y.

The boundary layer functions $Q_1^{(\pm)}(\xi)$ are the solutions of the equations

$$\frac{d^2 Q_1^{(\pm)}}{d\xi^2} = \frac{d}{d\xi}\left[A(\overline{u}_0^{(\pm)}(x_0, x_0) + Q_0^{(\pm)}, x_0, 0)[Q_1^{(\pm)} + \overline{u}_1^{(\pm)}(x_0, x_0)]\right] + \\ +x_1\left[\tilde{A}_u^{(\pm)}(\xi)\left(\frac{\partial \overline{u}_0^{(\pm)}}{\partial x}(x_0, x_0) + \frac{\partial \overline{u}_0^{(\pm)}}{\partial y}(x_0, x_0)\right) + \tilde{A}_x^{(\pm)}(\xi)\right]\frac{dQ_0^{(\pm)}}{d\xi} + H_1^{(\pm)}(\xi), \tag{29}$$

where $H_1^{(\pm)}(\xi) = R_1^{(\pm)}(\xi) + S_0^{(\pm)}(\xi)$, considered for $\xi \in \mathbb{R}^{\pm}$, correspondingly, and satisfying to the boundary conditions

$$Q_1^{(-)}(0) + \overline{u}_1^{(-)}(x_0, x_0) + x_1\left[\frac{\partial \overline{u}_0^{(-)}}{\partial x}(x_0, x_0) + \frac{\partial \overline{u}_0^{(-)}}{\partial y}(x_0, x_0)\right] = \\ = Q_1^{(+)}(0) + \overline{u}_1^{(+)}(x_0, x_0) + x_1\left[\frac{\partial \overline{u}_0^{(+)}}{\partial x}(x_0, x_0) + \frac{\partial \overline{u}_0^{(+)}}{\partial y}(x_0, x_0)\right] = 0, \tag{30}$$

$$Q_1^{(-)}(-\infty) = Q_1^{(+)}(+\infty) = 0. \tag{31}$$

To analyze problem (29)-(31) we shall use the same scheme as for problem (14)-(16) but in simpler form because of problem (29)-(31) is linear. ¿From Lemma 1 and the definition of $H_1(\xi)$ it immediately follows that $H_1(\xi)$ has estimate $\left|H_1^{(\pm)}(\xi)\right| \le Ce^{-\nu|\xi|}$ for all $\xi \in \mathbb{R}^{\pm}$. Then we can easy obtain like in Lemma 1 that $\frac{dQ_0^{(\pm)}}{d\xi}(\pm\infty) = 0$. Now it is obvious that, if the value x_1 is known, the linear problem (29)-(31) has a unique solution and this solution also has an exponential estimate at infinity.

To calculate value x_1 we shall use the first order C^1-matching condition (see (12))

$$\frac{dQ_1^{(-)}}{d\xi}(0) + \frac{\partial \overline{u}_0^{(-)}}{\partial x}(x_0, x_0) = \frac{dQ_1^{(+)}}{d\xi}(0) + \frac{\partial \overline{u}_0^{(+)}}{\partial x}(x_0, x_0). \tag{32}$$

Integrating Eq. (29) by ξ in interval $(-\infty, 0]$ we obtain

$$\frac{dQ_1^{(-)}}{d\xi}(0) = A(u^\star(x_0), x_0, 0)\left[Q_1^{(-)}(0) + \overline{u}_1^{(-)}(x_0, x_0)\right] - A(\overline{u}_0^{(-)}(x_0, x_0), x_0, 0)\overline{u}_1^{(-)}(x_0, x_0) + \\ +x_1\left[\frac{\partial \overline{u}_0^{(-)}}{\partial x}(x_0, x_0) + \frac{\partial \overline{u}_0^{(-)}}{\partial y}(x_0, x_0)\right]\left\{A(u^\star(x_0), x_0, 0) - A(\overline{u}_0^{(-)}(x_0, x_0), x_0, 0)\right\} + \\ +x_1\int\limits_{\overline{u}_0^{(-)}(x_0, x_0)}^{u^\star(x_0)} A_x(u, x_0, 0)du + \int\limits_{-\infty}^{0} H_1^{(-)}(\tau)d\tau.$$

By the same way we can obtain expression for $\dfrac{dQ_1^{(+)}}{d\xi}(0)$, namely

$$-\frac{dQ_1^{(+)}}{d\xi}(0) = A(\overline{u}_0^{(+)}(x_0,x_0),x_0,0)\overline{u}_1^{(+)}(x_0,x_0) - A(u^\star(x_0),x_0,0)\left[Q_1^{(+)}(0)+\overline{u}_1^{(+)}(x_0,x_0)\right]+$$
$$+x_1\left[\frac{\partial\overline{u}_0^{(+)}}{\partial x}(x_0,x_0)+\frac{\partial\overline{u}_0^{(+)}}{\partial y}(x_0,x_0)\right]\left\{A(\overline{u}_0^{(+)}(x_0,x_0),x_0,0)-A(u^\star(x_0),x_0,0)\right\}+$$
$$+x_1\int\limits_{u^\star(x_0)}^{\overline{u}_0^{(+)}(x_0,x_0)} A_x(u,x_0,0)du+\int\limits_0^{+\infty}H_1^{(+)}(\tau)d\tau.$$

Using last two expressions and formulae (30), (32), we can write

$$\frac{\partial\overline{u}_0^{(+)}}{\partial x}(x_0,x_0)-\frac{\partial\overline{u}_0^{(-)}}{\partial x}(x_0,x_0)=A(\overline{u}_0^{(+)}(x_0,x_0),x_0,0)\overline{u}_1^{(+)}(x_0,x_0)-$$
$$-A(\overline{u}_0^{(-)}(x_0,x_0),x_0,0)\overline{u}_1^{(-)}(x_0,x_0)+x_1\left\{A(\overline{u}_0^{(+)}(x_0,x_0),x_0,0)\left[\frac{\partial\overline{u}_0^{(+)}}{\partial x}(x_0,x_0)+\frac{\partial\overline{u}_0^{(+)}}{\partial y}(x_0,x_0)\right]-\right.$$
$$-A(\overline{u}_0^{(-)}(x_0,x_0),x_0,0)\left[\frac{\partial\overline{u}_0^{(-)}}{\partial x}(x_0,x_0)+\frac{\partial\overline{u}_0^{(-)}}{\partial y}(x_0,x_0)\right]+$$
$$\left.+\int\limits_{\overline{u}_0^{(-)}(x_0,x_0)}^{\overline{u}_0^{(+)}(x_0,x_0)}A_x(u,x_0,0)du\right\}+\int\limits_{-\infty}^{0}H_1^{(-)}(\tau)d\tau+\int\limits_0^{+\infty}H_1^{(+)}(\tau)d\tau.$$

After simplification of this expression we obtain

$$\frac{\partial\overline{u}_0^{(+)}}{\partial x}(x_0,x_0)-\frac{\partial\overline{u}_0^{(-)}}{\partial x}(x_0,x_0)=A(\overline{u}_0^{(+)}(x_0,x_0),x_0,0)\overline{u}_1^{(+)}(x_0,x_0)-$$
$$-A(\overline{u}_0^{(-)}(x_0,x_0),x_0,0)\overline{u}_1^{(-)}(x_0,x_0)+x_1J_x'(x_0)+\int\limits_{-\infty}^{0}H_1^{(-)}(\tau)d\tau+\int\limits_0^{+\infty}H_1^{(+)}(\tau)d\tau.$$

Because of $J_x'(x_0)\neq 0$ (see Condition III) we can define x_1 from the last equation.

