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Optimal control theory based design of elasto-magnetic metamaterial

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Abstract

A method to design a new type of metamaterial is presented. A two-step strategy to define an optimal long-range force distribution embedded in an elastic support to control wave propagation is considered.
The first step uses a linear quadratic regulator (LQR) to produce an optimal set of long-range interactions. In the second step, a least square passive approximation of the LQR optimal gains is determined. The paper investigates numerical solutions obtained by the previously described procedure. Finally, we discuss physical and engineering implications and practical use of the present study.

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Keywords: metamaterials, wave propagation, long-range, elastic metamaterials, electromagnetic metamaterials

1. Statement of the problem

Metamaterial science is gaining a wide interest in the scientific community [1–4]. In recent papers [5–7] the authors investigated some particular features of long-range metamaterials [1]. This physical context considers particles interacting not only by closest neighbour forces, but also through long-distance actions [8]. A mathematical model to describe the long-range interactions in mechanical systems takes into account nonlinear integro-differential equations: the second order differential part represents the standard closest neighbour forces, while the integral part models the long-distance interactions [6]. An alternative form, shown in [5], uses a higher order differential model to describe the particles motion. The larger the order of derivation, the longer is the distance of interaction between the particles. We analyse a rod structure that is equipped with long-range interaction forces. Such a system can be physically realised starting from a conventional elastic rod structure to which is added a network of particles, uniformly distributed over its length, that can interact each other through electrical or magnetic charges. Without giving details, this physical arrangement is the subject of an experimental investigation and setup in the Sapienza Labs.
The two systems have respective forms [5,6]:

$$\rho\frac{\partial^2 w}{\partial t^2} - E\frac{\partial^2 w}{\partial x^2} - g * w = 0 \tag{1}$$

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$$\rho \frac{\partial^2 w}{\partial t^2} - E\frac{\partial^2 w}{\partial x^2} + \sum_i \beta_i \frac{\partial^i w}{\partial x^i} = 0 \tag{2}$$

where, in general, the number of terms in the summation can be arbitrarily large, even infinite. Long-range effects in these equations are easily understood. While the term $\frac{\partial^2 w}{\partial x^2}$ accounts for the closest neighbour forces, the term $g * w$, or $\sum_i \beta_i \frac{\partial^i w}{\partial x^i}$, includes the long-distance interaction. In fact, the discretized form of $g * w \approx \sum_k g(x - \xi_k)w(\xi_k)$ makes clear how the convolution term brings at the point x all the force contributions coming from any point ξ_k along the waveguide that represents the long-range effect. Similarly, the set of even derivatives $\sum_i \beta_i \frac{\partial^{2i} w}{\partial x^{2i}} = \sum_i \frac{\beta_i}{\varepsilon^{2i}} \sum_{m=0}^{2i} (-1)^m \binom{2i}{m} w\left[(k - i + m)\,\varepsilon\right]$, when written by discretized finite differences, where ε is the discretization step along the x axis, substantially produces the same effect.

Long-range non-local effects can be obtained by different technologies. Two of them are investigated by our group and use magnets and direct elastic connections, suitably placed along the waveguides represented, for example, by equations (1) and (2). It is an obvious consideration that the embedding of magnets (as well as spring-like connection elements) in the material has a cost and technology complications. Neverthless, their benefit in terms of vibration control of some and selected parts of the structure has been outlined in [5–7]. Benefits and costs of magnets embedding claim for a trade-off analysis. Therefore, the natural questions arise: how strong would be the magnetic force, what an optimal collocation of the magnets, how many magnets we decide to embed? Of course the wider the embedding, the larger is the cost and the technological effort. Moreover, the process is physical not trivial to control, so also the benefit we expect to receive is not easy predictable.

The previous considerations strongly suggest that the idea of using long-range interaction forces must be necessarily accompanied by an optimization analysis. This permits, in the form proposed in the present paper, to make the best selection of the long-range forces, provided a given limit on the maximum allowed range for them. In the final part of the paper, the method finally allows for directions on the choice of magnets properties for the best embedding. The problem here is technically posed in terms of best selection of $g(x)$, or β_i's, to produce a material response able to minimize (or maximize) an objective function.

