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Kernels of adjoints of composition operators on Hilbert spaces of analytic functions

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KERNELS OF ADJOINTS OF COMPOSITION OPERATORS ON HILBERT SPACES OF ANALYTIC FUNCTIONS

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To the important people in my life.

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SYMBOLS

$\mathbb{C}$	complex plane, $\{z = a + bi \, : \, a, b \in \mathbb{R}\}$
$\widehat{\mathbb{C}}$	extended complex plane, $\mathbb{C} \cup \{\infty\}$
$\mathbb{D}$	complex unit disk, $\{z \in \mathbb{C} \, : \, \lvert z\rvert < 1\}$
$\partial\mathbb{D}$	complex unit circle, $\{z \in \mathbb{C} \, : \, \lvert z\rvert = 1\}$
$H^p(\mathbb{D})$	Hardy space
$H^2(\mathbb{D})$	classical Hardy space
$A^p(\mathbb{D})$	Bergman space
$A^2(\mathbb{D})$	classical Bergman space
$\mathcal{D}$	Dirichlet space
$H^2(\beta)$	weighted Hardy space
K_α	reproducing kernel function
C_φ	composition operator with symbol φ
$W_{\varphi,\psi}$	weighted composition operator with symbols φ and ψ
C_φ^*	adjoint of composition operator with symbol φ
$A^2_\alpha(\mathbb{D})$	weighted Bergman space
T_φ	Toeplitz operator with symbol φ

ABBREVIATIONS

LFT	linear fractional transformation

ABSTRACT

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This thesis contains a collection of results in the study of the adjoint of a composition operator and its kernel in weighted Hardy spaces, in particular, the classical Hardy, Bergman, and Dirichlet spaces. In 2006, Cowen and Gallardo-Gutiérrez laid the groundwork for an explicit formula for the adjoint of a composition operator with rational symbol acting on the Hardy space, and in 2008, Hammond, Moorhouse, and Robbins established such a formula. In 2014, Goshabulaghi and Vaezi obtained analogous formulas for the adjoint of a composition operator in the Bergman and Dirichlet spaces. While it is known that the kernel of the adjoint of a composition operator whose symbol is not univalent on the complex unit disk is infinite-dimensional, no classification has been given for functions in this kernel.

Chapter 1 introduces the relevant definitions in the study of composition operators and their adjoints. Chapter 2 provides the background for results obtained by Cowen and Gallardo-Gutiérrez, and Hammond, Moorhouse, and Robbins in the Hardy space. The results by Goshabulaghi and Vaezi for the Bergman and Dirichlet spaces are also given. Chapter 3 contains explicit descriptions of the kernel of the adjoint of a composition operator in a particular class on general weighted Hardy spaces. Chapter 4 uses the adjoint formula by Hammond, Moorhouse, and Robbins to give a functional equation that characterizes functions in the kernel of the adjoint of a composition operator with a rational symbol of degree two on the Hardy space. Chapters 5 and 6 use the adjoint formulas by Goshabulaghi and Vaezi to prove some results about the kernels of adjoints of composition operators on the Bergman and Dirichlet spaces.

1. Introduction

We begin by introducing Hilbert spaces of analytic functions, composition operators, and other definitions needed to develop the theory used to describe adjoints of composition operators from Cowen and MacCluer's [1] book.

1.1 Hilbert Spaces of Analytic Functions

Definition 1.1.1 *A Banach space of complex-valued functions on a set X is called a **functional Banach space on** X if the vector operations are the pointwise operations, $f(x) = g(x)$ for each x in X implies $f = g$, $f(x) = f(y)$ for each function in the space implies $x = y$, and the linear functional $f \mapsto f(x)$ is continuous for each x in X. A functional Banach space whose functions are analytic on the underlying set X is called a **Banach space of analytic functions**.*

Suppose $\mathcal{H}$ is a Hilbert space of analytic functions on X, i.e., a Banach space of analytic functions with an inner product. Because point evaluation for each x in X is a continuous linear functional, the Riesz Representation Theorem implies that there is a function K_x in $\mathcal{H}$ that induces this linear functional such that $f(x) = \langle f, K_x \rangle_{\mathcal{H}}$ for each f in $\mathcal{H}$.

Definition 1.1.2 *In a Hilbert space of analytic functions $\mathcal{H}$, the functions K_x are called the **reproducing kernel functions** and $\mathcal{H}$ is called a **reproducing kernel Hilbert space**.*

1.2 Some Special Hilbert Spaces of Analytic Functions

Now, we consider Hilbert spaces of analytic functions on the complex unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

1.2.1 The Hardy Space, $H^2(\mathbb{D})$

Definition 1.2.1 *For $0 < p < \infty$, the* ***Hardy space*** *$H^p(\mathbb{D})$ is the set of analytic functions on the complex unit disk for which*

$$\sup_{0<r<1} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^p \frac{d\theta}{2\pi} < \infty.$$

In fact, the supremum in the definition above is actually a limit as r tends to 1. Note that if $f^*\big(e^{i\theta}\big) = \lim_{r\to 1^-} f\left(re^{i\theta}\right)$, then $||f^*||_{L^p(\partial\mathbb{D})} = ||f||_{H^p(\mathbb{D})}$.

For $p = 2$, we recover the classical Hardy space $H^2(\mathbb{D})$ which is a Hilbert space with the inner product

$$\langle f, g \rangle_{H^2} = \int_0^{2\pi} f\left(e^{i\theta}\right) \overline{g\left(e^{i\theta}\right)} \frac{d\theta}{2\pi}.$$

Alternatively, if we write f and g as their Maclaurin series with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ where $z \in \mathbb{D}$ and $a_n, b_n \in \mathbb{C}$, then

$$\langle f, g \rangle_{H^2} = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

We also have that the square of the norm of f is

$$||f||_{H^2}^2 = \int_0^{2\pi} \left| f\left(e^{i\theta}\right) \right|^2 \frac{d\theta}{2\pi} = \sum_{n=0}^{\infty} |a_n|^2.$$

The reproducing kernel function, K_α, in the Hardy space $H^2(\mathbb{D})$ that evaluates at α in $\mathbb{D}$ is given by

$$K_\alpha(z) = \frac{1}{1 - \overline{\alpha} z} \quad \text{and} \quad ||K_\alpha||_{H^2} = \frac{1}{\sqrt{1 - |\alpha|^2}}.$$

1.2.2 The Bergman Space, $A^2(\mathbb{D})$

Definition 1.2.2 *For $0 < p < \infty$, the* ***Bergman space*** *$A^p(\mathbb{D})$ is the set of analytic functions on the complex unit disk for which*

$$\int_{\mathbb{D}} |f(z)|^p \frac{dA(z)}{\pi} < \infty$$

where $dA(z)$ is the Lebesgue area measure on the complex unit disk.

For $p = 2$, we recover the classical Bergman space $A^2(\mathbb{D})$ which is a Hilbert space with the inner product

$$\langle f, g \rangle_{A^2} = \int_{\mathbb{D}} f(z)\overline{g(z)} \frac{dA(z)}{\pi}.$$

Alternatively, if we write f and g as their Maclaurin series with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ where $z \in \mathbb{D}$ and $a_n, b_n \in \mathbb{C}$, then

$$\langle f, g \rangle_{A^2} = \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{n+1}.$$

We also have that the square of the norm of f is

$$||f||_{A^2}^2 = \int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi} = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.$$

The reproducing kernel function, K_α, in the Bergman space $A^2(\mathbb{D})$ that evaluates at α in $\mathbb{D}$ is given by

$$K_\alpha(z) = \frac{1}{(1 - \overline{\alpha} z)^2} \quad \text{and} \quad ||K_\alpha||_{A^2} = \frac{1}{1 - |\alpha|^2}.$$

1.2.3 The Dirichlet Space, $\mathcal{D}$

Definition 1.2.3 *The **Dirichlet space** $\mathcal{D}$ is the set of analytic functions on the complex unit disk for which*

$$\int_{\mathbb{D}} |f'(z)|^2 \frac{dA(z)}{\pi} < \infty$$

where $dA(z)$ is the Lebesgue area measure on the complex unit disk.

Writing f and g as their Maclaurin series with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ where $z \in \mathbb{D}$ and $a_n, b_n \in \mathbb{C}$, the inner product on $\mathcal{D}$ is

$$\langle f, g \rangle_{\mathcal{D}} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} \frac{dA(z)}{\pi} = a_0 \overline{b_0} + \sum_{n=1}^{\infty} n a_n \overline{b_n}$$

and the square of the norm of f is

$$||f||_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \frac{dA(z)}{\pi} = |a_0|^2 + \sum_{n=1}^{\infty} n |a_n|^2.$$

1.2.4 Weighted Hardy Spaces, $H^2(\beta)$

Definition 1.2.4 *A Hilbert space $\mathcal{H}$ whose vectors are analytic functions on the complex unit disk is called a* ***weighted Hardy space*** *if the monomials $1, z, z^2, \ldots$ constitute a complete orthogonal set of non-zero vectors in $\mathcal{H}$.*

Assuming that the norm on $\mathcal{H}$ satisfies $||1||_{\mathcal{H}} = 1$ and taking $\beta(n) = ||z^n||_{\mathcal{H}}$, then for f and g in $\mathcal{H}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ where $z \in \mathbb{D}$ and $a_n, b_n \in \mathbb{C}$, the inner product on $\mathcal{H}$ is given by

$$\langle f, g \rangle_{\mathcal{H}} = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta(n)^2$$

and the square of the norm of f is given by

$$||f||_{\mathcal{H}}^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta(n)^2.$$

The weighted Hardy space with weight sequence $\{\beta(n)\}_{n=0}^{\infty}$ will be denoted $H^2(\beta)$.

Remark 1.2.1 *We note that each weighted Hardy space $H^2(\beta)$ is a reproducing kernel Hilbert space with*

$$K_\alpha(z) = \sum_{n=0}^{\infty} \frac{\overline{\alpha}^n z^n}{\beta(n)^2}$$

for each point α in $\mathbb{D}$. For each positive integer m, evaluation of the m^{th} derivative of f in $H^2(\beta)$ at α is a bounded linear functional and $f^{(m)}(\alpha) = \left\langle f, K_\alpha^{(m)} \right\rangle_{H^2(\beta)}$ where

$$K_\alpha^{(m)}(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\overline{\alpha}^{n-m} z^n}{\beta(n)^2}.$$

Remark 1.2.2 *The classical Hardy space, the classical Bergman space, and the Dirichlet space are weighted Hardy spaces with $\beta(n) = 1$, $\beta(n) = (n+1)^{-1/2}$, and (up to an equivalent norm) $\beta(0) = 1$ and $\beta(n) = n^{1/2}$, respectively.*

We also define a class of weighted Hardy spaces called the weighted Bergman spaces (sometimes called the standard weight Bergman spaces).

Definition 1.2.5 *For $\alpha > -1$, the* ***weighted Bergman space*** $A^2_\alpha(\mathbb{D})$ *is the set of analytic functions on the complex unit disk for which*

$$\int_{\mathbb{D}} |f(z)|^2 \left(1 - |z|^2\right)^\alpha \frac{dA(z)}{\pi} < \infty$$

where $dA(z)$ is the Lebesgue area measure on the complex unit disk.

Note that the classical Bergman space $A^2(\mathbb{D})$ is recovered when $\alpha = 0$.

1.3 Composition Operators on Hilbert Spaces of Analytic Functions

We now give the definition of a composition operator on a Hilbert space of analytic functions.

Definition 1.3.1 *Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. The* ***composition operator*** C_φ*, with symbol φ, acting on $\mathcal{H}$ is defined by*

$$(C_\varphi f)(z) = f(\varphi(z))$$

for $z \in \mathbb{D}$ and $f \in \mathcal{H}$.

If, in addition, ψ is a complex-valued function defined on $\mathbb{D}$, the ***weighted composition operator*** $W_{\varphi,\psi}$ *is defined by*

$$(W_{\varphi,\psi} f)(z) = \psi(z) f(\varphi(z)).$$

Note that C_φ and $W_{\varphi,\psi}$ are linear operators.

Remark 1.3.1 *Composition operators with analytic symbols φ that map $\mathbb{D}$ to $\mathbb{D}$ acting on a Hardy space $H^p(\mathbb{D})$ for $p \geq 1$ or a weighted Bergman space $A^2_\alpha(\mathbb{D})$ for $\alpha > -1$ are well-behaved, i.e., bounded. Cowen and MacCluer [1] showed that on a Hardy space $H^p(\mathbb{D})$ for $p \geq 1$, the operator norm of C_φ where φ is an analytic map of $\mathbb{D}$ to $\mathbb{D}$ satisfies*

$$\left(\frac{1}{1 - |\varphi(0)|^2}\right)^{1/p} \leq ||C_\varphi|| \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right)^{1/p}.$$

Hurst [2] showed a similar inequality on a weighted Bergman space $A^2_\alpha(\mathbb{D})$ for $\alpha > -1$:

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{(\alpha+2)/2} \leq ||C_\varphi|| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{(\alpha+2)/2}.$$

However, composition operators with analytic symbols φ that map $\mathbb{D}$ to $\mathbb{D}$ acting on the Dirichlet space $\mathcal{D}$ are not bounded unless $\varphi \in \mathcal{D}$ and has bounded multiplicity [3]. Indeed, since $f(z) = z \in \mathcal{D}$, then $(C_\varphi f)(z) = \varphi(z)$ must be in $\mathcal{D}$ if C_φ is a bounded operator. The norm of a function $f \in \mathcal{D}$ measures the area of the image of f, counting multiplicity. If we consider the analytic function $\varphi(z) = e^{\frac{z+1}{z-1}}$, then φ is an infinite-to-one map of $\mathbb{D}$ to $\mathbb{D}$ but φ is not in $\mathcal{D}$. Therefore, C_φ is not bounded on $\mathcal{D}$.

