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Rank constrained homotopies of matrices and the Blackadar-Handelman conjectures on C*-algebras

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RANK CONSTRAINED HOMOTOPIES OF MATRICES AND THE BLACKADAR-HANDELMAN CONJECTURES ON C^*-ALGEBRAS

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To my parents and two brothers.

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ABSTRACT

De Silva, Kaushika PhD, Purdue University, May 2016. Rank Constrained Homotopies of Matrices and the Blackadar-Handelman Conjectures on C^*-algebras. Major Professor: Andrew Toms.

Rank constrained homotopies of matrices:

For any $n \geq k \geq l \in \mathbb{N}$, let $S(n,k,l)$ be the set of all non-negative definite matrices $a \in M_n(\mathbb{C})$ with $l \leq \text{rank } a \leq k$. We investigate homotopy equivalence of continuous maps from a compact Hausdorff space X into sets of the form $S(n,k,l)$. From [37] it is known that for any n, if $4 dim\, X \leq k-l$ where $dim\, X$ denote the covering dimension of X, then there is exactly one homotopy class of maps from X into $S(n,k,l)$. In Section 3.1 we improve this bound by a factor of 8 by confirming $C(X, S(n,k,l))$ to have exactly one homotopy class of maps when $\left\lfloor \frac{dim\, X}{2} \right\rfloor \leq k-l$. This in particular means $\pi_r(S(n,k,l)) = 0$ for every $r \leq 2(k-l)+1$.

In Section 3.2 using classical homotopy theory methods together with C^*-algebraic ideas we confirm that for any $d \in \mathbb{N}$, if $\pi_r(S(n,k,l)) = 0, \forall r \leq d$, then there is only one homotopy class of maps in $C(X, S(n,k,l))$ for any compact Hausdorff X with $dim\, X \leq d$.

Blackadar-Handelman conjectures on C^*-algebras:

Let $DF(A)$ denote the set of all dimension functions on a C^*-algebra A and let $LDF(A)$ be the set of all $s \in DF(A)$ which are lower semicontinuous. It is well known that $DF(A)$ is naturally identified with the state space of the Cuntz semigroup $W(A)$. From [6], $LDF(A)$ bijectively corresponds to the space of all normalized quasitraces $QT(A)$ through a continuous affine map. [6] conjectures $LDF(A)$ to be pointwise dense in $DF(A)$ and $DF(A)$ to be a Choquet simplex.

In Theorem 5.1.1 we provide an equivalent condition for the first of these conjectures for unital A. Applying this condition we confirm the first conjecture for all unital A for which either the radius of comparison is finite or the semigroup $W(A)$ is almost unperforated (Theorem 5.2.5). Our results are achieved through applications of the techniques developed in [8] and [33].

If $LDF(A)$ is dense in $DF(A)$ for an unital A that has only finitely many extreme points in $QT(A)$, through a simple application of Krein-Milman Theorem we note that $DF(A) = LDF(A)$ and that $DF(A)$ is affinely homeomorphic to $QT(A)$. Together with results on the first conjecture this confirms the second conjecture for a new class of C^*-algebras.

Possibility of extending these results to inductive limits remain an open question.

In general the second conjecture is true for any unital A for which (ordered) Grothendieck group ${K_0}^*(A)$ of $W(A)$ has Riesz interpolation property [15] and every known confirmation of the second conjecture is achieved by showing Riesz interpolation hold for ${K_0}^*(A)$ $[1, 9, 29]$. We consider a *stably approximate* version of interpolation that is weaker than the classical Riesz interpolation. In fact it is easily seen that this property is even weaker than the *asymptotic interpolation* property considered in [28]. In Corollary 6.4.3 we confirm $DF(A)$ to be a Choquet simplex for any unital A for which $W(A)$ satisfies this weaker notion of interpolation.

While Corollary 6.4.3 has the scope of confirming the second conjecture for a broader class of C^*-algebras, finding a ‘good’ class of C^*-algebras in which $W(A)$ exhibits stably approximate interpolation but does not satisfy Riesz interpolation remains open.

1. INTRODUCTION

This thesis presents results obtained in two deferent directions. In Chapters 2 and 3 we focus on developing homotopy properties of certain rank constrained subsets of complex matrices. In Chapters 4, 5 and 6 we direct our attention towards answering two conjectures on the geometry of the dimension functions on C^*-algebras that were posted in [6].

Chapter 2 starts with basics from C^*-algebra theory. The rest of Chapter 2 recalls some topics in vector bundle theory and introduce other background results and terminology that we require in approaching the first topic. We present our results on the first topic in Chapter 3.

In Chapter 4 we have included preliminaries related to the two conjectures mentioned above. A particular interest is given towards recalling theory of partially ordered abelian groups and semigroups that play a central role in our approach to answering these conjectures. Chapters 5 and 6 present the main results that we have obtained in this direction.

In what follows in this Chapter we stimulate interest for both the topics while summarizing the already known results. At the same time an attempt is taken to explain the main ideas behind our proofs and their origins.

1.1 Rank constrained homotopies of complex matrices

Understanding homotopy properties of topological spaces is a fundamental question in topology. The knowledge on the homotopy of a space has often contributed to establishing results of significant importance in many branches of mathematics.

For a given triple $n, k, l \in \mathbb{N}$ with $n \geq k \geq l$ we focus on establishing homotopy properties of the sets,

$$S(n,k,l) = \{b \in M_n(\mathbb{C})_+ : \, l \leq \operatorname{rank}(b) \leq k\},$$

where $M_n(\mathbb{C})_+$ denotes the set of non-negative definite matrices in $M_n(\mathbb{C})$.

In Theorem 3.1.4, we show that for any given triple $n, k, l \in \mathbb{N}$ and a compact Hausdorff space X with $\lfloor \frac{dim\, X}{2} \rfloor \leq k - l$, there is precisely one homotopy class of functions in $C(X, S(n,k,l))$. Here, by $dim\, X$ we mean the covering dimension of the space X and $\lfloor m \rfloor$ stands for the *floor* function.

From Theorem 3.1.4 it is immediate that $\pi_r(S(n,k,l)) = 0$ for all $r \leq 2(k-l)+1$, where π_r stands for the rth homotopy group (Corollary 3.1.5).

The spaces $S(n,k,l)$ are natural objects in their own rights. However, to our knowledge the first considerations of the homotopy properties of the spaces $S(n,k,l)$ originated from the relevance of the topic to the theory of C^*-algebras [31, 37–39]. Our work builds on and is highly influenced by the ideas developed in these studies.

[37, Theorem 3.4] shows that a unital simple ASH algebras with slow dimension growth is $\mathcal{Z}$-stable. The basic building blocks of ASH-algebras are of the type $M_n((C(X)))$ where X is compact Hausdorff and the proof of [37, Theorem 3.4] depends heavily on homotopy properties of the spaces $S(n,k,l)$. One of the main technical results of [37] ([37, Proposition, 2.5]) shows $C(X, S(n,k,l))$ to have a precisely one homotopy class of maps for any $n, k, l \in \mathbb{N}$ and a compact Hausdorff space X with $\lfloor \frac{dim\, X}{2} \rfloor \leq k - l$. Theorem 3.1.4 here directly improves this. However, we should note that this improvement does not impact the main result of [37].

To further motivate this study as well as to better dissect the ideas involved in our work, we now exhibit the connection of spaces $S(n,k,l)$ to complex Grassmannians.

Recall that for $1 \leq n \leq \infty$, $k \in \mathbb{N}$ with $k \leq n$, $G_k(\mathbb{C}^n)$ stands for the complex Grassmann variety of k-dimensional subspaces of $\mathbb{C}^n$. Identifying each subspace V of $\mathbb{C}^n$, with the orthogonal projection of $\mathbb{C}^n$ on V leads to a natural homeomorphism from $G_k(\mathbb{C}^n)$ to the space $\mathcal{P}_k(\mathbb{C}^n)$ that consists of all rank k projections in $M_n(\mathbb{C})$.

Note that the inclusion $\mathcal{P}_k(\mathbb{C}^n) \subset S(n,k,k)$ is a homotopy equivalence. Thus, with above identification, $G_k(\mathbb{C}^n)$ is homotopy equivalent to $S(n,k,k)$. In this sense, the spaces $S(n,k,l)$ can be viewed as generalizations of the Grassmann varieties, at least for homotopy interests.

For instance setting $k = l$ in Corollary 3.1.5 recover the classical result that states $G_k(\mathbb{C}^n)$ is simply connected for any pair of k, n. However also note that our work does not provide an alternate proof of this fact, rather we use stronger classical results in our proof.

Some of the major applications of Grassmann varieties appear in the theory of vector bundles. Given $f \in C(X, G_k(\mathbb{C}^n)\ )$ one may *pullback* the canonical k-dimensional vector bundle ${\gamma_k}^n$ over $G_k(\mathbb{C}^n)$ to a k-dimensional vector bundle $(f^*({\gamma_k}^n))$ over X. Moreover, it is well known that $G_k(\mathbb{C}^\infty)$ acts as the classifying space of k-dimensional complex vector bundles over paracompact spaces, thus each k-dimensional vector bundle over a paracompact X corresponds to a bundle of the form $f^*(\gamma_{kn})$ and isomorphisms classes of k-dimensional vector bundles over X bijectively correspond to homotopy classes of maps in $C(X, G_k(\mathbb{C}^n)\ )$.

In a somewhat similar vein to a map $f \in C(X, G_k(\mathbb{C}^n)\ )$ generates a vector bundle, a map $a \in C(X, S(n,k,l)\ )$ generates a *bundle* over X. Indeed, for each such map a let ξ_a be the triple $(E(a), \pi_1, X)$ where,

$$E(a) = \{(x, v) \in X \times \mathbb{C}^n : x \in X, v \in a(x)(\mathbb{C}^n)\}$$

and $\pi_1 : E(a) \to X$ is the restriction of the coordinate projection of $X \times \mathbb{C}^n$ on X to $E(a)$.

While each fiber of a bundle ξ_a is a vector subspace of $\mathbb{C}^n$, in general bundles of this form are not vector bundles as they fail to be locally trivial. In fact, a typical bundle of the form ξ_a does not even have a constant fiber. Nonetheless the restrictions of these bundles to certain subsets of X are vector bundles, and thus one may seek to apply classical vector bundle theory to establish certain useful structure properties of these bundles. From the [38] (c.f. [31]) such an approach to understanding the structure of bundles ξ_a works best when a is in a certain special class called *well* *supported* positive elements. It is an immediate consequence of [38, Theorem 3.9] that each $a \in C(X, S(n,k,l))$ is homotopic to some well supported $b \in C(X, S(n,k,l))$ (Lemma 3.1.1).

Following [37], we view $a, b \in C(X, S(n,k,l))$ as bundles over X in the above sense and use the structure of these bundles to establish the homotopy equivalence of a, b whenever $\lfloor \frac{dim\, X}{2} \rfloor \leq k - l$. Therein our method of proof of Theorem 3.1.4 involves two main ideas. For any $a \in C(X, S(n,k,l))$ we seek to find some $\tilde{a} \in C(X, S(n,k,l))$ which is homotopic to a and some *trivial* vector bundle η over X of dimension l such that η is a sub-bundle of $\xi_{\tilde{a}}$. Using the notion of well supported positive elements and stability properties of vector bundles we show that this is indeed possible. Modulo the identification of vector bundles over X with projections in matrix algebras of $C(X)$ ([34]), the proof of Theorem 3.1.4 is now complete if we establish that $G_l(\mathbb{C}^n)$ acts as the classifying space of l-dimensional vector bundles over X for all compact Hausdorff X with $\lceil \frac{dim\, X}{2} \rceil \leq n - l$ and our second step is to prove this.

Indeed from the first step (after identifying vector bundles with projections) for any given $a, b \in C(X, S(n,k,l)\ )$ with $\lfloor \frac{dim\, X}{2} \rfloor \leq k - l$ there is a pair of projections $p_a, p_b \in M_n(C(X))$ of rank l which generate isomorphic vector bundles such that a, b are homotopic to p_a, p_b respectively in $C(X, S(n,k,l)\ )$. Then, as we may assume $n > k$ we have $\lceil \frac{dim\, X}{2} \rceil \leq n - l$ and from the conclusion of the second step p_a is homotopic to p_b in $C(X, S(n,k,l)\ )$.

We note that the fact of $G_l(\mathbb{C}^n)$ being the classifying space of l-dimensional vector bundles over compact Hausdorff spaces X with $\lceil \frac{dim\, X}{2} \rceil \leq n - l$ may well be a already known result. After all for the special case of X been a CW-complex this fact is stated in [21]. However, we could not find a clear reference of the conclusion for general compact Hausdorff spaces and hence we include a proof in this generality in Proposition 3.1.3.

From a topological view point it would have been more natural to prove the conclusion of Corollary 3.1.5 independently and then attempt to use that conclusion (i.e. triviality of *lower* homotopy groups) to derive the conclusion of Theorem 3.1.4.

In Section 3.2 we show that such an approach would indeed have been possible and moreover apply it in a slightly wider scope. We chose not to take this approach in proving Theorem 3.1.4 for to two reasons. Firstly, it would not have any effect on the main technicalities of our augments. Secondly, taking such an approach in proving Theorem 3.1.4 would have by passed the use of Proposition 3.1.3, which we thought could be of independent interest.

The main Theorem of Section 3.2 show that for a fixed $d \in \mathbb{N}$, if $\pi_r(S(n,k,l)) = 0$ for all $r \leq d$, then $C(X, S(n,k,l))$ has precisely one homotopy class of maps for every compact Hausdorff space X with $dim\, X \leq d$. We achieve this by applying classical dimension theory and homotopy theory techniques [14, 27, 45] together with some C^*-algebraic ideas.

Given a triple (n,k,l), a question of natural interest now is to find a non vanishing homotopy group of $S(n,k,l)$, if such exists. In the two extreme cases $n = k$ or $l = 0$ space $S(n,k,l)$ is contractible and hence all homotopy groups vanish. If $k = l$, $S(n,k,l)$ is homotopy equivalent to the complex Grassmannian $G_k(\mathbb{C}^n)$ and thus there are non vanishing homotopy groups of $S(n,k,l)$. For an arbitrary triple (n,k,l) this question remains open.

1.2 Dimension functions on C^*-algebras and Blackadar-Handelman conjectures

The notion of *dimension functions* extends the concept of rank of a matrix to arbitrary operator algebras and has contributed to results of significant importance in the field. For von-Neumann algebras dimension functions were first considered in [26], where the notion played a crucial part in classifying *factors*. For the C^*-case the notion was initiated by Cuntz in [12] where he introduced dimension functions for simple unital C^*-algebras. We write $DF(A)$ to denote the space of all dimension functions of a C^*-algebra A.

One main motivations of [12] was to provide methods to incorporate the techniques developed in [17] to study the notion of dimension functions on C^*-algebras. In particular Cuntz looked to identify $DF(A)$ of a C^*-algebra A with the state space of a suitable *partially ordered abelian group* G and apply the theory on such state spaces developed in [17] to analyze dimension functions. Cuntz accomplished this by associating the semigroup $W(A)$ (*Cuntz semigroup*) to a (unital simple) C^*-algebra A and confirming that $DF(A)$ is naturally homeomorph to the state space of ${K_0}^*(A)$, the Grothendieck enveloping group of $W(A)$. Cuntz semigroup is a natural extension of the Murray-von Neumann semigroup of projections to positive elements in $M_\infty(A)$ and is equipped with a translation invariant partial order. $W(A)$ is now understood to be an important isomorphism invariant for C^*-algebras and unraveling its' structure has become a fundamental research topic.

The ideas of [12] extends naturally to arbitrary (i.e possibly non-simple or non-unital) C^*-algebras. Continuing from [12], Handelman [20] and later Blackdar and Handelman [6] developed a more general and a detailed theory for dimension functions on C^*-algebras. One important consequence of [6] is a representation theorem for *lower semicontinuous dimension functions* (LDF(A)) via *normalized quasitraces* ($QT(A)$).

Recall that a (normalized) quasitrace is a complex-valued function on a C^*-algebra having all the usual properties of a (normalized) positive trace, but with linearity assumed only on commutative C^*-subalgebras. From [19] for unital and exact A every (normalized) quaitrace is a (normalized) trace.

Given $\tau \in QT(A)$ there corresponds a lower semicontinuous dimension function $d_\tau : M_\infty(A)_+ \to [0, \infty)$ given by

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}), \forall a \in M_\infty(A)_+.$$

Blackadar and Handelman in [6] showed that the above assignment defines an affine bijection from $QT(A)$ onto $LDF(A)$ and that the map has a continuous inverse.

We focus on two conjectures proposed in [6];

Conjecture 1.2.1 *[6] For any C^*- algebra A, $LDF(A)$ is dense in $DF(A)$ in the topology of pointwise convergence.*

Conjecture 1.2.2 *[6] For any C^*- algebra A, the affine space $DF(A)$ is a simplex.*

From the aforementioned identification of $LDF(A)$, Conjecture 1.2.1, if true, allows to approximate arbitrary dimension functions through better behaved and more intrinsic ones that corresponds to quasitraces. On the other hand the notion of a *simplex* generalize the concept of a standard n-simplex. Compact simplexes are of particular importance and are commonly referred to as *Choquet* simplexes. There is a well developed theory on Choquet simplexes - see [15]. Note that Conjecture 1.2.2 if true, would imply that $DF(A)$ is Choquet for a unital A. Thus positive answers to either of the conjectures provide useful tools that could be applied to derive properties of $DF(A)$ which in turn could provide details on the structure of $W(A)$.

For non stably finite C^*-algebras the conjectures hold trivially, as in this case ${K_0}^*(A) = 0$ and $DF(A)$ is the empty set. In the stably finite case there are various regularity conditions on C^*-algebras that imply at least one of the conjectures to be true. We outline these in the proceeding paragraph.

Conjecture 1.2.2 holds for unital commutative A by [6, Theorem I.2.4]. In [29, Corollary 4.4], Conjecture 1.2.2 is confirmed for unital C^*-algebras of real rank zero and stable rank one. The most general results on the conjectures that we are aware of appear in [9]. Theorem B of [9] confirm both the conjectures for unital, simple, separable, stably finite C^*- algebras which are either exact and $\mathcal{Z}$-stable or are AH-algebras of slow dimension growth. Furthermore [9, Remark 6.5] show that for exact A Conjecture 1.2.1 holds if A is assumed to have strict comparison instead of $\mathcal{Z}$-stability. Applying the methods in [9], both the conjectures are confirmed for several classes of continuous fields of C^*-algebras in [1].

The above confirmations of the conjectures are more or less consequences of two structure theorems for $W(A)$ (i.e. [29, Theorem 2.8] and [9, Theorems 6.4 and 6.6]). Apart form their usefulness in establishing the conjectures these structure theorems have other important applications - see [10] for an example. However, for the purpose of confirming the conjectures such theorems are too strong requirements to ask for. To our knowledge there has not been any work focusing on the conjectures alone and we attempt to step in this direction. Note that we will be only considering conjectures in the unital case.

In Chapter 5 we mainly focus on Conjecture 1.2.1. We investigate the possibility of applying theory on state spaces of partially ordered semigroups developed mainly in [8] (c.f [17]) to confirm Conjecture 1.2.1 for a wider range of unital C^*-algebra. As it turns out this can be readily achieved. These techniques (of [8]) allow us to prove the following theorem which give an alternate form of Conjecture 1.2.1 for unital A.

Theorem 5.1.1 Let A be a unital C^*-algebra. Then $LDF(A)$ is dense in $DF(A)$ if and only if $\iota : (W(A), \langle 1_A \rangle) \to (LAff_b(QT(A))^+, 1)$ is a stable order embedding.

By $LAff_b(QT(A))^+$ we mean the partially ordered abelian semigroup of bounded non negative lower semicontinuous affine maps on $QT(A)$ and ι is given by $\iota(\langle a \rangle)(\tau) = d_\tau(a), \forall \tau \in QT(A)$. *Stable order embedding* is a notion introduced in [8].

In a sense Theorem 5.1.1 provides a weaker form of the representation of $W(A)$ given in [9, Theorem 6.4].

Using the above we confirm the conjectures in the following cases.

Theorem 5.2.5 Let A be any unital C^*-algebra. The following hold.