The problem can be formulated as follows:

$$\begin{cases} \text{Find } w(x,t) \text{ and } g(x) \text{ so that} \\ \quad \min J = \int_0^T \int_{\mathcal{R}^2} \left[s(\xi,\eta)w(\xi,t)w(\eta,t) + p(\xi,\eta)\dot{w}(\xi,t)\dot{w}(\eta,t) + r(\xi,\eta)g(\xi)g(\eta) \right] d\xi d\eta dt \\ \text{s.t.} \quad \rho \frac{\partial^2 w}{\partial t^2} - E\frac{\partial^2 w}{\partial x^2} + \sum_i \beta_i \frac{\partial^i w}{\partial x^i} = 0 \qquad \text{with } i \text{ even} \end{cases} \tag{3}$$

where the weighting functions $s(\xi,\eta)$, $p(\xi,\eta)$ and $r(\xi,\eta)$ are given.

The mechanical interpretation of this formulation is straight: we aim at introducing a long-range interaction $g(x)$, so that the objective function J is minimum and $w(x,t)$ and $g(x)$ satisfy the equation of motion (2). The objective function J is the integral over the structure of two quadratic forms: the first, $s(\xi,\eta)w(\xi,t)w(\eta,t) + p(\xi,\eta)\dot{w}(\xi,t)\dot{w}(\eta,t)$, is associated to some form of energy, while the second, $r(\xi,\eta)g(\xi)g(\eta)$, is associated to the long-range effort. In other words, we desire to minimize the energy along the structure, protecting some regions more than others, by suitably modulating $s(\xi,\eta)$ and $p(\xi,\eta)$. This minimization has a cost related to the long-range force intensity to produce the vibration reduction effect. The second quadratic form quantifies this effort and the objective function J expresses the balance between the enhancement of the material performance ($s(\xi,\eta)w(\xi,t)w(\eta,t) + p(\xi,\eta)\dot{w}(\xi,t)\dot{w}(\eta,t)$) and the cost in terms of long-range force ($r(\xi,\eta)g(\xi)g(\eta)$) to produce the desired material response.

The optimality problem (3) is faced using at first the Linear Quadratic Regulator method - LQR. However, this technique provides an active control solution, in which the controlled system can receive energy from the environment, as typical of active controls. Instead, the desired structure must perform a passive control response, associated only to long-range connections. Therefore, we define the form of the control system by additional forces that account only for passive long-range actions. A least square method - LSM is used to make these forces as far as possible close to the LQR control. Numerical simulations complete the analysis.

2. Outline of the control strategy

Aim of this work is the particle motion control in the previously introduced system, through long-range interactions. Dynamic systems can be operated at minimum cost by applying the optimal control theory, LQR in this specific case, which provides a solution for an active control technique. However, in our statement of the problem (3), the long-range forces must be of passive type. To merge what obtained by the LQR method, taken as a reference, and the passive behaviour, a least square procedure is then applied, trying to adapt a passive scheme to an active control. The result is able to imitate the active control as close as it is possible.

We start from a discrete version of problem (3). Introducing the vector $\mathbf{x}(t) = [w(x_1,t), w(x_2,t), ...w(x_N,t)]^T$, where $x_i = i\varepsilon$ (ε is the discretization step along the waveguide), $\mathbf{M}$ and $\mathbf{K}_{CN}$ are the mass and stiffness matrices, respectively, equation (2) takes the form:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}_{CN}\dot{\mathbf{x}} + \mathbf{K}_{CN}\mathbf{x} + \sum_h \beta_h \mathbf{K}_h \mathbf{x} = 0 \tag{4}$$

where, for generality, a damping matrix $\mathbf{C}_{CN}$ is also considered, associated to closest neighbour interaction. The last term $\sum_h \beta_h \mathbf{K}_h \mathbf{x}$ is the discrete counterpart of the continuous form $\sum_h \beta_h \frac{\partial^h w}{\partial x^h}$, when using central finite differences. In this case $\mathbf{K}_h$ are banded symmetric matrices of bandwidth $h+1$. The discretized form of equation (4) can replace equation (2) only for a certain range of excitation. This implies that the characteristic excitation wavelength λ must be much longer than ε, $\lambda \gg \varepsilon$, making (4) a good approximation for (2).

In engineering practice, we know that a certain amount of damping remains associated to any force. This fact induces to include a second long-range damping-related effect, which simply states damping as proportional to the stiffness. Therefore, reconsidering equation (2), it produces:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}_{CN}\dot{\mathbf{x}} + \mathbf{K}_{CN}\mathbf{x} + \sum_h \beta_h \mathbf{K}_h (\mathbf{x} + \gamma\dot{\mathbf{x}}) = 0 \tag{5}$$

where γ is the coefficient that controls the damping associated to $\sum_h \beta_h \mathbf{K}_h$ (Note that the β's in (5) are not the same used in (2), but they are simply proportional). Equation (5) suggests that our direct optimization goal is related to the best choice of the β_h's. However, the direct approach to optimization through the β_h implies the problem is nonlinear, since $\mathbf{x}$ is an unknown of the problem as β_h and they appear as a product in the equation of motion.

