4-2016

Monotonicity Formulas for Diffusion Operators on Manifolds and Carnot Groups, Heat Kernel Asymptotics and Wiener's Criterion on Heisenberg-type Groups

Kevin L. Rotz
Purdue University

Graduate School Form
30 Updated 12/26/2015

PURDUE UNIVERSITY
GRADUATE SCHOOL
Thesis/Dissertation Acceptance

This is to certify that the thesis/dissertation prepared

By Kevin Rotz

Entitled

MONOTONICITY FORMULAS FOR DIFFUSION OPERATORS ON MANIFOLDS AND CARNOT GROUPS, HEAT KERNEL ASYMPTOTICS AND WIENER'S CRITERION ON HEISENBERG-TYPE GROUPS

For the degree of Doctor of Philosophy

Is approved by the final examining committee:

Nicola Garofalo
Co-chair

Donatella Danielli
Co-chair

Fabrice Baudoin

N.K. Yip

To the best of my knowledge and as understood by the student in the Thesis/Dissertation Agreement, Publication Delay, and Certification Disclaimer (Graduate School Form 32), this thesis/dissertation adheres to the provisions of Purdue University's "Policy of Integrity in Research" and the use of copyright material.

Approved by Major Professor(s): Donatella Danielli

Approved by: David Goldberg 4/22/2016

Head of the Departmental Graduate Program Date

MONOTONICITY FORMULAS FOR DIFFUSION OPERATORS ON MANIFOLDS AND CARNOT GROUPS, HEAT KERNEL ASYMPTOTICS AND WIENER'S CRITERION ON HEISENBERG-TYPE GROUPS

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Kevin L. Rotz

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

May 2016

Purdue University

West Lafayette, Indiana

For my wife, Rachel,

and my father,

who have supported me through everything.

ACKNOWLEDGMENTS

The list of people who have contributed to my education – and thereby indirectly to this thesis – is too long to contain in such a short document. However, here is a partial list of those to whom I am indebted for their continued encouragement and guidance.

Professor Nicola Garofalo has been a great teacher and mentor while I have been at Purdue. I have taken numerous courses with him which helped shape my appreciation and enjoyment of analysis and partial differential equations. He has always encouraged my work, especially when I felt overwhelmed and that what I was working on was hopelessly difficult. For that I am extremely grateful.

I am thankful to Professors Donatella Danielli and Andrei Gabrielov who have both graciously offered travel support on multiple occasions, including travel to Albuquerque, Madrid, and Seattle. Thanks also to Steve Bell, the Graduate School, the Department of Mathematics, and the College of Education for research support during various semesters.

Thanks go to all of my committee members, Professors Garofalo, Donatella Danielli, Fabrice Baudoin, and Aaron Yip, for the interesting courses they have taught, including analysis, geometric measure theory, PDEs, gamma convergence, curvature dimension inequalities, variational calculus, and stochastic calculus. These courses have given me a well-rounded foundation in analysis, geometry, and even probability.

Thanks are due to my office mates in Math 609 – Xuejing Zhang, Jason Parker, Jimmy Vogel, and of course Reginald McGee – for keeping me sane as a graduate student and for just being great friends.

Going back even further, I would like to thank Dr. Richard Robinett in the Penn State physics department for being a great mentor and advisor as an undergraduate. Thanks also to Dr. C. Sean Bohun, who taught my first multi-variable calculus course and instilled a deep appreciation for applied mathematics and pedagogy. Thank you to my high school calculus teachers, Mr. Sean Kolanowski and Mrs. Becky Shubert.

I would like to thank my father for his constant encouragement and interest in what I am working on (even if he did not necessarily understand it).

And finally, thank you to my life-companion, Rachel.

TABLE OF CONTENTS

LIST OF FIGURES

SYMBOLS

$d_{\mathbb{G},CC}$	Carnot-Carathéodory distance
$(\delta_\lambda)_{\lambda>0}$	One-parameter group of non-isotropic dilations
Δ_H	sub-Laplacian
dg	Bi-invariant Haar measure on a Carnot group
$\mathfrak{g}$	Lie algebra of a Carnot group $\mathbb{G}$
$\mathbb{G}$	Carnot group
Γ	Carré du champ
Γ_2	Iterated carré du champ
$\mathfrak{h}$	Lie algebra of the Heisenberg group
h_n	Exhaustion sequence
$\mathbb{H}^n$	n-dimensional Heisenberg group
$J(z)$	Kaplan's mapping $J : V_2 \to \mathrm{End}(V_1)$
$\mathcal{L}$	A diffusion operator
N_H	Horizontal normal
$\nabla_H u$	Horizontal gradient of u
p_t	Heat kernel of a diffusion operator
P_t	Heat semigroup of a diffusion operator
Φ	Fundamental solution for $-\Delta_H$
Q	Homogeneous dimension
ρ	H-gauge (on Carnot groups) or Korányi gauge (on H-type groups)
σ_H	H-perimeter measure
V_j	jth layer of the Lie algebra of a Carnot group
$\mathcal{Z}$	Generator of non-isotropic dilations $(\delta_\lambda)_{\lambda>0}$

ABSTRACT

Rotz, Kevin L. PhD, Purdue University, May 2016. Monotonicity Formulas for Diffusion Operators on Manifolds and Carnot Groups, Heat Kernel Asymptotics and Wiener's Criterion on Heisenberg-type Groups . Major Professor: Nicola Garofalo, Donatella Danielli.

The contents of this thesis are an assortment of results in analysis and sub-Riemannian geometry, with a special focus on the Heisenberg group $\mathbb{H}^n$, Heisenberg-type (H-type) groups, and Carnot groups.

As we wish for this thesis to be relatively self-contained, the main definitions and background are covered in Chapter 1. This includes basic information about Carnot groups, $\mathbb{H}^n$, H-type groups, diffusion operators, and the curvature dimension inequality.

Chapter 2 incorporates excerpts from a paper by N. Garofalo and the author, [42]. In it, we propose a generalization of Almgren's frequency function $N : (0, 1) \to \mathbb{R}$ for solutions to the sub-elliptic Laplace equation $\Delta_H u = 0$ in the unit ball of a Carnot group of arbitrary step. If the function u has vanishing discrepancy, then the frequency is monotonically non-decreasing, and we are able to prove a form of strong unique continuation for such functions.

Chapter 3 grew out of the author seeking parabolic montonicity formulas in the same vein as Almgren's frequency. These include two types of monotonicity formulas, those of Struwe- and Poon-type [72], [67]. If a diffusion operator $\mathcal{L}$ on a complete manifold $\mathbb{M}$ satisfies the curvature dimension inequality $CD(\rho, n)$, then we are able to prove that for solutions to $\mathcal{L}u = u_t$ in $\mathbb{M} \times (0, T)$, Struwe's energy monotonicity holds, at least for time values close enough to T. We introduce a new condition, $C(\omega)$ where $\omega \in C^1(0, T)$, related to the Hessian of the heat kernel, and are able to prove a Poon-type frequency monotonicity formula when taking into account a weighting

factor depending on ω. We also give examples of manifolds satisfying $C(\omega)$, the most interesting of which includes the Ornstein-Uhlenbeck operator. Monotonicity of the weighted frequency also implies a form of strong-unique continuation.

In Chapter 4, we derive asymptotics for the heat kernel on H-type groups and generalize a gradient bound from a paper of Garofalo and Segala [43] to these groups. This gradient bound in turn implies a strong Harnack inequality and Wiener criterion similar to those found in [31] and [43].

1. Introduction and Definitions

1.1 Carnot groups

1.1.1 Step r Carnot groups

Definition 1.1.1 *A* Carnot group of step r *is a Lie group* $\mathbb{G}$ *which is diffeomorphic to* $\mathbb{R}^N$ *for some* $N \in \mathbb{N}$*, and whose Lie algebra* $\mathfrak{g}$ *can be decomposed as follows:*

1. *There exists an* $r \in \mathbb{N}$*,* $r \geq 2$ *such that* $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$
2. $[V_j, V_j] = V_{j+1}$*,* $j = 1, \ldots, r-1$
3. $[V_j, V_r] = 0$*,* $j = 1, \ldots, r$

We will write $\dim V_j = n_j$, so that $N = \sum_{j=1}^r n_j$.

As such $\mathbb{G}$ are necessarily nilpotent, the exponential mapping $\exp : \mathfrak{g} \to \mathbb{G}$ is an analytic diffeomorphism (see Theorem 1.2.1, [24]). We may therefore uniquely identify each point in $\mathbb{G}$ with an element of $\mathfrak{g}$. Fix a basis $\mathcal{B} = \{e_m^j : j = 1, \ldots, r,\ m = 1, \ldots, n_j\}$, where $e_m^j \in V_j$, $j = 1, \ldots, n$, and let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ which makes $\mathcal{B}$ into an orthonormal basis.

Definition 1.1.2 *Let* $g \in \mathbb{G}$*, and suppose that*

$$\exp^{-1}(g) = \sum_{j=1}^{r} \sum_{m=1}^{n_j} x_m^j e_m^j$$

where $x_j = (x_m^j) \in \mathbb{R}^{n_j}$*,* $j = 1, \ldots, r$ *are the components of* r *tuples in Euclidean spaces of different dimensions. We define the* exponential coordinates *of* g *with respect to the basis* $\mathcal{B}$ *as the* N*-tuple*

$$x = (x_1, \ldots, x_r), \qquad x_j \in \mathbb{R}^{n_j}.$$

When the basis has already been fixed, we will just call these the *exponential coordinates of* g. We will often identify g with its exponential coordinates.

As a consequence of exp being a diffeomorphism, we can lift the Lebesgue measure on $\mathfrak{g}$ to a *bi-invariant Haar measure* on $\mathbb{G}$, which we denote by dg.

As $\mathbb{G}$ is a Lie group, for each fixed $g \in \mathbb{G}$, the left-translation mapping $L_g : \mathbb{G} \to \mathbb{G}$ is smooth. We define the vector fields X^j_m on $\mathbb{G}$ by pushing forward the basis $\mathcal{B}$,

$$(X^j_m)_g = (L_g)_* e^j_m,$$

where $(L_g)_*$ is the differential of L_g. The inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ extends to $T\mathbb{G}$ by requiring X^j_m form an orthonormal basis at each $g \in \mathbb{G}$. More formally, we define

$$\langle \cdot, \cdot \rangle_g = (L_{g^{-1}})^* \langle \cdot, \cdot \rangle. \tag{1.1}$$

If S is an orientable, C^2 hypersurface in $\mathbb{G}$, we define the horizontal normal N_H and H-perimeter measure σ_H by

$$N_H = \operatorname{proj}_{V_1} \nu = \sum_{j=1}^{n_1} \langle \nu, X^1_j \rangle X^1_j \tag{1.2}$$

$$d\sigma_H = |N_H| \; d\sigma, \tag{1.3}$$

where ν is the outward unit normal and $\sigma = H^{N-1}$ is the $N-1$-dimensional Hausdorff measure of S. See [27] for relevant details. Note that N_H is simply the projection of ν onto V_1. If S is the level set of some C^2 function f, then

$$N_H = \frac{\nabla_H f}{|\nabla f|}, \qquad d\sigma_H = \frac{|\nabla_H f|}{|\nabla f|} \; d\sigma.$$

Given $\Omega \subset \mathbb{G}$ open and $u : \Omega \to \mathbb{G}$ smooth, we define the *sub-Laplacian* of u by

$$\Delta_H u = \sum_{j=1}^{n_1} (X^1_j)^2 u,$$

and the *horizontal gradient* of u as

$$\nabla_H u = \sum_{j=1}^{n_1} (X^1_j u) X^1_j.$$

Using the inner product, one finds

$$|\nabla_H u|^2 = \sum_{j=1}^{n_1} (X_j^1 u)^2.$$

Thanks to Hörmander's theorem [50], the sub-Laplacian on a Carnot group is always C^∞ hypoelliptic. In addition, there exists a fundamental solution $\Phi(g)$ for $-\Delta_H$ that left-inverts $-\Delta_H$. That is,

$$-\Delta_H(\Phi \circ L_{g^{-1}}) = \delta_g$$

in the sense of distributions (see [16]). By hypoellipticity, Φ is smooth away from the group identity e.

Assign to each layer V_j the formal degree j. Given $u \in V_j$, we define for $\lambda > 0$ the family of non-isotropic dilations $\Delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ by the formula,

$$\Delta_\lambda u = \lambda^j u.$$

This induces a mapping $\delta_\lambda : \mathbb{G} \to \mathbb{G}$,

$$\delta_\lambda = \exp \circ \Delta_\lambda \circ \exp^{-1}.$$

The action $\theta : \mathbb{R} \times \mathbb{G} \to \mathbb{G}$ given by

$$\theta_t(g) = \delta_{e^t}(g)$$

defines a one-parameter group of acting on $\mathbb{G}$. The infinitesimal generator $\mathcal{Z}$ of θ is given by the formula (see [18], chapter IV, section 3)

$$(Xu)(\theta_t(g)) = \frac{d}{dt}(u \circ \theta_t(g)).$$

If we call $r = e^t$, then unraveling the definitions and using the chain rule we have

$$(\mathcal{Z}u)(\delta_r g) = r\frac{d}{dr}(u \circ \delta_r(g)), \quad r > 0. \tag{1.4}$$

In exponential coordinates, one has

$$\delta_\lambda(x_1, x_2, \ldots, x_r) = (\lambda x_1, \lambda^2 x_2, \ldots, \lambda^r x_r)$$

so that

$$\mathcal{Z} = \langle x_1, \nabla_{x_1} \rangle + 2\langle x_2, \nabla_{x_2} \rangle + \cdots + r\langle x_r, \nabla_{x_r} \rangle, \tag{1.5}$$

where ∇_{x_j} denotes the Riemannian gradient with respect to the variables $x_j^1, \ldots, x_j^{n_j}$, that is,

$$\nabla_{x_j} u = \frac{\partial u}{\partial x_j^1} \frac{\partial}{\partial x_j^1} + \cdots + \frac{\partial u}{\partial x_j^{n_j}} \frac{\partial}{\partial x_j^{n_j}}.$$

The Riemannian divergence of $\mathcal{Z}$ is known as the *homogeneous dimension* of $\mathbb{G}$, and is given by

$$Q = \operatorname{div} \mathcal{Z} = \sum_{j=1}^{r} j n_j.$$

We refer the reader to Proposition 5.3.12 of [16] for a proof of the following.

Proposition 1.1.1 *Let $\mathbb{G}$ be a Carnot group and Φ be the fundamental solution of $-\Delta_H$ with pole at e. Then Φ is homogeneous of degree $2 - Q$ with respect to the dilations $(\delta_\lambda)_{\lambda>0}$.*

Euclidean space $\mathbb{R}^n$ can be viewed as a Carnot group of step 1. Indeed, it is a commutative Lie group with group action $x \circ y = x + y$ and its Lie algebra is isomorphic to $\mathbb{R}^n$. Its sub-Laplacian and horizontal gradient are the usual Laplacian and gradient on $\mathbb{R}^n$. The group dilations are given by $\delta_\lambda(x) = \lambda x$, and the homogeneous dimension Q agrees with the topological dimension. Theorems which hold for Carnot groups of arbitrary step also hold for $\mathbb{R}^n$.

Remark 1.1.1 *In the present work, we will often assume that $r = 2$. In that case we make the following simplifications to the notation: $\dim V_1 = n$, $\dim V_2 = m$; the basis $\mathcal{B}$ will be written $\mathcal{B} = \{e_1, \ldots, e_n, \varepsilon_1, \ldots, \varepsilon_m\}$, with $e_j \in V_1$, $j = 1, \ldots, n$ and $\varepsilon_k \in V_2$, $k = 1, \ldots, m$; the exponential coordinates will be denoted (x, z), where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$; and the vector fields X_m^j will be denoted $X_j^1 = X_j$, $j = 1, \ldots, n$, and $X_k^2 = Z_k$, $k = 1, \ldots, m$.*

1.1.2 The Heisenberg group, $\mathbb{H}^n$

Definition 1.1.3 *The n-dimensional* Heisenberg-group $\mathbb{H}^n$ *is* $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, *and endowed with the group law*

$$(x', y', z') \circ (x, y, z) = (x + x', y + y', z + z' + \frac{1}{2}(\langle x', y\rangle - \langle y', x\rangle)), \tag{1.6}$$

where $x, x', y, y' \in \mathbb{R}^n$, $z, z' \in \mathbb{R}$, *and* $\langle \cdot, \cdot \rangle$ *is the usual inner product on* $\mathbb{R}^n$.

By computing the differential of left-translation by an element $g = (x, y, z)$ and pushing forward the standard basis elements of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, we get the following representations of basis vectors for its Lie algebra $\mathfrak{h}$:

$$\begin{aligned} X_j &= (L_g)_* e_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial z}, \qquad j = 1, \ldots, n \\ Y_j &= (L_g)_* e_{n+j} = \frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial z}, \qquad j = 1, \ldots, n \\ Z &= (L_g)_* e_{2n+1} = \frac{\partial}{\partial z}. \end{aligned}$$

We easily see that

$$[X_i, Y_j] = \delta_{ij} Z, \qquad [V, Z] = 0, \ V \in \mathfrak{h}, \tag{1.7}$$

Hence $\mathbb{H}^n$ is a step-two Carnot group with $V_1 = \operatorname{span}\{X_j, Y_j : j = 1, \ldots, n\}$, $V_2 = \operatorname{span}\{Z\}$, and homogeneous dimension $Q = 2n + 2$. Figure 1.1 illustrates V_1 in the case $n = 1$.

We recall that, in the $r = 2$ case (c.f. [24], page 12) the Baker-Campbell-Hausdorff formula reads

$$\exp^{-1}(\exp U \exp V) = U + V + \frac{1}{2}[U, V]. \tag{1.8}$$

Taking

$$\begin{aligned} U &= \sum_{j=1}^{n} (x'_j X_j + y'_j Y_j) + z' Z \\ V &= \sum_{j=1}^{n} (x_j X_j + y_j Y_j) + z Z, \end{aligned}$$

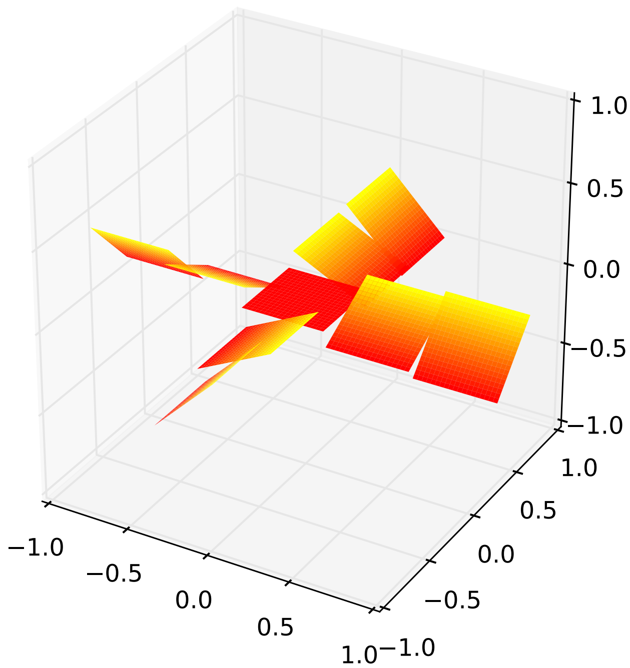


Figure 1.1. The horizontal distribution spanned by X and Y in the Heisenberg group $\mathbb{H}$ for various points in the xy-plane. Graphic produced in Python using matplotlib.

we have from (1.8) and (1.7)

$$\begin{aligned}\exp^{-1}(\exp U \circ \exp V) &= \sum_{j=1}^{n}[(x_j + x_j')X_j + (y_j + y_j')Y_j] + (z + z')Z \\ &\quad + \frac{1}{2}[\sum_{j=1}^{n}(x_j'X_j + y_j'Y_j) + z'Z, \sum_{k=1}^{n}(x_jX_j + y_jY_j) + zZ] \\ &= \sum_{j=1}^{n}[(x_j + x_j')X_j + (y_j + y_j')Y_j] + (z + z')Z \\ &\quad + \frac{1}{2}\sum_{j=1}^{n}(x_j'y_j - y_j'x_k)Z.\end{aligned}$$

In particular, the group law (1.6) is already given in exponential coordinates.

If one takes the sums of the squares of the X_j and Y_j, we get an exponential representation of the sub-Laplacian:

$$\begin{aligned}\sum_{j=1}^{n}(X_j^2+Y_j^2) &= \sum_{j=1}^{n}(\frac{\partial^2}{\partial x_j^2} - y_j\frac{\partial^2}{\partial x_j \partial z} + \frac{y_j^2}{4}\frac{\partial^2}{\partial z^2}) \\ &\quad + \sum_{j=1}^{n}(\frac{\partial^2}{\partial y_j^2} + x_j\frac{\partial^2}{\partial y_j \partial z} + \frac{x_j^2}{4}\frac{\partial^2}{\partial z^2}) \\ &= \Delta_{(x,y)} + \frac{r^2}{4}\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z}\Theta\end{aligned}$$

where we have denoted $\Delta_{(x,y)}$ to be the Laplacian in the variables x and y, $r^2 = |x|^2 + |y|^2$, and Θ is the vector field defined by

$$\Theta = \sum_{j=1}^{n}(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}). \tag{1.9}$$

1.1.3 Groups of Heisenberg type

Each step-two Carnot group comes with a mapping $J : V_2 \to \mathrm{End}(V_1)$ defined by

$$\langle J(z)x, x'\rangle = \langle [x, x'], z\rangle$$

where we have denoted

$$x = x_1e_1 + \cdots x_ne_n, \qquad x' = x_1'e_1 + \cdots + x_n'e_n', \qquad z = z_1\varepsilon_1 + \cdots + z_m\varepsilon_m,$$

and $\langle\cdot,\cdot\rangle$ is the left-invariant inner product defined in Section 1.1.1.

Definition 1.1.4 *If the inner product $\langle\cdot,\cdot\rangle$ can be chosen in such a way that for each $z \in V_2$ with $|z| = 1$ the mapping $J(z)$ is an orthogonal mapping, then the group $\mathbb{G}$ is said to be of* Heisenberg-type *(or* H-type *for short).*

The mapping J is called the Kaplan mapping, and was considered by Kaplan in [53]. It is useful, because it allows us to write down the left-invariant vector fields and sub-Laplacian in terms of the Kaplan mapping:

Lemma 1.1.1 *If $\mathbb{G}$ is a group of Heisenberg-type, then in exponential coordinates*

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{\ell=1}^{m} \langle J(\varepsilon_\ell)x, e_j\rangle \frac{\partial}{\partial z_\ell} \tag{1.10}$$

$$\Delta_H = \Delta_x + \frac{|x|^2}{4}\Delta_z + \sum_{\ell=1}^{m} \frac{\partial}{\partial z_\ell}\Theta_\ell, \tag{1.11}$$

where

$$\Theta_\ell = \sum_{j=1}^{n} \langle J(\varepsilon_\ell)x, e_j\rangle \frac{\partial}{\partial x_j}.$$

See [44] for the relevant calculations.

We note that the Heisenberg group itself is a group of Heisenberg-type, since $J(\varepsilon)$ is given by the block matrix

$$J(\varepsilon) = \begin{pmatrix} j & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & j & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & j \end{pmatrix},$$

where $\mathbf{0}$ represents the 2×2 zero matrix and j is a 2×2 matrix

$$j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In this case, $m = \ell = 1$, and a simple computation shows that $\Theta_1 = \Theta$, where Θ is given above in (1.9).

Distances in groups of H-type

Let $\mathbb{G}$ be a group of H-type. There are two distinct (though comparable) ways to quantify distances on $\mathbb{G}$.

The *Korányi gauge* $\rho : \mathbb{G} \to [0, \infty)$ is defined by

$$\rho(g) = (|x|^4 + 16|z|^2)^{1/4}. \tag{1.12}$$

Introduced by Korányi in [55], the Korányi gauge gives a distance function d_K : $\mathbb{G} \times \mathbb{G} \to [0, \infty)$ via the formula

$$d_K(g, h) = \rho(g^{-1}h) = \rho(h^{-1}g).$$

A proof of the triangle inequality for d_K can be found in [21].

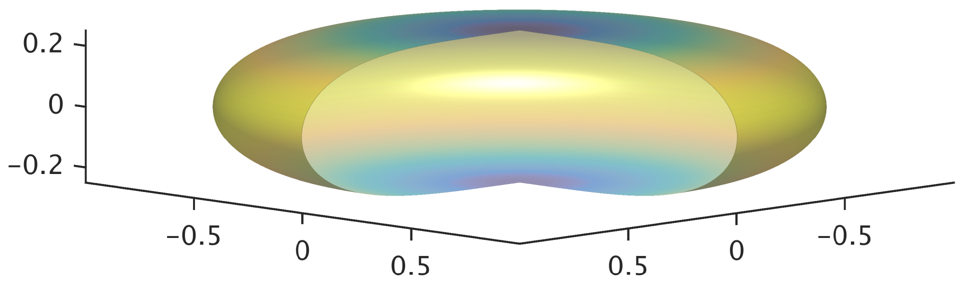


Figure 1.2. The unit "gauge-sphere" in $\mathbb{H}$, that is, $\{g \in \mathbb{H} : \rho(g) = 1\}$. The coloring is based on the Euclidean distance from the origin. The portion $\{x > 0, y > 0\}$ has been cut out to reveal more of the structure of the ball. Graphic produced in MATLAB.

Another useful distance to consider is known as the *Carnot-Carathéodory distance* (or CC-distance for short) and is defined as follows. We call a C^1 curve $\gamma : [0, 1] \to \mathbb{G}$ *horizontal* if, for each $t \in [0, 1]$ its velocity vector $\dot{\gamma}$ lies in V_1, that is,

$$\dot{\gamma}(t) \in \operatorname{span}\{(X_1)_{\gamma(t)}, \ldots, (X_n)_{\gamma(t)}\}.$$

The length of a horizontal curve γ is defined by the formula

$$L_{\mathbb{G}}(\gamma) = \int_0^1 |\dot{\gamma}(t)| \, dt,$$

where $|a| = \langle a, a \rangle^{1/2}$ and the inner product is the one given in (1.1). The distance $d_{\mathbb{G},CC} : \mathbb{G} \times \mathbb{G} \to [0, \infty)$ is then given by

$$d_{\mathbb{G},CC}(g, h) = \inf\{L_{\mathbb{G}}(\gamma) : \gamma \text{ is horizontal }, \gamma(0) = g, \gamma(1) = h\}.$$

Thanks to the Chow-Rashevsky theorem (see [22] or [70]), this is a true distance. The length minimizing curves are called *geodesics*. In the Heisenberg group, geodesics are spirals, as illustrated in Figure 1.3.

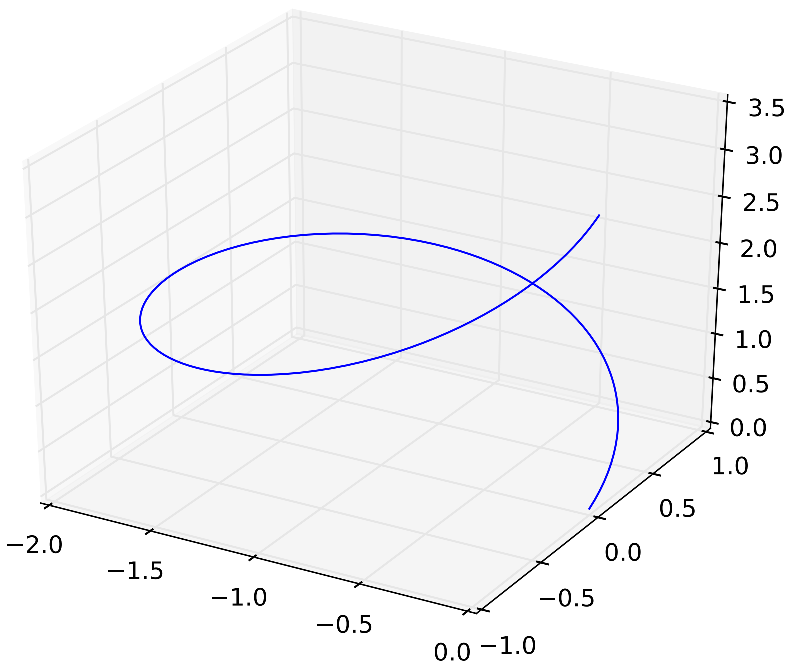


Figure 1.3. A geodesic connecting $(0,0,0)$ to $(0,0,\pi)$ in $\mathbb{H}$. Graphic produced with Python and the matplotlib library.

Although the Carnot-Carathéodory distance is defined on all Carnot groups in the same way described above, H-type groups are distinguished in that it is possible to write down the Carnot-Carathéodory distance for these groups exactly, see [12] and [20] for $\mathbb{H}^n$, and [30] for H-type groups. We define $\nu : [0,\pi] \to [0,\infty]$ by $\nu(0) = 0$, $\nu(\pi) = \infty$, and

$$\nu(\theta) = \frac{\theta}{\sin^2\theta} - \cot\theta = -\frac{d}{d\theta}(\theta\cot\theta), \qquad 0 < \theta < \pi.$$

ν is a strictly increasing diffeomorphism. Let $g = (x, z) \in \mathbb{G}$ be given and $\theta = \theta_g \in [0, \pi]$ be the unique solution to $\nu(\theta) = \frac{4|z|}{|x|^2}$. Then

$$d_{\mathbb{G},CC}(g, e) = \begin{cases} \sqrt{4\pi|z|} & \text{if } x = 0 \\ |x|\frac{\theta}{\sin\theta} & \text{if } x \neq 0. \end{cases} \tag{1.13}$$

Here, we use the convention $\frac{\theta}{\sin\theta} = 1$ when $\theta = 0$. Often, we will write $d(g) = d(x, z) = d_{\mathbb{G},CC}(g, e)$ whenever $g = (x, z)$ in exponential coordinates.

Figure 1.4 shows CC-spheres of varying radii in the Heisenberg group. Notice that the ball is "pinched" near the z-axis. This feature is more pronounced for spheres of larger radii.

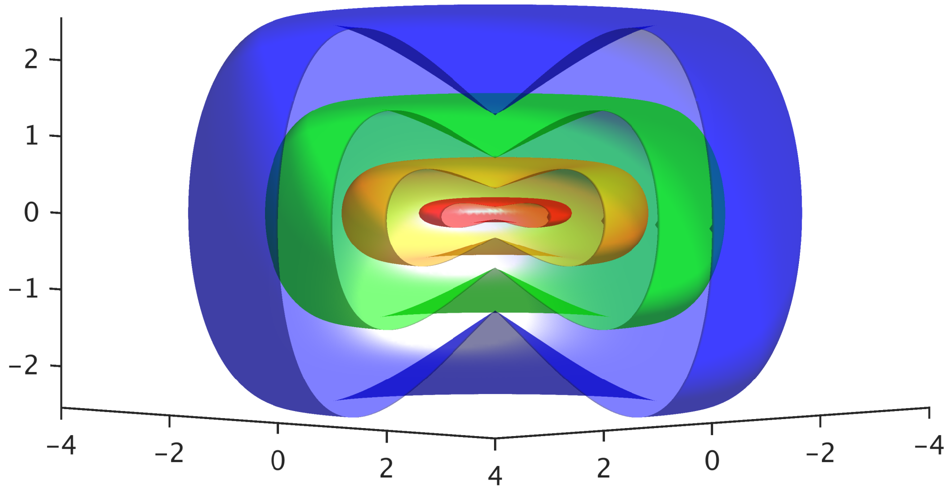


Figure 1.4. The Carnot-Carathéodory spheres $\{g \in \mathbb{H} : d_{\mathbb{G},CC}(g, e) = r\}$ of radii $r = 1, 2, 3, 4$. The portion $\{x > 0, y > 0\}$ has been cut out to reveal more of the structure of the ball. Graphic produced in MATLAB.

ρ and $d_{\mathbb{G},CC}(\cdot, e)$ give two different ways of measuring distance from the group identity, however they are comparable in the following sense: there exists $C \geq 1$ such that for every $g \in \mathbb{G}$,

$$C^{-1}\rho(g) \leq d_{\mathbb{G},CC}(g, e) \leq C\rho(g).$$

Such inequality follows from the 1-homogeneity of $d_{\mathbb{G},CC}$ and ρ, together with the compactness of the unit gauge balls.

As we will see in the next two sections, the Korányi gauge is connected to the Green function for H-type groups, whereas the Carnot-Carathéodory distance is related to asymptotics of the heat kernel. The relation of the heat kernel and Carnot-Carathéodory distance was studied extensively in [30] where Eldredge gave two-sided Gaussian bounds away from the group identity. This connection is further cemented in Chapter 4.

The fundamental solution on Carnot groups and the H-gauge

An interesting observation which leads to a formula for the fundamental solution on H-type groups is the following.

Proposition 1.1.2 *Suppose that $f : [0,\infty) \to \mathbb{R}$ and $u = f \circ \rho$. Then u solves $\Delta_H u = 0$ if and only if f satisfies*

$$f'' + \frac{Q-1}{\rho} f' = 0.$$

Before proving Proposition 1.1.2, we prove the following Lemma which is useful not only here, but also later on in Chapter 2.

Lemma 1.1.2 *Let $\mathbb{G}$ be of H-type. If u is any smooth function defined on a domain $\Omega \subset \mathbb{G}$, then in all of Ω one has*

$$\begin{aligned} \langle \nabla_H u, \nabla_H \rho \rangle &= \frac{|x|^2}{\rho^3} \mathcal{Z} u + \frac{1}{2} \langle J(\nabla_z \rho) x, \nabla_x u \rangle \\ &= \frac{|x|^2}{\rho^3} \mathcal{Z} u + \frac{4}{\rho^3} \langle J(z) x, \nabla_x u \rangle. \end{aligned} \tag{1.14}$$

In particular, taking $u = \rho$ and $\Omega = \mathbb{G}$ one has

$$|\nabla_H \rho|^2 = \frac{|x|^2}{\rho^2}.$$

Proof According to (1.10) we have

$$\begin{aligned}
\langle \nabla_H u, \nabla_H \rho \rangle &= \sum_{j=1}^{n} (\frac{\partial u}{\partial x_j} + \frac{1}{2} \sum_{\ell=1}^{m} \langle J(\varepsilon_\ell)x, e_j \rangle \frac{\partial u}{\partial z_\ell})(\frac{\partial \rho}{\partial x_j} + \frac{1}{2} \sum_{k=1}^{m} \langle J(\varepsilon_k)x, e_j \rangle \frac{\partial \rho}{\partial z_k}) \\
&= \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial \rho}{\partial x_j} + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{m} \frac{\partial u}{\partial x_j} \langle J(\varepsilon_k)x, e_j \rangle \frac{\partial \rho}{\partial z_k} \\
&\quad + \sum_{j=1}^{n} \sum_{\ell=1}^{m} \langle J(\varepsilon_\ell)x, e_j \rangle \frac{\partial u}{\partial z_\ell} \frac{\partial \rho}{\partial x_j} \\
&\quad + \frac{1}{4} \sum_{j=1}^{n} \sum_{\ell=1}^{m} \sum_{k=1}^{m} \langle J(\varepsilon_\ell)x, e_j \rangle \frac{\partial u}{\partial z_\ell} \langle J(\varepsilon_k)x, e_j \rangle \frac{\partial \rho}{\partial z_k} \\
&= \langle \nabla_x u, \nabla_x \rho \rangle + \frac{1}{2} \langle J(\nabla_z \rho)x, \nabla_x u \rangle + \frac{1}{2} \langle J(\nabla_z u)x, \nabla_x \rho \rangle \\
&\quad + \frac{1}{4} \sum_{j=1}^{n} \sum_{\ell=1}^{m} \sum_{k=1}^{m} \langle J(\varepsilon_\ell)x, e_j \rangle \frac{\partial u}{\partial z_\ell} \langle J(\varepsilon_k)x, e_j \rangle \frac{\partial \rho}{\partial z_k}. \qquad (1.15)
\end{aligned}$$

Note that

$$\sum_{j=1}^{n} \langle J(\varepsilon_\ell)x, e_j \rangle \langle J(\varepsilon_k)x, e_j \rangle = \langle J(\varepsilon_\ell)x, J(\varepsilon_k)x \rangle = \delta_{\ell k} |x|^2,$$

where the last equality holds by polarization and the H-type assumption. Hence

$$\frac{1}{4} \sum_{j=1}^{n} \sum_{\ell=1}^{m} \sum_{k=1}^{m} \langle J(\varepsilon_\ell)x, e_j \rangle \frac{\partial u}{\partial z_\ell} \langle J(\varepsilon_k)x, e_j \rangle \frac{\partial \rho}{\partial z_k} = \frac{1}{4} \langle \nabla_z u, \nabla_z \rho \rangle |x|^2. \qquad (1.16)$$

We easily see that

$$\nabla_x \rho = \frac{|x|^2}{\rho^3} x \qquad (1.17)$$

$$\nabla_z \rho = \frac{8}{\rho^3} z, \qquad (1.18)$$

and therefore

$$\langle J(\nabla_z u)x, \nabla_x \rho \rangle = \frac{|x|^2}{\rho^3} \langle J(\nabla_z u)x, x \rangle = \frac{|x|^2}{\rho^3} \langle [x, x], J(\nabla_z u) \rangle = 0. \qquad (1.19)$$

Inserting (1.16) and (1.19) into (1.15), we find

$$\langle \nabla_H u, \nabla_H \rho \rangle = \langle \nabla_x u, \nabla_x \rho \rangle + \frac{1}{4} |x|^2 \langle \nabla_z u, \nabla_z \rho \rangle + \frac{1}{2} \langle J(\nabla_z \rho)x, \nabla_x u \rangle. \qquad (1.20)$$

On the other hand, from (1.17) and (1.18),

$$\begin{aligned}\langle\nabla_x u,\nabla_x\rho\rangle+\frac{1}{4}|x|^2\langle\nabla_z u,\nabla_z\rho\rangle &= \frac{|x|^2}{\rho^3}\langle\nabla_x u,x\rangle+\frac{2|x|^2}{\rho^3}\langle\nabla_z u,z\rangle\\ &= \frac{|x|^2}{\rho^3}\mathcal{Z}u. \qquad (1.21)\end{aligned}$$

Substituting (1.21) into (1.19), we arrive at (1.14).

By inspection, one easily sees that the Korányi gauge is homogeneous of degree 1 with respect to the dilations $(\delta_\lambda)_{\lambda>0}$ and therefore $\mathcal{Z}\rho=\rho$. Inserting this information, together with (1.17) and (1.19) (with $u=\rho$) into (1.14), we find

$$|\nabla_H\rho|^2=\frac{|x|^2}{\rho^2},$$

completing the proof. ■

Proof of Proposition 1.1.2 (1.17) and (1.18) imply that for $\ell=1,\dots,m$,

$$\begin{aligned}\Theta_\ell\rho &= \sum_{j=1}^n\frac{|x|^2x_j}{\rho^3}\langle J(\varepsilon_\ell)x,e_j\rangle\\ &= \frac{|x|^2}{\rho^3}\sum_{j=1}^n\langle J(\varepsilon_\ell)x,x\rangle\\ &= 0\end{aligned}$$

where we have again used $[x,x]=0$. Also, we have

$$\begin{aligned}\Delta_x\rho &= \frac{|x|^2}{\rho^7}[(n+2)\rho^4-3|x|^4]\\ \Delta_z\rho &= \frac{8}{\rho^7}(m\rho^4-24|z|^2)\end{aligned}$$

and therefore

$$\begin{aligned}\Delta_H\rho &= \frac{|x|^2}{\rho^7}[(n+2)\rho^4-3|x|^4+2m\rho^4-48|z|^2)\\ &= \frac{|x|^2}{\rho^7}[(Q+2)\rho^4-3(|x|^4+16|z|^2))\\ &= (Q-1)\frac{|x|^2}{\rho^3}.\end{aligned}$$

If $u = f \circ \rho$, then

$$\begin{aligned}\Delta_H u &= f''(\rho)|\nabla_H \rho|^2 + f'(\rho)\Delta_H \rho \\ &= \frac{|x|^2}{\rho^2} f''(\rho) + (Q-1)\frac{|x|^2}{\rho^3} f'(\rho)\end{aligned}$$

from which the claim follows. ■

Using an integrating factor, it is easy to see that the general solution to the ODE in Proposition 1.1.2 is given by

$$f(\rho) = c_1 \rho^{2-Q} + c_2.$$

As we expect the Green function for the sub-Laplacian to decay to 0 at infinity, this suggests that $c_2 = 0$. In [53], Kaplan proved the following:

Theorem 1.1.1 *Define $C > 0$ by*

$$c_1^{-1} = n(Q-2)\int_{\mathbb{G}} \frac{dxdz}{\left((|x|^2+1)^2 + 16|z|^2\right)^{\frac{Q+2}{4}}}.$$

If $\mathbb{G}$ is of H-type, $G = c_1\rho^{2-Q}$ is a fundamental solution for $-\Delta_H$ with pole at $e \in \mathbb{G}$..

In particular, this gives a formula for the Green function on the Heisenberg group. However, this had previously been found in the special case of $\mathbb{H}^n$ by Folland in [35].

Theorem 1.1.1 motivates the following definition on Carnot groups of arbitrary step.

Definition 1.1.5 *Let $\mathbb{G}$ be any Carnot group and $\Phi : \mathbb{G} \setminus \{e\} \to (0, \infty)$ the fundamental solution for $-\Delta_H$ with pole at e. We define the H-gauge $\rho : \mathbb{G} \to \mathbb{R}$ by the formula*

$$\rho(g) = \begin{cases} 0 & \textit{if } g = e \\ \Phi(g)^{\frac{1}{2-Q}} & \textit{otherwise.} \end{cases} \tag{1.22}$$

The H-gauge is a smooth function defined on all of $\mathbb{G}$ and, due to Proposition 1.1.1, homogeneous of degree 1 with respect to the dilations $(\delta_\lambda)_{\lambda>0}$. On groups of

H-type, the H-gauge in (1.22) is a constant multiple of the Korányi gauge by Theorem 1.1.1; thus when $\mathbb{G}$ is of H-type, we will renormalize the H-gauge so that it agrees with the Korányi gauge. Remarkably, the H-gauge behaves rather like the Korányi gauge even on general Carnot groups as evidenced from the following (see Proposition 5.4.3 in [16]).

Proposition 1.1.3 *Let $\mathbb{G}$ be any Carnot group. Let $f : [0, \infty) \to \mathbb{R}$ be smooth. Then*

$$\Delta_H \rho = |\nabla_H \rho|^2 \frac{Q-1}{\rho},$$

and thus

$$\Delta_H (f \circ \rho) = |\nabla_H \rho|^2 [f''(\rho) + \frac{Q-1}{\rho} f'(\rho)].$$

The heat kernel for groups of H-type

Similar to the Laplacian $-\Delta$ in $\mathbb{R}^n$, the sub-Laplacian $-\Delta_H$ is a generator for a Markov process on $\mathbb{G}$ known as Brownian motion. The Brownian motion on $\mathbb{G}$ can be approximated as follows: consider n independent one-dimensional fair random walks (S^i), $i = 1, 2, \ldots$ having step size $\sqrt{h}$, where $h > 0$. Call the jumps on the kth step $S_k = (S_k^1, \ldots, S_k^n) \in \mathbb{R}^n$, so $S_k^i = \pm\sqrt{h}$ for $i = 1 \ldots, n$. This not only gives a discrete random walk in $\mathbb{R}^n$ by considering $X_k = S_1 + \cdots + S_k$ at time kh, but also one in the group $\mathbb{G}$ by treating $g_n = (X_n, 0) \in \mathbb{R}^{n+m}$ as an element of the group in exponential coordinates and letting

$$h_n = g_n \circ g_{n-1} \circ \ldots \circ g_2 \circ g_1.$$

denote a particle's position at time kh. Then by letting $h \to 0$, we recover Brownian motion on $\mathbb{G}$.