One important consequence of studying composition operators on weighted Hardy spaces is their relationship to unilateral weighted shifts. For example, consider the Hilbert space $\ell^2(\mathbb{N})$ regarded as a space of complex-valued functions on the set of non-negative integers $\mathbb{N}$ and define φ on $\mathbb{N}$ such that $\varphi(n) = n+1$. Then, the composition operator C_φ acting on $\ell^2(\mathbb{N})$ is

$$(f(0), f(1), f(2), \ldots) \mapsto (f(1), f(2), \ldots)$$

which is the backward shift of multiplicity one. In fact, $\ell^2(\mathbb{N})$ is isomorphic to the classical Hardy space $H^2(\mathbb{D})$ and multiplication by z on $H^2(\mathbb{D})$ gives rise to the forward shift operator on $\ell^2(\mathbb{N})$, the adjoint of the backward shift.

1.4 Adjoints of Composition Operators on Hilbert Spaces of Analytic Functions

Definition 1.4.1 *Let $\mathcal{H}$ be a Hilbert space of analytic functions and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. The **adjoint of the composition operator with symbol** φ acting on $\mathcal{H}$ is denoted C^*_φ and is defined by*

$$\left\langle C^*_\varphi f, g\right\rangle_{\mathcal{H}} = \langle f, C_\varphi g\rangle_{\mathcal{H}}$$

for $f, g \in \mathcal{H}$.

If $\mathcal{H}$ is the classical Hardy space or the classical Bergman space, the adjoint of a composition operator can be written immediately in integral form. Indeed,

$$(C_\varphi^* f)(z) = \left\langle C_\varphi^* f, K_z \right\rangle_{\mathcal{H}} = \langle f, C_\varphi K_z \rangle_{\mathcal{H}} \tag{1.1}$$

which gives

$$\text{in} \quad H^2(\mathbb{D}) : \qquad (C_\varphi^* f)(z) = \int_0^{2\pi} \frac{f\left(e^{i\theta}\right)}{1 - \overline{\varphi\left(e^{i\theta}\right)} z} \frac{d\theta}{2\pi}$$

$$\text{in} \quad A^2(\mathbb{D}) : \qquad (C_\varphi^* f)(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \overline{\varphi(w)} z)^2} \frac{dA(w)}{\pi}$$

Unfortunately, these adjoint formulas do not provide much insight to the structure of C_φ^*, but the work of Cowen and Gallardo-Gutiérrez [4] followed by Hammond, Moorhouse, and Robbins [5] led to a more explicit formula for C_φ^* in the Hardy space. In Chapter 5, we provide partial results leading to a more explicit formula for C_φ^* in the Bergman space. In 2014, Goshabulaghi and Vaezi [6] established analogous formulas for C_φ^* in the Bergman space and the Dirichlet space.

2. Background

In establishing a formula without using an integral for the adjoint of a composition operator on the Hardy space, Cowen and Gallardo-Gutiérrez [4] introduced compatible pairs of multiple-valued functions and multiple-valued weighted composition operators. We give these definitions and the results leading to the adjoint formula in the Hardy space finalized by Hammond, Moorhouse, and Robbins [5]. Then, Goshabulaghi and Vaezi [6] use this formula to recover analogous formulas for the adjoint of a composition operator in the Bergman and Dirichlet spaces.

From [4], we have the following definitions:

Definition 2.0.1 *Let $\Omega \subset \mathbb{C}$ be a domain and z_0 a point of Ω. Let K be a finite set in Ω that does not include z_0. Suppose σ and ψ are functions analytic in a simply-connected neighborhood of z_0 in $\Omega \backslash K$ and suppose they are arbitrarily continuable in $\Omega \backslash K$. We say (σ, ψ) is a* ***compatible pair of multiple-valued functions on*** *Ω if for any path γ in $\Omega \backslash K$ along which the continuation of σ yields the same branch as at the beginning, it is also the case that continuation of ψ along γ yields the same branch as at the beginning.*

As a consequence of this definition, if (σ, ψ) is a compatible pair of multiple-valued functions on Ω, then the number of branches of ψ at any point is a divisor of the number of branches of σ at that point.

Definition 2.0.2 *Suppose $\Omega \subset \mathbb{C}$ is a domain and K is a finite set in Ω. Suppose that σ is an n-valued analytic function that is arbitrarily continuable in $\Omega \backslash K$ and takes values in Ω. Assume that ψ is an m-valued (where m divides n) bounded analytic function that is arbitrarily continuable in $\Omega \backslash K$. The* ***multiple-valued weighted***

composition operator $W_{\sigma,\psi}$ *on a Hilbert space of analytic function* $\mathcal{H}$ *is the operator defined by*

$$\left(W_{\sigma,\psi}f\right)(z) = \sum \psi(z)f(\sigma(z))$$

for f *in* $\mathcal{H}$ *and where the sum is taken over all branches of the compatible pair* (σ,ψ) *for* z *in* $\Omega\backslash K$*.*

We note that if σ has a removable singularity at a point $\xi \in K$ and if ψ is bounded in a punctured neighborhood of ξ, then ψ also has a removable singularity at ξ. Indeed, each branch of σ is single-valued in a neighborhood of ξ and each branch of σ is associated with a particular branch of ψ. Therefore, ψ is also single-valued in a neighborhood of ξ.

2.1 An Adjoint Formula in the Hardy Space

From [5], we have the following explicit formula for the adjoint of a composition operator with rational symbol on the Hardy space $H^2(\mathbb{D})$:

Theorem 2.1.1 *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be a non-constant rational map, and let* C_φ *act on* $H^2(\mathbb{D})$*. Set* $\sigma(z) = \dfrac{1}{\overline{\varphi^{-1}(1/\overline{z})}}$, $\psi(z) = \dfrac{z\sigma'(z)}{\sigma(z)}$, *and* $\varphi(\infty) = \lim_{|z|\to\infty} \varphi(z)$*. Then,*

$$\left(C_\varphi^* f\right)(z) = \frac{f(0)}{1 - \overline{\varphi(\infty)}z} + \sum \psi(z)f(\sigma(z)) \tag{2.1}$$

where the sum is taken over the branches of σ*.*

In considering φ to be a map from the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself, note that the number of branches of φ^{-1} matches the degree of φ and hence, the number of branches of σ matches the degree of φ. Furthermore, the multiple-valued function σ has a corresponding ψ for which (σ,ψ) is a compatible pair of multiple-valued functions on $\mathbb{D}$.

Corollary 2.1.1 *Suppose that* $\varphi : \mathbb{D} \to \mathbb{D}$ *is a rational map, and let* C_φ *act on* $H^2(\mathbb{D})$*. If* $\varphi(\infty) = \infty$*, then* C_φ^* *is a (multiple-valued) weighted composition operator.*

2.2 An Adjoint Formula in the Bergman Space

In the formula for the adjoint of a composition operator with rational symbol on the Bergman space $A^2(\mathbb{D})$, Goshabulaghi and Vaezi [6] use the integral operator $Q : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ defined by $Qf = F$ where F is the antiderivative of f in $A^2(\mathbb{D})$ with $F(0) = 0$.

Theorem 2.2.1 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a non-constant rational map, and let C_φ act on $A^2(\mathbb{D})$. Set $\sigma(z) = \dfrac{1}{\overline{\varphi^{-1}(1/\overline{z})}}$, $u(z) = \dfrac{z^2\sigma'(z)}{\sigma(z)^2}$, and $\varphi(\infty) = \lim_{|z|\to\infty} \varphi(z)$. Then,*

$$\left(C_\varphi^* f\right)(z) = \frac{f(0)}{\left(1 - \overline{\varphi(\infty)}z\right)^2} + \sum u'(z)(Qf)(\sigma(z)) + \sum u(z)\sigma'(z)f(\sigma(z)) \qquad (2.2)$$

where each sum is taken over the branches of σ.

Note that each branch of σ has a corresponding u used in the sums of Equation 2.2.

2.3 An Adjoint Formula in the Dirichlet Space

Lastly, we give the formula for the adjoint of a composition operator with rational symbol on the Dirichlet space $\mathcal{D}$ [6].

Theorem 2.3.1 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a non-constant rational map, and let C_φ act on $\mathcal{D}$. Set $\sigma(z) = \dfrac{1}{\overline{\varphi^{-1}(1/\overline{z})}}$ and $\varphi(\infty) = \lim_{|z|\to\infty} \varphi(z)$. Then,*

$$\left(C_\varphi^* f\right)(z) = f(0)K_{\varphi(0)}(z) + \sum f(\sigma(z)) - \sum f(\sigma(0))$$

where each sum is taken over the branches of σ.

3. Results in Weighted Hardy Spaces

We develop results about the kernel of the adjoint of a composition operator whose symbol is rational acting on a weighted Hardy space and provide an explicit description of the kernels of some of these adjoints. First, we state the relationship between the range of a linear operator and the kernel of its adjoint [7].

Theorem 3.0.1 *Let $\mathcal{H}$ be a Hilbert space and let T be a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}$. Then*

$$\ker(T^*) = \operatorname{rng}(T)^{\perp}.$$

We now identify the range of a composition operator acting on a weighted Hardy space.

Lemma 3.0.1 *Let $H^2(\beta)$ be a weighted Hardy space and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. If C_φ is a bounded operator on $H^2(\beta)$, then $\operatorname{span}\{\varphi^n\}_{n=0}^{\infty}$ is dense in $\operatorname{rng}(C_\varphi)$.*

Proof Since $\{p_n(z) = z^n\}_{n=0}^{\infty}$ is a complete orthogonal set in $H^2(\beta)$, i.e., $\operatorname{span}\{p_n\}_{n=0}^{\infty}$ is dense in $H^2(\beta)$, we have that $\operatorname{span}\{C_\varphi p_n\}_{n=0}^{\infty} = \operatorname{span}\{\varphi^n\}_{n=0}^{\infty}$ is dense in $\operatorname{rng}(C_\varphi)$.

■

We also have the following lemma for the constant term of a function in the kernel of the adjoint of a composition operator acting on a weighted Hardy space.

Lemma 3.0.2 *Let $H^2(\beta)$ be a weighted Hardy space and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. If C_φ is a bounded operator on $H^2(\beta)$ and if f is in $\ker(C_\varphi^*)$, then $f(0) = 0$.*

Proof Since $\varphi^0 = 1$ is in $\operatorname{rng}(C_\varphi)$ by Lemma 3.0.1 and if f is in $\ker(C_\varphi^*)$, then $\langle f, 1\rangle_{H^2(\beta)} = 0$. Writing f as its Maclaurin series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have

$$\begin{aligned} 0 &= \langle f, 1\rangle_{H^2(\beta)} \\ &= a_0 \\ &= f(0) \end{aligned}$$

■

3.1 Composition Operators Whose Adjoints Have Trivial Kernels

We begin by identifying the kernel of the adjoint of a composition operator with linear fractional symbol on weighted Hardy spaces.