We prefer to split the problem into two easier steps: the first approaches the problem of determining a vector of control forces that replaces the term $\sum_h \beta_h \mathbf{K}_h (\mathbf{x} + \gamma\dot{\mathbf{x}})$ by $\mathbf{Du}$, where $\mathbf{D}$ is an assigned matrix of constant coefficients, while $\mathbf{u}$ in an unknown control variable. With this choice, the optimization problem collapses into a standard LQR formulation. This will lead to $\mathbf{u} = \mathbf{K}^*_{OPT}\mathbf{x} + \mathbf{C}^*_{OPT}\dot{\mathbf{x}}$. However, this form represents an active control and the energy balance of the material substantially differs from the one of a passive material. This means $\mathbf{K}^*_{OPT}$ does not reproduce, in general, a stiffness matrix, since it is not symmetric, and the signs of coefficients do not satisfy necessary conditions to be an elastic matrix.

To make the additional force $\mathbf{Du} = \mathbf{DK}^*_{OPT}\mathbf{x} + \mathbf{DC}^*_{OPT}\dot{\mathbf{x}} = \mathbf{f}^K_{LQR} + \mathbf{f}^C_{LQR}$ suitable to be adapted to a passive long-range force $\sum_h \beta_h \mathbf{K}_h \mathbf{x} + \sum_h \beta_h \gamma \mathbf{K}_h \dot{\mathbf{x}} = \mathbf{f}^K_{LR} + \mathbf{f}^C_{LR}$, we can ask the β_h's to minimize $\int_0^T \left(\left| \mathbf{f}^K_{LR} - \mathbf{f}^K_{LQR} \right|^2 + \left| \mathbf{f}^C_{LR} - \mathbf{f}^C_{LQR} \right|^2 \right) dt$ in a least square sense.

3. First step: LQR approach

To design a material structure engineered to control wave propagation, the LQR technique is considered. The controlled equation of motion takes the form:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}_{CN}\dot{\mathbf{x}} + \mathbf{K}_{CN}\mathbf{x} + \mathbf{Du} = 0 \tag{6}$$

$\mathbf{u}$ is the long-range control vector and $\mathbf{D}$ the control matrix.
In general, $\mathbf{u}$ can have a lower dimension with respect to $\mathbf{x}$ and it has not a direct physical interpretation, but it is the generator element of the long-distance force. The vector $\mathbf{Du}$ is indeed interpreted as a force, namely the control force due to the long-range interaction.
The functional J to be minimized is the discrete counterpart of J in equation (3):

$$J = \int_0^{+\infty} \left(\mathbf{x}^T\mathbf{S}\mathbf{x} + \dot{\mathbf{x}}^T\mathbf{P}\dot{\mathbf{x}} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt \tag{7}$$

where $\mathbf{P}$, $\mathbf{S}$ and $\mathbf{R}$ are in general arbitrarily chosen matrices, according to the requirements of the LQR method. However, we can select $\mathbf{P}$, $\mathbf{S}$ and $\mathbf{R}$ simply diagonal.

Before to proceed further, it is necessary to write the problem at the first order using $\mathbf{z} = \begin{Bmatrix} \dot{\mathbf{x}} \\ \mathbf{x} \end{Bmatrix}$. With obvious notation, we obtain:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} \qquad\qquad J = \int_0^{+\infty} \left(\mathbf{z}^T\mathbf{Q}\mathbf{z} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt \tag{8}$$

The optimal control problem min J, solved by the LQR, produces the solution $\mathbf{u} = -\mathbf{K}_{OPT}\mathbf{z}$, that substituted into the equation of motion (6) yields:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}_{CN}\dot{\mathbf{x}} + \mathbf{K}_{CN}\mathbf{x} - \mathbf{D}\mathbf{K}_{OPT} \begin{Bmatrix} \dot{\mathbf{x}} \\ \mathbf{x} \end{Bmatrix} = 0 \tag{9}$$