We can easily check that the problems for determining the terms $\overline{u}_k(x,y)$ and $Q_k(\xi)$ for $k\geq 2$ have the same structure as problem (24), (25) and (29)-(31), respectively. Thus, performing the above procedure by induction, we can prove that all these problems are always solvable as well, and the following estimates hold for all $k\geq 1$:

$$\left|Q_k^{(\pm)}(\xi)\right|\leq Ce^{-\nu|\xi|},\quad \left|\frac{dQ_k^{(\pm)}}{d\xi}\right|\leq Ce^{-\nu|\xi|},\quad \left|H_k^{(\pm)}(\xi)\right|\leq Ce^{-\nu|\xi|}\quad \text{for all}\quad \xi\in\mathbb{R}^{(\pm)}, \tag{33}$$

where $C>0$ and $\nu>0$ are certain constants independent of ξ.

3. Main Result

To validate the asymptotics constructed above, we invoke the method of differential inequalities [2], [8], [9]. Recall the classical definition of upper and lower solutions to problem (3), (4) (it can be given by analogy with definitions of more general problems presented in [2]).

Definition. *The function* $\beta(x) \in C[a,b] \cap C^2(a,x^\star] \cap C^2[x^\star,b)$, *where* $x^\star \in (a,b)$, *is called an upper solution to problem (3), (4) if*

1) $L[\beta] \le 0$ *for all* $x \in (a,x^\star) \cap (x^\star,b)$,

2) $\dfrac{d\beta}{dx}(x^\star+0) - \dfrac{d\beta}{dx}(x^\star-0) \le 0$,

3) $\beta(a) \ge u^a$ *and* $\beta(b) \ge u^b$.

Similarly, the function $\alpha(x)$ *belonging to the same class of smoothness is called a lower solution if it satisfies the conversed inequalities.*

The proof of the existence of a solution to problem (3), (4) relies on the following theorem on differential inequalities.

Theorem 1. *Assume that there exist the functions* $\alpha(x)$ *and* $\beta(x)$ *such that the following conditions are valid:*

(a) $\alpha(x)$ *and* $\beta(x)$ *are the lower and upper solutions to problem (3), (4), respectively;*

(b) $\alpha(x) \le \beta(x)$ *for all* $x \in [a,b]$;

(c) $A(u,x,\cdot) \in C^1([\alpha(x),\beta(x)] \times [a,b])$, $g(u,v,x,s,\cdot)$ *and* $g_u(\dots)$, $g_v(\dots) \in C([\alpha(x),\beta(x)] \times [\alpha(s),\beta(s)] \times [a,b]^2)$;

(d) $g_v(\dots) \le 0$ *for all* $(u,v,x,s) \in [\alpha(x),\beta(x)] \times [\alpha(s),\beta(s)] \times [a,b]^2$.

Then problem (3), (4) has at least one classical solution $u(x)$ *such that* $\alpha(x) \le u(x) \le \beta(x)$ *for all* $x \in [a,b]$.

We note once again that Theorem 1 can be used in our case only if the function g in Eq. (3) satisfies the following condition.

Condition VI. *The function* $g(u,v,x,s,\varepsilon)$ *is monotonically nonincreasing with respect to* v *for all admissible values of its arguments.*

The lower and upper solutions to problem (3), (4) are sought in the form

$$\begin{aligned} \alpha_n(x,\varepsilon) &= U_{(n+1)\alpha}(x,\varepsilon) + \varepsilon^{n+1}\left[-q(x,y) + w_\alpha(\xi_\alpha)\right] + \varepsilon^{n+2} Q_{(n+2)\alpha}(\xi_\alpha,\varepsilon), \\ \beta_n(x,\varepsilon) &= U_{(n+1)\beta}(x,\varepsilon) + \varepsilon^{n+1}\left[q(x,y) + w_\beta(\xi_\beta)\right] + \varepsilon^{n+2} Q_{(n+2)\beta}(\xi_\beta,\varepsilon), \end{aligned} \tag{34}$$

where $U_{(n+1)\alpha}(x,\varepsilon)$ and $U_{(n+1)\beta}(x,\varepsilon)$ are $(n+1)$-st order $(n \ge 0)$ partial sums of series (8) with $y = x^\star_\alpha(\varepsilon)$ and $y = x^\star_\beta(\varepsilon)$, respectively, and

$$x^\star_\alpha(\varepsilon) = \sum_{k=0}^{n+1} \varepsilon^k x_k - \varepsilon^{n+1}\delta, \quad x^\star_\beta(\varepsilon) = \sum_{k=0}^{n+1} \varepsilon^k x_k + \varepsilon^{n+1}\delta.$$

Besides stretched variable ξ we put $\xi_\alpha = [x - x^\star_\alpha(\varepsilon)]/\varepsilon$ in α_n, and we put $\xi_\beta = [x - x^\star_\beta(\varepsilon)]/\varepsilon$ in β_n. The functions $q(x,y)$, $w_\alpha(\xi)$, $w_\beta(\xi)$, $Q_{(n+2)\alpha}(\xi,\varepsilon)$, $Q_{(n+2)\beta}(\xi,\varepsilon)$ and parameter δ are defined below.

Taking into account the form of the upper solution (34) and the expansion (13) presented in Section 2, we obtain

$$
\begin{aligned}
L[\beta_n] = \varepsilon^n \Bigg\{ \frac{d^2 w_\beta}{d\xi^2} - \frac{d}{d\xi}\Big[A(\overline{u}_0(x_0,x_0)+Q_0,x_0,0)[w_\beta + q(x_0,x_0)]\Big] - \\
-\delta\left[\tilde{A}_u(\xi)\left(\frac{\partial\overline{u}_0}{\partial x}(x_0,x_0)+\frac{\partial\overline{u}_0}{\partial y}(x_0,x_0)\right)+\tilde{A}_x(\xi)\right]\frac{dQ_0}{d\xi}\Bigg\} - \\
-\varepsilon^{n+1}\left\{A(\overline{u}_0(x,y),x,0)\frac{\partial q}{\partial x}+P(x,y)q(x,y)+\int_a^b g_v(\overline{u}_0(x,y),\overline{u}_0(s,y),x,s,0)q(s,y)ds\right\} + \\
+\varepsilon^{n+1}\left\{\frac{d^2 Q_{(n+2)\beta}}{d\xi^2}-\frac{d}{d\xi}\Big[A(\overline{u}_0(x_0,x_0)+Q_0,x_0,0)Q_{(n+2)\beta}\Big]-H_{(n+2)\beta}(\xi)\right\}+o(\varepsilon^{n+1}).
\end{aligned}
$$

Here, $H_{(n+2)\beta}(\xi)$ is the function that has the same structure as $H_{n+2}(\xi)$ except for the changes incorporated into the $(n+1)$-st order of the asymptotic.

Let us assume that the functions $w_\beta(\xi)$ and $Q_{(n+2)\beta}(\xi,\varepsilon)$ are chosen such that the first and the third braces are equal to zero and the function $q(x,y)$ is a solution to the integro-differential equation

$$
A(\overline{u}_0(x,y),x,0)\frac{\partial q}{\partial x}+P(x,y)q(x,y)+\int_a^b g_v(\overline{u}_0(x,y),\overline{u}_0(s,y),x,s,0)q(s,y)ds = 1 \qquad (35)
$$

with boundary conditions

$$
q(a,y)=1, \qquad q(b,y)=1. \qquad (36)
$$

In this case, $L[\beta_n] \le -\varepsilon^{n+1}+o(\varepsilon^{n+1})$. Thus, for sufficiently small values of the parameter ε we obtain $L[\beta_n]<0$.