The dilations $(\delta_\lambda)_{\lambda>0}$ means that the steps behave like $\sqrt{h}$ in the horizontal directions, but h in the second layer. Thus the process — and hence the heat flow

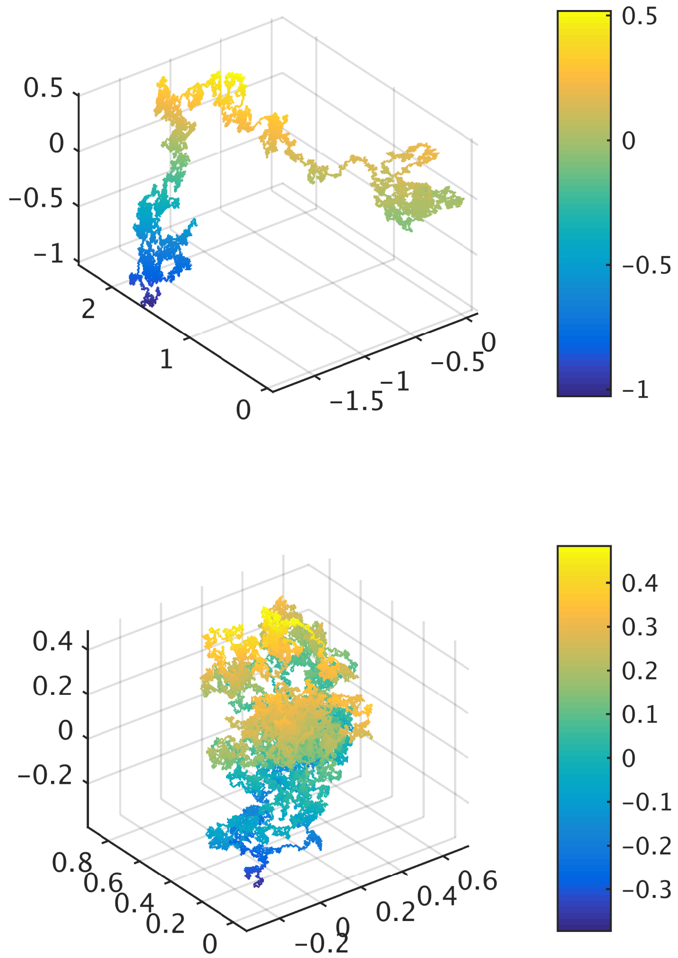


Figure 1.5. Brownian motion for $t \in [0, 1]$ starting at the origin. Top: In the Heisenberg group $\mathbb{H}$ using the sub-Laplacian as generator. Bottom: In $\mathbb{R}^3$ using the Laplacian as generator.

associated to the heat operator $\partial_t - \Delta_H$ — tends to spread out more in the horizontal directions since h is small. An illustration of this is given in Figure 1.5, and compared with Brownian motion in $\mathbb{R}^3$. Notice how the Heisenberg Brownian motion does not significantly rise or fall while it moves close to the y-axis. This reflects the fact that $Y = \partial_y$ when $x = 0$.

If $u : \mathbb{G} \times (0, \infty) \to \mathbb{R}$ solves the heat equation and $u(g,t) = v(\rho(g),t)$ for some v, then v solves

$$|\nabla_H \rho|^2 (v_{\rho\rho} + \frac{Q-1}{\rho} v_\rho) = v_t.$$

In the elliptic case, the absence of the v_t term allows us to divide by $|\nabla_H \rho|^2$ and solve the corresponding ODE, but this is no longer possible in the parabolic case. Thus unlike the harmonic case, one cannot solve the heat equation directly to find the heat kernel in terms of the Korányi gauge. In fact, the heat kernel on H-type groups is more closely related to the Carnot-Carathéodory distance. This is graphically evident from Figure 1.6, which shows a thermal cross-section of the Heisenberg heat kernel when $t = 1$. The horizontal axis represents $|x|$ (and reflected about the vertical axis), and the vertical axis z.

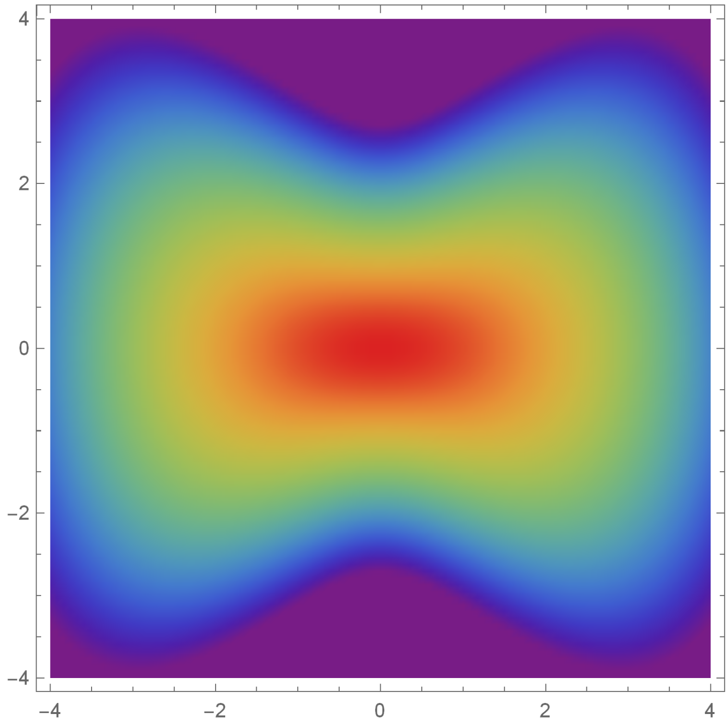


Figure 1.6. Thermal diagram of the Heisenberg heat kernel at $t = 0.1$. Compare to the CC-spheres given in Figure 1.4. Graphic produced in Mathematica.

Regardless of the qualitative similarity between Figures 1.4 and 1.6, it is *not* true that $p_t(g, g')$ is Gaussian with respect to the Carnot-Carathéodory distance. However, two-sided Gaussian bounds of the following form apply: there exists a constant $M > 0$ depending only on the homogeneous dimension

$$M^{-1}t^{-Q/2}\exp(-\frac{Md(g)^2}{t}) \leq p_t(g) \leq Mt^{-Q/2}\exp(-\frac{Md(g)^2}{t}). \tag{1.23}$$

(1.23) holds for all Carnot groups and, more generally, sum-of-squares operators satisfying Hörmander's condition, see [52]. If d is replaced by any other equivalent homogeneous norm, e.g. the Korányi gauge, one gets a similar two-sided Gaussian estimate, perhaps with a different constant M.

The most explicit formula available for the H-type heat kernel with pole at e is given in exponential coordinates by

$$p_t((x,z),(0,0)) = (4\pi t)^{-Q/2}\int_{\mathbb{R}^m} \exp(i\langle\frac{2\xi}{t}, z\rangle)\exp(-\frac{|x|^2}{4t}\cdot\frac{2|\xi|}{\tanh 2|\xi|})(\frac{2|\xi|}{\sinh 2|\xi|})^{n/2}\, d\xi. \tag{1.24}$$

(1.24) is attributed to Hulanicki [51] and Gaveau [45], see also [25], [69], and [30]. Due to the group structure of $\mathbb{G}$ and left-invariance of $-\Delta_H$, we have $p_t(g', g) = p_t(g^{-1}g', e)$. Therefore, we typically speak of the function $p_t(g) = p_t(g, e)$.

Notice from (1.24) that $p_t(x, z)$ is homogeneous of degree $-Q$ with respect to the space-time dilations $(x, z, t) \mapsto (\lambda x, \lambda^2 z, \lambda^2 t)$. Thus

$$p_t(x,z) = t^{-Q/2}p_1(\frac{x}{\sqrt{t}}, \frac{z}{t^2}).$$

This also implies that $p_t(x, z)$ solves the first-order PDE

$$\mathcal{Z}p_t + 2t\partial_t p_t = -Qp_t,$$

where $\mathcal{Z}$ is the usual generator of the non-isotropic dilations in (1.5). This will be a helpful observation in Chapter 4.

1.2 Diffusion operators and the Γ calculus

Let $\mathbb{M}$ be a smooth manifold with smooth measure μ and an elliptic second order diffusion operator $\mathcal{L}$ defined on smooth functions on $\mathbb{M}$. We assume that $\mathcal{L}1 = 0$ and that $\mathcal{L}$ is symmetric on $C_0^\infty(\mathbb{M})$ with respect to the $L^2(\mathbb{M},\mu)$ inner product. Many of the definitions that follow can also be found in [5] and [7].

1.2.1 The *carré du champ*

Definition 1.2.1 *The* carré du champ *is a symmetric first-order bilinear differential operator defined on smooth functions, given by the formula*

$$\Gamma(u,v) = \frac{1}{2}(\mathcal{L}(uv) - u\mathcal{L}v - v\mathcal{L}u). \tag{1.25}$$

The iteration of the carré du champ is a symmetric second-order bilinear differential operator defined on smooth functions. It is defined similar to (1.25)*, but uses* Γ *as the multiplication:*

$$\Gamma_2(u,v) = \frac{1}{2}(\mathcal{L}\Gamma(u,v) - \Gamma(u,\mathcal{L}v) - \Gamma(\mathcal{L}u,v)). \tag{1.26}$$

We write $\Gamma(u)$ as short-hand for $\Gamma(u,u)$, and similarly for Γ_2. It is important to note that $\Gamma(u) \geq 0$ since $\mathcal{L}$ is a diffusion operator. This fact can be found in [5].

The definition of the carré du champ allows us to perform a version of integration by parts whenever u or v is compactly supported:

$$\int_{\mathbb{M}} \Gamma(u,v) = \frac{1}{2}\int_{\mathbb{M}} [\mathcal{L}(uv) - u\mathcal{L}v - v\mathcal{L}u]\; d\mu = -\int_{\mathbb{M}} u\mathcal{L}v\; d\mu = -\int_{\mathbb{M}} v\mathcal{L}u\; d\mu.$$

The last equality holds since $\mathcal{L}$ is assumed to by symmetric over $C_0^\infty(\mathbb{M})$, whereas

$$\int_{\mathbb{M}} \mathcal{L}(uv)\; d\mu = 0$$

since the integration is performed over a compact set. In particular, we may take a cutoff function which is identically equal to 1 on the domain of integration and use the symmetry of the operator $\mathcal{L}$. Since $\mathcal{L}(1) = 0$, the formula follows.

The carré du champ can be used to extend $\mathcal{L}$ to a self-adjoint operator on $\mathcal{D}(\mathcal{L}) \subset L^2(\mathbb{M})$ via the Friedrichs extension. We briefly recall the construction. One defines the Dirichlet form $\mathcal{E}$ and the Dirichlet norm $\|\cdot\|_{\mathcal{E}}$ on C_0^∞ :

$$\mathcal{E}(u,v) = \int_{\mathbb{M}} \Gamma(u,v)\; d\mu$$

$$\|u\|_{\mathcal{E}}^2 = \|u\|_{L^2(\mathbb{M})}^2 + \mathcal{E}(u,u).$$

We let $\mathcal{D}(\mathcal{E})$ be the completion of $C_0^\infty(\mathbb{M})$ with respect to this norm, which imbeds into $L^2(\mathbb{M})$. The domain $\mathcal{D}(\mathcal{L})$ is then defined by the set of $u \in \mathcal{D}(\mathcal{E})$ such that there exists a finite constant $C = C(u) < \infty$ with

$$\mathcal{E}(u,v) \le C \|v\|_{L^2(\mathbb{M})}$$

for every $v \in \mathcal{D}(\mathcal{E})$. In this case, we define $\mathcal{L}u$ by extending the integration by parts formula

$$\int_{\mathbb{M}} v\mathcal{L}u \, d\mu = -\int_{\mathbb{M}} \Gamma(u,v) \, d\mu, \qquad u \in \mathcal{D}(\mathcal{L}),\ v \in \mathcal{D}(\mathcal{E}). \tag{1.27}$$

We emphasize that $\mathcal{D}(\mathcal{L}) \subset \mathcal{D}(\mathcal{E})$ by definition.

1.2.2 The curvature-dimension inequality

Let $\mathbb{M}$ be an n-dimensional Riemannian manifold with metric g and Laplace-Beltrami operator $\mathcal{L} = \Delta$. We recall the formula

$$\Delta(uv) = u\Delta v + v\Delta u + g(\nabla u, \nabla v)$$

where ∇u is the Riemannian gradient of u, that is, $\nabla u = (du)^\sharp$, where $\sharp : T^*(\mathbb{M}) \to T(\mathbb{M})$ is the musical isomorphism, see [59]. From this it follows that

$$\Gamma(u,v) = g(\nabla u, \nabla v). \tag{1.28}$$

We also recall the Bochner formula [14],

$$\Gamma_2(u) = \left\|\nabla^2 u\right\|_{HS}^2 + \mathrm{Ric}(\nabla u, \nabla u), \tag{1.29}$$

where $\|A\|_{HS}$ is the Hilbert-Schmidt norm, $\nabla^2 u$ is the Hessian of u, and Ric is the Ricci curvature tensor. If the Ricci curvature is bounded below in the sense of bilinear forms by some $\rho \in \mathbb{R}$, then by the Cauchy-Schwarz inequality,

$$\Gamma_2(u) \ge \frac{1}{n}(\Delta u)^2 + \rho g(\nabla u, \nabla u).$$

This observation for Riemannian manifolds motivates the following definition for arbitrary manifolds with measure and diffusion operator.

Definition 1.2.2 *We say that the pair* ($\mathbb{M}$, $\mathcal{L}$) *satisfies the* curvature-dimension inequality $CD(\rho, n)$ *for* $\rho \in \mathbb{R}$ *and* $n \in (0, \infty]$ *if, for each* $u \in C^\infty(\mathbb{M})$,

$$\Gamma_2(u) \geq \frac{1}{n}(\mathcal{L}u)^2 + \rho\Gamma(u). \qquad CD(\rho, n)$$

In $CD(\rho, n)$, *we treat* $\frac{1}{n}$ *as 0 when* $n = \infty$.

Based on the discussion above, n-dimensional Riemannian manifolds with $\mathrm{Ric} \geq \rho$ satisfy $CD(\rho, n)$. In fact, Bakry proved in [3] that an n-dimensional Riemannian manifold satisfies $CD(\rho, n)$ if and only if $\mathrm{Ric} \geq \rho$.

What is not so obvious is that there are other manifolds with diffusion operator $\mathcal{L}$ satisfying $CD(\rho, n)$. Many examples can be found in [5]. Two interesting ones:

- *Bessel operator*: Fix $\alpha \geq 0$. Let $\mathbb{M} = (0, \infty)$, and define $\mathcal{L}_\alpha = \frac{d^2}{dx^2} + \frac{\alpha}{x}\frac{d}{dx}$. Then the pair $\mathbb{M}$ and $\mathcal{L}$ satisfy $CD(0, \alpha + 1)$. Notice that when $\alpha > 0$, the "dimension" part of the curvature-dimension inequality is *strictly larger than* the topological dimension of $\mathbb{M}$.

- *Ornstein-Uhlenbeck operator*: Fix $\rho \in \mathbb{R}$. Let $\mathbb{M} = \mathbb{R}^n$ and $\mathcal{L}_\rho = \Delta - \rho\langle x, \nabla\rangle$. The operator $\mathcal{L}_\rho$ is the generator of a stochastic process given by a time-change of Brownian motion. This pair satisfies $CD(\rho, \infty)$. Interestingly, if $\rho \neq 0$, one can show that $(\mathbb{M}, \mathcal{L}_\rho)$ cannot satisfy $CD(\rho', n)$ for any choice of ρ' when n is finite – the operator is intrinsically infinite dimensional.

1.2.3 The heat semigroup for $\mathcal{L}$

As $\mathcal{L}$ is a diffusion operator, it possesses a heat semigroup $(P_t)_{t\geq 0}$ defined on $L^2(\mathbb{M}, \mu)$ which satisfies the following properties, see [7].

- $P_{t+s} = P_t \circ P_s$

- $P_0 = Id$

- P_t is *sub-Markov* and *positivity preserving*, that is if $0 \leq f \leq 1$ then $0 \leq P_t f \leq 1$

- There exists a smooth positive kernel $p_t : \mathbb{M} \times \mathbb{M} \to (0, \infty)$ such that

$$P_t f(x) = \int_{\mathbb{M}} p_t(x, y) f(y)\, d\mu(y).$$

Furthermore, $p_t(x, y) = p_t(y, x)$ since $\mathcal{L}$ is symmetric, and p_t satisfies the Chapman-Kolmogorov identity

$$p_t(x, y) = \int_{\mathbb{M}} p_t(x, z) p_t(z, y)\, d\mu(z). \tag{1.30}$$

1.2.4 Completeness for diffusion operators

Saying that a Riemannian manifold $(\mathbb{M}, g)$ is complete means to say that it is complete as a metric space when the metric is the geodesic metric. Equivalently by the Hopf-Rinow theorem, $\mathbb{M}$ is complete if it is geodesically complete, that is, every maximally extended geodesic is defined on all of $\mathbb{R}$ [18]. A third, equivalent, notion that applies to diffusion operators is the following:

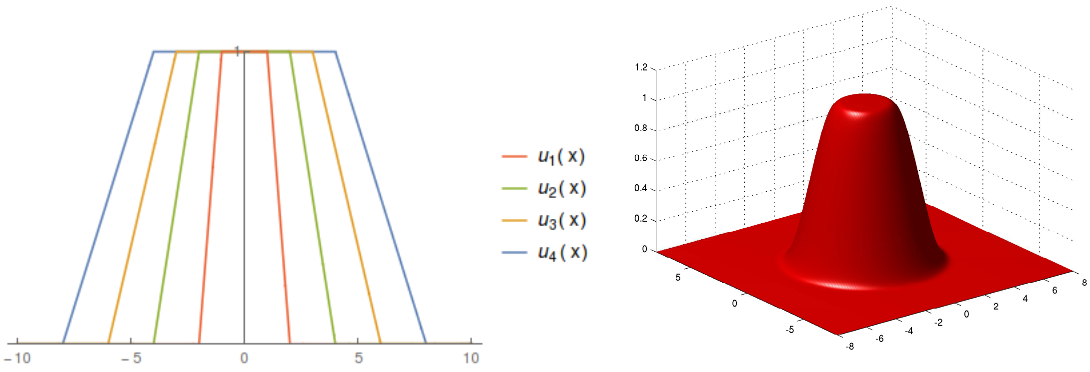
Definition 1.2.3 *We will say that the pair* $(\mathbb{M}, L)$ *is complete if there exists an* exhaustion sequence *of functions* $\{h_n\}_{n=1}^{\infty}$*, such that (1)* $h_n \in C_0^\infty(\mathbb{M})$ *for each* n*, (2)* $0 \le h_n \le 1$*, (3)* $h_n \nearrow 1$ *pointwise as* $n \to \infty$*, and (4)* $\|\Gamma(h_n)\|_{L^\infty(\mathbb{M})} \to 0$ *as* $n \to \infty$*.*

It is tempting to include the measure μ in the definition of completeness since the $L^\infty(\mathbb{M})$ norm is used, but this is unnecessary since the sequence h_n is assumed to be smooth.

For example, in $\mathbb{R}^n$, one can construct an exhaustion sequence as follows. Let

$$u_n(x) = \begin{cases} 1 & |x| \in [0, n] \\ 2 - \frac{1}{n}|x| & |x| \in [n, 2n] \\ 0 & |x| \in [2n, \infty), \end{cases}$$

and define $h_n(x)$ as a mollified version of $u_n(|x|)$. More generally on a Riemannian manifold $(\mathbb{M}, g)$, one uses a mollified form of $u_n(d(x, \mathcal{O}))$, where $d(\cdot, \cdot)$ is the Riemannian geodesic distance induced by the metric g, and $\mathcal{O}$ is an arbitrary fixed point in $\mathbb{M}$.



(a) (b)

Figure 1.7. (a): $u_n(|x|)$, $n = 1, 2, 3, 4$ (produced in Mathematica). (b) h_2 in the exhaustion sequence for $\mathbb{R}^2$ (produced in MATLAB).

1.3 Layout of the remainder of this thesis

The remainder of this document contains three projects. Below is a short list of the topics, including the portions of the present chapter required to follow the indicated chapter.

- Chapter 2 – Almgren-type frequency on Carnot groups. This chapter refers to Sections 1.1.1–1.1.3, excluding the Carnot-Carathéodory distance and heat kernel on H-type groups.

- Chapter 3 – Struwe- and Poon-type functionals for symmetric diffusion operators. This makes use primarily of Sections 1.2.1 – 1.2.4, but also briefly works with the heat kernel and heat semi-group on H-type groups (among other sub-Riemannian manifolds) in Section 3.7. Thus we also recommend reading Section 1.1.3.

- Chapter 4 – Heat kernel asymptotics and Wiener criterion for groups of H-type. As the chapter's name suggests, Sections 1.1.1–1.1.3 are useful here, including the Carnot-Carathéodory distance and heat kernel for H-type groups.

2. Almgren-type frequency on Carnot groups

2.1 History of Almgren's frequency

Let $B_r = \{x \in \mathbb{R}^n : |x| < r\}$. In [2], Almgren showed that the frequency of u given by

$$N(r) = \frac{r \int_{B_r} |\nabla u|^2 \, dx}{\int_{\partial B_r} u^2 \, d\sigma} = \frac{rD(r)}{H(r)}, \quad 0 < r < R \tag{2.1}$$

is monotonically non-decreasing in r when u is harmonic in B_R. In [40] and [41], Garofalo and Lin showed that monotonicity of Almgren's frequency implies doubling of the height function

$$H(r) = \int_{\partial B_r} u^2 \, d\sigma,$$

which in turn implies the doubling of the integral of u^2 over solid balls and the strong unique continuation property (sucp) for the Laplacian. Recently, the frequency has been applied to the thin obstacle problem, see [66] for this aspect.

Garofalo and Lanconelli [39] gave a definition of the Almgren's frequency on the Heisenberg-group $\mathbb{H}^n$ using the Korányi gauge and the horizontal gradient of u. Specifically, they defined the height and Dirichlet energy by

$$H_u(r) = \int_{\{\rho(g)=r\}} u^2 \frac{|\nabla_H \rho|^2}{|\nabla \rho|} dH^{2n}$$
$$D_u(r) = \int_{\{\rho(g)<r\}} |\nabla_H u|^2 \, dg,$$

where H^{2n} is the Hausdorff measure of the set $\{\rho(g) = r\}$. Almgren's frequency on $\mathbb{H}^n$ is then still defined in terms of the height and Dirichlet functions by (2.1). Garofalo and Lanconelli were interested in sucp for the Heisenberg group. An important quantity in their analysis was the vector field

$$\Theta = \sum_{j=1}^{n} y_j \partial_{x_j} - x_j \partial_{y_j}, \tag{2.2}$$

which is the same vector field appearing in the sub-Laplacian Δ_H given in (1.9).

In this chapter, we recall the main results of the paper [42] by the author and Garofalo. There, the authors gave a definition of Almgren's frequency on Carnot groups of arbitrary step, and gave a sufficient condition for such frequency to be monotonic. This condition became known as the function u having vanishing discrepancy at the identity. They also studied the local boundedness of the frequency and proved a sufficient condition for N to be locally bounded. Finally, a relationship between discrepancy and one-parameter monotonicity formulas of Weiss- and Monneau-type was established. In particular, these are monotonic if u has vanishing discrepancy at e.

The layout of the chapter will be rather similar to [42].

2.2 Preliminaries: Almgren's monotonicity in $\mathbb{R}^n$

Before we begin, we would like to recall an outline of the proof of Almgren's monotonicity in $\mathbb{R}^n$. This provides a basis for the proof in Carnot groups. Assume throughout that u is classically harmonic.

1. Integrating by parts, one has
$$rD(r) = \int_{\partial B_r} u\mathcal{Z}u \, d\sigma,$$
where $\mathcal{Z}u = \langle x, \nabla u\rangle$ is the generator of the (isotropic) dilations on $\mathbb{R}^n$ applied to u.

2. By differentiating $H(r)$, we arrive at the differential equation
$$H'(r) = \frac{n-1}{r}H(r) + 2D(r).$$

3. Using the Rellich identity and the co-area formula, the derivative of the Dirichlet function is given by
$$D'(r) = \frac{n-2}{r}D(r) + 2\int_{\partial B_r} (\frac{\mathcal{Z}u}{r})^2 \, d\sigma. \tag{2.3}$$

4. Using the first three steps and inserting them into the logarithmic derivative of $N(r)$, the non-decreasingness of the frequency follows from the Cauchy-Schwarz inequality.

The main steps of our study of Almgren's frequency will follow along the same lines. However, we will first prove an equivalent of Step 2, and Step 1 will follow from the proof Step 2 with little effort. The quantity known as discrepancy comes into play when completing the equivalent of Step 3. The final part of the proof, Step 4, is given in Section 2.5.

2.3 The height, energy, and frequency on Carnot groups

Let $\mathbb{G}$ be a Carnot group of arbitrary step. If ρ denotes the H-gauge (Definition 1.1.5), we set for each $r > 0$

$$B_r = \{g \in \mathbb{G} : \rho(g) < r\}, \qquad S_r = \{g \in \mathbb{G} : \rho(g) = r\}.$$

B_r (respectively S_r) is called the gauge ball (respetively sphere) of radius r centered at the identity.

Fix $R > 0$ and let u be a function defined on B_R.

Definition 2.3.1 *(a) The* height integral *of u at e is given by*

$$H(u, r) = \int_{S_r} u^2 |\nabla_H \rho| \, d\sigma_H \tag{2.4}$$

where $r \in (0, R)$.

(b) The Dirichlet integral *of u at e is defined for $r \in (0, R)$ by*

$$D(u, r) = \int_{B_r} |\nabla_H u|^2 \, dg. \tag{2.5}$$

(c) The frequency *of u at e is the quotient*

$$N(u, r) = \frac{rD(u, r)}{H(u, r)} \tag{2.6}$$

for each $r \in (0, R)$ such that $H(u, r) \neq 0$.

See Chapter 1 for definitions of the relevant quantities such as ∇_H, dg, and σ_H. When the function u has been fixed or is otherwise understood, we write $H(r)$, $D(r)$, and $N(r)$.

Remark 2.3.1 *One may object to dividing by the height functional in* (2.6)*, as it is not guaranteed a priori that $H(r) \neq 0$ on $(0, R)$. We will, however, show in Corollary 2.5.1 that this is not a problem when $\Delta_H u = 0$.*

Remark 2.3.2 *One may also define the height integral, Dirichlet integral, and frequency at an arbitrary point g_0 if u is defined in an open neighborhood of g_0 by replacing the H-gauge ρ with the quantity $d(\cdot, g_0) = \rho(\cdot \circ g_0^{-1})$. However, by the left-translation invariance of the Haar and H-perimeter measures, these can all be related to height, Dirichlet, and frequency functionals at e. See [42] for details.*

Remark 2.3.3 *Let $u : B_R \to \mathbb{R}$. Both the Haar measure and H-perimeter measure of the gauge sphere scale well with respect to the dilations [27],*

$$d(\delta_\lambda(g)) = \lambda^Q \; dg$$
$$d\sigma_H(\delta_\lambda(g)) = \lambda^{Q-1} d\sigma_H(g).$$

These behaviors are somewhat expected if one compares these quantities to how the Lebesgue and Hausdorff measure of the ball and sphere dilate in $\mathbb{R}^n$. Combining these with the 0-homogeneity of $|\nabla_H \rho|$, it follows that

$$N(u \circ \delta_\lambda, r) = N(u, r\lambda), \qquad \lambda \in (0, \frac{R}{r}).$$

It is because of this observation that we hereafter assume that $R = 1$.

2.4 First derivatives of height and energy; discrepancy

2.4.1 The derivative of the height function

The following lemma generalizes Step 2 of the Euclidean proof.

Lemma 2.4.1 *Assume that u is harmonic in B_1. Then*

$$2D(r) = r^{Q-1}\frac{d}{dr}(r^{1-Q}H(r)). \tag{2.7}$$

Equivalently, performing the differentiation,

$$H'(r) = \frac{Q-1}{r}H(r) + 2D(r). \tag{2.8}$$

In order to prove Lemma 2.4.1, we need the following representation formula for a function v involving the Green function having pole at $e \in \mathbb{G}$ (see [23]).

Lemma 2.4.2 *Let $r > 0$, $g \in \mathbb{G}$, and $v \in C^\infty(\mathbb{G})$. Then*

$$v(g) = \int_{S_r} v\frac{|\nabla_H\Gamma(\cdot,g)|^2}{|\nabla\Gamma(\cdot,g)|}\, dH^{N-1} - \int_{B_r} \Delta_H v[\Gamma(g',g) - r^{2-Q}]\, dg', \tag{2.9}$$

where $\Gamma(g,g')$ is the fundamental solution of $-\Delta_H$ with pole at g'.

Proof of Lemma 2.4.1 Take $g = e$ in (2.9). Recalling that $\rho^{2-Q} = \Phi$ and $\Phi(g) = \Gamma(g,e)$, we compute that

$$\frac{\nabla_H\Phi}{\Phi} = \rho^{Q-2}\nabla_H(\rho^{2-Q}) = \rho^{Q-2}(2-Q)\rho^{1-Q}\nabla_H\rho = \frac{2-Q}{\rho}\nabla_H\rho.$$

and hence

$$\frac{|\nabla_H\Phi|}{\Phi} = (Q-2)\frac{|\nabla_H\rho|}{\rho}.$$

Notice that we have used the fact that $Q \geq 2$. A similar formula holds for the length of the Riemannian gradients. Consequently,

$$\begin{aligned}
\frac{|\nabla_H\Phi|^2}{|\nabla\Phi|} &= \frac{|\nabla_H\Phi|^2/\Phi^2}{|\nabla\Phi|/\Phi}\Phi \\
&= \frac{(Q-2)^2|\nabla_H\rho|^2/\rho^2}{(Q-2)|\nabla\rho|/\rho}\rho^{2-Q} \\
&= \frac{(Q-2)|\nabla_H\rho|^2}{|\nabla\rho|}\rho^{1-Q}
\end{aligned}$$

Inserting this into the representation formula (2.9),

$$\begin{aligned}
v(e) &= \int_{S_r} v\frac{(Q-2)|\nabla_H\rho|^2}{|\nabla\rho|}\rho^{1-Q}\, dH^{N-1} - \int_{B_r}\Delta_H v[\rho^{2-Q} - r^{2-Q}]\, dg \\
&= (Q-2)r^{1-Q}\int_{S_r} v|\nabla_H\rho|\, d\sigma_H - \int_{B_r}\Delta_H v[\rho^{2-Q} - r^{2-Q}]\, dg,
\end{aligned}$$

where we have used the definition of the H-perimeter measure and the fact that $\rho = r$ on S_r. We now choose $v = u^2$ to arrive at

$$\begin{aligned} u^2(e) &= (Q-2)r^{1-Q}\int_{S_r} u^2|\nabla_H\rho|\ d\sigma_H - \int_{B_r}\Delta_H(u^2)[\rho^{2-Q} - r^{2-Q}]\ dg \\ &= (Q-2)r^{1-Q}H(r) - 2\int_{B_r}|\nabla_H u|^2[\rho^{2-Q} - r^{2-Q}]\ dg, \end{aligned} \tag{2.10}$$

where now we have used the definition of the height functional, the formula

$$\Delta_H(u^2) = 2u\Delta_H u + 2|\nabla_H u|^2,$$

and the fact that u is harmonic. We now differentiate each side of (2.10) with respect to r:

$$\begin{aligned} (Q-2)\frac{d}{dr}(r^{1-Q}H(r)) &= 2\frac{d}{dr}(\int_{B_r}|\nabla_H u|^2[\rho^{2-Q} - r^{2-Q}]\ dg) \\ &= 2\int_{B_r}|\nabla_H u|^2\frac{d}{dr}(-r^{2-Q})\ dg \\ &= 2(Q-2)r^{1-Q}\int_{B_r}|\nabla_H u|^2\ dg \\ &= 2(Q-2)r^{1-Q}D(r). \end{aligned}$$

Note that the surface term vanishes when taking the derivative since the integrand is zero when $\rho = r$. ■

2.4.2 The derivative of the Dirichlet function

We next wish to establish an analogue of Step 3 for Carnot groups. However, in doing so we need a replacement for the Rellich identity. The proof of the following for an arbitrary domain is from [44].

Proposition 2.4.1 *Assume that u and ζ are a smooth function and vector field defined in a neighborhood of B_r. Then one has*

$$2\int_{S_r}\zeta u\langle\nabla_H u, N_H\rangle dH^{N-1}+\int_{B_r}\operatorname{div}\zeta|\nabla_H u|^2\,dg$$
$$=2\sum_{i=1}^{n}\int_{B_r}X_i u[X_i,\zeta]u\,dg+2\int_{B_r}\zeta u\Delta_H u\,dg+\int_{S_r}|\nabla_H u|^2\langle\zeta,\nu\rangle\,dH^{N-1} \tag{2.11}$$

In (2.11), div is the Riemannian gradient and ν the Riemannian unit normal. N_H was defined previously in (1.2).

It should be noted that Proposition 2.4.1 holds with less stringent regularity requirements: ζ only need be C^1, B_r and $S_r=\partial B_r$ may be replaced with an arbitrary bounded open C^1 domain Ω and $\partial\Omega$, and u can be taken to belong to the Folland-Stein class $\Gamma^2(\overline{\Omega})$ (see [36]). However, the statement given is sufficient for our purposes by hypoellipticity.

We wish to take $\zeta=\mathcal{Z}$. In doing so, we need to use the following properties for $\mathcal{Z}$, which may be found in [26]:

Lemma 2.4.3 *Let $\mathcal{Z}$ be the generator of the non-isotropic dilations $(\delta_\lambda)_{\lambda>0}$.*

(i) $\operatorname{div}\mathcal{Z}=Q$

(ii) For $i=1,\ldots,n$, $[X_i,\mathcal{Z}]=X_i$.

Combining Proposition 2.4.1 and Lemma 2.4.3, one arrives at:

Lemma 2.4.4 *Assume that u is harmonic on the unit gauge ball. Then*

$$D'(r)=2\int_{S_r}\frac{\mathcal{Z}u}{r}\langle\nabla_H u,\nabla_H\rho\rangle\frac{dH^{N-1}}{|\nabla\rho|}+\frac{Q-2}{r}D(r). \tag{2.12}$$

Proof According to the co-area formula,

$$D(r)=\int_0^r\Big(\int_{\rho=t}|\nabla_H u|^2\frac{dH^{N-1}}{|\nabla\rho|}\Big)\,dt,$$

whence

$$D'(r) = \int_{S_r} |\nabla_H u|^2 \frac{dH^{N-1}}{|\nabla \rho|}. \tag{2.13}$$

We now take $\zeta = \mathcal{Z}$ in Proposition 2.4.1, using Lemma 2.4.3 and the fact that $\nu = \frac{\nabla \rho}{|\nabla \rho|}$:

$$2 \int_{S_r} \mathcal{Z}u \langle \nabla_H u, N_H \rangle dH^{N-1} + (Q-2) \int_{B_r} |\nabla_H u|^2 \; dg = \int_{S_r} |\nabla_H u|^2 \mathcal{Z}\rho \; \frac{dH^{N-1}}{|\nabla \rho|}. \tag{2.14}$$

As ρ is homogeneous of dimension 1, $\mathcal{Z}\rho = \rho = r$ on S_r. Combining this fact with the definition of $D(r)$, (2.13), and (2.14) show that

$$2 \int_{S_r} \mathcal{Z}u \langle \nabla_H u, N_H \rangle dH^{N-1} + (Q-2) D(r) = r D'(r).$$

Finally, we use the definition of the horizontal normal (using again that $\nu = \frac{\nabla \rho}{|\nabla \rho|}$) to conclude the proof.

■

2.4.3 Discrepancy at the group identity

When observing the derivative of the Dirichlet functional in $\mathbb{R}^n$ (2.3), the ideal situation to be in would be if

$$\begin{aligned} D'(r) &= \frac{Q-2}{r} D(r) + 2 \int_{S_r} (\frac{\mathcal{Z}u}{r})^2 |\nabla_H \rho| \; d\sigma_H \\ &= \frac{Q-2}{r} D(r) + 2 \int_{S_r} (\frac{\mathcal{Z}u}{r})^2 |\nabla_H \rho|^2 \; \frac{dH^{N-1}}{|\nabla \rho|}. \end{aligned} \tag{2.15}$$

Introducing and removing the surface integral in (2.15) from our formula (2.12) gives

$$D'(r) = \frac{Q-2}{r} D(r) + 2 \int_{S_r} (\frac{\mathcal{Z}u}{r})^2 |\nabla_H \rho|^2 \; \frac{dH^{N-1}}{|\nabla \rho|} \tag{2.16}$$

$$+ 2 \int_{S_r} \frac{\mathcal{Z}u}{r} \left(\langle \nabla_H u, \nabla_H \rho \rangle - \frac{\mathcal{Z}u}{r} |\nabla_H \rho|^2 \right) \frac{dH^{N-1}}{|\nabla \rho|}. \tag{2.17}$$

Comparing this ideal situation with (2.12), we introduce the following quantity:

Definition 2.4.1 *Let* $u : B_1 \to \mathbb{R}$*. The* discrepancy of u *(at the identity* e*) is given by*

$$E_u := \langle \nabla_H \rho, \nabla_H u \rangle - \frac{\mathcal{Z}u}{r} |\nabla_H \rho|^2.$$

With Definition 2.4.1 in mind, Lemma 2.12 reads as follows:

$$D'(r) = \frac{Q-2}{r} D(r) + 2 \int_{S_r} (\frac{\mathcal{Z}u}{r})^2 |\nabla_H \rho| \; d\sigma_H + 2 \int_{S_r} \frac{\mathcal{Z}u}{r} E_u \frac{dH^{N-1}}{|\nabla \rho|} \tag{2.18}$$

2.5 Properties of the frequency function

We are now in a position to study closer the frequency of a harmonic function u, in particular its well-definedness and monotonicity properties in regard to discrepancy. Our first goal is to show that it is safe to divide by the height function when $u \not\equiv 0$.

Proposition 2.5.1 *Assume that* $\Delta_H u = 0$ *in* B_R*. Then*

$$rD(r) = \int_{S_r} u \mathcal{Z} u |\nabla_H \rho| \; d\sigma_H. \tag{2.19}$$

Proof Ironically, our starting point is to differentiate the height function. As ρ is homogeneous of degree 1, $|\nabla_H \rho|^2$ is homogeneous of degree 0. On the other hand, recalling how the H-perimeter measure of the gauge sphere dilates (see Remark 2.3.3), if we write $g = \delta_r(\hat{g})$ where $\hat{g} = \delta_{1/r}(g) \in S_1$, we see

$$\begin{aligned} H(r) &= \int_{S_r} u^2(g) |\nabla_H \rho|(g) \; d\sigma_H(g) \\ &= r^{Q-1} \int_{S_1} u^2(\delta_r(\hat{g})) |\nabla_H \rho|(\hat{g}) \; d\sigma_H(\hat{g}). \end{aligned}$$

Hence if we differentiate and use the fact that $\mathcal{Z}$ is the generator of the dilations $(\delta_\lambda)_{\lambda>0}$, specifically (1.4),

$$\begin{aligned} \frac{d}{dr}(r^{1-Q} H(r)) &= \int_{S_1} 2u(\delta_r(\hat{g})) \frac{d}{dr}(u(\delta_r(\hat{g}))) |\nabla_H \rho|(\hat{g}) \; d\sigma_H(\hat{g}) \\ &= \frac{2}{r} \int_{S_1} u(\delta_r(\hat{g})) (\mathcal{Z}u) \circ \delta_r(\hat{g}) |\nabla_H \rho|(\hat{g}) \; d\sigma_H(\hat{g}) \\ &= 2r^{-Q} \int_{S_r} u(g) (\mathcal{Z}u)(g) |\nabla_H \rho|(g) \; d\sigma_H(g). \end{aligned} \tag{2.20}$$

But we know from (2.7) that

$$\frac{d}{dr}(r^{1-Q}H(r)) = 2r^{1-Q}D(r). \tag{2.21}$$

From (2.20) and (2.21) follows the proposition.

■

Proposition 2.5.1 implies the following.

Corollary 2.5.1 *Assume that $\Delta_H u = 0$ in B_R. If $H(r_0) = 0$, then $u \equiv 0$ on B_{r_0}.*

Proof If $H(r_0) = 0$ for some $r_0 \in (0, R)$, then $u \equiv 0$ on S_{r_0}. Then $D(r_0) = 0$ by Proposition 2.5.1. But then by (2.5), $\nabla_H u \equiv 0$ on B_{r_0}. As $\mathfrak{g}$ is generated by V_1, this means that u must be constant in B_{r_0}. Finally, since $u = 0$ on S_{r_0}, it follows that $u \equiv 0$ in B_{r_0}. ■

We hereafter assume that $u \not\equiv 0$ on B_r for any $r \in (0, R)$ in which case N is well-defined on $(0, R)$. A consequence of Proposition 2.5.1 is the following.

Corollary 2.5.2 *Assume that u is harmonic and homogeneous of degree κ with respect to $(\delta_\lambda)_{\lambda>0}$. Then $N(r) \equiv \kappa$.*

Proof If u is homogeneous of degree κ, then $\mathcal{Z}u = \kappa u$. This, (2.19), and the definition of $H(r)$, $N(r)$ give the statement. ■

Theorem 2.5.1 *Let $u : B_1 \to \mathbb{R}$ be harmonic on a Carnot group of arbitrary step. Assume in addition that u has vanishing discrepancy. Then $r \mapsto N(r)$ is monotonically non-decreasing in the unit interval.*

Proof The logarithmic derivative of u is given by

$$\frac{N'(r)}{N(r)} = \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)}.$$

We use Lemmas 2.4.1 and 2.4.4 (specifically (2.18) which relates $D'(r)$ to the discrepancy of u) to find

$$\begin{aligned}\frac{N'(r)}{N(r)} &= \frac{1}{r} + \frac{Q-2}{r} + \frac{2}{D(r)}\int_{S_r}(\frac{\mathcal{Z}u}{r})^2|\nabla_H\rho|^2\,\frac{dH^{N-1}}{|\nabla\rho|} \\ &\quad + \frac{2}{D(r)}\int_{S_r}\frac{\mathcal{Z}u}{r}E_u\frac{dH^{N-1}}{|\nabla\rho|} - \frac{Q-1}{r} - \frac{2D(r)}{H(r)} \\ &= \frac{2}{r^2D(r)H(r)}\Big[H(r)\int_{S_r}(\mathcal{Z}u)^2|\nabla_H\rho|^2\,\frac{dH^{N-1}}{|\nabla\rho|} \\ &\qquad + rH(r)\int_{S_r}\mathcal{Z}uE_u\frac{dH^{N-1}}{|\nabla\rho|} - r^2D(r)^2\Big]. \qquad (2.22)\end{aligned}$$

By Cauchy-Schwarz and Proposition 2.5.1,

$$\begin{aligned}(rD(r))^2 &= (\int_{S_r}u\mathcal{Z}u|\nabla_H\rho|\;d\sigma_H)^2 \\ &\le (\int_{S_r}u^2|\nabla_H\rho|\;d\sigma_H)(\int_{S_r}(\mathcal{Z}u)^2|\nabla_H\rho|\;d\sigma_H) \\ &= H(r)\int_{S_r}(\mathcal{Z}u)^2|\nabla_H\rho|\;d\sigma_H \qquad (2.23)\end{aligned}$$

Inserting (2.23) into (2.22),

$$\frac{N'(r)}{N(r)} \ge \frac{2}{rD(r)}\int_{S_r}\mathcal{Z}uE_u\frac{dH^{N-1}}{|\nabla\rho|}.$$

Thus if $E_u \equiv 0$, it follows that $N'(r) \ge 0$.