The partition $\mathbf{K}_{OPT} = \begin{bmatrix} \mathbf{C}^*_{OPT} & \mathbf{K}^*_{OPT} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ finally determines the controlled equation of motion:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}_{CN}\dot{\mathbf{x}} + \mathbf{K}_{CN}\mathbf{x} - \mathbf{D}\mathbf{C}^*_{OPT}\dot{\mathbf{x}} - \mathbf{D}\mathbf{K}^*_{OPT}\mathbf{x} = 0 \tag{10}$$

4. Second step: LSM passive approximation

The goal here is to reproduce the active control of equation (10), in terms of a long-range interaction, as it is described by equation (5). This operation is achieved acting on the coefficients β_h's.
The criterion to match the two forms has been outlined in section 2:

$$\min_{\beta_h} Y = \min_{\beta_h} \left[\int_0^T \left(\left|\mathbf{f}^K_{LR} - \mathbf{f}^K_{LQR}\right|^2 + \left|\mathbf{f}^C_{LR} - \mathbf{f}^C_{LQR}\right|^2 \right) dt \right] \tag{11}$$

where, on the basis of the results of section 3, we obtain:

$$\begin{aligned} \left|\mathbf{f}^K_{LR} - \mathbf{f}^K_{LQR}\right|^2 &= \left| \sum_h \mathbf{K}_h\mathbf{x}(\mathbf{t}) - \mathbf{K}_{LQR}\mathbf{x}(\mathbf{t}) \right|^2 \\ \left|\mathbf{f}^C_{LR} - \mathbf{f}^C_{LQR}\right|^2 &= \left| \gamma \sum_h \mathbf{K}_h\mathbf{x}(\mathbf{t}) - \mathbf{C}_{LQR}\mathbf{x}(\mathbf{t}) \right|^2 \end{aligned} \tag{12}$$

With the notation $\boldsymbol{\beta} = [\beta_1, \beta_2, ..., \beta_M]^T$, we have to solve the least square system $\dfrac{\partial Y}{\partial \boldsymbol{\beta}} = 0$.
Let:

$$\begin{aligned} T^{(k)}_{rs} &= \int_0^T \left(\mathbf{x}^T\mathbf{K}_r^T\mathbf{K}_s\mathbf{x}\right) dt & T^{(c)}_{rs} &= \gamma^2 \int_0^T \left(\dot{\mathbf{x}}^T\mathbf{K}_r^T\mathbf{K}_s\dot{\mathbf{x}}\right) dt \\ V^{(k)}_{r} &= \int_0^T \left(\mathbf{x}^T\mathbf{K}_r^T\mathbf{K}_{LQR}\mathbf{x}\right) dt & V^{(c)}_{r} &= \gamma \int_0^T \left(\dot{\mathbf{x}}^T\mathbf{K}_r^T\mathbf{C}_{LQR}\dot{\mathbf{x}}\right) dt \\ W^{(k)}_{s} &= \int_0^T \left(\mathbf{x}^T\mathbf{K}_{LQR}^T\mathbf{K}_s\mathbf{x}\right) dt & W^{(c)}_{s} &= \gamma \int_0^T \left(\dot{\mathbf{x}}^T\mathbf{C}_{LQR}^T\mathbf{K}_s\dot{\mathbf{x}}\right) dt \end{aligned} \tag{13}$$

The linear system generated by $\frac{\partial Y}{\partial \boldsymbol{\beta}} = 0$ produces:

$$\boldsymbol{\beta} = \frac{1}{2}\left(\mathbf{T}^{(k)} + \mathbf{T}^{(c)}\right)^{-1}\left(\mathbf{V}^{(k)} + \mathbf{V}^{(c)} + \mathbf{W}^{(k)} + \mathbf{W}^{(c)}\right) \quad (14)$$

that definitely solves the posed problem.

5. Numerical Simulations

The simulated system is a one-dimensional discrete longitudinal waveguide with uniform masses, each one connected to its first neighbours through equal dampers and springs. A simulation time over an observation time much smaller than the first natural period of the elastic rod allows to neglect boundary conditions, assimilating the system to an infinite rod. The damping coefficient has been set as the 5% of the critical damping. To perform the simulations, an initial non-zero displacement is imposed to the waveguide, in the form of a damped sine-shape decaying from the left end.