By analogy with (24), (25), one can reduce problem (35), (36) to the problem, considered in Appendix. It will be clear from the following consideration that we are not concerned with arbitrary solutions to (35), (36) but only with those solutions that for y from vicinity of x_0 are positive everywhere on the interval $x \in [a,b]$. Therefore, the problem is to determine conditions under which (35), (36) has a positive solution $q(x,y)$.

Condition II and Condition VI imply that operator T_y, corresponding to (35), (36), is positive (see Appendix). Let us introduce new condition.

Condition V*. *Suppose that all eigenvalues λ of the eigenvalue problem*

$$
\begin{aligned}
\lambda v^{(-)}(x) &= \int_a^{x_0} N^{(--)}(x,s,x_0)v^{(-)}(s)\,ds + \int_{x_0}^b N^{(-+)}(x,s,x_0)v^{(+)}(s)\,ds, \ a \le x \le y \\
\lambda v^{(+)}(x) &= \int_a^{x_0} N^{(+-)}(x,s,x_0)v^{(-)}(s)\,ds + \int_{x_0}^b N^{(++)}(x,s,x_0)v^{(+)}(s)\,ds, \ y \le x \le b
\end{aligned} \qquad (37)
$$

satisfy the inequality $|\lambda|<1$.

Obviously, Condition V* implies Condition V. Moreover, from Lemmas A_1-A_5 we obtain the next result.

Lemma 3. *If Condition V* and Condition VI are fulfilled, then there exist constants $\Delta>0$ and $q_0>0$ such that for every $y\in[x_0-\Delta,x_0+\Delta]$ problem (35), (36) has positive solution $q(x,y)$ satisfying $q(x,y)\ge q_0$ for all $(x,y)\in[a,b]\times[x_0-\Delta,x_0+\Delta]$.*

The function $w_\beta(\xi)$ is determined by solving the problem

$$\begin{gathered}\frac{d^2 w_\beta^{(\pm)}}{d\xi^2} = \frac{d}{d\xi}\left[A(\overline{u}_0^{(\pm)}(x_0,x_0) + Q_0^{(\pm)}, x_0, 0)[w_\beta^{(\pm)} + q^{(\pm)}(x_0,x_0)]\right] + \\ +\delta\left[\tilde{A}_u^{(\pm)}(\xi)\left(\frac{\partial \overline{u}_0^{(\pm)}}{\partial x}(x_0,x_0) + \frac{\partial \overline{u}_0^{(\pm)}}{\partial y}(x_0,x_0)\right) + \tilde{A}_x^{(\pm)}(\xi)\right]\frac{dQ_0^{(\pm)}}{d\xi}, \quad \xi \in \mathbb{R}^\pm, \\ w_\beta^{(-)}(0) + q^{(-)}(x_0,x_0) + \delta\left[\frac{\partial \overline{u}_0^{(-)}}{\partial x}(x_0,x_0) + \frac{\partial \overline{u}_0^{(-)}}{\partial y}(x_0,x_0)\right] = \\ = w_\beta^{(+)}(0) + q^{(+)}(x_0,x_0) + \delta\left[\frac{\partial \overline{u}_0^{(-)}}{\partial x}(x_0,x_0) + \frac{\partial \overline{u}_0^{(-)}}{\partial y}(x_0,x_0)\right] = 0, \\ w_\beta^{(-)}(-\infty) = w_\beta^{(+)}(+\infty) = 0\end{gathered} \tag{38}$$

Problem (38) can be investigated by analogy with problem (29)-(31). Obviously, it has unique solution for all real δ. Moreover, it is possible to find that

$$\frac{dw_\beta^{(+)}}{d\xi}(0) - \frac{dw_\beta^{(-)}}{d\xi}(0) = A(\overline{u}_0^{(-)}(x_0,x_0), x_0, 0)q^{(-)}(x_0,x_0) - A(\overline{u}_0^{(+)}(x_0,x_0), x_0, 0)q^{(+)}(x_0,x_0) - \delta J_x'(x_0).$$

This difference will be negative for sufficiently large positive values of $\delta J_x'(x_0)$. The last is possible only if

$$\operatorname{sign}[\delta] = \operatorname{sign}\left[J_x'(x_0)\right]. \tag{39}$$

Suppose that this fact is true. Then

$$\frac{d\beta}{dx}(x_\beta^\star(\varepsilon)+0) - \frac{d\beta}{dx}(x_\beta^\star(\varepsilon)-0) = \varepsilon^n\left\{\frac{dw_\beta^{(+)}}{d\xi}(0) - \frac{dw_\beta^{(-)}}{d\xi}(0)\right\} + O(\varepsilon^{n+1}) < 0$$

for sufficiently small $\varepsilon > 0$.

The function $Q_{(n+2)\beta}(\xi)$ is determined by solving the problem

$$\begin{gathered}\frac{d^2 Q_{(n+2)\beta}^{(\pm)}}{d\xi^2} = \frac{d}{d\xi}\left[A(\overline{u}_0^{(\pm)}(x_0,x_0) + Q_0^{(\pm)}, x_0, 0)Q_{(n+2)\beta}^{(\pm)}\right] + H_{(n+2)\beta}^{(\pm)}(\xi), \quad \xi \in \mathbb{R}^\pm, \\ Q_{(n+2)\beta}^{(-)}(0) = 0, \quad Q_{(n+2)\beta}^{(+)}(0) = d_\beta(\varepsilon), \\ Q_{(n+2)\beta}^{(-)}(-\infty) = Q_{(n+2)\beta}^{(+)}(+\infty) = 0,\end{gathered} \tag{40}$$

where

$$\begin{gathered}d_\beta(\varepsilon) = \varepsilon^{-(n+2)}\left\{U_{(n+1)\beta}(x_\beta^\star(\varepsilon)-0,\varepsilon) + \varepsilon^{n+1}\left[q^{(-)}(x_\beta^\star(\varepsilon), x_\beta^\star(\varepsilon)) + w_\beta^{(-)}(0)\right] - \right. \\ \left. -U_{(n+1)\beta}(x_\beta^\star(\varepsilon)+0,\varepsilon) - \varepsilon^{n+1}\left[q^{(+)}(x_\beta^\star(\varepsilon), x_\beta^\star(\varepsilon)) + w_\beta^{(+)}(0)\right]\right\}.\end{gathered}$$

Definition of $U_{(n+1)\beta}(x,\varepsilon)$ implies that $d_\beta(\varepsilon) = O(1)$ when $\varepsilon \to +0$. Thus, since the function $H_{(n+2)\beta}(\xi)$ has the same exponential estimate at infinity as the function $H_{n+2}(\xi)$ (see formula (33)), then problem (40) is solvable by analogy with problem (29)-(31). Its solution is unique and has exponential estimate like (33) uniformly by ε from some interval $(0, \varepsilon_0]$.

Now it is easy to see that built function $\beta_n(x,\varepsilon)$ is upper solution for problem (3), (4) for sufficiently small values of ε. Similarly, it can be shown that, under the above conditions, functions $w_\alpha(\xi)$ and $Q_{(n+2)\alpha}(\xi)$ can be chosen such that $\alpha_n(x,\varepsilon)$ is lower solution for problem (3), (4).