■

2.6 Analysis of discrepancy

In view of the original result of Almgren and Theorem 2.5.1, one might expect that every function has vanishing discrepancy in the case where $\mathbb{G} = \mathbb{R}^n$, $n \ge 3$. Although Theorem 2.5.1 gives only a sufficient condition for Almgren's monotonicity, it is in this case necessary as well, as demonstrated by the following.

Proposition 2.6.1 *Let $\mathbb{G} = \mathbb{R}^n$ with the usual group structure. If $u : B_1 \to \mathbb{R}$, then u has vanishing discrepancy at e.*

Proof Recall that (see e.g. [33]), up to a multiplicative constant, the H-gauge $\rho(x) = |x|$. The proof is then a simple computation. ■

Theorem 2.5.1 is not worth much if the class of functions having vanishing discrepancy is empty. Thus the next proposition shows that there exist in general functions having vanishing discrepancy.

Proposition 2.6.2 *The function $g \mapsto \rho(g)$ has vanishing discrepancy at e. In particular, any function depending only on the H-gauge distance of g to the identity also has vanishing discrepancy at e.*

Proof Due to the fact that Φ is homogeneous of degree $2-Q$ with respect to the dilations $(\delta_\lambda)_{\lambda>0}$, ρ is homogeneous of degree 1. This means that $\mathcal{Z}\rho = \rho$, in which case

$$E_\rho = \langle \nabla_H \rho, \nabla_H \rho \rangle - \frac{\mathcal{Z}\rho}{\rho} |\nabla_H \rho|^2 \equiv 0.$$

The rest follows from the chain rule. ■

2.6.1 Discrepancy in H-type groups

In general, it is impossible to have an explicit representation of discrepancy without knowing the fundamental solution. As mentioned in Theorem 1.1.1, such formula was provided by Folland for $\mathbb{H}^n$ and Kaplan for the case of H-type groups – a constant multiple of the $(2-Q)$th power of the Korányi gauge. As such, up to rescaling the domain of the frequency, we may assume that the gauge being used is the Korányi gauge, that is,

$$\rho(x, z) = (|x|^4 + 16|z|^2)^{1/4}$$

in exponential coordinates.

Let $\mathbb{G}$ be a group of H-type (see Section 1.1.3 for the definition and notation). Lemma 2.6.1 gives a representation of the discrepancy of a function at e.

Lemma 2.6.1 *If $u : B_1 \to \mathbb{R}$, then*

$$E_u = \frac{4}{\rho^3}\langle J(z)x, \nabla_x u\rangle \tag{2.24}$$

$$= \frac{4}{\rho^3}\sum_{\ell=1}^{m} z_\ell \Theta_\ell u. \tag{2.25}$$

In particular, u has vanishing discrepancy if and only if

$$\langle J(z)x, \nabla_x u\rangle = \sum_{\ell=1}^{m} z_\ell \Theta_\ell u \equiv 0.$$

Proof We recall Lemma 1.1.2, which states that for any u defined in the unit gauge ball

$$\langle \nabla_H u, \nabla_H \rho\rangle = \frac{|x|^2}{\rho^3}\mathcal{Z}u + \frac{1}{2}\langle J(\nabla_z \rho)x, \nabla_x u\rangle$$

$$|\nabla_H \rho|^2 = \frac{|x|^2}{\rho^2}.$$

Combining these equations yields

$$\langle \nabla_H u, \nabla_H \rho\rangle = \frac{\mathcal{Z}u}{\rho}|\nabla_H \rho|^2 + \frac{1}{2}\langle J(\nabla_z \rho)x, \nabla_x u\rangle.$$

Hence by definition of discrepancy,

$$E_u := \langle \nabla_H u, \nabla_H \rho\rangle - \frac{\mathcal{Z}u}{\rho}|\nabla_H \rho|^2 = \frac{1}{2}\langle J(\nabla_z \rho)x, \nabla_x u\rangle.$$

In view of (1.18), we arrive at (2.24). To get (2.25), we write $z = \sum_{\ell=1}^{m} z_\ell \varepsilon_\ell$, $\nabla_x u = \sum_{k=1}^{n} (\partial_{x_k} u) e_k$ and use the definition of Θ_ℓ given in the proof of Lemma 1.1.1.

∎

A consequence of Lemma 2.6.1 and the discussion at the end of Section 1.1.3 produces the following corollary. This shows that our generalization of Almgren's frequency also properly generalizes that of Garofalo and Lanconelli from [39].

Corollary 2.6.1 *Let $\mathbb{G} = \mathbb{H}^n$. A function u has vanishing discrepancy at e if and only if $\Theta u = 0$, where Θ is given in* (2.2).

Lemma 2.6.1 also gives us a way to produce many more examples of functions with vanishing discrepancy in H-type groups.

Definition 2.6.1 *Let $\mathbb{G}$ be of H-type and $u : B_1 \to \mathbb{R}$.*

(a) If there exists $v : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}$ such that $u(x, z) = v(|x|, z)$ in exponential coordinates, then we say that u has cylindrical symmetry.

(b) Assume that $\mathbb{G} = \mathbb{H}^n$ for some n and set $r_j = |x_j + iy_j|$, $j = 1, \ldots, n$. If $u(x, y, z) = v(r_1, \ldots, r_n, z)$ for some function v, then u is said to be polyradial.

(c) Fix $U, V \in V_1$, the first layer of the Lie algebra of $\mathbb{G}$. Define

$$f_{UV}(x, z) = \langle J(z)x, U\rangle\langle x, V\rangle - \langle J(z)x, V\rangle\langle x, U\rangle.$$

If u is in the envelope of functions generated by the functions f_{UV}, then u is said to be a generalized polyradial function.

Note that cylindrical functions are always polyradial in the case of $\mathbb{G} = \mathbb{H}^n$.

Proposition 2.6.3 *Suppose that u is any of the types of functions in Definition 2.6.1. Then u has vanishing discrepancy at e.*

Proof First, suppose that u has cylindrical symmetry and write $r = |x|$. Then

$$\nabla_x u = \frac{1}{r}\frac{\partial v}{\partial r}x,$$

hence from (2.24)

$$\langle J(z)x, \nabla_x u\rangle = \frac{1}{r}\frac{\partial v}{\partial r}\langle J(z)x, x\rangle = 0.$$

For polyradial functions, it is simpler to use Corollary 2.6.1 than (2.24). As Θ and the r_j are independent of $z \in \mathbb{R}$, it suffices to show that $\Theta r_j = 0$ for $j = 1, \ldots, n$. But

$$\begin{aligned}
\Theta r_j &= \sum_{i=1}^{n}(x_i\frac{\partial r_j}{\partial y_i} - y_i\frac{\partial r_j}{\partial x_i}) \\
&= x_j\frac{y_j}{r_j} - y_j\frac{x_j}{r_j} \\
&= 0.
\end{aligned}$$

Finally, we prove that generalized polyradial functions have vanishing discrepancy by showing that each f_{UV} has vanishing discrepancy. By the product rule,

$$\nabla_x f_{UV} = -\langle x, V\rangle J(z)U + \langle J(z)x, U\rangle V - \langle J(z)x, V\rangle U + \langle x, U\rangle J(z)V.$$

hence

$$\begin{aligned}\langle J(z)x, \nabla_x f_{UV}\rangle &= -\langle x, V\rangle\langle J(z)x, J(z)U\rangle + \langle J(z)x, U\rangle\langle J(z)x, V\rangle \\ &\quad - \langle J(z)x, V\rangle\langle J(z)x, U\rangle + \langle x, U\rangle\langle J(z)x, J(z)V\rangle \\ &= -\langle x, V\rangle\langle J(z)x, J(z)U\rangle + \langle x, U\rangle\langle J(z)x, J(z)V\rangle.\end{aligned}$$

Now, for any $W, W' \in V_1$ we have

$$\begin{aligned}\langle J(z)W, J(z)W'\rangle &= |z|^2\langle W, W'\rangle \\ \langle J(z)W, W'\rangle &= -\langle W, J(z)W'\rangle.\end{aligned}$$

Thus

$$\begin{aligned}\langle J(z)x, \nabla_x f_{UV}\rangle &= -|z|^2\langle x, V\rangle\langle x, U\rangle + |z|^2\langle x, U\rangle\langle x, V\rangle \\ &= 0.\end{aligned}$$

We now appeal to the second half of Lemma 2.6.1. ■

2.7 Strong unique continuation property for Carnot groups

Having studied the frequency and its derivative, we can now focus on the question of unique continuation on Carnot groups. In $\mathbb{R}^n$, the arguments we are about to follow were first introduced by Garofalo and Lin in [41]. Let us start with the relevant definition.

Definition 2.7.1 *Let u be harmonic in B_1. We say that u vanishes to infinite order at e if for each k,*

$$\int_{B_r} u^2 \, dg = O(r^k) \text{ as } r \to 0^+.$$

The proof of Garofalo and Lin first uses the frequency function to prove that the height function satisfies a doubling condition, that is, $H(2r) \leq CH(r)$, where $C > 0$ is a constant independent of r (but depending on u). Then, via the coarea formula, this doubling can be transferred to the solid integrals of u^2 over balls. Finally, an inductive argument is used to show that if u vanishes to infinite order, its L^2-norm over B_1 – hence u itself – must be identically zero.

First, let us prove that the height function satisfies a doubling condition when u is harmonic and has vanishing discrepancy.

Lemma 2.7.1 *Assume that u is harmonic on B_1 and fix $0 < r_0 < 1$. Assume also that u has vanishing discrepancy. There exists a constant $C_1 = C_1(r_0, u, Q)$ such that for any $0 < r \leq \frac{1}{2}r_0$, we have*

$$H(2r) \leq 2^{2N(r_0)+Q-1} H(r). \tag{2.26}$$

Proof Let us begin by rewriting (2.7) as

$$\frac{d}{dr}(\ln r^{1-Q} H(r)) = \frac{2D(r)}{H(r)} = \frac{2N(r)}{r}.$$

If we integrate between r and $2r$, we find that

$$\begin{aligned}
\ln \frac{2^{1-Q} H(2r)}{H(r)} &= 2 \int_r^{2r} \frac{N(t)}{t}\, dt \\
&\leq 2N(r_0) \int_r^{2r} \frac{1}{t}\, dt \\
&= 2N(r_0) \ln 2.
\end{aligned}$$

In the middle line, we have used the non-decreasingness of the frequency to majorize $N(t)$ by $N(2r)$, then by $N(r_0)$. By rearranging this inequality, we find that we can take $C_1 = 2^{2N(r_0)+Q-1}$. ■

At this point, we would like to transfer the doubling property from the height function to the integrals of u^2 over the solid balls. However, the weighting factor

$\frac{|\nabla_H\rho|^2}{|\nabla\rho|}$ inside of the height integrals poses an obstacle. Rather, integrating (2.26) from 0 to r, we obtain by the coarea formula

$$\int_0^r H(2t)\ dt = \int_0^r \int_{\partial B_{2t}} u^2|\nabla_H\rho|^2 \frac{dH^{N-1}}{|\nabla\rho|}$$
$$= \frac{1}{2}\int_{B_{2r}} u^2|\nabla_H\rho|^2\ dg$$
$$\int_0^r H(t)\ dt = \int_{B_r} u^2|\nabla_H\rho|^2\ dg,$$

and hence

$$\int_{B_{2r}} u^2|\nabla_H\rho|^2\ dg \le 2^{2N(r_0)+Q}\int_{B_r} u^2|\nabla_H\rho|^2\ dg,$$

which is not quite what we want. It is of course possible to bound

$$\sup_{B_r}|\nabla_H\rho|^2 = \sup_{B_1}|\nabla_H\rho|^2 =: C_2$$

since $|\nabla_H\rho|$ is homogeneous of degree 0 with respect to $(\delta_\lambda)_{\lambda>0}$ and the unit ball is compact, from this we obtain

$$\int_{B_{2r}} u^2|\nabla_H\rho|^2\ dg \le C_3\int_{B_r} u^2\ dg \tag{2.27}$$

for some constant C_3. However, removing $|\nabla_H\rho|^2$ inside the left-hand side of this inequality takes some work. The way to deal with this is to introduce the following mean value operators.

Definition 2.7.2 *Fix a continuous function u on $\mathbb{G}$. Given $r > 0$ and $g_0 \in \mathbb{G}$, we define $M_r u(g_0)$ by*

$$M_r u(g_0) = \frac{Q-2}{Q} r^{-Q}\int_{B_r} u(g)|\nabla_H\rho(g\circ g_0^{-1})|^2\ dg.$$

We should emphasize the importance of the dependence of $M_r u(g_0)$ on the point g_0 as this will be crucial in removing the factor of $|\nabla_H\rho|^2$ from the left-hand side of (2.27).

If u is a harmonic function, we recall by the representation formula (2.9) (and an appropriate repetition of the first part of the proof of Lemma 2.4.1)

$$u(g) = (Q-2)r^{1-Q}\int_{S_r} v\frac{|\nabla_H\rho(h^{-1}\circ g)|^2}{|\nabla\rho(h^{-1}\circ g)|}\ dH^{N-1}(h).$$

Hence integrating from 0 to r and using the coarea formula,

$$u(g) = \frac{Q-2}{Q}\int_{B_r} v|\nabla_H \rho(h^{-1}\circ g)|^2\, dh = M_r u(g). \tag{2.28}$$

This equality is valid for any harmonic u and $r > 0$.

We need to use the following lemma:

Lemma 2.7.2 *Let $v \geq 0$ be a continuous function on $\mathbb{G}$. There exist positive constants $\tilde{C}, \lambda, \Lambda$ depending only on the group $\mathbb{G}$ such that for every $r > 0$ one can find $g_0 \in \mathbb{G}$ such that the following hold:*

(a) $g_0 \in S_{\lambda r}$

(b) $M_r v(g) \leq \tilde{C} M_{\Lambda r} v(g_0)$ for all $g \in B_r$

(c) $M_r v(g_0) \leq \tilde{C} M_{\Lambda r} v(g)$ for all $g \in B_r$.

The proof of Lemma 2.7.2 can be found in [15], specifically Lemma 3.1. The important part is that the constants λ, Λ and C are all independent of u, r. The power of Lemma 2.7.2 is that it allows us to compare the averages over balls of a function v to averages over nearby balls of comparable radii.

We now prove the following.

Lemma 2.7.3 *Assume that u is harmonic on B_1. Then there exists a constant $C_4 > 0$ depending only on $\mathbb{G}$ such that for every $r > 0$ one has*

$$C_4^{-1}\int_{B_r} u^2\, dg \leq \int_{B_r} u^2 |\nabla_H \rho|^2\, dg \leq C_4 \int_{B_r} u^2\, dg.$$

Proof As previously mentioned, the second inequality follows from the compactness of the unit ball and 0-homogeneity of $|\nabla_H \rho|^2$, that is,

$$\int_{B_r} u^2 |\nabla_H \rho|^2\, dg \leq \max_{B_1} |\nabla_H \rho|^2 \int_{B_r} u^2\, dg$$

Conversely,

$$\int_{B_r} u^2\, dg \leq |B_r| \sup_{B_r} u^2 = r^Q |B_1| \sup_{B_r} u^2,$$

where we have denoted by $|B_r|$ the Haar measure of the gauge-ball of radius r. Choose $g_1 \in \overline{B_r}$ such that

$$u(g_1)^2 = \sup_{B_r} u^2.$$

For every $\alpha > 0$ (to be fixed presently) we have from (2.28)

$$|u(g_1)| = |M_{\alpha r} u(g_1)| \leq M_{\alpha r}|u|(g_1),$$

the second inequality following from the defintion of the mean value operators as an integral.

We now apply Lemma 2.7.2 to get positive constants $\tilde{C}, \Lambda, \lambda$ and a $g_0 \in B_{\alpha r}$. Combining parts (b) and (c) of that Lemma, we find

$$|u(g_1)| \leq M_{\alpha r}|u|(g_1) \leq \tilde{C} M_{\Lambda \alpha r}|u|(g_0) \leq \tilde{C}^2 M_{\Lambda^2 \alpha r}|u|(e), \quad g_1 \in B_{\alpha r}.$$

As we have control over $\alpha > 0$, we choose $\alpha = \Lambda^{-2}$. Then,

$$\begin{aligned}
\int_{B_r} u^2 \, dg &\leq |B_1| r^Q \sup_{B_r} u^2 \\
&= |B_1| r^Q u(g_1)^2 \\
&\leq |B_1| r^Q \tilde{C}^4 (M_r|u|(e))^2 \\
&= |B_1| r^Q \tilde{C}^4 (r^{-Q} \frac{Q-2}{Q} \int_{B_r} |u(g)||\nabla_H \rho(g)|^2)^2 \, dg)^2 \\
&\leq |B_1| \tilde{C}^4 (\frac{Q-2}{Q})^2 r^{-Q} (\int_{B_r} |\nabla_H \rho|^2 \, dg)(\int_{B_r} u(g)^2 |\nabla_H \rho(g)|^2 \, dg) \\
&= |B_1|^2 \tilde{C}^4 (\frac{Q-2}{Q})^2 \max_{B_1} |\nabla_H \rho|^2 \int_{B_r} u(g)^2 |\nabla_H \rho(g)|^2 \, dg.
\end{aligned}$$

We thus take $C_4 = \max_{B_1} |\nabla_H \rho|^2 \max\{1, |B_1|^2 \tilde{C}^4 (\frac{Q-2}{Q})^2\}$ to conclude.

■

We can now prove the strong unique continuation property for Carnot groups.

Theorem 2.7.1 *Let u be harmonic on B_1 with vanishing discrepancy. If u vanishes to infinite order at $e \in \mathbb{G}$, then $u \equiv 0$ on all of B_1.*

Proof Fix $r_0 \in (0,1)$. Combining Lemmas 2.7.1 with 2.7.3 we infer the existence of a constant $C_5 > 0$ depending on u, r_0, and $\mathbb{G}$ such that for every $r \in (0, r_0]$

$$\int_{B_r} u^2 \, dg \le C_5 \int_{B_{2^{-1}r}} u^2 \, dg.$$

Beginning with $r = r_0$, we have by induction that for each $j \in \mathbb{N}$ and any arbitrary $\gamma > 0$,

$$\begin{aligned}
\int_{B_{r_0}} u^2 \, dg &\le C_5^j \int_{B_{2^{-j}r_0}} u^2 \, dg \\
&= C_5^j |B_{2^{-j}r_0}|^{\gamma} |B_{2^{-j}r_0}|^{-\gamma} \int_{B_{2^{-j}r_0}} u^2 \, dg \\
&= C_5^j |B_1|^{\gamma} (2^{-j} r_0)^{Q\gamma} |B_{2^{-j}r_0}|^{-\gamma} \int_{B_{2^{-j}r_0}} u^2 \, dg \\
&= (2^{-Q\gamma} C_5)^j |B_{r_0}|^{\gamma} |B_{2^{-j}r_0}|^{-\gamma} \int_{B_{2^{-j}r_0}} u^2 \, dg.
\end{aligned}$$

At this point, we choose $\gamma = \frac{\ln C_5}{Q \ln 2}$ so that

$$\int_{B_{r_0}} u^2 \, dg \le |B_{r_0}|^{\gamma} |B_{2^{-j}r_0}|^{-\gamma} \int_{B_{2^{-j}r_0}} u^2 \, dg \tag{2.29}$$

for each $j \in \mathbb{N}$. In particular, since u vanishes to infinite order at e, the right-hand side of (2.29) converges to 0 as $j \to \infty$. Consequently, $u \equiv 0$ in B_{r_0}. Finally, since $r_0 \in (0,1)$ was arbitrary, we conclude that $u \equiv 0$ in B_1.

■

Remark 2.7.1 *We should mention that the proof of Lemma 2.7.1 truly only depends on the* local boundedness *of the frequency of u. Hence, it is possible to prove a version of Theorem 2.7.1 where one removes the assumption of vanishing discrepancy and replaces it with the assumption that the frequency of u belongs to $L^\infty_{loc}(0,1)$. More generally, if we assume the following growth condition on the discrepancy of u:*

$$|E_u| \le \frac{f(\rho)}{\rho} |\nabla_H \rho|^2 |u| \qquad \text{in } B_1.$$

where f is a positive Dini-integrable function defined on $(0,1)$, then the frequency is locally bounded. This aspect is explored more fully in Section 7 of [42].

2.8 One parameter Weiss-type monotonicity formulas for Carnot groups

Let u be harmonic in the unit gauge ball of a Carnot group $\mathbb{G}$. Fix $\kappa > 0$, and set

$$\begin{aligned}\mathcal{W}_\kappa(r) &= r^{2-2\kappa-Q}D(r) - \kappa r^{1-Q-2\kappa}H(r)\\ &= \frac{r^{1-2\kappa-Q}}{H(r)}(N(r)-\kappa), \quad 0<r<1.\end{aligned}$$

Theorem 2.8.1 *Let u be harmonic on the unit gauge ball. If in addition u has vanishing discrepancy, then $r \mapsto \mathcal{W}_\kappa(r)$ is monotonically non-decreasing.*

Proof By computation,

$$\begin{aligned}\frac{d}{dr}\mathcal{W}_\kappa &= \frac{2-Q-2\kappa}{r}r^{2-2\kappa-Q}D(r) + r^{2-2\kappa-Q}D'(r)\\ &\quad - \frac{\kappa(1-Q-2\kappa)}{r}r^{1-Q-2\kappa}H(r) - \kappa r^{1-Q-2\kappa}H'(r)\\ &= r^{2-2\kappa-Q}\Big[D'(r) - \frac{Q-2}{r}D(r) - \frac{2\kappa}{r}D(r)\\ &\quad + \frac{2\kappa^2}{r^2}H(r) + \frac{\kappa(Q-1)}{r^2}H(r) - \frac{\kappa}{r}H'(r)\Big].\end{aligned}$$

Inserting (2.8) and (2.18),

$$\begin{aligned}\frac{d}{dr}\mathcal{W}_\kappa = r^{2-2\kappa-Q}\Big[&2\int_{S_r}(\frac{\mathcal{Z}u}{r})^2|\nabla_H\rho|\,d\sigma_H + 2\int_{S_r}\frac{\mathcal{Z}u}{r}E_u\frac{dH^{N-1}}{|\nabla\rho|} - \frac{2\kappa}{r}D(r)\\ &+ \frac{2\kappa^2}{r^2}H(r) - \frac{2\kappa}{r}D(r)\Big].\end{aligned}$$

Recalling the definition of $H(r) = \int_{S_r} u^2|\nabla_H\rho|\,d\sigma_H$, and (2.19),

$$\begin{aligned}\frac{d}{dr}\mathcal{W}_\kappa &= r^{2-2\kappa-Q}\Big[2\int_{S_r}(\frac{\mathcal{Z}u}{r})^2|\nabla_H\rho|\,d\sigma_H + 2\int_{S_r}\frac{\mathcal{Z}u}{r}E_u\frac{dH^{N-1}}{|\nabla\rho|}\\ &\quad + \frac{2\kappa^2}{r^2}\int_{S_r}u^2|\nabla_H\rho|\,d\sigma_H - \frac{4\kappa}{r^2}\int_{S_r}u\mathcal{Z}u|\nabla_H\rho|\,d\sigma_H\Big]\\ &= 2r^{-2\kappa-Q}\Big[\int_{S_r}(\mathcal{Z}u)^2|\nabla_H\rho|\,d\sigma_H + r\int_{S_r}\mathcal{Z}uE_u\frac{dH^{N-1}}{|\nabla\rho|}\\ &\quad + \kappa^2\int_{S_r}u^2|\nabla_H\rho|\,d\sigma_H - 2\kappa\int_{S_r}u\mathcal{Z}u|\nabla_H\rho|\,d\sigma_H\Big]\\ &= 2r^{-2\kappa-Q}\Big[\int_{S_r}(\mathcal{Z}u-\kappa u)^2|\nabla_H\rho|\,d\sigma_H + r\int_{S_r}\mathcal{Z}uE_u\frac{dH^{N-1}}{|\nabla\rho|}\Big].\end{aligned}$$

$E_u \equiv 0$, $\mathcal{W}_\kappa'(r)$ is the integral of a perfect square, hence $\mathcal{W}_\kappa$ is non-decreasing. ■

Corollary 2.8.1 *Assume that u is harmonic with vanishing discrepancy. Then $\mathcal{W}_\kappa$ is constant if and only if u is homogeneous of degree κ with respect to the dilations $(\delta_\lambda)_{\lambda>0}$.*

Proof $\mathcal{W}_\kappa$ is constant if and only if $\mathcal{Z}u = \kappa u$ on S_r for every $r \in (0,1)$. ■

3. Struwe- and Poon-type functionals for symmetric diffusion operators

3.1 Introduction

In [72], M. Struwe studied a weighted energy for solutions to $\Delta u = u_t$ in $\mathbb{R}^n \times (0, T)$, given by

$$e_x(t) = (T-t) \int_{\mathbb{R}^n} |\nabla u|^2(y,t) p_{T-t}(x,y)\ dy$$

where u solves $\Delta u = u_t$, $x \in \mathbb{R}^n$ is fixed, and $p_t(x,y)$ is the heat kernel on $\mathbb{R}^n$ with pole at $(y, 0) \in \mathbb{R}^n \times (0, \infty)$. Provided that the gradient of u is unformly bounded in both space and time variables, Struwe proved that his weighted energy is monotonically decreasing in t, and then used this result to prove partial regularity of weak solutions. During the course of the computation of the derivative of Struwe's energy, one finds that

$$\frac{1}{2} e'(t) = -(T-t) \int_{\mathbb{M}} p_{T-t} [\Delta u + \langle \nabla \ln p_{T-t}, \nabla u \rangle]^2\ dy + G(t), \tag{3.1}$$

where $G(t)$ is defined by

$$G(t) = -(T-t) \int_{\mathbb{R}^n} [\frac{1}{2(T-t)} g_{ij} + (\ln p_{T-t})_{ij}] p_{T-t} u_i u_j\ dy. \tag{3.2}$$

and $g = (g_{ij})$ is the standard Euclidean metric on $\mathbb{R}^n$. The crucial aspect of the proof is that the Euclidean heat kernel satisfies the following differential equation:

$$(\ln p_t)_{x_i x_j} + \frac{1}{2t} \delta_{ij} = 0 \tag{3.3}$$

so that $G(t) \equiv 0$ and the derivative of Struwe's energy is the negative of the integral of a perfect square.

When tracing (3.3), one finds $\Delta p_t + \frac{n}{2t} = 0$, which is the optimal case of the Li-Yau inequality found in [62] when the Ricci curvature of the manifold is non-negative.

Thus one may begin to expect that, given curvature assumptions, a differential matrix inequality similar to (3.3) exists. This fact was discovered and proven by R. Hamilton in [47]. Specifically, he proved that, for compact connected Riemannian manifolds which are Ricci parallel and which possess weakly-positive sectional curvatures, the heat kernel for the Laplace-Beltrami operator satisfies

$$(\ln p_t)_{ij} + \frac{1}{2t} g_{ij} \geq 0.$$

Hamilton then immediately used this matrix inequality to extend Struwe's monotonicity result to Riemannian manifolds with those same curvature assumptions in [48], replacing the Euclidean aspects of Struwe's result with their appropriate Riemannian ones.

A few years later, C. Poon worked on a parabolic version of Almgren's frequency function in [67] for $\mathbb{R}^n$. Using computations performed in [72] and the Cauchy-Schwarz inequality, Poon proved that the function

$$n(t) = (T-t)\frac{\int_{\mathbb{R}^n} |\nabla u|^2(y,t) p_{T-t}(x,y)\ dy}{\int_{\mathbb{R}^n} u^2(y,t) p_{T-t}(x,y)\ dy}$$

is monotonically non-increasing whenever u solves the heat equation in $\mathbb{R}^n \times (0,T)$ and is uniformly bounded and possesses uniformly bounded derivatives. Again, (3.3) played a crucial role, but Poon remarked that such computations would still work in the manifolds considered by Hamilton.

Our present work is to use the semigroup and carré du champ $\Gamma(\cdot,\cdot)$ (see Definition 1.2.1) to extend the definition of Struwe's energy and Poon's frequency to smooth connected manifolds $\mathbb{M}$ with elliptic diffusion operator $\mathcal{L}$ and a measure μ with respect to which $\mathcal{L}$ is symmetric over $C_0^\infty(\mathbb{M})$. We are able to prove the following in Section 3.4.3.

Theorem 3.1.1 *Fix $T > 0$ and $x \in \mathbb{M}$. Suppose that $(\mathbb{M}, \mathcal{L})$ is complete and satisfies the curvature-dimension inequality $CD(\rho, n)$ for some $\rho \in \mathbb{R}$. Suppose that $u \in \mathcal{D}(\mathcal{L})$ solves the heat equation $\mathcal{L}u = u_t$ on $\mathbb{M} \times (0,T)$ and set*

$$e(t) = (T-t)P_{T-t}\Gamma(u).$$

Then e is monotonically non-increasing on (T_m, T), where $T_m = \max\{T + \frac{1}{2\rho}, 0\}$ if $\rho < 0$ and $T_m = 0$ if $\rho \geq 0$. In particular, if $\rho \geq 0$, then e is non-increasing on all of $(0, T)$.

For the definition of $CD(\rho, n)$ and completeness, we refer the reader to Definitions 1.2.2 and 1.2.3.

For general diffusion operators satisfying $CD(\rho, n)$, although there is a notion of Hessian, there is no known analogue of Hamilton's matrix Harnack inequality. For Poon's frequency monotonicity, we do not currently know of a way to get around this, and we suspect that Poon's frequency may not be decreasing in this case. Instead, we introduce a new Hessian condition $C(\omega)$ which generalizes Hamilton's matrix Harnack inequality to smooth connected manifolds with diffusion operator $\mathcal{L}$. This condition, given in Definition 3.2.2, is satisfied in many cases, including some cases where the triple $(\mathbb{M}, \mathcal{L}, \mu)$ satisfies the curvature dimension inequality $CD(\rho, n)$ such as the Ornstein-Uhlenbeck operator. In Definition 3.5.1, we give modifications of the Struwe and Poon functions which, in Section 3.5.2, are shown to retain their monotonicity properties when u solves the heat equation, and, in case $\mathbb{M} = \mathbb{R}^n$, $\mathcal{L} = \Delta$, and μ is the Lebesgue measure, reduce to the Euclidean originals. Specifically, we prove that

$$\mathfrak{e}(t) = e^{\omega(T-t)} P_{T-t}\Gamma(u)$$
$$\mathfrak{n}(t) = e^{\omega(T-t)} \frac{P_{T-t}\Gamma(u)}{P_{T-t}(u^2)}$$

are monotonically non-increasing under $C(\omega)$ and $CD(\rho, n)$. See Theorems 3.5.1 and 3.5.2 for specifics.

3.2 Preliminaries

Let $\mathbb{M}$ be a connected manifold with elliptic diffusion operator $\mathcal{L}$ and measure μ with respect to which $\mathcal{L}$ is symmetric in $C_0^\infty(\mathbb{M})$ with the L^2-norm. We will denote this by the triple $(\mathbb{M}, \mathcal{L}, \mu)$. We may study the associated heat equation for $T \in (0, \infty]$,

$$\mathcal{L}u = u_t, \qquad \text{in } \mathbb{M} \times (0, T). \tag{3.4}$$

Throughout, we will assume that the manifold $\mathbb{M}$ is *complete* and satisfies the curvature dimension inequality. See Definitions 1.2.3 and 1.2.2 in Chapter 1 for the definitions and discussions of these properties.

We begin with a definition from [5].

Definition 3.2.1 *Given u, v, w smooth, the* Hessian *of the function u is given by the formula*

$$H_u(v,w) = \frac{1}{2}\left[\Gamma(v,\Gamma(u,w)) + \Gamma(w,\Gamma(u,v)) - \Gamma(u,\Gamma(v,w)\right]. \tag{3.5}$$

Above, $\Gamma(\cdot,\cdot)$ is the carré du champ of Definition 1.2.1.

The reason for the name Hessian is clear if one looks at the Riemannian and Euclidean cases, for

$$H_u(v,w) = \nabla^2 u(\nabla v, \nabla w) \qquad \text{Riemannian manifolds}$$

$$H_u(v,w) = \langle \nabla^2 u \nabla v, \nabla w\rangle \qquad \text{Euclidean space}$$

where $\nabla^2 u$ is the Hessian tensor of u (or Hessian matrix $(\frac{\partial^2 u}{\partial x_j \partial x_i})_{1\le i,j,\le n}$ in $\mathbb{R}^n$) and ∇v, is the Riemannian (Euclidean) gradient of v (likewise for w), and $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^n$. As with the carré du champ, we will write $H_u(v,v) = H_u(v)$ for short.

Let us now introduce the following new Hessian condition.

Definition 3.2.2 *Let $\mathbb{M}$ be a smooth manifold with diffusion operator $\mathcal{L}$ and symmetric measure $d\mu$. Let $\mathcal{S}$ be a subset of smooth functions defined on $\mathbb{M}$, $x_0 \in \mathbb{M}$, and $\omega(t) : (0,\infty) \to \mathbb{R}$ be a C^1 function. We will say that $\mathbb{M}$ satisfies the condition $C(\omega, x_0)$ for the set $\mathcal{S}$ if for every $u \in \mathcal{S}$, the inequality*

$$H_{\ln p_t(x_0,\cdot)}(u,u) + \frac{1}{2}\omega'(t)\Gamma(u) \ge 0, \tag{3.6}$$

holds on all of $\mathbb{M}$. In terms of the carré du champ, (3.6) *reads*

$$\Gamma(u,\Gamma(u,\ln p_t)) - \frac{1}{2}\Gamma(\ln p_t, \Gamma(u)) + \frac{1}{2}\omega'(t)\Gamma(u) \ge 0.$$

If (3.6) *holds for each $x_0 \in \mathbb{M}$ and $\mathcal{S} = C^\infty(\mathbb{M})$, then we shall simply say that $\mathbb{M}$ satisfies $C(\omega)$.*

In Section 3.3, we give examples of manifolds and operators which satisfy $C(\omega, x_0)$.

Remark 3.2.1 *In $\mathbb{R}^n$, (3.6) is in fact equality when $\omega(t) = \ln t$. We will give more examples of manifolds satisfying $C(\omega)$ below in Section 3.3.*

Remark 3.2.2 *Assume $\mathbb{M}$ is a Lie group and $\mathcal{L}$ is left-invariant with respect to left-translation – that is, $\mathcal{L}(u \circ L_g) = (\mathcal{L}u) \circ L_g$ for each $g \in \mathbb{M}$. Due to the left-invariance of the heat kernel p_t, the condition $C(\omega, x_0)$ for some $x_0 \in \mathbb{M}$ implies that $C(\omega)$ holds.*

We are now ready to define the various functions of interest throughout this paper.

Definition 3.2.3 *(Height, Struwe energy, and Poon functions) Fix a solution to (3.4) and $(x_0, T) \in \mathbb{M} \times (0, \infty)$.*

(a) The height function $h : (0, T) \to \mathbb{R}$ *of u at x_0 by*

$$h_{x_0}(t) = h(t) = P_{T-t}[u^2](x_0),$$

provided that this quantity makes sense, for example, if $u \in L^p(\mathbb{M})$ for some $p \in [1, \infty]$.

(b) Struwe's energy function $e_{x_0} : (0, T) \to \mathbb{R}$ *of u is a weighted version of the Dirichlet energy and is given by the formula*

$$e_{x_0}(t) = e(t) = (T - t) P_{T-t}[\Gamma(u)](x_0),$$

provided that this quantity makes sense.

(c) We define Poon's frequency function $n_{x_0} : (0, T) \to \mathbb{R}$ *of u at $x_0 \in \mathbb{M}$ by the quotient*

$$n_{x_0}(t) = n(t) = \frac{e(t)}{h(t)}.$$

provided both h and e make sense and $h(t) \neq 0$ in $(0, T)$.

Finally, we define a weighting factor which will become useful when studying $C(\omega, x_0)$ manifolds.

Definition 3.2.4 *If $\mathbb{M}$ satisfies $C(\omega, x_0)$, define the weight $\eta_{x_0}(t) = \eta(t)$ by*

$$\eta(t) = \frac{e^{\omega(t)}}{t}.$$

In Section 3.5, we will work with the function η to prove monotonicity of the functions $\eta(T-t)e_{x_0}(t)$ and $\eta(T-t)n_{x_0}(t)$ whenever $\mathbb{M}$ satisfies $C(\omega, x_0)$. We will show below that in the case where $\mathbb{M} = \mathbb{R}^n$, $\mathcal{L} = \Delta$, and μ is the Lebesgue measure, $\eta(t) = 1$, thus this result encompasses original counterparts found in [72] and [67].

3.3 Examples of manifolds satisfying and $CD(\rho, n)$ and $C(\omega, x_0)$

The purpose of this section is to provide motivation for the definition of the Hessian condition $C(\omega, x_0)$. Our first example is the simplest of all.

Example 3.3.1 *($\mathbb{R}^n$ with Laplacian)* Suppose that $\mathbb{M} = \mathbb{R}^n$, μ the Lebesgue measure, and $\mathcal{L} = \Delta$. The heat kernel is well-known, and satisfies

$$(\ln p_t(x, y))_{ij} = -\frac{\delta_{ij}}{2t}.$$

Hence regardless of the function u and the point $x \in \mathbb{R}^n$, we find

$$\langle \nabla_y^2 \ln p_t(x, y) \nabla_y u, \nabla_y u \rangle = -\frac{1}{2t} |\nabla_y u|^2.$$

As mentioned above, we have

$$\Gamma(u, \Gamma(\ln p_t, u)) - \frac{1}{2}\Gamma(\ln p, \Gamma(u)) = \langle \nabla^2 \ln p_t(x, y) \nabla u, \nabla u \rangle$$

since the derivatives commute. This shows that $(\mathbb{R}^n, \Delta, dx)$ satisfies $C(t \mapsto \ln t)$. The generalized and modified Struwe and Poon functions e, n agree with their original counterparts, and the weighting factor is $\eta(t) = 1$.

Example 3.3.2 *(Hamilton's manifolds)* More generally, let $\mathbb{M}$ be a compact Riemannian manifold which is Ricci parallel and whose sectional curvatures are weakly positive. It remains true that

$$\Gamma(u, \Gamma(\ln p_t, u) - \frac{1}{2}\Gamma(\ln p_t, \Gamma(u)) = \nabla^2(\ln p_t)(\nabla u, \nabla u),$$

where $\nabla^2 p_t$ is the Hessian tensor of p_{T-t}. As previously mentioned, Hamilton proved in [47] that the components of the Hessian tensor satisfy

$$\nabla^2(\ln p_t)_{ij} + \frac{g_{ij}}{2t} \geq 0$$

in relation to the metric g. Thus as in the Euclidean case we may take $\omega(t) = \ln t$ in $C(\omega)$ for each $x_0 \in \mathbb{M}$. Also as before, the generalized and modified Struwe and Poon functions agree with their Riemannian counterparts in [67] and [47], and the weighting factor is $\eta(t) = 1$.

The following example is one that motivated and inspired the definition of the Hessian inequality $C(\omega, x_0)$. We will use it as a concrete model to demonstrate the theorems of this chapter.

Example 3.3.3 *(Ornstein-Uhlenbeck operator on $\mathbb{R}^n$)* Let again $\mathbb{M} = \mathbb{R}^n$. Fix $\rho \in \mathbb{R}$, and let $\mathcal{L} = \mathcal{L}_\rho$ be the Ornstein-Uhlenbeck operator,

$$\mathcal{L}_\rho u = \Delta u - \rho\langle x, \nabla u\rangle. \tag{3.7}$$

The derivatives in (3.7) are taken with respect to x, and, in contrast to (3.12) below, the inner product with ∇u involves the same variable as the differentiation. The appropriate measure for this operator is

$$d\mu(x) = e^{-\rho\frac{\|x\|^2}{2}}\,dx,$$

and the heat kernel is a modification of the Mehler kernel (see [5] for the case where $\rho = 1$),

$$p_t(x,y) = \left(\frac{\rho e^{\rho t}}{4\pi\sinh(\rho t)}\right)^{1/2} \exp\left(-\rho\frac{\|x\|^2 + \|y\|^2 - 2\langle x, y\rangle e^{\rho t}}{4e^{\rho t}\sinh(\rho t)}\right). \tag{3.8}$$

It is well-known that the triple $(\mathbb{R}^n, \mathcal{L}, d\mu)$ satisfy $CD(\rho, \infty)$. In fact, if $\rho \neq 0$, then $(\mathbb{R}^n, \mathcal{L}, d\mu)$ cannot satisfy $CD(\rho, n)$ for any finite n.

Let us also show that $(\mathbb{R}^n, \mathcal{L}, d\mu)$ also satisfies $C(\omega)$ for some function ω. One may easily compute

$$\Gamma(u, v) = \langle\nabla u, \nabla v\rangle$$

and hence

$$\Gamma(u, \Gamma(u, \ln p_t)) - \frac{1}{2}\Gamma(\ln p_t, \Gamma(u)) = \langle \nabla^2 \ln p_t \nabla u, \nabla u \rangle.$$

But from (3.8)

$$(\nabla^2 \ln p_t)_{ij} = -\rho \frac{\delta_{ij}}{2e^{\rho t}\sinh(\rho t)}, \tag{3.9}$$

so that

$$\Gamma(u, \Gamma(u, \ln p_t)) - \frac{1}{2}\Gamma(\ln p_t, \Gamma(u)) + \rho \frac{\Gamma(u)}{2e^{\rho t}\sinh(\rho t)} = 0.$$

Therefore, if we choose

$$\omega(t) = \ln\left(\frac{1 - e^{-2\rho t}}{2\rho}\right).$$

then the operator $\mathcal{L}$ satisfies $C(\omega)$. The corresponding weighting factor is given by

$$\eta(t) = \frac{1 - e^{-2\rho t}}{2\rho t}.$$

Note that, in the limit as $\rho \to 0$, we have

$$\lim_{\rho \to 0} \eta(t) = 1.$$

Thus the modified Struwe and Poon's functions reduce to their Euclidean counterparts in the limit. This is expected since the operator and measure $d\mu$ reduce to the Laplacian and the Lebesgue measure, respectively.

Remark 3.3.1 *In the previous example, the function*

$$f(x) = \frac{x}{e^x \sinh x}$$

is a decreasing function of x. Therefore, if $\rho' \leq \rho$, we have

$$\frac{-\rho}{2e^{\rho t}\sinh(\rho t)} \geq \frac{-\rho'}{e^{\rho' t}\sinh(\rho' t)}.$$

In particular, if $\rho \geq 0$, we may take

$$\omega(t) = \ln t.$$

The modified Struwe and Poon functions reduce to the original Struwe and Poon functions for the Ornstein-Uhlenbeck operator with parameter $\rho \geq 0$.