1. If A has finite radius of comparison then $LDF(A)$ is dense in $DF(A)$.
2. If $W(A)$ almost unperforated then $LDF(A)$ is dense in $DF(A)$.
3. If $\partial_e(QT(A))$ is a finite set and if either of the assumptions above (in 1,2) holds for A then $DF(A) = LDF(A)$ and $DF(A)$ is affiinely homeomorphic to $QT(A)$. In particular $DF(A)$ is a Choquet simplex.

To prove (1) and (2), we verify that the alternate form of Conjecture 1.2.1 provided in Theorem 5.1.1 hold in the respective classes. In (1) this is done by applying techniques of [8] once more while in the second case this is done by following the ideas of [33]. Combining the conclusions of parts 1 and 2 with Lemma 5.2.4 which is mainly a consequence of Krein-Milman Theorem, we prove (3).

On the one hand these results do not assume simplicity or exactness as in [9] and on the other hand finite radius of comparison is a considerably weaker assumption than any of the regularity assumptions considered in [9]. Most of the continuous fields considered in [1] are also known to have finite radius of comparison.

In particular, the counter examples for Elliott's classification conjecture constructed in [36] and Villadsen algebras of type I [41] have finite radius of comparison but are not covered by [9]. Furthermore Villadsen algebras of type II [42] are of finite radius comparison with a unique quasitrace, and thus satisfy both conjectures from Theorem 5.2.5. This means that for each $n \in \mathbb{N}$, we now know that there are unital algebras of stable rank n which satisfy the conjectures.

For simple C^*-algebras, almost unperforation of $W(A)$ is equivalent to strict comparison (i.e zero radius of comparison) and thus the second case may seem redundant when compared to 1. However, in general (without simplicity) it is not clear how the two properties relate to each other.

From [15, Theorem 10.17] the state space of an *interpolation group* is a Choquet simplex. Thus, for any unital C^*-algebra A if the ordered group ${K_0}^*(A)$ is an interpolation group then Conjecture 1.2.2 holds for A. In fact, each of the confirmations of Conjecture 1.2.2 that are known to us goes through this fact. In other words, in each of those cases the group $K_0^*(A)$ was shown to have the Riesz interpolation property [1, 9, 29].

The existing techniques that can be applied to show that ${K_0}^*(A)$ is an interpolation group are limited in scope. For instance, in [9] and [29] the interpolation property of ${K_0}^*(A)$ is derived yet again from the respective structure theorems for $W(A)$ (i.e [9, Theorem] and [29, Theorem 2.8]) and as pointed out before both these theorems require the C^*-algebra A to have strong regularity properties. Moreover, the interpolation assumption on $W(A)$ is in itself a strong regularity condition for a C^*-algebra A. Hence, to confirm Conjecture 1.2.2 for a larger class of unital C^*-algebras it is both natural and desirable to find out the ways in which the hypothesis of [15, Theorem 10.17] could be relaxed.

In Chapter 6 we investigate a class of scaled partially ordered abelian groups that satisfy a weaker notion of interpolation (Definition 6.2.3). We name such groups as ordered groups with *stably approximate interpolation*. Analogues to the case of interpolation groups this weaker notion of interpolation has equivalent characterizations of *decomposition* and *refinement* types (Proposition 6.2.2). In Corollary 6.3.7, which is the main result of the chapter, we show $S(G, u)$ to be a Choquet simplex for any scaled ordered group G which belongs to this class of groups. Our proof is based very much on the ideas of [15]. We simply show that the techniques used in [15] in proving Theorem 6.1.4 for interpolation groups can be adopted to our case after some modifications. The class of ordered groups we consider is strictly larger than the class of interpolation groups. In particular these include the *asymptotic interpolation groups* considered in [28] (see Example 6.2.4).

Corollary 6.3.8 provides a method of confirming Conjecture 1.2.2 to a boarder class of C^*-algebras. There are C^*-algebras where the respective ${K_0}^*$ groups are not interpolation groups but sill exhibit stably approximate interpolation (see proof of 6.4.4). However, our work is not complete in this regard as we do not have an example for a class of C^*-algebras where $K_0^*(A)$ has stably approximate interpolation but does not have asymptotic interpolation (in the sense of [28]). Apart from its relevance in confirming the second Conjecture, the notion of stably approximate interpolation can potentially generalize certain other results from the theory of interpolation groups. At the end of Chapter 6 we point out some observations and possibilities for future work, in these directions.

2. PRELIMINARIES AND NOTATIONS

2.1 C^*-algebras.

We recall some basic definitions, results and terminology from C^*-algebra theory. Our main references for this section are [23] and [25].

Let A be an associative algebra over $\mathbb{C}$. Let $* : A \to A$ be a conjugate linear involuntary operation. We will denote the image of $a \in A$ under $*$ by a^*. If $*$ is such that $(ab)^* = b^*a^*, \forall a, b \in A$, $*$ is called an *adjoint* operation on A and the pair $(A, *)$ is called a $*$-*algebra*. A sub-$*$-algebra of $(A, *)$ is a subalgebra of A which is closed under the $*$-operation.

Recall that a complex *Banach algebra* is an associative complex algebra A together with Banach space norm $||\cdot||$ which satisfy $||ab|| \leq ||a||||b||, \forall a, b \in A$.

Definition 2.1.1 *Let $(A, ||\cdot||)$ be a complex Banach algebra and $(A, *)$ be a $*$-algebra. Then $(A, *, ||\cdot||)$, or simply A when $*$ and $||\cdot||$ are clear, is said to be a C^*-algebra if $||a^*a|| = ||a||^2$ for all $a \in A$.*

A C^*-algebra A is called unital if A is unital as an algebra. If A is unital we denote the unit by 1_A or simply 1 when the context is clear.

It follows that for all a in a C^*-algebra A, $||a|| = ||a^*||$.

The condition $||a^*a|| = ||a||^2$ is called the C^*-identity of the norm. A norm (not necessarily complete) on a $*$-algebra A which satisfies the above with $||ab|| \leq ||a||||b|| \, \forall a, b \in A$ is called a C^*-*norm*. There is at most one complete C^*-norm on a $*$-algebra.

A non empty subset B of a C^*-algebra A is a sub-C^*-algebra of A if B together with operations and norm inherited by A form a C^*-algebra of its own right. Thus, B is a sub-C^*-algebra of A if and only if B is norm closed sub-$*$-algebra of A.

Examples:

i). $\mathbb{C}$ together with usual operations and norm.

ii). $M_n(\mathbb{C})$ with usual operations and operator norm on matrices.

iii). The set of all bounded linear operators $B(H)$, on a Hilbert space H, together with the usual operations and operator norm.

iv). $C_0(X)$ - the algebra of complex valued continuous functions on a locally compact Hausdorff space X that vanish at infinity, together with the *sup* norm $||\cdot||_\infty$ and point-wise conjugation as the adjoint.

In the above examples, all but (iv) are unital and (iv) is unital if and only if X is compact. From 2.2.9 any abstract commutative C^*-algebra is of the form (iv).

Definition 2.1.2 *Let A, B be C^*-algebras. $\phi : A \to B$ is called a $*$-homomorphism if ϕ is linear, multiplicative and satisfy $\phi(a^*) = \phi(a)^*, \forall a \in A$. If in addition ϕ is bijective then its called a $*$-isomorphism. If A and B are unital and ϕ is such that $\phi(1_A) = 1_B$ then ϕ is called an unital $*$-homomorphism.*

Any abstract C^*-algebra can be faithfully represented as a sub-C^*-algebra of $B(H)$ for some Hilbert space H.

Theorem 2.1.3 *(Gelfand-Naimark) For each C^*-algebra A there is a Hilbert space H and an isometric $*$-homomorphism $\phi : A \to B(H)$. If A is seperable (i.e. contain a countable dense subset) then H can chosen to be separable.*

Let A be a C^*-algebra and I be a two sided (algebraic) ideal of A. Then I is automatically $*$-closed. Hence, its a sub-C^*-algebra iff its norm closed.

In this case (I norm closed), the quotient algebra $A/I = \{a + I : a \in A\}$ can be equipped with a complete norm given by $||a + I|| = \inf\{||a + x|| : x \in I\}$ and a well defined adjoint operation given by $(a + I)^* = a^* + I$. In this way A/I becomes a C^*-algebra and the quotient mapping $\pi : A \to A/I$ is a $*$-homomorphism with $\mathrm{Ker}(\pi) = I$.

A *unital extension* of a C^*-algebra A is any unital C^*-algebra which contain A as a closed ideal. For every C^*-algebra A there is a unitization $\tilde{A}$ with $\tilde{A}/A = \mathbb{C}$.

$\tilde{A}$ is unique (up to isomorphism) with respect to this property and is the minimal nontrivial unitization of A. That is, for any unitzation B of A with $B/A \neq 0$ there is an injective $*$-homomorphism $\iota : \tilde{A} \to B$ such that $\iota(a) = a, \forall a \in A$. In particular, we may construct $\tilde{A}$ by setting $\tilde{A}$ to be $A \oplus \mathbb{C}$ as a vector space and defining multiplication and adjoint operation by $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$ and $(a, \lambda)^* = (a^*, \overline{\lambda})$, respectively. With these operations $A \oplus \mathbb{C}$ form a $*$-algebra that contains A as an ideal and the norm of A extends to a (necessarily) unique norm on $A \oplus \mathbb{C}$ that makes $A \oplus \mathbb{C}$ a C^*-algebra.

If $\phi : A \to B$ is a $*$-homomorphism then there is a unique unital $*$-homomorphism $\tilde{\phi} :\to \tilde{A} \to \tilde{B}$ that extends ϕ.

2.2 Functional calculus for normal elements in a C^*-algebra.

If A is an unital Banach algebra, $a \in A$ is said to be *invertible* if there is $b \in A$ such that $ab = ba = 1$. Set of all invertible elements of A is denoted by $Inv(A)$. $Inv(A)$ is an open set in A.

If $a \in A$ with A unital, *the spectrum* of a is the set $\{\lambda \in \mathbb{C} : \lambda 1_A - a \notin Inv(A)\}$ and is denoted by $\sigma_A(a)$. For any untial Banach algebra A, $\sigma_A(a)$ is a non-empty compact subset of $\mathbb{C}$ for every $a \in A$.

If A is a non unital C^*-algebra A, we embed A in $\tilde{A}$ and define spectrum of a by viewing a as an element in $\tilde{A}$.

From the definition, the spectrum of an element in a Banach algebra $\sigma(a)$ depends on the ambient algebra A. However, in the C^*-case this dependence is superficial.

Theorem 2.2.1 *Let B be sub-C^*-algebra of C^*-algebra A and let $a \in B$. The following hold;*

1. *If A is unital and $1_A \in B$ (hence B is also unital) then $\sigma_A(a) = \sigma_B(a)$.*
2. *If A is non unital or is unital with $1_A \notin B$, then $\sigma_A(a) = \sigma_B(a) \cup \{0\}$.*

This independence of the spectrum of a on the ambient C^*-algebra is vital property in C^*-algebra theory.

Theorem 2.2.2 *Let A,B be C^*-algebra and $\phi : A \to B$ be a $*$-homomorphism. Then, $\sigma(\phi(a)) \cup \{0\} \subset \sigma(a) \cup \{0\}, \forall a \in A$. Moreover if A, B and ϕ are unital then $\sigma(\phi(a)) \subset \sigma(a), \forall a \in A$.*

Proof Suppose first that A, B and ϕ are unital. Then, $\phi(Inv(A)) \subset Inv(A)$ and hence if $\lambda 1_B - \phi(a)$ is not invertible in B then $\lambda 1_A - \phi(a)$ is not invertible in A. Thus, $\sigma(\phi(a)) \subset \sigma(a)$.

In general if $\phi : A \to B$ is a $*$-homomorphism consider the unique unital $*$-homomorphism $\tilde{\phi}(a) : \tilde{A} \to \tilde{B}$ that extends ϕ. Now from 2.2.1, $\sigma_{\tilde{A}}(a) = \sigma_A(a) \cup \{0\}$ and $\sigma_{\tilde{B}}(\phi(a)) = \sigma_B(\phi(a)) \cup \{0\}$. Since $\tilde{\phi}$ is unital, using what we showed above, we get $\sigma(\phi(a)) \cup 0 \subset \sigma(a) \cup \{0\}$. ■

Definition 2.2.3 *Let a be an element in a C^*-algebra. Then the supremum of $\sigma(a)$ is called the spectral radius of a. We will denote this by $r(a)$.*

Note that $r(a) \leq ||a||$ for every $a \in A$.

Theorem 2.2.4 *If $a \in A$, where A is a C^*-algebra, $r(a) = \lim_{n\to\infty} ||a^n||^{1/n}$.*

Theorem 2.2.3 holds for unital Banach algebras [25].

An element a in a C^*-algebra is called *self adjoint* if $a^* = a$. Set of all self adjoint elements is denoted by A_{sa}. For $a \in A_{sa}$, $\sigma(a) \subset \mathbb{R}$. a is said to be *normal* if $a^*a = aa^*$. In particular any self adjoint element or any element in a abelian C^*-algebra is normal.

Following simple result paves the way for *Gelfand representation* (Theorem 2.2.9) of abelian C^*-algebras and hence, for the functional calculus for normal elements in arbitrary C^*-algebras. Functional calculus extends the spectral theorem for normal matrices over $\mathbb{C}$. It is one of the most important and widely used theorems in C^*-algebra theory. We will encounter several applications of the Theorem in Chapter 3.

Theorem 2.2.5 *If $a \in A$ is self adjoint then $r(a) = ||a||$.*

Proof Suppose $a = a^*$. By C^*-identity of the norm we get, $||a||^2 = ||a^*a|| = ||a^2||$. Moreover, by induction $||a||^{2n} = ||a^{2n}||, \forall n \in \mathbb{N}$. Hence, by 2.2.5,

$$r(a) = \lim_{n\to\infty} ||a^n||^{1/n} = \lim_{n\to\infty} ||a^{2n}||^{1/2n} = \lim_{n\to\infty} (||a||^{2n})^{1/2n} = ||a||.$$

■

Remark: From Corollary 2.2.5, if a $*$-algebra admits a complete C^*-norm then that norm is unique.

Corollary 2.2.6 *Let $\phi : A \to B$ be a $*$-homomorphism. Then ϕ is norm decreasing.*

Proof By Theorem 2.2.2, $r(\phi(c)) \leq r(c), \forall c \in A$. Thus, by Theorem 2.2.5,

$$||\phi(a)^*\phi(a)|| = r(\phi(a)^*\phi(a)) \leq r(a^*a) = ||a^*a||.$$

By C^*-identity of the norms, $||\phi(a)||^2 \leq ||a||^2, \forall a \in A$ and ϕ is norm decreasing. ■

A *character* on a Banach algebra A is a non zero homomorphism (i.e. a multiplicative linear map) from A into $\mathbb{C}$. Set of all characters on A is denoted by $\Omega(A)$.

Theorem 2.2.7 *Let A be an abelian Banach algebra and let $a \in A$.*

1. *If A is untial, $\sigma(a) = \{\gamma(a) : \gamma \in \Omega(A)\}$.*
2. *If A is non unital, $\sigma(a) = \{\gamma(a) : \gamma \in \Omega(A)\} \cup \{0\}$.*

Each character γ is norm decreasing, and $||\gamma|| = 1$ if A is unital. In particular, for every Banach algebra A, $\Omega(A) \subset A^*$ where A^* is the Banach space dual of A.

If A is a C^*-algebra, each $\gamma \in \Omega(A)$ necessarily preserves adjoints, Thus, each γ is a $*$-homomorphism into $\mathbb{C}$. If $\gamma \in \Omega(A)$ then $\overline{\gamma} : \tilde{A} \to \mathbb{C}$ given by $\overline{\gamma}(a+\lambda 1) = \gamma(a)+\lambda$ is a character on $\Omega(\tilde{A})$ and is the unique character on $\tilde{A}$ which extends γ.

From 2.2.7 and 2.2.5, $\Omega(A)$ is non empty for any C^*-algebra A. This is not true for non unital Banach algebras.

Theorem 2.2.8 *Let A be a non zero abelian C^*-algebra. Then $\Omega(A)$ is a non empty locally compact Hausdorff space of A^* in weak*-topology. $\Omega(A)$ is compact if A is unital.*

Let A be a Banach algebra with $\Omega(A) \neq \emptyset$. For each $a \in A$, let $\hat{a} : \Omega(A) \to \mathbb{C}$ denote the evaluation function given by $\hat{a}(\gamma) = \gamma(a), \forall \gamma \in \Omega(A)$. It is easily seen that $\hat{a}$ is well defined and moreover is a function in $C_0(\Omega(A))$. This $\hat{a}$ is called the *Gelfand transform* of a. The map taking a to its Gelfand transform is a homomorphism but in general its neither injective nor surjective. In the C^*-case this map is always defined and is moreover an $*$-isomorphism. We call this the *Gelfand representation* of a abelian C^*-algebra.

Theorem 2.2.9 *(Gelfand) Every abelian C^*-algebra is isometrically isomorphic to the C^*-algebra $C_0(X)$ for some locally compact Hausdorff space X.*

Proof Set $X = \Omega(A)$. By 2.2.8 X is non empty locally compact Hausdorff space and $\psi : A \to C_0(X)$ given by $a \mapsto \hat{a}$ gives a $*$-homomorphism from A to $C_0(X)$.

By C^*-identity and 2.2.5, $||a||^2 = ||a^*a|| = r(a^*a)$. On the other hand, from 2.2.7, $r(a^*a) = \sup_{\gamma \in \Omega(A)} ||\gamma(a^*a)|| = ||\widehat{(a^*a)}||_\infty$. Thus, $||a||^2 = ||\hat{a}||^2{}_\infty$ and the mapping $a \mapsto \hat{a}$ is an isometry.

Then $\psi(A) \subset C_0(X)$ is a $|| \cdot ||_\infty$-closed sub-$*$-algebra that separates points of X and does not vanish identically at any point of X. Hence, by Stone-Weierstrass Theorem $\psi(A)$ is whole of $C_0(X)$. Thus $a \mapsto \hat{a}$ is indeed a isometric $*$-isomorphism.

■

Note from the above that A is unital if and only if $\Omega(A)$ is compact and in this case A is isomorphic to $C(X)$.

We recall the (continuous) functional calculus for normal elements in a C^*-algebra. For $a_1, a_2, ..a_k$ in a C^*-algebra A, $C^*(a_1, a_2, ..., a_k)$ denotes the sub-C^*-algebra of A generated by $a_1, a_{,2}, ..., a_k$.

Theorem 2.2.10 *Let A be a unial C^*-algebra and let $a \in A$ be a normal element. Then there exists an unique (unital isometric) $*$-isomorphism $\varphi : C(\sigma(a)) \to C^*(a, 1)$ with $\varphi(\iota) = a$ where $\iota : \sigma(a) \to \mathbb{C}$ is given by $\iota(z) = z, \forall z \in \sigma(a)$.*

Proof $B = C^*(a,1)$ is an unital abelian C^*-algebra. Let $\psi : B \to \Omega(B)$ be the Gelfand representation of B. $\hat{a} : \Omega(B) \to \sigma(a)$ is a homeomorphism and hence, $f \mapsto f \circ \hat{a}$ is a $*$-isomorphism from $C(\sigma(a))$ to $C(\Omega)$. Clearly $\varphi : C(\sigma(a)) \to B$ defined by, $f \mapsto \psi^{-1}(f \circ \hat{a})$ is a $*$-isomorphism. Since $\psi(a)(\gamma) = \gamma(a)$ and $\iota \circ \hat{a}(\gamma) = \iota(\hat{a}(\gamma)) = \iota(\gamma(a)) = \gamma(a)$ for all $\gamma \in \Omega(A)$, we have $\varphi(\iota) = a$. Obviously φ is unital. From Stone-Weierstrass Theorem $C(\sigma(a))$ is the C^*-algebra generated by ι and 1. Hence, φ is the unique $*$-isomorphism from $C(\sigma(a))$ to $C^*(a,1)$ with $\varphi(\iota) = a$. ∎

Remarks:

1. φ provided in Theorem 2.2.10 is called the *functional calculus* at a. For any $f \in \sigma(a)$ it is customary to write $f(a)$ to denote $\varphi(f)$. This notation agrees with the definition of $f(a)$ when a is a polynomial and $f(a) \in C^*(a)$ iff $f \in C_0(\sigma(a))$.