Let us investigate now sign of the difference $\beta_n(x,\varepsilon)-\alpha_n(x,\varepsilon)$. For this purpose divide interval $[a,b]$ in three parts:

$$I_1 = [a, \min\{x^\star_\alpha(\varepsilon), x^\star_\beta(\varepsilon)\}], \quad I_2 = [\min\{x^\star_\alpha(\varepsilon), x^\star_\beta(\varepsilon)\}, \max\{x^\star_\alpha(\varepsilon), x^\star_\beta(\varepsilon)\}],$$
$$I_3 = [\max\{x^\star_\alpha(\varepsilon), x^\star_\beta(\varepsilon)\}, b].$$

On the interval I_2 from definitions (34) and C-match condition on asymptotic we obtain

$$\beta_n(x,\varepsilon) - \alpha_n(x,\varepsilon) = -\varepsilon^n 2\delta \frac{dQ_0^{(-)}}{d\xi}(0) + O(\varepsilon^{n+1})$$

Comparing (23), (39) and the last expression we conclude that $\beta_n(x,\varepsilon)-\alpha_n(x,\varepsilon) > 0$ on the interval I_2 only if the next condition is fulfilled.

Condition VII. *Let inequality* $\dfrac{J'_x(x_0)}{\varphi^{(+)}(x_0,x_0) - \varphi^{(-)}(x_0,x_0)} < 0$ *is satisfied.*

On the interval I_1 one can write

$$\beta_n(x,\varepsilon) - \alpha_n(x,\varepsilon) = \left[\overline{u}_0^{(-)}(x, x^\star_\beta(\varepsilon)) - \overline{u}_0^{(-)}(x, x^\star_\alpha(\varepsilon))\right] +$$
$$+\left\{Q_0^{(-)}\left(\frac{x - x^\star_\beta(\varepsilon)}{\varepsilon}\right) - Q_0^{(-)}\left(\frac{x - x^\star_\alpha(\varepsilon)}{\varepsilon}\right)\right\} + \varepsilon\left\{Q_1^{(-)}\left(\frac{x - x^\star_\beta(\varepsilon)}{\varepsilon}\right) - Q_1^{(-)}\left(\frac{x - x^\star_\alpha(\varepsilon)}{\varepsilon}\right)\right\} +$$
$$+\varepsilon^{n+1}\left\{q^{(-)}(x, x^\star_\beta(\varepsilon)) + w_\beta^{(-)}\left(\frac{x - x^\star_\beta(\varepsilon)}{\varepsilon}\right) + q^{(-)}(x, x^\star_\alpha(\varepsilon)) - w_\alpha^{(-)}\left(\frac{x - x^\star_\alpha(\varepsilon)}{\varepsilon}\right)\right\} + O(\varepsilon^{n+2}).$$

Using mean value formula one can rewrite this as

$$\beta_n(x,\varepsilon) - \alpha_n(x,\varepsilon) = \varepsilon^n 2\delta \frac{\partial \varphi^{(-)}}{\partial y}(x, \tilde{y}) - \varepsilon^n 2\delta \frac{\partial Q_0^{(-)}}{\partial \xi}(\zeta_1) - \varepsilon^{n+1} 2\delta \frac{\partial Q_1^{(-)}}{\partial \xi}(\zeta_2) +$$
$$+\varepsilon^{n+1}\left\{q^{(-)}(x, x^\star_\beta(\varepsilon)) + w_\beta^{(-)}\left(\frac{x - x^\star_\beta(\varepsilon)}{\varepsilon}\right) + q^{(-)}(x, x^\star_\alpha(\varepsilon)) - w_\alpha^{(-)}\left(\frac{x - x^\star_\alpha(\varepsilon)}{\varepsilon}\right)\right\} + O(\varepsilon^{n+2}).$$

where $\tilde{y}$ is number from interval between $x^\star_\alpha(\varepsilon)$ and $x^\star_\beta(\varepsilon)$, and ζ_1, ζ_2 are some numbers from the interval between $[x - x^\star_\alpha(\varepsilon)]/\varepsilon$ and $[x - x^\star_\beta(\varepsilon)]/\varepsilon$.

Let us estimate the first term in the last formula. Taking partial derivative by y from

Eq. (6) we obtain coupled system

$$
\begin{aligned}
&A(\varphi^{(-)}(x,y),x,0)\frac{\partial}{\partial x}\left[\frac{\partial\varphi^{(-)}}{\partial y}\right]+P^{(-)}(x,y)\frac{\partial\varphi^{(-)}}{\partial y}+\\
&+\int\limits_a^y g_v(\varphi^{(-)}(x,y),\varphi^{(-)}(s,y),x,s,0)\frac{\partial\varphi^{(-)}}{\partial y}(s,y)\,ds+\\
&+\int\limits_y^b g_v(\varphi^{(-)}(x,y),\varphi^{(+)}(s,y),x,s,0)\frac{\partial\varphi^{(+)}}{\partial y}(s,y)\,ds+\\
&+g(\varphi^{(-)}(x,y),\varphi^{(-)}(y,y),x,y,0)-g(\varphi^{(-)}(x,y),\varphi^{(+)}(y,y),x,y,0)=0,\quad a<x<y,\\
&A(\varphi^{(+)}(x,y),x,0)\frac{\partial}{\partial x}\left[\frac{\partial\varphi^{(+)}}{\partial y}\right]+P^{(+)}(x,y)\frac{\partial\varphi^{(+)}}{\partial y}+\\
&+\int\limits_a^y g_v(\varphi^{(+)}(x,y),\varphi^{(-)}(s,y),x,s,0)\frac{\partial\varphi^{(-)}}{\partial y}(s,y)\,ds+\\
&+\int\limits_y^b g_v(\varphi^{(+)}(x,y),\varphi^{(+)}(s,y),x,s,0)\frac{\partial\varphi^{(+)}}{\partial y}(s,y)\,ds+\\
&+g(\varphi^{(+)}(x,y),\varphi^{(-)}(y,y),x,y,0)-g(\varphi^{(+)}(x,y),\varphi^{(+)}(y,y),x,y,0)=0,\quad y<x<b,
\end{aligned}
\tag{41}
$$

Obviously, derivative $\frac{\partial\varphi^{(\pm)}}{\partial y}(x,y)$ have to be a solution of system (41) with boundary conditions

$$
\frac{\partial\varphi^{(-)}}{\partial y}(a,y)=0,\qquad \frac{\partial\varphi^{(+)}}{\partial y}(b,y)=0. \tag{42}
$$

Problem (41), (42) can be analyzed by analogy with (35), (36). Taking into account Condition II, Condition VI, Condition VII and formula (39) we conclude $2\delta\frac{\partial\varphi^{(-)}}{\partial y}(x,\tilde{y})\geq 0$ for all $x\in I_1$.