Example 3.3.4 *(Doob's h transform)*

Let $\mathbb{M}$ be a Riemannian manifold with Riemannian measure λ and Laplace-Beltrami operator Δ, and assume that $(\mathbb{M}, \Delta, d\lambda)$ satisfies $C(\omega, x_0)$ for some $x_0 \in \mathbb{M}$. Suppose that there exists a smooth function $h : \mathbb{M} \to (0, \infty)$ satisfying

$$\Delta h + \alpha h = 0$$

and define

$$\begin{aligned} \mathcal{L} &= \Delta - 2\langle \frac{\nabla h}{h}, \nabla \cdot \rangle, \\ d\mu &= h^2 \; d\lambda. \end{aligned} \tag{3.10}$$

If p_t is the heat kernel for $(\mathbb{M}, \Delta, \lambda)$, then the heat kernel for $\mathcal{L}$ is given by Doob's h-transform (see chapter 9 of [46]):

$$\tilde{p}_t(x, y) = e^{\alpha t} \frac{p_t(x, y)}{h(x)h(y)}. \tag{3.11}$$

Obviously, we have

$$\nabla^2 \ln \tilde{p}_t = \nabla^2 \ln p_t - \nabla^2 \ln h.$$

where the derivatives are taken with respect to the y variable. If

$$-\nabla^2 \ln h(y) \geq -\frac{1}{2}\beta g$$

in the sense of bilinear forms for some $\beta \in \mathbb{R}$, and if $(\mathbb{M}, \Delta, dx)$ satisfies (3.6), then $(\mathbb{M}, \mathcal{L}, d\mu)$ satisfies $C(\tilde{\omega}, x_0)$, with

$$\tilde{\omega}(t) = \omega(t) + \beta t.$$

Let us compare the two Struwe energies for the operators Δ and $\mathcal{L}$. We denote e the Struwe energy for $(\mathbb{M}, \Delta, d\lambda)$ and $\tilde{e}$ the Struwe energy for $(\mathbb{M}, \mathcal{L}, d\mu)$, both at x_0, and similarly for η and $\tilde{\eta}$. We have

$$\begin{aligned} e_{x_0}(t) &= (T-t) \int_{\mathbb{M}} |\nabla u|^2 p_{T-t}(x_0, \cdot) \; d\lambda \\ \tilde{e}_{x_0}(t) &= \frac{e^{\alpha(T-t)}}{h(x_0)} (T-t) \int_{\mathbb{M}} |\nabla u|^2 p_{T-t}(x_0, \cdot) h \; d\lambda, \end{aligned}$$

where in the second line we have used (3.10) and (3.11). Also,

$$\eta(t) = e^{\omega(t)}, \qquad \tilde{\eta}(t) = \eta(t)e^{\beta t}.$$

Let us give an example of such h. If $\mathbb{M} = \mathbb{R}^n$, we can fix a point $z \in \mathbb{R}^n$ and define $h(x) = e^{\langle x,z\rangle}$. Such function satisfies

$$\Delta h - |z|^2 h = 0$$

and

$$\nabla^2 \ln h = 0.$$

Thus the operator $\mathcal{L}$ given by

$$\mathcal{L}u = \Delta u - 2\langle z, \nabla u\rangle \tag{3.12}$$

together with the measure $d\mu(x) = e^{2\langle x,z\rangle}dx$ satisfies $C(t \mapsto \ln t)$. We would like to emphasize that – contrary to the Ornstein-Uhlenbeck operator – the derivatives are all taken with respect to the variable x; z is fixed and independent of x. The Struwe energy for a solution to $\mathcal{L}u = u_t$ is given by

$$e_{x_0}(t) = (T-t)e^{-\langle z,x_0\rangle - |z|^2(T-t)} \int_{\mathbb{R}^n} |\nabla u|^2 p_{T-t}(x_0, y)e^{\langle z,y\rangle}\ dy.$$

where p_t is the usual Euclidean heat kernel.

Example 3.3.5 *(The Bessel operator with parameter α)* Let $\mathbb{M} = (0, \infty)$ and let $\alpha \geq 0$. The Bessel operator with parameter α is given by

$$\mathcal{B}_\alpha = \frac{d^2}{dx^2} + \frac{\alpha}{x}\frac{d}{dx}, \qquad x > 0.$$

This operator is symmetric with respect to the measure $d\mu = y^\alpha\ dy$. If we set $n = \alpha + 1$, then the triple $(\mathbb{M}, \mathcal{B}_\alpha, d\mu)$ satisfies $CD(0, n)$. The heat kernel for this operator is explicitly known (see for example Exercise 3.8 of [7]), and is given by

$$p_t(x, y) = (2t)^{-n/2}\left(\frac{xy}{2t}\right)^{-\nu} I_\nu(\frac{xy}{2t}) \exp(-\frac{x^2 + y^2}{4t}), \tag{3.13}$$

where I_ν is the modified Bessel function of the first kind and of order ν and $\nu = \frac{n}{2} - 1$.

Let us show that $(\mathbb{M}, \mathcal{B}_\alpha, d\mu)$ also satisfies the Hessian condition $C(\omega)$. We begin by writing

$$\ln p_t(x,y) = f(x,t) + \ln(z^{-\nu} I_\nu(z)) - \frac{y^2}{4t}$$

where $f(x,t)$ is a function depending only on x and t and $z = \frac{xy}{2t}$. When differentiating,

$$\frac{\partial^2}{\partial y^2} \ln p_t(x,y) = -\frac{1}{2t} + \frac{x^2}{4t^2} \frac{d^2}{dz^2} \ln \left(z^{-\nu} I_\nu(z)\right)$$

We now use a result from [65]: the function $z \mapsto z^{-\nu} I_\nu(z)$ is logarithmically convex whenever $\nu > -\frac{1}{2}$, and when $\nu = -\frac{1}{2}$,

$$z^{-1/2} I_{-1/2}(z) = \sqrt{\frac{2}{\pi}} \frac{\cosh z}{z}$$

which is easily seen to be logarithmically convex. Thus

$$\frac{\partial^2}{\partial y^2} \ln p_t(x,y) \geq -\frac{1}{2t}, \qquad \nu \geq -\frac{1}{2}. \tag{3.14}$$

Therefore, as in the Euclidean case we may take $\omega(t) = \ln t$. The generalized Struwe energy is then

$$e_{x_0}(t) = (T-t) \int_0^\infty |\nabla u|^2(y,t) p_{T-t}(x_0, y) y^\alpha dy,$$

where p_t is given in (3.13).

Remark 3.3.2 *We should note that when $\alpha = n-1 \in \mathbb{N}_0$, the Bessel operator corresponds to the radial part of the Laplacian on $\mathbb{R}^n$, or equivalently as the generator of the stochastic process $(\|B_t\|)_{t\geq 0}$, where $(B_t)_{t\geq 0}$ is an n-dimensional Brownian motion. In such case, one may integrate out the angular dependence of the height function h and Struwe energy e_x using polar coordinates centered at x together with the identity*

$$\int_{\mathbb{S}^{n-1}} e^{\langle \zeta, \omega \rangle} \, d\sigma(\omega) = |\zeta|^{1-\frac{n}{2}} (2\pi)^{n/2} I_{\frac{n}{2}-1}(|\zeta|).$$

Thus in this case, the monotonicity of e *could be surmised directly from Struwe's original result in [72]. The novelty in this example is that* $C(t \mapsto \ln t)$ *even holds when* $\alpha \geq 0$ *is not an integer, i.e. when* B_α *does not correspond to the radial part of the Laplacian.*

Example 3.3.6 *(Radial functions on the hyperbolic model spaces* $\mathbb{H}^n$*)*

Let $\mathbb{H}^n$ be the model hyperbolic manifold, which as a manifold is given by $\mathbb{R}^n$ and endowed with the metric (written in polar coordinates $(r, \Omega) \in \mathbb{R}_+ \times \mathbb{S}^{n-1}$)

$$g = dr^2 + \sinh^2 r g_{\mathbb{S}^{n-1}}.$$

Here, $g_{\mathbb{S}^{n-1}}$ is the usual metric on the $(n-1)$-sphere. In this coordinate system, the Laplace-Beltrami operator Δ and Riemannian measure $d\lambda$ are given by (see [46])

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial r^2} + (n-1)\coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}} \\ d\lambda &= \sinh^{n-1} r \; dr d\Omega_{\mathbb{S}^{n-1}}, \end{aligned}$$

where $\Delta_{\mathbb{S}^{n-1}}$ and $d\Omega_{\mathbb{S}^{n-1}}$ are the Laplacian and Riemannian measure on $\mathbb{S}^{n-1}$.

Let $p_t^n(x, y)$ denote the heat kernel for $\mathbb{H}^n$ with pole at $x \in \mathbb{H}^n$. Formulas for $p_t^n(x, y)$ are known explicitly, and p_t^n is always a function of the geodesic distance r between the points x and y. Therefore we write $p_t^n(x, y) = p_t^n(r(x, y))$. Then the heat kernels satisfy the following recurrence relation for $n \geq 1$:

$$p_t^{n+2}(r) = -\frac{1}{2\pi} e^{-nt} \frac{1}{\sinh r} \frac{\partial}{\partial r} p_t^n(r) \tag{3.15}$$

where

$$p_t^1(r) = (4\pi t)^{-1/2} \exp(-\frac{r^2}{4t}) \tag{3.16}$$

$$p_t^2(r) = \frac{\sqrt{2}}{(4\pi t)^{3/2}} e^{-\frac{1}{4}t} \int_r^\infty \frac{s e^{-\frac{s^2}{4t}}}{\sqrt{\cosh s - \cosh r}} \, ds. \tag{3.17}$$

According to [28], (3.15) is due to Millson in an unpublished paper, while (3.17) can be found in [64]. From the recurrence relation, the next odd-dimensional heat kernel is given by

$$p_t^3(r) = (4\pi t)^{-3/2} \frac{r}{\sinh r} e^{-\frac{r^2}{4t} - t}. \tag{3.18}$$

Fix a point $x \in \mathbb{H}^n$ and suppose that $u : \mathbb{H}^n \to \mathbb{R}$ is a function of the radial distance from the point x, i.e. $u(y) = v(d(x, y))$ for some function $v : [0, \infty) \to \mathbb{R}$. Then, denoting $r = d(x, y)$, one can compute

$$H(\ln p_t)(u, u) = (\ln p_t)_{rr} u_r^2.$$

Since $\Gamma(u) = u_r^2$, in order get a lower bound for $H(p_t)(u, u)$ in terms of u_r^2 whenever u is radial at x, it suffices to find a lower bound on $(\ln p_t)_{rr}$.

If $n = 1$, then $\mathbb{H}^1$ is isometric to $\mathbb{R}$ with the usual metric, so nothing interesting happens, i.e. $\mathbb{H}^1$ satisfies $C(t \mapsto \ln t)$. For the $n = 3$, a little computation based on (3.18) cedes the equation

$$(\ln p_t^3)_{rr} = \operatorname{csch}^2 r - \frac{1}{r^2} - \frac{1}{2t},$$

whose minimum occurs at $r = 0$:

$$\min_{r \geq 0} (\ln p_t^3)_{rr} = -\frac{1}{3} - \frac{1}{2t}.$$

Thus we see that, over the set $\mathcal{S}$ of radial functions at x_0, $\mathbb{H}^3$ satisfies $C(\omega, x_0)$ over $\mathcal{S}$ with

$$\omega(t) = \ln t + \frac{2}{3} t.$$

Computations of higher odd-dimensional hyperbolic manifolds give other interesting lower bounds on the second logarithmic radial derivative of the heat kernel. We list the first few here:

$$\begin{aligned}
\min_{r \geq 0} (\ln p_t^5)_{rr} &= -\frac{1}{2t} - \frac{4}{5} + \frac{2}{5(2t+3)} \\
\min_{r \geq 0} (\ln p_t^7)_{rr} &= -\frac{1}{2t} - \frac{9}{7} + \frac{32t + 30}{7(16t^2 + 30t + 15)} \\
\min_{r \geq 0} (\ln p_t^9)_{rr} &= -\frac{1}{2t} - \frac{16}{9} + \frac{288t^2 + 392t + 140}{9(96t^3 + 196t^2 + 140t + 35)} \\
\min_{r \geq 0} (\ln p_t^{11})_{rr} &= -\frac{1}{2t} - \frac{25}{11} + \frac{12288t^3 + 19680t^2 + 10920t + 2100}{11(3072t^4 + 6560t^3 + 5460t^2 + 2100t + 315)}.
\end{aligned}$$

The pattern appears to be, for $n \geq 3$ odd,

$$\min_{r\geq 0}(\ln p_t^n)_{rr} = -\frac{1}{2t} - \frac{(n-1)^2}{4n} + \frac{P_n'(t)}{nP_n(t)} \tag{3.19}$$

where $P_n(t)$ is a polynomial of degree $\frac{n-3}{2}$. However, at this time the author does not know a general formula for $P_n(t)$, or even a recursive relationship between them. If (3.19) does hold in general, then the odd-dimensional hyperbolic spaces are found to satisfy $C(\omega, x_0)$ for radial functions at x_0, where

$$\omega(t) = \ln\left(\frac{t}{P_n(t)^{2/n}}\right) + \frac{(n-1)^2}{2n}t.$$

Remark 3.3.3 *If u is a radially symmetric function on $\mathbb{H}^n$, then after a long computation we find that*

$$\Gamma_2(u) = u_{rr}^2 + (n-1)\operatorname{csch}^2 r u_r^2 \geq 0.$$

Thus if we restrict our attention to radial functions, we have the curvature-dimension condition $CD(0,\infty)$ for $\mathbb{H}^n$.

3.4 Monotonicity results for manifolds satisfying $CD(\rho, n)$

3.4.1 Some heat kernel bounds

Before we begin the analysis of the derivatives of the height function h, it is necessary to establish a few heat kernel bounds. The purpose of this section is twofold: (1) For each fixed $x \in \mathbb{M}$, and $t > 0$, $y \mapsto p_t(x, y)$ is bounded, and (2) under the assumption $CD(\rho, n)$, for each fixed $x \in \mathbb{M}$ and $t > 0$, $y \mapsto \Gamma(p_t)(x, y)$ is bounded. The results of this section allow us to integrate by parts when we compute h', h'' in Section 3.4.2.

Because of completeness, for each $1 \leq p < \infty$, the heat semigroup is an $L^p(\mathbb{M}, \mu)$ contraction. Therefore, for each $t > 0$, P_t maps $L^1(\mathbb{M})$ to itself. Furthermore, since $u(x,t) = P_t f(x)$ solves the Cauchy problem in $\mathbb{M}$,

$$\begin{cases} \mathcal{L}u = u_t & \text{in } \mathbb{M} \times (0,\infty) \\ u(x,0) = f(x) \end{cases}$$

the function $x \mapsto P_t f(x)$ is continuous, hence $P_t f(x)$ always defined and finite when $f \in L^1(\mathbb{M})$.

Heat kernel bound in one variable

The following is an exercise from [46] in the Riemannian case. We give a proof for diffusion operators case for the sake of completeness.

Lemma 3.4.1 *Let $\mathbb{M}$ be complete. For every $x \in \mathbb{M}$ and for every $t > 0$, the function $y \mapsto p_t(x, y)$ is bounded.*

Proof For simplicity, set $f(y) = p_t(x, y)$ and suppose that f is not bounded on $\mathbb{M}$. Then for each $n \in \mathbb{N}$, there exists a $y_n \in \mathbb{M}$ such that $f(y) = n$. Since $f(y)$ is continuous, the set $\{y_n\}_{n \geq 1}$ must be discrete. Therefore, for each n, there exists an $r_n > 0$ such that

$$f(y) \geq n - \frac{1}{2} \text{ on } B(y_n, r_n), \qquad B(y_n, r_n) \cap B(y_m, r_m) = \emptyset, \qquad n \neq m.$$

We consider the function

$$g(z) = \sum_{n=1}^{\infty} \frac{1}{n^2 \mu(B(y_n, r_n))} \mathbb{1}_{B(y_n, r_n)}(z).$$

One may easily check that $\|g\|_{L^1(\mathbb{M})} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. But

$$\begin{aligned} P_t g(x) &= \sum_{n=1}^{\infty} \frac{1}{n^2 \mu(B(y_n, r_n))} \int_{B(y_n, r_n)} f(y) \, d\mu(y) \\ &\geq \sum_{n=1}^{\infty} \frac{1}{n^2 \mu(B(y_n, r_n))} \int_{B(y_n, r_n)} (n - \frac{1}{2}) \, d\mu(y) \\ &= \sum_{n=1}^{\infty} \frac{n - \frac{1}{2}}{n^2} \end{aligned}$$

which is a divergent series. This contradicts the continuity of the mapping $x \mapsto P_t g(x)$ since $g \in L^1(\mathbb{M})$. ■

Next, we use the heat kernel bound in one variable to get a uniform bound on $y \mapsto \Gamma(p_t)(x,y)$ for fixed $x \in \mathbb{M}$, $t > 0$.

In [54], B. Kim proved that, if $\mathbb{M}$ satisfies the generalized curvature dimension inequality $CD(-K, \rho_2, \kappa, d)$ introduced in [9], then any bounded positive solution u to the subelliptic heat equation $\mathcal{L}u = u_t$ on $\mathbb{M} \times (0, \infty)$ satisfies a modification of Hamilton's gradient estimate [47] on $\mathbb{M} \times (0, \infty)$:

$$t\Gamma(\ln u(x,t)) \leq (1 + \frac{2\kappa}{\rho_2} + 2Kt) \ln(\frac{\|u\|_\infty}{u(x,t)}). \tag{3.20}$$

In this present situation, we may take $\kappa = 0$, $\rho_2 = 1$, and $d = n$ if $\mathbb{M}$ satisfies $CD(0, n)$ (note that $n = \infty$ is valid since (3.20) does not depend on d). In fact, the choice of ρ_2 is arbitrary, since it appears in the generalized curvature dimension inequality only with the Γ^Z term, which we take to be zero in our case. Hence we have the following.

Lemma 3.4.2 *Fix $x \in \mathbb{M}$, $t > 0$ and suppose that $\mathbb{M}$ satisfies $CD(\rho, n)$ with $\rho \leq 0$. Then*

$$\Gamma_y(p_t(x,y)) \leq \frac{1}{t}(1 - 2\rho t)\frac{2M}{e^{3/2}} p_t(x,y).$$

In particular, $y \mapsto \Gamma_y(p_t)(x,y)$ is bounded on $\mathbb{M}$.

Proof For simplicity, let $p_t(x,y) = u(y,t)$. Let $M = \|u(\cdot,t)\|_{L^\infty(\mathbb{M})} < \infty$ by the previous lemma. u solves the heat equation, so by Hamilton's estimate (3.20),

$$\Gamma(u(y,t)) \leq \frac{1}{t}(1 - 2\rho t) \ln(\frac{M}{u(y,t)}) u^2(y,t)$$

Consider the function $f : [0, M] \to [0, \infty)$ given by

$$f(s) = \ln(\frac{M}{s}) s^2.$$

We have the bound

$$f(s) \leq \frac{2M}{e^{3/2}} s$$

for $s \in [0, M]$. Therefore

$$\Gamma(u(y,t)) \leq \frac{1}{t}(1 - 2\rho t)\frac{2M}{e^{3/2}} u(y,t) \tag{3.21}$$

in which case the boundedness of $y \mapsto \Gamma(p_t)(x,y)$ follows from that of $y \mapsto p_t(x,y)$.

■

3.4.2 Derivatives of the height function

We are now ready to compute the first and second derivatives of the height function h. The key ingredients are the exhaustion sequence $\{h_n\}_{n\geq 1}$ of Section 1.2.4 and the heat kernel estimates of the previous section.

Proposition 3.4.1 *Let $\mathbb{M}$ be complete and satisfy $CD(\rho, n)$. Suppose that $\mathcal{L}u = u_t$ and that $u \in \mathcal{D}(\mathcal{L})$ for each $t \in (0, T)$. Then the first and second derivatives of the height function h are given by*

$$h'(t) = -2P_{T-t}[\Gamma(u)](x) \tag{3.22}$$

$$h''(t) = 4P_{T-t}[\Gamma_2(u)](x). \tag{3.23}$$

In particular, h is a decreasing function. If $\rho \geq 0$, then h is also convex.

Proof We begin with a preliminary observation. Due to Proposition 3.3.17 in [5], the assumptions of completeness and $CD(\rho, n)$ together with $u \in \mathcal{D}(\mathcal{L})$ imply that u^2, $\Gamma(u)$, and $\Gamma_2(u)$ belong to $L^1(\mathbb{M})$ for each $t \in (0, T)$.

Let $\{h_n\}_{n=1}^{\infty} \subset C_0^{\infty}(\mathbb{M})$ be an exhaustion sequence. Then by differentiation,

$$\begin{aligned}
\partial_t P_{T-t}[u^2 h_n] &= P_{T-t}[2uh_n\mathcal{L}u - \mathcal{L}(u^2 h_n)] \\
&= P_{T-t}[2uh_n\mathcal{L}u - u^2\mathcal{L}(h_n) - h_n\mathcal{L}(u^2) - 2\Gamma(u^2, h_n)] \\
&= -2P_{T-t}[\Gamma(u)h_n] - P_{T-t}[u^2\mathcal{L}(h_n) + 2\Gamma(u^2, h_n)].
\end{aligned} \tag{3.24}$$

Focusing on the term $P_{T-t}[\Gamma(u^2, h_n)]$, the Cauchy-Schwarz inequality implies

$$\begin{aligned}
P_{T-t}[\Gamma(u^2, h_n)] &= \int_{\mathbb{M}} 2u\Gamma(u, h_n)p_{T-t}\, d\mu \\
&\leq 2\int_{\mathbb{M}} |u|\sqrt{\Gamma(u)}\sqrt{\Gamma(h_n)}p_{T-t}\, d\mu \\
&\leq 2\left\|\sqrt{\Gamma(h_n)}\right\|_{L^\infty(\mathbb{M})} \int_{\mathbb{M}} |u|\sqrt{\Gamma(u)}p_{T-t}\, d\mu \\
&\leq 2\left\|\sqrt{\Gamma(h_n)}\right\|_{L^\infty(\mathbb{M})} \sqrt{P_{T-t}[u^2]}\sqrt{P_{T-t}[\Gamma(u)]}
\end{aligned}$$

which converges to 0 as $n \to \infty$. On the other hand,

$$
\begin{aligned}
|\int_{\mathbb{M}} u^2 \mathcal{L}(h_n) p_{T-t}\, d\mu| &= |\int_{\mathbb{M}} \Gamma(h_n, p_{T-t}) u^2\, d\mu + \int_{\mathbb{M}} \Gamma(h_n, u^2) p_{T-t}\, d\mu| \\
&\le |\int_{\mathbb{M}} \Gamma(h_n, p_{T-t}) u^2\, d\mu| + |P_{T-t}[\Gamma(h_n, u^2)]| \\
&\le \|\Gamma(h_n)\|^{1/2}_{L^\infty(\mathbb{M})} \|\Gamma(p_{T-t}(x,\cdot))\|^{1/2}_{L^\infty(\mathbb{M})} \|u\|^2_{L^2(\mathbb{M})} \\
&\qquad + |P_{T-t}[\Gamma(h_n, u^2)]|
\end{aligned}
$$

which converges to zero as $n \to \infty$ by the first part of the proof and Lemma 3.4.2. This proves (3.22).

To prove (3.23), we again use the exhaustion sequence,

$$
\begin{aligned}
\partial_t P_{T-t}[u^2 h_n] &= P_{T-t}[h_n \partial_t \Gamma(u) - \mathcal{L}(\Gamma(u) h_n)] \\
&= P_{T-t}[2h_n \Gamma(u, \mathcal{L}u) - \mathcal{L}(\Gamma(u)) h_n - \Gamma(u)\mathcal{L}h_n - 2\Gamma(\Gamma(u), h_n)] \\
&= -2P_{T-t}[\Gamma_2(u) h_n] - P_{T-t}[2\Gamma(\Gamma(u), h_n) + \Gamma(u)\mathcal{L}h_n]. \qquad (3.25)
\end{aligned}
$$

where we have used the facts that $\partial_t \Gamma(u) = 2\Gamma(u, u_t)$ and that u solves $\mathcal{L}u = u_t$. Using the inequality

$$\Gamma(\Gamma(u)) \le 4\Gamma(u)(\Gamma_2(u) - \rho\Gamma(u))$$

when $\mathcal{L}$ satisfies $CD(\rho, n)$ (see, for example, section C.6 of [5]), the Cauchy-Schwarz inequality implies that

$$P_{T-t}[\sqrt{\Gamma(\Gamma(u)})] \le 2(P_{T-t}[\Gamma(u)] P_{T-t}[\Gamma_2(u) - \rho\Gamma(u)])^{1/2} < \infty.$$

Hence we get

$$
\begin{aligned}
|P_{T-t}[\Gamma(\Gamma(u), h_n)]| &= \int_{\mathbb{M}} \sqrt{\Gamma(\Gamma(u))} \sqrt{\Gamma(h_n)} p_{T-t}\, d\mu \\
&\le \left\| \sqrt{\Gamma(h_n)} \right\|_{L^\infty(\mathbb{M})} \int_{\mathbb{M}} \sqrt{\Gamma(\Gamma(u))} p_{T-t}\, d\mu
\end{aligned}
$$

which converges to 0 as $n \to \infty$. For the last term in (3.25), we integrate by parts to get

$$P_{T-t}[\Gamma(u)\mathcal{L}h_n] = -\int_{\mathbb{M}} \Gamma(h_n, \Gamma(u)) p_{T-t}\, d\mu - \int_{\mathbb{M}} \Gamma(h_n, p_{T-t}) \Gamma(u)\, d\mu$$

which can be treated in a similar fashion to the computation of h'. ■

Remark 3.4.1 *If $\mathbb{M}$ is a complete Riemannian manifold with Ricci curvature bounded from below, then Proposition 3.4.1 holds for functions of the type $u = P_t f$, where $f \in L^2(\mathbb{M})$.*

Remark 3.4.2 *A natural question to ask following Proposition 3.4.1 is whether h is logarithmically convex if $\mathbb{M}$ satisfies $CD(0,n)$ for some n. We posit that this is not the case, even in Euclidean space. As a counter-example, consider the function $f(x) = \sin x$ on $\mathbb{M} = \mathbb{R}$ and set $u(x,t) = P_t f(x) = e^{-t}\sin x$. If we fix $x_0 = 0$ and consider $h = h_0$, then one may compute*

$$h(t) = e^{-2T}\sinh(2(T-t)).$$

However, this function is not logarithmically convex. We find that

$$\frac{d^2}{dt^2}\ln h(t) = -4\operatorname{csch}^2(2(T-t)).$$

In fact, the height function in this case turns out to be logarithmically concave*!*

Corollary 3.4.1 *One may express the energy and frequency functions in terms of the height function and its derivatives. Namely, for $u \in \mathcal{D}(\mathcal{L})$,*

$$e(t) = -\frac{1}{2}(T-t)h'(t) \tag{3.26}$$
$$n(t) = -\frac{1}{2}(T-t)\frac{h'(t)}{h(t)}. \tag{3.27}$$

3.4.3 Small-time monotonicity for generalized Struwe function on manifolds satisfying $CD(\rho, n)$

We can now prove Theorem 3.1.1, that is, our generalization of Struwe's energy is monotonically decreasing in time.

Proof of Theorem 3.1.1 Differentiating (3.26) and using Proposition 3.4.1, we have

$$e'(t) = -P_{T-t}\Gamma(u) - 2(T-t)P_{T-t}\Gamma_2(u). \tag{3.28}$$

By the assumption $CD(\rho, n)$, $\Gamma_2(u) \geq \rho\Gamma(u)$ and hence, by the sub-Markov property of the semi-group, one has

$$e'(t) \leq -[1 + 2\rho(T - t)]P_{T-t}\Gamma(u). \tag{3.29}$$

Because $P_{T-t}\Gamma(u) \geq 0$, a sufficient condition for $e' \leq 0$ is

$$1 + 2\rho(T - t) \geq 0.$$

If $\rho \geq 0$ this is always true. Otherwise, it happens whenever $t \geq T + \frac{1}{2\rho}$. ■

Remark 3.4.3 *In the case of Riemannian manifolds, Theorem 3.1.1 removes the restriction of being Ricci parallel, and relaxes the curvature assumption from having weakly positive sectional curvatures to merely having non-negative Ricci curvature. We also relax compactness to completeness, but as compensation require more of the function u.*

Remark 3.4.4 *We would also like to remark that Struwe used the monotonicity of the function $E(r) = e(T - r^2)$. Monotonicity of $e(t)$ for values of t near T implies that of E near 0, and this is enough.*

3.4.4 Monotonicity for generalized Poon frequency on Riemannian manifolds

In [67], Poon proved that the frequency function n is monotonically decreasing in the space $\mathbb{R}^n$. After proving the theorem, he remarks that by following the same steps as in the Euclidean case, one may extend the theorem to Hamilton's manifolds (i.e. compact, Ricci parallel, weakly positive sectional curvatures). Here, we flesh out a few more details of his proof in this case using the height function.

Theorem 3.4.1 *Let $\mathbb{M}$ be a complete Riemannian manifold having weakly positive sectional curvatures and which is Ricci parallel. Then for each solution u to the heat equation, the frequency n is monotonically decreasing.*

Proof By differentiating Definition 3.2.3(c), the theorem will follow if

$$h(t)e'(t) \le h'(t)e(t). \tag{3.30}$$

Inserting the result of Proposition 3.4.1, (3.30) is equivalent to

$$h(t)e'(t) \le -\frac{1}{2}(T-t)(h'(t))^2. \tag{3.31}$$

But if we write Definition 3.2.3(a) as an integral involving the heat kernel and differentiate, we have

$$\begin{aligned} h'(t) &= \int_{\mathbb{M}} [-\Delta p_{T-t} u^2 + 2p_{T-t} u \Delta u] \, d\mu \\ &= \int_{\mathbb{M}} [2u\langle \nabla u, \nabla p_{T-t}\rangle + 2p_{T-t} u \Delta u] \, d\mu \\ &= 2\int_{\mathbb{M}} u p_{T-t} [\frac{\langle \nabla p_{T-t}, \nabla u\rangle}{p_{T-t}} + \Delta u] \, d\mu \\ &\le 2(\int_{\mathbb{M}} u^2 p_{T-t} \, d\mu)^{1/2} (\int_{\mathbb{M}} p_{T-t} [\frac{\langle \nabla p_{T-t}, \nabla u\rangle}{p_{T-t}} + \Delta u]^2 \, d\mu)^{1/2} \\ &= 2\sqrt{h(t)} \cdot \sqrt{\frac{2G(t) - e'(t)}{2(T-t)}}, \end{aligned} \tag{3.32}$$

where we have use the Cauchy-Schwarz inequality and (3.1) in the fourth and fifth lines, respectively. (3.2) and $p_{T-t} > 0$ imply that $G(t) \le 0$. This fact and (3.32) together imply (3.31). ■

We would like to state that, at the moment, we do not know whether the assumptions in Theorem 3.4.1 are optimal, i.e. whether or not if one may relax the assumptions to those of Theorem 3.1.1 or even to Riemannian manifolds with $\mathrm{Ric} \ge \rho$. We strongly suspect that on Riemannian manifolds, because the asymptotics of the heat kernel depends on the sectional curvatures of the manifold (see, for instance, [71]), that the assumptions are the best that one could hope for. However, furnishing a possible counter-example has proven to be difficult. Even in the model hyperbolic space $\mathbb{H}^3$, for which the heat kernel at the vertex is known explicitly, the values $(P_t u)(x)$ can only easily be computed at the vertex of the hyperboloid, whereas to compute the height function h one needs $P_t u$ on all of $\mathbb{H}^3$.

To finish this section, we would like to state a sufficient condition for monotonicity of the frequency.

Proposition 3.4.2 *If the height function h is logarithmically convex, then n is monotonically decreasing.*

Proof Combining equations (3.22), (3.29), and (3.31), we find that n is decreasing if and only if

$$(T-t)\left[(h'(t))^2 - h(t)h''(t)\right] \leq -h(t)h'(t).$$

But this is easily seen to be true because the left-hand side is negative by logarithmic convexity, and the right-hand side is positive because $h \geq 0$, $h' \leq 0$. ■

3.5 Monotonicity results for manifolds satisfying $C(\omega, x_0)$

As shown in the previous section, the fact that Poon's frequency is monotonically decreasing on manifolds of Hamilton-type is a direct result of Hamilton's matrix Harnack inequality. A replacement for this inequality is given by $C(\omega, x_0)$, and under this assumption, we show in this section that the Struwe and Poon functionals can be reweighted to retain their monotonicity properties.

Definition 3.5.1 *(Weighted Struwe energy and Poon frequency for $\mathcal{L}$). Assume the triple $(\mathbb{M}, \mathcal{L}, \mu)$ satisfies $C(\omega, x_0)$. Fix a solution to $\mathcal{L}u = u_t$ on $\mathbb{M} \times (0,T)$. Define the weighted Struwe energy and Poon frequency on $(0,T)$ by*

$$\mathfrak{e}_{x_0}(t) = \eta(t) e_{x_0}(t) = e^{\omega(T-t)} P_{T-t}[\Gamma(u)](x_0)$$
$$\mathfrak{n}_{x_0}(t) = \eta(t) n_{x_0}(t) = e^{\omega(T-t)} \frac{P_{T-t}[\Gamma(u)](x_0)}{h_{x_0}(t)},$$

provided all of the quantities make sense. Above, h_{x_0}, e_{x_0} and n_{x_0} are the usual functions given in Definition 3.2.3 and $\eta = \eta_{x_0}$ is from Definition 3.2.4,

3.5.1 An integration by parts identity

Before we begin the analysis of $\mathfrak{e}$ and $\mathfrak{n}$, we need to prove that we can still perform the required integration by parts. The first is a simple consequence of integrating by parts.

Lemma 3.5.1 *Fix $t > 0$ and $x \in \mathbb{R}^n$. If $u \in C_0^\infty(\mathbb{R}^n)$, then*

$$P_t[\Gamma_2(u)](x) = \int_{\mathbb{R}^n} p_t(x,\cdot)[\mathcal{L}u + \Gamma(u, \ln p_t(x,\cdot))]^2 \, d\mu \\ + \int_{\mathbb{R}^n} p_t(x,\cdot) H_{\ln p_t}(u,u)(x,\cdot) \, d\mu. \tag{3.33}$$

In (3.33)*, all derivatives are taken with respect to the variable of integration.*

The goal of this section is to extend (3.33) to functions $u \in \mathcal{D}(\mathcal{L})$. The main idea of the proof is to show that each of the integrals in (3.33) are continuous under the topology of $\mathcal{D}(\mathcal{L})$, then appeal to the fact that $C_0^\infty(\mathbb{R}^n)$ is a core for $\mathcal{D}(\mathcal{L})$ since the operator is essentially self-adjoint. Lemma 3.5.2 is the first step in this direction.

Lemma 3.5.2 *Let $t > 0$ and $x_0 \in \mathbb{M}$ be fixed. Assume that $\mathcal{L}$ is a complete diffusion operator on $\mathbb{M}$ with measure μ, which satisfies $CD(\rho, \infty)$.*

(a) The bilinear operator Γ_2 extends to a continuous bilinear operator on $\mathcal{D}(\mathcal{L}) \times \mathcal{D}(\mathcal{L})$ which satisfies

$$\int_{\mathbb{M}} \Gamma_2(u) \, d\mu = \int_{\mathbb{M}} (\mathcal{L}u)^2 \, d\mu$$

for each $u \in \mathcal{D}(\mathcal{L})$.

(b) Consider the linear mapping $T = T_{x_0,t} : \mathcal{D}(\mathcal{E}) \to L^2(\mathbb{M})$ given by

$$Tu = \sqrt{p_t(x_0,\cdot)}\Gamma(u, \ln p_t(x_0,\cdot)),$$

where the derivatives are taken with respect to the variable y. Then T is continuous.

(c) Consider the symmetric bilinear mapping $B = B_{x_0,t} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to L^1(\mathbb{M})$*,*

$$B(u,v) = \frac{1}{2}(H_{p_t(x_0,\cdot)}(u,v) + H_{p_t(x_0,\cdot)}(v,u))$$

If either $y \mapsto \Gamma_2(p_t)(x_0,y)$ *or* $y \mapsto p_t(y,y)$ *is bounded, then* B *is continuous.*

Remark 3.5.1 *One of the two conditions in part (c) are met in many different cases. For example, the heat kernel for the Ornstein-Uhlenbeck operator satisfies* $y \mapsto \Gamma_2(p_t)(x,y) \in L^\infty(\mathbb{R}^n)$ *(but if* $\rho > 0$*,* p_t *is not bounded on the diagonal). On the other hand,* p_t *is bounded on its diagonal in many examples, including Euclidean space (more generally, on Carnot groups), and under various assumptions, e.g. ultracontractivity or the Faber-Krahn inequalitiy with* $\Lambda(v) = cv^{-2/n}$ *(see [46] Corollary 15.17).*

Proof of Lemma 3.5.2 Part (a) is Proposition 3.3.16 in [5].

For (b), we have

$$\begin{aligned}
\|Tu\|^2_{L^2(\mathbb{M})} &= \int_{\mathbb{M}} p_t \Gamma(u, \ln p_t)^2 \, d\mu \\
&= \int_M p_t \Gamma(u) \Gamma(\ln p_t) \, d\mu \\
&\le \|p_t \Gamma(\ln p_t)\|_{L^\infty(\mathbb{M})} \int_{\mathbb{M}} \Gamma(u) \, d\mu \\
&\le \|p_t \Gamma(\ln p_t)\|_{L^\infty(\mathbb{M})} \|u\|^2_{\mathcal{D}(\mathcal{E})} .
\end{aligned}$$

Since we assume $CD(\rho, \infty)$, we have the Hamilton bound (3.20),

$$\Gamma(\ln p_t) \le \frac{1}{t}(1 - 2\rho t) \ln\left(\frac{\|p_t\|_{L^\infty(\mathbb{M})}}{p_t}\right),$$

whereas the function $f : [0, M] \to [0, \infty)$ given by $f(s) = s \ln(\frac{M}{s})$ achieves its maximum value of $\frac{M}{e}$ at $s = \frac{M}{e}$. In particular,

$$p_t \Gamma(\ln p_t) \le \frac{1}{t}(1 - 2\rho t) \frac{\|p_t\|_{L^\infty(\mathbb{M})}}{e} = C(x_0, t) < \infty,$$

where $C(x_0, t)$ is a constant depending on x_0, t

To prove (c), we can polarize and show continuity on the diagonal since B is symmetric and bilinear. For this, we have the following bound (equation (3.3.7) in [5])

$$H_{p_t}(u,u)^2 \leq [\Gamma_2(p_t) - \rho\Gamma(p_t)]\Gamma(u)^2.$$

Thus

$$\|B(u,u)\|_{L^1(\mathbb{M})} \leq \left\| \sqrt{\Gamma_2(p_t) - \rho\Gamma(p_t)} \right\|_{L^\infty(\mathbb{M})} \mathcal{E}(u).$$

This proves (c) in the case where $y \mapsto \Gamma_2(p_t)(x_0, y)$ is bounded on $\mathbb{M}$.

On the other hand, assume that p_t is bounded on the diagonal. By the Chapman-Kolmogorov equation (1.30),

$$B_{x_0,2t}(u,u) = \int_{\mathbb{M}} \int_{\mathbb{M}} p_t(x_0, z) H^y_{p_t(z,y)}(u,u)\ d\mu(z)\ d\mu(y), \tag{3.34}$$

where $H^y_{p_t(z,y)}$ means the derivatives are taken with respect to the variable y. Assuming the curvature-dimension equality, we have (see equation (3.3.7) of the book by Bakry, Gentil, and Ledoux)

$$H^y_{p_t(z,y)}(u,u) \leq \sqrt{\Gamma^y_2(p_t(z,y)) - \rho\Gamma^y(p_t(z,y))}\Gamma^y(u),$$

where Γ^y and Γ^y_2 denote derivatives with respect to y. Combining this with (3.34) and the Cauchy-Schwarz inequality,

$$\begin{aligned} |B_{x_0,2t}(u,u)| \leq{}& \sqrt{p_{2t}(x_0,x_0)} \\ &\times \int_{\mathbb{M}} \Gamma^y(u) \Big(\int_{\mathbb{M}} [\Gamma^y_2(p_t(z,y)) - \rho\Gamma^y(p_t(z,y))]\ d\mu(z)\Big)^{1/2}\ d\mu(y), \end{aligned} \tag{3.35}$$

where we have used

$$\int_{\mathbb{M}} p_t(x_0,z)^2\ d\mu(z) = p_{2t}(x_0,x_0).$$

Therefore, if we knew that

$$\sup_{y\in\mathbb{M}} \int_{\mathbb{M}} [\Gamma^y_2(p_t(z,y)) - \rho\Gamma^y(p_t(z,y))]\ d\mu(z) < \infty,$$

we would be done. Since we assume $CD(\rho, n)$, it is sufficient to prove that

$$\sup_{y\in\mathbb{M}} \int_{\mathbb{M}} \Gamma_2^y p_t(z,y)\ d\mu(z) = \sup_{y\in\mathbb{M}} \int_{\mathbb{M}} (\mathcal{L}^y p_t(z,y))^2\ d\mu(z) < \infty.$$

We now use the spectral resolution and follow along the lines of proof of Theorem 7.6 in [46]. Specifically, we can argue as in equation (7.22) from there to find that

$$\|\mathcal{L}P_t f\|_{L^2(\mathbb{M})} \leq \frac{1}{t}e^{-1}\|f\|_{L^2(\mathbb{M})}.$$

Since $p_t(z,y) = (P_{t/2}p_{t/2}(\cdot,y))(z)$, it follows with $f = p_{t/2}(\cdot,y)$ that

$$\int_{\mathbb{M}} (\mathcal{L}^y p_t(z,y))^2\ d\mu(z) \leq \frac{1}{t}e^{-1}\left\|p_{t/2}(\cdot,y)\right\|_{L^2(\mathbb{M})} = \frac{1}{t}e^{-1}p_t(y,y). \tag{3.36}$$

Combining (3.35) and (3.36), we have

$$|B_{x_0,2t}(u,u)| \leq \frac{1}{t}e^{-1}\sqrt{p_{2t}(x_0,x_0)}\mathcal{E}(u,u)\sup_{y\in\mathbb{M}} p_t(y,y).$$

Thus if $y \mapsto p_t(y,y)$ is bounded on $\mathbb{M}$, we are done.