2. If a is a normal element in a non unital C^*-algebra A, we view a as an element in $\tilde{A}$ and define the functional calculus for a. Then, for $f \in C(\sigma(a))$, $f(a)$ is in general an element in $\tilde{A}$ and $f(a) \in A$ if and only if $f(0) = 0$.

3. Let A, B be untial C^*-algebras and $\phi : A \to B$ be an unital $*$-homomorphism. Then form 2.2.2, $\sigma(\phi(a)) \subset \sigma(a)$. Thus for all $f \in C(\sigma(a))$, $f(\phi(a))$ is well defined. Moreover, we have $f(\phi(a)) = \phi(f(a))$. In particular, if $\gamma : A \to \mathbb{C}$ is a character then $\gamma(f(a)) = f(\gamma(a))$.

An element a in a C^*-algebra A is said to be *positive* if a is self adjoint and $\sigma(a) \subset [0, \infty)$. Set of all positive elements of A is denoted by A_+. From Theorem 2.2.10, $a \in A_+$ if and only if $a = b^*b$ for some $b \in A$. In particular from 2.2.10, for each $a \in A_+$ there is a unique element $c \in A_+$ such that $c^2 = a$. Indeed, we simply set $c = f(a)$ where $f \in C(\sigma(a))$ is the function given by $f(t) = t^{1/2}$. Such c is called the *square root* of A and is unique for given a. The subset A_+ form a cone in A.

Theorem 2.2.11 *Let A be an untial C^*-algebras and $a \in A$ a normal element. Then, for every $f \in C(\sigma(a))$, $\sigma(f(a)) = f(\sigma(a))$ and for all $g \in C(f(a))$, $g(f(a)) = (g \circ f)(a)$*

Proof Let $B = C^*(a, 1)$. Then $f(a) \in B$ and $\sigma(f(a)) = \{\gamma(f(a)) : \gamma \in \Omega(B)\} = \{f(\gamma(a)) : \gamma \in \Omega(B)\} = f(\sigma(a))$. Note that the second equality holds by remark 2 above.

To show the second part, set $C = C^*(f(a), 1)$. Then $C \subset B$ is an unital sub-C^*-algebra and each $\gamma \in \Omega(B)$ restricts to character γ_C on C. By applying remark 2 above again, for each $g \in \sigma(f(a))$, $\gamma((g \circ f)(a)) = (g \circ f)(\gamma(a)) = g(f(\gamma(a))) = g(\gamma(f(a)) = g(\gamma_C(f(a)) = \gamma_C(g(f(a))) = \gamma(g(f(a)))$. As this holds for all $\gamma \in \Omega(B)$, $(g \circ f) = g(f(a))$. ■

In some sense the following Lemma [23, Lemma 2.2.3] provide a uniform continuity result for the functional calculus. We find this to be useful.

Lemma 2.2.12 *[23, Lemma 2.2.3] Let $K \subset \mathbb{R}$ be compact and non empty. Suppose $f : K \to \mathbb{C}$ is a continuous function. Let A be an unital C^*-algebra and F_K be the set of all $a \in A_{sa}$ with $\sigma(a) \subset K$. Then the function from F_K to A induced by f via the functional calculus (i.e. given by $a \mapsto f(a)$) is continuous.*

For $\epsilon \geq 0$, let $f_\epsilon : [0, \infty) \to [0, \infty)$ be defined by,

$$f_\epsilon(t) = \max\{\epsilon, t\}, \forall t \in [0, \infty).$$

If A is any C^*-algebra and $a \in A_+$ we will use $(a - \epsilon)_+$ to denote the positive element $f_\epsilon(a)$ given by the functional calculus of a. Note that $||a - (a - \epsilon)_+|| < \epsilon$ and approximating a by $(a - \epsilon)_+$ is a useful technical tool in C^*-algebra theory that applies in many instances. We will encounter various applications of the technique.

Proposition 2.2.13 *[33, Proposition 2.2] Let A be a C^*-algebra and $a, b \in A$. Let $\epsilon > 0$ and suppose that $||b - a|| < \epsilon$. Then there exists $c \in A$ such that,*

$$(a - \epsilon)_+ = c^*bc.$$

A pair of positive elements $a, b \in A$ are said to be Murray-von Neumann equivalent if there is some $v \in A$ such that $a = v^*v$ and $b = vv^*$.

Lemma 2.2.14 *([13, Lemma 3.3] c.f. [22]) Suppose $a, b \in A_+$ are Murray-von Neumann equivalent. Then for any continuous function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(a)$ and $f(b)$ are Murray-von Neumann equivalent.*

2.3 Matrix algebras

Let A be a C^*-algebra. For $n \in \mathbb{N}$ let $M_n(A)$ denote the set of all $n \times n$ matrices over A. Equip $M_n(A)$ with usual $\mathbb{C}$-algebra structure and define $[a_{ij}]^* = [b_{ij}]$ where $b_{ij} = (a_{ji})^*, \forall 1 \leq i, j \leq n$ for each matrix $[a_{ij}] \in M_n(A)$. One can define a C^*-norm on $M_n(A)$ as follows.

Recall from the Gelfand-Naimrak Theorem (Theorem 2.1.3) there is a Hilbert space H and an isometric $*$-homomorphism $\varphi : A \to B(H)$. Choose any such φ and define $\varphi_n : M_n(A) \to B(H^n)$, where H^n is the n-fold direct sum of H with it self, by setting

$$\varphi_n([a_{ij}])((\xi_i)_{1\times n}) = [\varphi_n(a_{i1})(\xi_1) + \varphi_n(a_{i2})(\xi_2) + ... + \varphi_n(a_{in})(\xi_n)]_{1\times n},$$

for all $v \in H^n$. Define a norm on $M_n(A)$ by $||a|| = ||\varphi_n(a)||, \forall a \in M_n(A)$. It is easily verified that this norm is indeed a C^*-norm on the $*$-algebra A and is complete. Combining all these we get a C^*-algebraic structure on $M_n(A)$. Since there is at most one choice for a norm that makes a given $*$-algebra a C^*-algebra, the norm on $M_n(A)$ defined above is independent from the choice of the isometric representation φ.

Given $a_1, a_2, ..., a_n$ in a C^*-algebra A, we write $\text{diag}(a_1, a_2, ..., a_n)$ to represent the *diagonal matrix* which has $a_1, a_2, ..., a_n$ in the main diagonal.

$M_n(A)$ is unital iff A is unital and $\text{diag}(1, 1, .., 1)_{n\times n}$ is the unit in this case.

If A, B are C^*-algebras and $\phi : A \to B$ is a $*$-homomorphism then there is a natural $*$-homomorphism $\phi_n : M_n(A) \to M_n(B)$ given by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$. We will simply write ϕ in place of ϕ_n when the context is clear.

For any C^*-algebra A and $n, m \in \mathbb{N}$ then one has the natural identification $(M_m(A))_n \cong M_{mn}(A)$ and one can view $M_n(A)$ to be a sub-C^*-algebra of $M_{n+m}(A)$ via the *upper left hand corner* embedding. That is, we identify $a \in M_n(A)$ with

$$\begin{pmatrix} a & 0_{n\times m} \\ 0_{m\times n} & 0_{m\times m} \end{pmatrix}$$

which is an element in $M_{m+n}(A)$. We write $a \oplus 0_{m\times m}$ to represent this matrix. More generally for $a \in M_n(A)$ and $b \in M_m(A)$, we write $a \oplus b$ to denote the matrix

$$\begin{pmatrix} a & 0_{n\times m} \\ 0_{m\times n} & b \end{pmatrix}.$$

Using the above identifications let $M_\infty(A) = \bigcup\limits_{n\in\mathbb{N}} M_n(A)$. $M_\infty(A)$ is a $*$-algebra and the canonical norm it inherits is clearly a C^*-norm, but it is not a complete norm.

2.4 Equivalence of projections in C^*-algebras and Murray-von Neumann semigroup of projections

An element p in a $*$-algebra A is called a *projection* if $p = p^2 = p^*$. We will write $\mathcal{P}(A)$ to denote the set of all projections in A. If A is C^*-algebra a projection is always positive and is of norm 1 if its non zero. In fact for every $p \in \mathcal{P}(A)$, $\sigma(p) \subset \{0, 1\}$. If $\phi : A \to B$ is a $*$-homomorphism and $p \in \mathcal{P}(A)$ then $\phi(p) \in \mathcal{P}(B)$.

There are three important equivalence relations on the set $\mathcal{P}(A)$. In this section we aim to define these relations, recall how they compare with each other and then introduce the *Murray-von Neumann semigroup* of projections for C^*-algebras. In chapter 3 we look to reduce the question of homotopy equivalence of maps in $C(X, S(n, k, l))$ to a question about *homotopy equivalence* of projections in $M_n(C(X))$. Our main purpose in this and the next two sections is to build the background for that.

$u \in Inv(A)$ is called a *unitary* if $u^* = u^{-1}$. The set of all unitaries of an unital C^*-algebra A is denoted by $\mathcal{U}(A)$. A *path* in a C^*-algebra A is continuous map $f : [0, 1] \to A$.

Definition 2.4.1 *Let A be a C^*-algebra and $p, q \in \mathcal{P}(A)$.*

1. p is Murray von Neumann equivalent to q ($p \sim q$), if $p = v^*v$ and $q = vv^*$ for some $v \in A$.

2. p is unitary equivalent to q ($p \sim_u q$), if $p = u^*qu$ for some $u \in \mathcal{U}(\tilde{A})$.

3. p is homotopy equivalent to q ($p \sim_h q$), if $p = f(0)$ and $q = f(1)$ for a projection valued path f.

Remarks:

1. All three relations defined above are equivalence relations.

2. If A is an untial C^*-algebra and $p, q \in \mathcal{P}(A)$ with $p \sim_u q$ then there is a unitary $u \in A$ such that $p = u^*qu$.

Proposition 2.4.2 explain how the three relations compare with each other.

Proposition 2.4.2 *[23, Proposition 2.2.7] Let p, q be projections in a C^*-algebra A.*

1. If $p \sim q$ then $p \sim_u q$.

2. If $p \sim_u q$ then $p \sim_h q$.

The following gives useful criteria to determine the equivalence of projections.

Proposition 2.4.3 *[23, Proposition 2.2.4] If $p, q \in A$ are projections in a C^*-algebra A and $||p - q|| < 1$ then $p \sim_h q$.*

In the simplest case of $A = \mathbb{C}$ all three relations are equivalent to saying that ranks of the projections p, q are the same. However neither of the reverse implications hold in general.

Examples: [23]

1. An *isometry* in an unital C^*-algebra is an element s such that $s^*s = 1$. An isometry is invertible iff it is an unitary element. There are C^*-algebras which contain non unitary isometries. If s is such an isometry in some unital A one has $1 = s^*s \sim ss^*$, but 1 and ss^* are not unitary equivalent. In particular, if s is the *unilateral shift* in $B(l^2(\mathbb{N}))$ given by $s(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...)$ then $ss^* \sim 1$ but $ss^* \nsim_u 1$.

2. C^*-algebra $C(\mathbb{T}^3)$, contain a pair of projections p, q with $p \sim_u q$ but $p \nsim_h q$. The verification of this fact utilize K-theoretic arguments and can be found in [23].

While the above examples do show the three relations to be distinct in $\mathcal{P}(A)$, when the relations are extended to include projections in all matrix algebras over A the relations are equivalent to each other.

Proposition 2.4.4 *[23, Proposition 2.2.8] Let p, q be projections in a C^*-algebra A.*

1. If $p \sim q$ then $(p \oplus 0) \sim_u (q \oplus 0)$ in $M_2(A)$.

2. If $p \sim_u q$ then $(p \oplus 0) \sim_h (q \oplus 0)$ in $M_2(A)$.

Let $\mathcal{P}_\infty(A) = \mathcal{P}(M_\infty(A)) = \bigcup\limits_{n \in \mathbb{N}} \mathcal{P}(M_n(A))$.

Definition 2.4.5 *Let A be a C^*-algebra. Let $p, q \in \mathcal{P}_\infty(A)$.*

1. p is Murray-von Neumann equivalent to q if there is $m \in \mathbb{N}$ such that there are p' ,q' in $M_m(A)$ that represent p,q respectively with $p' \sim q'$ in $M_m(A)$.

2. p is unitary equivalent to q if there is $m \in \mathbb{N}$ such that there are p' ,q' in $M_m(A)$ that represent p,q respectively with $p' \sim_u q'$ in $M_m(A)$.

3. p is homotopy equivalent to q if there is $m \in \mathbb{N}$ such that there are p' ,q' in $M_m(A)$ that represent p,q respectively with $p' \sim_h q'$ in $M_m(A)$.

From Proposition 2.4.4 it is immediate that the there equivalence relations above are equivalent. We write $\sim_0$ to denote either of these identical relations. The equivalence class of a projection $p \in \mathcal{P}_\infty(A)$ with respect the relation $\sim_0$ is denoted by $[p]_0$.

Murray-von Neumann semigroup of a C^*-algebra.

For a C^*-algebra A, let $\mathcal{D}(A) = \{[p]_0 : p \in \mathcal{P}_\infty(A)\}$. For $[p]_0$,$[q]_0$ in $\mathcal{D}(A)$ let $[p]_0 + [q]_0 = [p \oplus q]_0$. The equivalence class $[p \oplus q]_0$ does not depend on the choice of the representations of the equivalence classes in left hand side. Thus $+$ is a well defined operation and is associative. The *Murray-von Neumann* semigroup of A is the pair $(\mathcal{D}(A), +)$. For simplicity we will write $\mathcal{D}(A)$ to denote the semgroup as well. $\mathcal{D}(A)$ is abelian and $[0]_0$ acts as an identity in $\mathcal{D}(A)$.

If $A = M_n(C(X))$ for some compact Hausdorff space X and $n \in \mathbb{N}$, equivalence classes of projections in $\mathcal{P}_\infty(A)$ naturally correspond to isomorphism classes of vector bundles over the space X. This correspondence is essentially a consequence of a classical Theorem of Swan [34]. In the next two sections we recall the theory of vector bundles and then proceed to give a brief outline of the above correspondence. Our ultimate goal is to use this identification to avail vector bundle techniques (in particular stability properties and homotopy classification) in constructing certain rank constrained homotopies in the positive cone of $M_n(C(X))$ in Chapter 3.

2.5 Complex Vector Bundles.

Most of the results of this and the proceeding section are well known. Our main references for vector bundle theory is Husemoller [21]. The identification of vector bundles with projections in $\mathcal{P}_\infty(C(X))$ is essentially contained in the work of Swan [34]. For the sake of completeness, we include proofs in cases where we did not find clear references in the literature.

Unless stated otherwise throughout the chapter X will denote a compact Hausdorff space. The base field of all vector spaces we consider is the field of complex numbers $\mathbb{C}$.

Definition 2.5.1 *[21] A bundle is a triple (E, π, B) where B, E are topological spaces and $\pi : E \to B$ is a continuous map. Space B is called the base space of the bundle and E is called the total space.*

We will use Greek letters such as $\xi, \zeta, \gamma, ..$ to denote bundles.

Examples:

1. Let $\xi = (B \times F, \pi, B)$ where B, F are topological spaces and $\pi : B \times F :\to B$ be the projection on the first coordinate. ξ is a bundle over B and is called a *product* bundle.

2. Let $a : B \to M_n(\mathbb{C})$ be continuous. Let $E = \{(x, v) : v \in a(x)(\mathbb{C}^n)\} \subset B \times \mathbb{C}^n$ and $\pi : E \to B$ be the restriction of the coordinate projection of $B \times \mathbb{C}^n$ onto B to the set E. Then $\xi_a = (E, \pi, B)$ is a bundle over B.

A bundle (F, q, B') is a *subbundle* of a bundle (E, p, B) if $F \subset E$, $B' \subset B$ and $q = p|_F$.

Definition 2.5.2 *Let $\xi = (E, p, B)$ and $\zeta = (F, q, B')$ be bundles. A morphism of bundles from ξ to ζ is a pair of continuous maps (u, f) with $u : E \to F$, $f : B \to B'$ and $uq = pf$. If both f, u are homeomorphisms then the pair (u, f) is said to be an isomorphism of bundles and the two bundles are said to be isomorphic.*

If p in (E, p, B) is a surjection, then the map u of a bundle morphism (u, f) is uniquely determined by the map f. If ξ and ζ are bundles over a space B a bundle morphism (u, f) is called a B-morphism if $f = 1_B$.

For a bundle (E, p, B) and $b \in B$, *fiber* over b mean the set $p^{-1}(b)$ and we will denote this set by E_b.

Few bundle constructions:

1. The *Whitney* sum $\xi_1 \oplus \xi_2$ of the two bundles $\xi_1 = (E_1, p_1, B)$ and $\xi_2 = (E_2, p_2, B)$ is the triple (E, q, B), where $E = \{(x, x') \in E_1 \times E_2 : p_1(x) = p_2(x')\} \subset E_1 \times E_2$ and $q : E \to B$ is the map given by $q(x, x') = p_1(x) = p_2(x')$. The Whitney sum is also called the *fiber product*. Note that $E_b = (E_1)_b \times (E_2)_b \subset E_1 \times E_2$ for all $b \in B$.

2. If $\xi = (E, p, B)$ and $B' \subset B$, the *restriction* of ξ to B' (denoted $\xi|'_B$) is the bundle (E', q, B') where $E' = \{x \in E : p(x) \in B'\}$ and $q = p|'_E$.

3. Let $\xi = (E, p, B)$ be a bundle and $f : B' \to B$ continuou. The *pullback bundle* $f^*(\xi)$ of ξ under f is the triple (E', q, B') where $E' = \{(b', x) : f(b') = p(x)\}$ and $q : E' \to B'$ is the map given by $q(b', x) = b'$.

A space F is called *the fiber of the bundle* (E, p, B) if for every $b \in B$ the fiber E_b is homeomorphic to F. A bundle (E, p, B) is said to be *trivial* with fiber F provided (E, p, B) is B-isomorphic to the product bundle $(B \times F, p, B)$.

A complex vector bundle is a bundle in which each fiber is homeomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$ and is *locally* trivial. We recall the precise definition.

Definition 2.5.3 *Let B be a topological space. A complex vector bundle over B is a bundle (E, π, B) such that for every $b \in B$;*

1. The fiber E_b admits a finite dimensional complex vector space structure.

2. There is an open neighborhood $U \subset B$ of b, $n \in \mathbb{N}$ and a homeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{C}^n$ such that h restricts to a vector space isomorphism on E_x for each $x \in U$ and $h \circ \pi = \pi_1$, where π_1 is the canonical coordinate projection of $U \times \mathbb{C}^n$ on U.

Condition (2) of Definition 2.5.3 is called the local triviality condition.

Given $\xi = (E, \pi, B)$, if there is some $k \in \mathbb{N}$ such that for each $b \in B$ the vector space dimension of E_b is k then ξ is said to be a vector bundle of dimension k. This is the case if X is connected.

A sub-bundle η of vector bundle ξ is a vector bundle of its own right when each fiber of the sub-bundle is given the vector space structure it inherits by being a subspace of the corresponding fiber of ξ.

Definition 2.5.4 *A morphism from a vector bundle $\xi = (E, p, B)$ to a vector bundle $\eta = (E', q, B')$ is pair (u, f) which is a morphism of underlying bundles where in addition $u|_{E_b} : E_b \to E_{f(b)}$ is a $\mathbb{C}$-linear map for all $b \in B$. Isomorphisms, B-morphisms and B-isomorphisms of vector bundles are similarly defined. We write $\xi \cong \eta$ if the bundles ξ and η are isomorphic and $\xi \cong_B \eta$ if they are B-isomorphic.*

We will use $\mathrm{Vect}(X)$ to denote the set of all vector bundles over a space X, and $\mathrm{Vect}_k(X)$ to denote the set of all $\xi \in \mathrm{Vect}_k(X)$ of dimension k. A bundle $\gamma \in \mathrm{Vect}_k(X)$ is called a *trivial* bundle if its X-isomorphic to the product bundle θ^k.

Examples :

1. Example 1 following 2.5.1 is a vector bundle if $F = \mathbb{C}^n$ for some n. We will denote this product bundle by θ^n.