Now from Lemma 1 we can choose $C_1>0$ and $\nu_1>0$ such that

$$
-2\delta\frac{\partial Q_0^{(-)}}{\partial\xi}(\zeta_1)\geq C_1\exp\left\{\nu_1\frac{x-\max\left[x_\alpha^\star(\varepsilon),x_\beta^\star(\varepsilon)\right]}{\varepsilon}\right\}.
$$

Besides, estimates (33) imply that there exist constants $C_2>0$ and $\nu_2>0$ such that

$$
\left|2\delta\frac{\partial Q_1^{(-)}}{\partial\xi}(\zeta_2)-w_\beta^{(-)}\left(\frac{x-x_\beta^\star(\varepsilon)}{\varepsilon}\right)+w_\alpha^{(-)}\left(\frac{x-x_\alpha^\star(\varepsilon)}{\varepsilon}\right)\right|\leq C_2\exp\left\{\nu_2\frac{x-\min\left[x_\alpha^\star(\varepsilon),x_\beta^\star(\varepsilon)\right]}{\varepsilon}\right\}.
$$

Thus, for sufficiently small ε we have

$$
-\varepsilon^n 2\delta\frac{\partial Q_0^{(-)}}{\partial\xi}(\zeta_1)-\varepsilon^{n+1}\left\{2\delta\frac{\partial Q_1^{(-)}}{\partial\xi}(\zeta_2)-w_\beta^{(-)}\left(\frac{x-x_\beta^\star(\varepsilon)}{\varepsilon}\right)+w_\alpha^{(-)}\left(\frac{x-x_\alpha^\star(\varepsilon)}{\varepsilon}\right)\right\}\geq-\varepsilon^{n+1}q_0,
$$

where q_0 is constant from Lemma 3. Therefore, for such values of ε we obtain

$$
\beta_n(x,\varepsilon)-\alpha_n(x,\varepsilon)\geq\varepsilon^{n+1}q_0+O(\varepsilon^{n+2})>0
$$

for all $x \in I_1$. Interval I_3 is analyzed by analogy with I_1.

Thus, we found conditions which guarantee us that $\alpha_n(x,\varepsilon)$ and $\beta_n(x,\varepsilon)$ satisfy all conditions of Theorem 1. Therefore, problem (3), (4) has classical solution $u(x,\varepsilon)$ such that $\alpha_n(x,\varepsilon) \le u(x,\varepsilon) \le \beta_n(x,\varepsilon)$. Taking into account that $\beta_n(x,\varepsilon) - \alpha_n(x,\varepsilon) = O(\varepsilon^n)$ we find that $u(x,\varepsilon) = \alpha_n(x,\varepsilon) + O(\varepsilon^n)$. In the case when $n \ge 1$, we can neglect inessential terms and write $u(x,\varepsilon) = U_{n-1}(x,\varepsilon) + O(\varepsilon^n)$ with $x^\star(\varepsilon) = \sum_{k=0}^{n} \varepsilon^k x_k$. Thus, we have proved the following theorem.

Theorem 2. *Assume that $A \in C^{n+2}(\mathbb{R} \times [a,b] \times \mathbb{R}_0^+)$, $g \in C^{n+2}(\mathbb{R}^2 \times [a,b]^2 \times \mathbb{R}_0^+)$ $(n \ge 0)$, and Conditions I-IV, V* and VI-VII are fulfilled. Then for sufficiently small values of the parameter $\varepsilon > 0$ problem (3), (4) has a classical solution $u = u(x,\varepsilon)$ such that*

$$\lim_{\varepsilon\to 0} u(x,\varepsilon) = \begin{cases} \varphi^{(-)}(x,x_0) & \text{for} \quad a \le x < x_0, \\ \varphi^{(+)}(x,x_0) & \text{for} \quad x_0 < x \le b. \end{cases} \tag{43}$$

Moreover, if smoothness condition on coefficients A and g are fulfilled for $n \ge 1$ then $u(x,\varepsilon) = U_{n-1}(x,\varepsilon) + O(\varepsilon^n)$ where

$$U_{n-1}(x,\varepsilon) = \begin{cases} \sum\limits_{k=0}^{n-1} \varepsilon^k \left[\overline{u}_k^{(-)}(x, x^\star(\varepsilon)) + Q_k^{(-)}(\xi) \right] & \text{for} \quad a \le x \le x^\star(\varepsilon), \\ \sum\limits_{k=0}^{n-1} \varepsilon^k \left[\overline{u}_k^{(+)}(x, x^\star(\varepsilon)) + Q_k^{(+)}(\xi) \right] & \text{for} \quad x^\star(\varepsilon) < x \le b \end{cases}$$

with $\xi = [x - x^\star(\varepsilon)]/\varepsilon$ and $x^\star(\varepsilon) = \sum_{k=0}^{n} \varepsilon^k x_k$.

4. Asymptotic stability and local uniqueness of internal layer solutions

Any solution $u_s(x,\varepsilon)$ to problem (3), (4) is at the same time a stationary solution to the associated parabolic problem (1), (2) with the initial condition $u(x,0,\varepsilon) = u_s(x,\varepsilon)$. This raises the question of stability of such a solution. Below, we show that for sufficiently small ε every solution $u_s(x,\varepsilon)$ to problem (3), (4) given by Theorem 2 is asymptotically stable on a certain set D in the sense of the following definition.

Definition 2. *Let D be an arbitrary subset of functions in $C^2[a,b]$. The solution $u_s(x,\cdot)$ to problem (3), (4) is said to be asymptotically stable on D if, for any function $v(x) \in D$ satisfying the boundary conditions (4), problem (1), (2) with the initial condition $u(x,0,\cdot) = v(x)$ has a unique solution $u(x,t,\cdot)$ defined for all $t \ge 0$ and*

$$\lim_{t\to+\infty} \max_{x\in[a,b]} |u(x,t,\cdot) - u_s(x,\cdot)| = 0.$$

To prove this fact, we use the method proposed in [10] for the analysis of similar problems. However, we first obtain an estimate for the first derivative of the solution to problem (3), (4) required for further consideration. To do this, we use the following properties of the upper and lower solutions constructed in Section 3:

$$
\begin{gathered}
L[\alpha_n] = O(\varepsilon^{n+1}) > 0, \qquad\qquad L[\beta_n] = O(\varepsilon^{n+1}) < 0, \\
|\alpha_n(x,\varepsilon) - u(x,\varepsilon)| = O(\varepsilon^n), \quad |\beta_n(x,\varepsilon) - u(x,\varepsilon)| = O(\varepsilon^n), \\
|\beta_n(x,\varepsilon) - \alpha_n(x,\varepsilon)| = O(\varepsilon^n).
\end{gathered} \tag{44}
$$

Using the same arguments as in the work [7] (see Lemma 2 therein) we can prove the following statement.

Lemma 4. *Suppose the conditions of Theorem 2 are fulfilled. Then, for any solution $u(x,\varepsilon)$ to problem (3), (4) such that $\alpha_n(x,\varepsilon) \le u(x,\varepsilon) \le \beta_n(x,\varepsilon)$, the following relationships are valid:*

$$
\frac{du}{dx} - \frac{d\alpha_n}{dx} = O(\varepsilon^{n-1}), \quad \frac{du}{dx} - \frac{d\beta_n}{dx} = O(\varepsilon^{n-1}).
$$

We now turn to the justification of the asymptotic stability of solutions given by Theorem 2. Let $u = u_s(x,\varepsilon)$ be one such solution. Then, consider the following two functions:

$$
\begin{aligned}
\alpha(x,t,\varepsilon) &= u_s(x,\varepsilon) - r_\alpha(x,\varepsilon)e^{-\varepsilon Kt}, \qquad r_\alpha(x,\varepsilon) = u_s(x,\varepsilon) - \alpha_n(x,\varepsilon), \\
\beta(x,t,\varepsilon) &= u_s(x,\varepsilon) + r_\beta(x,\varepsilon)e^{-\varepsilon Kt}, \qquad r_\beta(x,\varepsilon) = \beta_n(x,\varepsilon) - u_s(x,\varepsilon),
\end{aligned} \tag{45}
$$

where K is a positive constant.