Definition 3.1.1 *If a, b, c, d are complex numbers such that $ad-bc \neq 0$, the mapping*

$$z \mapsto \frac{az+b}{cz+d}$$

*is called a **linear fractional transformation** (or **LFT**).*

Lemma 3.1.1 *Let $H^2(\beta)$ be a weighted Hardy space, and let $\varphi(z) = az + b$ be a non-constant map such that $\varphi : \mathbb{D} \to \mathbb{D}$. If C_φ is a bounded operator on $H^2(\beta)$, then*

$$\ker(C_\varphi^*) = \{0\}.$$

Proof Note that $\varphi(z) = az + b$ is in $\operatorname{span}\{1, z\}$. Since $\operatorname{span}\{\varphi^n\}_{n=0}^{\infty}$ is dense in $\operatorname{rng}(C_\varphi)$ by Lemma 3.0.1, then $\operatorname{span}\{\varphi^n\}_{n=0}^{\infty} = \operatorname{span}\{z^n\}_{n=0}^{\infty}$ is dense in $\operatorname{rng}(C_\varphi)$. Therefore,

$$\ker(C_\varphi^*) = \operatorname{rng}(C_\varphi)^\perp = \{0\}.$$

■

Lemma 3.1.2 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an LFT with $\varphi(z) = \dfrac{az+b}{cz+d}$. If $c \neq 0$ and $\overline{\alpha} = -\dfrac{c}{d}$, then φ^n is in $\operatorname{span}\left\{1, K_\alpha^{(m)}\right\}_{m=0}^{n-1}$ where $K_\alpha(z) = \dfrac{1}{1-\overline{\alpha}z}$ is the reproducing kernel* *function at α in the classical Hardy space $H^2(\mathbb{D})$ and* $K_\alpha^{(m)}(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!}\overline{\alpha}^{n-m} z^n$ *is the kernel function that evaluates the m^{th} derivative of a function in $H^2(\mathbb{D})$ at α as in Remark 1.2.1.*

Proof Observe that

$$\begin{aligned}
K_\alpha^{(m)}(z) &= \sum_{n=m}^{\infty} \frac{n!}{(n-m)!}\overline{\alpha}^{n-m} z^n \\
&= z^m \overline{\alpha}^{-m} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!}\overline{\alpha}^{n} z^{n-m} \\
&= z^m \overline{\alpha}^{-m} \frac{d^m}{dz^m} \sum_{n=0}^{\infty} \overline{\alpha}^{n} z^{n} \\
&= z^m \overline{\alpha}^{-m} \frac{d^m}{dz^m} \frac{1}{1-\overline{\alpha}z} \\
&= z^m \overline{\alpha}^{-m} \overline{\alpha}^{m} \frac{m!}{(1-\overline{\alpha}z)^{m+1}} \\
&= \frac{m! z^m}{(1-\overline{\alpha}z)^{m+1}}
\end{aligned}$$

Since φ maps $\mathbb{D}$ to $\mathbb{D}$, then $d \neq 0$ and we can rewrite φ as

$$\varphi(z) = \frac{az+b}{d\left(1-\left(-\frac{c}{d}\right)z\right)} = \frac{az+b}{d(1-\overline{\alpha}z)}$$

which is in span $\{1, K_\alpha\}$. Now, consider $\varphi(z)^n = \dfrac{(az+b)^n}{d^n\left(1-\overline{\alpha}z\right)^n}$ and rewrite $\varphi(z)^n$ as

$$\begin{aligned}
\varphi(z)^n &= A_0 + \frac{A_1}{1-\overline{\alpha}z} + \frac{A_2 z}{(1-\overline{\alpha}z)^2} + \cdots + \frac{A_n z^{n-1}}{(1-\overline{\alpha}z)^n} \\
&= A_0 + A_1 K_\alpha(z) + B_2 K_\alpha'(z) + \cdots + B_n K_\alpha^{(n-1)}(z)
\end{aligned}$$

which is in span $\left\{1, K_\alpha^{(m)}\right\}_{m=0}^{n-1}$. ■

Theorem 3.1.1 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an LFT with $\varphi(z) = \dfrac{az+b}{cz+d}$. If C_φ acts on $H^2(\mathbb{D})$, then $\ker(C_\varphi^*) = \{0\}$.*

Proof Since φ maps $\mathbb{D}$ to $\mathbb{D}$, then $d \neq 0$. If $c = 0$, then φ is a polynomial of degree one and Lemma 3.1.1 shows $\ker(C_\varphi^*) = \{0\}$. Otherwise, if $c \neq 0$, we can rewrite φ as $\varphi(z) = \dfrac{az+b}{d\left(1-\left(-\frac{c}{d}\right)z\right)} = \dfrac{az+b}{d(1-\overline{\alpha}z)}$ where $\overline{\alpha} = -\dfrac{c}{d}$ which is in span $\{1, K_\alpha\}$. Since span $\{\varphi^n\}_{n=0}^\infty$ is dense in $\text{rng}(C_\varphi)$ by Lemma 3.0.1, then span $\left\{1, K_\alpha^{(n)}\right\}_{n=0}^\infty$ is dense in $\text{rng}(C_\varphi)$ by Lemma 3.1.2. For f in $\ker(C_\varphi^*) = \text{rng}(C_\varphi)^\perp$, we have from Remark 1.2.1 that

$$0 = \langle f, K_\alpha^{(n)} \rangle_{H^2} = f^{(n)}(\alpha)$$

for $n = 0, 1, 2, \ldots$ which implies that $f(z) = \displaystyle\sum_{n=0}^\infty \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n = 0$. Therefore, $\ker(C_\varphi^*) = \{0\}$. ■

In fact, we have the same result for C_φ with linear fractional symbol acting on weighted Bergman spaces and we give a different proof from Theorem 3.1.1. Before proceeding, we state the definition of an analytic Toeplitz operator.

Definition 3.1.2 *Let $\mathcal{H}$ be the Hardy space $H^2(\mathbb{D})$ or a weighted Bergman space $A_\alpha^2(\mathbb{D})$ and let φ be a function in $H^\infty(\mathbb{D})$. The **analytic Toeplitz operator** T_φ, with symbol φ, acting on $\mathcal{H}$ is defined by*

$$(T_\varphi h)(z) = (\varphi h)(z)$$

for $z \in \mathbb{D}$, and $h \in \mathcal{H}$.

Following Cowen's [9] work that established a formula for the adjoint of a composition operator with linear fractional symbol on the Hardy space $H^2(\mathbb{D})$, Hurst [2] proved an analogous formula for the adjoint of the same type of composition operator acting on a weighted Bergman space.

Theorem 3.1.2 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an LFT with $\varphi(z) = \dfrac{az+b}{cz+d}$. Then, the map $\sigma(z) = \dfrac{\overline{a}z-\overline{c}}{-\overline{b}z+\overline{d}}$ takes $\mathbb{D}$ to $\mathbb{D}$, $g(z) = \left(-\overline{b}z+\overline{d}\right)^{-1}$ and $h(z) = cz+d$ are in $H^\infty(\mathbb{D})$, and as operators on the Hardy space $H^2(\mathbb{D})$,*

$$C_\varphi^* = T_g C_\sigma T_h^*.$$

Theorem 3.1.3 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an LFT with $\varphi(z) = \dfrac{az+b}{cz+d}$ and let $\alpha > -1$. Then, $\sigma(z) = \dfrac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}$ maps $\mathbb{D}$ to $\mathbb{D}$, $g(z) = \left(-\overline{b}z + \overline{d}\right)^{-(\alpha+2)}$ and $h(z) = (cz+d)^{\alpha+2}$ are in $H^\infty(\mathbb{D})$, and as operators on the weighted Bergman space $A^2_\alpha(\mathbb{D})$,*

$$C^*_\varphi = T_g C_\sigma T^*_h.$$

For φ in these two theorems, we show that C^*_φ has trivial kernel in the Hardy space $H^2(\mathbb{D})$ and the weighted Bergman spaces $A^2_\alpha(\mathbb{D})$ for $\alpha > -1$.

Theorem 3.1.4 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an LFT with $\varphi(z) = \dfrac{az+b}{cz+d}$. If C_φ acts on the Hardy space $H^2(\mathbb{D})$ or a weighted Bergman space $A^2_\alpha(\mathbb{D})$ for $\alpha > -1$, then*

$$\ker(C^*_\varphi) = \{0\}.$$

Proof We have from Theorems 3.1.2 and 3.1.3 that $C^*_\varphi = T_g C_\sigma T^*_h$. Note that g and h are non-zero, and T^*_h is invertible if T_h is invertible. Indeed, since φ maps $\mathbb{D}$ to $\mathbb{D}$, then $\dfrac{1}{cz+d}$ is bounded on $\mathbb{D}$ and so h is bounded above and bounded below by a non-zero constant on $\mathbb{D}$. Therefore, T_h is invertible.

Now, if f is in $\ker(C^*_\varphi)$, then for $z \in \mathbb{D}$

$$\begin{aligned} 0 &= (C^*_\varphi f)(z) \\ &= (T_g C_\sigma T^*_h f)(z) \\ &= g(z)(C_\sigma T^*_h f)(z) \\ 0 &= (C_\sigma T^*_h f)(z) \\ 0 &= (T^*_h f)(z) \end{aligned}$$

We have $(T^*_h f)(z) = 0$ because $\ker(C_\sigma) = \{0\}$. Therefore, $f \equiv 0$ since $\ker(T^*_h) = \{0\}$. ■

In addition to adjoints of composition operators with linear fractional symbols having trivial kernels, those with univalent polynomial symbols on $\mathbb{D}$ also have trivial kernels, that is, polynomials that are one-to-one on $\mathbb{D}$. We introduce a couple of definitions and a proposition from Sarason's [10] paper on *Weak-Star Generators of $H^\infty(\mathbb{D})$*.

Definition 3.1.3 *If B is a bounded domain in the plane, then the **Carathéodory hull** (or $\mathscr{C}$-hull) of B is the complement of the closure of the unbounded component of the complement of the closure of B.*

The $\mathscr{C}$-hull of B is denoted B^*.

Definition 3.1.4 *A conformal map φ of $\mathbb{D}$ onto a bounded simply connected domain G is a **sequential generator** if every function in $H^\infty(\mathbb{D})$ is the weak-star limit of a sequence of polynomials in φ.*

Proposition 3.1.1 *Let φ be a conformal map of $\mathbb{D}$ onto a bounded simply connected domain G. The function φ is a sequential generator if and only if G is a component of its $\mathscr{C}$-hull.*

We combine these notions to prove the following.

Lemma 3.1.3 *If φ is a polynomial univalent on $\mathbb{D}$, then φ is a weak-star generator of $H^\infty(\mathbb{D})$.*

Proof Since φ is a polynomial that is univalent on $\mathbb{D}$, then φ is a conformal map of $\mathbb{D}$ onto $G = \varphi(\mathbb{D})$ and $\varphi : \mathbb{D} \to G$ is one-to-one and onto. Now, G is a component of its $\mathscr{C}$-hull, and in fact, $G = G^*$. Therefore, φ is a sequential generator by Proposition 3.1.1. ■

Theorem 3.1.5 *Let $\mathcal{H}$ be the Hardy space $H^2(\mathbb{D})$ or a weighted Bergman space $A^2_\alpha(\mathbb{D})$ for $\alpha > -1$, and let φ be a polynomial that is univalent on $\mathbb{D}$. If C_φ acts on $\mathcal{H}$, then* $\ker(C_\varphi^*) = \{0\}$.

Proof If φ is a polynomial univalent on $\mathbb{D}$, then φ is a weak-star generator of $H^\infty(\mathbb{D})$ by Lemma 3.1.3. Since φ is a weak-star generator of $H^\infty(\mathbb{D})$, then any polynomial can be approximated by polynomials in φ which are dense in $\mathcal{H}$. Therefore, $\ker(C_\varphi^*) = \text{rng}(C_\varphi)^\perp = \{0\}$. ■

Example 3.1.1 *Let $\varphi(z) = \dfrac{z^2 + 10z + 9}{20}$. If $z \in \mathbb{D}$ with $\varphi(z) = \varphi(w)$, then either $w = z \in \mathbb{D}$ or $w = -z - 10 \notin \mathbb{D}$ which shows that φ is univalent on $\mathbb{D}$. Therefore, φ is a weak-star generator of $H^\infty(\mathbb{D})$ and the kernel of C_φ^* is $\{0\}$ by Theorem 3.1.5.*

3.2 Composition Operators Whose Adjoints Have Non-trivial Kernels

We give results on some composition operators whose adjoints have non-trivial kernels and are, in fact, spanned by a subset of the monomials $\{z^n\}_{n=0}^{\infty}$.

Theorem 3.2.1 *Let $\psi : \mathbb{D} \to \mathbb{D}$ be analytic, let d be an integer with $d \geq 2$, and consider $\varphi(z) = \psi(z^d)$ and $M = \overline{\text{span}\{z^k : k \notin d\mathbb{N}\}}$. If C_φ acts on the Hardy space $H^2(\mathbb{D})$ or a weighted Bergman space $A^2_\alpha(\mathbb{D})$ for $\alpha > -1$, then*

$$\ker(C_\varphi^*) \supset M.$$

Furthermore, if $\ker(C_\psi^*) = \{0\}$, then $\ker(C_\varphi^*) = M$.

Proof Let $\mathcal{H}$ be the Hardy space $H^2(\mathbb{D})$ or a weighted Bergman space $A^2_\alpha(\mathbb{D})$ for $\alpha > -1$. If $f \in \mathcal{H}$, then

$$\begin{aligned} (C_\varphi f)(z) &= f(\psi(z^d)) \\ &= \left(C_{z^d}(f \circ \psi)\right)(z) \\ &= \left(C_{z^d} C_\psi f\right)(z) \end{aligned}$$

and so $C_\varphi = C_{z^d} C_\psi$ and $C_\varphi^* = C_\psi^* C_{z^d}^*$. Now, we have that $\ker(C_\varphi^*) \supset \ker(C_{z^d}^*)$ and $\ker(C_{z^d}^*) = \text{rng}(C_{z^d})^\perp = \text{span}\left\{z^{dn}\right\}_{n=0}^{\infty\ \perp} = \overline{\text{span}\left\{z^k : k \neq dn\right\}} = M$.

If $\ker(C_\psi^*) = \{0\}$, then $\overline{\text{rng}(C_\psi)} = \mathcal{H}$, i.e., C_ψ has dense range in $\mathcal{H}$. Therefore, C_φ has dense range in $C_{z^d}\mathcal{H}$, and

$$\ker(C_\varphi^*) = \text{rng}(C_\varphi)^\perp = \text{span}\left\{z^{dn}\right\}_{n=0}^{\infty\ \perp} = \overline{\text{span}\left\{z^k : k \neq dn\right\}} = M.$$

■

Theorem 3.2.2 *Let φ be a rational map from $\mathbb{D}$ to $\mathbb{D}$ such that $\varphi(z) = z^d \psi(z)$ for some integer $d \geq 2$ and ψ an analytic map with $\psi(0) \neq 0$. Let $\mathcal{H}$ be the Hardy space $H^2(\mathbb{D})$ or a weighted Bergman space $A^2_\alpha(\mathbb{D})$ for $\alpha > -1$, let C_φ act on $\mathcal{H}$, and set $N = \ker(C_\varphi^*)$, $P_m = \{$polynomials of degree $\leq m\}$, and $N_m = N \cap P_m$. Then, $\dim(N_m) = m$ if $1 \leq m \leq d-1$ and $\dim(N_m) = (d-1)k$ if $m = dk-1$ for $k = 1, 2, \ldots$.*

Proof Write φ as its Maclaurin series $\varphi(z) = \sum_{n=d}^{\infty} a_n z^n$ where $a_d \neq 0$ and

$$\varphi(z)^l = \sum_{n=dl}^{\infty} b_n z^n$$

where $b_{dl} = a_d^l \neq 0$.