2. Example 2 following 2.5.1 is a vector bundle if $a : B \to M_n(\mathbb{C})$ is projection valued.

3. Let $G_k(\mathbb{C}^n)$ be the Grassman variety of k-dimensional subspaces of $\mathbb{C}^n$ for $n, k \in \mathbb{N}$. That is, $G_k(\mathbb{C}^n)$ is the set of all k-dimensional subspaces of $\mathbb{C}^n$ with the quotient topology given by the map from the set $\{(v_1, v_2, ..., v_k) \in \mathbb{C}^n \ : \langle v_i, v_j\rangle = \delta_{ij}\}$ onto $G_k(\mathbb{C}^n)$ that maps each tuple $(v_1, v_2, ..., v_k)$ to the k-dimensional vector subspace of $\mathbb{C}^n$ spanned by the tuple. $G_k(\mathbb{C}^n)$ is a compact Hausdorff space and the canonical k-dimensional vector bundle over $G_k(\mathbb{C}^n)$, denoted by ${\gamma_k}^n$, is the sub-bundle of the product bundle $(G_k(\mathbb{C}^n) \ \times \mathbb{C}^n \ , \pi, G_k(\mathbb{C}^n)\ )$ where the total space $E({\gamma_k}^n)$ of ${\gamma_k}^n$ is the subspace of $G_k(\mathbb{C}^n) \ \times \mathbb{C}^n$ which consists of all the pairs $(V, v) \in G_k(\mathbb{C}^n) \ \times \mathbb{C}^n$ where $v \in V$. For a fixed $k \in \mathbb{N}$, let $G_k(\mathbb{C}^\infty) \ = \bigcup\limits_{k\leq n} G_k(\mathbb{C}^n)$ and equip this set with the inductive topology coming from the natural inclusions $G_k(\mathbb{C}^n) \ \hookrightarrow G_k(\mathbb{C}^{n+1})$. Similarly there are natural inclusions $E({\gamma_k}^n) \hookrightarrow E({\gamma_k}^{n+1})$. Set $E_\infty = \bigcup\limits_{k\leq n} E({\gamma_k}^n)$ to be the associated inductive space. Then $\gamma_k = (E_\infty, p, G_k(\mathbb{C}^\infty)\ )$ is a vector bundle over $G_k(\mathbb{C}^\infty)$.

Remark: For any pair $n, k \in \mathbb{N}$, $G_k(\mathbb{C}^n)$ is naturally homeomorphic to the set of all $p \in \mathcal{P}(M_n)$ of rank k. If $p \in \mathcal{P}(M_n(C(X)))$ is of rank k then $\xi_p \cong_X p^*({\gamma_k}^n)$ where in the second bundle p is viewed as a map from X to $G_k(\mathbb{C}^n)$ using the first identification.

Vector bundle constructions

1. If $\xi \in \mathrm{Vect}(X)$ and $Y \subset X$ then $\xi|_Y \in \mathrm{Vect}(X)$ where $\xi|_Y$ is the restriction of ξ to Y as a bundle.

2. If $\xi_1 = (E_1, p_1, X) \in \mathrm{Vect}_l(X), \xi_2 = (E_2, p_2, X) \in \mathrm{Vect}_l(X)$ then $\xi_1 \oplus \xi_2 \in \mathrm{Vect}_{k+l}(X)$. Indeed, each fiber of $\xi_1 \oplus \xi_2$ is of the form $(E_1)_b \times (E_2)_b$ and hence

admits a natural vector space structure. The local triviality for $\xi_1 \oplus \xi_2$ follows from the local triviality of ξ_1 and ξ_2.

3. If $\xi = (E, p, X) \in \mathrm{Vect}_k(X)$ and $f : Y \to X$ is continuous, then $f^*(\xi) \in \mathrm{Vect}_k(Y)$. For each $y \in Y$ fiber of $f^*(\xi)$ over y is the set $\{(y, v) : y \in Y, v \in p^{-1}(f(y))\}$, and thus has a vector space structure which is naturally isomorphic to that of $E_{f(y)}$. Local triviality of ξ makes $f^*(\xi)$ locally trivial.

Let $\xi, \gamma \in \mathrm{Vect}(X)$. If there is some $\eta \in \mathrm{Vect}(X)$ with $\xi = \gamma \oplus \eta$ then γ is said to be a *direct summand* of ξ. Every sub-bundle of ξ is a direct summand of ξ .

Let us now recall the following stability property of vector bundles. This will be one of the main ingredients of the proof of our main Theorem of Chapter 3.

Theorem 2.5.5 *[16,21] Let X be a finite dimensional compact Hausdorff space. Let $\xi, \gamma \in Vect(X)$ and θ^n denote the product bundle of dimension n over X. Let k be the dimension of ξ and write $m = \lfloor \frac{dim X}{2} \rfloor$. The following hold,*

1. $\xi \cong \eta \oplus \theta^{k-m}$ for some $\eta \in Vect_m(X)$.

2. If $k \geq \lceil \frac{dim X}{2} \rceil$ and $\xi \oplus \delta \cong \gamma \oplus \delta$, where $\delta \in Vect(X)$ is trivial, then $\xi \cong \gamma$.

A proof of the conclusions of the Theorem for finite CW-complexes is provided in [21, Chap. 9, Theorems 1.2 and 1.5]. By applying dimension theory techniques the statement for compact Hausdorff spaces can be reduced to the CW case and thus the theorem holds in this generality [16, Theorem 2.5].

Next we recall the homotopy classification of vector bundles. For any $\xi \in \mathrm{Vect}(X)$ we will write $\langle \xi \rangle$ to denote the X-isomorphism class of ξ. By $\langle \mathrm{Vect}(X) \rangle$ we denote the set of all X-isomorphism classes of $\mathrm{Vect}(X)$ and by $\langle \mathrm{Vect}_k(X) \rangle$ we denote the set of all isomorphism classes of $\mathrm{Vect}_k(X)$.

For continuous maps $f, g : X \to Y$, we write $f \sim_h g$ to mean they are are homotopic and write $[X, Y]$ to denote the set of all homotopy classes of continuous maps from X to Y.

Theorem 2.5.6 *[21, Corollary 5.6 and Theorem 6.2] Let X be any compact Hausdorff space. Fix $k \in \mathbb{N}$.*

1. *If $\xi \in Vect_k(X)$ there is a continuous map $f : X \to G_k(\mathbb{C}^\infty)$ with $\xi \cong_X f^*(\gamma_k)$.*
2. *For two continuous maps $f, g : X \to G_k(\mathbb{C}^\infty)$, $f^*(\gamma_k) \cong_X g^*(\gamma_k)$ iff $f \sim_h g$.*

From the Theorem there is bijective map from $\langle \mathrm{Vect}_k(X) \rangle$ to $[X, G_k(\mathbb{C}^\infty)]$ for every compact Hausdorff space X and $k \in \mathbb{N}$. In fact [21] proves the Theorem for all paracompact spaces. Owing to Theorem 2.5.6, space $G_k(\mathbb{C}^\infty)$ is also called the *classifying* space (of vector bundles of rank k) for the class of all paracompact spaces.

For the class of CW-complexes wiith dim $X \leq d$ for a fixed d we have the following stronger version of Theorem 2.5.6.

Theorem 2.5.7 *[21, Chapter 8, Theorem 7.2] [21, Chap. 8, Theorem 7.2] Let X be a CW-complex and n, k be non-negative integers. Then, the function that assigns to each homotopy class $[f] : X \to G_k(\mathbb{C}^n)$ the isomorphism class of the k-dimensional vector bundle $f^*(\gamma_k^n)$ over X is a bijection, if $n \geq k + \lceil \frac{dim\, X}{2} \rceil$.*

To prove Theorem 3.1.4 we need a similar result to hold for compact Hausdorff spaces. In Proposition 3.1.3, based on the proof of [16, Theorem 2.5] we apply dimension theory results [14, 27] and few C^*-algebraic techniques to prove that 2.5.7 indeed extends to include compact Hausdorff spaces.

2.6 Vector bundles and projections

If $p \in M_n(C(X))_+$ is a projection, then the bundle ξ_p defined in the preceding section is a vector bundle over X. In fact this assignment $p \mapsto \xi_p$ induce a natural bijection from $D(C(X))$ to $\langle \mathrm{Vect}(X) \rangle$. This correspondence is well known and is in by and large a consequence of the following classical Theorem by Swan [34] (c.f. [35]).

Theorem 2.6.1 *[34, Corollary 5] Let X be a compact Hausdorff space and $\xi \in Vect(X)$. Then ξ is direct summand of a trivial bundle.*

Corollary 2.6.2 *(c.f. [34]) Given $\xi \in$ Vect(X), there is some $m \in \mathbb{N}$ and a projection $p = p_\xi \in M_m(C(X))$ such that $\xi_p \cong \xi$.*

Proof Suppose ξ is a vector bundle over X, with total space E and the fiber at $x \in X$ being $E_{x.}$ From 2.6.1, there is some $n \in \mathbb{N}$ and a bundle $\xi^\perp$ over X such that $\xi \oplus \xi^\perp \cong \theta^n$, where θ^n is the n dimensional product bundle. Thus, for each $x \in X$, we may assume that $E_x \oplus F_x = \mathbb{C}^n$, where F is the total space of $\xi^\perp$. Let $p_\xi(x)$ denote the orthogonal projection of $\mathbb{C}^n$ on the fiber $E_x \subset \mathbb{C}^n$. Then, by the local triviality of ξ the map $x \mapsto p_\xi(x)$ is continuous and clearly $\xi_p \cong_X \xi$.

■

In chapter 3 we always work inside $M_n(C(X))$ for a fixed n. Thus, when associating vector bundles to projections we want to ensure that, for fixed k there is n such that for any compact Hausdorff space X with *dim* X small enough and for each $\xi \in \text{Vect}_k(X)$, there is a projection p inside $M_n(C(X))$ with $\xi_p \cong \xi$. While Corollary 2.6.2 does not guarantee this directly, combining 2.6.2 with stability properties for vector bundles (Theorem 2.5.5) addresses the issue.

We write $\mathcal{P}_k(M_n(C(X)))$ to denote the set of all $p \in \mathcal{P}(M_n(C(X)))$ of constant rank k.

Theorem 2.6.3 *(Theorem 2.5.5 and [34]) Let X be a compact Hausdorff space of dimension $d < \infty$. Suppose $m \geq k + \lfloor \frac{d}{2} \rfloor$. Then there is a natural bijection between the set of all Murray - von Neumann classes of projections in $\mathcal{P}_k(M_m(C(X)))$ and the set of all isomorphism classes of vector bundles in Vect$_k(X)$, induced by $p \mapsto \xi_p \, \forall p \in \mathcal{P}_k(M_n(C(X))$.*

Proof It is easy to see that the induced map is well defined and injective. We verify that the assignment $[p]_0 \mapsto \langle \xi_p \rangle$ is a surjection. For this, it suffices from the proof of the preceding Corollary to show that for any $\xi \in \text{Vect}_k(X)$ there is some $\gamma \in \text{Vect}_{\lfloor \frac{d}{2} \rfloor}(X)$ such that $\xi \oplus \gamma \cong \theta^n$.

By [34, Corollary 5] there is a bundle δ, say of dimension t, so that $\xi \oplus \delta \cong \theta^{k+t}$. Suppose $t > \lfloor \frac{d}{2} \rfloor$, then by part 1 of Theorem 2.5.5, $\delta = \gamma \oplus \theta^{t - \lfloor \frac{d}{2} \rfloor}$ for some $\gamma \in$

$\mathrm{Vect}_{\lfloor \frac{d}{2} \rfloor}(X)$. If dimension of $\xi \oplus \gamma$ is k_1, then $k_1 = k + \lfloor \frac{d}{2} \rfloor \geq \lceil \frac{d}{2} \rceil$. Hence, by part 2 of 2.5.5, $\xi \oplus \gamma \cong \theta^{k+\lfloor \frac{d}{2} \rfloor}$. For fixed $n, k \in \mathbb{N}$, if $\xi \in \mathrm{Vect}_k(X)$ and $n \geq k + \lfloor \frac{d}{2} \rfloor$ then the above observation allows us to choose the projection p_ξ corresponding to ξ, so that it is an element in $M_n(C(X))$.

■

Definition 2.6.4 (Trivial Projections) *([37] c.f. [31,38]) A projection $p \in M_n(C(X))$ is called a trivial projection, if the corresponding vector bundle ξ_p is a trivial vector bundle over X.*

Note that if $p, q \in \mathcal{P}(M_\infty(C(X)))$ with p trivial then q is trivial iff $q \sim p$.

2.7 Bundles associated to positive elements in $M_n(C(X))$ and *well supported* positive elements.

Throughout the thesis the identification $M_n(C(X)) \cong C(X, M_n(\mathbb{C}))$ is used freely. Given a function f on X and $Z \subset X$, $f|_Z$ denotes the restriction of f to Z.

Positive elements in $M_n(C(X))$ (in the C^*-sense) are simply the continuous functions $a : X \to M_n(\mathbb{C})$ where the image $a(x)$ at each x is a non-negative definite matrix. For $a \in M_n(C(X))$, we denote the *rank function* of a by r_a and define $r_a(x)$ to be the rank of the matrix $a(x)$ for each $x \in X$. $r_a : X \to \mathbb{N}$ is a lower semicontinuous map.

For $a \in M_n(C(X))_+$, the projection that maps each x to orthogonal projection of $\mathbb{C}^n$ on $a(x)(\mathbb{C}^n)$ is called the *support projection* of a.

Let us write p_a to denote this map. Note that p_a, in general is not continuous. However, if E is any subset of X on which r_a is constant, then $p_a|_E$ is continuous on E. Thus for any such E, the bundle $\xi_a|_E$ is a vector bundle over E. Indeed it is the vector bundle $\xi_{(p_a|_E)}$.

Thus, if rank values of a are $n_1, n_2, n_3, ...n_k$ and we set $E_i = \{x \in X : r_a(x) = n_i\}$, then $\xi_a|_{E_i}$ is a vector bundle over E_i for every i. In this manner we can partition X into a finite collection of subsets, such that the restriction of ξ_a to each subset is a vector bundle of constant dimension. A typical subset E_i formed here can be highly irregular. On the other hand, classical structure theorems in vector bundle theory assumes the base space to have regularity properties such as compactness. Therefore, this partition of X in general does not allow us to apply techniques of classical vector bundle theory to analyze the structure of $\xi_{p_i} = \xi_a|_{E_i}$. We can overcome this issue if a is such that the projection $p_a|_{E_i}$ (i.e. the support projection of a on E_i) continuously extends to a projection p_i defined on the closure $\overline{E}_i \subset X$ of E_i, for each i. Moreover, if we also have the extended projections p_i and p_j to be comparable on $\overline{E}_i \cap \overline{E}_j$, for any two distinct i, j with $\overline{E}_i \cap \overline{E}_j \neq \emptyset$, then we can attempt to suitably extend the local structure of ξ_a (i.e. structure properties of various $\xi_{p_i}s$) to derive global structure properties of ξ_a. The notion of *well supported* positive elements considered in [38] (c.f. [31]) represents positive elements of this special type.

Definition 2.7.1 *([39] c.f. [31]) Let X be a compact Hausdorff space and let $a \in M_n(C(X))_+$. Suppose that $n_1 < n_2 < \cdots < n_k$ and $E_i, 1 \leq i \leq k$ are as in the preceding paragraph. We say that a is well supported if for each $1 \leq i \leq k$ there is a projection $p_i \in M_n(C(\overline{E}_i))$ such that $\lim_{r\to\infty} a(x)^{1/r} = p_i(x), \forall x \in E_i$, and $p_i(x) \leq p_j(x)$ whenever $x \in \overline{E}_i \cap \overline{E}_j$, and $i \leq j$.*

It is not hard to construct $a \in M_n(C(X))_+$ for which p_i's do not even extend to $\overline{E}_i$'s, where p_i, E_i are as in 2.7.1. However, the following Theorem [38] shows that arbitrary positive elements are approximated by well supported positive elements up to any given tolerance. As an immediate consequence of Theorem 2.7.2 we get any $a \in C(X, S(n, k, l))$ to be homotopic to a well supported positive element in $C(X, S(n, k, l))$, where $S(n, k, l) = \{a \in M_n(\mathbb{C}) : l \leq \text{rank } a \leq k\}$ (see 3.1.1). This is crucial to the proof of our main theorem of Chapter 3.

Theorem 2.7.2 *[38, Theorem 3.9] Let X be a compact Hausdorff space and let $a \in M_n(C(X))_+$. Then, for every $\delta > 0$, there exists a well supported element $b \in M_n(C(X))_+$ such that $b \leq a$ and $||a - b|| < \delta$, with the range of r_b equal to the range of r_a.*

Remark: Theorem stated above is only a part of [38, Theorem3.9]. There, in the hypothesis X is assumed to be a finite simplicial complex. However, the simplicial structure of X is required only for the second part of the Theorem which guarantees that each $\overline{E}_i$ corresponding to b (as defined in 2.7.1) can assumed to be a sub-complex of X. For the Theorem as stated here, X being compact and Hausdorff suffices.

2.8 Extending projections and positive elements subject to constrains

In order to derive global structure of a bundle ξ_a using its local structure, where $a \in M_n(C(X))_+$, we need techniques to extend projection (positive) valued functions defined on closed subsets of X to functions on X of the same type. Moreover, we require these extensions to preserve certain constrains that are already true in the respective subsets. In this section we recall such techniques developed in the literature. In some instances proofs are given to ensure that the results are stated in the context we require. However, all the results in this section appear in (or are direct consequences of) previous work, mainly of [31, 37, 38].

Theorem 2.8.1 *[31, Proposition 4.2 (1)] Let X be a compact Hausdorff space of dimension $d < \infty$, and let $Y \subset X$ be closed. Let $p, q \in M_n(C(X))$ be projections with the property that rank $q(x) + \lfloor \frac{d}{2} \rfloor \leq$ rank $p(x)$, $\forall x \in X$. Let $s_0 \in M_n(C(Y))$ be such that $s_0^* s_0 = q|_Y$ and $s_0 s_0^* \leq p|_Y$. It follows that there is $s \in M_n(C(X))$ such that $s^* s = q$, $ss^* \leq p$, and $s_0 = s|_Y$.*

In [39] the following is proven as a Corollary to 2.8.1

Corollary 2.8.2 *[39, Corollary 2.7] Let X be a compact Hausdorff space of dimension $d < \infty$, and let $E_1, ..., E_k$ be a cover of X by closed sets. Let $q \in M_n(C(X))$ and for each $i \in 1, ..., k$ let $p_i \in M_n(C(E_i))$ be a projection of constant rank n_i. Assume that $n_1 < n_2 < \cdots < n_k$ and $p_i(x) \leq p_j(x)$ whenever $x \in E_i \cap E_j$ and $i \leq j$. Finally, suppose that $n_i -$ rank $q \geq \lfloor \frac{d}{2} \rfloor$ for every i. Then the following hold:*

If $Y \subset X$ is closed, $q|_Y$ is trivial and $\forall y \in Y$,

$$q(y) \leq \bigwedge_{\{i|y \in E_i\}} p_i(y),$$

then $q|_Y$ can be extended to a projection $\widetilde{q}$ on X which is also trivial and satisfies,

$$\widetilde{q}(x) \leq \bigwedge_{\{i|x \in E_i\}} p_i(x), \forall x \in X.$$

Corollary 2.8.3 *Let X be a compact Hausdorff space of dimension $d < \infty$, and let $Y \subset X$ be closed. Then any trivial projection $r \in M_n(C(Y))$ extends to a trivial projection on X, provided that rank $r \leq n - \left\lfloor \frac{d}{2} \right\rfloor$.*

Proof Suppose that the rank of r is k and take q to be any constant projection in M_n of rank k. Then q defines a trivial projection in $M_n(C(X))$ and since r is also trivial on Y, from (2.6.3) there exists $s_0 \in M_n(C(Y))$ such that ${s_0}^* s_0 = q|_Y$ and $r = s_0 {s_0}^*$. Let p be the unit of $M_n(C(X))$. Then, as $s_0 {s_0}^* \leq p|_Y$ and rank $q \leq$ rank $p - \left\lfloor \frac{d}{2} \right\rfloor$, by Theorem 2.8.1 s_0 extends to $s \in M_n(C(X))$ such that, $s^* s = q$. Thus, $\tilde{r} = ss^*$ is a trivial projection on X with $\tilde{r}|_Y = s_0 {s_0}^* = r$.