It is clear that the functions α and β satisfy the inequalities

$$
\begin{gathered}
\alpha(a,t,\varepsilon) \le u^a \le \beta(a,t,\varepsilon), \quad \alpha(b,t,\varepsilon) \le u^b \le \beta(b,t,\varepsilon) \quad \text{for all} \quad t \in [0,+\infty), \\
\alpha(x,t,\varepsilon) \le \beta(x,t,\varepsilon) \quad \text{for all} \quad (x,t) \in [a,b] \times [0,+\infty).
\end{gathered} \tag{46}
$$

Let us now consider the expression

$$
\begin{aligned}
-\frac{\partial\beta}{\partial t} + L[\beta] = e^{-\varepsilon Kt} \{ & \varepsilon K r_\beta + L[\beta_n] + O((\beta_n - u_s)(\beta_n - u_s)'_x) + \\
& + O((\beta_n)'_x(\beta_n - u_s)^2) + O([(u_s)'_x + (r_\beta)'_x] r_\beta^2 + r_\beta^2) \} .
\end{aligned}
$$

Taking into account that $\dfrac{\partial \beta_n}{\partial x} = O(\varepsilon^{-1})$ and using results of Lemma 4, we obtain

$$
\begin{aligned}
-\frac{\partial\beta}{\partial t} + L[\beta] &= e^{-\varepsilon Kt} \{ \varepsilon K r_\beta + L[\beta_n] + O(\varepsilon^{2n-1}) + O(\varepsilon^{2n-1}) + O(\varepsilon^{2n-1}) \} = \\
&= e^{-\varepsilon Kt} \{ \varepsilon K r_\beta + L[\beta_n] + O(\varepsilon^{2n-1}) \} .
\end{aligned}
$$

The first pair of inequalities in (44) implies that there exists a constant $C > 0$, such that the estimate $L[\beta_n] \le -C\varepsilon^{n+1}$ is valid for sufficiently small $\varepsilon > 0$. Similarly, definition (45) and the second pair of inequalities in (44) imply that $r_\beta(x,\varepsilon) = O(\varepsilon^n)$. Thus, choosing a sufficiently small value K, we can satisfy the inequality $\varepsilon K r_\beta + L[\beta_n] < 0$. taking into account that $\varepsilon K r_\beta + L[\beta_n] = O(\varepsilon^{n+1})$, we obtain that, for any $n > 2$ and sufficiently small $\varepsilon > 0$, the inequality $-\dfrac{\partial\beta}{\partial t} + L[\beta] < 0$ holds for all $(x,t) : a \le x \le b$ and $t \ge 0$. Consequently, $\beta(x,t,\varepsilon)$ is an upper solution for problem (1), (2) with the initial condition $u(x,t,\varepsilon) = u_s(x,\varepsilon)$ at $t = 0$. Similarly, it can be shown that $\alpha(x,t,\varepsilon)$ is a lower solution for the same problem. Hence, on the basis of the theorem for parabolic integro-differential equations analogous to Theorem 1 (see [2], [8], [9] and the remark presented after Theorem 2), we conclude that any solution $u(x,t,\varepsilon)$ to problem (1), (2) with the initial condition $\alpha_n(x,\varepsilon) \le u(x,0,\varepsilon) \le \beta_n(x,\varepsilon)$ satisfies the inequality

$$\alpha(x,t,\varepsilon) \le u(x,t,\varepsilon) \le \beta(x,t,\varepsilon) \quad \forall (x,t) \in [a,b] \times [0,+\infty). \tag{47}$$

However, since $\beta(x,t,\varepsilon) - \alpha(x,t,\varepsilon) = (r_\beta - r_\alpha)e^{-Kt}$, it obviously follows that the solution $u_s(x,\varepsilon)$ to problem (3), (4) is asymptotically stable in the sense of Definition 2 on the set $D_\varepsilon \equiv \{u(x) \in C^2[a,b] : \alpha_n(x,\varepsilon) \le u(x) \le \beta_n(x,\varepsilon)\}$. Furthermore, the above estimate implies the local uniqueness of this solution on the set D_ε. Indeed, assume that, in addition to $u_s(x,\varepsilon)$, one more solution to problem (3), (4), say, $\tilde{u}(x,\varepsilon)$, exists on the set D_ε. Then, by virtue of the above analysis, it satisfies estimate (47) as well as $u_s(x,\varepsilon)$ does. Therefore, the following inequality holds for all $t \ge 0$

$$|u_s(x,\varepsilon) - \tilde{u}(x,\varepsilon)| \le \beta(x,t,\varepsilon) - \alpha(x,t,\varepsilon) \quad \forall x \in [a,b].$$

However, this is possible only in the case when $\tilde{u}(x,\varepsilon) \equiv u_s(x,\varepsilon)$. Thus, we have proved the following theorem.

Theorem 3. *Suppose the conditions of Theorem 2 are fulfilled at $n = 3$. Then, for sufficiently small $\varepsilon > 0$ solution $u_s(x,\varepsilon)$ to problem (3), (4) lying between the barriers $\alpha_3(x,\varepsilon)$ and $\beta_3(x,\varepsilon)$ is unique and asymptotically stable on the set*

$$D_\varepsilon = \{u(x) \in C^2[a,b],\ \alpha_3(x,\varepsilon) \le u(x) \le \beta_3(x,\varepsilon),\ u(a) = u^a,\ u(b) = u^b\}.$$

5. Example.

As an example let us consider BVP for Burger's equation with nonlocal reaction term

$$\varepsilon u'' = uu' + \int_0^1 [u(x) - k^2u^2(x)u(s)]ds, \quad x \in (0,1), \tag{48}$$

$$u(0) = B, \quad u(1) = -B, \tag{49}$$

where k and B are numerical parameters. This problem correspond to the problem (3), (4) with $A(u,x,\varepsilon) = u$ and $g(u,v,x,s,\varepsilon) = u - k^2u^2v$. Note that function $g(\cdot)$ satisfies Condition VI.

Reduced equation for BVP (48), (49) has form

$$uu' + \int_0^1 [u(x) - k^2u^2(x)u(s)]ds = 0.$$

Excluding trivial solution $u(x) \equiv 0$ we obtain

$$u' + 1 - k^2 u(x) \int\limits_0^1 u(s)ds = 0. \tag{50}$$

The last equation can be represented as

$$u' - au = -1, \tag{51}$$

where a is certain constant depending on solution $u(x)$. Now one can write discontinuous solution to Eq. (51) satisfying boundary conditions (49)

$$\varphi(x,\cdot) = \begin{cases} \varphi^{(-)}(x,\cdot) = Be^{ax} + \dfrac{1-e^{ax}}{a}, & 0 \le x < y, \\ \varphi^{(+)}(x,\cdot) = -Be^{a(x-1)} + \dfrac{1-e^{a(x-1)}}{a}, & y < x \le 1. \end{cases} \tag{52}$$

Comparing Eq. (50) with Eq. (51) we obtain

$$a = k^2 \int\limits_0^1 u(s)ds = k^2 \left\{ \int\limits_0^y \left[\frac{1}{a} + \left(B - \frac{1}{a} \right) e^{as} \right] ds + \int\limits_y^1 \left[\frac{1}{a} - \left(B + \frac{1}{a} \right) e^{a(s-1)} \right] ds \right\}.$$

or after transformation

$$y = \frac{1}{a} \ln \left\{ \left[2B - 1 + \frac{a^2}{k^2} \right] \left[B(1+e^{-a}) + \frac{e^{-a}-1}{a} \right]^{-1} \right\}. \tag{53}$$

Eq. (53) represents in implicit form functional dependence $a(y)$. Thus, formulae (52), (53) really define a family of discontinuous solutions to Eq. (51).