The long-range interaction force is determined using the procedure illustrated in sections (3) and (4). The matrix ex-

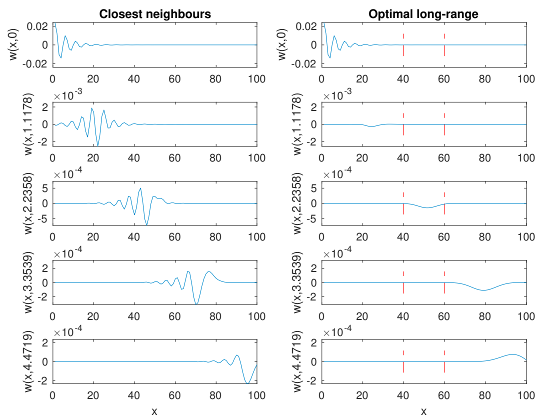

Fig. 1. Controlled waveguide simulations. The dashed vertical lines (red in digital version) show the controlled region

pansion $\sum_h \beta_h \mathbf{K}_h$ has been limited only to two terms, confining the range of interaction in equation (2) up to the fourth order derivatives. The plots of Figure 1 show the comparison between the classical waveguide and the controlled one by long-range forces. The decay of the response in the second case is self-evident. The choice of the **Q** matrix has weights not uniform, emphasising the reduction in a the region (identified by the vertical dashed lines).

The strong decrease of the response is physically due also to the extra-damping induced by the long-range. However, the optimization process guarantees this is the best result one can obtain connecting the masses of the simulated chain in a range up to 2ε distance.

In other words, if we have a linear chain of N masses, decide to limit interactions to the range 2ε means we put a limit on the number of additional connections to be activated. This number is $4(N-1)$. With this upper bound, the outlined procedure guarantees the best result.

6. Physical and engineering remarks

One of the final goals of this work is the design of a new type of elasto-magnetic metamaterial, defining the distribution and the intensity of magnetic dipoles embedded within an elastic support to obtain remarkable effects on the capability of controlling wave propagation. The previous results find a general solution to this problem, independently of the nature of the considered force. In other words, we found the best $g(x)$ in equation (1), whatever the physical origin for $g(x)$, but making it possible to realize it by a passive system of long-range connections.

A Gaussian-like interaction is considered here, since it retraces the long-range decay with the distance, typical of the magnetic forces and can suitably interpolate magnetic interaction. As in [7], $f(r) = \mu r e^{-\sigma r^2}$, assumed $r = x - \xi + w(x) - w(\xi)$ and $\xi = i\varepsilon$; the coefficients β_h's can then be analytically expressed as functions of μ, σ and ε. Limiting the interaction at the second neighbours, the related coefficients are:

$$\begin{aligned} \beta_2 &= \mu(1 - 2\beta\varepsilon^2)e^{-\beta\varepsilon^2} + 4\mu(1 - 8\beta\varepsilon^2)e^{-4\beta\varepsilon^2} \\ \beta_4 &= \mu(1 - 8\beta\varepsilon^2)e^{-4\beta\varepsilon^2} \end{aligned} \tag{15}$$

Given equation (15) and the values of β_2 and β_4 as obtained by the LSM procedure and for a fixed ε, the coefficients μ and σ can be evaluated and they lead to the Gaussian-like force represented below. Therefore, the present analysis

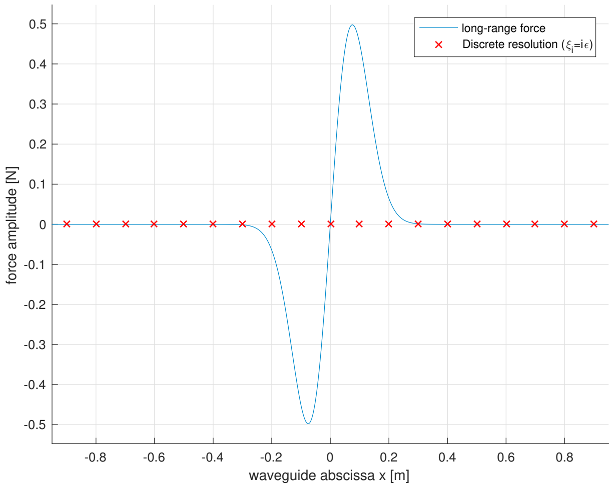

Fig. 2. Identified long-range force

is able to provide the general structure of the best long-range force, once the designer puts a limit on the range of the force. This is a practical, reasonable, engineering bound that confines the cost and the technological effort. If the long-range force would be built by mechanical elastic connections, an element of practical simplicity can suggest this bound. If a magnetic force would be generated by embedding small magnets, a practical bound is related to the intensity, i.e. the force range of the used magnets.

The theory of long-range interaction developed in [5–7] and the optimality criteria for an effective design proposed in the present paper are going to be used in an experimental campaign in the labs of the Dept. of Mechanical and Aerospace Engineering at Sapienza, University of Rome.

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