■

Remark: In the hypothesis of 2.8.2, its assumed that q is defined on X. However, 2.8.3 implies that the conclusion of 2.8.2 is valid even when q is defined only on Y. In chapter 3, we will use this observation without further mention.

We need one more extension result, Proposition 2.8.7. Before we present this, we introduce some terminology and point out few properties concerning the rank functions that we find to be useful.

From Section 2.2 recall that for any $\epsilon \geq 0$ and $a \in M_n(C(X))_+$, $(a - \epsilon)_+$ denotes the element $f_\epsilon(a) \in M_n(C(X))_+$ given by the functional calculus of a where,

$$f_\epsilon(t) = \max\{\epsilon, t\}, \forall t \in [0, 1].$$

Recall that by support projection of $d \in M_n(C(X))$ we mean the function (not necessarily continuous) which maps x to the orthogonal projection of $\mathbb{C}^n$ onto $d(x)(\mathbb{C}^n)$. We denote the support projection of $f_\epsilon(a)$ by $\chi_{(\epsilon,1]}(a)$. This is same as the *spectral projection* of a that corresponds to the set $(\epsilon, ||a||]$ from general C^*-algebra theory.

Give $a \in M_n(C(X))_+$, let $\Gamma_a : X \to [0, ||a||]^n$ be the map given by

$$\Gamma_a(x) = (\lambda_1(x),\ \lambda_2(x)........,\ \lambda_n(x)),$$

where $\lambda_1(x), \lambda_2(x), ..., \lambda_n(x)$ are the eigenvalues of $a(x)$ in non decreasing order. The following is a straightforward consequence of the continuity of Γ_a on X.

Lemma 2.8.4 *Let $a \in M_n(C(X))_+$ and $\eta \geq 0$. The map $x \mapsto$ rank$[\chi_{(\eta,||a||]}(a(x))]$ where $\chi_{(\eta,||a||]}$ denotes the characteristic map on $(\eta, ||a||]$, is lower semi-continuous.*

Proof It is clear that Γ_a is continuous on X for any $a \in M_n(C(X))_+$. Let $x \in X$ and suppose $\text{rank}[\chi_{(\eta,||a||]}(a(x))] = m$. Then, there are exactly m (with possible repetitions) eigenvalues of $a(x)$, which are grater than η. Moreover, as $\lambda_i(x)$ are in increasing order, $\lambda_{n-m+1}(x), \lambda_{n-m+2}(x), ...\lambda_n(x)$ are exactly the eigenvalues of $a(x)$ which are greater than η. Set $\epsilon = \lambda_{n-m+1}(x) - \eta > 0$ and by continuity of Γ_a choose a neighborhood U_x of x such that

$$||\Gamma_a(x) - \Gamma_a(y)|| < \epsilon, \forall y \in X.$$

For all $1 \leq i \leq n$,

$$|\lambda_i(x) - \lambda_i(y)| < \epsilon$$

Therefore by the choice of ϵ for each $y \in U_x$ and for all i,

$$n - m + 1 \leq i \leq n \implies \lambda_i(y) > \eta.$$

Thus, for each $y \in U_x$,

$$\text{rank}[\chi_{(\eta,||a||]}(a(y))] \geq m = \text{rank}[\chi_{(\eta,||a||]}(a(x))].$$

$\blacksquare$

Lemma (2.8.5) is a weaker version of [37, Lemma 2.1]. We include a proof for two reasons. Firstly the proof in this weaker version is technically much simpler. The other reason is that [37, Lemma 2.1] quotes the result only for compact metric spaces, but essentially the same techniques works for any compact Hausdorff space.

Lemma 2.8.5 *[37, Lemma 2.1] Let X be a compact Hausdorff space and suppose that $a \in M_n(C(X))_+$. Let $l \in \mathbb{N}$ be such that rank $a(x) \geq l$, $\forall x \in X$. Then, there is some $\eta > 0$ such that for each $x \in X$, the spectral projection $\chi_{(\eta,\infty)}(a(x))$ has rank at least l.*

Proof For each $x \in X$, let $\eta_x = \frac{1}{2}\min\{\lambda \in \text{spec } a(x){:}\lambda > 0\}$. Note that since $l > 0$, η_x exists for each $x \in X$. Then,

$$\text{rank } a(x) = \text{rank}[\chi_{(\eta_x,\infty)}(a(x))], \forall x \in X.$$

By Lemma 2.8.4, for each $x \in X$, the map $y \mapsto \text{rank}[\chi_{(\eta_x,\infty)}(a(y))]$ is lower semi-continuous. So, for each $x \in X$, there exists an open neighborhood U_x of x such that,

$$\text{rank}[\chi_{(\eta_x,\infty)}(a(y))] \geq \text{rank}[\chi_{(\eta_x,\infty)}(a(x))], \forall y \in U_x.$$

By compactness of X, choose some finite set of points $\{x_1, x_2, ...x_L\}$ such that

$$X = \bigcup_{1 \leq i \leq L} U_{x_i}.$$

By setting $\eta = \min_{1 \leq i \leq L}(\eta_{x_i}) > 0$, for every $y \in U_{x_i}$ we get,

$$\begin{aligned}
\text{rank}[\chi_{(\eta,\infty)}(a(y))] &\geq \text{rank}[\chi_{(\eta_{x_i},\infty)}(a(y))] \\
&\geq \text{rank}[\chi_{(\eta_{x_i},\infty)}(a(x))] \\
&= \text{rank } a(x) \\
&\geq l.
\end{aligned}$$

Since $X = \bigcup_{1 \leq i \leq L} U_{x_i}$, this completes the proof. ■

Lemma 2.8.6 is again a weaker version of result in [37]. We provide the proof for same reasons we had for 2.8.5.

Lemma 2.8.6 *[37, Lemma 2.7] Let $n, k, l \in \mathbb{N}$ with $n \geq k \geq l$ and let X be a compact Hausdorff space. Suppose $Y \subset X$ is closed and $a \in C(Y, S(n, k, l)))$. Then there is a open set U in X containing Y and $\tilde{a} \in C(U, S(n, k, l))$ such that $\tilde{a}|_Y = a$.*

Proof By Tietze extension theorem there exists an open set W in X containing Y and $a_1 \in M_n(C(X))_+$ such that $a_1|_Y = a$.

Since the rank function is lower semi-continuous and rank $a_1(y) \geq l$ for all $y \in Y$, by using the compactness of Y choose an open set V with $Y \subset V \subset W$ such that

$$\text{rank } a_1(z) \geq l, \, \forall z \in V. \tag{2.1}$$

Moreover, using the compactness of X and shrinking V if necessarily, we may assume that above inequality holds for all z in $\overline{V}$. Then by the previous Lemma there is $\eta > 0$ such that

$$\text{rank } [\chi_{(\eta,\infty)}(a_1(z))] \geq l, \, \forall z \in \overline{V}. \tag{2.2}$$

By continuity of a_1 for each $n \in \mathbb{N}$ choose an open set $U_n \subset V$ with $Y \subset U_n \subset V$ such that,

$$\forall z \in U_n, \exists y \in Y \ \text{ with } \ ||a_1(z) - a(y)|| \leq \frac{\eta}{2^n}.$$

By suitably shrinking U_n let us assume $\overline{U_{n+1}} \subset U_n$. Thus, using the normality of X we can define a continuous function $f : \overline{U_1} \to [0, 1]$ such that $f(y) = 0, \forall y \in Y$ and

$$\eta > f(x) \geq \frac{\eta}{2^n}, \, \forall x \in \overline{U_1}. \tag{2.3}$$

Define $\tilde{a} \in M_n(C(U_1))$ by

$$\tilde{a} = (a_1(x) - f(x))_+$$

. Then because of 2.3 and Lemma 2.8.5,

$$\text{rank } \ \tilde{a}(x) \geq l, \forall x \in U_1.$$

To get the upper rank bound note that if $x \in U_n \setminus \overline{U_{n+1}}$ then there is some $y \in Y$ such that

$$||a_1(x) - a(y)|| < \frac{\eta}{2^n}$$

.

Thus, by 2.3 and Proposition 2.2.13

$$\begin{aligned}
\text{rank } \tilde{a}(x) &= \text{rank } (a_1(x) - f(x))_+ \\
&\leq \text{rank } (a_1(x) - \frac{\eta}{2^n})_+ \\
&\leq \text{rank } a(y) \\
&\leq k
\end{aligned}$$

■

Recall that $a, b \in M_n(C(X))_+$ are said to be Murray-von Neumann equivalent if there is some $v \in M_n(C(X))$ such that $a = v^*v$ and $b = vv^*$.

Proposition 2.8.7 *Let X be a compact Hausdorff space and $Y \subset X$ be closed. Suppose $a, b \in C(X, S(n,k,l))$ are Murray-von Neumann equivalent. Then there is an open set U in X with $Y \subset U$ and Murray-von Neumann equivalent $\tilde{a}, \tilde{b} \in C(U, S(n,k,l))_+$, extending a and b respectively. If in addition $a = p, b = q$ are projections then we may choose the extensions to be Murray-von Neumann equivalent projections.*

Proof To prove the first part, suppose $a = v^*v$ and $b = vv^*$ for some $v \in M_n(C(X))$. Use Tietze extension Theorem to extend v to some $v_1 \in M_n(C(\overline{W})$ where W is as in the proof of 2.8.6. Set $a_1 = {v_1}^*v_1$ and $b_1 = v_1{v_1}^*$ and choose V such that 2.1 holds for both a_1 and b_1. Choose $\eta > 0$ such that 2.2 holds for $\chi_\eta(a_1)$ as well as for $\chi_\eta(b_1)$. Now construct $\tilde{a}$, $\tilde{b}$ and U as in the proof of 2.8.6. Since $a_1 \sim b_1$, from Lemma 2.2.14 it follows that $\tilde{a} \sim \tilde{b}$.

To prove the second part, use the first part to extend p, q to $\tilde{a}, \tilde{b} \in C(U, S(n,k,l))$ with $\tilde{a} \sim \tilde{b}$. Now since $\sigma(p), \sigma(q) \subset \{0,1\}$ use the continuity of Γ_a and Γ with compactness of Y to choose an open set U_1 of X such that $Y \subset U_1 \subset \overline{U}_1 \subset U$ and,

$$\begin{aligned} \sigma\tilde{a}|_{\overline{U}_1}) &\subset [0, \frac{1}{3}] \cup [\frac{2}{3}, \frac{4}{3}] \\ \sigma(\tilde{b}|_{\overline{U}_1}) &\subset [0, \frac{1}{3}] \cup [\frac{2}{3}, \frac{4}{3}]. \end{aligned}$$

Let $f : [0, \frac{4}{3}] \to [0,1]$ be any continuous function which is zero on $[0, \frac{1}{3}]$, one on $[\frac{2}{3}, \frac{4}{3}]$. Then $f(\tilde{a}|_{\overline{U}_1})$, $f(\tilde{b}|_{\overline{U}_1})$ are projections and are Murray-von Neumann equivalent by 2.2.14. Setting $U = U_1$, $\tilde{p} = f(\tilde{a}|_{U_1})$ and $\tilde{q} = f(\tilde{b}|_{U_1})$ completes the proof. ■

3. HOMOTOPY EQUIVALENCE OF MAPS IN $C(X, S(n, k, l))$

Recall that for any $n, k, l \in \mathbb{N}$ with $n \geq k \geq l$ we write $S(n, k, l)$ to denote the set $\{a \in M_n(\mathbb{C})_+ : l \leq \text{rank } a \leq k\}$.

3.1 Homotopy equivalence of maps in $C(X, S(n, k, l))$ when $\left\lfloor \frac{dim\, X}{2} \right\rfloor \leq k - l$

The goal of this is to prove Theorem 3.1.4. The proof of 3.1.4 is achieved through two main steps; Lemma 3.1.2 and Proposition 3.1.3. Lemma 3.1.2 is our main technical result and reduces proving 3.1.4 to extending Theorem 2.5.7 [21, Chapter 8, Themorem 7.2] to all compact Hausdorff spaces. As mentioned in chapter 2, this extension may be known. However there is no clear reference of the fact and in Proposition 3.1.3 we provide a proof that follows a pattern similar to the proof of [16, Theorem 2.5]. We essentially use the identification of a compact Hausdorff space as an inverse limit of compact metric spaces and the identification of a compact metric spaces as an inverse limit of CW-complexes to show that conclusion of Proposition 3.1.3 follows form 2.5.7. This reduction arguments we provide use some C^*-algebraic techniques we recalled in Chapter 2.

Theorem 3.1.4 follows immediately by combining Lemma 3.1.2 with Proposition 3.1.3.

We need the following direct consequence of Theorem 2.7.2 .

Lemma 3.1.1 *Let X be a compact Hausdorff space and $a \in M_n(C(X))_+$. Then there exists a continuous path $t \mapsto a_t$ in $M_n(C(X))_+$ which connects $a_0 = a$ to a_1 which is well supported in the sense of Definition 2.7.1 and has the same rank values as that of a. The path is such that rank $a_t(x) =$ rank $a(x), \forall t \in [0, 1), \forall x \in X$.*

Proof Applying Theorem 2.7.2, choose a well supported positive element $b \leq a$ such that b has the same rank values as that of a and set $a_t = (1-t)a + tb$. We only have to verify that rank $a_t(x) =$ rank $a(x)$ for every $t \in (0,1)$ and $x \in X$. But this is immediate.

Since $b \leq a$, if $0 < t < 1$,

$$(1-t)a \leq a_t \leq (1-t)a + ta = a.$$

Therefore, rank $a_t(x) =$ rank $a(x)$, $\forall t \in (0,1), \forall x \in X$. ■

Lemma 3.1.2 *Let X be a compact Hausdorff space with $dim\ X < \infty$. Suppose $n, k, l \in \mathbb{N}$ are such that $k \leq n$ and $k - l \geq \left\lfloor \frac{dim\,X}{2} \right\rfloor$. Let $a \in C(X, S(n,k,l))$. Then there is a continuous path $h : [0,1] \to C(X, S(n,k,l))$ such that $h(0) = a$ and $h(1)$ is a trivial projection of rank l.*

Proof Let X, n, k, l and a be as given in the hypothesis. By Lemma 3.1.1 we can clearly assume that a is well supported.

Let the rank values of a be $n_1 < n_2 < ..... < n_L$ and let $E_1, E_2 \ldots, E_L$ and $p_1, p_2, \ldots, p_L$ be as in Definition 2.7.1. For convenience we will write $F_i = \overline{E}_i$ and $d = dim\,X$.

We first consider the case $n_L \leq \left\lfloor \frac{d}{2} \right\rfloor$. Then, choose $p \in M_n(C(X))$ to be any trivial projection of rank l and let

$$h(t) = (1-t)a + tp.$$

Now for each $t \in [0,1], x \in X$,

$$\begin{aligned} \text{rank}\,[h(t)(x)] &\leq \text{rank}\,a(x) + \text{rank}\,p \\ &\leq n_L + l \\ &\leq \left\lfloor \frac{d}{2} \right\rfloor + l \\ &\leq k, \end{aligned}$$

and clearly rank $[h(t)(x)] \geq l$. Thus, we get the required path.

Now let us assume $n_L > \left\lfloor \frac{d}{2} \right\rfloor$.

Fix r such that $n_r > \left\lfloor \frac{d}{2} \right\rfloor$ and $n_{r-1} \leq \left\lfloor \frac{d}{2} \right\rfloor$, where we allow the possibility $r = 1$ and set $n_0 = 0,\ F_0 = \emptyset$.

In what proceeds, we construct a trivial projection $R \in M_n(C(X))$ of rank l such that,

$$\text{rank}\,(R + a)(x) \quad \leq \quad k, \forall x \in X.$$

Once we have such R, we define $h : [0, 1] \to M_n(C(X))$ by,

$$h(t) = (1 - t)a + tR, \forall t \in [0, 1].$$

It is almost immediate that this path remains in side $C(S(n, k, l))$.

We focus on constructing R. To this end, we follow an inductive argument to define a trivial projection $q_L \in M_n(C(\bigcup\limits_{r \leq j \leq L} F_j))$ such that,

$$\text{rank } q_L = n_L - \left\lfloor \frac{d}{2} \right\rfloor$$

and

$$\text{rank } (a + q_L)(x) \leq n_L, \forall x \in \bigcup_{r \leq j \leq L} F_j.$$

Since F_r is compact Hausdorff with $dim F_r \leq d$ and $p_r \in M_n(C(F_r))$ is a projection of rank $n_r > \left\lfloor \frac{d}{2} \right\rfloor$, use Theorem 2.5.5 (1) and Swans' correspondence (2.6.3) to find a trivial projection $q_r \in M_n(C(F_r))$ such that,

$$\text{rank } q_r = n_r - \left\lfloor \frac{d}{2} \right\rfloor$$

and $q_r \leq p_r$.

By well supportedness of a, each $p_i \in M_n(C(F_i))$ is of constant rank n_i and whenever $r \leq i \leq j$ with $F_i \cap F_j \neq \emptyset$,

$$p_i(x) \leq p_j(x), \forall x \in F_i \cap F_j. \tag{3.1}$$

Also for all $j \geq r$,

$$\begin{aligned} \text{rank } p_j - \text{rank } q_r &\geq n_r - \text{rank } q_1 \\ &\geq \left\lfloor \frac{d}{2} \right\rfloor. \end{aligned}$$

Hence, we apply Corollary 2.8.2 with $X = \bigcup_{r \leq j \leq L} F_j$, $Y = F_1$, $q = q_r$ (by the remark following Corollary 2.8.3, q in 2.8.2 need not be defined on X) to extend q_r to a trivial projection in $M_n(C(\bigcup_{r \leq j \leq L} F_j)))$ - which we again call q_r - such that whenever $r \leq j$,

$$q_r(x) \leq p_j(x), \forall x \in F_j.$$

Then, for each $r \leq j \leq L$,

$$\text{rank } (q_r + a)(x) \leq n_j, \forall x \in F_j.$$

If $r = L$ then we are done (defining q_L).

Thus, let us assume $r < L$.

Suppose that for some t with $t \leq t < L$ we have defined a trivial projection $q_t \in M_n(C(\bigcup_{r \leq j \leq L} F_j))$ such that the following hold,

$$\begin{aligned}
\text{rank } q_t &= n_t - \left\lfloor \frac{d}{2} \right\rfloor, &(3.2)\\
q_t(x) &\leq p_j(x), \forall x \in F_j, \forall t \leq j \leq L, &(3.3)\\
\text{rank } (q_t + p_j) &\leq n_t, \forall r \leq j \leq t. &(3.4)
\end{aligned}$$

Then whenever $t+1 \leq j \leq L$, $(p_j - q_t) \restriction_{F_{t+1}} \in M_n(C(F_{i+1}))$ is a projection of constant rank $(n_j - n_t) + \lfloor \frac{d}{2} \rfloor$.

Thus, since $dim\, F_{t+1} \leq d$, by applying Theorems 2.5.5 and 2.6.3 we choose a trivial projection $q_{t,t+1} \in M_n(C(F_{t+1}))$ such that,

$$\text{rank } q_{t,t+1} = n_{t+1} - n_t$$

and $q_{t,t+1} \leq p_{t+1} - q_t$.

By applying Corollary 2.8.2 with $X = \bigcup_{t+1 \leq j \leq L} F_j$, $Y = F_{t+1}$ and $q = q_{t,t+1}$ we extend $q_{t,t+1}$ to a trivial projection in $M_n(C(\bigcup_{t+1 \leq j \leq L} F_j))$ (which we again name $q_{t,t+1}$) such that whenever $j \geq t+1$,

$$q_{t,t+1}(x) \leq p_j(x) - q_t(x), \forall x \in F_j. \tag{3.5}$$

Set $q_{t+1} = q_t + q_{t,t+1}$.

Since q_t, $q_{t,t+1}$ are orthogonal trivial projections, q_{t+1} is a trivial projection in $M_n(C(\bigcup_{t+1\le j\le L} F_j))$.