■

By the chain rule,

$$\begin{aligned} p_t H_{\ln p_t}(u,u) &= H_{p_t}(u,u) - p_t\Gamma(\ln p_t, u)^2 \\ &= H_{p_t}(u,u) - (Tu)^2. \end{aligned}$$

We combine this with Lemma 3.5.2 to arrive at the following:

Proposition 3.5.1 *Fix $t > 0$ and $x \in \mathbb{R}^n$. If $u \in \mathcal{D}(\mathcal{L})$, then*

$$\begin{aligned} P_t[\Gamma_2(u)](x) = &\int_{\mathbb{R}^n} p_t(x,\cdot)[\mathcal{L}u + \Gamma(u, \ln p_t(x,\cdot))]^2\ d\mu \\ &+ \int_{\mathbb{R}^n} p_t(x,\cdot) H_{\ln p_t(x_0,\cdot)}(u,u)\ d\mu. \end{aligned} \tag{3.37}$$

3.5.2 Monotonicity for reweighted Struwe energy

Theorem 3.5.1 (Modified Struwe monotonicity) *Fix $x_0 \in \mathbb{M}$ and assume that $\mathcal{L}$ satisfies $CD(\rho, n)$, $C(\omega, x_0)$ over a set $\mathcal{S}$. Let p_t denote the heat kernel for $\mathcal{L}$ with* *pole at* $(x_0, 0)$*. Assume that* $u \in \mathcal{S}$ *solves the heat equation* (3.4) *on* $\mathbb{M} \times (0, T)$*, and that for each* $t \in (0, T)$*,* $u \in \mathcal{D}(\mathcal{L})$*. Then*

$$\mathfrak{e}_{x_0}(t) = \eta(T-t)e_{x_0}(t) = e^{\omega(T-t)} P_{T-t}[\Gamma(u)](x_0)$$

is a non-increasing function of t*.*

Proof The computations are similar to those in [72] and [47]. Due to Proposition 3.4.1, we can write

$$\mathfrak{e}(t) = -\frac{1}{2} e^{\omega(T-t)} h'(t).$$

Then

$$\begin{aligned}
\mathfrak{e}'(t) &= \frac{1}{2}\omega'(T-t)e^{\omega(T-t)}h'(t) - \frac{1}{2}e^{\omega(T-t)}h''(t) \\
&= \frac{1}{2}e^{\omega(T-t)}[h'(t)\omega'(T-t) - h''(t)] \\
&= -2e^{\omega(T-t)}[\frac{1}{2}\omega'(T-t)P_{T-t}\Gamma(u) + P_{T-t}\Gamma_2(u)].
\end{aligned}$$

Writing $P_{T-t}\Gamma(u) = \int_{\mathbb{R}^n} p_{T-t}\Gamma(u)\, d\mu$ and combining with Proposition 3.5.1,

$$\begin{aligned}
\mathfrak{e}'(t) = -2e^{\omega(T-t)} &\int_{\mathbb{M}} [\frac{1}{2}\omega'(T-t)\Gamma(u) + H_{\ln p_{T-t}}(u,u)] p_{T-t}\, d\mu && (3.38) \\
- 2e^{\omega(T-t)} &\int_{\mathbb{M}} p_{T-t}[\mathcal{L}u + \Gamma(u, \ln p_{T-t})]^2\, d\mu. && (3.39)
\end{aligned}$$

Thus by the assumption $C(\omega, x_0)$,

$$\mathfrak{e}'(t) \le -2e^{\omega(T-t)} \int_{\mathbb{M}} p_{T-t}[\mathcal{L}u + \Gamma(u, \ln p_{T-t})]^2\, d\mu \le 0. \tag{3.40}$$

■

3.5.3 Monotonicity for reweighted Poon frequency

Theorem 3.5.2 (Modified Poon monotonicity) *Assume that* $(\mathbb{M}, \mathcal{L}, \mu)$ *satisfies* $CD(\rho, n)$ *and* $C(\omega, x_0)$ *over* $\mathcal{S}$*, and suppose that* $u \in \mathcal{S}$ *satisfies the conditions of Theorem 3.5.1. Then*

$$\mathfrak{n}_{x_0}(t) = e^{\omega(T-t)} \frac{P_{T-t}[\Gamma(u)](x_0)}{P_{T-t}[u^2](x_0)}$$

is monotonically non-increasing in t*.*

Proof As in Theorem 3.5.1, the computations are similar to that of their Euclidean counterparts. $\mathfrak{n}$ is decreasing as soon as

$$-h(t)\mathfrak{e}'(t) \geq -\mathfrak{e}(t)h'(t). \tag{3.41}$$

As in Theorem 3.4.1, one can show (via integration by parts) that

$$h'(t) = 2\int_{\mathbb{M}} [\mathcal{L}u + \Gamma(u, \ln p_{T-t})]up_{T-t}\, d\mu. \tag{3.42}$$

Applying the Cauchy-Schwarz inequality we see that

$$(h'(t))^2 \leq 4(\int_{\mathbb{M}} u^2 p_{T-t}\, d\mu)(\int_{\mathbb{M}} p_{T-t}[\mathcal{L}u + \Gamma(u, p_{T-t})]^2\, d\mu). \tag{3.43}$$

Using (3.40), we identify the right-hand side of (3.43) as being

$$4(\int_{\mathbb{M}} u^2 p_{T-t}\, d\mu)(\int_{\mathbb{M}} p_{T-t}[\mathcal{L}u + \Gamma(u, p_{T-t})]^2\, d\mu) \leq -2e^{-\omega(T-t)}\mathfrak{e}'(t)h(t). \tag{3.44}$$

Chaining (3.43) and (3.44) together and multiplying by $\frac{1}{2}e^{\omega(T-t)}$ yields

$$\frac{1}{2}e^{\omega(T-t)}(h'(t))^2 \leq -\mathfrak{e}'(t)h(t).$$

Finally, recalling that $\mathfrak{e}(t) = -\frac{1}{2}e^{\omega(T-t)}h'(t)$, we arrive at (3.41). ∎

Strong unique continuation for solutions of $\mathcal{L}u = u_t$

As in [67], which was modeled off of the elliptic case in [40], one may use the reweighted Poon frequency to establish a form of unique continuation. In what follows, we fix $x_0 \in \mathbb{M}$ and set

$$\begin{aligned}
H_{x_0}(r) &= H(r) = h(T-r^2) = \int_{\mathbb{M}} u^2(y, T-r^2)p_{r^2}(x,y)\, d\mu(y)\\
E_{x_0}(r) &= E(r) = \mathfrak{e}(T-r^2) = \exp(\omega(r^2))\int_{\mathbb{M}} \Gamma(u)(y, T-r^2)p_{r^2}(x,y)\, d\mu(y)\\
N_{x_0}(r) &= N(r) = \frac{E(r)}{H(r)}.
\end{aligned}$$

each of which are defined whenever $T - r^2 \in (0,T)$, that is, whenever $r \in (0, \sqrt{T})$. Since $N(r) = \mathfrak{n}(T-r^2)$ and $\mathfrak{n}$ is decreasing, it follows that N is an increasing function of r.

Lemma 3.5.3 *Assume that $H(r_0) = 0$. Then $H(r) \equiv 0$ for $0 \leq r < r_0$.*

Proof If $H(r_0) = 0$, then necessarily $u(y, T - r_0^2) \equiv 0$ for all $y \in \mathbb{M}$. Since u solves the heat equation, it follows that $u(y,t) \equiv 0$ for all $y \in \mathbb{M}$ and $t \in (T - r_0^2, T)$. If $0 < r < r_0$, then $u(y, T - r^2) \equiv 0$, hence $H(r) = 0$. ■

Henceforth, we assume that $H(r) \neq 0$ for any $r \in (0, \sqrt{T})$, so that the frequency N is well-defined on the interval $(0, \sqrt{T})$.

Fix $r_0 \in (0, \sqrt{T})$. Then we have

$$H'(r) = -2rh'(T - r^2) = 4r \exp(-\omega(r^2))E(r)$$

hence

$$\frac{H'(r)}{H(r)} = 4r \exp(-\omega(r^2))N(r) \leq 4r \exp(-\omega(r^2))N(r_0).$$

Integrating now each side for $r \in (0, r_0)$, we arrive at

$$\ln \frac{H(r_0)}{H(r)} \leq 4N(r_0) \int_r^{r_0} s \exp(-\omega(s^2))\ ds,$$

which is equivalent to

$$H(r) \geq H(r_0) \exp\left(-4N(r_0) \int_r^{r_0} s \exp(-\omega(s^2))\ ds\right). \tag{3.45}$$

We therefore get the following theorem.

Theorem 3.5.3 *Suppose that u is a solution to $\mathcal{L}u = u_t$ in $\mathbb{M} \times (0,T)$ satisfying the conditions of Theorem 3.5.2. Suppose that there exists a constant C such that for each $k \in \mathbb{N}$,*

$$H(r) \leq C \exp(-k \int_r^{\sqrt{T}} s \exp(-\omega(s^2))\ ds). \tag{3.46}$$

Then $u \equiv 0$.

Proof If $0 < r_0 < \sqrt{T}$, then (3.46) still holds for each k if we integrate on the interval (r, r_0) rather than $(r, \sqrt{T})$. This contradicts (3.45) for k large enough, hence $H(r_0) = 0$. By Lemma 3.5.3, H vanishes on $(0, r_0)$, hence u vanishes for $t \in (T - r_0^2, T)$. Letting $r_0 \nearrow \sqrt{T}$ finishes the proof. ■

Example 3.5.1 *(Unique continuation for Ornstein-Uhlenbeck)* Let us give a specific example of this theorem. Consider the Ornstein-Uhlenbeck operator, as in Example 3.3.3. We recall that

$$\omega(t) = \ln\left(\frac{1-e^{-2\rho t}}{2\rho}\right)$$

In this case, one can easily compute that

$$\int_r^{r_0} s\exp(-\omega(s^2))\ ds = \frac{1}{2}\ln\left(\frac{1-e^{2\rho r^2}}{1-e^{2\rho r_0^2}}\right),$$

and therefore (3.46) reads

$$H(r) \le C\left(\frac{1-e^{2\rho r^2}}{1-e^{2\rho T}}\right)^{\frac{k}{2}},\ k = 1, 2, \dots$$

In the limit where $\rho_0 \to 0$, this reads

$$H(r) \le C(\frac{r}{\sqrt{T}})^k, \qquad k = 1, 2, \dots,$$

from which we recover Poon's unique continuation theorem for the Laplacian in $\mathbb{R}^n$.

3.6 A sub-Riemannian Struwe energy for manifolds satisfying the generalized curvature dimension inequality

We now discuss the monotonicity of Struwe's energy function in the sub-Riemannian setting. As evidenced in Theorem 3.1.1, Struwe's energy monotonicity (*without* the correction factor $\eta(t)$) follows if $\Gamma_2(u)$ is non-negative. In the sub-Riemannian setting, this is no longer always true. For example, the first difficulty in studying the energy function in the setting of Carnot groups is when one computes the analogue of the Bochner identity for the Heisenberg group $\mathbb{H}^n$:

$$\Gamma_2(u) = \left\|\nabla_H^2 u\right\|_{HS}^2 + \frac{n}{2}(Zu)^2 + 2\langle\nabla_H u, J_b\nabla_H Zu\rangle,$$

where $\nabla_H^2 u$ is the symmetrized horizontal Hessian, e.g. if $n = 1$, $\nabla_H^2 u$ is given by

$$\nabla_H^2 u = \begin{pmatrix} X^2 u & (XY+YX)u \\ (XY+YX)u & Y^2 u \end{pmatrix},$$

and J_b is the usual complex structure on $T_{1,0}(\mathbb{H}^n)$, see [29]. The non-negativity or even boundedness of $\Gamma_2(u)$ is no longer guaranteed because of the presence of the final term involving $J-b$. A possible replacement criterion for sub-Riemannian manifolds having transverse symmetries is the generalized curvature dimension inequality of Baudoin and Garofalo, denoted $CD(\rho_1, \rho_2, \kappa, d)$. See [9] for the original definition, which we recall in Defintion 3.6.1 below.

In this section, we propose a modified energy function e_{sub} on sub-Riemannian manifolds $\mathbb{M}$ with transverse symmetries which possesses two important properties: (1) it reduces to the original definition in the case of Riemannian manifolds, and (2) if $\mathbb{M}$ satisfies the generalized curvature dimension inequality, then e_{sub} is monotonically decreasing on $(0, T)$. We then compare this sub-Riemannian Struwe energy function with the Struwe energy function associated to the tamed Riemannian metric. In the case of the Heisenberg group $\mathbb{H}^n$, we show that these in general are two different functions.

The set-up is identical to [9]. Let $\mathbb{M}$ be a smooth manifold having a smooth measure μ and a smooth (sub-)elliptic operator $\mathcal{L}$ such that (a) $\mathcal{L}1 = 0$ and (b) $\mathcal{L}$ is symmetric over $C_0^\infty(\mathbb{M})$ with respect to μ. As in the Riemannian case, we define the carré du champ and its iteration Γ_2 by Definition 1.2.1:

$$\Gamma(u,v) = \frac{1}{2}(\mathcal{L}(uv) - u\mathcal{L}v - v\mathcal{L}u)$$
$$\Gamma_2(u,v) = \frac{1}{2}(\mathcal{L}\Gamma(u,v) - \Gamma(v,\mathcal{L}u) - \Gamma(u,\mathcal{L}v)).$$

We further assume that $\mathbb{M}$ has a first-order, bilinear, symmetric differential operator, Γ^Z, for which

$$\Gamma^Z(uv,w) = u\Gamma^Z(v,w) + v\Gamma^Z(u,w).$$

We assume that $\Gamma^Z(u) \geq 0$, that $\mathbb{M}$ is complete in the sense of Hypothesis 1.1 in [9], and the following commutativity relation: given $u \in C^\infty(\mathbb{M})$,

$$\Gamma(u, \Gamma^Z(u)) = \Gamma^Z(u, \Gamma(f)).$$

The iteration of Γ^Z, denoted Γ_2^Z, is given by

$$\Gamma_2^Z(u,v) = \frac{1}{2}(\mathcal{L}\Gamma^Z(u,v) - \Gamma^Z(u,\mathcal{L}v) - \Gamma^Z(v,\mathcal{L}u)).$$

The Heisenberg group $\mathbb{H}^n$ satisfies $CD(0, \frac{n}{2}, 2n, 1)$ with respect to $\mathcal{L} = \Delta_H$ (the sub-Laplacian on $\mathbb{H}^n$) and $\Gamma^Z(u,v) = ZuZv$, where Z is the Reeb vector field. More generally on step-two Carnot groups, we may write $\Gamma^Z(u,v)$ as

$$\Gamma^Z(u,v) = \sum_{i=1}^{k} Z_i u Z_i v \tag{3.47}$$

for some vector fields $Z_i \in V_2$, where V_2 is the second layer of the Lie-algebra and $k = \dim V_2$.

We assume the following new hypothesis about functions on $\mathbb{M} \times I$, where I is a (possibly unbounded) sub-interval of $\mathbb{R}$: for each $u, v \in \mathbb{C}^\infty(\mathbb{M} \times I)$,

$$\frac{\partial}{\partial t}\Gamma^Z(u,v) = \Gamma^Z(u,v_t) + \Gamma^Z(u_t,v). \tag{3.48}$$

We should emphasize that (3.48) is a perfectly reasonable assumption considering (3.47).

Definition 3.6.1 *We say that $\mathbb{M}$ satisfies the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ with respect to $\mathcal{L}$ and Γ^Z if there exist $\rho_1 \in (-\infty, \infty)$, $\rho_2 > 0$, $\kappa \geq 0$, and $0 < d \leq \infty$ such that for each $\nu > 0$ and each $u \in C^\infty(\mathbb{M})$, we have*

$$\Gamma_2(u) + \nu\Gamma_2^Z(u) \geq \frac{1}{d}(\mathcal{L}u)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(u) + \rho_2\Gamma^Z(u). \tag{3.49}$$

In the case of a Riemannian manifold, (3.49) with $\mathcal{L} = \Delta$, $\kappa = 0$, $\Gamma^Z = 0$, $d = n = \dim \mathbb{M}$, reads

$$\Gamma_2(u) \geq \frac{1}{n}(\Delta u)^2 + \rho_1\Gamma(u),$$

which is exactly the original curvature dimension inequality that follows from the Bochner identity and a lower bound ρ_1 on the Ricci curvature tensor.

We are now in a position to introduce a modified energy function.

Definition 3.6.2 *Let $\mathbb{M}$ satisfy $CD(\rho_1, \rho_2, \kappa, d)$. Fix $x \in \mathbb{M}$, and $T > 0$. Let u solve the heat equation $\mathcal{L}u = u_t$ in $\mathbb{M} \times (0, T)$, and assume that u, $\Gamma(u)$, $\Gamma_2(u)$, $\Gamma^Z(u)$ and $\Gamma_2^Z(u)$ are all uniformly bounded. We define the sub-Riemannian energy function $e_{sub} : (0, T) \to \mathbb{R}$ by*

$$e_{sub}(t) = e_{sub,x}(t) = \frac{1}{2}(T-t)P_{T-t}[\Gamma(u)](x) + \kappa(T-t)^2 P_{T-t}[\Gamma^Z(u)](x). \tag{3.50}$$

Remark 3.6.1 *If $\mathbb{M}$ is a Riemannian manifold with Ricci curvature which is bounded below by ρ, then $\mathbb{M}$ satisfies $CD(\rho, 1, 0, n)$ and hence (3.50) reduces down to Struwe's original energy function.*

The following theorem uses the generalized curvature dimension inequality and partially generalizes Theorem 3.1.1.

Theorem 3.6.1 *Fix $x \in \mathbb{M}$. Let $\mathbb{M}$ satisfy $CD(\rho_1, \rho_2, \kappa, d)$, where $\rho_1 \geq 0$. Then $t \mapsto e_{sub}$ is monotonically decreasing.*

Proof By computations similar to Proposition 3.4.1

$$\partial_t P_{T-t}[\Gamma(u)] = -2P_{T-t}[\Gamma_2(u)]. \tag{3.51}$$

Furthermore,

$$\begin{aligned}
\partial_t P_{T-t}[\Gamma^Z(u)] &= P_{T-t}[-\mathcal{L}\Gamma^Z(u) + \partial_t \Gamma^Z(u)] \\
&= P_{T-t}[-\mathcal{L}\Gamma^Z(u) + 2\Gamma^Z(u, \mathcal{L}u)] \\
&= -2P_{T-t}[\Gamma_2^Z(u)].
\end{aligned} \tag{3.52}$$

where in the second line we have used (3.48). From (3.50), (3.51), (3.52), we get

$$\begin{aligned}
e'_{sub}(t) &= P_{T-t}\left[-\frac{1}{2}\Gamma(u) - (T-t)\Gamma_2(u) - 2\kappa(T-t)\Gamma^Z(u) - 2\kappa(T-t)^2\Gamma_2^Z(u)\right] \\
&= -(T-t)P_{T-t}\left[\Gamma_2(u) + \frac{1}{2(T-t)}\Gamma(u) + 2\kappa(T-t)\Gamma_2^Z(u)\right] \qquad (3.53) \\
&\quad - 2\kappa(T-t)P_{T-t}[\Gamma^Z(u)] \\
&\leq -(T-t)P_{T-t}\left[\Gamma_2(u) + \frac{1}{2(T-t)}\Gamma(u) + 2\kappa(T-t)\Gamma_2^Z(u)\right] \qquad (3.54)
\end{aligned}$$

by $\Gamma^Z(u) \geq 0$ and the sub-Markov property.

We now break into two cases.

- $\kappa = 0$: Then (3.54) reads

$$e'_{sub}(t) \leq -(T-t)P_{T-t}[\Gamma_2(u)].$$

$CD(\rho_1, \rho_2, 0, d)$ and $\rho_1 \geq 0$ imply that

$$\Gamma_2(u) + \nu\Gamma_2^Z(u) \geq 0, \qquad \nu > 0.$$

By choosing ν as small as we want, this means that $\Gamma_2(u) \geq 0$ pointwise. Thus $e'_{sub}(t) \leq 0$ follows from the positivity-preserving property of the semi-group.

- $\kappa > 0$: In this case, as a result of $\rho_1 \geq 0$ and $\rho_2 > 0$, $CD(\rho_1, \rho_2, \kappa, d)$ yields

$$\Gamma_2(u) + \nu\Gamma_2^Z(u) + \frac{\kappa}{\nu}\Gamma(u) \geq 0 \tag{3.55}$$

Choosing $\nu = 2\kappa(T-t) > 0$ in (3.55),

$$\Gamma_2(u) + 2\kappa(T-t)\Gamma_2^Z(u) + \frac{1}{2(T-t)}\Gamma(u) \geq 0. \tag{3.56}$$

$e'_{sub}(t) \leq 0$ now follows from (3.54), (3.56), and, once again, the positivity-preserving property.

■

Remark 3.6.2 *There is one somewhat dissatisfying aspect of Theorem 3.6.1: if $\kappa > 0$, there does not appear to be a way to extend this to the case where $\rho_1 < 0$, which would be the analogue of Theorem 3.1.1. Using $CD(\rho_1, \rho_2, \kappa, d)$ in this case, we find that*

$$\Gamma_2(u) + \frac{1}{2(T-t)}\Gamma(u) + 2\kappa(T-t)\Gamma_2^Z(u)$$
$$\geq \left(\frac{1}{2(T-t)} + \rho_1 - \frac{\kappa}{\nu}\right)\Gamma(u) + (2\kappa(T-t) - \nu)\Gamma_2^Z(u)$$

We could force the right-hand side to be non-negative if we required

$$2\kappa(T-t) \geq \nu, \qquad \text{and} \qquad \nu \geq \frac{2\kappa(T-t)}{1 + 2(T-t)\rho_1},$$

but this is not possible whenever $\kappa > 0$ and $\rho_1 < 0$.

3.6.1 Comparison of the sub-Riemannian energy function with the tamed energy function

The purpose of this section is to demonstrate that Theorem 3.6.1 does not follow in an obvious way from Theorem 3.1.1. Throughout this section, we will let $\mathbb{M}$ be a sub-Riemannian manifold with sub-Riemannian metric h and sub-Laplacian $\mathcal{L}$. We will also fix $T > 0$ and a solution to the heat equation $\mathcal{L}u = u_t$ on $\mathbb{M} \times (0,T)$. Let us begin with the appropriate definition from [13].

Definition 3.6.3 *A Riemannian metric g is said to* tame *a sub-Riemannian metric h if h is the restriction of g to to the horizontal bundle.*

For example, the Webster metric g_θ on any CR manifold $\mathbb{M}$ tames its own restriction to the Levi distribution $H(\mathbb{M})$. More generally, if the tangent bundle can be decomposed as $T(\mathbb{M}) = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{H}$ is the horizontal distribution of $\mathbb{M}$, then one may always tame the metric h by choosing any metric on $\mathcal{V}$ and making $\mathcal{H}$ and $\mathcal{V}$ orthogonal. However, such completion is not unique and depends on the choice of metric on $\mathcal{V}$.

Fix a metric g which tames h and let $\tilde{\mathcal{L}}$ denote the Laplace-Beltrami operator of the tamed metric. This (now non-degenerate) elliptic operator has a corresponding carré du champ, which we denote by $\tilde{\Gamma}$, and also an iteration, denoted $\tilde{\Gamma}_2$.

The Laplace-Beltrami operator has a self-adjoint extension from $C_0^\infty(\mathbb{M})$, the Friedrichs extension. We let $\tilde{P}_t$ be the semigroup obtained from this self-adjoint extension. Finally, we denote, for fixed $x \in \mathbb{M}$, Struwe's original energy function by the usual notation, i.e.

$$e_x(t) = \frac{1}{2}(T-t)\tilde{P}_{T-t}[\tilde{\Gamma}(u)](x).$$

We would like to note that, if the Ricci curvature of the tamed metric is bounded from below, then by Theorem 3.1.1 $e(t)$ is decreasing whenever $T - t$ is small.

Two obvious questions about e and e_{sub} are the following: is the monotonicity of the sub-Riemannian energy function e_{sub} just a restatement of the monotonicity of the tamed energy function e (in other words, is $e = e_{sub}$?) If not, can we find a direct comparison of the two energy functions?

We have the following proposition, which is a first step in answering the first question.

Proposition 3.6.1 *Suppose that u is of the form $u(x,t) = (P_t f)(x)$ for some $f \in L^2(\mathbb{M})$. Then*

$$\lim_{t\to T^-} \frac{e_{sub}(t)}{T-t} = \frac{1}{2}\Gamma(P_T f)$$
$$\lim_{t\to T^-} \frac{e(t)}{T-t} = \frac{1}{2}\tilde{\Gamma}(P_T f).$$

The proof is immediate based on the definitions. As a result, if $\Gamma(P_T f) \neq \tilde{\Gamma}(P_T f)$, then $e_{sub} \neq e$.

3.6.2 e and e_{sub} on the Heisenberg group $\mathbb{H}^n$

Let g_θ denote the Webster metric on $\mathbb{H}^n$, see [29]. This metric tames the sub-Riemannian metric defined on the horizontal bundle of the Heisenberg group. We will denote a point $g \in \mathbb{H}^n$ by its exponential coordinates $g = (x, y, z)$, where $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$. If we consider the basis $\{X_j, Y_j, Z : j = 1, \ldots, n\}$, where

$$X_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial z}, \qquad Y_j = \frac{\partial}{\partial y_j} - \frac{x_j}{2}\frac{\partial}{\partial z},$$

then this set is orthonormal under g_θ. Thus the corresponding Laplace-Beltrami operator is given by

$$\Delta = \sum_j (X_j^2 + Y_j^2) + Z^2 = \Delta_H + Z^2.$$

The completed carré du champ is given by

$$\tilde{\Gamma}(u, v) = \Gamma(u, v) + \Gamma^Z(u, v).$$

As a result of Proposition 3.6.1, we have the following:

Corollary 3.6.1 *In general, $e_{sub}(t) \neq e(t)$.*

Proof Let $f \in L^2(\mathbb{H}^n)$ and $T > 0$ be such that $\Gamma^Z(P_T f) \neq 0$. Since $\Gamma(P_T f) = \tilde{\Gamma}(P_T f)$ if and only if $\Gamma^Z(P_T f) = 0$, it follows that the corresponding sub-Riemannian and tamed energies are not the same. ■

We would now like to find an explicit comparison between the sub-Riemannian and tamed energy functions in terms of the heat kernel p_t. Because Z^2 and Δ_H commute, we can formally write

$$e^{t\Delta_H} e^{tZ^2} = e^{t\Delta}.$$

Thus one might expect the following.

Proposition 3.6.2 *Let $p_t(x, y, z)$ denote the heat kernel of Δ_H with pole at $(0, 0)$, see (1.24). Let $h_t(z) = (4\pi t)^{-1/2} \exp(-\frac{z^2}{4t})$. Then*

$$\tilde{p}_t(x, y, z) = [h_t * p_t(x, y, \cdot)](z)$$

is the heat kernel for the Laplace-Beltrami operator $\Delta_H + Z^2$.

Proof Let $\Delta_{(x,y,z)}$ denote the Laplace-Beltrami operator with respect to the variables (x, y, z), $\Delta_{H,(x,y,z)}$ the sub-Laplacian with respect to the variables (x, y, z), and $\Delta_{(x,y)}$ denote the Euclidean Laplacian with respect to the variables (x, y). By differentiating under the integral,

$$\begin{aligned}(\Delta_{(x,y,z)} - \partial_t)\tilde{p}_t(x, y, z) = \int_{\mathbb{R}} \Big[& p_t(x, y, \zeta)\{\Delta_{(x,y,z)} h_t(z - \zeta) - \partial_t h_t(z - \zeta)\} \\ & + h_t(z - \zeta)\{\Delta_{(x,y,z)} p_t(x, y, \zeta) - \partial_t p_t(x, y, \zeta)\} \\ & + 2\Gamma_{(x,y,z)}(h_t(z - \zeta), p_t(x, y, \zeta)) \Big]\, d\zeta \end{aligned} \tag{3.57}$$

Simple calculations show that

$$(\Delta_{(x,y,z)} - \partial_t) h_t(z - \zeta) = \frac{x^2 + y^2}{4} \partial_z^2 h_t(z - \zeta) = \frac{x^2 + y^2}{4} \partial_\zeta^2 h_t(z - \zeta). \tag{3.58}$$

On the other hand, looking at the p_t part under the integral,

$$\begin{aligned}\Delta_{(x,y,z)}p_t(x,y,\zeta) &= \Delta_{(x,y)}p_t(x,y,\zeta)\\ &= (\Delta_{H,(x,y,\zeta)})p_t(x,y,\zeta) - \frac{r^2}{4}\partial_\zeta^2 p_t(x,y,\zeta)\\ &= \partial_t p_t(x,y,\zeta) - \frac{r^2}{4}\partial_\zeta^2 p_t(x,y,\zeta) \qquad (3.59)\end{aligned}$$

Thus, inserting (3.57) and (3.59) into (3.58), one has

$$\begin{aligned}(\Delta_{(x,y,z)} - \partial_t)\tilde{p}_t(x,y,z) &= \frac{r^2}{4}\int_{\mathbb{R}} \Big[p_t(x,y,\zeta)\partial_\zeta^2 h_t(z-\zeta) - h_t(z-\zeta)\partial_\zeta^2 p_t(x,y,\zeta)\ d\zeta\\ &\qquad + 2\int_{\mathbb{R}} \Gamma_{(x,y,z)}(h_t(z-\zeta), p_t(x,y,\zeta))\Big]\ d\zeta\\ &= 2\int_{\mathbb{R}} \Gamma_{(x,y,z)}(h_t(z-\zeta), p_t(x,y,\zeta))\ d\zeta\end{aligned}$$

after integrating by parts on the first term. To take care of this last term, we note that

$$\begin{aligned}X_{(x,y,z)}p_t(x,y,\zeta) &= \partial_x p_t(x,y,\zeta)\\ Y_{(x,y,z)}p_t(x,y,\zeta) &= \partial_y p_t(x,y,\zeta)\\ \partial_z p_t(x,y,\zeta) &= 0\end{aligned}$$

and also

$$\begin{aligned}X_{(x,y,z)}h_t(z-\zeta) &= -\frac{y}{2}\partial_z h_t(z-\zeta)\\ Y_{(x,y,z)}h_t(z-\zeta) &= \frac{x}{2}\partial_z h_t(z-\zeta)\end{aligned}$$

thus

$$\begin{aligned}\Gamma_{(x,y,z)}(h_t(z-\zeta), p_t(x,y,\zeta)) &= (-\frac{y}{2}\partial_z h_t(z-\zeta))(\partial_x p_t(x,y,\zeta))\\ &\qquad + (\frac{x}{2}\partial_z h_t(z-\zeta))(\partial_y p_t(x,y,\zeta))\\ &= \frac{1}{2}\partial_z h_t(z-\zeta)\,(x\partial_y - y\partial x)\,p_t(x,y,\zeta)\\ &= 0\end{aligned}$$

since p_t is radial in (x, y). Thus $\tilde{p}_t$ at least solves the Riemannian heat equation. To finish, note that letting $t \to 0^+$, we have $h_t \to \delta_0$ in $\mathscr{D}'(\mathbb{R})$, and $p_t \to \delta_0$ in $\mathscr{D}'(\mathbb{R}^3)$. Thus because the convolution of Dirac deltas is another Dirac delta, we get $\tilde{p}_t \to \delta_0$ in $\mathscr{D}'(\mathbb{R}^3)$. ■

The previous proposition allows us to write the tamed energy function on $\mathbb{H}^n$ in terms of the sub-Riemannian heat kernel. Writing $g = (x, y, z)$,

$$e(t) = \frac{1}{2}(T-t)\int_{\mathbb{H}^n}[\Gamma(u) + \Gamma^Z(u)](g)(\int_{\mathbb{R}} h_{T-t}(z-\zeta)p_{T-t}(x, y, \zeta)\ d\zeta)\ dg$$

whereas the sub-Riemannian energy is given by

$$e_{sub}(t) = \frac{1}{2}(T-t)\int_{\mathbb{H}^n}[\Gamma(u) + 2(T-t)\Gamma^Z(u)](g)p_{T-t}(g)\ dg.$$

3.7 Struwe's energy monotonicity on sub-Riemannian manifolds

We end this chapter by briefly commenting on a few special cases where Struwe's energy function is monotonic for sub-classes of functions on sub-Riemannian manifolds. As mentioned in the previous section, Struwe's energy function e is monotonically non-increasing if $\Gamma_2(u)$ is non-negative. While this is not true in general, one can impose additional restrictions on the function u to achieve $\Gamma_2(u) \geq 0$.

This section is primarily motivated from an explicit computation of $\Gamma_2(u)$ that the author performed for cylindrical functions on the Heisenberg group $\mathbb{H}^n$ (see Definition 2.6.1(b)). Later, the author learned that such computations had already been performed for $\mathbb{H}^n$, [68], and that similar results existed in the literature for $SU(2)$, [8], and $SL(2, \mathbb{R})$, [17]. Now, we show that such results hold on even more general spaces – the CR unit sphere $\mathbb{S}^{2n+1}$, H-type groups, and the anti-de Sitter spaces – if one restricts to functions which have cylindrical symmetries when the notion of having cylindrical symmetry is appropriately defined.

3.7.1 Cylindrically invariant functions on the CR Sphere, $\mathbb{S}^{2n+1}$, and $SU(2)$

The $(2n+1)$-dimensional unit sphere is a strictly-pseudoconvex CR manifold and posesses a corresponding sub-Riemannian structure. In [10], Baudoin and Wang used the fibration $\mathbb{S}^{2n+1} \to \mathbb{CP}^n$ induced from the action of the unit circle $\mathbb{S}^1$ on $\mathbb{S}^{2n+1}$ to produce a coordinate system which makes the study of the heat kernel for the sub-Laplacian of $\mathbb{S}^{2n+1}$ amenable. The authors showed that the so-called "cylindrical" part of the sub-Laplacian is given by

$$\widetilde{\Delta_H} = \frac{\partial^2 u}{\partial r^2} + ((2n-1)\cot r - \tan r)\frac{\partial u}{\partial r} + \tan^2 r \frac{\partial^2 u}{\partial \theta^2}.$$

The carré du champ for cylindrically invariant functions is immediate from the quadratic form of the sub-Laplacian:

$$\Gamma(u) = (\frac{\partial u}{\partial r})^2 + \tan^2 r (\frac{\partial u}{\partial \theta})^2.$$

If one computes Γ_2 for functions possessing this cylindrical invariance, one finds that

$$\begin{aligned}
\Gamma_2(u) = {} & (\frac{\partial^2 u}{\partial r^2})^2 + \tan^4 r (\frac{\partial^2 u}{\partial \theta^2})^2 - 2\sec^2 r \tan r \frac{\partial^2 u}{\partial \theta^2}\frac{\partial u}{\partial r} \\
& + (\sec^2 r + (2n-1)\csc^2 r)(\frac{\partial u}{\partial r})^2 + 2\sec^2 r(\sec^2 r + (n-1))(\frac{\partial u}{\partial \theta})^2 \\
& + 4\sec^2 r \tan r \frac{\partial u}{\partial \theta}\frac{\partial^2 u}{\partial r \partial \theta} + 2\tan^2 r (\frac{\partial^2 u}{\partial r \partial \theta})^2 \\
= {} & (\frac{\partial^2 u}{\partial r^2})^2 + 2(n-1)[\csc^2 r (\frac{\partial^2 u}{\partial r^2})^2 + \sec^2 r (\frac{\partial u}{\partial \theta})^2] \\
& + \left(\tan^2 r \frac{\partial^2 u}{\partial \theta^2} - 2\csc(2r)\frac{\partial u}{\partial r}\right)^2 + 2\left(\sec^2 r \frac{\partial u}{\partial \theta} + \tan r \frac{\partial^2 u}{\partial r \partial \theta}\right)^2 \\
\geq {} & 0.
\end{aligned}$$

Setting $n = 1$, we recover the result of Baudoin and Bonnefont in [8], where the authors proved that cylindrically invariant functions on $SU(2)$ possess the positive curvature property.

3.7.2 Cylindrically invariant functions on the anti-de Sitter spaces $\mathbb{H}_{2n+1}$ and SL(2,$\mathbb{R}$)

Using the universal covering of $SL(2,\mathbb{R})$, Bonnefont in [17] introduced cylindrical coordinates on $SL(2,\mathbb{R})$ and showed that the heat kernel is only depends on the radial component of these coordinates. In fact, if u is radial

$$\Gamma_2(u) = (\frac{\partial^2 u}{\partial r^2})^2 + (\frac{2}{\sinh 2r}\frac{\partial u}{\partial r} - \tanh^2 r\frac{\partial u}{\partial z^2})^2 + 2(\frac{1}{\cosh^2 r}\frac{\partial u}{\partial z} + \tanh r\frac{\partial^2 u}{\partial r\partial z})^2$$

More generally, on the n-dimensional anti-de Sitter spaces, which we denote by $\mathbb{H}_{2n+1}$, Wang gave a representation of the cylindrical component of the sub-Laplacian in [73]:

$$\widetilde{\Delta_H} = \frac{\partial^2}{\partial r^2} + ((2n-1)\coth r + \tanh r)\frac{\partial}{\partial r} + \tanh^2 r\frac{\partial^2}{\partial \theta^2}.$$

As in the $\mathbb{S}^{2n+1}$ case, the carré du champ of a cylindrically invariant function is immediate from the quadratic form of the sub-Laplacian:

$$\Gamma(u) = (\frac{\partial u}{\partial r})^2 + \tanh^2 r(\frac{\partial u}{\partial \theta})^2.$$

A long computation also shows that

$$\begin{aligned}\Gamma_2(u) &= (\frac{\partial^2 u}{\partial r^2})^2 + \tanh^4 r(\frac{\partial^2 u}{\partial \theta^2})^2 - 2\operatorname{sech}^2 r\tanh r\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial \theta^2}\\
&\quad + ((2n-1)\operatorname{csch}^2 r - \operatorname{sech}^2 r)(\frac{\partial u}{\partial r})^2 + 2\tanh^2 r(\frac{\partial^2 u}{\partial r\partial\theta})^2\\
&\quad + 2\operatorname{sech}^2 r(\operatorname{sech}^2 r + n - 1)(\frac{\partial u}{\partial \theta})^2 + 4\operatorname{sech}^2 r\tanh r(\frac{\partial u}{\partial \theta})(\frac{\partial^2 u}{\partial r\partial\theta})\\
&= (\frac{\partial^2 u}{\partial r^2})^2 + 2(n-1)\left[\operatorname{sech}^2 r(\frac{\partial u}{\partial \theta})^2 + \operatorname{csch}^2 r(\frac{\partial u}{\partial r})^2\right]\\
&\quad + (\tanh^2 r\frac{\partial^2 u}{\partial \theta^2} - \frac{2}{\sinh(2r)}\frac{\partial u}{\partial r})^2 + 2(\operatorname{sech}^2 r\frac{\partial u}{\partial \theta} + \tanh r\frac{\partial^2 u}{\partial r\partial\theta})^2\\
&\geq 0.\end{aligned}$$

Similar to how setting $n = 1$ in the $\mathbb{S}^{2n+1}$ case recovers the $SU(2)$ case, if we here set $n = 1$ we recover the $SL(2,\mathbb{R})$ case.

3.7.3 Cylindrically symmetric functions on H-type groups

See Definition 2.6.1 for the definition of cylindrical functions. The fact that $\Gamma_2(u) \geq 0$ for cylindrically symmetric u was first noted by Bakry, Baudoin, Bonnefont and Chafaï in [4] for the case $n = 1$. Later, Qian extended this to the n-dimensional Heisenberg group $\mathbb{H}^n$ in [68]. In the proposition below, we give an extension of this property to H-type groups.

Proposition 3.7.1 *Let $\mathbb{G}$ be of Heisenberg-type. If u has cylindrical symmetry, then $\Gamma_2(u) \geq 0$.*

It is easily seen that the vector fields Θ_ℓ annihilate cylindrically symmetric functions, $\ell = 1, \ldots, m$. Thus, whenever u is cylindrical, (1.11) reads

$$\Delta_H u = \Delta_x u + \frac{|x|^2}{4}\Delta_z u, \qquad u \text{ cylindrically symmetric.} \tag{3.60}$$

Before proving Proposition 3.7.1, we need a few lemmas.

Lemma 3.7.1 *Let $\mathbb{G}$ be a group of Heisenberg type. Then* $\dim V_2 < \dim V_1$.

Proof Our starting point is Remark 18.1.6 in [16]: there exists a Heisenberg-type group with $n = \dim V_1$ and $m = \dim V_2$ if and only if $m < \rho(n)$, where $\rho(n)$ is the Radon-Hurwitz number given by

$$\rho(n) = 8p + q,$$

when n is written as $n = k2^{4p+q}$, where $k \geq 1$ is odd and $0 \leq q \leq 3$. We will show that $\rho(n) \leq n$ for all n, from which it follows that necessarily $m < n$.

Note that, if $x \geq 1$, we have the bound

$$8x \leq 16^x - 8,$$

which follows by basic calculus. Hence for $p \geq 1$,

$$\rho(n) = 8p + q \leq 8p + 3 \leq 16^p - 5 \leq 2^{4p}2^q \leq n.$$

On the other hand, if $p = 0$, then instead the bound $2^x \geq x$ for $x \geq 0$ gives

$$\rho(n) = q \leq 2^q \leq n.$$

$\blacksquare$

Lemma 3.7.2 *Let $\mathbb{G}$ be of Heisenberg-type. If u has cylindrical symmetry, then so do $|\nabla_H u|^2$ and $\Delta_H u$.*

Proof Repeating the calculations in Lemma 1.1.2, specifically (1.15) and (1.16), we have for any smooth u defined on a subset of $\mathbb{G}$ that

$$\Gamma(u) = |\nabla_H u|^2 = |\nabla_x u|^2 + \langle J(\nabla_z u)x, \nabla_x u\rangle + \frac{|x|^2}{4}|\nabla_z u|^2.$$

If $u(x,z)$ is any cylindrically symmetric function, then $\nabla_x u = \frac{x}{|x|}\frac{\partial u}{\partial |x|}$, hence

$$\langle J(\nabla_z u)x, \nabla_x u\rangle = 0.$$

From this we infer that

$$|\nabla_H u|^2 = |\nabla_x u|^2 + \frac{|x|^2}{4}|\nabla_z u|^2 \tag{3.61}$$

$$\Delta_H u = \Delta_x u + \frac{|x|^2}{4}\Delta_z u, \tag{3.62}$$

which are obviously cylindrically symmetric since u is. $\blacksquare$

Proof of Proposition 3.7.1 In the computations that follow, we assume that u has cylindrical symmetry. By combining Lemma 3.7.2 (specifically (3.61)) with (3.60), we find

$$\begin{aligned}
\Gamma_2(u) &= \frac{1}{2}\Delta_H(|\nabla_H u|^2) - \langle \nabla_H u, \nabla_H \Delta_H u\rangle \\
&= \frac{1}{2}(\Delta_x + \frac{|x|^2}{4}\Delta_z)(|\nabla_x u|^2 + \frac{|x|^2}{4}|\nabla_z u|^2) \\
&\quad - \langle \nabla_x u, \nabla_x(\Delta_x u + \frac{|x|^2}{4}\Delta_z u)\rangle - \frac{|x|^2}{4}\langle \nabla_z u, \nabla_z(\Delta_x u + \frac{|x|^2}{4}\Delta_z u)\rangle \\
&= \frac{1}{2}\Delta_x|\nabla_x u|^2 - \langle \nabla_x u, \nabla_x(\Delta_x u)\rangle + \frac{|x|^4}{16}\left(\frac{1}{2}\Delta_z(|\nabla_z u|^2) - \langle \nabla_z u, \nabla_z \Delta_z u\rangle\right) \\
&\quad + \frac{1}{2}\Delta_x(\frac{|x|^2}{4}|\nabla_z u|^2) + \frac{1}{2}\frac{|x|^2}{4}\Delta_z|\nabla_x u|^2 - \langle \nabla_x u, \nabla_x(\frac{|x|^2}{4}\Delta_z u)\rangle \\
&\quad - \frac{|x|^2}{4}\langle \nabla_z u, \nabla_z \Delta_x u\rangle.
\end{aligned}$$

Using the standard Bochner formula on Euclidean space,

$$\Gamma_2(u) = \left\|\nabla_x^2 u\right\|_{HS}^2 + \frac{|x|^4}{16}\left\|\nabla_z^2 u\right\|_{HS}^2 + \frac{1}{2}\Delta_x(\frac{|x|^2}{4}|\nabla_z u|^2) \tag{3.63}$$

$$+ \frac{1}{2}\frac{|x|^2}{4}\Delta_z|\nabla_x u|^2 - \langle\nabla_x u, \nabla_x(\frac{|x|^2}{4}\Delta_z u)\rangle - \frac{|x|^2}{4}\langle\nabla_z u, \nabla_z\Delta_x u\rangle, \tag{3.64}$$

where $\nabla_x^2 u$ and $\nabla_z^2 u$ are the Euclidean Hessians with respect to the variables x and z. Similar calculations to the Euclidean Bochner formula show that

$$\begin{aligned} \frac{1}{2}\Delta_z|\nabla_x u|^2 &= \left\|\nabla_{xz}^2 u\right\|_{HS}^2 + \langle\nabla_x u, \nabla_x\Delta_z u\rangle \\ \frac{1}{2}\Delta_x|\nabla_z u|^2 &= \left\|\nabla_{xz}^2 u\right\|_{HS}^2 + \langle\nabla_z u, \nabla_z\Delta_x u\rangle \\ \frac{1}{2}\Delta_x(\frac{|x|^2}{4}|\nabla_z u|^2) &= \frac{1}{2}\frac{|x|^2}{4}\Delta_x|\nabla_z u|^2 + \frac{n}{4}|\nabla_z u|^2 + \langle\nabla_x(\frac{|x|^2}{4}), \nabla_x|\nabla_z u|^2\rangle \end{aligned} \tag{3.65}$$

where $\nabla_{xz}^2 u$ is the $n \times m$ matrix

$$\nabla_{xz}^2 u = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1 \partial z_1} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial z_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial x_n \partial z_1} & \cdots & \frac{\partial^2 u}{\partial x_n \partial z_m} \end{pmatrix}.$$

Combining equations (3.65) with (3.64) and simplifying, we find that

$$\begin{aligned} \Gamma_2(u) = {} & \left\|\nabla_x^2 u\right\|_{HS}^2 + \frac{|x|^4}{16}\left\|\nabla_z^2 u\right\|_{HS}^2 + \frac{|x|^2}{2}\left\|\nabla_{xz}^2 u\right\|_{HS}^2 \\ & + \frac{n}{4}|\nabla_z u|^2 + \langle\nabla_x(\frac{|x|^2}{4}), \nabla_x|\nabla_z u|^2\rangle - \Delta_z u\langle\nabla_x u, \nabla_x(\frac{|x|^2}{4})\rangle. \end{aligned} \tag{3.66}$$

By the Cauchy-Schwarz inequality

$$\frac{|x|^4}{16}\left\|\nabla_z^2 u\right\|_{HS}^2 \geq \frac{|x|^4}{16m}(\Delta_z u)^2, \tag{3.67}$$

and by completing the square, one has

$$\frac{|x|^4}{16m}a^2 - ab = m\left(\frac{|x|^2}{4m}a - \frac{2}{|x|^2}b\right)^2 - \frac{4m}{|x|^4}b^2. \tag{3.68}$$

Taking $a = \Delta_z u$ and $b = \langle\nabla_x u, \nabla_x(\frac{|x|^2}{4})\rangle$ in (3.68) and combining with (3.66) and (3.67) gives us the inequality

$$\begin{aligned} \Gamma_2(u) \geq {} & \left\|\nabla_x^2 u\right\|_{HS}^2 + m\left(\frac{|x|^2}{4m}\Delta_z u - \frac{2}{|x|^2}\langle\nabla_x u, \nabla_x(\frac{|x|^2}{4})\rangle\right)^2 - \frac{4m}{|x|^4}\langle\nabla_x u, \nabla_x(\frac{|x|^2}{4})\rangle^2 \\ & + \frac{|x|^2}{2}\left\|\nabla_{xz}^2 u\right\|_{HS}^2 + \frac{n}{4}|\nabla_z u|^2 + \langle\nabla_x(\frac{|x|^2}{4}), \nabla_x|\nabla_z u|^2\rangle \end{aligned} \tag{3.69}$$

We now use the function v for which $u(x,z) = v(|x|,z)$. Denoting $r = |x|$, it follows that

$$
\begin{aligned}
\left\|\nabla_x^2 u\right\|_{HS}^2 &= v_{rr}^2 + \frac{n-1}{r^2} v_r^2 \\
\left\|\nabla_{xz}^2 u\right\|_{HS}^2 &= |\nabla_z v_r|^2 \\
\langle \nabla_x (\frac{r^2}{4}), \nabla_x u \rangle &= \frac{1}{2} r v_r \\
|\nabla_x u|^2 &= v_r^2 \\
u_{x_i} &= \frac{x_i}{r} v_r.
\end{aligned}
\tag{3.70}
$$

Inserting these formulas into (3.69) gives

$$
\begin{aligned}
\Gamma_2(u) \geq v_{rr}^2 &+ \frac{n-1}{r^2} v_r^2 + m\left(\frac{r^2}{4m}\Delta_z v - \frac{1}{r} v_r\right)^2 - \frac{m}{r^2} v_r^2 \\
&+ \frac{r^2}{2}|\nabla_z v_r|^2 + \frac{n}{4}|\nabla_z v|^2 + \frac{1}{2} r (|\nabla_z v|^2)_r \\
= v_{rr}^2 &+ \frac{n-1-m}{r^2} v_r^2 + m\left(\frac{r^2}{4m}\Delta_z v - \frac{1}{r} v_r\right)^2 \\
&+ \frac{r^2}{2}|\nabla_z v_r|^2 + \frac{n}{4}|\nabla_z v|^2 + r\langle \nabla_z v, \nabla_z v_r\rangle.
\end{aligned}
\tag{3.71}
$$

We complete the square once more:

$$
\frac{n}{4}|a|^2 + r\langle a, b\rangle = \frac{n}{4}|a + \frac{2r}{n} b|^2 - \frac{r^2}{n}|b|^2
$$

for any two vectors $a, b \in \mathbb{R}^{m+n}$. Taking now $a = \nabla_z v$ and $b = \nabla_z v_r$ and inserting into (3.71), we find after simplifying that

$$
\begin{aligned}
\Gamma_2(u) \geq v_{rr}^2 &+ \frac{n-1-m}{r^2} v_r^2 + m\left(\frac{r^2}{4m}\Delta_z v - \frac{1}{r} v_r\right)^2 \\
&+ \frac{n-2}{2n} r^2 |\nabla_z v_r|^2 + \frac{n}{4}|\nabla_z (v + \frac{2r}{n} v_r)|^2
\end{aligned}
\tag{3.72}
$$

whenever u is cylindrical. In particular, since $n \geq 2$ and $m < n$ by Lemma 3.7.1, $\blacksquare$

We would like to emphasize that the assumption that u is cylindrical is important to the computation. In equation (3.69), there are two terms that are possibly negative, however the equations (3.70) provide ways of combining those terms with other non-negative terms, which results in all positive terms.