Direct calculation with formula (22) gives

$$J(x) = \int\limits_{\varphi^{(-)}(x,x)}^{\varphi^{(+)}(x,x)} u\,du = \tfrac{1}{2} \left[\varphi^{(+)}(x,x) - \varphi^{(-)}(x,x) \right] \left[\varphi^{(+)}(x,x) + \varphi^{(-)}(x,x) \right] =$$
$$= \tfrac{1}{2} \left[-\left(B - \tfrac{1}{a} \right) e^{ax} - \left(B + \tfrac{1}{a} \right) e^{a(x-1)} \right] \left[\tfrac{2}{a} + \left(B - \tfrac{1}{a} \right) e^{ax} - \left(B + \tfrac{1}{a} \right) e^{a(x-1)} \right] \Big|_{a=a(x)}.$$

As x and a are connected with Eq. (53) then $J(x) = \widetilde{J}(a)$, where

$$\widetilde{J}(a) = -\frac{1}{2} \left\{ 2B - 1 + \frac{a^2}{k^2} \right\} \left\{ \frac{2}{a} + \left[2B - 1 + \frac{a^2}{k^2} \right] \left[B(1-e^{-a}) - \frac{1+e^{-a}}{a} \right] \left[B(1+e^{-a}) + \frac{e^{-a}-1}{a} \right]^{-1} \right\}.$$

Obviously, solutions of equation $\widetilde{J}(a) = 0$ correspond in one to one to the solutions of equation $J(x) = 0$. Now using easy but long transformations one can obtain

$$\widetilde{J}(a) = -\frac{1}{2} \left[\frac{1}{6} - \frac{2}{k^2} + B(2B-1) \right] a + O(a^2) \quad \text{for} \quad a \to 0.$$

This means that $a = 0$ is a solution of the equation $\widetilde{J}(a) = 0$. From formula (53) we also found that

$$x_0 = \lim_{a\to 0} \frac{1}{a} \ln \left\{ \left[2B - 1 + \frac{a^2}{k^2}\right] \left[B(1+e^{-a}) + \frac{e^{-a}-1}{a}\right]^{-1} \right\} = \frac{1}{2}$$

is corresponding solution of the equation $J(x) = 0$. Moreover,

$$\lim_{a\to 0} \frac{dy}{da} = \frac{\dfrac{1}{k^2} - \dfrac{B}{4} + \dfrac{1}{24}}{2B-1}.$$

Thus,

$$J'(1/2) = \widetilde{J}'(0) : \left.\frac{dy}{da}\right|_{a=0} = \left\{1 - \frac{k^2}{2}\left[\frac{1}{6} + B(2B-1)\right]\right\} \frac{2B-1}{1 - \dfrac{k^2}{4}\left(B - \dfrac{1}{6}\right)}. \tag{54}$$

For $a = 0$ from (52) we have

$$\varphi(x, 1/2) = \begin{cases} \varphi^{(-)}(x, 1/2) = B - x, & 0 \le x < 1/2, \\ \varphi^{(+)}(x, 1/2) = -B + 1 - x, & 1/2 < x \le 1. \end{cases} \tag{55}$$

Obviously, if $B > 1$ then Condition II is satisfied.

From definitions given in Condition V we obtain

$$N^{(--)}(x, s, 1/2) = N^{(-+)}(x, s, 1/2) = k^2 \left[Bx - \tfrac{x^2}{2}\right],$$
$$N^{(+-)}(x, s, 1/2) = N^{(++)}(x, s, 1/2) = k^2 \left[B - \tfrac{1}{2} + (1-B)x - \tfrac{x^2}{2}\right].$$

Thus, spectral problem for coupled system (37) with $x_0 = 1/2$ has form

$$\lambda v^{(-)}(x) = k^2 \left[Bx - \tfrac{x^2}{2}\right] \left\{ \int_0^{1/2} v^{(-)}(s)\, ds + \int_{1/2}^{1} v^{(+)}(s)\, ds \right\},$$
$$\lambda v^{(+)}(x) = k^2 \left[B - \tfrac{1}{2} + (1-B)x - \tfrac{x^2}{2}\right] \left\{ \int_0^{1/2} v^{(-)}(s)\, ds + \int_{1/2}^{1} v^{(+)}(s)\, ds \right\}.$$

Direct analysis of this system gives two eigenvalues

$$\lambda_1 = 0, \qquad \lambda_2 = \frac{k^2}{4}\left(B - \frac{1}{6}\right). \tag{56}$$

From (54) and (55) we have

$$\frac{J'(1/2)}{\varphi^{(+)}(1/2, 1/2) - \varphi^{(-)}(1/2, 1/2)} = -\frac{1 - \dfrac{k^2}{2}\left[\dfrac{1}{6} + B(2B-1)\right]}{1 - \dfrac{k^2}{4}\left(B - \dfrac{1}{6}\right)}. \tag{57}$$

Now taking into account formulae (56) and (57), Theorems 2 and 3 imply the next statement.

Theorem 4. *Assume that* $B > 1$ *and inequality* $k^2 < 2\left[B(2B-1)+\frac{1}{6}\right]^{-1}$ *is fulfilled. Then for sufficiently small values of the parameter* $\varepsilon > 0$ *BVP (48), (49) has a classical solution* $u = u(x,\varepsilon)$ *such that*

$$\lim_{\varepsilon\to+0} u(x,\varepsilon) = \begin{cases} B - x & \text{for } \; 0 \le x < 1/2, \\ -B+1-x & \text{for } \; 1/2 < x \le 1. \end{cases} \tag{58}$$

Moreover, this solution is locally unique and asymptotically stable.

Appendix.

Let us consider the system of two coupled linear integro-differential equations, which depend parametricaly on $y \in (a,b)$,

$$\begin{aligned} &\frac{du^{(-)}}{dx} + p^{(-)}(x,y)u^{(-)} + \int_a^y K^{(--)}(x,s,y)u^{(-)}(s)\,ds + \\ &\qquad\qquad + \int_y^b K^{(-+)}(x,s,y)u^{(+)}(s)\,ds = f^{(-)}(x,y), \quad a < x < y, \\ &\frac{du^{(+)}}{dx} + p^{(+)}(x,y)u^{(+)} + \int_a^y K^{(+-)}(x,s,y)u^{(-)}(s)\,ds + \\ &\qquad\qquad + \int_y^b K^{(++)}(x,s,y)u^{(+)}(s)\,ds = f^{(+)}(x,y), \quad y < x < b \end{aligned} \tag{59}$$

with boundary conditions

$$u^{(-)}(a) = v_a, \qquad u^{(+)}(b) = v_b. \tag{60}$$

Here

$$\begin{aligned} p^{(-)}, f^{(-)} &\in C(\{(x,y) : a \le x \le y \le b\}), \\ p^{(+)}, f^{(+)} &\in C(\{(x,y) : a \le y \le x \le b\}), \\ K^{(--)} &\in C(\{(x,s,y) : a \le x \le y \le b,\; a \le s \le y\}), \\ K^{(-+)} &\in C(\{(x,s,y) : a \le x \le y \le b,\; y \le s \le b\}), \\ K^{(+-)} &\in C(\{(x,s,y) : a \le y \le x \le b,\; a \le s \le y\}), \\ K^{(++)} &\in C(\{(x,s,y) : a \le y \le x \le b,\; y \le s \le b\}), \end{aligned} \tag{61}$$

and v_a, v_b are some constants. For given $y \in (a,b)$, we shall call pair of functions

$$u^{(-)} \in C^1([a,y]),\; u^{(+)} \in C^1([y,b])$$

solution to problem (59), (60), if these functions satisfy equations (59) and conditions (60) pointwise.