If $1 \leq m \leq d-1$, then $\dim(N_m) = m$. Indeed, if $f(z) = \sum_{j=0}^{m} c_j z^j$ is in N_m, then $c_0 = 0$ by Lemma 3.0.2 and $\langle f, \varphi^l \rangle_{\mathcal{H}} = 0$ for each $l = 1, 2, \ldots$ and for any $c_j \in \mathbb{C}$, $j = 1, 2, \ldots, m$.

If $m = dk - 1$ for $k = 1, 2, \ldots$, proceed by induction on k. If $k = 1$, then $m = d - 1$ and $\dim(N_{d-1}) = d - 1$. Now, assume that $\dim(N_{dk-1}) = (d-1)k$. Consider $f \in N_{d(k+1)-1}$ and write $f(z) = \sum_{j=1}^{d(k+1)-1} c_j z^j$. Note that $c_0 = 0$, and the matrix of coefficients for the system of equations $\left\{\langle f, \varphi^l \rangle_{\mathcal{H}} = 0\right\}_{l=1}^{k}$ has rank k. For $l > k$, $\langle f, \varphi^l \rangle_{\mathcal{H}} = 0$ since the degree of f is less than dl. The Rank-Nullity Theorem gives $\dim(N_{k+1}) = (d(k+1) - 1) - k = (d-1)(k+1)$. ■

Corollary 3.2.1 *With hypotheses as in Theorem 3.2.2, if $m = dk$ for $k = 1, 2, \ldots$, then* $\dim(N_m) = (d-1)k$.

Proof Note that $P_{dk-1} \subset P_{dk}$ and hence $N_{dk-1} \subset N_{dk}$.

Suppose there exists $f \in P_{dk}$ of degree dk and write $f(z) = \sum_{j=1}^{dk} c_j z^j$ where $c_{dk} \neq 0$. Then, $\langle f, \varphi^k \rangle_{\mathcal{H}} = c_{dk}\overline{a_{dk}}\beta(dk)^2 \neq 0$ and so $f \notin N$, i.e., N does not contain any polynomials of degree dk. Therefore, $N_{dk-1} = N_{dk}$. ■

4. Results in the Hardy Space

We turn our attention to adjoints of composition operators with kernels that are non-trivial, specifically those composition operators whose symbols are rational of degree two. Our main result is a functional equation that is satisfied by functions in the kernel of the adjoint of a composition operator of this type acting on the classical Hardy space $H^2(\mathbb{D})$.

4.1 Composition Operators with Rational Symbols of Degree Two

Consider composition operators whose symbols are not univalent on the complex unit disk. The kernel of the adjoint of a composition operator in this class is non-trivial and, in fact, is infinite-dimensional.

Lemma 4.1.1 *Let $\mathcal{H}$ be the Hardy space $H^2(\mathbb{D})$ or a weighted Bergman space $A^2_\alpha(\mathbb{D})$ for $\alpha > -1$. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are n distinct points in $\mathbb{D}$, then the set of reproducing kernels $\{K_{\alpha_1}, K_{\alpha_2}, \ldots, K_{\alpha_n}\}$ is linearly independent.*

Proof Consider $q(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$ and $q_j(z) = \dfrac{q(z)}{z-\alpha_j}$. Note that $q_j(\alpha_j) \neq 0$. Now, if $c_1K_{\alpha_1} + c_2K_{\alpha_2} + \cdots + c_nK_{\alpha_n} = 0$ for some constants $c_1, c_2, \ldots, c_n \in \mathbb{C}$, then $\langle f, c_1K_{\alpha_1} + c_2K_{\alpha_2} + \cdots + c_nK_{\alpha_n}\rangle_{\mathcal{H}} = 0$ for any $f \in \mathcal{H}$. In particular, for each $j = 1, \ldots, n$,

$$\begin{aligned} 0 &= \langle q_j, c_1K_{\alpha_1} + c_2K_{\alpha_2} + \cdots + c_nK_{\alpha_n}\rangle_{\mathcal{H}} \\ &= \overline{c_j} q_j(\alpha_j) \\ 0 &= c_j \end{aligned}$$

Therefore, $\{K_{\alpha_1}, K_{\alpha_2}, \ldots, K_{\alpha_n}\}$ is linearly independent. ■

Theorem 4.1.1 *Let $\mathcal{H}$ be the Hardy space $H^2(\mathbb{D})$ or a weighted Bergman space $A^2_\alpha(\mathbb{D})$ for $\alpha > -1$. If φ is not univalent on $\mathbb{D}$, then $\ker(C_\varphi^*)$ is infinite-dimensional.*

Proof Note that for every α in $\mathbb{D}$, $C_\varphi^* K_\alpha(z) = K_{\varphi(\alpha)}(z)$ since

$$\begin{aligned} C_\varphi^* K_\alpha(z) &= \left\langle C_\varphi^* K_\alpha, K_z \right\rangle_{\mathcal{H}} \\ &= \langle K_\alpha, C_\varphi K_z \rangle_{\mathcal{H}} \\ &= \overline{C_\varphi K_z(\alpha)} \\ &= \overline{K_z(\varphi(\alpha))} \\ &= \left\langle K_{\varphi(\alpha)}, K_z \right\rangle_{\mathcal{H}} \\ &= K_{\varphi(\alpha)}(z) \end{aligned}$$

Now, take α, β in $\mathbb{D}$ such that $\alpha \neq \beta$, $\varphi(\alpha) = \varphi(\beta)$. Then $K_\alpha - K_\beta$ is in $\ker(C_\varphi^*)$ since $\left(C_\varphi^*(K_\alpha - K_\beta)\right)(z) = K_{\varphi(\alpha)}(z) - K_{\varphi(\beta)}(z) = 0$. Because φ is not univalent on $\mathbb{D}$, there exist an open set in $\mathbb{D}$ containing α, say U, and an open set in $\mathbb{D}$ containing β, say V, such that $U \cap V = \varnothing$ and $\varphi(U) = \varphi(V)$. Now, the set $S = \{K_\alpha - K_\beta : \varphi(\alpha) = \varphi(\beta), \alpha \in U, \beta \in V\}$ is linearly independent by Lemma 4.1.1 and hence, $\overline{\text{span}(S)}$ is an infinite-dimensional subspace of $\ker(C_\varphi^*)$. ■

While the kernel of the adjoint of a composition operator with a non-univalent symbol on the complex unit disk has been well-known to be infinite-dimensional, no classification has been given for functions in this kernel. Using Equation 2.1, we give a functional equation that characterizes functions in the kernel of the adjoint of a composition operator with a rational symbol of degree two acting on the classical Hardy space $H^2(\mathbb{D})$.

4.1.1 Main Results

Let φ be a rational map of degree two from $\mathbb{D}$ to $\mathbb{D}$ with $\varphi(z) = \dfrac{a_1z^2 + b_1z + c_1}{a_2z^2 + b_2z + c_2}$. When referring to such a degree two rational map, we assume that the numerator and denominator have no common factors. Recall from Theorem 2.1.1,

$$\left(C_\varphi^* f\right)(z) = \frac{f(0)}{1 - \overline{\varphi(\infty)}z} + \sum_{j=1}^{2} \psi_j(z) f(\sigma_j(z))$$

where $\sigma(z) = \dfrac{1}{\overline{\varphi^{-1}\left(1/\overline{z}\right)}}$ and $\psi(z) = \dfrac{z\sigma'(z)}{\sigma(z)}$. Since φ is a rational map of degree two, φ^{-1} has two branches and therefore σ has two branches, say σ_1 and σ_2, defined on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Furthermore, $\psi_1(z) = \dfrac{z\sigma_1'(z)}{\sigma_1(z)}$ and $\psi_2(z) = \dfrac{z\sigma_2'(z)}{\sigma_2(z)}$, and (σ, ψ) is a compatible pair of multiple-valued functions on $\mathbb{D}$.

Theorem 4.1.2 *Let φ be a rational map of degree two from $\mathbb{D}$ to $\mathbb{D}$ with*

$$\varphi(z) = \frac{a_1z^2 + b_1z + c_1}{a_2z^2 + b_2z + c_2}.$$

There exists a function ζ, not the identity, such that

$$\zeta(z) = \frac{1}{\overline{\varphi^{-1}(\varphi(1/\overline{z}))}},$$

$\zeta(\sigma_1) = \sigma_2$ where σ_1 and σ_2 are the branches of σ in Theorem 2.1.1, and $\zeta \circ \zeta = \mathrm{id}$. Furthermore,

$$\zeta(z) = -\frac{\left(\overline{a_1b_2} - \overline{b_1a_2}\right) + \left(\overline{a_1c_2} - \overline{c_1a_2}\right)z}{\left(\overline{a_1c_2} - \overline{c_1a_2}\right) + \left(\overline{b_1c_2} - \overline{c_1b_2}\right)z}.$$

Proof Solve for $\zeta(z)$ by solving for w in $\varphi\left(\dfrac{1}{\overline{z}}\right) = \varphi\left(\dfrac{1}{\overline{w}}\right)$:

$$\begin{aligned}
\frac{a_1/\overline{z}^2 + b_1/\overline{z} + c_1}{a_2/\overline{z}^2 + b_2/\overline{z} + c_2} &= \frac{a_1/\overline{w}^2 + b_1/\overline{w} + c_1}{a_2/\overline{w}^2 + b_2/\overline{w} + c_2} \\
\frac{a_1 + b_1\overline{z} + c_1\overline{z}^2}{a_2 + b_2\overline{z} + c_2\overline{z}^2} &= \frac{a_1 + b_1\overline{w} + c_1\overline{w}^2}{a_2 + b_2\overline{w} + c_2\overline{w}^2} \\
(a_1 + b_1\overline{z} + c_1\overline{z}^2)(a_2 + b_2\overline{w} + c_2\overline{w}^2) &= (a_1 + b_1\overline{w} + c_1\overline{w}^2)(a_2 + b_2\overline{z} + c_2\overline{z}^2)
\end{aligned}$$

$$(a_1 + b_1\overline{z} + c_1\overline{z}^2)(a_2 + b_2\overline{w} + c_2\overline{w}^2) - (a_1 + b_1\overline{w} + c_1\overline{w}^2)(a_2 + b_2\overline{z} + c_2\overline{z}^2) = 0$$

$$(\overline{w}-\overline{z})\left[a_1b_2-b_1a_2+(a_1c_2-c_1a_2)(\overline{w}+\overline{z})+(b_1c_2-c_1b_2)\overline{zw}\right]=0$$

Therefore, the two branches of $\dfrac{1}{\overline{\varphi^{-1}(\varphi(1/\overline{z}))}}$ are $\zeta_0(z)=w=z$ and

$$\zeta(z)=w=-\frac{\left(\overline{a_1b_2}-\overline{b_1a_2}\right)+(\overline{a_1c_2}-\overline{c_1a_2})z}{(\overline{a_1c_2}-\overline{c_1a_2})+\left(\overline{b_1c_2}-\overline{c_1b_2}\right)z}$$

which are both single-valued functions on $\widehat{\mathbb{C}}$.

To show that $\zeta(\sigma_1)=\sigma_2$, suppose the two branches of φ^{-1} are φ_1^{-1} and φ_2^{-1} and write σ_1 and σ_2 from Theorem 2.1.1 as follows:

$$\sigma_1(z)=\frac{1}{\overline{\varphi_1^{-1}(1/\overline{z})}}\qquad\qquad \sigma_2(z)=\frac{1}{\overline{\varphi_2^{-1}(1/\overline{z})}}$$

Solving for $1/\overline{z}$ gives

$$\frac{1}{\overline{z}}=\varphi\left(\frac{1}{\overline{\sigma_1(z)}}\right)=\varphi\left(\frac{1}{\overline{\sigma_2(z)}}\right).$$

Since ζ is not the identity, then

$$\zeta(\sigma_1(z))=\frac{1}{\overline{\varphi_2^{-1}(\varphi(1/\overline{\sigma_1(z)}))}}=\frac{1}{\overline{\varphi_2^{-1}(\varphi(1/\overline{\sigma_2(z)}))}}=\sigma_2(z)$$

and it follows that $\zeta\circ\zeta=\mathrm{id}$. ■

Our main result follows from Theorems 2.1.1 and 4.1.2.