The property of cylindrical symmetry is one that is preserved by the semigroup. Indeed, the heat kernel p_t posesses cylindrical symmetry, see (1.24), and this implies the preservation of this property by the semi-group. Thus Corollary 3.7.1 applies to heat solutions of the form $u = P_t f$ where $f \in L^2(\mathbb{G})$ is cylindrically symmetric.

We end by briefly commenting on the case where the function u does not have vanishing discrepancy. In the case of $\mathbb{H}^1$, one can compute

$$\begin{aligned}\Gamma_2(u) = \left\|\nabla_x^2 u\right\|_{HS}^2 + \left(\frac{r^2}{4}u_{zz} - \frac{1}{r^2}(xu_x + yu_y)\right)^2 - \frac{1}{r^4}(xu_x + yu_y)^2 \\ + \frac{1}{2}[u_z + (xu_{xz} + yu_{yz})]^2 - \Theta u_z(\Delta_H u) \\ + 2\Gamma(u_z, \Theta u) - \frac{1}{2}u_{zz}\Theta^2 u \end{aligned} \tag{3.73}$$

If $\Theta u \neq 0$, then the last three terms may be positive or negative, and we lose all control over the sign of $\Gamma_2(u)$.

Corollary 3.7.1 *Let $\mathbb{G}$ be of Heisenberg type and $u : \mathbb{G} \times (0,T) \to \mathbb{R}$ be a solution to the sub-elliptic heat equation $\Delta_H u = u_t$ which is also a cylindrical function in the space variables, i.e. $u(x,z,t) = v(|x|,z,t)$. Then if $u \in \mathcal{D}(\Delta_H)$, the Struwe's energy associated to u*

$$\begin{aligned} e(t) &= (T-t)P_{T-t}[|\nabla_H u|^2](g) \\ &= (T-t)\int_{\mathbb{G}} |\nabla_H u|^2(g')p_{T-t}(g,g')\ dg \end{aligned}$$

is a monotonically non-increasing function of t.

Proof By Proposition 3.7.1, we see that Δ_H satisfies $CD(0,\infty)$ for cylindrical functions, hence by (the proof of) Theorem 3.1.1, Struwe's energy is decreasing for all $t \in (0,T)$. ∎

Remark 3.7.1 *One can of course make similar statements about the non-increasingness of Struwe's energy for cylindrical functions on the sub-elliptic model spaces $\mathbb{S}^{2n+1}$ and anti-de Sitter spaces.*

4. Heat kernel asymptotics and Wiener criterion for groups of Heisenberg-type

4.1 Statement of the problem

Let $p_t^e : \mathbb{R}^n \to (0,\infty)$ denote the Euclidean heat kernel with pole at $0 \in \mathbb{R}^n$, that is,

$$p_t^e(x) = (4\pi t)^{-n/2} \exp(-\frac{|x|^2}{4t}).$$

We obviously have

$$\nabla \ln p_t^e = -\frac{x}{2t}, \quad \partial_t \ln p_t^e = -\frac{n}{2t} + \frac{|x|^2}{4t^2}.$$

Thus, given $\Theta > 1$, the bound

$$|\nabla \ln p_t^e|^2 \leq \Theta \partial_t \ln p_t^e \tag{4.1}$$

holds if and only if

$$|x|^2 \geq \frac{2n\Theta}{\Theta - 1} t.$$

In [31], Evans and Gariepy connected the bound (4.1) to a strong Harnack inequality. This Harnack inequality was vital to their proof of Wiener's criterion for the heat equation. This bound was extended to parabolic operators with smooth variable coefficients by Garofalo and Lanconelli in [37], and Garofalo and Segala gave the first proof of Theorem 4.1.1 in the sub-Riemannian setting in the case of the Heisenberg group $\mathbb{H}^n$ [43].

Let $\mathbb{G}$ be a group of Heisenberg-type with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$. We let $p_t : \mathbb{G} \times (0,\infty) \to \mathbb{R}$ be the heat kernel with pole at the group identity e associated to the sub-elliptic heat equation, $\Delta_H u = u_t$. We prove the following theorem.

Theorem 4.1.1 *Given $\Theta > 1$, there exists $\delta > 0$ depending on $\dim V_1$, $\dim V_2$, and Θ such that*

$$|\nabla_H \ln p_t|^2 \leq \Theta \partial_t \ln p_t \tag{4.2}$$

whenever $0 < t \leq \delta\rho(x,z)^2$. Here $\rho(x,z)$ denotes the Korányi gauge.

As an application, we briefly indicate in section 4.6 how Theorem 4.1.1 can be used to develop a corresponding extension of the strong Harnack inequality and Wiener criterion to H-type groups, see in particular Theorems 4.6.1 and 4.6.2.

4.2 Preliminary reductions to Theorem 4.1.1

Let $\mathbb{G}$ be a step-two Carnot group of Heisenberg-type. We write $\mathfrak{g} = V_1 \oplus V_2$ for the Lie algebra. Since the Kaplan mapping $J : V_2 \to \mathrm{End}(V_1)$ induces a complex structure on first layer, V_1 is necessarily even-dimensional. Thus we write $2n = \dim V_1$ and $m = \dim V_2$, hence the homogeneous dimension is $Q = 2n + 2m$.

Let $X_1, \ldots, X_{2n}$ be a basis for V_1 and $Z_1, \ldots, Z_m$ be a basis for V_2, and $\langle \cdot, \cdot \rangle$ a left-invariant inner product making the collection of these vector fields orthonormal. In exponential coordinates, we will write $g = (x, z)$, where we have identified $x = x_1X_1 + \cdots + x_{2n}X_{2n}$, $z = z_1Z_1 + \cdots z_mZ_m$.

As mentioned in Section 1.1.3, explicit formulas for the heat kernel exist, which we now recall:

$$p_t(x,z) = 2^m(4\pi t)^{-Q/2} \int_{\mathbb{R}^m} \exp(i\langle \frac{\xi}{t}, z\rangle) \exp(-\frac{|x|^2}{4t} \cdot \frac{|\xi|}{\tanh|\xi|})(\frac{|\xi|}{\sinh|\xi|})^n \, d\xi, \tag{4.3}$$

We note that the integrand of (1.24) differs from that of (4.3), due to our change of notation $\dim V_1 = 2n$ and the change of variables $\xi \mapsto \frac{1}{2}\xi$.

By inspection, p_t is homogeneous of degree $-Q$ with respect to the non-isotropic space-time dilations $(\tilde{\delta}_\lambda)_{\lambda>0}$ on $\mathbb{G} \times (0, \infty)$ given by

$$\tilde{\delta}_\lambda((x,z),t) = (\delta_\lambda(x,z), \lambda^2 t),$$

where $(\delta_\lambda)_{\lambda>0}$ is the usual one-parameter group of dilations on a Carnot group. Thus we have for every $t > 0$

$$p_t(x, z) = t^{-Q/2} p_1 \circ \delta_{1/\sqrt{t}}(x, z).$$

Also, $\frac{1}{t}\rho(x, z) = \rho \circ \delta_{1/t}(x, z)$. Due to these observations, we can turn Theorem 4.1.1 into an equivalent statement for p_1: Given $\Theta > 1$, there exists $\delta > 0$ such that $d(x, z) \geq \frac{1}{\delta}$ implies

$$|\nabla_H p_1|^2 \leq -\frac{\Theta}{2} p_1(Qp_1 + \mathcal{Z}p_1). \tag{4.4}$$

In (4.4), $\mathcal{Z}$ is the usual generator of the non-isotropic dilations $(\delta_\lambda)_{\lambda>0}$, i.e.

$$\mathcal{Z} = \langle x, \nabla_x \rangle + 2\langle z, \nabla_z \rangle.$$

Note that (4.4) is quadratic in p_1 on each side. Since

$$p_1(x, z) = 2^m (4\pi)^{-Q/2} \int_{\mathbb{R}^m} \exp(i\langle \xi, z \rangle) \exp(-\frac{|x|^2}{4} \cdot \frac{|\xi|}{\tanh |\xi|})(\frac{|\xi|}{\sinh |\xi|})^n \, d\xi,$$

we ignore the factor of $2^m(4\pi)^{-Q/2}$ in front of the integral representation of p_1 and work instead with the function

$$h(x, z) = \int_{\mathbb{R}^m} \exp(i\langle \xi, z \rangle) \exp(-\frac{|x|^2}{4} \cdot \frac{|\xi|}{\tanh |\xi|})(\frac{|\xi|}{\sinh |\xi|})^n \, d\xi. \tag{4.5}$$

Considering all of these observations, Theorem 4.1.1 will follow from the following.

Proposition 4.2.1 *Theorem 4.1.1 is equivalent to the following statement: Given* $\Theta > 1$, *there exists* $\delta > 0$ *such that* $\rho(x, z) \geq \frac{1}{\delta}$ *implies*

$$|\nabla_H h|^2 \leq -\frac{\Theta}{2} h(Qh + \mathcal{Z}h). \tag{4.6}$$

Since the Carnot-Carathéodory distance and Korányi gauge are equivalent, one could also state Theorem 4.1.1 for the Carnot-Carathéodory distance $d(x, z)$, albeit with a different δ. However, due to the equivalence of the distances, these theorems would imply each other. Therefore we will use the distances d and ρ interchangeably in the proof presented below.

Set $r = |x|$ and $s = |z|$. By an abuse of notation, we will often write $h(x, z) = h(r, s)$ since the function h is invariant under orthogonal translations in the horizontal and vertical variables. By this, we mean that if $\mathcal{O}_1 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $\mathcal{O}_2 : \mathbb{R}^m \to \mathbb{R}^m$ are orthogonal transformations, then $h(\mathcal{O}_1 x, \mathcal{O}_2 z) = h(x, z)$. It should be emphasized, however, that h is not invariant under orthogonal transformations of $\mathbb{R}^{2n+m}$.

The proof we give of Theorem 4.1.1 is reminiscent of that of [43], broken into the following steps and cases:

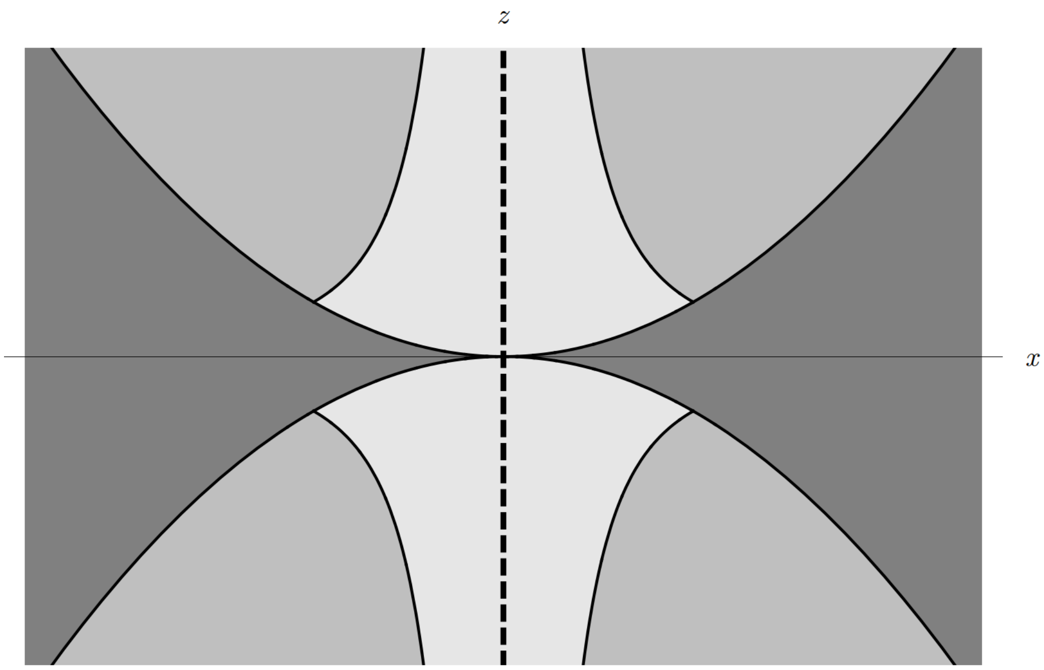


Figure 4.1. The various regions of study and their subcases.

- Step 1: $\frac{|z|}{|x|^2}$ is bounded (dark gray in Figure 4.1). This step uses an m-dimensional version of the method of steepest descent, specifically Theorem 1.1 of [34].

- Step 2: $\frac{|z|}{|x|^2}$ is unbounded. Step 2 must be further into two cases: when m is odd and m is even.

- Case 1 (m odd): We are able to use a formula in [30] which relates the integral representation of h over $\mathbb{R}^m$ to a sum of one-dimensional Fourier integrals. This allows us to, by breaking into four sub-cases, use techniques from [43] on estimating residues to give asymptotics for each of the terms in the summation and relate those back to p_1 itself:
  - ∗ Subcase (a): $\frac{|z|}{|x|^2}$ and $|x|^2|z|$ large (medium gray in Figure 4.1)
  - ∗ Subcase (b): $\frac{|z|}{|x|^2}$ and $|x|^2|z|$ bounded (light gray red in Figure 4.1)
  - ∗ Subcase (c): $\frac{|z|}{|x|^2}$ and $x = 0$ (vertical dashed line in Figure 4.1)
- Case 2 (m even): As the formula for p_1 makes sense for any pair of natural numbers n, m (regardless of whether or not an H-type group of the corresponding dimensions actually exists), we use a method of descent argument to add in an auxiliary dimension to the center. The asymptotics derived in Case 1 apply, and we then remove the extra dimension by using an observation from [30] which relates the case of the center having dimension $m+1$ to the case where the center has dimension m. The bound (4.6) then follows from the asymptotics in the same way as in Case 1.

We will make use of Theorem 4.2.1 below on multiple occasions to directly derive asymptotics in two of the regions (Step 1 and Step 2, sub-case 1). For its proof, see [34] (specifically, Theorem 1.1 on page 417 and Proposition 1.1 on page 418), or "saddle point method" in [49]. It is a higher-dimensional analogue of the special contours chosen in [43].

Theorem 4.2.1 *Let $\gamma \subset \mathbb{C}^m$ be an m-dimensional smooth, compact, real manifold with boundary, and λ a large, positive parameter. Assume that $f(\zeta)$ and $S(\zeta)$ are holomorphic in a neighborhood of γ. Define*

$$F(\lambda) = \int_\gamma f(\zeta) e^{\lambda S(\zeta)}\, d\zeta,$$

Assume that $\max_\gamma \operatorname{Re} S(\zeta)$ *is attained at a single point* $\zeta_0 \in \gamma$*, and that* ζ_0 *is a non-singular saddle point, that is,* $\nabla S(\zeta_0) = 0$ *and* $\det(\nabla^2 S(\zeta_0)) \neq 0$*. Then*

$$F(\lambda) = (\frac{2\pi}{\lambda})^{m/2}(\det(-\nabla^2 S(\zeta_0)))^{-1/2} \exp(\lambda S(\zeta_0))[f(\zeta_0) + O(\lambda^{-1})].$$

Let f and g be two functions, and $g \neq 0$. We write $f(w) \sim g(w)$ as $w \to \infty$ to mean

$$\lim_{w\to\infty} \frac{f(w)}{g(w)} = 1.$$

By carrying out our expansions in the various regions of interest, we come to the following lemma which is the major stepping stone in the proof of Theorem 4.1.1.

Lemma 4.2.1 *Let* $\mathbb{G}$ *be a group of H-type. Then as* $\frac{d(x,z)}{\sqrt{t}} \to \infty$*, we have the following asymptotic relations:*

(a) $\frac{|z|}{|x|^2}$ *bounded:*

$$|\nabla_H p_t|^2 \sim \frac{d(x,z)^2}{4t^2} p_t^2, \quad \mathcal{Z} p_t \sim -\frac{d(x,z)^2}{2t} p_t, \quad \frac{\partial p_t}{\partial |z|} \sim -\frac{\theta}{t} p_t$$

If we avoid the paraboloid $\frac{4|z|}{|x|^2} = \nu(\frac{\pi}{2}) = \frac{\pi}{2}$*, then also*

$$\frac{\partial p_t}{\partial |x|} \sim -\cos\theta \frac{d(x,z)}{4t} p_t.$$

(b) $\frac{|z|}{|x|^2} \to \infty$*,* $|x|^2|z|/t^2 \to \infty$*:*

$$\frac{\partial p_t}{\partial |x|} \sim \frac{d(x,z)}{4t} p_t, \quad \frac{\partial p_t}{\partial |z|} \sim -\frac{\pi}{t} p_t$$

(c) $\frac{|z|}{|x|^2} \to \infty$*,* $|x|^2|z|/t^2$ *bounded,* $x \neq 0$*:*

$$\frac{\partial p_t}{\partial |x|} \sim \frac{d(x,z)}{4t} \frac{I_n(\frac{\sqrt{\pi|x|^2|z|}}{t})}{I_{n-1}(\frac{\sqrt{\pi|x|^2|z|}}{t})} p_t, \quad \frac{\partial p_t}{\partial |z|} \sim -\frac{\pi}{t} p_t$$

(d) $x = 0$*:*

$$\frac{\partial p_t}{\partial |x|} = 0, \quad \frac{\partial p_t}{\partial |z|} \sim -\frac{\pi}{t} p_t.$$

In Lemma 4.2.1 part (a), the asymptotics provided for $\frac{\partial p_t}{\partial |x|}$ are only valid if we avoid $\theta = \frac{\pi}{2}$ since $\cos\theta$ vanishes along this paraboloid. However, the asymptotics given in part (a) for $|\nabla_H p_t|^2$ and $\mathcal{Z}p_t$ and $\frac{\partial p_t}{\partial |z|}$ work for any value of θ assuming $\frac{|z|}{|x|^2}$ is bounded.

We should point out that asymptotics for p_t itself derived in steps 1 and 2 exist already in the literature, see [60] for H-type groups, and [61] for the Baouendi-Grushin operator for $(x, z) \in \mathbb{R}^n \times \mathbb{R}$,

$$\mathcal{B}_1 = \Delta_x + \frac{|x|^2}{4}\frac{\partial^2}{\partial z^2},$$

when $\frac{|z|}{|x|^2}$ is bounded. The author would like to thank F. Baudoin for bringing the work of H.-Q. Li, especially [60], to his attention. Our method of expansion differs in some places from that of Li's. For example, instead of using properties of the Bessel functions and the representation of p_t due to Randall in [69], we follow the methods of [43] and apply the saddle-point theorem to the m-dimensional integral (4.5) in the case where $\nu(\theta)$ is bounded. The present work also requires asymptotic expansions of the derivatives of p_t, $\frac{\partial p_t}{\partial |x|}$ and $\frac{\partial p_t}{\partial |z|}$ as in Lemma 4.2.1, and these are a new contribution of this chapter.

4.3 Step 1: $0 \le \nu(\theta) \le M$

We begin by assuming that $\nu(\theta) = \frac{4|z|}{|x|^2} \le M$ (recall the definition of ν given in Section 1.1.3). In this case,

$$|x| \le \rho(x, z) \le (1 + M^2)^{1/4}|x|,$$

so the Korányi gauge, Carnot-Carathéodory distance, and $r = |x|$ are all equivalent distances. It is therefore sufficient to prove Theorem 4.1.1 for $|x| \ge \frac{1}{\delta}$ for some $\delta > 0$. We also note that ν is strictly increasing function of θ, it follows that $0 \le \theta \le \theta_0 = \nu^{-1}(M)$.

The main goal of this section is to derive the following asymptotics for h and its derivatives in this region:

$$h(x,z) \sim (\frac{8\pi}{|x|^2})^{m/2}(\det - \nabla_\zeta^2 S(0,\theta,\hat{z}))^{-1/2}(\frac{\theta}{\sin\theta})^n \exp(-\frac{d(x,z)^2}{4})$$

$$\frac{\partial h}{\partial r} \sim -\frac{r}{2}\frac{\theta}{\tan\theta}h$$

$$\frac{\partial h}{\partial s} \sim -\theta h,$$

where S is a function which we will define presently in (4.8).

The flow of the argument is similar to the one presented in [43]. The major difference is use of Theorem 4.2.1 since we are now dealing with a higher-dimensional integral. In order to use Theorem 4.2.1, we adopt the notation and procedure of [30]. Since h is the inverse Fourier transform in ξ of a function only depending on $|\xi|$, the result only depends on $s = |z|$. Thus we may write

$$h(r,s) = \int_{\mathbb{R}^m} \exp[\frac{r^2}{4}\varphi(\zeta,\theta,\hat{z})] f_0(\zeta)\ d\zeta,$$

where $\hat{z} \in \mathbb{S}^{m-1}$ is an arbitrary fixed point and

$$\varphi(\zeta,\theta,\hat{z}) = i\langle\zeta,\hat{z}\rangle\nu(\theta) - \frac{\sqrt{\zeta^2}}{\tanh\sqrt{\zeta^2}} \tag{4.7}$$

$$f_0(\zeta) = (\frac{\sqrt{\zeta^2}}{\sinh\sqrt{\zeta^2}})^n.$$

In (4.7), $\sqrt{\cdot}$ is a holomorphic square root such that $\sqrt{\zeta} > 0$ for positive real ζ and $\operatorname{Im}\sqrt{\zeta} \geq 0$. Also, we have denoted $\zeta^2 = \langle\zeta,\zeta\rangle$ is the usual bilinear inner product on $\mathbb{R}^m$, extended to $\mathbb{C}^m$. We emphasize that it is *not* the complex inner product on $\mathbb{C}^m$. In using the bilinear inner product, φ and f are holomorphic on their domains.

By Lemma 5.4 in [30], it is possible to shift the integration to the complex hyperplane $\zeta + i\theta\hat{z} \in \mathbb{C}^m$:

$$h(r,s) = \int_{\mathbb{R}^m} \exp[\frac{r^2}{4}\varphi(\zeta + i\theta\hat{z},\theta,\hat{z})] f_0(\zeta + i\theta\hat{z})\ d\zeta.$$

Set

$$S(\zeta,\theta,\hat{z}) = \varphi(\zeta + i\theta\hat{z},\theta,\hat{z}) - \varphi(i\theta\hat{z},\theta,\hat{z}) = \varphi(\zeta + i\theta\hat{z},\theta,\hat{z}) + \frac{\theta^2}{\sin^2\theta}, \tag{4.8}$$

so that

$$h(r,s) = \exp(-\frac{r^2\theta^2}{4\sin^2\theta}) \int_{\mathbb{R}^m} \exp[\frac{r^2}{4} S(\zeta,\theta,\hat{z})] f_0(\zeta + i\theta\hat{z})\ d\zeta$$

By Lemma 5.4 in [30], we can focus on integrating over a ball of radius ε, for some $\varepsilon > 0$:

$$h(r,s) = \exp(-\frac{r^2\theta^2}{4\sin^2\theta}) \left\{ \int_{B(0,r)} \exp[\frac{r^2}{4} S(\zeta,\theta,\hat{z})] f_0(\zeta + i\theta\hat{z})\ d\zeta + O(r^{-2m}) \right\}. \quad (4.9)$$

It is important to note that the function $O(r^{-2m})$ and $r > 0$ are uniform in θ, $\hat{z}$. We are almost ready to apply Theorem 4.2.1.

Lemma 4.3.1 *If $\zeta \neq 0$, then*

$$\operatorname{Re} S(\zeta,\theta,\hat{z}) < \operatorname{Re} S(0,\theta,\hat{z}) = 0.$$

Proof By Lemma 5.7 of [30], for any $\zeta \neq 0$,

$$\operatorname{Re} \frac{\sqrt{(\zeta + i\theta\hat{z})^2}}{\tanh \sqrt{(\zeta + i\theta\hat{z})^2}} > \frac{\theta}{\tan\theta}.$$

Therefore

$$\begin{aligned} \operatorname{Re} S(\zeta,\theta,\hat{z}) &= \operatorname{Re}(i\langle \zeta + i\theta\hat{z}, \hat{z}\rangle \nu(\theta) + \frac{\theta^2}{\sin^2\theta}) - \operatorname{Re} \frac{\sqrt{(\zeta + i\theta\hat{z})^2}}{\tanh \sqrt{(\zeta + i\theta\hat{z})^2}} \\ &< -\theta\nu(\theta) + \frac{\theta^2}{\sin^2\theta} - \frac{\theta}{\tan\theta} \\ &= 0. \end{aligned}$$

■

Lemma 4.3.2 *The function $\zeta \mapsto S(\zeta,\theta,\hat{z})$ has a saddle point at $\zeta_0 = 0$. Furthermore,*

$$(\frac{2}{3})^m \leq \det(-\nabla_\zeta^2 S(0,\theta,\hat{z})) \leq (\nu'(\theta))^m,$$

the lower bound being uniform in $\theta, \hat{z}$. Thus the saddle point is non-singular.

Proof It is equivalent to show the same for $\varphi(\zeta,\theta,\hat{z})$ at $\zeta_0 = i\theta\hat{z}$. Differentiating φ,

$$\nabla_\zeta \varphi = i\nu(\theta)\hat{z} - i\nu(-i\sqrt{\zeta^2})\hat{\zeta}$$
$$\nabla_\zeta^2 \varphi = -i\nu(-i\sqrt{\zeta^2})\frac{I_m}{\sqrt{\zeta^2}} + \frac{i}{2\sqrt{\zeta^2}}\nu(-i\sqrt{\zeta^2})\hat{\zeta}\otimes\hat{\zeta} - \nu'(-i\sqrt{\zeta^2})\hat{\zeta}\otimes\hat{\zeta}.$$

where I_m is the $m\times m$ identity, $\hat{\zeta} = \frac{\zeta}{\sqrt{\zeta^2}}$, and $a\otimes b$ is the $m\times m$ matrix with entries $a_i b_j$, $1\le i,j\le m$ whenever $a,b\in\mathbb{C}^m$. Obviously, $\nabla_\zeta\varphi(i\theta\hat{z},\theta,\hat{z}) = 0$. Also,

$$-\nabla_\zeta^2\varphi(i\theta\hat{z},\theta,\hat{z}) = \frac{\nu(\theta)}{\theta}I_m - \frac{\nu(\theta)}{\theta}\hat{z}\otimes\hat{z} + \nu'(\theta)\hat{z}\otimes\hat{z}$$

This is a fully real matrix. If $\xi\in\mathbb{R}^m$ is any non-zero vector, then

$$\langle -\nabla_\zeta^2\varphi(i\theta\hat{z})\xi,\xi\rangle = \frac{\nu(\theta)}{\theta}|\xi|^2 + (\nu'(\theta) - \frac{\nu(\theta)}{\theta})\langle\hat{z},\xi\rangle^2.$$

It is easy to check that

$$\nu'(\theta) \ge \frac{\nu(\theta)}{\theta} \ge \frac{2}{3}.$$

with equality if and only if $\theta = 0$ in each case. Thus by Cauchy-Schwarz,

$$\frac{2}{3}|\xi|^2 \le \langle -\nabla_\zeta\varphi(i\theta\hat{z})\xi,\xi\rangle \le \nu'(\theta)|\xi|^2.$$

Therefore, the eigenvalues of $-\nabla_\zeta^2\varphi(i\theta\hat{z},\theta,\hat{z})$ are all larger than $\frac{2}{3}$, uniformly in $\theta,\hat{z}$. ∎

Proof of Theorem 4.1.1 when $\nu(\theta)$ is bounded Combining Lemmas 4.3.1 and 4.3.2, the integral in (4.9) is ready to have Theorem 4.2.1 applied to it, with $\gamma = B(0,\varepsilon)$, $\lambda = \frac{r^2}{4}$, $\zeta_0 = 0$, and $f = f_0$. Thus, since $\frac{\sin\theta}{\theta}\le 1$ for all $\theta\in\mathbb{R}$,

$$\begin{aligned} h(r,s) &= \exp(-\frac{r^2\theta^2}{4\sin^2\theta})(\frac{8\pi}{r^2})^{m/2}(\det -\nabla_\zeta^2 S(0,\theta,\hat{z}))^{-1/2}(\frac{\theta}{\sin\theta})^n\left\{1+O(r^{-2})\right\} \\ &= \exp(-\frac{d(x,z)^2}{4})(\frac{8\pi}{r^2})^{m/2}(\det -\nabla_\zeta^2 S(0,\theta,\hat{z}))^{-1/2}(\frac{\theta}{\sin\theta})^n\left\{1+O(r^{-2})\right\}. \end{aligned}$$

We would like to emphasize that the $O(r^{-2m})$ part in (4.9) may either absorbed into $O(r^{-2})$ or ignored completely since $m\ge 1$, and because $\det -\nabla^2 S(0,\theta,\hat{z})$ and $\frac{\theta}{\sin\theta}$

are bounded uniformly in $\theta, \hat{z}$ (recall that $0 \leq \theta \leq \theta_0 < \pi$). In the second line, we have also used $d(x,z) = \frac{\theta}{\sin\theta}|x|$ (recall (1.13)) when $x \neq 0$.

As

$$\begin{aligned}
\frac{\partial h}{\partial r} &= -\frac{r}{2}\int_{\mathbb{R}^m} \exp[\frac{r^2}{4}\varphi(\zeta,\theta,\hat{z})] f_0(\zeta)(\frac{\sqrt{\zeta^2}}{\tanh\sqrt{\zeta^2}})\, d\zeta, \\
\frac{\partial h}{\partial s} &= \int_{\mathbb{R}^m} \exp[\frac{r^2}{4}\varphi(\zeta,\theta,\hat{z})] f_0(\zeta)(i\langle\zeta,\hat{z}\rangle)\, d\zeta,
\end{aligned}$$

we can use the same arguments as above to show that

$$\begin{aligned}
\frac{\partial h}{\partial r} &= -\frac{r}{2}h\left\{\frac{\theta}{\tan\theta} + O(r^{-2})\right\} \\
\frac{\partial h}{\partial s} &= h\left\{-\theta + O(r^{-2})\right\} \sim -\theta h
\end{aligned}$$

as $r \to \infty$. Above, we have used the notation $f \sim g$ as $w \to \infty$ to mean $\lim_{w\to\infty} \frac{f(w)}{g(w)} = 1$ for the $\frac{\partial h}{\partial s}$. We have also crucially used the fact that $\nu(\theta)$ – hence θ and $\frac{\theta}{\tan\theta}$ – are bounded functions. It is tempting to write $\partial_r h \sim -\frac{r}{2}\frac{\theta}{\tan\theta}h$, but this is only valid if $\theta_0 = \nu^{-1}(M) < \frac{\pi}{2}$, otherwise the function $\frac{\theta}{\tan\theta}$ may vanish.

Because of the symmetry of the function h (i.e. it only depends on $r = |x|$ and $s = |z|$), we have that

$$\begin{aligned}
|\nabla_H h|^2 &= (\frac{\partial h}{\partial r})^2 + \frac{r^2}{4}(\frac{\partial h}{\partial s})^2 \\
\mathcal{Z}h &= r\frac{\partial h}{\partial r} + 2s\frac{\partial h}{\partial s} \\
&= r\frac{\partial h}{\partial r} + \frac{r^2}{2}\nu(\theta)\frac{\partial h}{\partial s}.
\end{aligned}$$

It thus follows that as $r \to \infty$,

$$|\nabla_H h|^2 \sim \frac{d(x,z)^2}{4}h^2 \tag{4.10}$$

$$\mathcal{Z}h \sim -\frac{d(x,z)^2}{2}h, \tag{4.11}$$

In (4.10) we have used the Pythagorean identity, and in (4.11) we have used the fact that $\nu(\theta)$ is bounded. We have again used (1.13) to write the right-hand side of (4.10) and (4.11) in terms of the Carnot-Carathéodory distance. In order to prove Theorem 4.1.1 for this case, it is sufficient to prove

$$\frac{d(x,z)^2}{4} \leq -\frac{\Theta}{2}(Q - \frac{d(x,z)^2}{2})$$

as $r \to \infty$. This may be rearranged to

$$d(x,z)^2 \geq \frac{2Q\Theta}{\Theta - 1}, \tag{4.12}$$

as $r \to \infty$. Given $\Theta > 1$, (4.12) obviously holds for large enough r and $d(x,z)$. Since the Carnot-Carathéodory distance and Korányi distance are equivalent metrics, Theorem 4.1.1 follows in the case where $\nu(\theta) = \frac{4|z|}{|x|^2}$ is bounded. ∎

Remark 4.3.1 *The region where $\nu(\theta)$ is bounded is the most "Euclidean-like." In $\mathbb{R}^n$, the Carnot-Carathéodory distance agrees with the usual Euclidean distance. Thus, for example, there is a clear comparison of* (4.10) *and* (4.11) *with their Euclidean counterparts:*

$$|\nabla p_1|^2 = \frac{|x|^2}{4} p_1$$
$$\mathcal{Z} p_1 = \langle x, \nabla p_1 \rangle = -\frac{|x|^2}{2} p_1.$$

Even (4.12) *is reminiscent of the Euclidean requirement for Theorem 4.1.1 to hold.*

Remark 4.3.2 *Fix $a, b \in \mathbb{R}$ and $u \in \mathbb{S}^{m-1}$. One may show that the eigenvalues of a matrix of the form $M = aI_m + bu \otimes u$ are $\lambda_1 = a + b$ and $\lambda_2, \ldots, \lambda_m = a$. In fact, assuming $u = (u_1, \ldots, u_m)$ with $u_1 \neq 0$, an eigen-basis for M is given by*

$$\xi_1 = u,\ \xi_k = (-\frac{u_k}{u_1}, 0, \ldots, 0, 1, 0, \ldots, 0), \quad k = 1, \ldots, m-1,$$

where in ξ_k, $k = 2, \ldots, m$, the 1 is taken to be in the the kth position. Hence,

$$-\det \nabla^2 S(0, \theta, \hat{z}) = (\frac{\nu(\theta)}{\theta})^{m-1} \nu'(\theta).$$

Using this observation and the asymptotic expansion of $h(x,z) = 2^{-m}(4\pi)^{Q/2} p_1(x,z)$ given above, we obtain the asymptotic expansion of p_1 given in Theorem 1.4 of [60].

4.4 Step 2: $\nu(\theta)$ unbounded, m odd

We now study the case where $\nu(\theta)$ is allowed to diverge to infinity. In [43], Garofalo and Segala shifted the contour of (4.3) from $\mathbb{R}$ to $\mathbb{R} + \frac{3\pi}{2}i$ using the residue theorem, showed that the integral over $\mathbb{R} + \frac{3\pi}{2}i$ is negligible, and gave asymptotics for the residue of the integrand at $\zeta = i\pi$. However, as we are now integrating over $\mathbb{R}^m$, we first have to reduce to one-dimensional integrals if we wish to apply this type of argument.

The key is to use the following formula for h, valid for m odd and $z \neq 0$:

$$h(x,z) = \sum_{k=0}^{\frac{m-1}{2}} c_{m,k}|z|^{k-m+1} \operatorname{Re} \int_{\mathbb{R}} \exp(i\rho|z| - \frac{|x|^2\rho}{4\tanh\rho})(\frac{\rho}{\sinh\rho})^n(-i\rho)^k \, d\rho. \tag{4.13}$$

Here, the $c_{m,k}$ are non-negative constants depending on m and k:

$$c_{m,k} = \begin{cases} 1 & \text{if } m = 1, k = 0 \\ 0 & \text{if } m \geq 3, k = 0 \\ (2\pi)^{\frac{m-1}{2}} 2^{k - \frac{m-1}{2}} \binom{\frac{m}{2}-1}{\frac{m-1}{2}-k} & \text{if } m \geq 3, k = 1, \ldots, \frac{m-1}{2}. \end{cases}$$

Note that $c_{m,k} > 0$ when $m \geq 3$ and $k \geq 1$, and $c_{m,\frac{m-1}{2}} = (2\pi)^{\frac{m-1}{2}}$, $m \geq 1$.

(4.13) can be found in [30] and [60], and is arrived at by integrating out the angular dependence of the heat kernel formula. The resulting integral involves Bessel functions $J_{m/2-1}$. Then, by writing $J_{m/2-1} = \operatorname{Re}(J_{m/2-1} + iY_{m/2-1}) = \operatorname{Re} H^{(1)}_{m/2-1}$, where $H^{(1)}_\nu$ is the Hankel function of the first kind of order ν, and applying a closed-form formula for $H^{(1)}_{m/2-1}$ when m is odd (see Section 3.3 of [63]), one arrives at (4.13).

We note that in fact the integrals in (4.13) are fully real. Indeed, when $k = 2\ell + 1$ is odd, we have

$$\exp(i\rho s)(-i\rho)^k = (-1)^{\ell+1}\sin(\rho s)\rho^{2\ell+1} + i(-1)^\ell \cos(\rho s)\rho^{2\ell+1}.$$

Due to the fact that $\exp(-\frac{r^2\rho}{4\tanh\rho})$ is even in ρ, the imaginary part integrates to zero since $\cos(\rho s)\rho^k$ is odd in ρ. If, on the other hand, $k = 2\ell$ is even, then

$$\exp(i\rho s)(-i\rho)^k = (-1)^\ell \cos(\rho s)\rho^{2\ell} + i(-1)^\ell \sin(\rho s)\rho^{2\ell}.$$

Again, the imaginary part is odd in ρ, so it integrates out to zero. Thus, we drop the Re in (4.13).

We write

$$h(x,z) = \sum_{k=0}^{\frac{m-1}{2}} c_{m,k}|z|^{k-m+1} h_k(x,z), \tag{4.14}$$

where we have denoted

$$\begin{aligned} h_k(x,z) &= \int_{\mathbb{R}} \exp(\varphi(\rho,x,z)) f_k(\rho)\, d\rho \\ \varphi(\rho,x,z) &= i\rho|z| - \frac{|x|^2\rho}{4\tanh\rho} \\ f_k(\rho) &= (\frac{\rho}{\sinh\rho})^n(-i\rho)^k \end{aligned} \tag{4.15}$$

We will use the same abuse of notation with the h_k's as we do with h itself and write $h_k(x,z) = h_k(r,s)$, similarly $\varphi(\rho,x,z) = \varphi(\rho,r,s)$. By the chain and product rules, we can compute

$$\frac{\partial h}{\partial r} = -\frac{r}{2}\sum_{k=0}^{\frac{m-1}{2}} c_{m,k}s^{k-m+1}\int_{\mathbb{R}} \exp(\varphi(\rho,x,z))(\frac{\rho}{\sinh\rho})^n(-i\rho)^k\frac{\rho}{\tanh\rho}\, d\rho \tag{4.16}$$

$$\frac{\partial h}{\partial s} = \sum_{k=0}^{\frac{m-1}{2}}(k-m+1)c_{m,k}s^{k-m}h_k(r,s) - \sum_{k=0}^{\frac{m-1}{2}} c_{m,k}s^{k-m+1}h_{k+1}(r,s) \tag{4.17}$$

As we are now interested in what happens to h when the quantities $|z||x|^2$ and $\frac{|z|}{|x|^2}$ are of various sizes, we will write throughout this entire case

$$R = \frac{r^2}{4}, \quad \mu = \sqrt{\frac{R}{\pi s}} = \sqrt{\frac{1}{\pi\nu(\theta)}}, \quad \lambda = \sqrt{\pi R s}.$$

Since $\nu(\theta) = \frac{4|z|}{|x|^2}$ is taken to be large, we have

$$4|z| \leq \rho(x,z)^2 = (|x|^4 + 16|z|^2)^{1/2} = 4|z|(1 + \frac{1}{\nu(\theta)})^{1/2}.$$

Therefore, in order to prove Theorem 4.1.1 for this region, it is suffucient to let $s = |z| \to \infty$.