It is well-known fact that linear IVP

$$\frac{dv}{dx} + p(x)v = r(x), \qquad v(x_0) = v_0$$

with continuous coefficients $p(x)$ and $r(x)$ has a unique solution, which can be represented in the integral form

$$v(x) = v_0 \exp\left\{ -\int_{x_0}^{x} p(\xi)d\xi \right\} + \int_{x_0}^{x} r(z) \exp\left\{ -\int_{z}^{x} p(\xi)d\xi \right\} dz.$$

Using that it is easy to obtain the next equivalence result.

Lemma A_1**.** *The integro-differential system (59), (60) is equivalent to the integral equation*

$$u(x) - F(x,y) = \int_a^b N(x,s,y)u(s)\, ds \equiv (T_y u)(x), \tag{62}$$

where

$$N(x,s,y) = \begin{cases} -\int_a^x K^{(--)}(z,s,y) \exp\left\{ -\int_z^x p^{(-)}(\xi,y)d\xi \right\} dz, & x,s \le y, \\ -\int_a^x K^{(-+)}(z,s,y) \exp\left\{ -\int_z^x p^{(-)}(\xi,y)d\xi \right\} dz, & x \le y < s, \\ \int_x^b K^{(+-)}(z,s,y) \exp\left\{ -\int_z^x p^{(+)}(\xi,y)d\xi \right\} dz, & s \le y < x, \\ \int_x^b K^{(++)}(z,s,y) \exp\left\{ -\int_z^x p^{(+)}(\xi,y)d\xi \right\} dz, & y < x,s \end{cases}$$

and

$$F(x,y) = \begin{cases} v_a \exp\left\{ -\int_a^x p^{(-)}(\xi,y)d\xi \right\} + \int_a^x f^{(-)}(s,y) \exp\left\{ -\int_s^x p^{(-)}(\xi,y)d\xi \right\} ds, & x \le y, \\ v_b \exp\left\{ -\int_b^x p^{(+)}(\xi,y)d\xi \right\} - \int_x^b f^{(+)}(s,y) \exp\left\{ -\int_s^x p^{(+)}(\xi,y)d\xi \right\} ds, & y < x. \end{cases}$$

It is well known that the integral operators T_y, introduced in (62), are linear compact operators on $L^2(a,b)$. Moreover, if for a given $y \in (a,b)$ we have $N(x,s,y) \ge 0$ for all $(x,s) \in [a,b]^2$, then T_y is a positive operator, i.e. it maps every nonnegative function v in a nonnegative function $T_y v$ (analogously, it maps nonpositive functions in nonpositive functions). Using Fredholm's theory for such systems it is easy to prove the next results (where we denote by I the identity map in $L^2(a,b)$):

Lemma A_2**.** *If $I - T_y$ is injective, then the system (59), (60) has a unique solution for every pair of right hand sides $f^{(-)} \in C(\{(x,y) : a \le x \le y \le b\})$, $f^{(+)} \in C(\{(x,y) : a \le y \le x \le b\})$ and for any boundary values v_a and v_b.*

Lemma A_3**.** *Suppose that T_y is a positive operator and that its spectral radius is less than one.*

Then system (59), (60) with $f^{(-)}(x,y) \geq 0$ for all $x \in [a,y]$, $f^{(+)}(x,y) \geq 0$ for all $x \in [y,b]$ and $v_a, v_b > 0$ has a unique nonnegative solution (i.e. $u^{(-)}(x,y) \geq 0$ for all $x \in [a,y]$ and $u^{(+)}(x,y) \geq 0$ for all $x \in [y,b]$). Analogously, system (59), (60) with $f^{(-)}(x,y) \leq 0$ for all $x \in [a,y]$, $f^{(+)}(x,y) \leq 0$ for all $x \in [y,b]$ and $v_a, v_b < 0$, has a unique nonpositive solution.

The following results show that these solvability properties of system (59), (60) are preserved under small changes of the parameter y:

Lemma A_4**.** *Operator family T_y is continuous (in the uniform operator topology corresponding to $L^2(a,b)$) with respect to $y \in (a,b)$.*

Proof. We have

$$\begin{aligned}\|T_y u - T_{y_0} u\|^2_{L^2(a,b)} &= \int_a^b \left(\int_a^b \left(N(x,s,y) - N(x,s,y_0)\right) u(s)\, ds \right)^2 dx \leq \\ &\leq \int_a^b \int_a^b |N(x,s,y) - N(x,s,y_0)|^2 \, ds\, dx\, \|u\|^2_{L^2(a,b)}.\end{aligned}$$

Hence, we have to show that $\int_a^b \int_a^b |N(x,s,y) - N(x,s,y_0)|^2\, ds\, dx$ can be made arbitrarily small if y is chosen sufficiently close to y_0. Let us suppose that $y \leq y_0$ (the case $y \geq y_0$ can be treated similarly). The integrals

$$\int_a^y \left(\int_a^y |N^{(--)}(x,s,y) - N^{(--)}(x,s,y_0)|^2\, ds + \int_{y_0}^b |N^{(-+)}(x,s,y) - N^{(-+)}(x,s,y_0)|^2\, ds \right) dx,$$

and

$$\int_{y_0}^b \left(\int_a^y |N^{(+-)}(x,s,y) - N^{(+-)}(x,s,y_0)|^2\, ds + \int_{y_0}^b |N^{(++)}(x,s,y) - N^{(++)}(x,s,y_0)|^2\, ds \right) dx$$

are small for y close to y_0 because the functions $N^{(--)}$, $N^{(-+)}$, $N^{(+-)}$ and $N^{(++)}$, defined as in formulae (28), are uniformly continuous. And the integrals

$$\int_y^{y_0} \int_a^b |N(x,s,y) - N(x,s,y_0)|^2\, ds\, dx, \quad \int_a^b \int_y^{y_0} |N(x,s,y) - N(x,s,y_0)|^2\, ds\, dx$$

and

$$-\int_y^{y_0} \int_y^{y_0} |N(x,s,y) - N(x,s,y_0)|^2\, ds\, dx$$

are small for y close to y_0 because the functions $N^{(--)}$, $N^{(-+)}$, $N^{(+-)}$ and $N^{(++)}$ and, hence, N, are bounded. But $\int_a^b \int_a^b |N(x,s,y) - N(x,s,y_0)|^2 \, ds \, dx$ is the sum of all these integrals.

Lemma A_5**.** *If $I - T_{y_0}$ is injective, then $I - T_y$ is also injective for all y close to y_0.*

Proof. The operators $I - T_y$ are Fredholm of index zero because the operators T_y are compact. Hence, $I - T_y$ is injective if and only if it is bijective, i.e. if and only if it is an isomorphism. But the set of isomorphisms is open in the uniform operator topology.

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