Theorem 4.1.3 *Let φ be a rational map of degree two mapping $\mathbb{D}$ into $\mathbb{D}$ and let C_φ act on the Hardy space $H^2(\mathbb{D})$. For ζ as in Theorem 4.1.2, f is in $\ker(C_\varphi^*)$ if and only if*

$$\zeta(z)f(z)+z\zeta'(z)f(\zeta(z))=0. \tag{4.1}$$

Proof Recall that $f(0)=0$ for $f\in\ker(C_\varphi^*)$ by Lemma 3.0.2. Using Theorem 2.1.1,

$$\begin{aligned}0 &= C_\varphi^* f(z)\\ &= \frac{f(0)}{1-\overline{\varphi(\infty)}z}+\sum_{j=1}^{2}\psi_j(z)f(\sigma_j(z))\\ &= \psi_1(z)f(\sigma_1(z))+\psi_2(z)f(\sigma_2(z))\end{aligned}$$

Set $w = \sigma_1(z)$. Then $z = \sigma_1^{-1}(w)$. From Theorem 4.1.2, $\zeta(w) = \zeta(\sigma_1(z)) = \sigma_2(z)$. Rewrite ψ_1 and ψ_2 as follows:

$$\begin{aligned}
\psi_1(z) &= \frac{z\sigma_1'(z)}{\sigma_1(z)} \\
&= \frac{\sigma_1^{-1}(w)\sigma_1'(\sigma_1^{-1}(w))}{w} \\
\psi_2(z) &= \frac{z\sigma_2'(z)}{\sigma_2(z)} \\
&= \frac{\sigma_1^{-1}(w)\zeta'(w)\sigma_1'(\sigma_1^{-1}(w))}{\zeta(w)} \\
&= \frac{\sigma_1^{-1}(w)\sigma_1'(\sigma_1^{-1}(w))\zeta'(w)}{\zeta(w)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 &= \psi_1(z)f(\sigma_1(z)) + \psi_2(z)f(\sigma_2(z)) \\
&= \frac{\sigma_1^{-1}(w)\sigma_1'(\sigma_1^{-1}(w))}{w}f(w) + \frac{\sigma_1^{-1}(w)\sigma_1'(\sigma_1^{-1}(w))\zeta'(w)}{\zeta(w)}f(\zeta(w)) \\
0 &= \frac{1}{w}f(w) + \frac{\zeta'(w)}{\zeta(w)}f(\zeta(w)) \\
0 &= \zeta(w)f(w) + w\zeta'(w)f(\zeta(w)).
\end{aligned}$$

This shows that if f is in $\ker(C_\varphi^*)$, then f satisfies Equation 4.1.

Now, if f satisfies Equation 4.1, we show that $f(0) = 0$ and f is in $\ker(C_\varphi^*)$. Solving for $f(z)$ in Equation 4.1 gives

$$f(z) = -\frac{z\zeta'(z)}{\zeta(z)}f(\zeta(z)).$$

If $\zeta(0) \neq 0$, then $f(0) = -\dfrac{0 \cdot \zeta'(0)}{\zeta(0)} f(\zeta(0)) = 0$. Otherwise, if $\zeta(0) = 0$, then ζ has the form $\zeta(z) = \dfrac{az}{cz+d}$ since ζ is an LFT by Theorem 4.1.2. Therefore, $\zeta'(z) = \dfrac{ad}{(cz+d)^2}$ and

$$\begin{aligned}
f(0) &= \lim_{z\to 0}\left(-\frac{z\zeta'(z)}{\zeta(z)} f(\zeta(z))\right) \\
&= \lim_{z\to 0}\left(-\frac{d}{cz+d} f(\zeta(z))\right) \\
&= -\frac{d}{c\cdot 0 + d} f(\zeta(0)) \\
&= -f(0) \\
f(0) &= 0
\end{aligned}$$

To see that $f \in \ker(C_\varphi^*)$, set $w = \sigma_1(z)$ and $\zeta(w) = \zeta(\sigma_1(z)) = \sigma_2(z)$ to transform Equation 4.1 back to

$$\begin{aligned}
0 &= \psi_1(z) f(\sigma_1(z)) + \psi_2(z) f(\sigma_2(z)) \\
&= \frac{f(0)}{1 - \overline{\varphi(\infty)} z} + \sum_{j=1}^{2} \psi_j(z) f(\sigma_j(z)) \\
&= C_\varphi^* f(z)
\end{aligned}$$

■

Corollary 4.1.1 *Let φ be a rational map of degree two mapping $\mathbb{D}$ into $\mathbb{D}$ and let C_φ act on the Hardy space $H^2(\mathbb{D})$. If f is in $\ker(C_\varphi^*)$ and $\mathbb{D} \cap \zeta(\mathbb{D}) \neq \varnothing$, then f can be extended to be analytic on $\mathbb{D} \cup \varphi(\mathbb{D})$.*

Proof From Theorem 4.1.2, we have that ζ is an LFT, and write $\zeta(z) = \dfrac{az+b}{cz+d}$ and $\zeta'(z) = \dfrac{ad-bc}{(cz+d)^2}$. If f is in $\ker(C_\varphi^*)$, then $f(0) = 0$ from Lemma 3.0.2 and therefore

f is in $zH^2(\mathbb{D})$, i.e., there is $g \in H^2(\mathbb{D})$ such that $f(z) = zg(z)$. From Equation 4.1, we have that $\zeta(z)f(z) + z\zeta'(z)f(\zeta(z)) = 0$, or solving for $f(\zeta(z))$ yields

$$\begin{aligned} f(\zeta(z)) &= -\frac{\zeta(z)}{z\zeta'(z)} f(z) \\ &= -\frac{(az+b)(cz+d)^2}{(ad-bc)(cz+d)z} zg(z) \\ &= -\frac{(az+b)(cz+d)}{ad-bc} g(z) \end{aligned}$$

where $-\dfrac{(az+b)(cz+d)}{ad-bc} g(z)$ is analytic for $z \in \mathbb{D}$. Therefore, $f(\zeta(z))$ is analytic for $z \in \mathbb{D}$ or f is analytic on $\zeta(\mathbb{D})$. Thus, f is analytic on $\mathbb{D}$ and $\zeta(\mathbb{D})$. ■

4.1.2 Examples

Example 4.1.1 *Let* $\varphi(z) = \dfrac{z^2+1}{2}$, *a map for which* $\varphi(\mathbb{D}) \subset \mathbb{D}$ *and, in fact, is two-to-one on* $\mathbb{D}$. *Then* $\zeta(z) = -z$ *and by Theorem 4.1.3,*

$$\ker(C_\varphi^*) = \left\{ f \in H^2(\mathbb{D}) \,\middle|\, f(z) = -f(-z) \right\}$$

which agrees with Corollary 3.2.1. Let $W = \overline{\operatorname{span}\{K_\alpha - K_\beta \mid \varphi(\alpha) = \varphi(\beta), \alpha, \beta \in \mathbb{D}\}}$. *We show that* $W = \ker(C_\varphi^*)$. *From Theorem 4.1.1, we have that* $W \subset \ker(C_\varphi^*)$. *Proceeding by contradiction, assume* $W \neq \ker(C_\varphi^*)$. *There exists* $f \in \ker(C_\varphi^*)$ *such that* $f = g + h$ *where* $g \in W$, $h \in W^\perp$, *and* $h \neq 0$. *Now,* $h = f - g \in \ker(C_\varphi^*)$. *Suppose* $\varphi(\alpha) = \varphi(\beta)$. *If* $\beta \neq \alpha$, *then* $\beta = -\alpha$. *For every* $\alpha \in \mathbb{D}$,

$$\begin{aligned} 0 &= \langle h, K_\alpha - K_\beta \rangle_{H^2} \\ &= \langle h, K_\alpha - K_{-\alpha} \rangle_{H^2} \\ &= h(\alpha) - h(-\alpha) \\ &= 2h(\alpha) \\ 0 &= h(\alpha) \end{aligned}$$

Therefore, $h = 0$ *which is a contradiction, and so* $W = \ker(C_\varphi^*)$. *In fact, the same conclusion holds in any weighted Bergman space* $A_\alpha^2(\mathbb{D})$ *for* $\alpha > -1$.

Example 4.1.2 *Let* $\varphi(z) = \dfrac{z^2+z}{2}$*, a map for which* $\varphi(\mathbb{D}) \subset \mathbb{D}$*, is two-to-one on* $S = \{z \in \mathbb{D} : \mathrm{Re}\,(z) > -1/2\}$*, and is univalent on* $\mathbb{D}\backslash S$*. Then* $\zeta(z) = -\dfrac{z}{z+1}$*. Theorem 4.1.3 gives* $f \in \ker(C_\varphi^*)$ *if and only if*

$$-\frac{z}{z+1}f(z) + z\frac{-1}{(z+1)^2}f\left(-\frac{z}{z+1}\right) = 0$$

$$(z+1)f(z) + f\left(-\frac{z}{z+1}\right) = 0 \tag{4.2}$$

If $f \in H^2(\mathbb{D})$*, then* $(z+1)f \in H^2(\mathbb{D})$ *and hence,* $(z+1)f$ *is analytic for* $z \in \mathbb{D}$*. Note that for some* $z \in \mathbb{D}$*, we have* $\zeta(z) \notin \mathbb{D}$*, and in fact, the linear fractional transformation* ζ *maps* $\mathbb{D}$ *onto the half-plane* $\{w : \mathrm{Re}\,(w) > -1/2\}$*. Therefore, the result from Theorem 4.1.3 shows that if* $f \in \ker(C_\varphi^*)$*, then* f *is analytic in a set larger than* $\mathbb{D}$*, namely in* $\mathbb{D} \cup \{w : \mathrm{Re}\,(w) > -1/2\}$*. Indeed, for* $w \in \mathbb{C}$ *with* $\mathrm{Re}\,(w) > -1/2$*, we have that* $\zeta^{-1}(w) = \zeta(w) \in \mathbb{D}$*. Rearranging Equation 4.2 gives*

$$f(w) = -(\zeta(w)+1)f(\zeta(w))$$

and so f *is analytic in the half-plane* $\{w : \mathrm{Re}\,(w) > -1/2\}$ *as well as in* $\mathbb{D}$*.*

4.2 More Composition Operators Whose Adjoints Have Trivial Kernels

We describe some more composition operators whose adjoints have trivial kernels.

Theorem 4.2.1 *Let* φ *be a rational map of degree two mapping* $\mathbb{D}$ *into* $\mathbb{D}$ *and let* C_φ *act on the Hardy space* $H^2(\mathbb{D})$*. Consider* ζ *as in Theorem 4.1.2. If* $\zeta(z_0) = \infty$ *for some* $z_0 \in \mathbb{D}$ *and* $\mathbb{D} \cup \zeta(\mathbb{D}) = \widehat{\mathbb{C}}$*, then* $\ker(C_\varphi^*) = \{0\}$*.*

Proof Since ζ is an LFT, we can write ζ as $\zeta(z) = \dfrac{az+b}{z-z_0}$ where $az_0+b \neq 0$ and $\zeta'(z) = -\dfrac{az_0+b}{(z-z_0)^2}$. If f is in $\ker(C_\varphi^*)$, then f satisfies Equation 4.1:

$$\begin{aligned} 0 &= \zeta(z)f(z) + z\zeta'(z)f(\zeta(z)) \\ 0 &= \frac{az+b}{z-z_0}f(z) - \frac{z(az_0+b)}{(z-z_0)^2}f(\zeta(z)) \end{aligned} \tag{4.3}$$

If $z_0 = 0$, then multiplying both sides of Equation 4.3 by z gives

$$0 = (az + b)f(z) - bf(\zeta(z)) \tag{4.4}$$

and evaluating at $z = 0$ results in $f(\infty) = 0$ since $f(0) = 0$ by Lemma 3.0.2.
If $z_0 \neq 0$, then multiplying both sides of Equation 4.3 by $(z - z_0)^2$ gives

$$0 = (az + b)(z - z_0)f(z) - z(az_0 + b)f(\zeta(z))$$

and evaluating at $z = z_0$ results in $f(\infty) = 0$.

Note that $\zeta(\mathbb{D})$ is an open set in $\widehat{\mathbb{C}}$ and is the complement of a closed disk in $\mathbb{D}$. Indeed, since $\mathbb{D} \cup \zeta(\mathbb{D}) = \widehat{\mathbb{C}}$, then $\mathbb{D} \cap \zeta(\mathbb{D})$ is open and non-empty, and as an LFT, ζ maps the unit circle $\partial\mathbb{D}$ to a circle contained in $\mathbb{D}$. Now, there exists $0 < r_0 < 1$ such that the circle of radius r_0 centered at the origin is contained in $\mathbb{D} \cap \zeta(\mathbb{D})$. Consider the compact sets $\overline{r\mathbb{D}}$ and $\zeta\left(\overline{r\mathbb{D}}\right)$ for $r_0 \leq r < 1$. Since f is a function in $H^2(\mathbb{D})$, then f is bounded on compact subsets of $\mathbb{D}$, and thus f is bounded on $\overline{r\mathbb{D}}$. For $z \in \zeta(\mathbb{D})$, there is $w \in \mathbb{D}$ such that $\zeta(w) = z$ and $f(z) = f(\zeta(w)) = \dfrac{(z - z_0)(az + b)}{z(az_0 + b)} f(w)$ from Equation 4.4 shows that f is bounded on $\zeta\left(\overline{r\mathbb{D}}\right)$. Therefore, f is a bounded and analytic function on $\widehat{\mathbb{C}}$, and $f \equiv 0$ by Liouville's Theorem. ■

Example 4.2.1 *Let* $\varphi(z) = \dfrac{3z}{4 - z^2}$*, a map for which* $\varphi(\mathbb{D}) \subset \mathbb{D}$*, and in fact, is univalent on* $\mathbb{D}$*. Then,* $\zeta(z) = \dfrac{1}{4z}$ *maps* $\mathbb{D}$ *to* $\widehat{\mathbb{C}} \backslash \frac{1}{4}(\mathbb{D} \cup \partial\mathbb{D})$ *and so* $\mathbb{D} \cup \zeta(\mathbb{D}) = \widehat{\mathbb{C}}$*. Therefore,* $\ker(C_\varphi^*) = \{0\}$ *by Theorem 4.2.1.*

Theorem 4.2.2 *For* $\varphi(z) = az^2 + bz + c$ *mapping* $\mathbb{D}$ *to* $\mathbb{D}$*,* φ *is univalent on* $\mathbb{D}$ *if and only if* $\left|\dfrac{b}{a}\right| \geq 2$*. Furthermore, if* C_φ *acts on the Hardy space* $H^2(\mathbb{D})$ *or a weighted Bergman space* $A^2_\alpha(\mathbb{D})$ *for* $\alpha > -1$*, then* $\ker(C_\varphi^*) = \{0\}$*.*

Proof Suppose $\varphi(z) = \varphi(w)$ and solve for w:

$$\begin{aligned} az^2 + bz + c &= aw^2 + bw + c \\ az^2 - aw^2 + bz - bw &= 0 \\ a(z + w)(z - w) + b(z - w) &= 0 \\ (z - w)(az + aw + b) &= 0 \end{aligned}$$

Either $w = z$ or $w = -\dfrac{az+b}{a} = -\left(z + \dfrac{b}{a}\right)$. Now, φ is univalent on $\mathbb{D}$ if and only if $-\left(z + \dfrac{b}{a}\right) \notin \mathbb{D}$ if and only if $\left|\dfrac{b}{a}\right| \geq 2$.