Moreover, by equation (3.2),

$$\begin{aligned}\text{rank } q_{t+1} &= (n_t - \left\lfloor \frac{d}{2} \right\rfloor) + (n_{t+1} - n_t) \\ &= n_{t+1} - \left\lfloor \frac{d}{2} \right\rfloor\end{aligned}$$

and by (3.3) and (3.5) whenever $j \ge t+1$, $\forall x \in F_j$,

$$\begin{aligned}q_{t+1}(x) &= [q_t(x) + q_{t,t+1}(x)] \\ &\le p_j(x).\end{aligned}$$

Finally, by (3.4) for each $r \le j \le t+1$,

$$\begin{aligned}\text{rank } (q_{t+1} + p_j) &\le \text{rank } \; q_{t,t+1} + \text{rank } \; (q_t + p_j) \\ &\le (n_{t+1} - n_t) + n_t \\ &= n_{t+1}.\end{aligned}$$

By proceeding in this manner construct a trivial projection $q_L \in M_n(C(\bigcup_{r\le j\le L} F_j))$ of rank $n_L - \lfloor \frac{d}{2} \rfloor$ such that,

$$\text{rank } (q_L + p_j)(x) \le n_L, \forall x \in F_j, \forall r \le j \le L$$

and

$$q_L(x) \le p_L(x), \forall x \in F_L.$$

Choose $R_1 \in M_n(C(X))$ to be any trivial projection (of rank $n_L - \lfloor \frac{d}{2} \rfloor$) which extends q_L. Note that such R_1 exists by Corollary 2.8.3.

By the choice of r, whenever $j < r$, $\forall x \in F_j$,

$$\begin{aligned}\text{rank}\,(R_1 + a)(x) &\le (n_L - \left\lfloor \frac{d}{2} \right\rfloor) + \left\lfloor \frac{d}{2} \right\rfloor \\ &\le n_L.\end{aligned}$$

Thus, since $R_1 \upharpoonright_{F_j} = q_L \upharpoonright_{F_j}$ whenever $j \geq r$ we conclude,

$$\text{rank}\,(R_1 + a)(x) \leq n_L, \forall x \in X.$$

If $n_L = k$, then

$$\text{rank } R_1 = k - \left\lfloor \frac{d}{2} \right\rfloor \geq l,$$

and we choose R to be any trivial sub-projection of rank l .

Hence we are left with the case $k > n_L$.

In this case,

$$\begin{aligned} \text{rank } (1_n - R_1) &= n - (n_L - \left\lfloor \frac{d}{2} \right\rfloor) \\ &\geq (k - n_L) + \left\lfloor \frac{d}{2} \right\rfloor, \end{aligned}$$

and we apply Theorem 2.5.5 (1) and Theorem 2.6.3 for one last time to choose a trivial projection $R_2 \in M_n(C(X))$ of rank $k - n_L$ with $R_2 \leq (1_n - R_1)$.

Now $R_1 + R_2$ is a trivial projection of rank $k - \lfloor \frac{d}{2} \rfloor$ and

$$\text{rank } (R_1 + R_2 + a)(x) \leq k, \forall x \in X.$$

To complete the proof we choose R to be a trivial sub-projection of $R_1 + R_2$ of rank l. ■

Combining 2.5.7 with Lemma 3.1.2 directly proves Theorem 3.1.4 for all *CW*-complexes. We now prove Proposition 3.1.3.

Recall that if A, B are C^*-algebras and $\phi : A \to B$ is a $*$-homomorphism, then for any $n \in \mathbb{N}$ we have an induced $*$-homomorphism from $M_n(A)$ to $M_n(B)$ given by $[a_{ij}] \mapsto [\phi(a_{ij})]$.

Proposition 3.1.3 *Let X be a compact Hausdorff space with dim $X < \infty$ and suppose $n, k \in \mathbb{N}$ with $n - k \geq \lceil \frac{dim\, X}{2} \rceil$. Then the isomorphism classes of k-dimensional locally trivial complex vector bundles over X are in bijective correspondence with the homotopy classes of maps $p\colon X \to \mathcal{P}_k(\mathbb{C}^n)$.*

Proof Let $[X, P_k(\mathbb{C}^n)]$ stand for the homotopy classes of maps in $\mathcal{P}_k(M_n(C(X)))$ and let $\langle \mathrm{Vect}_k(X) \rangle$ denote the set of all isomorphic classes of locally trivial k-dimensional vector bundles over X. Throughout the proof we will use the natural identification of $G_k(\mathbb{C}^n)$ with $\mathcal{P}_k(\mathbb{C}^n)$ and the identification $p^*({\gamma_k}^n) \cong \xi_p$, where $p^*({\gamma_k}^n)$ is the pull back of the canonical k-dimensional bundle ${\gamma_k}^n$ over $G_k(\mathbb{C}^n)$ to X by viewing p as a map from X to $G_k(\mathbb{C}^n)$ and ξ_p is the vector bundle corresponding to p considered in Chapter 2.

Define $\psi : [X, P_k(\mathbb{C}^n)] \to \langle \mathrm{Vect}_k(X) \rangle$ by $\psi([p]) = \langle \xi_p \rangle$, where $p \in \mathcal{P}_k(M_n(C(X)))$. From results in chapter 2 and the above identifications, it is clear that the map ψ is a well defined surjection.

To complete the proof, we have to show that if the vector bundles associated with two projections in $\mathcal{P}_k(M_n(C(X)))$ are isomorphic then the two projections are homotopic in $\mathcal{P}_k(M_n(C(X)))$.

Let us first assume that X is a compact metric space.

Then by [27, Chapter 27, Theorem 8], X is homeomorphic to an inverse limit of finite simplicial complexes X_α, with $dim\, X_\alpha \leq dim\, X$ for each α.

Let $\psi_\alpha : X \to X_\alpha$ be the corresponding maps. We have the induced homomorphisms,

$$\psi_\alpha^T : C(X_\alpha) \to C(X)$$

given by $\psi_\alpha^T(f) = f \circ \psi_\alpha$.

Moreover, by the inverse limit structure of X,

$$\overline{\bigcup_\alpha \psi_\alpha^T(C(X_\alpha))} = C(X), \tag{3.6}$$

i.e. $\bigcup\limits_\alpha \psi_\alpha^T(C(X_\alpha))$ is a $||\cdot||_\infty$ dense $*$-subalgebra of $C(X)$. Note that if $\alpha < \beta$ then

$$\psi_\alpha^T(C(X_\alpha)) \subset \psi_\beta^T(C(X_\beta)), \tag{3.7}$$

and hence $\bigcup\limits_\alpha \psi_\alpha^T(C(X_\alpha))$ is indeed a $*$-subalgebra.

Suppose $p, q \in \mathcal{P}_k(M_n(C(X)))$ are such that $\xi_p \cong \xi_q$. By Theorem 2.6.3, there is some $v \in M_n(C(X))$ such that $p = v^*v, q = vv^*$.

Fix $0 < \epsilon < 1/3$.

By (3.6), there is some α and $\tilde{v}_\alpha \in M_n(C(X_\alpha)))$ with $||\tilde{v}_\alpha|| \leq 1$ such that,

$$||\psi_\alpha^T(\tilde{v}_\alpha) - v|| < \epsilon/2.$$

Write $Y_\alpha = \psi_\alpha(X)$ and $v_\alpha = \tilde{v}_\alpha \restriction_{Y_\alpha}$, $a_\alpha = v_\alpha^* v_\alpha$, $b_\alpha = v_\alpha v_\alpha^*$.

Then $a_\alpha \in M_n(C(Y_\alpha))_+$ with

$$||a_\alpha - p|_{Y_\alpha}|| \leq \epsilon. \tag{3.8}$$

It follows,

$$\sigma(a_\alpha) \subset [0, \epsilon) \cup (1 - \epsilon, 1].$$

Similarly $||b_\alpha - q|_{Y_\alpha}|| \leq \epsilon$ and

$$\sigma(b_\alpha) \subset [0, \epsilon) \cup (1 - \epsilon, 1].$$

Let $f\colon [0, 1] \to [0, 1]$ be the continuous function which vanishes on $[0, \epsilon]$, is equal to 1 on $[1 - \epsilon, 1]$ and is linear on $(\epsilon, 1 - \epsilon)$. Then $f(a_\alpha)$, $f(b_\alpha)$ are projections in $M_n(C(Y_\alpha))$ with

$$||a_\alpha - f(a_\alpha)|| < \epsilon, ||b_\alpha - f(b_\alpha)|| < \epsilon. \tag{3.9}$$

Moreover, $f(a_\alpha)$ and $f(b_\alpha)$ are Murray-von Neumann equivalent projections by Lemma 2.2.14. Thus, since Y_α is a closed in X_α, by applying Proposition 2.8.7 choose some open neighborhood U of Y_α so that $f(a_\alpha), f(b_\alpha)$ extends to $p_\alpha, q_\alpha \in \mathcal{P}_k(M_n(C(U)))$ respectively, with $p_\alpha \sim q_\alpha$.

Since X_α is a finite simplicial complex, after a finite simplicial refinement of X_α via barycentric subdivisions, choose a sub complex Z of X_α with $Y \subset Z \subset U$. For convenience let us denote the restrictions of p_α, q_α to Z by p_α, q_α.

Since p_α and q_α generate isomorphic vector bundles over Z, each of rank k, form Theorem 2.5.7 and the identification of $G_k(\mathbb{C}^n)$ with $\mathcal{P}_k(M_n((\mathbb{C}))$, there is a continuous path,

$$t \mapsto h_\alpha(t) \in \mathcal{P}_k(M_n(C(Z))),$$

such that $h_\alpha(0) = p_\alpha$, $h_\alpha(1) = q_\alpha$.

This gives a path $t \mapsto h(t) \in \mathcal{P}_k(M_n(C(X)))$, given by $h(t)(x) = h_\alpha(t)(\psi_\alpha(x))$.

Note that for all $x \in X$,

$$\begin{aligned} ||p(x) - h(0)(x)|| &= ||p(x) - p_\alpha(\psi_\alpha(x))|| \\ &= ||p(x) - f(a_\alpha)(\psi_\alpha(x))|| \\ &\leq ||p(x) - a_\alpha(\psi_\alpha(x))|| + ||a_\alpha(\psi_\alpha(x)) - f(a_\alpha)(\psi_\alpha(x))|| \\ &\leq 2\epsilon \quad \text{(by 3.8 and 3.9)} \end{aligned}$$

Thus, $||p - h(0)|| < 1$ and similarly $||q - h(1)|| < 1$.

Therefore, from 2.4.3 [23, Proposition 2.2.4]

$$p \sim_h h(0) \sim_h h(1) \sim_h q.$$

This completes the proof for compact metric spaces.

Now suppose X is an arbitrary compact Hausdorff space. Then X is an inverse limit of compact metric spaces X_λ, with $dim\, X_\lambda \leq dim\, X$ for each λ. From the first step the conclusion of the Proposition holds for each X_λ. Thus the conclusion for X follows from essentially the same argument we presented in proving the first step.

■

Theorem 3.1.4 *Let X be a compact Hausdorff space X with $\lfloor \frac{dim\, X}{2} \rfloor \leq k-l$. There is only one homotopy class of continuous maps $f : X \to S(n,k,l)$, i.e. $C(X, S(n,k,l))$ is path connected.*

Proof Observe that the case $n = k$ is straightforward. For any $a \in C(X, S(n,k,l))$ we have the linear path $t \mapsto (1-t)a + 1_n$ connecting a to 1_n.

So we assume $n > k$.

Let $a, b \in C(X, S(n,k,l))$. Since $\lfloor \frac{dim\, X}{2} \rfloor \leq k - l$, by applying Lemma 3.1.2 choose trivial projections p, q of rank l such that there are paths inside $C(X, S(n,k,l))$ connecting a to p and b to q. Since $n > k$ and $k - l \geq \lfloor \frac{dim\, X}{2} \rfloor$, $n - l \geq \lceil \frac{dim\, X}{2} \rceil$ and we may apply the preceding Proposition. In particular since $\xi_p \cong \theta^l \cong \xi_q$, there is a path inside $C(X, S(n,k,l))$ connecting p and q by Proposition 3.1.3. Thus, there is a path connecting a and b in $C(X, S(n,k,l))$. ■

Corollary 3.1.5 *For every* $r \leq 2(k-l)+1$, $\pi_r(S(n,k,l)) = 0$.

Proof Follows directly from Theorem 3.1.4 as $dim\, S^r = r$. ■

3.2 Homotopy equivalence of maps in $C(X, S(n,k,l))$ and homotopy groups of $S(n,k,l)$

In Theorem 3.2.4 we prove that for a fixed integer d if $\pi_r(S(n,k,l)) = 0$ for all $r \leq d$, then $C(X, S(n,k,l))$ is path connected for every compact Hausdorff space X with $dim\, X \leq d$.

The proof of Theorem 3.2.4 involves two main steps. In Lemma 3.2.1 we apply classical homotopy theory results (see [45]) to prove the conclusion of 3.2.4 when X is a finite simplicial complex. Then in Lemma 3.2.3 we use a dimension theory argument which is similar in flavor to the proof of Proposition 3.1.3 to reduce the proof of Theorem 3.2.4 to proving it in the case of if X being a finite simplicial complex.

We recall some terminology from homotopy theory. Our main reference here is [45].

A topological space X is said to be *compactly generated* if a subspace A of X is closed in X if and only if $A \cap K$ is closed in K for all compact subspaces K of X. In particular $S(n,k,l)$ is compactly generated since its a metric space. Lemma 3.2.1 follows by a simple application of fact in Theorem [45].

Lemma 3.2.1 *Suppose X is a finite simplicial complex of covering $dim\, X \leq d$. If $\pi_r(S(n,k,l)) = 0$ for each $r \leq d$ then $C(X, S(n,k,l))$ is path connected.*

Proof We will use induction on the number of simplexes in the complex K.

If K consists of a single simplex then result is true since K is contractible.

Suppose that result is true for every simplicial complex which contains r number of simplexes.

To complete the inductive step, let $K = L \cup \{s\}$ where L is a sub complex of K containing r number of simplexes and s is a n-simplex for some $n \leq d$.

Since is $\{K, L\}$ is a *NDR* pair and $S(n,k,l)$ is compactly generated, following sequence is exact in the category of sets with base points by [45, Diagram 6.3].

$$[C(L, S(n,k,l))] \xleftarrow{i_*} [C(K, S(n,k,l))] \xleftarrow{p_*} [C((K/L), S(n,k,l))] \tag{3.10}$$

where K/L is the quotient space and $[C(Z, S(n,k,l))]$ mean the set of all homotopy classes of maps in $C(Z, S(n,k,l))$. The maps i_* and p_* are the maps induced by the inclusion $i : L \to K$ and the quotient map $p : K \to K/L$. As the respective base points of the three sets $[C(L, S(n,k,l))], [C(K, S(n,k,l))]$ and $C(K/L, S(n,k,l))]$ we may choose $[f], [f \circ p], [f \circ p \circ \iota]$ with f being any constant map $z \mapsto a$, for a fixed $a \in S(n,k,l)$.

By the induction hypothesis $[C(L, S(n,k,l))]$ consists of a single point. Since $(K/L) \cong S^n$ and $\pi_n(S(n,k,l)) = 0$ by assumption, $[C((K/L), S(n,k,l))]$ is also a singleton. Thus by exactness at the middle of (3.10), $[C(K, S(n,k,l))]$ contains only one point, i.e. $C(K, S(n,k,l))$ is path connected.

■

Lemma 3.2.2 *Let $X = \varprojlim X_\alpha$, for a inverse system of compact Hausdorff spaces $(X_\alpha, \psi_{\alpha\beta})$. Let $\psi_\alpha : X \to X_\alpha$ be the natural maps and $\epsilon > 0$. Fixed $a \in C(X, S(n,k,l))$, there is some index α and $c \in C(X_\alpha, (M_n(\mathbb{C}))_+)$ such that $\psi_\alpha^T(c) \in C(X, S(n,k,l))$ and there is a path in $C(X, S(n,k,l))$ connecting a to $\psi_\alpha^T(c)$.*

Proof Let $a \in M_n(C(X))_+$ and note that w.l.o.g we may assume $||a|| = 1$.

Use Lemma 2.8.5 to pick $0 < \eta < 1$ such that,

$$\text{rank } [\chi_{(2\eta,1]}(a(x))] \geq l, \forall x \in X. \tag{3.11}$$

Set $\epsilon = \frac{\eta}{3}$. Since a is positive and $||a|| \leq 1$, there is some $c \in M_n(C(X))_+$ such that $c^2 = a$ and $||c|| \leq 1$. By the inductive limit structure of X choose some index α and $d \in M_n(C(X_\alpha))$ such that,

$$||c - \psi_\alpha^T(d)|| < \epsilon. \tag{3.12}$$

On the other hand since $c^* = c$,

$$\begin{aligned} ||c - \psi_\alpha^T(d^*))|| &= ||(c^* - \psi_\alpha^T(d^*))|| \\ &= ||(c - \psi_\alpha^T(d))^*|| \\ &= ||c - \psi_\alpha^T(d)|| \\ &< \epsilon \end{aligned} \tag{3.13}$$

Put $b = d^*d$ then $b \in M_n(C(X_\alpha))_+$ and moreover,

$$\begin{aligned} ||a - \psi_\alpha^T(b))|| &= ||c^2 - \psi_\alpha^T(d^*d)|| \\ &= ||c^2 - \psi_\alpha^T(d^*)\psi_\alpha^T(d)|| \\ &\leq ||c^2 - c \cdot \psi_\alpha^T(d)|| + ||c \cdot \psi_\alpha^T(d) - \psi_\alpha^T(d^*)\psi_\alpha^T(d)|| \\ &\leq ||c|| \cdot ||c - \psi_\alpha^T(d)|| + ||c - \psi_\alpha^T(d^*)|| \cdot ||\psi_\alpha^T(d)|| \\ &< 3\epsilon \\ &= \eta \end{aligned} \tag{3.14}$$

The last inequality follows from (3.12), (3.13) and bounds $||c|| \leq 1, ||\psi_\alpha^T(d)|| \leq 2$.

Let us write $a_\alpha = \psi_\alpha^T(b)$.

From Proposition 2.2.13 and (3.14) there is some $d \in M_n(C(X))$ with,

$$(a_\alpha - \eta)_+ = d^*ad.$$

Therefore, for every $x \in X$,

$$\begin{aligned} \text{rank } (a_\alpha - \eta)_+(x) &\leq \text{rank } (a(x)) \\ &\leq k. \end{aligned} \tag{3.15}$$

From (3.14) and the functional calculus of a_α,

$$\begin{aligned} ||(a_\alpha - \eta)_+ - a|| &\leq ||(a_\alpha - \eta)_+ - a_\alpha|| + ||a_\alpha - a|| \\ &< \eta + \eta \\ &= 2\eta. \end{aligned}$$

Therefore, by Proposition 2.2.13 it follows that,

$$\text{rank } [(a - 2\eta)_+(x)] \leq \text{rank } [(a_\alpha - \eta)_+(x)], \forall x \in X.$$

Now, from (3.11) and since rank $(a_\alpha - \eta)_+(x) = \text{rank } \chi_{(2\eta,1]}(a(x))$,

$$\text{rank } (a_\alpha - \eta)_+(x) \geq l, \forall x \in X. \tag{3.16}$$

Put $c = \psi_\alpha^T((b - \eta)_+)$.

Then,

$$c = (b - \eta)_+ \circ \psi_\alpha = ((b \circ \psi_\alpha) - \eta)_+ = (a_\alpha - \eta)_+.$$

Thus by (3.15) and (3.16), $c \in C(X, S(n, k, l))$.

Now consider $h : [0, 1] \to C(X, S(n, k, l))$ given by,

$$h(t) = [((1 - t)a + ta_\alpha) - \eta]_+.$$

Note that h is continuous by Lemma 2.2.12 [23].

We have $h(0) = (a - \eta)_+$ and $h(1) = c$ and,

$$||a - h(t)|| = ||a - ((1 - t)a + ta_\alpha)|| = t||a - a_\alpha|| < \eta, \forall t \in [0, 1].$$

Thus, by an argument similar to the one used to show $c \in C(X, S(n, k, l))$,

$$h(t) \in C(X, S(n, k, l)), \forall t \in [0, 1].$$

Proof of the Lemma is now complete since a is homotopic to $(a - \eta)_+$ as maps in $C(X, S(n, k, l))$. Indeed observe that the linear path $t \mapsto (1 - t)a + t(a - \eta)_+$ is contained in $C(X, S(n, k, l))$. ■

Lemma 3.2.3 *Suppose for each finite simplicial complex Z with $dim\, Z \leq d$, the function space $C(Z, S(n,k,l))$ is path connected. Then, for every compact Hausdorff space X of covering dimension d, space $C(X, S(n,k,l))$ is path connected.*

Proof Like in the proof of 3.1.3, we first prove the result for the case of X being a compact metric space. In this case $X = \varprojlim X_\alpha$, where $(X_\alpha, \psi_{\alpha\beta})$ is a inverse system finite simplicial complexes with $dim\, X_\alpha \leq d$. Let $\psi_\alpha : X \to X_\alpha$ be the natural maps.