4.4.1 Sub-case 1: $\mu \to 0^+$ and $\lambda \to \infty$

Our main goal of this section is to prove the following.

Proposition 4.4.1 *As $\mu \to 0^+$ and $\lambda \to \infty$, we have the following asymptotic relations.*

$$\begin{aligned} h(x,z) &\sim \pi(2\pi)^{\frac{m-1}{2}}\sqrt{\frac{\pi}{\lambda}}(\frac{\mu}{\lambda})^{\frac{m-1}{2}}\mu^{1-n}\exp(-\frac{d(x,z)^2}{4}) \\ \frac{\partial h}{\partial s} &\sim -\pi h \\ \frac{\partial h}{\partial r} &\sim \sqrt{\pi|z|}h. \end{aligned}$$

We note that $\rho \mapsto \varphi(\rho, x, z)$ and the f_k in (4.15) are meromorphic in a neighborhood of the strip $\Omega = \{z \in \mathbb{C} : \operatorname{Im} z \in [0, \frac{3\pi}{2}]\}$, each with singularities at $\rho = i\pi$. Thus by the residue theorem, we have

$$h_k(r,s) = \int_{\operatorname{Im}\rho=\frac{3\pi}{2}} \exp(\varphi(\rho,x,z))f_k(\rho)\ d\rho + 2\pi i\operatorname{Res}_{i\pi}[\exp(\varphi(\cdot,x,z))f_k]$$

$$\begin{aligned} \int_{\mathbb{R}} \exp(\varphi(\rho,x,z))f_k(\rho)\frac{\rho}{\tanh\rho}\ d\rho &= \int_{\operatorname{Im}\rho=\frac{3\pi}{2}} \exp(\varphi(\rho,x,z))f_k(\rho)\frac{\rho}{\tanh\rho}\ d\rho \\ &\quad + 2\pi i\operatorname{Res}_{i\pi}[\exp(\varphi(\cdot,x,z))f_k\frac{\cdot}{\tanh\cdot}] \end{aligned}$$

Let us first get asymptotics for the residue terms. This is a generalization of the discussion in Step 2, Case 1 of [43]. Many of the same ideas are still present, but we apply Theorem 4.2.1 instead of estimating the residue directly.

Lemma 4.4.1 *Let g be a meromorphic function near $z = i\pi$ and having a pole of order ℓ at $i\pi$ for some $\ell \geq 0$. Set*

$$F(\rho) = \exp(i\rho s - \frac{r^2\rho}{4\tanh\rho})(\frac{\rho}{\sinh\rho})^n g(\rho). \tag{4.18}$$

Then

$$\begin{aligned} \operatorname{Res}[F; i\pi] = -\frac{i}{2}\frac{c_{-\ell}}{(-i\pi)^\ell}\mu^{-\ell-n+1}\exp(-\frac{d(x,z)^2}{4})\sqrt{\frac{\pi}{\lambda}} \\ \times\{1 + O(\mu) + O(\lambda^{-1})\} \end{aligned} \tag{4.19}$$

as $\mu \to 0^+$ and $\lambda \to \infty$, where $c_{-\ell}$ is the $-\ell$ coefficient in the Laurent series expansion of g near $i\pi$, i.e.

$$c_{-\ell} = \lim_{z \to i\pi} (z - i\pi)^\ell g(z).$$

Proof We have

$$\begin{aligned}
\mathrm{Res}[F; i\pi] &= i\pi \,\mathrm{Res}[F(i\pi(1+\cdot)); 0] \\
&= i\pi \frac{1}{2\pi i} \int_{C_\mu} F(i\pi(\zeta+1))\ d\zeta \\
&= \frac{1}{2}(-1)^n \exp(-\pi s) \\
&\quad \times \int_{C_\mu} \exp(-\pi\zeta s - \frac{r^2(1+\zeta)}{4\tan\pi\zeta})(\frac{\pi(1+\zeta)}{\sin\pi\zeta})^n g(i\pi(1+\zeta))\ d\zeta
\end{aligned} \tag{4.20}$$

where we have followed the notation of [43] and written $C_\mu = \{-\mu e^{i\zeta} : -\pi \le \zeta \le \pi|\}$. By peeling off the singularity at $\zeta = 0$ inside the phase of (4.20), we may write

$$\begin{aligned}
\mathrm{Res}[F; i\pi] &= -\frac{i}{2} \exp(-\pi s - \frac{r^2}{4}) \\
&\quad \times \int_{-\pi}^{\pi} \exp[\lambda\psi(\zeta,\mu)](\frac{\pi(1-\mu e^{i\zeta})}{\sin\pi\mu e^{i\zeta}})^n g(i\pi(1-\mu e^{i\zeta}))\mu e^{i\zeta}\ d\zeta,
\end{aligned} \tag{4.21}$$

where $\psi(\zeta,\mu) = 2\cos\zeta - \mu q(\pi\mu e^{i\zeta})$ and

$$q(z) = \frac{z-\pi}{\tan z} + \frac{\pi}{z} - 1 \tag{4.22}$$

is holomorphic in a neighborhood of zero, and having a simple zero at 0. Since we assume that that g has a pole of order ℓ at $i\pi$, we have the expansion

$$g(i\pi(1-w)) = w^{-\ell}(\frac{c_{-\ell}}{(-i\pi)^\ell} + wA(w))$$

where A is holomorphic near $w = 0$ and $c_{-\ell}$ is the $-\ell$-coefficient in the Laurent expansion of g near $i\pi$. This in turn implies that

$$(\frac{\pi(1-w)}{\sin\pi w})^n g(i\pi(1-w))w = w^{-\ell-n+1}(\frac{c_{-\ell}}{(-i\pi)^\ell} + wB(w))$$

for a suitable B holomorphic function near 0. Hence

$$\begin{aligned}
&\int_{-\pi}^{\pi} \exp[\lambda\psi(\zeta,\mu)](\frac{\pi(1-\mu e^{i\zeta})}{\sin\pi\mu e^{i\zeta}})^n g(i\pi(1-\mu e^{i\zeta}))\mu e^{i\zeta}\ d\zeta \\
&\qquad = \int_{-\pi}^{\pi} \exp[\lambda\psi(\zeta,\mu)](\mu e^{i\zeta})^{1-\ell-n}(\frac{c_{-\ell}}{(-i\pi)^\ell} + \mu e^{i\zeta}B(\mu e^{i\zeta})\ d\zeta \\
&\qquad = \mu^{1-\ell-n}\frac{c_{-\ell}}{(-i\pi)^\ell}\int_{-\pi}^{\pi} \exp[\lambda\psi(\zeta,\mu)](e^{i\zeta})^{1-\ell-n}\ d\zeta \\
&\qquad\quad + \mu^{2-\ell-n}\int_{-\pi}^{\pi} \exp[\lambda\psi(\zeta,\mu)](e^{i\zeta})^{2-\ell-n}B(\mu e^{i\zeta})\ d\zeta
\end{aligned} \tag{4.23}$$

We now wish to apply Theorem 4.2.1 to (4.23). Claim: The mapping $\zeta \mapsto \psi(\zeta,\mu)$ has a non-degenerate critical point (which depends on μ and we thus denote by $\zeta(\mu)$) when

$$\zeta(\mu) = \frac{i}{2}\ln(\frac{\pi}{\nu(\theta)(\pi-\theta)^2})$$

for $0 < \theta \leq \pi$, the value of $\zeta(0) = 0$ being defined as a left-hand limit when $\theta = \pi$. Above, we have used the fact that a value of $\mu = \frac{1}{\sqrt{\pi\nu(\theta)}}$ may be put into one-to-one correspondence with a value of $\theta \in [0,\pi]$, i.e. $\theta = \theta(\mu) = \nu^{-1}(\frac{1}{\pi\mu^2})$.

To prove the claim, we note that

$$\psi_\zeta(\zeta,\mu) = -2\sin\zeta - i\mu^2\pi e^{iz}q'(\mu\pi e^{i\zeta}). \tag{4.24}$$

Now,

$$\exp(i\zeta(\mu)) = (\pi-\theta)\sqrt{\frac{\nu(\theta)}{\pi}} = \frac{\pi-\theta}{\mu\pi}, \tag{4.25}$$

and because of the identity

$$2\sin(\frac{i}{2}\ln x) = i(x^{1/2} - x^{-1/2}), \quad x > 0,$$

we have

$$2\sin(\zeta(\mu)) = \sin(\frac{i}{2}\ln(\frac{\pi}{\nu(\theta)(\pi-\theta)^2})) = i(\frac{1}{\pi-\theta}\sqrt{\frac{\pi}{\nu(\theta)}} - (\pi-\theta)\sqrt{\frac{\nu(\theta)}{\pi}}). \tag{4.26}$$

Combining (4.24)–(4.26) and recalling that $\mu = \sqrt{\frac{1}{\pi\nu(\theta)}}$

$$\begin{aligned}\psi_\zeta(\zeta(\mu),\mu) &= i((\pi-\theta)\sqrt{\frac{\nu(\theta)}{\pi}} - \frac{1}{\pi-\theta}\sqrt{\frac{\pi}{\nu(\theta)}}) - i(\pi-\theta)\sqrt{\frac{1}{\pi\nu(\theta)}}q'(\pi-\theta) \\ &= \frac{i}{(\pi-\theta)\sqrt{\pi\nu(\theta)}}\left[(\pi-\theta)^2\nu(\theta) - \pi - (\pi-\theta)^2 q'(\pi-\theta)\right]. \qquad (4.27)\end{aligned}$$

From (4.22),

$$q(z) = \frac{z}{\tan z} + \pi\cot z + \frac{\pi}{z} - 1,$$

therefore, recalling that $\nu(\theta) = -\frac{d}{d\theta}(\theta\cot\theta)$,

$$q'(z) = -\nu(z) - \pi\csc^2 z - \frac{\pi}{z^2}.$$

Since $\nu(\pi-\theta) = \pi\csc^2\theta - \nu(\theta)$ and $\csc^2(\pi-\theta) = \csc^2\theta$, it follows that

$$\begin{aligned}q'(\pi-\theta) &= -\nu(\pi-\theta) - \frac{\pi}{(\pi-\theta)^2} \\ &= \nu(\theta) - \frac{\pi}{(\pi-\theta)^2}. \qquad (4.28)\end{aligned}$$

From (4.27) and (4.28), we conclude that $\psi_\zeta(\zeta(\mu),\mu) = 0$.

We can also compute $\psi(\zeta(\mu),\mu)$ in a comparable fashion. Similar to before, we have the identity

$$2\cos(\frac{i}{2}\ln x) = x^{1/2} + x^{-1/2}, \quad x > 0.$$

Thus

$$\begin{aligned}\psi(\zeta(\mu),\mu) &= 2\cos\zeta(\mu) - \sqrt{\frac{1}{\pi\nu(\theta)}}q(\mu\pi e^{i\zeta(\mu)}) \\ &= \frac{1}{\pi-\theta}\sqrt{\frac{\pi}{\nu(\theta)}} + (\pi-\theta)\sqrt{\frac{\nu(\theta)}{\pi}} - \sqrt{\frac{1}{\pi\nu(\theta)}}q(\pi-\theta) \\ &= \frac{1}{\sqrt{\pi\nu(\theta)}}\left[\frac{\pi}{\pi-\theta} + (\pi-\theta)\nu(\theta) - q(\pi-\theta)\right].\end{aligned}$$

Inasmuch as $\lambda = \frac{r^2}{4}\sqrt{\pi\nu(\theta)}$ and $q(\pi-\theta) = \theta\cot\theta + \frac{\theta}{\pi-\theta}$, it follows that

$$\begin{aligned}\lambda\psi(\zeta(\mu),\mu) &= \frac{r^2}{4}\left[\frac{\pi}{\pi-\theta} + (\pi-\theta)\nu(\theta) - q(\pi-\theta)\right] \\ &= \frac{r^2}{4}\left[1 + \pi\nu(\theta) - \frac{\theta^2}{\sin^2\theta}\right].\end{aligned}$$

Finally, recall that $\nu(\theta) = \frac{4|z|}{|x|^2}$, implying

$$\lambda\psi(\zeta(\mu), \mu) = \frac{r^2}{4} + \pi s - \frac{d(x,z)^2}{4}, \tag{4.29}$$

where we've used – as in Step 1 – the fact that the Carnot-Carathéodory distance satisfies $d(x,z) = |x|\frac{\theta}{\sin\theta}$ when $x \neq 0$.

As for the non-degeneracy of $\zeta(\mu)$, we simply note that $\zeta(0)$ is obviously non-degenerate since $\psi(\zeta, 0) = 2\cos\zeta$, thus for μ sufficiently small $\zeta(\mu)$ is also non-degenerate.

We may connect $\pm\pi$ with a C^2 path contained in the $\operatorname{Re}\psi(\zeta, 0) < 2$ except when $\zeta = 0$. By continuity we can, for μ small enough, still connect $-\pi$ to π with a C^2 path $\gamma = \gamma_\mu$ contained in the region $\operatorname{Re}\psi(\zeta,\mu) < \operatorname{Re}\psi(\zeta(\mu),\mu)$, except at $\zeta(\mu)$, and such that $\gamma_\mu(0) = z(\mu)$. Refer to Figure 4.2 for some sample paths.

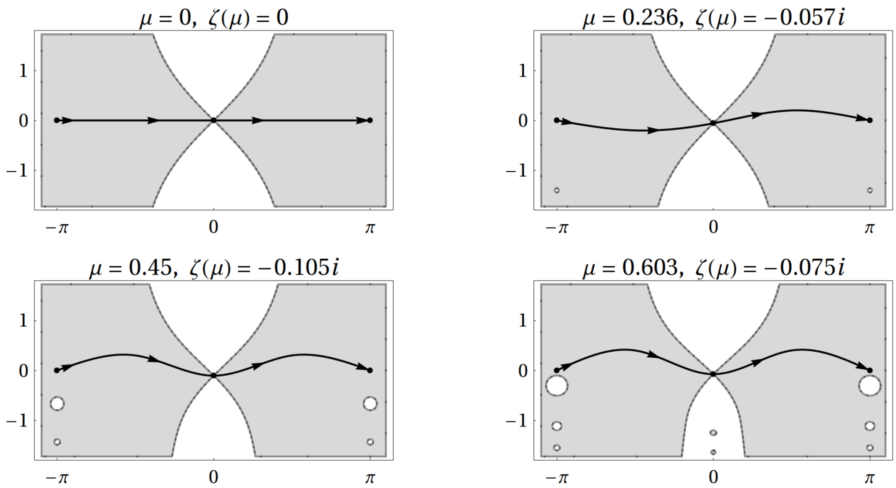


Figure 4.2. Sample paths connecting $-\pi$ to π contained in the region $\operatorname{Re}\psi(\zeta,\mu) \leq \operatorname{Re}\psi(\zeta(\mu),\mu)$, with equality only at the critical point of the mapping $\zeta \mapsto \psi(\zeta,\mu)$.

We may thus apply Theorem 4.2.1 with $S(\zeta) = \psi(\zeta, \mu)$, $\zeta_0 = \zeta(\mu)$ and $\gamma = \gamma_\mu$ to each of the integrals above to obtain

$$\begin{aligned} \int_{-\pi}^{\pi} e^{\lambda\psi(\zeta,\mu)} e^{ik\zeta}\ d\zeta &= \int_{\gamma_\mu} e^{\lambda\psi(\zeta,\mu)} e^{ik\zeta}\ d\zeta \\ &= (\frac{2\pi}{\lambda})^{1/2}(-\psi_{\zeta\zeta}(\zeta(\mu),\mu))^{-1/2} e^{\lambda\psi(\zeta(\mu),\mu)}[e^{ik\zeta(\mu)} + O(\lambda^{-1})] \quad (4.30) \end{aligned}$$

But for small μ we have

$$\begin{aligned} \zeta(\mu) &= \zeta(0) + O(\mu) = O(\mu) \\ e^{ik\zeta(\mu)} &= e^{ik\zeta(0)} + O(\mu) = 1 + O(\mu) \\ \psi_{\zeta\zeta} &= -2\cos\zeta + O(\mu^2) \end{aligned}$$

therefore $\psi_{\zeta\zeta}(\zeta(\mu), \mu) = -2 + O(\mu)$. Hence by the binomial expansion,

$$(-\psi_{\zeta\zeta}(\zeta(\mu),\mu))^{-1/2} = 2^{-1/2} + O(\mu). \quad (4.31)$$

Combining (4.31) with (4.30), we find that

$$\int_{-\pi}^{\pi} e^{\lambda\psi(\zeta,\mu)} e^{ik\zeta}\ d\zeta = (\frac{\pi}{\lambda})^{1/2} e^{\lambda\psi(\zeta(\mu),\mu)}(1 + O(\mu) + O(\lambda^{-1})).$$

Inserting this into (4.23) and combining with (4.21) and remembering that $\mu \to 0^+$ and B is bounded on C_μ uniformly in $\zeta \in [-\pi, \pi]$ and $\mu \ll 1$, we arrive at

$$\begin{aligned} \mathrm{Res}[F; i\pi] = &-\frac{i}{2}\frac{c_{-\ell}}{(-i\pi)^\ell}\mu^{-\ell-n+1}\exp(-\pi s - R + \lambda\psi(z(\mu),\mu))\sqrt{\frac{\pi}{\lambda}} \quad (4.32) \\ &\times \{1 + O(\mu) + O(\lambda^{-1})\}. \end{aligned}$$

Finally, we use (4.29) in (4.32) to arrive at (4.19).

■

The next lemma from [30] shows that the integrals over the set $\{z \in \mathbb{C} : \mathrm{Im}\, z = \frac{3\pi}{2}\}$ are negligible when compared to the residue terms.

Lemma 4.4.2 *There exists $M > 0$ such that whenever $\frac{4|z|}{r^2} \geq M$ we have*

$$\int_{\mathrm{Im}\,\rho = \frac{3\pi}{2}} \exp(\varphi(\rho, x, z)) f_k(\rho) g(\rho)\ d\rho \leq C \exp(-\frac{d(x,z)^2}{8})$$

provided $|f_k(\rho + \frac{3\pi}{2}i)g(\rho + \frac{3\pi}{2}i)|$ is an integrable function.

Corollary 4.4.1 *For each $k \in \mathbb{N}_0$,*

$$h_k(r,s) \sim \pi^{k+1}\mu^{-n+1}\sqrt{\frac{\pi}{\lambda}}\exp(-\frac{d(x,z)^2}{4})\{1+O(\mu)+O(\frac{1}{\lambda})\}$$

as $\mu \to 0^+$ and $\lambda \to \infty$.

Proof Take $g(\rho) = (-i\rho)^k$ in Lemma 4.4.1, with $\ell = 0$ and $c_0 = g(i\pi) = \pi^k$. By Lemma 4.4.2 the integral along $\operatorname{Im}\rho = \frac{3\pi}{2}$ is negligible compared to the residue term as $s \to \infty$. Thus

$$\begin{aligned} h_k(r,s) &\sim 2\pi i \operatorname{Res}[\exp(\varphi(\cdot,x,z))(\frac{\cdot}{\sinh \cdot})^n(-i\cdot)^k] \\ &\sim \pi^{k+1}\mu^{1-n}\sqrt{\frac{\pi}{\lambda}}\exp(-\frac{d(x,z)^2}{4}) \end{aligned}$$

as $\mu \to 0^+$ and $\lambda \to \infty$.

■

Remark 4.4.1 *Note that in the case where $m = 1$, it follows that $k = 0$ in the series expansion of $h(r,s)$. Setting $k = 0$ in Corollary 4.4.1 reproduces (2.42) of [43].*

Corollary 4.4.2 *For each $k \in \mathbb{N}_0$, as $\mu \to 0^+$ and $\lambda \to \infty$,*

$$\int_{\mathbb{R}} \exp(\varphi(\rho,x,z))(\frac{\rho}{\sinh\rho})^n(-i\rho)^k\frac{\rho}{\tanh\rho}\, d\rho \sim -\pi^{k+1}\mu^{-n}\sqrt{\frac{\pi}{\lambda}}\exp(-\frac{d(x,z)^2}{4}).$$

Proof As in the previous corollary, the integral along $\operatorname{Im}\rho = \frac{3\pi}{2}$ is negligible, therefore

$$\begin{aligned} &\int_{\mathbb{R}} \exp(\varphi(\rho,x,z))(\frac{\rho}{\sinh\rho})^n(-i\rho)^k\frac{\rho}{\tanh\rho}\, d\rho \\ &\qquad \sim 2\pi i \operatorname{Res}[\exp(\varphi(\cdot,x,z))(\frac{\cdot}{\sinh \cdot})^n(-i\cdot)^k\frac{\cdot}{\tanh \cdot}; i\pi]. \end{aligned}$$

Again apply Lemma 4.4.1, this time with $g(\rho) = \frac{\rho}{\tanh\rho}(-i\rho)^k$. Then $\ell = 1$ since $\tanh\rho$ has a pole of order 1 at $i\pi$. This means

$$\begin{aligned} c_{-1} &= \lim_{\rho\to i\pi}(\rho - i\pi)\frac{\rho}{\tanh\rho}(-i\rho)^k \\ &= i\pi^{k+1}. \end{aligned}$$

Inserting this information into Lemma 4.4.1 finishes the proof. ■

We can now produce asymptotics for h and its derivatives. Combining (4.14) with our asymptotics for h_k, and noting that $s = \frac{\lambda}{\pi\mu}$,

$$\begin{aligned} h(r,s) &= \sum_{k=0}^{\frac{m-1}{2}} c_{m,k} s^{k-m+1} h_k(r,s) \\ &= \sum_{k=0}^{\frac{m-1}{2}} c_{m,k} s^{k-m+1} \int_{\operatorname{Im}\rho=\frac{3\pi}{2}} \exp(\varphi(\rho,x,z)) f_k(\rho)\, d\rho \\ &\quad + \pi^m \sqrt{\frac{\pi}{\lambda}} \mu^{1-n} \exp(-\frac{d(x,z)^2}{4}) \\ &\qquad \times \sum_{k=0}^{\frac{m-1}{2}} c_{m,k} (\frac{\mu}{\lambda})^{m-k-1} \{1 + O(\mu) + O(\frac{1}{\sqrt{\lambda}}) + O(\lambda^{-\infty})\} \end{aligned}$$

as $\mu \to 0^+$ and $\lambda \to \infty$. By Lemma 4.4.2, the first summation is negligible compared to the second as s becomes large. In the second summation, the dominating term will be when $m - 1 - k$ is minimized, that is, when $k = \frac{m-1}{2}$. In this case, we arrive at

$$h(r,s) \sim \pi^m (2\pi)^{\frac{m-1}{2}} \sqrt{\frac{\pi}{\lambda}} (\frac{\mu}{\lambda})^{\frac{m-1}{2}} \mu^{1-n} \exp(-\frac{d(x,z)^2}{4}) \tag{4.33}$$

as $s \to \infty$, $\mu \to 0^+$ and $\lambda \to \infty$ since $c_{m,\frac{m-1}{2}} = (2\pi)^{\frac{m-1}{2}}$.

Now let us focus on $\frac{\partial h}{\partial r}$. We recall that

$$\begin{aligned} \frac{\partial h}{\partial r} &= -\frac{r}{2} \sum_{k=0}^{\frac{m-1}{2}} c_{m,k} s^{k-m+1} \int_{\mathbb{R}} \exp(\varphi(\rho,x,z)) (\frac{\rho}{\sinh\rho})^n (-i\rho)^k \frac{\rho}{\tanh\rho}\, d\rho \\ &= -\frac{r}{2} \sum_{k=0}^{\frac{m-1}{2}} c_{m,k} s^{k-m+1} \int_{\operatorname{Im}\rho=\frac{3\pi}{2}} \exp(\varphi(\rho,x,z)) (\frac{\rho}{\sinh\rho})^n (-i\rho)^k \frac{\rho}{\tanh\rho}\, d\rho \\ &\quad - \frac{r}{2} \sum_{k=0}^{\frac{m-1}{2}} c_{m,k} s^{k-m+1} 2\pi i \operatorname{Res}[\exp(\varphi(\cdot,x,z)) (\frac{\cdot}{\sinh\cdot})^n (-i\cdot)^k \frac{\cdot}{\tanh\cdot}; i\pi]. \end{aligned}$$

Again by Lemma 4.4.2, the integral parts in the first summation are negligible. Combining this with Corollary 4.4.2,

$$\begin{aligned}\frac{\partial h}{\partial r} &= O((\frac{\mu}{\lambda})^{-\infty}) + \frac{r}{2}\sum_{k=0}^{\frac{m-1}{2}} c_{m,k}(\frac{\lambda}{\pi\mu})^{k-m+1}\pi^{k+1}\mu^{-n}\sqrt{\frac{\pi}{\lambda}}\exp(-\frac{d(x,z)^2}{4}) \\ &\qquad \times\{1+O(\mu)+O(\frac{1}{\sqrt{\lambda}})+O(\lambda^{-\infty})\} \\ &= O((\frac{\mu}{\lambda})^{-\infty}) + \frac{r}{2}\pi^m\mu^{-n}\sqrt{\frac{\pi}{\lambda}}\exp(-\frac{d(x,z)^2}{4}) \\ &\qquad \times\sum_{k=0}^{\frac{m-1}{2}} c_{m,k}(\frac{\mu}{\lambda})^{m-1-k}\{1+O(\mu)+O(\frac{1}{\sqrt{\lambda}})+O(\lambda^{-\infty})\}.\end{aligned}$$

As before, the dominating term in the summation will be the $k = \frac{m-1}{2}$ term, hence

$$\begin{aligned}\frac{\partial h}{\partial r} &\sim \frac{r}{2}\pi^m(2\pi)^{\frac{m-1}{2}}\mu^{-n}\sqrt{\frac{\pi}{\lambda}}(\frac{\mu}{\lambda})^{\frac{m-1}{2}}\exp(-\frac{d(x,z)^2}{4}) \\ &\sim \frac{r}{2\mu}h.\end{aligned}$$

The second asymptotic follows from (4.33).

Finally, to estimate the $\frac{\partial h}{\partial s}$ term, we can again apply Lemma 4.4.2 to get that the dominating terms are the residues. Hence by Corollary 4.4.1,

$$\begin{aligned}\frac{\partial h}{\partial s} &= \sum_{k=0}^{\frac{m-1}{2}}(k-m+1)c_{m,k}s^{k-m}h_k(r,s) - \sum_{k=0}^{\frac{m-1}{2}} c_{m,k}s^{k-m+1}h_{k+1}(r,s) \\ &= O((\frac{\mu}{\lambda})^{-\infty}) + \pi\mu^{1-n}\exp(-\frac{d(x,z)^2}{4})\sqrt{\frac{\pi}{\lambda}} \\ &\quad \times\Big[\sum_{k=0}^{\frac{m-1}{2}}(k-m+1)c_{m,k}s^{k-m}\pi^k\{1+O(\mu)+O(\frac{1}{\sqrt{\lambda}})+O(\lambda^{-\infty})\} \\ &\quad -\sum_{k=0}^{\frac{m-1}{2}} c_{m,k}s^{k-m+1}\pi^{k+1}\{1+O(\mu)+O(\frac{1}{\sqrt{\lambda}})+O(\lambda^{-\infty})\}\Big] \\ &= O((\frac{\mu}{\lambda})^{-\infty}) + \pi^{m+1}\mu^{-n+1}\exp(-\frac{d(x,z)^2}{4})\sqrt{\frac{\pi}{\lambda}} \\ &\quad \times\Big[\sum_{k=0}^{\frac{m-1}{2}}(k-m+1)c_{m,k}(\frac{\mu}{\lambda})^{m-k}\{1+O(\mu)+O(\frac{1}{\sqrt{\lambda}})+O(\lambda^{-\infty})\} \\ &\quad -\sum_{k=0}^{\frac{m-1}{2}} c_{m,k}(\frac{\mu}{\lambda})^{m-1-k}\{1+O(\mu)+O(\frac{1}{\sqrt{\lambda}})+O(\lambda^{-\infty})\}\Big].\end{aligned}$$

As μ is small and λ large, the leading terms in the summations again come from the case when $k = \frac{m-1}{2}$. However, this time the leading term only occurs in the second summation. Thus,

$$\frac{\partial h}{\partial s} \sim -\pi^{m+1}(2\pi)^{\frac{m-1}{2}}\mu^{1-n}(\frac{\mu}{\lambda})^{\frac{m-1}{2}}\sqrt{\frac{\pi}{\lambda}}\exp(-\frac{d(x,z)^2}{4}). \tag{4.34}$$

We now collect our asymptotics. As $s \to \infty$, $\mu \to 0^+$ and $\lambda \to \infty$,

$$\frac{\partial h}{\partial s} \sim -\pi h \tag{4.35}$$

$$\frac{\partial h}{\partial r} \sim \frac{r}{2\mu}h \tag{4.36}$$

Remark 4.4.2 *If $m = 1$, then since*

$$\frac{\partial h}{\partial r} = \frac{\partial R}{\partial r}\frac{\partial h}{\partial R} = \frac{r}{2}\frac{\partial h}{\partial R},$$

(4.35) *and* (4.36) *recover equations (2.46) in [43].*

Recall, we wish to show that

$$|\nabla_H h|^2 \leq -\frac{\Theta}{2}h(Qh + \mathcal{Z}h),$$

which is equivalent to

$$(\frac{\partial h}{\partial r})^2 + \frac{r^2}{4}(\frac{\partial h}{\partial s})^2 \leq -\frac{\Theta}{2}h(Qh + r\frac{\partial h}{\partial r} + 2s\frac{\partial h}{\partial s}) \tag{4.37}$$

since

$$|\nabla_H h|^2 = |\nabla_x h|^2 + \frac{r^2}{4}|\nabla_H h|^2$$

$$\mathcal{Z}h = \langle x, \nabla_x h\rangle + 2\langle z, \nabla_z h\rangle$$

on H-type groups.

By (4.35) and (4.36), (4.37) is equivalent to asymptotically having

$$\frac{r^2}{4}(\frac{1}{\mu^2} + \pi^2) \leq -\frac{\Theta}{2}(Q + \frac{r^2}{2\mu} - 2s\pi).$$

But $\pi s = \frac{\lambda}{\mu}$ and $\frac{r^2}{4} = \mu\lambda$, so this reduces to

$$\frac{(\mu^2\pi^2 + 1)}{1 - \mu(1 + \frac{Q}{2\lambda})} \leq \Theta. \tag{4.38}$$

In the limit as $\mu \to 0^+$ and $\lambda \to \infty$, the left-hand side of (4.38) converges to 1. As $\Theta > 1$ is fixed, (4.38) obviously holds if $\lambda, \frac{1}{\mu}$ are large enough.

4.4.2 Sub-case 2: $\mu \to 0^+$ and λ bounded

We now assume that λ is bounded (yet still $\mu \to 0^+$). Let us first prove the following two lemmas.

Lemma 4.4.3 *Let n be a non-negative integer. Then for real z,*

$$I_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{z\cos\theta} e^{in\theta}\, d\theta, \tag{4.39}$$

where I_n is the modified Bessel function of the first kind of order n.

Proof We start with (3.6.1) of [63] for integer n:

$$J_n(z) = \frac{1}{2\pi} i^{-n} \int_{-\pi}^{\pi} e^{iz\cos\theta} e^{in\theta}\, d\theta.$$

We have by definition of the modified Bessel function of the first kind that, for integer n,

$$I_n(-z) = i^{-n} J_n(-iz) = \frac{1}{2\pi}(-1)^n \int_{-\pi}^{\pi} e^{z\cos\theta} e^{in\theta}\, d\theta.$$

To conclude, we just note that $I_n(-z) = (-1)^n I_n(z)$ for integer n (this is evident from the series expansion given in section 3.1 of [63]). ∎

Lemma 4.4.4 *Let g be as in Lemma 4.4.1, and F be defined as in* (4.18)*. Then if $\lambda > 0$ is bounded,*

$$2\pi i \operatorname{Res}[F; i\pi] \sim 2\pi^2 \mu^{1-n-\ell} \frac{c_{-\ell}}{(-i\pi)^\ell} I_{n+\ell-1}(2\lambda) \exp(-\pi s - R).$$

Proof Arguing as in Lemma 4.4.1,

$$\begin{aligned} 2\pi i \operatorname{Res}[F; i\pi] &= \pi \exp(-\pi s - R) \\ &\quad \times \int_{-\pi}^{\pi} \exp(\lambda\psi(\zeta,\mu)) \left(\frac{\pi(1-\mu e^{i\zeta})}{\sin(\pi\mu e^{i\zeta})}(g(i\pi(1-\mu e^{i\zeta}))\right)(\mu e^{i\zeta})\, d\zeta. \end{aligned} \tag{4.40}$$

Recall that $\psi(\zeta,\mu) = 2\cos\zeta - \lambda\mu q(\pi\mu e^{i\zeta})$. Because λ is bounded, we may Taylor expand and write

$$\exp(\lambda\psi(\zeta,\mu)) = e^{2\lambda\cos\zeta}\exp(-\lambda\mu q(\pi\mu e^{i\pi})) = e^{2\lambda\cos\zeta}(1+\mu C(\mu,R,\zeta))$$

for some uniformly bounded function C. Also, we have as in Lemma 4.4.1

$$\frac{\pi(1-w)}{\sin\pi w} g(i\pi(1-w))w = w^{-\ell-n+1}(\frac{c_{-\ell}}{(-i\pi)^{\ell}} + wB(w))$$

for a function B holomorphic near 0. Therefore

$$\begin{aligned}
&\int_{-\pi}^{\pi} \exp(2\lambda\cos\zeta - \mu\lambda q(\pi\mu e^{i\zeta}))(\frac{\pi(1-\mu e^{i\zeta})}{\sin\pi\mu e^{i\zeta}})^n g(i\pi(1-\mu e^{i\zeta}))\mu e^{i\zeta}\ d\zeta \\
&\quad = \int_{-\pi}^{\pi} \exp(2\lambda\cos\zeta)(1+rC(\mu,R,\zeta))(\mu e^{i\zeta})^{-\ell-n+1}(\frac{c_{-\ell}}{(-i\pi)^{\ell}} + \mu e^{i\zeta}B(\mu e^{i\zeta}))\ d\zeta \\
&\quad = \mu^{-\ell-n+1}\frac{c_{-\ell}}{(-i\pi)^{\ell}}\int_{-\pi}^{\pi}\exp(2\lambda\cos\zeta)e^{i(1-n-\ell)\zeta}(1+rD(\mu,R,\zeta))\ d\zeta
\end{aligned}$$

for a suitable function D which is uniformly bounded for $\mu \ll 1$, $R > 0$, and $\zeta \in [-\pi,\pi]$. By this and Lemma 4.4.3, we conclude that

$$\begin{aligned}
&\int_{-\pi}^{\pi} \exp(2\lambda\cos\zeta - \mu\lambda q(\pi\mu e^{i\zeta}))(\frac{\pi(1-\mu e^{i\zeta})}{\sin\pi\mu e^{i\zeta}})^n g(i\pi(1-\mu e^{i\zeta}))\mu e^{i\zeta}\ d\zeta \\
&\qquad\qquad = 2\pi\mu^{1-n-\ell}\frac{c_{-\ell}}{(-i\pi)^{\ell}} I_{n+\ell-1}(2\lambda)(1+O(\mu))
\end{aligned}$$

as $\mu \to 0$. Combining this with (4.40) finishes the proof. ∎

Remark 4.4.3 *It would be nice to have a geometric interpretation of the asymptotic in Lemma 4.4.4 as in Step 1 and Step 2, Case 1, Sub-case 1. However at this time the author is not aware of a way to do this.*

Corollary 4.4.3 *For $\lambda > 0$, we have the following asymptotic relations.*

$$\begin{aligned}
h &\sim 2\pi^{m+1}(2\pi)^{\frac{m-1}{2}}\mu^{1-n}(\frac{\mu}{\lambda})^{\frac{m-1}{2}} I_{n-1}(2\lambda)\exp(-\pi s - R) \\
\frac{\partial h}{\partial s} &\sim -2\pi^{m+2}(2\pi)^{\frac{m-1}{2}}\mu^{1-n}(\frac{\mu}{\lambda})^{\frac{m-1}{2}} I_{n-1}(2\lambda)\exp(-\pi s - R) \\
\frac{\partial h}{\partial r} &\sim r\pi^{m+1}(2\pi)^{\frac{m-1}{2}}(\frac{\mu}{\lambda})^{\frac{m-1}{2}}\mu^{-n} I_n(2\lambda)\exp(-\pi s - R).
\end{aligned}$$

Proof Each of the relations is an application of Lemmas 4.4.2 and 4.4.4. For h, we use the functions $g(\rho) = (-i\rho)^k$ for $k = 0, 1, 2, \ldots, \frac{m-1}{2}$ to obtain asymptotics for each h_k:

$$h_k(r, s) \sim 2\pi^{2+k}\mu^{1-n}I_{n-1}(2\lambda)\exp(-\pi s - R).$$

Then, in the expression (4.14), the dominating term is again when $k = \frac{m-1}{2}$ since s is large. Hence

$$\begin{aligned} h(r, s) &\sim 2\pi^{2+\frac{m-1}{2}}(2\pi)^{\frac{m-1}{2}}\mu^{1-n}s^{-\frac{m-1}{2}}I_{n-1}(2\lambda)\exp(-\pi s - R) \\ &= 2\pi^{m+1}(2\pi)^{\frac{m-1}{2}}\mu^{1-n}(\frac{\mu}{\lambda})^{\frac{m-1}{2}}I_{n-1}(2\lambda)\exp(-\pi s - R) \end{aligned}$$

as $\mu \to 0^+$. As before, we've used $c_{m,\frac{m-1}{2}} = (2\pi)^{\frac{m-1}{2}}$.

Finding the asymptotics for $\frac{\partial h}{\partial s}$ is very similar, although as in sub-case 1, the dominating term only happens in the second summation of (4.17).

Finally, for the asymptotics of $\frac{\partial h}{\partial r}$, we use now the functions $g = g_k$,

$$g_k(\rho) = (-i\rho)^k\frac{\rho}{\tanh\rho}, \qquad k = 0, 1, \ldots, \frac{m-1}{2}.$$

Each has a simple pole at $i\pi$. Here, $c_{-1} = i\pi^{1+k}$. Once again, the $k = \frac{m-1}{2}$ term dominates, hence from (4.16) and Lemma 4.4.4,

$$\begin{aligned} \frac{\partial h}{\partial r} &\sim -\frac{r}{2}(2\pi)^{\frac{m-1}{2}}s^{-\frac{m-1}{2}}[2\pi^2\mu^{-n}\frac{c_{-1}}{(-i\pi)}I_n(2\lambda)\exp(-\pi s - R)] \\ &= -\frac{r}{2}(2\pi)^{\frac{m-1}{2}}(\frac{\lambda}{\pi\mu})^{-\frac{m-1}{2}}[2\pi^2\mu^{-n}\frac{i\pi^{1+\frac{m-1}{2}}}{(-i\pi)}I_n(2\lambda)\exp(-\pi s - R)] \\ &= r\pi^{m+1}(2\pi)^{\frac{m-1}{2}}(\frac{\mu}{\lambda})^{\frac{m-1}{2}}\mu^{-n}I_n(2\lambda)\exp(-\pi s - R) \end{aligned}$$

■

As a consequence, we arrive at

$$\frac{\partial h}{\partial s} \sim -\pi h \tag{4.41}$$

$$\frac{\partial h}{\partial r} \sim \frac{r}{2\mu}(\frac{I_n(2\lambda)}{I_{n-1}(2\lambda)})h \tag{4.42}$$

We should emphasize that the second asymptotic relation relies crucially on the fact that $I_{n-1}(2\lambda)$ has no zeroes since R, hence λ, is taken to be strictly positive.

Now returning to (4.6), our desired estimate requires that

$$\frac{r^2}{4r^2}((\frac{I_n(2\lambda)}{I_{n-1}(2\lambda)})^2 + \mu\pi^2) \le -\frac{\Theta}{2}(Q + \frac{r^2}{2\mu}(\frac{I_n(2\lambda)}{I_{n-1}(2\lambda)}) - 2\pi s)$$

as $\mu \to 0^+$ and $s \to \infty$. This rearranges into having

$$\frac{(\frac{I_n(2\lambda)}{I_{n-1}(2\lambda)})^2 + \mu^2\pi^2}{1 - \frac{Q\mu}{2\lambda} - \mu(\frac{I_n(2\lambda)}{I_{n-1}(2\lambda)})} \le \Theta. \tag{4.43}$$

Taking into account the bound

$$\frac{I_n(2\lambda)}{I_{n-1}(2\lambda)} < 1, \quad \lambda > 0,$$

see (3.16.3) of [63], (4.43) would hold if we had

$$\frac{1 + \mu^2\pi^2}{1 - \frac{Q\mu}{2\lambda} - \mu} \le \Theta. \tag{4.44}$$

Finally, recalling that $\frac{\mu}{\lambda} = \frac{1}{\pi s}$, it is obvious that, given $\Theta > 1$, (4.44) holds provided μ is small enough and s is large enough.

Remark 4.4.4 *It is interesting to point out that one can compare the asymptotics in sub-case 3 to those for sub-case 2. Due to the asymptotic behavior [1],*

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}, \qquad z \in \mathbb{R}, z \to \infty,$$

it follows that

$$\frac{I_n(2\lambda)}{I_{n-1}(2\lambda)} \sim 1$$

as $\lambda \to \infty$. This allows us to compare (4.42) *with* (4.36)*. However, in deriving* (4.42) *we use the boundedness of λ, whereas* (4.36) *uses Theorem 4.2.1 which specifically requires λ to be large. Therefore this comparison is only formal.*

4.4.3 Sub-case 3: $x = 0$

This case is perhaps the simplest of them all. We are led to estimate

$$\begin{aligned}2\pi i \operatorname{Res}[\exp(i \cdot s)(\frac{\cdot}{\sinh \cdot})^n(-i\cdot)^k; i\pi] \\ = -2\pi^2(-1)^n e^{-\pi s} \operatorname{Res}[\exp(-\pi \cdot s)(\frac{\pi(\cdot + 1)}{\sin \pi \cdot})^n(\pi(\cdot + 1))^k; 0]\end{aligned}$$

for $k \in \mathbb{N}_0$. The estimate for these is given in the following lemma.

Lemma 4.4.5 *Let $k \in \mathbb{N}_0$. Then*

$$2\pi i \operatorname{Res}_{i\pi}[\exp(i \cdot s)(\frac{\cdot}{\sinh \cdot})^n(-i\cdot)^k] = 2\pi^{n+k+1} e^{-\pi s} \frac{1}{(n-1)!} s^{n-1} \left\{1 + O(\frac{1}{s})\right\}$$

as $s \to \infty$. In particular,

$$h_k(0, z) \sim 2\pi^{n+k+1} e^{-\pi s} \frac{1}{(n-1)!} s^{n-1}$$

as $z \to \infty$.