If φ is univalent on $\mathbb{D}$ and if C_φ acts on the Hardy space $H^2(\mathbb{D})$ or a weighted Bergman space $A^2_\alpha(\mathbb{D})$ for $\alpha > -1$, then $\ker(C_\varphi^*) = \{0\}$ by Theorem 3.1.5. ∎

4.3 Composition Operators with Monomial Symbols

We now investigate a couple of composition operators with special rational symbols.

Theorem 4.3.1 *Let n be an integer with $n \geq 2$ and $\varphi(z) = z^n$. Let C_φ act on $H^2(\mathbb{D})$. Then, f is in $\ker(C_\varphi^*)$ if and only if*

$$\sum_{j=1}^{n} f\left(\zeta_j(z)\right) = 0$$

where $\zeta(z) = e^{2\pi ik/n}z$, k and n are relatively prime, $\zeta_j = \underbrace{\zeta \circ \zeta \circ \ldots \circ \zeta}_{j \text{ times}}$, and $z \in \mathbb{D}$.

Proof Observe the branches of $\varphi^{-1}(z) = e^{2\pi im/n}z^{1/n}$ for $m = 1, 2, \ldots, n$, and the branches of σ from Theorem 2.1.1 are $\sigma(z) = \dfrac{1}{\overline{\varphi^{-1}(1/\overline{z})}} = e^{2\pi im/n}z^{1/n}$ with the corresponding maps

$$\psi(z) = \frac{z\sigma'(z)}{\sigma(z)} = \frac{z\frac{1}{n}e^{2\pi im/n}z^{1/n-1}}{e^{2\pi im/n}z^{1/n}} = \frac{1}{n}.$$

Also, note the branches of

$$\zeta(z) = \frac{1}{\overline{\varphi^{-1}(\varphi(1/\overline{z}))}} = \sigma\left(\frac{1}{\overline{\varphi(1/\overline{z})}}\right) = e^{2\pi im/n}z.$$

If $1 \leq k < n$ is relatively prime to n, then $e^{2\pi ik/n}$ is a primitive n^{th} root of unity and $\zeta_n = \text{id}$. Furthermore, for $\zeta(\sigma(z)) = e^{2\pi i(k+m)/n}z^{1/n}$, a different branch of σ, for $m = 1, 2, \ldots, n$.

Now, if $f \in \ker(C_\varphi^*)$, then $f(0) = 0$ from Lemma 3.0.2, and we have from Theorem 2.1.1 that

$$\begin{aligned}
0 &= (C_\varphi^* f)(z) \\
&= \frac{f(0)}{1 - \overline{\varphi(\infty)}z} + \sum \psi(z) f(\sigma(z)) \\
&= \sum \frac{1}{n} f(\sigma(z)) \\
0 &= \sum f(\sigma(z)) \qquad (4.5)
\end{aligned}$$

where the sum is taken over the branches of σ. For a particular branch of σ, set $w = \sigma(z)$ in Equation 4.5 and so

$$\sum_{j=1}^{n} f(\zeta_j(w)) = 0.$$

■

5. Results in the Bergman Space

Before Goshabulaghi and Vaezi's [6] paper, we established some preliminary results describing more explicitly the adjoint of a composition operator on the classical Bergman space $A^2(\mathbb{D})$.

5.1 Some Useful Calculations

We first give a few calculations that will help prove these results.

5.1.1 Partial Fraction Decomposition of a Rational Map

For α in $\mathbb{D}\backslash\{0\}$ and a polynomial $p(z)$ for which $p(1/\overline{\alpha}) \neq 0$, consider the fraction $\dfrac{1}{(1-\overline{\alpha}z)^2 p(z)^2}$ and the following partial fraction decomposition:

$$\frac{1}{(1-\overline{\alpha}z)^2 p(z)^2} = \frac{A}{(1-\overline{\alpha}z)^2} + \frac{B}{1-\overline{\alpha}z} + \frac{C}{p(z)^2} + \frac{D}{p(z)} \tag{5.1}$$

where A, B are constants in $\mathbb{C}$ and C, D are polynomials in z. We solve for A by multiplying each side of Equation 5.1 by $(1-\overline{\alpha}z)^2$ and evaluating at $z = 1/\overline{\alpha}$:

$$\begin{aligned}
\frac{(1-\overline{\alpha}z)^2}{(1-\overline{\alpha}z)^2 p(z)^2} &= \frac{A(1-\overline{\alpha}z)^2}{(1-\overline{\alpha}z)^2} + \frac{B(1-\overline{\alpha}z)^2}{1-\overline{\alpha}z} + \frac{C(1-\overline{\alpha}z)^2}{p(z)^2} + \frac{D(1-\overline{\alpha}z)^2}{p(z)} \\
\frac{1}{p(z)^2} &= A + B(1-\overline{\alpha}z) + \frac{C(1-\overline{\alpha}z)^2}{p(z)^2} + \frac{D(1-\overline{\alpha}z)^2}{p(z)} \\
\frac{1}{p(1/\overline{\alpha})^2} &= A + B\cdot 0 + \frac{C\cdot 0}{p(1/\overline{\alpha})^2} + \frac{D\cdot 0}{p(1/\overline{\alpha})} \\
\frac{1}{p(1/\overline{\alpha})^2} &= A
\end{aligned} \tag{5.2}$$

Now we have

$$\begin{aligned}
\frac{1}{(1-\overline{\alpha}z)^2p(z)^2} &= \frac{1}{p(1/\overline{\alpha})^2(1-\overline{\alpha}z)^2}+\frac{B}{1-\overline{\alpha}z}+\frac{C}{p(z)^2}+\frac{D}{p(z)} \\
\frac{1}{(1-\overline{\alpha}z)^2p(z)^2}-\frac{1}{p(1/\overline{\alpha})^2(1-\overline{\alpha}z)^2} &= \frac{B}{1-\overline{\alpha}z}+\frac{C}{p(z)^2}+\frac{D}{p(z)} \\
\frac{p(1/\overline{\alpha})^2-p(z)^2}{p(1/\overline{\alpha})^2(1-\overline{\alpha}z)^2p(z)^2} &= \frac{B}{1-\overline{\alpha}z}+\frac{C}{p(z)^2}+\frac{D}{p(z)} \qquad (5.3)
\end{aligned}$$

and we solve for B by multiplying each side of Equation 5.3 by $1-\overline{\alpha}z$ and evaluating at $z=1/\overline{\alpha}$:

$$\begin{aligned}
\frac{\left(p(1/\overline{\alpha})^2-p(z)^2\right)(1-\overline{\alpha}z)}{p(1/\overline{\alpha})^2(1-\overline{\alpha}z)^2p(z)^2} &= \frac{B(1-\overline{\alpha}z)}{1-\overline{\alpha}z}+\frac{C(1-\overline{\alpha}z)}{p(z)^2}+\frac{D(1-\overline{\alpha}z)}{p(z)} \\
\frac{p(1/\overline{\alpha})^2-p(z)^2}{p(1/\overline{\alpha})^2(1-\overline{\alpha}z)p(z)^2} &= B+\frac{C(1-\overline{\alpha}z)}{p(z)^2}+\frac{D(1-\overline{\alpha}z)}{p(z)} \\
\frac{p(1/\overline{\alpha})+p(z)}{p(1/\overline{\alpha})^2p(z)^2}\cdot\frac{p(1/\overline{\alpha})-p(z)}{1-\overline{\alpha}z} &= B+\frac{C(1-\overline{\alpha}z)}{p(z)^2}+\frac{D(1-\overline{\alpha}z)}{p(z)} \qquad (5.4)
\end{aligned}$$

In evaluating $\frac{p(1/\overline{\alpha})-p(z)}{1-\overline{\alpha}z}$ at $z=1/\overline{\alpha}$, we use l'Hôspital's rule in finding the limit:

$$\begin{aligned}
\lim_{z\to 1/\overline{\alpha}}\frac{p(1/\overline{\alpha})-p(z)}{1-\overline{\alpha}z} &= \lim_{z\to 1/\overline{\alpha}}\frac{-p'(z)}{-\overline{\alpha}} \\
&= \frac{p'(1/\overline{\alpha})}{\overline{\alpha}}
\end{aligned}$$

Finally, evaluating Equation 5.4 at $z=1/\overline{\alpha}$, we obtain

$$\begin{aligned}
\frac{p(1/\overline{\alpha})+p(1/\overline{\alpha})}{p(1/\overline{\alpha})^2p(1/\overline{\alpha})^2}\cdot\frac{p'(1/\overline{\alpha})}{\overline{\alpha}} &= B+\frac{C\cdot 0}{p(1/\overline{\alpha})^2}+\frac{D\cdot 0}{p(1/\overline{\alpha})} \\
\frac{2p'(1/\overline{\alpha})}{p(1/\overline{\alpha})^3} &= B \qquad (5.5)
\end{aligned}$$

Now, let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be n distinct points in $\mathbb{D}\backslash\{0\}$ and consider

$$q(z)=(1-\overline{\alpha_1}z)(1-\overline{\alpha_2}z)\cdots(1-\overline{\alpha_n}z).$$

Setting $q_j(z)=\frac{q(z)}{1-\overline{\alpha_j}z}$, we use partial fractions to expand $1/q(z)^2$ as above:

$$\begin{aligned}
\frac{1}{q(z)^2} &= \frac{1}{(1-\overline{\alpha_j}z)^2q_j(z)^2} \\
&= \frac{A_j}{(1-\overline{\alpha_j}z)^2}+\frac{B_j}{1-\overline{\alpha_j}z}+\sum_{\substack{1\le k\le n\\ k\neq j}}\left(\frac{A_k}{(1-\overline{\alpha_k}z)^2}+\frac{B_k}{1-\overline{\alpha_k}z}\right)
\end{aligned}$$

Using Equations 5.2 and 5.5, we solve for A_j and B_j:

$$A_j = \frac{1}{q_j(1/\overline{\alpha_j})^2} \tag{5.6}$$

$$B_j = \frac{2q_j'(1/\overline{\alpha_j})}{q_j(1/\overline{\alpha_j})^3} \tag{5.7}$$

5.1.2 Roots of Unity

Let ξ be an n^{th} root of unity, i.e., $\xi = e^{2\pi ik/n}$ for some $k = 1, 2, \ldots, n$. Consider the polynomial $1 - z^n$ and rewrite it as follows:

$$1 - z^n = 1 - \xi^n z^n$$

$$1 - z^n = (1 - \xi z)(1 + \xi z + \xi^2 z^2 + \cdots + \xi^{n-1} z^{n-1}) \tag{5.8}$$

Dividing each side of Equation 5.8 by $1 - \xi z$ and evaluating the limit as $z \to 1/\xi$ results in

$$\lim_{z \to 1/\xi} \frac{1 - z^n}{1 - \xi z} = 1 + \xi \cdot \frac{1}{\xi} + \xi^2 \cdot \left(\frac{1}{\xi}\right)^2 + \cdots + \xi^{n-1} \cdot \left(\frac{1}{\xi}\right)^{n-1}$$

$$\lim_{z \to 1/\xi} \frac{1 - z^n}{1 - \xi z} = n \tag{5.9}$$

5.1.3 An Integral Operator

We define an integral operator and show some of its properties acting on a Hilbert space of analytic functions.

Definition 5.1.1 *Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$. The **integral operator** A acting on $\mathcal{H}$ is defined by*

$$(Af)(z) = \int_0^z f(w)dw$$

for $z \in \mathbb{D}$ and $f \in \mathcal{H}$.