Suppose $a, b \in C(X, S(n,k,l))$. We need to construct a path in $C(X, S(n,k,l))$ connecting a and b. From Lemma 3.2.2 we may assume that $a = \psi_\alpha^T(c)$, $b = \psi_\beta^T(d)$, for some $\alpha = \beta$ and $c, d \in M_n(C(X_\alpha))_+$.

Put $Y = \psi_\alpha(X) \subset X_\alpha$. Then, Y is closed and for each $y = \psi_\alpha(x) \in Y$,

$$\text{rank } (c(y)) = \text{rank } (c(\psi_\alpha(x))) = \text{rank } (a(x))$$

Hence,

$$l \leq \text{rank } (c(y)) \leq k, \forall y \in Y.$$

Similarly,

$$l \leq \text{rank } (d(y)) \leq k, \forall y \in Y.$$

Therefore, there is some open neighborhood U of Y in X_α and $\tilde{c}, \tilde{d} \in C(U, S(n,k,l))$ such that $\tilde{c}, \tilde{d}$ are extensions of c, d respectively by Lemma 2.8.6.

After a refinement of the simplicial structure of X_α, choose a finite sub complex Z of X_α such that, $Y \subset Z \subset U$ and view $\tilde{c}, \tilde{d}$ to be maps in $C(Z, S(n,k,l))$. Now, as Z is a finite simplicial complex with $dim\, Z \leq d$, by the hypothesis there is a path $\tilde{g} : [0,1] \to C(Z, S(n,k,l))$ such that $\tilde{g}(0) = \tilde{c}$ and $\tilde{g}(1) = \tilde{d}$.

Define a path $g : [0,1] \to C(X, M_n)$ by,

$$g(t)(x) = \tilde{g}(t)(\psi_\alpha(x)).$$

Clearly that $g(0) = \psi_\alpha^T(c) = a$, $g(1) = \psi_\alpha^T(d) = b$ and $g(t) \in C(X, S(n,k,l))$.

This proves the result in the case of X being a compact metric space with $dim\, X \leq d$. If X is an arbitrary compact Hausdorff space, write $X = \varprojlim X_\alpha$ where X_α is compact metric with $dim\, X_\alpha \leq d$. From what we have just seen $C(X_\alpha, S(n,k,l))$ is path connected for each α. Thus, following a similar argument to that in the metric case gives the result for X. ■

Combining Lemma 3.2.1 and Lemma 3.2.3 we get;

Theorem 3.2.4 *Let X be compact Hausdorff with $\dim X \leq d$. If $\pi_r(S(n,k,l)) = 0$ for each $r \leq d$ then, $C(X, S(n,k,l))$ is path connected.*

4. FURTHER PRELIMINARIES AND BLACKADAR-HANDELMAN CONJECTURES

All groups we consider are abelian and all semigroups are abelian monoids. Unless stated otherwise an order relation is assumed to be a partial order.

4.1 Partially ordered abelian semigroups and their state spaces

Our main reference for this section is [8].

Definition 4.1.1 *A partially ordered semigroup is a pair* $(M, \leq)$ *where* M *is an abelian semigroup and* $\leq$ *is a partial order on* M *with* $a \leq b \implies a + c \leq b + c$*,* $\forall a, b, c \in M$*. We also assume that* $0 \leq a$ *for all* $a \in M$*.*

Remark: The term partially ordered semigroup is used even without assuming $0 \leq a, \forall a \in M$. In such instances the term *positively ordered semigroup* is used for ones which in addition satisfy $0 \leq a, \forall a \in M$. We do not distinguish the two cases.

All order relations we consider will be partial orders and for convenience we write *ordered semigroup* to mean a partially ordered semigroup in the sense of 4.1.1.

An element u in $(M, \leq)$ is called an *order unit* if for each $x \in M$ there is some $n \in \mathbb{N}$ with $x \leq nu$. A triple $(M, \leq, u)$ where $(M, \leq)$ and u are as above is called a *scaled ordered semigroup*. If the order and the order unit are clear we may write M to denote $(M, \leq, u)$.

Given $(M, \leq, u)$ to $(N, \leq, v)$, a *morphism of scaled ordered semigroups* is a map $\phi : M \to N$ which is additive and order preserving with $\phi(0) = 0$ and $\phi(u) = v$.

A *state* on $(M, \leq, u)$ is a morphism from $(M, \leq, u)$ to $(\mathbb{R}^+, \leq, 1)$ where $\mathbb{R}^+$ is the additive semigroup of non negative real numbers and $\leq$ is as usual. The set of all states of $(M, \leq, u)$ will be denoted by $S(M, \leq, u)$, or by $S(M)$ if the choice of order unit and order are clear. $S(M)$ is a convex subset of the space of all real valued functions on M. The natural topology of $S(M)$ is the topology of point-wise convergence and it is compact in this topology.

A morphism $\phi : (M, \leq, u) \to (N, \leq, v)$ as above induce a continuous affine map $\phi^\sharp : S(N) \to S(M)$ via composition.

As our eventual plan is to study the state space of the *Cuntz semigroup* (4.2.1) of a C^*-algebra, maps on the level of state spaces are of interest to us. In [8] Blackadar and Rødram introduce the following class of morphisms of scaled ordered semigroups and study the maps on the level of state spaces induced by morphisms in this class.

Definition 4.1.2 *[8, Definition 2.2] Let $\phi : (M, \leq, u) \to (N, \leq, v)$ be a morphism of scaled ordered semigroups. ϕ is called a stable order embedding if for any $x, y \in M$, there is $n \in \mathbb{N}$ and $z \in M$ with $nx + z + u \leq ny + z$ if and only if there is $m \in \mathbb{N}$ and $w \in N$ with $m\phi(x) + v + w \leq m\phi(y) + w$.*

We recall some useful results from [8].

Lemma 4.1.3 *[8, Lemma 2.8] Let $(M, \leq, u)$ be a scaled ordered semigroup. Then for any given $x, y \in M$, $s(x) < s(y)$ for all $s \in S(M)$ if and only if there is some $n \in \mathbb{N}$ and $z \in M$ such that $nx + z + u \leq ny + z$.*

Lemma 4.1.4 *[8, Lemma 2.9] Let $(M, \leq, u)$ be a scaled ordered semigroup and K be a nonempty compact convex subset $S(M)$. Suppose that for any $a, b \in M$ if $s(a) < s(b)$ for all $s \in K$ then $s(a) < s(b)$ for all $s \in S(M)$. Then $K = S(M)$.*

Theorem 4.1.5 *[8, Theorem 2.6] Let $\phi : (M, \leq, u) \to (N, \leq, v)$ be a morphism of scaled ordered semigroups. Then ϕ is a stable order embedding iff*

$$S(M) = \{g \circ \phi : \; g \in S(N)\}.$$

Remark: In [8] the above results are shown to hold in grater generality. Namely, when M, N are pre-ordered. As we will only be considering partially ordered semigroups, we limit to the above versions.

4.2 The Cuntz semigroup of a C^*-algebra and its order relation

Let A be a C^*-algebra and $M_n(A)$ denote the $n \times n$ matrices over A. Recall that $M_\infty(A) = \bigcup_{n \in \mathbb{N}} M_n(A)$ subject to the identifications coming from the natural inclusions $\iota_n : M_n(A) \to M_{n+1}(A)$ given by,

$$\varphi_n(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $M_\infty(A)_+ = \bigcup_{n \in \mathbb{N}} M_n(A)_+$. For every $a, b \in M_\infty(A)_+$, a is said to be *Cuntz subequivalent* to b - written $a \preccurlyeq b$, if there is a sequence $v_n \in M_\infty(A)$ such that

$$\lim_n ||v_n^* b v_n - a|| = 0.$$

If $a \preccurlyeq b$ and $b \preccurlyeq a$ then a and b are said to be *Cuntz equivalent* ($a \sim b$).

The relation $\sim$ is an equivalence relation and $\langle a \rangle$ denotes the equivalence class of $a \in M_\infty(A)_+$.

The set of equivalence classes

$$W(A) = \{\langle a \rangle : a \in M_\infty(A)_+\}$$

together with the operation

$$\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$$

is an abelian semigroup and the relation $\leq$ on $W(A)$ defined by

$$\langle a \rangle \leq \langle b \rangle \iff a \preccurlyeq b,$$

is a partial order. Moreover the pair $(W(A), \leq)$ is a ordered semigroup in the sense of Definition 4.1.1.

Definition 4.2.1 *If A is a C^*-algebra the pair $(W(A), \leq)$ is called the Cuntz semigroup of A.*

In the case of an unital C^*-algebra A, the equivalence class $\langle 1_A \rangle$ is an order unit for $W(A)$. In general the equivalence class of any full positive element of A (i.e. any positive element which generates A as an ideal) is an order unit for A. Henceforth, we mainly consider unital C^*-algebras and unless stated otherwise we will take the order unit of $W(A)$ to be the one generated by 1_A.

The Cuntz semigroup construction is functorial. Let $\phi : A \to B$ be an unital $*$-homomorphism between unital C^*-algebras A and B. Let $W(\phi) : W(A) \to W(B)$ be given by

$$W(\phi)(\langle a \rangle) = \langle \phi(a) \rangle, \forall a \in M_\infty(A)_+,$$

where we abuse the notation slightly and denote the lift of ϕ to matrix algebras over A by ϕ as well. Then $W(\phi)$ is a well defined morphism of scaled ordered semigroups and this construction is functional.

We find the following Proposition [33, Proposition 2.7] regarding the order of $W(A)$ to be extremely useful.

Recall that for any C^*-algebra B, $a \in B_+$ and $\epsilon > 0$, $(a - \epsilon)_+$ denotes the element of $C^*(a)$ which corresponds (via the functional calculus of a) to the function

$$f_\epsilon(t) = max\{t - \epsilon, 0\},\ t \in \sigma(a),$$

where $\sigma(a)$ is the spectrum of a.

Proposition 4.2.1 *[33, Proposition 2.7] Let A be a C^*-algebra and $a, b \in M_\infty(A)_+$. The following are equivalent;*

1. $\langle a \rangle \leq \langle b \rangle$.
2. *For every $\epsilon > 0$, $\langle (a - \epsilon)_+ \rangle \leq \langle b \rangle$.*

4.3 Partially ordered abelian groups and the group ${K_0}^*(A)$

In this section we recall the definition of the group ${K_0}^*(A)$ introduced by Cuntz in [12]. ${K_0}^*(A)$ is the Grothendieck enveloping group of $W(A)$ and inherits a natural order from the order of $W(A)$ that makes ${K_0}^*(A)$ a *partially ordered abelian group*. In the next section we proceed to outline the well known identification of *dimension functions* of a C^*-algebra A with the *state* space of ${K_0}^*(A)$ - (see (Lemma 4.4.2) which provides a convenient technical framework to study dimension functions. It should be noted that the main motivation for introducing ${K_0}^*(A)$ in [12] was to materialize this identification. The identification is natural and some times in the literature dimension functions on A are defined as states of ${K_0}^*(A)$.

Definition 4.3.1 *[15] Let G be an abelian group and 0 be denote the identity element in G. Let $0 \in G_+ \subset G$ be such that $a + b \in G_+, \forall a, b \in G_+$ and $G_+ \cap (-G_+) = \{0\}$. The pair (G, G_+) is called a partially ordered abelian group and its said to be a directed if $G = G_+ - G_+$*

The subset G_+ above is called the *positive cone* of G. For any pair $a, b \in G$ we write $a \leq b$ iff $b - a \in G_+$. This defines a partial order on G.

Conversely if G is a group and $\leq$ is partial order on G which is translation invariant (i.e. for all $a, b, c \in G$, $a \leq b \implies a + c \leq b + c$), then $G_+ = \{g : g \in G, 0 \leq g\}$ satisfies the conditions given in Definition 4.3.1 and $a \leq b \iff b - a \in G_+$. Hence, specifying a subset G_+ as in Definition 4.3.1 is equivalent to specifying a translation invariant partial order on G.

An element $u \in G_+$ is called an *order unit* for (G, G_+) if it is the case that for all $a \in G$ there exists $n \in \mathbb{N}$ such that $-nu \leq a \leq nu$.

If u is as above, for all $g \in G$ one has $g = nu - (nu - g)$ where $n \in \mathbb{N}$ is such that $g \leq nu$. In particular, any partially ordered group which has an order unit is directed.

We will often write *ordered group* to mean a partially ordered abelian group in the sense of 4.3.1 and write G in place of the pair (G, G_+) when the set G_+ is clear.

A state s on (G, G_+, u) where (G, G_+) is an ordered group and u is an order unit, is an additive map $s : G \to \mathbb{R}$ satisfying $s(G_+) \subset [0, \infty)$ and $s(u) = 1$. Set of all states on (G, G_+, u) is denoted by $S(G, G_+, u)$ or simply by $S(G, u)$ when G_+ is clear. As in the semigroup case $S(G, G_+, u)$ is a compact convex subset of the space of all real valued maps on G where the topology on $S(G, G_+, u)$ is the topology of point-wise convergence. For detailed discussion on partially ordered abeilan groups and their states see [15].

Following the notation of [12] let us write $K_0^*(A)$ to denote the Grothendieck group of $W(A)$ and set

$$K_0^*(A)_{++} = \{\gamma(y) - \gamma(x) : x, y \in W(A) \text{ and } x \leq y\}$$

where $\gamma : W(A) \to {K_0}^*(A)$ denotes the natural map given by Grothendieck construction. Its not hard to observe the following ([6] c.f [12]).

The pair $(K_0^*(A), K_0^*(A)_{++})$ form a partially ordered abeilan group and $\gamma(\langle 1_A \rangle)$ is a order unit for $(K_0^*(A), K_0^*(A)_{++})$. Note also that $(K_0^*(A), K_0^*(A)_{++})$ is directed; i.e. $K_0^*(A) = K_0^*(A)_{++} - K_0^*(A)_{++}$.

It should be noted that there is another natural ordering on ${K_0}^*(A)$. Namely one could give ${K_0}^*(A)$ an ordered group structure by setting the positive cone to be $\{\gamma(x) : x \in W(A)\}$. Its customary to associate the notation ${K_0}^*(A)\ _+$ to denote this positive cone. Note that this construction does not contain any details of the order on $W(A)$ and work for the Grothendieck enveloping group of any abelian semigroup. There is no natural order unit for the ordered group $({K_0}^*(A)\ , {K_0}^*(A)\ _+)$ - in fact its not true that $({K_0}^*(A)\ {K_0}^*(A)\ _+)$ has an order unit in general. Thus, the ordered groups structure on ${K_0}^*(A)$ given by given by taking ${K_0}^*(A)\ _{++}$ as the cone of positive elements seems more useful. Most importantly with this choice the resulting order structure on ${K_0}^*(A)$ contain (some) details of the order on $W(A)$ and hence facilitates the said identification (Lemma 4.4.2) of the *dimension functions* of A with states on $({K_0}^*(A)\ , {K_0}^*(A)\ _{++})$.

4.4 Dimension functions, lower semicontinuous dimension functions and quasitraces on a C^*-algebra

Definition 4.4.1 *(c.f. [6, 21]) A dimension function on an unital C^*-algebra A is a function $d : M_\infty(A)_+ \to [0, \infty)$ which satisfies the following conditions;*

1. $d(1) = 1$.
2. $d(a + b) = d(a) + d(b)$ *for all* $a, b \in M_\infty(A)_+$ *with* $a \perp b$.
3. $d(a) \leq d(b)$ *for all* $a, b \in M_\infty(A)_+$ *with* $a \precsim b$.

The set of all dimension functions on A is denoted by $DF(A)$ and it is given topology of point-wise convergence.

Remarks:

1. By replacing (1) above with the condition $\sup\{d(a) : a \in A_+\} = 1$, one can extend the definition of dimension functions to include non unital algebras.

2. In [6] the range of a dimension function is taken to be the set $M_\infty(A)$, not just the positive cone $M_\infty(A)_+$. It is easily seen that the two definitions are equivalent. A dimension function in the sense of 4.4.1 extends to all of $M_\infty(A)$ by setting $d(a) = d(a^*a)$. .

3. The notion of a dimension function in [12] is the one give in Definition 4.4.1 with the additional assumption $d(a) = 0$ iff $a = 0$. For simple C^*-algebras this additional condition follows from the conditions 1-3 of the definition here. Note that [12] mainly concerns simple C^*-algebras.

Lemma 4.4.2 *([6] c.f. [12]) For an unital C^*-algebra A, $DF(A)$ is in bijective correspondence with $S(K_0^*(A), K_0^*(A)_{++}, \gamma(\langle 1_A \rangle))$ and thus with $S = S(W(A), \leq, \langle 1_A \rangle)$.*

Proof We outline the identifications involved.

Any $s \in S(W(A))$ uniquely determines a state s' on $K_0^*(A)$ given by $s'(\gamma(x) - \gamma(y)) = s(x) - s(y)$. Conversely if $s' \in S(K_0^*(A))$ then $s(x) = s'(\gamma(x))$ is a state on $W(A)$.

On the other hand if $f \in DF(A)$ then $s_f(\langle a \rangle) = f(a)$ is a state on $W(A)$ and if $s \in S(W(A))$ then f_s defined on $M_\infty(A)_+$ by $f_s(a) = s(\langle a \rangle)$ is in $DF(A)$. ■

The above correspondences are homeomorphisms with respect to the point-wise topologies on each set. Due to the identifications given in 4.4.2, it is customary to call either of these three spaces as dimension functions. From here on we will use this identification freely.

A dimension function s is said to be *lower semicontinuous* if for each $a \in M_\infty(A)_+$

$$s(\langle a \rangle) \leq \liminf_n s_n(\langle a_n \rangle)$$

whenever (a_n) is a sequence in $M_\infty(A)_+$ converging to a in norm. In other words, this means that when s is identified as a dimension function in the original sense (i.e. 4.4.1) s is lower semicontinuous in the norm topology of $M_\infty(A)_+$. The set of all lower semicontinuous dimension functions of A is denoted by $LDF(A)$.

Lemma 4.4.3 *For an unital C^* -algebra A and $s \in S(W(A), \langle 1_A \rangle)$, s is lower semicontinuous if and only if $s(\langle a \rangle) = \sup_{\epsilon>0} s(\langle (a-\epsilon)_+ \rangle)$*

Proof Note that for any positive ϵ, $(a-\epsilon)_+ \leq a$ and therefore

$$\sup_{\epsilon>0} s((a-\epsilon)_+) \leq s(a), \forall s \in S.$$

To show the reverse implication let $s \in LDF(A)$. For any decreasing sequence of positive numbers (ϵ_n) converging to 0, $((a-\epsilon_n)_+)_{n\in\mathbb{N}}$ converges to a in norm. Since s is lower semicontinuous,

$$s(\langle a \rangle) \leq \liminf_n s_n(\langle (a-\epsilon_n)_+ \rangle \leq \sup_{\epsilon>0} s((a-\epsilon)_+) \leq s(a).$$

To prove the forward implication suppose $s(\langle a \rangle) = \sup_{\epsilon>0} s(\langle (a-\epsilon)_+ \rangle)$.

Let $(a_n)_{n\in\mathbb{N}}$ be any sequence in $M_\infty(A)_+$ converging to a in norm. Let $\epsilon > 0$. Then choose $k \in \mathbb{N}$ such that

$$||a - a_n|| < \epsilon, \forall n \geq k.$$

Now by Proposition 2.2.13 for all $n \geq k$, $(a-\epsilon)_+ \precsim a_n$ and thus,

$$s(\langle (a-\epsilon)_+ \rangle) \leq \liminf_n s(\langle a_n \rangle)$$

Hence, since ϵ is arbitrary,

$$s(\langle a \rangle) = \sup_{\epsilon > 0} s(\langle (a-\epsilon)_+ \rangle) \leq \liminf_n s(\langle a_n \rangle)$$

and s is lower semicontinuous.