Proof Write

$$(\frac{\pi(\zeta + 1)}{\sin \pi\zeta})^n(\pi(\zeta + 1))^k = \sum_{j=-n}^{\infty} c_j \zeta^j = \zeta^{-n} \sum_{j=0}^{\infty} c_{j-n} \zeta^j$$

as a Laurent series expansion about 0. Then since

$$\exp(-\pi s\zeta) = \sum_{\ell=0}^{\infty} \frac{(-\pi s)^\ell}{\ell!} \zeta^\ell,$$

it follows from the formula for multiplication of power series that

$$\begin{aligned}\exp(-\pi\zeta s)(\frac{\pi(\zeta + 1)}{\sin \pi\zeta})^n(\pi(\zeta + 1))^k &= \zeta^{-n}(\sum_{j=0}^{\infty} c_{j-n}\zeta^j)(\sum_{\ell=0}^{\infty} \frac{(-\pi s)^\ell}{\ell!} z^\ell) \\ &= \sum_{m=0}^{\infty}(\sum_{j=0}^{m} \frac{(-\pi s)^j}{j!} c_{m-j-n})\zeta^{m-n}\end{aligned}$$

The residue term occurs whenever $m = n - 1$. Therefore

$$\begin{aligned}2\pi i \operatorname{Res}_{i\pi}[\exp(i\zeta s)(\frac{\zeta}{\sinh \zeta})^n(-i\zeta)^k] &= -2\pi^2(-1)^n e^{-\pi s} \sum_{j=0}^{n-1} \frac{(-\pi)^i}{j!} c_{-1-j} s^j \\ &= 2\pi^{n+1} e^{-\pi s} \frac{1}{(n-1)!} c_{-n} s^{n-1}(1 + O(\frac{1}{s}))\end{aligned}$$

as $s \to \infty$. To finish, we just note that $c_{-n} = \pi^k$. ∎

Corollary 4.4.4 *As $s \to \infty$,*

$$h(0,z) \sim 2\pi^{n+\frac{m+1}{2}}(2\pi)^{\frac{m-1}{2}}e^{-\pi s}\frac{1}{(n-1)!}s^{n-1-\frac{m-1}{2}} \tag{4.45}$$

$$\frac{\partial h}{\partial s}(0,z) \sim -2\pi^{n+\frac{m+1}{2}+1}(2\pi)^{\frac{m-1}{2}}e^{-\pi s}\frac{1}{(n-1)!}s^{n-1-\frac{m-1}{2}}. \tag{4.46}$$

In particular,

$$\frac{\partial h}{\partial s}(0,s) \sim -\pi h(0,s). \tag{4.47}$$

Proof Once again, we use (4.14), (4.17), and Lemma 4.4.2, together now with Lemma 4.4.5. In both (4.14) and (4.17), the leading term still occurs when $k = \frac{m-1}{2}$ since s is large. In (4.17), the largest power of s occurs in the second summation, hence the negative and extra factor of π. In both (4.45) and (4.46), we have as before used $c_{m,\frac{m-1}{2}} = (2\pi)^{\frac{m-1}{2}}$. ■

Remark 4.4.5 *To mimic Step 1 and sub-case 1, we can put more geometric content in Corollary 4.4.4 by noting that $\frac{d(0,z)^2}{4} = \pi|z|$. Hence we can write* (4.45) *and* (4.46) *as*

$$h(0,z) \sim 2\pi^{n+\frac{m+1}{2}}(2\pi)^{\frac{m-1}{2}}\frac{1}{(n-1)!}s^{n-1-\frac{m-1}{2}}e^{-\frac{d(0,z)^2}{4}}$$

$$\frac{\partial h}{\partial s}(0,z) \sim -2\pi^{n+\frac{m+1}{2}+1}(2\pi)^{\frac{m-1}{2}}\frac{1}{(n-1)!}s^{n-1-\frac{m-1}{2}}e^{-\frac{d(0,z)^2}{4}}$$

Proof of Theorem 4.1.1 when $x = 0$ and $\nu(\theta)$ is unbounded Since $x = 0$, $|\nabla_H h| = 0$. Thus if we wish (4.6) to hold, we need

$$0 \le -\frac{\Theta}{2}(Q - 2s\pi),$$

which is true whenever $s > \frac{Q}{2\pi}$, regardless of $\Theta > 1$. ■

4.5 Step 2: $\nu(\theta)$ unbounded, m even

Having established Theorem 4.1.1 for H-type groups with odd (topological) dimension, we turn our attention to the even case. Our specific point of injection will be the asymptotic relations (4.35)–(4.36) (sub-case 1), (4.41)–(4.42) (sub-case 2), and (4.47) (sub-case 3). As soon as we show that these remain valid for m even, the rest of the argument remains the same.

Before continuing, we employ the following notation. We assume that m is even throughout. Given $z \in \mathbb{R}^{m+1}$, we write $z = (z', z_{m+1})$ where $z' \in \mathbb{R}^m$. Similarly, we write

$$\lambda = \sqrt{\frac{\pi|x|^2|z|}{4}}, \quad \lambda' = \sqrt{\frac{\pi|x|^2|z'|}{4}}$$
$$\mu = \sqrt{\frac{|x|^2}{4\pi|z|}}, \quad \mu' = \sqrt{\frac{|x|^2}{4\pi|z'|}}$$

i.e. λ (respectively, μ) is the same as defined in Step 2, Case 1, and λ' (respectively μ') is the corresponding quantity of one fewer vertical dimension.

Remark 4.5.1 *Obviously, $\frac{1}{\lambda} \leq \frac{1}{\lambda'}$ (respectively $\mu \leq \mu'$), thus a function which is $O(\frac{1}{\lambda})$ (respectively $O(\mu)$) is automatically $O(\frac{1}{\lambda'})$ (respectively $O(\mu')$). We also note that $\lambda \sim \lambda'$, $\mu \sim \mu'$ for fixed z_{m+1} as $|z'| \to \infty$.*

The method we employ is based off of observations in section 7 of [30]. Specifically, let $h_{n,m}$ denote the function h which is a multiple of p_1 on $\mathbb{R}^n \times \mathbb{R}^m$, that is,

$$h_{n,m}(x,z) = \int_{\mathbb{R}^m} \exp[i\langle\lambda, z\rangle - \frac{|x|^2}{4}\frac{|\lambda|}{\tanh|\lambda|}](\frac{|\lambda|}{\sinh|\lambda|})^n \, d\lambda, \quad x \in \mathbb{R}^n, z \in \mathbb{R}^m.$$

Then since $\mathscr{F}\{1\} = 2\pi\delta_0$ in the sense of tempered distributions, we have by Fubini's theorem

$$h_{m,n}(x,z) = \frac{1}{2\pi}\int_{\mathbb{R}} h_{n,m+1}(x,(z,z_{m+1})) \, dz_{m+1}. \tag{4.48}$$

Proposition 4.5.1 *The results of Step 2 also holds for H-type groups with even-dimensional center.*

Proof Assume that m is even. The arguments in Section 4.3 are valid if $\frac{4|z|}{|x|^2}$ is bounded, so assume that $\frac{4|z|}{|x|^2}$ is unbounded. Then as $|z'| \to \infty$,

$$\begin{aligned}
\frac{\partial h_{n,m}}{\partial |z'|} &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\partial h_{n,m+1}}{\partial |z'|} \, dz_{m+1} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\partial h_{n,m+1}}{\partial |z|} \frac{\partial |z|}{\partial |z'|} \, dz_{m+1} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\partial h_{n,m+1}}{\partial |z|} \frac{|z'|}{|z|} \, dz_{m+1} \\
&\sim -\frac{1}{2} \int_{\mathbb{R}} h_{n,m+1}(x, (z, z_{m+1})) \frac{|z'|}{|z|} \, dz_{m+1}
\end{aligned}$$

where we have used (4.35), (4.41), or (4.47) – whichever is appropriate – in the final line. The use of $\sim$ is justified by (4.48) and Remark 4.5.1. For fixed $z_m \in \mathbb{R}$, $|z| \sim |z'|$ as $|z'|$ becomes large. Thus by the dominated convergence theorem,

$$\begin{aligned}
\frac{\partial h_{n,m}}{\partial |z'|} &\sim -\frac{1}{2} \int_{\mathbb{R}} h_{n,m+1}(x, z) \, dz_{m+1} \\
&= -\pi h_{n,m}(x, z')
\end{aligned}$$

as $|z'| \to \infty$.

For $\frac{\partial h_{n,m}}{\partial |x|}$, we split into two cases, one where $\lambda' \to \infty$ and one where λ' is bounded. In the former case, for $|z'| \to \infty$ and the dominated convergence theorem

$$\begin{aligned}
\frac{\partial h_{n,m}}{\partial |x|} &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\partial h_{n,m+1}}{\partial |x|} \, dz_{m+1} \\
&\sim \frac{1}{2\pi} \int_{\mathbb{R}} \frac{|x|}{2\mu} h_{n,m+1}(x, z) \, dz_{m+1} \\
&\sim \frac{1}{2\pi} \int_{\mathbb{R}} \frac{|x|}{2\mu'} h_{n,m+1}(x, z) \, dz_{m+1} \\
&= \frac{|x|}{2\mu'} h_{n,m}(x, z')
\end{aligned}$$

where in the second line we have used (4.36), and the third the fact that $\mu \sim \mu'$ as $|z'| \to \infty$. In the case where λ' is bounded, we instead get

$$\begin{aligned}
\frac{\partial h_{n,m}}{\partial |x|} &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\partial h_{n,m+1}}{\partial |x|}\, dz_{m+1} \\
&\sim \frac{1}{2\pi}\int_{\mathbb{R}} \frac{|x|}{2\mu} h_{n,m+1}(x,z)\frac{I_n(2\lambda)}{I_{n-1}(2\lambda)}\, dz_{m+1} \\
&\sim \frac{1}{2\pi}\int_{\mathbb{R}} \frac{|x|}{2\mu'} h_{n,m+1}(x,z)\frac{I_n(2\lambda')}{I_{n-1}(2\lambda')}\, dz_{m+1} \\
&= \frac{|x|}{2\mu'}\frac{I_n(2\lambda')}{I_{n-1}(2\lambda')} h_{n,m}(x,z').
\end{aligned}$$

Here, we have used (4.42) in the second line, $\mu \sim \mu'$, $\lambda \sim \lambda'$ as $|z'| \to \infty$, together with the fact that the quotient $\frac{I_n}{I_{n-1}}$ is continuous and bounded for positive arguments.

■

4.6 Strong Harnack inequality and Wiener's criterion for H-type groups

As an application of Theorem 4.1.1, we can now extend two theorems to groups of Heisenberg-type. Since this theory is well-known for the elliptic case, we take an expository approach and refer the reader to the relevant proofs when appropriate.

First, we need some notation. For brevity, we denote the heat operator $\mathcal{H}_H = \Delta_H - \partial_t$. Solutions to $\mathcal{H}_H u = 0$ are called *caloric* functions. Throughout, we let $G \subset \mathbb{G} \times \mathbb{R}$ be an open bounded set.

Definition 4.6.1 *The Dirichlet problem for G and $\mathcal{H}_H$ is the following: given $\varphi : \partial G \to \mathbb{R}$ continuous, find a u which is caloric in G and such that $u|_{\partial G} = \varphi$.*

Given $\varphi \in C(\partial G)$, such u always exists in the sense of Perron-Wiener-Brelot-Bauer, and is defined as follows (see [11], [31]):

$$H_\varphi^G(g,t) = \inf_K \{v(g,t)\}$$

where K is the set of so-called *supertemperatures* v satisfying

$$\liminf_{G \ni (g,t) \to (g_0,t_0)} v(g,t) \geq \varphi(g_0,t_0) \text{ for every } (g_0,t_0) \in \partial G.$$

Furthermore, $u = H^G_\varphi$ is smooth in G. However, the function u may not be continuous up to the boundary.

Definition 4.6.2 *We say that* $(g_0, t_0) \in \partial G$ *is* $\mathcal{H}_H$*-regular if for every* $\varphi \in C(\partial G)$*, the Perron-Wiener-Brelot-Bauer solution* H^G_φ *always continuously takes up the values of* φ*, that is,*

$$\lim_{G \ni (g,t) \to (g_0,t_0)} H^G_\varphi(g,t) = \varphi(g_0, t_0).$$

Definition 4.6.3 *The* heat ball (or $\mathcal{H}_H$-ball) centered at $(e, 0) \in \mathbb{G} \times (0, \infty)$ and of radius r *is the set* Ω_r *defined by*

$$\Omega_r = \{(g,t) \in \mathbb{G} \times (-\infty, 0) : p_t(g) > (4\pi r)^{-Q/2}\}.$$

The corresponding heat sphere (or $\mathcal{H}_H$-sphere) centered at $(e, 0)$ of radius r *is the boundary of* Ω_r *together with the point* $(e, 0)$*, that is,*

$$\partial\Omega_r = \{(g,t) \in \mathbb{G} \times (-\infty, 0) : p_{-t}(g) > (4\pi r)^{-Q/2}\} \cup \{(e, 0)\}.$$

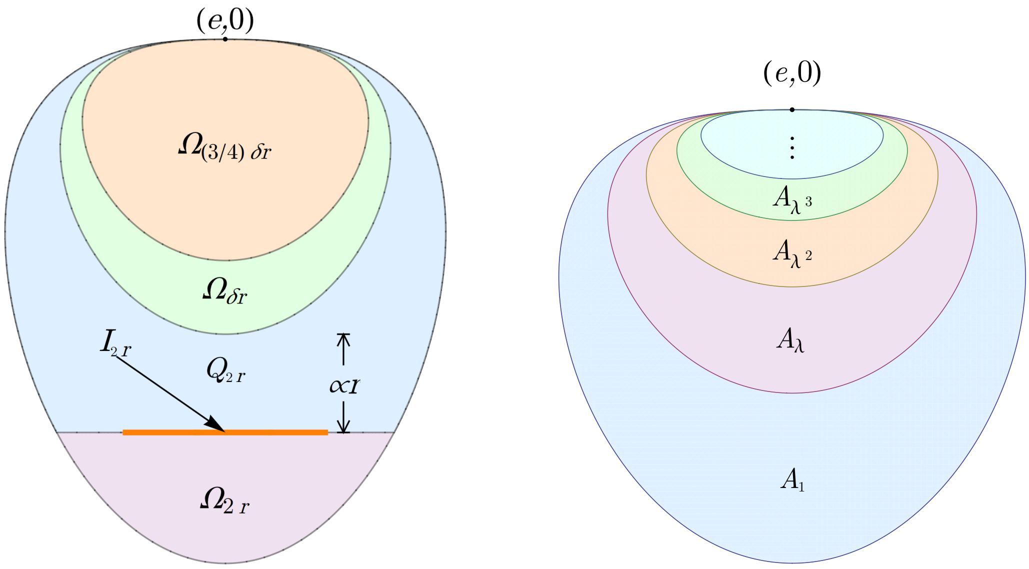
Remark 4.6.1 *It is also possible to define the* $\mathcal{H}_H$*-ball and* $\mathcal{H}_H$*-sphere centered at any point* $(g,t) \in \mathbb{G} \times \mathbb{R}$ *by using the group's left-translation and the usual translation in* $\mathbb{R}$ *for the time component.*

Remark 4.6.2 *By the two-sided Gaussian estimates in* (1.23)*, the heat-balls for* $\mathbb{G}$ *are always trapped between two Gaussian heat-balls.*

Assume that three positive constants β, γ, δ depending on the homogeneous dimension have been fixed in such a way that if we define

$$Q_r = \{(g,t) \in \Omega_r : t > -\frac{\gamma r}{2}\}$$
$$I_r = \{(g,t) \in \Omega_r : \sigma = -\frac{\gamma r}{2}, \rho(g,t)^2 \leq \frac{\beta r}{2}\}$$

then $\Omega_{\delta r} \subset Q_{2r}$, and there exists a time gap between $\Omega_{\delta r}$ and I_{2r} proportional to r. The situation is illustrated in Figure 4.3(a). We have the following strong version of the Harnack inequality:



(a) Arrangement of heat balls in Theorem 4.6.1. (b) The A_{λ^k} rings.

Figure 4.3. The vertical dimension is time, the horizontal is $g \in \mathbb{G}$.

Theorem 4.6.1 *There exists positive Λ depending only on the homogeneous dimension of $\mathbb{G}$ such that if $u \in C(\overline{Q_{2r}} \setminus \{(0,0)\})$ is a positive caloric function in Q_{2r}, then*

$$\fint_{I_{2r}} u(g, -\gamma r)\, dg \le \Lambda \inf_{\Omega_{(3/4)\delta r}} u. \tag{4.49}$$

The proof of Theorem 4.6.1 follows in the same way as [43] to which we refer the interested reader. See also [31] and [56]. The key step – which is most relevant to the current discussion and follows from Theorem 4.1.1 – is the construction of a specific comparison function $v : \mathbb{G} \times (-\infty, 0) \to \mathbb{R}$ which is supercaloric in the set Ω_δ and below a space-time paraboloid.

More specifically, the following lemma is key.

Lemma 4.6.1 *Define for $x \ge 0$ and $(g,t) \in \Omega_\delta$*

$$\psi(x) = \tan^{-1}(x + 16) - \tan^{-1}(16)$$
$$v(g,t) = \psi \circ \ln\left[(4\pi\delta)^{Q/2} p_{-t}(g))\right].$$

Then there exists a constant $\lambda_0 > 0$ *such that* $\mathcal{H}_H v(g,t) \geq 0$ *whenever* $-\lambda_0 \rho(g)^2 \leq t < 0$ *and* $(g,t) \in \Omega_\delta$.

Proof In what follows, ψ, ψ', and ψ'' are always evaluated at $\ln((4\pi\delta)^{Q/2} p_{-t}(g))$. Obviously,

$$\partial_t v = \psi' \partial_t \ln p_{-t} \tag{4.50}$$

whereas if $f : \mathbb{R} \to \mathbb{R}$ is C^2, then due to the formula

$$\Delta_H f(u) = f'(u)\Delta_H u + f''(u)|\nabla_H u|^2,$$

we have

$$\begin{aligned}
\Delta_H v &= \psi' \Delta_H \ln(p_{-t}) + \psi'' |\nabla_H \ln(p_{-t})|^2 \\
&= \psi' \left(\frac{\Delta_H p_{-t}}{p_{-t}} - \frac{|\nabla_H p_{-t}|^2}{p_{-t}^2}\right) + \psi'' \frac{|\nabla_H p_{-t}|^2}{p_{-t}^2} \\
&= -\psi' \frac{\partial_t p_{-t}}{p_{-t}} + (\psi'' - \psi') \frac{|\nabla_H p_{-t}|^2}{p_{-t}^2} \qquad (4.51) \\
&= -\psi' \partial_t \ln p_{-t} + (\psi'' - \psi')|\nabla_H \ln p_{-t}|^2. \qquad (4.52)
\end{aligned}$$

Note that in (4.51), we've used the fact that p_t solves the heat equation on $\mathbb{G} \times (0, \infty)$, therefore p_{-t} solves the *backward* heat equation on $\mathbb{G} \times (-\infty, 0)$. Combining (4.50), (4.52) with the definition of the heat operator $\mathcal{H}_H$, we find

$$\mathcal{H}_H v = (\psi'' - \psi')|\nabla_H \ln p_{-t}|^2 - 2\psi' \partial_t \ln p_{-t}. \tag{4.53}$$

Note that $\psi'' - \psi' < 0$ for every $x \in \mathbb{R}$. Thus $\mathcal{H}_H v \geq 0$ if and only if

$$|\nabla_H \ln p_{-t}|^2 \leq \frac{2\psi'}{\psi'' - \psi'} \partial_t \ln p_{-t}. \tag{4.54}$$

Using the substitution $\tau = -t > 0$, this reads

$$|\nabla_H \ln p_\tau|^2 \leq \frac{2\psi'}{\psi' - \psi''} \partial_\tau \ln p_\tau. \tag{4.55}$$

By virtue of the fact that we are restricting attention to $(g,t) \in \Omega_\delta$, the argument inside of ψ', ψ'' satisfies $\ln[(4\pi\delta)^{Q/2} p_{-t}] \geq 0$. Furthermore,

$$\inf_{x \geq 0} \frac{2\psi'(x)}{\psi'(x) - \psi''(x)} = \frac{2\psi'(0)}{\psi'(0) - \psi''(0)} = \frac{514}{298} =: \Theta > 1. \tag{4.56}$$

By Theorem 4.1.1 and (4.56), there exists $\lambda_0 > 0$ such that (4.55) holds when $0 < \tau = -t \leq \lambda_0 \rho(g)^2$ and $(g,t) \in \Omega_\delta$, hence so do (4.54) and (4.53). ■

Before stating the Wiener criterion, we need two more definitions. The first is simply a set definition, see Figure 4.3(b), whereas the second is a parabolic version of a quantity often found in potential theory.

Definition 4.6.4 *Fix $\lambda \in (0,1)$. We define the ring-like sets A_{λ^k}, $k \in \mathbb{N}$ by the formula*

$$A_{\lambda^k} = \overline{\Omega_{\lambda^k}} \setminus \Omega_{\lambda^{k+1}}.$$

Definition 4.6.5 *Let $F \subset \mathbb{G} \times \mathbb{R}$ be a closed set, and $M^+(F)$ the set of non-negative Radon measures having support in F. Given $\mu \in M^+(\mathbb{G} \times \mathbb{R})$, the* heat-potential of *μ, $\Gamma_\mu : \mathbb{G} \times \mathbb{R} \to [0,\infty)$, is defined by the integral*

$$\Gamma_\mu(g,t) = \int_{\mathbb{G} \times (-\infty,t)} p_{t-t'}(g' \circ g^{-1}) \, d\mu(g',t').$$

The heat capacity *(or* thermal capacity*) of F is then given by*

$$\mathrm{cap}_{\mathcal{H}_H}(F) = \sup\{\mu(\mathbb{G} \times \mathbb{R}) : \mu \in M^+(F), \Gamma_\mu \leq 1\}.$$

The definition of heat capacity is in direct analogy to the classical Wiener capacity in $\mathbb{R}^n$ (see, for example [58], Section 4). Let $G : \mathbb{R}^n \times \mathbb{R}^n \to (0,\infty]$ denote the Green function for an elliptic operator and set

$$\Gamma_\mu(x) = \int_{\mathbb{R}^n} G(x,y) \, d\mu(y)$$

when $\mu \in M^+(\mathbb{R}^n)$. The Wiener capacity of a closed set is then defined by

$$\mathrm{cap}(F) = \sup\{\mu(\mathbb{R}^n) : \mu \in M^+(F), \Gamma_\mu \leq 1\}.$$

Capacity offers a way to "measure" subsets of $\mathbb{G} \times \mathbb{R}$ which are in some sense small – such as the ring-like sets A_{λ^k} when k becomes large. However, it should be noted that the heat capacity is only an outer measure, not a true measure.

We can now state the Wiener criterion for H-type groups.

Theorem 4.6.2 *Let $G \subset \mathbb{G} \times \mathbb{R}$ be open and bounded and $(e, 0) \in \partial G$. The point $(e, 0)$ is $\mathcal{H}_H$-regular if and only if for every $0 < \lambda < 1$ the series*

$$\sum_{k=1}^{\infty} \lambda^{-kQ/2} \operatorname{cap}_{\mathcal{H}_H}(G^c \cap A_{\lambda^k})$$

is divergent.

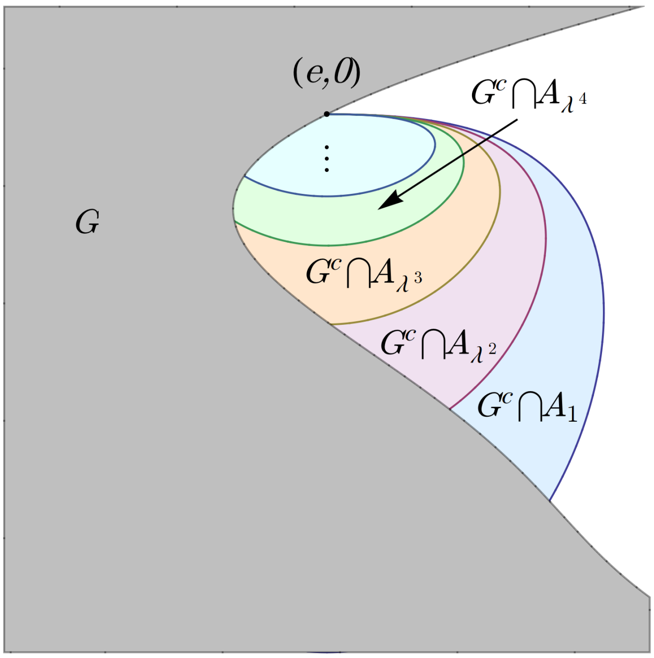


Figure 4.4. The Wiener criterion.

The proof of Theorem 4.6.2 follows mutatis mutandis from the proof given in [43]. For the details of the proof, we refer the interested reader to said paper. Instead, we outline the main ideas present in the proof of Theorem 4.6.2.

Key to the proof is that the heat capacity behaves in many ways similar to the elliptic 2-capacity, the properties of which are well-known and can be found in [32].

These are, in particular: the ability to construct a heat equilibrium measure V_K for any compact set K; $\mathrm{cap}_{\mathcal{H}_H}$ is monotonic with respect to set inclusion; and $\mathrm{cap}_{\mathcal{H}_H}$ is homogeneous of degree Q with respect to the space-time dilations $\tilde{\delta_\lambda}$ on $\mathbb{G} \times \mathbb{R}$. This latter property is due to the aforementioned $-Q$-homogeneity of the heat kernel p_t for H-type groups.

Also important to the proof is the following equivalences to regularity which simplifies the geometry greatly. Suppose that we have fixed a constant M depending only on the homogeneous dimension. More precisely, M comes from two-sided Gaussian bounds on p_t with respect to the Korányi gauge, see (1.23) and the comments following it. We define the space-time cylinders

$$C_r = \left\{ (g,t) \in \mathbb{G} \times \mathbb{R} : \rho(g)^2 \leq \frac{4\pi Q M^{1+2/Q}}{2e} r, t \in (-4\pi M^{2/Q}, 0) \right\}.$$

Both of the conditions in Theorem 4.6.2 are equivalent to the following two statements:

- $(e,0)$ is regular for G if and only if there exists $r > 0$ such that $V_{C_r \setminus G}(e,0) = 1$.
- $(e,0)$ is regular for G if and only if for every $r, c > 0$, $(e,0)$ is regular for the set $G \cup \mathring{C}_r \cap (\Omega_r^c \setminus \partial \Omega_r)$

For the first equivalence, one may see [11], [19], or [57]. For the second, we refer to [38].

5. Summary, Conclusions, and Future Work

The extension of Almgren's frequency monotonicity in Chapter 2 has given an interesting measure of the non-commutativity of step-r Carnot groups, $r \geq 2$, that is, the discrepancy vector field in Definition 2.4.1. The growth properties of the discrepancy of a function u are tied to the strong unique continuation property in such setting. Whether discrepancy appears in other contexts remains to be seen.

One may ask the question of whether Struwe's and Poon's monotonicity formulas of Chapter 3 hold on Carnot groups if u has vanishing discrepancy. At this point, there is no clear answer to this question. The main difficulty lies in the fact that the heat kernel is not Gaussian with respect to the H-gauge ρ as is the case in $\mathbb{R}^n$, whereas discrepancy is a quantity whose very definition depends on the H-gauge. Although Section 3.7.3 gives an extension of Struwe's energy monotonicity for a subclass of functions having vanishing discrepancy on H-type groups, the proof given relies crucially on the fact that $\Gamma_2(u) \geq 0$ for such functions, something which is not true for discrepancy-free functions in general. Moreover, this proof does not in an obvious way give Poon's monotonicity.

A result which is somewhat "close" to that of Poon and Struwe and which takes into account discrepancy is the following. Assume that u solves the degenerate parabolic PDE

$$\Delta_H u = |\nabla_H \rho|^2 u_t \text{ on } \mathbb{G} \times (0, T) \tag{5.1}$$

and define for $(g, t) \in \mathbb{G} \times (0, T)$

$$f_g(t) = (T - t) \int_{\mathbb{G}} |\nabla_H u|^2 (g') \Phi(g^{-1} \circ g', T - t) \; dg'$$
$$h_g(t) = \frac{f_g(t)}{\int_{\mathbb{G}} u(g')^2 \Phi(g^{-1} \circ g', T - t) \; dg'}$$

where

$$\Phi(g,t) = (4\pi t)^{-Q/2}\exp(-\frac{\rho(g)^2}{4t})$$

is a modification of the Euclidean heat kernel. If $\mathbb{G}$ is a *polarizable* Carnot group in the sense of Z. Balogh and J. Tyson (see [6]), in other words if the ∞-sub-Laplacian of the H-gauge ρ satisfies

$$\Delta_{H,\infty}\rho = \sum_{i,j=1}^{\dim V_1} X_j\rho X_i\rho X_i X_j\rho = 0,$$

and in addition u has vanishing discrepancy at the origin, then f_e and h_e are both monotonically non-decreasing functions. All H-type groups are polarizable, but these are the only known examples of polarizable Carnot groups. Balogh and Tyson have produced an example of a non-polarizable Carnot group. This result is dissatisfying as the PDE (5.1) has no appearance or application in the literature (outside of the case $\mathbb{R}^n$ in which it reduces after a time-rescaling to the usual heat equation $\Delta u = u_t$).

Another interesting question is whether $CD(\rho, n)$ implies a Hessian inequality $C(\omega)$. Attempts at proving such a result have been fruitless. In [47], Hamilton shows that a certain bilinear form is a heat super-solution, that is,

$$(\partial_t - \Delta)N \geq 0, \qquad N = \mathrm{Hess}\ln p_t + \frac{1}{2t}g$$

in the sense of matrices, then uses the maximum principle to conclude. As no metric is available for $CD(\rho, n)$ and the Hessian is defined by its action on functions rather than vector fields, one instead is led to consider the quantity

$$(\partial_t - \mathcal{L})N, \qquad N = H_{\ln p_t}(u,u) + \frac{1}{2}\omega'(t)\Gamma(u), \tag{5.2}$$

where u is a smooth function and ω' is to be determined. In (5.2), there are many terms that are seemingly uncontrollable by just $CD(\rho, n)$, for example,

$$\Gamma(u, \Gamma_2(u, \ln p_t)).$$

It may be that the Hamilton matrix Harnack inequality need not hold globally in order for Poon's frequency monotonicity to hold. Using tools developed in Chapter 4, preliminary analysis indicates that the eigenvalues of $\nabla_H^2 \ln p_1(\cdot, e)$ are asymptotically at least $-\frac{1}{2}$ outside of a gauge ball centered at $e \in \mathbb{H}$. This indicates that, at least far from the group identity, the Hamilton matrix Harnack inequality holds (or equivalently $C(t \mapsto \ln t)$ holds). We would like to study this further in the future and what the implications of such asymptotic behavior are.

Other suggested problems to work on are the following: J. Tyson has suggested studying whether Almgren's frequency monotonicity can be adjusted for the (perhaps sub-elliptic) p-Laplacian

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad p > 1$$

Another question, suggested by D. Vassilev, is to study Almgren's frequency in the setting of CR manifolds, perhaps producing a similar quantity to discrepancy.

REFERENCES

REFERENCES

[1] Milton Abramowitz and Irene A Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*, volume 55. Courier Corporation, 1964.

[2] Frederick J Almgren Jr. Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents. In *Minimal Submanifolds and Geodesics (Proc. Japan–United States Sem., Tokyo, 1977), North-Holland, Amsterdam–New York*, pages 1–6, 1979.

[3] D Bakry. L'hypercontractivité et son utilisation en théorie des semigroupes. Lectures on probability theory (Saint-Flour, 1992), 1–114. *Lecture Notes in Math*, 1581, 1985.

[4] Dominique Bakry, Fabrice Baudoin, Michel Bonnefont, and Djalil Chafaï. On gradient bounds for the heat kernel on the Heisenberg group. *Journal of Functional Analysis*, 255(8):1905–1938, 2008.

[5] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348. Springer Science & Business Media, 2013.

[6] Zoltán M Balogh and Jeremy T Tyson. Polar coordinates in carnot groups. *Mathematische Zeitschrift*, 241(4):697–730, 2002.

[7] Fabrice Baudoin. *Diffusion Processes and Stochastic Calculus.* European Mathematical Society, Zurich, Switzerland, 2014.

[8] Fabrice Baudoin and Michel Bonnefont. The subelliptic heat kernel on $SU(2)$: representations, asymptotics and gradient bounds. *Mathematische Zeitschrift*, 263(3):647–672, 2009.

[9] Fabrice Baudoin and Nicola Garofalo. Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries. *Journal of the European Mathematical Society (to appear).*

[10] Fabrice Baudoin and Jing Wang. The subelliptic heat kernel on the CR sphere. *Mathematische Zeitschrift*, 275(1-2):135–150, 2013.

[11] Heinz Bauer. Harmonische räume und ihre potential theorie. *Lecture Notes in Mathematics*, 22, 1966.

[12] Richard Beals, Bernard Gaveau, and Peter C Greiner. Hamilton–Jacobi theory and the heat kernel on Heisenberg groups. *Journal de mathématiques pures et appliquées*, 79(7):633–689, 2000.

[13] Andre Bellache and Jean-Jacques Risler. Sub-riemannian geometry. *Progress in Mathematics*, 144, 1996.

[14] S Bochner. Vector fields and Ricci curvature. *Bulletin of the American Mathematical Society*, 52(9):776–797, 1946.

[15] Andrea Bonfiglioli and Ermanno Lanconelli. Liouville-type theorems for real sub-Laplacians. *manuscripta mathematica*, 105(1):111–124, 2001.

[16] Andrea Bonfiglioli, Ermanno Lanconelli, and Francesco Uguzzoni. *Stratified Lie groups and potential theory for their sub-Laplacians*. Springer Science & Business Media, 2007.

[17] Michel Bonnefont. The subelliptic heat kernels on $\mathbf{SL}(2,\mathbb{R})$ and on its universal covering $\widetilde{\mathbf{SL}(2,\mathbb{R})}$: Integral representations and some functional inequalities. *Potential analysis*, 36(2):275–300, 2012.

[18] W.M. Boothby. *An Introduction to Differentiable Manifolds and Riemannian Geometry*. Pure and Applied Mathematics. Academic Press, 2003.

[19] Marcel Brelot. On topologies and boundaries in potential theory. 1971.

[20] Ovidiu Calin, Der-Chen Chang, and Peter Greiner. *Geometric analysis on the Heisenberg group and its generalizations*. American Mathematical Soc., 2008.

[21] Luca Capogna, Donatella Danielli, Scott D Pauls, and Jeremy Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, volume 259. Springer Science & Business Media, 2007.

[22] Wei-Liang Chow. Über systeme von linearen partiellen Differentialgleichungen erster Ordnung. *Math. Ann*, 117(1):98–105, 1939.

[23] Giovanna Citti, Nicola Garofalo, and Ermanno Lanconelli. Harnack's inequality for sum of squares of vector fields plus a potential. *American Journal of Mathematics*, pages 699–734, 1993.

[24] L.J. Corwin and F.P. Greenleaf. *Representations of Nilpotent Lie Groups and Their Applications: Basic theory and examples*. Number v. 1 in Cambridge studies in advanced mathematics. Cambridge University Press, 1990.

[25] Jacek Cygan. Heat kernels for class 2 nilpotent groups. *Studia Mathematica*, 3(64):227–238, 1979.

[26] Donatella Danielli and Nicola Garofalo. Geometric properties of solutions to subelliptic equations in nilpotent Lie groups. *Lecture Notes in Pure and Applied Mathematics*, pages 89–106, 1997.

[27] Donatella Danielli, Nicola Garofalo, and Duy-Minh Nhieu. Sub-Riemannian calculus on hypersurfaces in Carnot groups. *Advances in Mathematics*, 215(1):292–378, 2007.

[28] Amédée Debiard, Bernard Gaveau, and Edmond Mazet. Théoremes de comparaison en géométrie riemannienne. *Publications of the Research Institute for Mathematical Sciences*, 12(2):391–425, 1976.

[29] Sorin Dragomir and Guiseppe Tomassini. Differential geometry and analysis on cr manifolds. *Progr. Math*, 246, 2006.

[30] Nathaniel Eldredge. Precise estimates for the subelliptic heat kernel on h-type groups. *Journal de mathématiques pures et appliquées*, 92(1):52–85, 2009.

[31] Lawrence C Evans and Ronald F Gariepy. Wiener's criterion for the heat equation. *Archive for Rational Mechanics and Analysis*, 78(4):293–314, 1982.

[32] Lawrence Craig Evans and Ronald F Gariepy. *Measure theory and fine properties of functions*. CRC press, 2015.

[33] L.C. Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 2010.

[34] MV Fedoryuk. *Asimptotica: Integrali i ryadi.* Moscow: Nauka)(in Russian), 1987.

[35] GB Folland. A fundamental solution for a subelliptic operator. *Bulletin of the American Mathematical Society*, 79(2):373–376, 1973.

[36] Gerald B Folland. Subelliptic estimates and function spaces on nilpotent Lie groups. *Arkiv för matematik*, 13(1):161–207, 1975.

[37] Nicola Garofalo and Ermanno Lanconelli. Wieners criterion for parabolic equations with variable coefficients and its consequences. *Transactions of the American Mathematical Society*, 308(2):811–836, 1988.

[38] Nicola Garofalo and Ermanno Lanconelli. Asymptotic behavior of fundamental solutions and potential theory of parabolic operators with variable coefficients. *Mathematische Annalen*, 283(2):211–239, 1989.

[39] Nicola Garofalo and Ermanno Lanconelli. Frequency functions on the heisenberg group, the uncertainty principle and unique continuation. *Annales de l'institut Fourier*, 40(2):313–356, 1990.

[40] Nicola Garofalo and Fang-Hua Lin. Monotonicity properties of variational integrals, A_p weights and unique continuation. *Indiana University Mathematics Journal*, 35(2):245–268, 1986.

[41] Nicola Garofalo and Fang-Hua Lin. Unique continuation for elliptic operators: A geometric-variational approach. *Communications on pure and applied mathematics*, 40(3):347–366, 1987.

[42] Nicola Garofalo and Kevin Rotz. Properties of a frequency of Almgren type for harmonic functions in carnot groups. *Calculus of Variations and Partial Differential Equations*, pages 1–42, 2015.

[43] Nicola Garofalo and Fausto Segala. Estimates of the fundamental solution and Wiener's criterion for the heat equation on the Heisenberg group. *Indiana University mathematics journal*, 39(4):1155–1196, 1990.

[44] Nicola Garofalo and Dimiter Vassilev. Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups. *Mathematische Annalen*, 318(3):453–516, 2000.

[45] Bernard Gaveau. Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents. *Acta mathematica*, 139(1):95–153, 1977.

[46] Alexander Grigor'yan et al. *Heat kernel and analysis on manifolds*, volume 47. American Mathematical Soc., 2012.

[47] Richard Hamilton. A matrix Harnack estimate for the heat equation. *Comm. Anal. Geom*, 1(1):113–126, 1993.

[48] Richard Hamilton. Monotonicity formulas for parabolic flows on manifolds. *Comm. Anal. Geom*, 1(1):127–137, 1993.

[49] Michiel Hazewinkel. *Encyclopaedia of Mathematics: Reaction-Diffusion Equation – Stirling Interpolation Formula*, volume 8. Springer Science & Business Media, 2012.

[50] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Mathematica*, 119(1):147–171, 1967.

[51] Andrzej Hulanicki. The distribution of energy in the Brownian motion in the Gaussian field and analytic-hypoellipticity of certain subelliptic operators on the Heisenberg group. *Studia Mathematica*, 56(2):165–173, 1976.

[52] David S Jerison and Antonio Sánchez-Calle. Estimates for the heat kernel for a sum of squares of vector fields. *Indiana University mathematics journal*, 35(4):835–854, 1986.

[53] Aroldo Kaplan. Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. *Transactions of the American Mathematical Society*, 258(1):147–153, 1980.

[54] Bumsik Kim. Poincaré inequality and the uniqueness of solutions for the heat equation associated with subelliptic diffusion operators. *arXiv preprint arXiv:1305.0508*, 2013.

[55] Adam Korányi. Kelvin transforms and harmonic polynomials on the Heisenberg group. *Journal of Functional Analysis*, 49(2):177–185, 1982.

[56] Shigeo Kusuoka. Applications of the Malliavin calculus, part iii. 1987.

[57] Ermanno Lanconelli. Sul problema di Dirichlet per l'equazione del calore. *Annali di matematica pura ed applicata*, 97(1):83–114, 1973.

[58] Naum Samoĭlovich Landkof. *Foundations of modern potential theory*, volume 180. Springer, 1972.

[59] John M Lee. *Riemannian manifolds: an introduction to curvature*, volume 176. Springer Science & Business Media, 2006.

[60] Hong-Quan Li. Estimations optimales du noyau de la chaleur sur les groupes de type heisenberg. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2010(646):195–233, 2010.

[61] Hong-Quan Li. Estimations asymptotiques du noyau de la chaleur pour l'opérateur de Grushin. *Communications in Partial Differential Equations*, 37(5):794–832, 2012.

[62] Peter Li and Shing Tung Yau. On the parabolic kernel of the Schrödinger operator. *Acta Mathematica*, 156(1):153–201, 1986.

[63] Wilhelm Magnus, Fritz Oberhettinger, and Raj Soni. *Formulas and theorems for the special functions of mathematical physics*, volume 52. Springer Science & Business Media, 2013.

[64] Henry P McKean et al. An upper bound to the spectrum of Δ on a manifold of negative curvature. *Journal of Differential Geometry*, 4(3):359–366, 1970.

[65] Edward Neuman. Inequalities involving modified Bessel functions of the first kind. *Journal of mathematical analysis and applications*, 171(2):532–536, 1992.

[66] Arshak Petrosyan, Henrik Shahgholian, and Nina N Ural'ceva. *Regularity of free boundaries in obstacle-type problems*, volume 136. American Mathematical Soc., 2012.

[67] Chi-Cheung Poon. Unique continuation for parabolic operators. *Communications in Partial Differential Equations*, 21(3-4):521–539, 1996.

[68] Bin Qian. Positive curvature property for some hypoelliptic heat kernels. *Bulletin des sciences mathematiques*, 135(3):262–278, 2011.

[69] Jennifer Randall. The heat kernel for generalized Heisenberg groups. *The Journal of Geometric Analysis*, 6(2):287–316, 1996.

[70] PK Rashevsky. Any two points of a totally nonholonomic space may be connected by an admissible line. *Uch. Zap. Ped. Inst. im. Liebknechta, Ser. Phys. Math*, 2:83–94, 1938.

[71] Daniel W Stroock. An estimate on the Hessian of the heat kernel. *Journées équations aux dérivées partielles*, pages 1–4, 1995.

[72] Michael Struwe. On the evolution of harmonic maps in higher dimensions. *Journal of differential geometry*, 28(3):485–502, 1988.

[73] Jing Wang. The subelliptic heat kernel on the CR hyperbolic spaces. *arXiv preprint arXiv:1204.3642*, 2012.

VITA

VITA

Kevin was born and raised in a small town named Greencastle in south-central Pennsylvania. In December 2008, he received his bachelor's degrees in physics and mathematics from the Pennsylvania State University in University Park, PA. In August 2009, he moved to Lafayette, IN to pursue a PhD in mathematics at Purdue University. While at Purdue, he taught recitations and lectures for six years, and received the Bilsland Fellowship during his seventh and final year.

Kevin met his wife Rachel in high school, and they got married in August 2008. He and his wife and have two cats and two dogs. Kevin's hobbies include listening to music and podcasts, reading, tinkering with computers, and watching various science fiction television shows such as Doctor Who and the Star Trek and Stargate franchises. He also enjoys hiking, but in flat Indiana it's more like “hiking”.

Kevin has accepted a position as a Senior Systems Engineer at Raytheon Missile Systems in Tucson, Arizona. His long-term goal is to return to teaching at the college level after gaining experience in industry. While at Raytheon, he would like to remain partially active in math research.