Recall that the reproducing kernel function at α in $A^2(\mathbb{D})$ is $K_\alpha(z) = \dfrac{1}{(1-\overline{\alpha}z)^2}$. Then,

$$\begin{aligned}
(AK_\alpha)(z) &= \int_0^z \frac{1}{(1-\overline{\alpha}w)^2}dw \\
&= \frac{1}{\overline{\alpha}} \cdot \frac{1}{1-\overline{\alpha}z} \\
\overline{\alpha}(AK_\alpha)(z) &= \frac{1}{1-\overline{\alpha}z} \qquad (5.10)
\end{aligned}$$

Suppose f is in $A^2(\mathbb{D})$. Write f as $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\overline{\alpha}(AK_\alpha)(z) = \sum_{n=0}^{\infty} \overline{\alpha}^n z^n$. Now, consider $\langle f, \overline{\alpha}AK_\alpha \rangle_{A^2}$:

$$\begin{aligned}
\langle f, \overline{\alpha}AK_\alpha \rangle_{A^2} &= \alpha \frac{a_n \alpha^n}{n+1} \\
&= \frac{a_n \alpha^{n+1}}{n+1} \\
\langle f, \overline{\alpha}AK_\alpha \rangle_{A^2} &= (Af)(\alpha) \qquad (5.11)
\end{aligned}$$

5.2 Preliminary Results

We turn to preliminary results obtained in attempting to find an explicit formula for the adjoint of a composition operator acting on the Bergman space $A^2(\mathbb{D})$. Recall Equation 1.1:

$$(C_\varphi^* f)(\alpha) = \left\langle C_\varphi^* f, K_\alpha \right\rangle_{A^2} = \langle f, C_\varphi K_\alpha \rangle_{A^2}$$

Lemma 5.2.1 *Let $\varphi(z) = z^2$ and let C_φ act on $A^2(\mathbb{D})$. Then,*

$$C_\varphi^* = \frac{1}{4}C_{\sqrt{z}} + \frac{1}{4}C_{-\sqrt{z}} + \frac{1}{4}C_{\sqrt{z}}A + \frac{1}{4}C_{-\sqrt{z}}A.$$

Proof Consider $C_\varphi K_\alpha(z)$:

$$\begin{aligned} C_\varphi K_\alpha(z) &= K_\alpha(\varphi(z)) \\ &= K_\alpha(z^2) \\ &= \frac{1}{(1-\overline{\alpha}z^2)^2} \\ &= \frac{1}{\left[\left(1-\sqrt{\overline{\alpha}}z\right)\left(1+\sqrt{\overline{\alpha}}z\right)\right]^2} \\ &= \frac{A}{\left(1-\sqrt{\overline{\alpha}}z\right)^2} + \frac{B}{1-\sqrt{\overline{\alpha}}z} + \frac{C}{\left(1+\sqrt{\overline{\alpha}}z\right)^2} + \frac{D}{1+\sqrt{\overline{\alpha}}z} \end{aligned}$$

From Equations 5.6 and 5.7, we compute the constants A, B, C, D:

$$A = \frac{1}{\left(1+\sqrt{\overline{\alpha}}\cdot\frac{1}{\sqrt{\overline{\alpha}}}\right)^2} = \frac{1}{4} \qquad B = \frac{2\sqrt{\overline{\alpha}}}{\sqrt{\overline{\alpha}}\left(1+\sqrt{\overline{\alpha}}\cdot\frac{1}{\sqrt{\overline{\alpha}}}\right)^3} = \frac{1}{4}$$

$$C = \frac{1}{\left(1-\sqrt{\overline{\alpha}}\cdot\frac{1}{-\sqrt{\overline{\alpha}}}\right)^2} = \frac{1}{4} \qquad D = \frac{-2\sqrt{\overline{\alpha}}}{-\sqrt{\overline{\alpha}}\left(1-\sqrt{\overline{\alpha}}\cdot\frac{1}{-\sqrt{\overline{\alpha}}}\right)^3} = \frac{1}{4}$$

Now,

$$\begin{aligned} C_\varphi K_\alpha(z) &= \frac{1/4}{\left(1-\sqrt{\overline{\alpha}}z\right)^2} + \frac{1/4}{1-\sqrt{\overline{\alpha}}z} + \frac{1/4}{\left(1+\sqrt{\overline{\alpha}}z\right)^2} + \frac{1/4}{1+\sqrt{\overline{\alpha}}z} \\ &= \frac{1/4}{\left(1-\sqrt{\overline{\alpha}}z\right)^2} + \frac{1/4}{\left(1+\sqrt{\overline{\alpha}}z\right)^2} + \frac{1/4}{1-\sqrt{\overline{\alpha}}z} + \frac{1/4}{1+\sqrt{\overline{\alpha}}z} \end{aligned}$$

Using Equation 5.10, we obtain

$$C_\varphi K_\alpha(z) = \frac{1}{4}K_{\sqrt{\alpha}}(z) + \frac{1}{4}K_{-\sqrt{\alpha}}(z) + \frac{1}{4}\sqrt{\overline{\alpha}}(AK_{\sqrt{\alpha}})(z) + \frac{1}{4}\left(-\sqrt{\overline{\alpha}}\right)(AK_{-\sqrt{\alpha}})(z).$$

Therefore, using Equation 5.11, we have that

$$\begin{aligned} (C_\varphi^* f)(\alpha) &= \langle f, C_\varphi K_\alpha\rangle_{A^2} \\ &= \frac{1}{4}f\left(\sqrt{\alpha}\right) + \frac{1}{4}f\left(-\sqrt{\alpha}\right) + \frac{1}{4}(Af)\left(\sqrt{\alpha}\right) + \frac{1}{4}(Af)\left(-\sqrt{\alpha}\right) \\ &= \frac{1}{4}(C_{\sqrt{z}}f)(\alpha) + \frac{1}{4}(C_{-\sqrt{z}}f)(\alpha) + \frac{1}{4}(C_{\sqrt{z}}Af)(\alpha) + \frac{1}{4}(C_{-\sqrt{z}}Af)(\alpha) \end{aligned}$$

■

Lemma 5.2.2 *Let $\varphi(z) = z^3$ and let C_φ act on $A^2(\mathbb{D})$. Then,*

$$C_\varphi^* = \sum_{k=1}^{3} \frac{1}{9} C_{\sigma_k} + \sum_{k=1}^{3} \frac{2}{9} C_{\sigma_k} A$$

where $\sigma_k(z) = e^{2\pi i k/3} z^{1/3}$.

Proof Let $a_k = \sigma_k(\alpha) = e^{2\pi i k/3}\alpha^{1/3}$ for $k = 1, 2, 3$, and consider $C_\varphi K_\alpha(z)$:

$$\begin{aligned} C_\varphi K_\alpha(z) &= K_\alpha\left(\varphi(z)\right) \\ &= K_\alpha(z^3) \\ &= \frac{1}{(1-\overline{\alpha} z^3)^2} \\ &= \frac{1}{\left[\,(1-\overline{a_1}z)\,(1-\overline{a_2}z)\,(1-\overline{a_3}z)\,\right]^2} \\ &= \frac{A_1}{(1-\overline{a_1}z)^2} + \frac{B_1}{1-\overline{a_1}z} + \frac{A_2}{(1-\overline{a_2}z)^2} + \frac{B_2}{1-\overline{a_2}z} + \frac{A_3}{(1-\overline{a_3}z)^2} + \frac{B_3}{1-\overline{a_3}z} \end{aligned}$$

Using Equations 5.6 and 5.7, we compute the constants $A_1, A_2, A_3, B_1, B_2, B_3$ where $q_j(z) = \dfrac{1-\overline{\alpha}z^3}{1-\overline{a_j}z}$:

$$A_1 = \frac{1}{9} \qquad B_1 = \frac{2}{9}$$

$$A_2 = \frac{1}{9} \qquad B_2 = \frac{2}{9}$$

$$A_3 = \frac{1}{9} \qquad B_3 = \frac{2}{9}$$

Now,

$$\begin{aligned} C_\varphi K_\alpha(z) &= \frac{1/9}{(1-\overline{a_1}z)^2} + \frac{2/9}{1-\overline{a_1}z} + \frac{1/9}{(1-\overline{a_2}z)^2} + \frac{2/9}{1-\overline{a_2}z} + \frac{1/9}{(1-\overline{a_3}z)^2} + \frac{2/9}{1-\overline{a_3}z} \\ &= \frac{1/9}{(1-\overline{a_1}z)^2} + \frac{1/9}{(1-\overline{a_2}z)^2} + \frac{1/9}{(1-\overline{a_3}z)^2} + \frac{2/9}{1-\overline{a_1}z} + \frac{2/9}{1-\overline{a_2}z} + \frac{2/9}{1-\overline{a_3}z} \end{aligned}$$

Using Equation 5.10, we obtain

$$\begin{aligned} C_\varphi K_\alpha(z) &= \frac{1}{9}K_{a_1}(z) + \frac{1}{9}K_{a_2}(z) + \frac{1}{9}K_{a_3}(z) \\ &\quad + \frac{2\overline{a_1}}{9}(AK_{a_1})(z) + \frac{2\overline{a_2}}{9}(AK_{a_2})(z) + \frac{2\overline{a_3}}{9}(AK_{a_3})(z) \end{aligned}$$

Therefore, using Equation 5.11, we have that

$$\begin{aligned}(C_\varphi^* f)(\alpha) &= \langle f, C_\varphi K_\alpha \rangle_{A^2} \\ &= \frac{1}{9} f(a_1) + \frac{1}{9} f(a_2) + \frac{1}{9} f(a_3) + \frac{2}{9}(Af)(a_1) + \frac{2}{9}(Af)(a_2) + \frac{2}{9}(Af)(a_3) \\ &= \frac{1}{9}(C_{\sigma_1} f)(\alpha) + \frac{1}{9}(C_{\sigma_2} f)(\alpha) + \frac{1}{9}(C_{\sigma_3} f)(\alpha) \\ &\quad + \frac{2}{9}(C_{\sigma_1} Af)(\alpha) + \frac{2}{9}(C_{\sigma_2} Af)(\alpha) + \frac{2}{9}(C_{\sigma_3} Af)(\alpha)\end{aligned}$$

■

Theorem 5.2.1 *Let $\varphi(z) = z^n$ and let C_φ act on $A^2(\mathbb{D})$. Then,*

$$C_\varphi^* = \sum_{k=1}^{n} \frac{1}{n^2} C_{\sigma_k} + \sum_{k=1}^{n} \frac{n-1}{n^2} C_{\sigma_k} A$$

where $\sigma_k(z) = e^{2\pi i k/n} z^{1/n}$.

Proof Let $q(z) = 1 - \overline{\alpha} z^n = \prod_{k=1}^{n} (1 - \overline{a_k} z)$ and $q_j(z) = \dfrac{q(z)}{1 - \overline{a_j} z}$ where

$$a_k = \sigma_k(\alpha) = e^{2\pi i k/n} \alpha^{1/n}$$

for $1 \le k \le n$. Then, for $1 \le j \le n$,

$$\begin{aligned}(C_\varphi K_\alpha)(z) &= \prod_{k=1}^{n} \frac{1}{(1 - \overline{a_k} z)^2} \\ &= \frac{A_j}{(1 - \overline{a_j} z)^2} + \frac{B_j}{1 - \overline{a_j} z} + \sum_{\substack{1 \le k \le n \\ k \ne j}} \left[\frac{A_k}{(1 - \overline{a_k} z)^2} + \frac{B_k}{1 - \overline{a_k} z} \right]\end{aligned}$$

We compute A_j and B_j using Equations 5.6 and 5.7. Consider

$$q_j(z) = \prod_{\substack{1 \le k \le n \\ k \ne j}} (1 - \overline{a_k} z)$$

and

$$
\begin{aligned}
q_j(1/\overline{a_j}) &= \prod_{\substack{1\le k\le n\\ k\ne j}} \left(1 - \frac{\overline{a_k}}{\overline{a_j}}\right) \\
&= \prod_{\substack{1\le k\le n\\ k\ne j}} \left(1 - e^{-2\pi i(k-j)/n}\right) \\
&= n
\end{aligned}
$$

using Equation 5.9. Now,

$$
\begin{aligned}
q_j'(z) &= \frac{(1-\overline{a_j}z)q'(z) + \overline{a_j}q(z)}{(1-\overline{a_j}z)^2} \\
&= \frac{(1-\overline{a_j}z)(-n\overline{\alpha}z^{n-1}) + \overline{a_j}(1-\overline{\alpha}z^n)}{(1-\overline{a_j}z)^2} \\
&= \frac{-n\overline{\alpha}z^{n-1} + n\overline{a_j}\overline{\alpha}z^n + \overline{a_j} - \overline{a_j}\overline{\alpha}z^n}{(1-\overline{a_j}z)^2} \\
&= \frac{\overline{a_j} - n\overline{\alpha}z^{n-1} + (n-1)\overline{a_j}\overline{\alpha}z^n}{(1-\overline{a_j}z)^2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
q_j'(1/\overline{a_j}) &= \lim_{z\to 1/a_j} \frac{\overline{a_j} - n\overline{\alpha}z^{n-1} + (n-1)\overline{a_j}\overline{\alpha}z^n}{(1-\overline{a_j}z)^2} \\
&= \lim_{z\to 1/\overline{a_j}} \frac{-n(n-1)\overline{\alpha}z^{n-2} + n(n-1)\overline{a_j}\overline{\alpha}z^{n-1}}{-2\overline{a_j}(1-\overline{a_j}z)} \\
&= \lim_{z\to 1/\overline{a_j}} \frac{-n(n-1)(n-2)\overline{\alpha}z^{n-3} + n(n-1)^2\overline{a_j}\overline{\alpha}z^{n-2}}{2\overline{a_j}^2} \\
&= \lim_{z\to 1/\overline{a_j}} \frac{-n(n-1)\overline{\alpha}z^{n-3}\left(n-2-(n-1)\overline{a_j}z\right)}{2\overline{a_j}^2} \\
&= \frac{-n(n-1)\overline{\alpha}(1/\overline{a_j})^{n-3}(-1)}{2\overline{a_j}^2} \\
&= \frac{n(n-1)\overline{a_j}^n\overline{a_j}^{3-n}\overline{a_j}^{-2}}{2} \\
&= \frac{n(n-1)\overline{a_j}}{2}
\end{aligned}
$$

We obtain

$$A_j = \frac{1}{q_j(1/\overline{a_j})^2} = \frac{1}{n^2}$$

and

$$B_j = \frac{2q_j'(1/\overline{a_j})}{q_j(1/\overline{a_j})^3} = \frac{n-1}{n^2}.$$

Thus,

$$(C_\varphi K_\alpha)(z) = \sum_{k=1}^{n} K_{a_k}(z) + \sum_{k=1}^{n} \frac{n-1}{n^2}\overline{a_k}(AK_{a_k})(z)$$

and

$$\begin{aligned}(C_\varphi^* f)(z) &= \langle f, C_\varphi K_\alpha \rangle_{A^2} \\ &= \sum_{k=1}^{n} \frac{1}{n^2}(C_{\sigma_k} f)(\alpha) + \sum_{k=1}^{n} \frac{n-1}{n^2}(C_{\sigma_k} Af)(\alpha)\end{aligned}$$

■

Corollary 5.2.1 *Let $\varphi(z) = z^n$ and let C_φ act on $A^2(\mathbb{D})$. Then, $C_\varphi K_\alpha$ is not a linear combination of kernel functions K_{a_j}.*

Proof Rewrite $(C_\varphi K_\alpha)(z)$ as

$$\begin{aligned}(C_\varphi K_\alpha)(z) &= \sum_{k=1}^{n} \frac{1/n^2}{(1-\overline{a_k}z)^2} + \sum_{k=1}^{n} \frac{(n-1)/n^2}{1-\overline{a_k}z} \\ &= \sum_{k=1}^{n} \frac{1}{n^2} \cdot \frac{1+(n-1)(1-\overline{a_k}z)}{(1-\overline{a_k}z)^2} \\ &= \sum_{k=1}^{n} \left(\frac{1}{n} - \frac{n-1}{n^2}\overline{a_k}z\right) K_{a_k}(z)\end{aligned}$$

■

5.3 Relating the Adjoint of a Composition Operator on the Hardy Space and Bergman Space

In arriving at formulas for the adjoint of a composition operator with rational symbol on the Bergman and Dirichlet spaces, Goshabulaghi and Vaezi [6] introduce an operator T that maps one weighted Hardy space to another.

Definition 5.3.1 *Let $H^2(\gamma)$ and $H^2(\beta)$ be weighted Hardy spaces with weight sequences $\{\gamma(n) = \gamma_n\}_{n=0}^{\infty}$ and $\{\beta(n) = \beta_n\}_{n=0}^{\infty}$, respectively. Then $T : H^2(\beta) \to H^2(\gamma)$ is defined by*

$$T\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_n \frac{\beta_n}{\gamma_n} z^n.$$

If $H^2(\gamma) = H^2(\mathbb{D})$ and $H^2(\beta) = A^2(\mathbb{D})$, then $T : A^2(\mathbb{D}) \to H^2(\mathbb{D})$ and

$$T\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} \frac{a_n}{\sqrt{n+1}} z^n.$$

Letting S_0 denote the adjoint of C_φ on $H^2(\mathbb{D})$ and S_1 denote the adjoint of C_φ on $A^2(\mathbb{D})$, Goshabulaghi and Vaezi [6] show in their proof of the formula for S_1 that S_0 and S_1 are related by

$$(S_1 f)(z) = (z S_0(T(Tf)))'(z).$$

Furthermore, we show how the kernels of S_0 and S_1 are related.

Theorem 5.3.1 *If f is in $\ker(S_1)$, then $T(Tf)$ is in $\ker(S_0)$, i.e.,*

$$T(T(\ker(S_1))) \subset \ker(S_0).$$

Proof Let f be in $\ker(S_1)$. Then

$$(z S_0(T(Tf)))'(z) = 0$$

and so

$$(z S_0(T(Tf)))(z) = c$$

for some constant c in $\mathbb{C}$. Since S_0 is a map of $H^2(\mathbb{D})$ to $H^2(\mathbb{D})$, $(z S_0(T(Tf)))(z)$ is in $zH^2(\mathbb{D})$ and hence $c = 0$. Therefore, $T(Tf)$ is in $\ker(S_0)$. ■

In her 2007 paper, Wahl [11] gave a complete characterization of functions in the kernel of C_φ^* acting on $H^2(\mathbb{D})$ with symbols $\varphi(z) = \dfrac{(1-2c)z^2}{1-2cz}$ for $0 < c < 1/2$:

Theorem 5.3.2 *Let* $\varphi(z) = \dfrac{(1-2c)z^2}{1-2cz}$ *for* $0 < c < 1/2$. *Suppose* f *is in* $H^2(\mathbb{D})$ *with constant term equal to* 0. *Then,* f *is in* $\ker(S_0)$ *if and only if* f *satisfies the successive derivative condition*

$$\left\langle f, \frac{K_c^{(2j)}}{(2j)!} - c\frac{K_c^{(2j+1)}}{(2j+1)!} \right\rangle_{H^2} = 0 \text{ for } j = 0, 1, 2, \ldots.$$

Applying Theorem 5.3.1 proves the following corollary:

Corollary 5.3.1 *Suppose* f *is in* $A^2(\mathbb{D})$ *with constant term equal to* 0. *If* f *is in* $\ker(S_1)$, *then* $T(Tf)$ *satisfies the successive derivative condition*

$$\left\langle T(Tf), \frac{K_c^{(2j)}}{(2j)!} - c\frac{K_c^{(2j+1)}}{(2j+1)!} \right\rangle_{H^2} = 0 \text{ for } j = 0, 1, 2, \ldots.$$

6. Results in the Dirichlet Space

From the adjoint formula on the Dirichlet space given in Goshabulaghi and Vaezi's [6] paper, we give an analogous characterization of functions in the kernel of C_φ^* to Theorem 4.1.3. Recall that the formula for the adjoint of a composition operator on the Dirichlet space is

$$(C_\varphi^* f)(z) = f(0)K_{\varphi(0)}(z) + \sum f(\sigma(z)) - \sum f(\sigma(0))$$

where the each sum is taken over the branches of σ as defined in Theorem 2.1.1. Also, if f is in $\ker(C_\varphi^*)$, then $f(0) = 0$ by Lemma 3.0.2.

Note that if $\varphi : \mathbb{D} \to \mathbb{D}$ is rational, then C_φ is bounded on D. Indeed, if the degree of φ is the positive integer n, then φ is at most n-to-1 on $\mathbb{D}$ and the norm of φ is at most n times the image of φ from Remark 1.3.1. Therefore, $\varphi \in \mathcal{D}$, has bounded multiplicity, and C_φ is bounded.

6.1 Main Results

Theorem 6.1.1 *Let φ be a rational map of degree two mapping $\mathbb{D}$ into $\mathbb{D}$ and let C_φ act on the Dirichlet space $\mathcal{D}$. Suppose $\varphi(\infty) = \infty$, i.e., $\varphi(z) = \dfrac{a_1 z^2 + b_1 z + c_1}{b_2 z + c_2}$ where $a_1 \neq 0$. For ζ as in Theorem 4.1.2, f is in $\ker(C_\varphi^*)$ if and only if*

$$f(z) + f(\zeta(z)) = 0.$$

Proof Consider $\sigma_k(0)$ for $k = 1, 2$ where $\sigma_k(0) = \lim_{z \to 0} \dfrac{1}{\overline{\varphi^{-1}(1/\overline{z})}}$. Since $\varphi(\infty) = \infty$ and φ is a rational map of degree two, then the only preimage of ∞ under φ is ∞. Therefore, $\sigma_1(0) = 0$ and $\sigma_2(0) = 0$.

For f in $\ker(C_\varphi^*)$,

$$\begin{aligned} 0 &= (C_\varphi^* f)(z) \\ &= f(\sigma_1(z)) + f(\sigma_2(z)) - f(\sigma_1(0)) - f(\sigma_2(0)) \\ &= f(\sigma_1(z)) + f(\sigma_2(z)) \end{aligned}$$

Setting $w = \sigma_1(z)$, we have $\zeta(w) = \sigma_2(z)$ and therefore, $f(w) + f(\zeta(w)) = 0$. ■

Theorem 6.1.2 *Let φ be a rational map of degree two mapping $\mathbb{D}$ into $\mathbb{D}$ and let C_φ act on the Dirichlet space $\mathcal{D}$. Suppose $\varphi(\infty) \neq \infty$. For ζ as in Theorem 4.1.2, f is in* $\ker(C_\varphi^*)$ *if and only if*

$$f(z) + f(\zeta(z)) = f(\sigma_1(0)) + f(\sigma_2(0)).$$

Proof For f in $\ker(C_\varphi^*)$,

$$\begin{aligned} 0 &= (C_\varphi^* f)(z) \\ &= f(\sigma_1(z)) + f(\sigma_2(z)) - f(\sigma_1(0)) - f(\sigma_2(0)) \end{aligned}$$

Setting $w = \sigma_1(z)$, we have $\zeta(w) = \sigma_2(z)$ and therefore,

$$f(w) + f(\zeta(w)) - f(\sigma_1(0)) - f(\sigma_2(0)) = 0.$$

■

Theorem 6.1.3 *Let φ be a rational map of degree two mapping $\mathbb{D}$ into $\mathbb{D}$ and let C_φ act on the Dirichlet space $\mathcal{D}$. If f is in* $\ker(C_\varphi^*)$, *then f is analytic on $\mathbb{D} \cup \zeta(\mathbb{D})$.*

Proof If $\varphi(\infty) = \infty$, then $f(z) = -f(\zeta(z))$ by Theorem 6.1.1. Since $f(z)$ is analytic on $\mathbb{D}$, then $f(\zeta(z))$ is analytic in $\zeta(\mathbb{D})$ and so f is analytic on $\mathbb{D} \cup \zeta(\mathbb{D})$.

If $\varphi(\infty) \neq \infty$, then $f(z) = -f(\zeta(z)) + f(\sigma_1(0)) + f(\sigma_2(0))$ by Theorem 6.1.2. Since $f(z)$ is analytic in $\mathbb{D}$, then $-f(\zeta(z)) + f(\sigma_1(0)) + f(\sigma_2(0))$ is analytic in $\zeta(\mathbb{D})$ and so f is analytic on $\mathbb{D} \cup \zeta(\mathbb{D})$. ■

6.2 Relating the Adjoint of a Composition Operator on the Hardy Space and Dirichlet Space

Using Definition 5.3.1, we take $H^2(\gamma)$ to be the Dirichlet space and $H^2(\beta)$ to be the Hardy space so that $T : H^2(\mathbb{D}) \to \mathcal{D}$, that is to say,

$$T\left(\sum_{n=0}^{\infty} a_n z^n\right) = a_0 + \sum_{n=1}^{\infty} a_n \sqrt{n} z^n$$

for $\sum_{n=0}^{\infty} a_n z^n \in H^2(\mathbb{D})$. Note that T is an isometry. Let S_0 be the adjoint of a composition operator on $\mathcal{D}$ and let S_1 be the adjoint of the same composition operator on $H^2(\mathbb{D})$. In their proof of the adjoint of a composition operator on the Dirichlet space, Goshabulaghi and Vaezi [6] relate S_0 and S_1 by

$$(S_0 f)(z) = T(T(S_1(T^{-1}(T^{-1} f))))(z) \tag{6.1}$$

where f is a polynomial.

Using Equation 6.1, we relate the kernel of S_0 and the kernel of S_1.

Theorem 6.2.1 *If f is a polynomial in $\ker(S_0)$, then $T^{-1}(T^{-1} f)$ is in $\ker(S_1)$.*

Proof If f is in $\ker(S_0)$, then

$$\begin{aligned} 0 &= (S_0 f)(z) \\ &= T(T(S_1(T^{-1}(T^{-1} f))))(z) \end{aligned}$$

Since T is an isometry between $\mathcal{D}$ and $H^2(\mathbb{D})$, we have that $S_1(T^{-1}(T^{-1} f)) = 0$ or $T^{-1}(T^{-1} f)$ is in $\ker(S_1)$. ∎

7. Summary

We have seen that in the Hardy space, perhaps the most-studied Hilbert space of analytic functions, characterizing functions in $\ker\left(C_\varphi^*\right)$ is non-trivial for non-univalent symbols on $\mathbb{D}$, and there is still more to be done for rational symbols of higher degrees. The characterization as developed in Chapter 4 becomes more complex due to the additional branches of φ^{-1} that come with a rational φ of degree larger than 2. Even less is known in the Bergman and Dirichlet spaces, but results obtained in the Hardy space will hopefully lead to analogous results in other weighted Hardy spaces.

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