■

Given a dimension function $s \in DF(A)$ define $\overline{s} : M_\infty(A)_+ \to [0, \infty)$ by

$$\overline{s}(a) = \sup_{\epsilon > 0} s((a-\epsilon)_+), \forall a \in M_\infty(A)_+.$$

Proposition 4.4.4 *[33, Proposition 4.1] Let A be a C^*-algebra and let $s \in DF(A)$. Then $\overline{s}$ defined above is a well defined lower semicontinuous dimension function and $\overline{s}(\langle a \rangle) \leq s(\langle a \rangle)$ for all $a \in M_\infty(A)_+$.*

We recall the Definition of a quasitrace as given in [6].

Definition 4.4.5 *[6, Definition II.1.] A quasitrace on a C^*-algebra is a map $\tau : A \to \mathbb{C}$ such that,*

1. *$\tau(x^*x) = \tau(xx^*)$.*
2. *τ is linear on commutative $*$-subalgebras of A.*
3. *If $x = a + ib$ with a, b self adjoint then, $\tau(x) = \tau(a) + i\tau(b)$.*
4. *For each $n \in \mathbb{N}$, τ extends to a map on $M_n(A)$ that satisfies (1)-(3) in $M_n(A)$.*

A quasitrace τ is said to be normalized if $||\tau|| = \sup\{\tau(a) : a \in A_+, ||a|| \leq 1\} = 1$. In the case that A is unital this is equivalent to $\tau(1) = 1$. The set of all normalized quasitraces of A is denoted by $QT(A)$. In topology of point-wise convergence $QT(A)$ is a compact subset of all complex valued functions on A, provided A is unital. Moreover from [6], $QT(A)$ is a *simplex* for unital A.

Remarks:

1. In [6] a quaistrace (2-quasitrace as termed there) is required to satisfy condition (4) only for $n = 2$. However, from [6, Proposition II.4.1] a quasitrace in that sense extends to $M_n(A)$, $\forall n \in \mathbb{N}$.

2. The extension of a quasitrace τ to $M_n(A) = M_n(\mathbb{C}) \otimes A$ is uniquely determined for each n and is moreover is equal $Tr \otimes \tau$ where Tr is the canonical non normalized trace on respective $M_n(\mathbb{C})$.

3. A *state* τ on a C^*-algebra is a linear functional on A with $\tau(A_+) \subset [0, \infty)$ and $||\tau|| = 1$. For an unital C^*-algebra this is equivalent to saying τ is linear and $\tau(1_A) = 1$. τ is a *tracial state* if its a state with $\tau(ab) = \tau(ba), \forall a, b \in A$. Note that a tracial state is a linear quasitrace. Converse is true for unital and *exact* C^*-algebras from [19].

One important fact about lower semicontinuous dimension functions is that they are represented by quasitraces in a natural and a bijective way [6]. This makes the lower semicontinuous dimension functions much more tractable compared to arbitrary dimension functions. Conversely this representation can be used to prove properties of $QT(A)$ [6, Section II.2].

Given $\tau \in QT(A)$ define $d_\tau : W(A) \to [0, \infty)$ by

$$d_\tau(\langle a \rangle) = \lim_{n \to \infty} (a^{1/n}),$$

where we slightly abuse the notation and use τ to denote $Tr \otimes \tau$ as well. The map d_τ is well defined and moreover we have the following from [6].

Theorem 4.4.6 *[6, Theorems II.2.2 and II.3.1] Let A be a C^*-algebra. The map d_τ is a lower semicontinuous dimension function on A for each $\tau \in QT(A)$. The assignment $\tau \mapsto d_\tau$ gives an affine bijection from $QT(A)$ onto $LDF(A)$ which has a continuous inverse with respect to the pointwise topologies on both ends.*

For unital A, $QT(A)$ is a *simplex* [6, Theorem II.4.4]. Given a simplex K, the set of all lower semicontinuous affine maps from K into $\mathbb{R}$ that are non negative valued and bounded is denoted by $LAff_b(K)^+$. With pointwise addition and pointwise ordering $LAff_b(K)^+$ form an ordered semigroup.

For $\langle a\rangle \in W(A)$ define $\iota(\langle a\rangle) : QT(A) \to [0, \infty)$ by

$$\iota(\langle a\rangle)(\tau) = d_\tau(\langle a\rangle),\ \forall \tau \in QT(A).$$

Clearly $\iota(\langle a\rangle)$ is well defined with and $\iota(\langle a\rangle) \in LAff_b(QT(A))^+$ for all $\langle a\rangle \in W(A)$. Note that ι defines a morphism from $(W(A), \leq, \langle 1_A\rangle)$ to $(LAff_b(QT(A))^+, \leq, 1)$ where 1 is the constant function 1 on $QT(A)$ which is an order unit for $LAff_b(QT(A))^+$.

4.5 The Blackadar-Handelman conjectures

Before we recall the Conjectures few Definitions are in order.

A *partially ordered vector space* is a real vector space E together with a partial order $\leq$ which makes the additive group E a partially ordered group and is such that $x \leq y \implies \alpha x \leq \alpha y, \forall x, y$ and $\forall \alpha \in \mathbb{R}^+$.

Definition 4.5.1 *[15, Chapter 10] A convex cone in a real vector space E is any convex subset of E that is also a cone in the abelian group E. In other words a convex cone in E is a subset C of E such that $0 \in C$ and $\alpha_1 x_1 + \alpha_2 x_2 \in C$ for any $x_1, x_2 \in C$, $\alpha_1, \alpha_2 \in \mathbb{R}^+$. A convex cone C is called a strict convex cone if $x, -x \in C \implies x = 0, \forall x \in E$.*

If E is a partially ordered vector space, $\{x \in E : x \geq 0\}$ is a strict convex cone and is called the positive cone of E. Conversely if $C \subset E$ is a strict convex cone in E then $(E, \leq_C)$ where $x \leq_C y \iff y - x \in C$, is a partially ordered vector space with positive cone C.

A strict convex cone C is called a *lattice cone* if C is such that $(C, \leq_C)$ is a lattice.

If $C \subset E$ is a non-zero convex cone then a *base* for C is any convex subset K of C such that for every non-zero $x \in C$ there is a unique pair $(b, \alpha) \in K \times \mathbb{R}^+$ with $x = \alpha b$. By convention the empty set is considered a base for the zero cone. From [15, Lemma 10.1] if C has a base, then C is a strict cone.

Definition 4.5.2 *[15, Chapter 10] A simplex in a real vector space E is a convex subset K of E that is affinely isomorphic to a base for a lattice cone in some real vector space*

Conjecture 4.5.3 *[6] For any C^*- algebra A, the set of lower semicontinuous dimension functions $LDF(A)$ is dense in $DF(A)$ in the topology of point-wise convergence.*

Conjecture 4.5.4 *[6] The affine space $DF(A)$ is a simplex for any C^*- algebra A.*

Remarks:

1. We defined dimension functions for an unital C^*-algebra A. But as remarked after Definition 4.4.1, dimension functions are defined for non - unital A. Thus, the conjecture make sense in general. However, we will only be interested in the unital case. In fact all the known confirmations (see 4.5.5) of the conjectures assumes A to be unital.

2. A *Choquet simplex* is by definition a compact *simplex* in a locally convex Hausdorff space. Thus for unital A, if $DF(A)$ is a simplex then $DF(A)$ is a Choquet simplex. For a detailed discussion on simplexes - in particular Choquet simplexes, read [15].

Theorem given bellow lists the instances that we found in our literature survey where either of the conjectures had already been proven.

Theorem 4.5.5 *Let A be an unital C^*-algebra. The following hold.*

1. [6, Theorem I.2.4.] Conjecture 4.5.3 holds if A is commutative.

2. [29, Corollary 4.4] Conjecture 4.5.4 holds if A is has real rank zero and stable rank 1.

3. [9, Theorem B] If A is simple finite and is either exact and $\mathcal{Z}$-stable or AH of slow dimension growth then both the conjectures hold for A.

4. [1, Theorem 4.1] Let X be a finite dimensional, compact metric space, and let A be separable and unital. Conjecture 4.5.4 holds in the following cases:

(i) dim $X \leq 1$ *and* A *is a continuous field such that for all* $x \in X$, A_x *has stable rank one with trivial* K_1 *and is either of real rank zero or else simple and* $\mathcal{Z}$*-stable.*

(ii) X *is an arc-like space and* $A = C(X, B)$ *where* B *is simple, and either has real rank zero and finite radius of comparison, or else is* $\mathcal{Z}$*-stable.*

(iii) dim $X \leq 2$ *with vanishing second Čech cohomology group* $\check{\mathrm{H}}^2(X, \mathbb{Z}) = 0$, *and* $A = C(X, B)$ *with* B *an infinite simple AF-algebra.*

(iv) $A = C(X, B)$, *where* B *is a non-type I, unital simple, ASH-algebra with slow dimension growth.*

5. [1, Theorem 4.5] Let X *be a finite dimensional, compact metric space, and let* $A = C(X, B)$ *where* B *is an unital, separable,infinite dimensional and exact with stable rank one such that* $T(B)$ *is a Bauer simplex. Then Conjecture 4.5.3 holds for* A *in the following cases;*

(i) dim $X \leq 1$ *and* B *is simple,* $K_1(B) = 0$ *and* B *has strict comparison.*

(ii) X *is arc-like,* B *is simple, has real rank zero and strict comparison.*

(iii) dim $X \leq 2$ *and* $\check{\mathrm{H}}^2(X, \mathbb{Z}) = 0$, *with* B *an AF-algebra.*

(iv) B *is a non-type I, simple, unital ASH algebra with slow dimension growth.*

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5. NEW CONFIRMATIONS OF CONJECTURE 4.5.3 FOR UNITAL C^*-ALGEBRAS

All C^*-algebras are assumed to be unital and stably finite. Note that for unital A stably finiteness implies $DF(A) \neq \emptyset$. The converse hold for simple (unital) A.

5.1 An alternate criterion for density of $LDF(A)$ in $DF(A)$ for unital A

Recall from last chapter that for a C^*-algebra A, $\iota : W(A) \to LAff_b(QT(A))^+$ denotes the function given by $\iota(a)(\tau) = d_\tau(a), \forall \tau \in QT(A)$.

Theorem 5.1.1 *Let A be an unital stably finite C^*-algebra. Then $LDF(A)$ is dense in $DF(A)$ if and only if $\iota : (W(A), \leq, \langle 1_A \rangle) \to (LAff_b(QT(A))^+, \leq, 1)$ is a stable order embedding, where $LAff_b(QT(A))^+$ defined in the previous section and 1 denote the constant function 1.*

Proof Suppose ι is a stable order embedding.

Let K denote the pointwise closure of $LDF(A)$ in $DF(A)$. Then K is a compact convex subset of $DF(A)$. Suppose $x, y \in W(A)$ are such that $d(x) < d(y)$ for all $d \in K$. The function defined on K given by $d \mapsto d(y) - d(x)$ for every $d \in K$ is strictly positive and continuous on K in pointwise topology. Since K is compact the function attains a minimum $\delta > 0$ on K.
Choose some $n \in \mathbb{N}$ large enough so that $n\delta \geq 1$.

Then,

$$nd(x) + 1 \leq nd(y), \forall d \in K.$$

In particular,

$$nd_\tau(x) + 1 \leq nd_\tau(y), \forall \tau \in QT(A).$$

Therefore,

$$n\iota(x) + 1 \le n\iota(y).$$

Hence, as ι is a stable order embedding, there is some $m \in \mathbb{N}$ and $z \in W(A)$ such that,

$$mnx + z + \langle 1_A \rangle \le mny + z.$$

Thus,

$$s(x) < s(y), \forall s \in DF(A).$$

Therefore, by Lemma 4.1.4 $K = DF(A)$, i.e. $LDF(A)$ is dense in $DF(A)$.

Now suppose $LDF(A)$ is dense in $DF(A)$.

In general ι is an order preserving homomorphism. To verify it is a stable order embedding let $x, y \in W(A)$ and suppose that there is some $n \in \mathbb{N}$ such that,

$$n\iota(x) + 1 \le n\iota(y).$$

Then for all $\tau \in QT(A)$,

$$d_\tau(nx + \langle 1_A \rangle) \le d_\tau(ny).$$

Therefore, since $LDF(A)$ is dense in $DF(A)$,

$$s(nx + \langle 1_A \rangle) \quad \le \quad s(ny), \forall s \in DF(A)$$

Thus, $s(nx) < s(ny), \forall s \in DF(A)$ and hence, by Lemma 4.1.3, there is some $m \in \mathbb{N}$ and $z \in W(A)$ such that,

$$mnx + \langle 1_A \rangle + z \le mny + z.$$

Therefore, ι is a stable order embedding. ■

In the next section we apply Theorem 5.1.1 to provide new verifications of Conjecture 4.5.3.

5.2 New confirmations of Conjecture 1.2.2

We confirm Conjecture 4.5.3 for any unital C^*-algebra A which either has *finite radius of comparison* or has an almost *unperforated* $W(A)$.

Definition 5.2.1 *[40, Definition 6.1] Let A be a C^*-algebra. A has finite radius of comparison if there is some real number $r > 0$ such that the following hold for all $a, b \in M_\infty(A)_+$;*

$$(d_\tau(\langle a\rangle) + r \leq d_\tau(\langle b\rangle), \forall \tau \in QT(A)) \implies a \preccurlyeq b. \tag{5.1}$$

If A is of finite radius of comparison, the radius of comparison of A ($rc(A)$) is the infimum of all r as in (5.1). If not the radius of comparison is infinite and we write $rc(A) = \infty$. Note that $rc(A) = 0$ iff A has strict comparison.

Definition 5.2.2 *An ordered semigroup $(M, \leq)$ is said to be almost unperforated if $kx \leq k'y \implies x \leq y$ for all $x, y \in M$ and each $k, k' \in \mathbb{N}$ with $k > k'$*

To confirm the conjecture in the latter case we need the following Proposition ([33] c.f. [5, Theorem 6.8.51]).

Proposition 5.2.3 *[33, Proposition 3.2] Let $(M, \leq)$ be almost unperforated and $u, x \in M$. If u is an order unit and $s(x) < s(u)$ for all $s \in S(M, \leq, u)$ then $x < u$.*

Lemma 5.2.4 is mainly a consequence of Krein-Milman Theorem and show that if $QT(A)$ has only finitely many extremal points then Conjecture 4.5.3 implies Conjecture 4.5.4.

Lemma 5.2.4 *Suppose A is an unital C^*-algebra with $\partial_e(QT(A))$ finite and non empty. Then $LDF(A)$ is compact and moreover the map $g : QT(A) \to LDF(A)$ given by $\tau \mapsto d_\tau$ is an affine homeomorphism. If its also the case that $LDF(A)$ is dense in $DF(A)$ then $DF(A) = LDF(A)$ and $DF(A)$ is affinely homeomorphic to $QT(A)$.*

Proof From Theorem 4.4.6 [6], $g : QT(A) \to LDF(A)$ is an affine bijection and g^{-1} is continuous. Since $QT(A)$ is compact and convex, by Krein–Milman Theorem $QT(A)$ is the closure of the convex hull of $\partial_e(QT(A))$. As $\partial_e(QT(A))$ is assumed to be finite, its convex hull $co(\partial_e(QT(A)))$ is compact and therefore we have;

$$QT(A) = \overline{co(\partial_e(QT(A)))} = co(\partial_e(QT(A))).$$

Thus,

$$LDF(A) = g(QT(A)) = g(co(\partial_e(QT(A))))$$

and since g is an affine bijection,

$$g(\partial_e(QT(A))) = \partial_e(LDF(A)).$$

Therefore,

$$LDF(A) = g(co(\partial_e(QT(A)))) = co(\partial_e(LDF(A))).$$

In particular, since $\partial_e(LDF(A)) = g(co(\partial_e(QT(A))))$ is a non empty finite set, $LDF(A)$ is compact and g is a homeomorphism.

Now if $LDF(A)$ is dense in $DF(A)$, then $DF(A) = \overline{LDF(A)}$. As we had just noted, $LDF(A)$ is compact and so it is equal to its own closure. Thus $DF(A) = LDF(A)$ and is affinely homeomorphic to $QT(A)$ from the preceding paragraph. ■

Theorem 5.2.5 *Let A be any unital C^*-algebra. The following hold.*

1. If A has finite radius of comparison then $LDF(A)$ is dense in $DF(A)$.

2. If $W(A)$ almost unperforated then $LDF(A)$ is dense in $DF(A)$.

3. If $\partial_e(QT(A))$ is a finite set and if either of the assumptions above (in 1,2) hold for A then $DF(A) = LDF(A)$ and $DF(A)$ is affiinely homeomorphic to $QT(A)$. In particular $DF(A)$ is a Choquet simplex.

Proof Proof of 1:

Let $rc(A) = r < \infty$.

By Theorem 5.1.1 we only have to show that ι is a stable order embedding.

Let $x, y \in W(A)$ and suppose that there is some $n \in \mathbb{N}$ with,

$$n\iota(x) + 1 \leq n\iota(y).$$

Then for all $\tau \in QT(A)$,

$$d_\tau(nx + \langle 1_A \rangle) \leq d_\tau(ny).$$

Choose some $m \in \mathbb{N}$ large enough so that $m > r + 1$.

Then for all $\tau \in QT(A)$,

$$\begin{aligned} d_\tau(mnx + \langle 1_A \rangle) + r &= d_\tau(mnx) + 1 + r \\ &< d_\tau(mnx) + m \\ &= md_\tau(nx + \langle 1_A \rangle) \\ &\leq md_\tau(ny) \end{aligned}$$

Therefore, since $rc(A) = r$,

$$mnx + \langle 1_A \rangle \leq mny.$$

Proof of 2:

Again we only have to show that ι is a stable order embedding.

So let $a, b \in M_\infty(A)_+$ and suppose that there is some $n \in \mathbb{N}$ such that,

$$n\iota(\langle a \rangle) + 1 \leq n\iota(\langle b \rangle)$$

In particular, for all $\tau \in QT(A)$,

$$2nd_\tau(\langle a \rangle) + 1 < 2nd_\tau(\langle b \rangle) \tag{5.2}$$

Fix $\epsilon > 0$ and let $s \in DF(A)$ be arbitrary.

Then $\overline{s} \in LDF(A)$, where $\overline{s}$ is as in Proposition 4.4.4.

Thus, by equation (5.2)

$$2n\overline{s}(\langle a \rangle) + 1 < 2n\overline{s}(\langle b \rangle). \tag{5.3}$$

Note that by definition of $\overline{s}$ we have,

$$s(\langle(a-\epsilon)_+\rangle) \leq \overline{s}(\langle a\rangle).$$

Combining this with (5.3) we have,

$$\begin{array}{rcl} s(2n\langle(a-\epsilon)_+\rangle + 2\langle 1_A\rangle) & = & 2ns(\langle(a-\epsilon)_+\rangle) + 2 \\ & \leq & 2n\overline{s}(\langle a\rangle) + 1 + 1 \\ & < & 2n\overline{s}(\langle b\rangle) + 1 \\ & \leq & 2ns(\langle b\rangle) + 1 \\ & = & s(2n\langle b\rangle + \langle 1_A\rangle) \end{array}$$

Since $2n\langle b\rangle + \langle 1_A\rangle$ is an order unit for $W(A)$ and s is arbitrary, we apply Proposition 5.2.3 to conclude

$$2n\langle(a-\epsilon)_+\rangle + 2\langle 1_A\rangle \leq 2n\langle b\rangle + \langle 1_A\rangle.$$

Note that ϵ is arbitrary and in particular does not depend on n.

Thus by [33, Proposition 2.4] it follows that,

$$2n\langle a\rangle + 2\langle 1_A\rangle \leq 2n\langle b\rangle + \langle 1_A\rangle.$$

In particular for $z = \langle 1_A\rangle \in W(A)$,

$$2n\langle a\rangle + \langle 1_A\rangle + z \leq 2n\langle b\rangle + z$$

and we conclude that $\iota : W(A) \to LAff_b(QT(A))^+$ is a stable order embedding.

Proof of 3: First part follows directly from Lemma 5.2.4 and parts 1 and 2 above. To see that $DF(A)$ is a Choquet simplex recall $QT(A)$ is Choquet.

■

Examples:

1. C^*-algebras considered in [9, Theorem B] have finite radius of comparison. In particular, C^*-algebras in [9, Theorem B] have $rc(A) = 0$.

2. Most of the continuous fields considered in [1] are also known to have finite radius of comparison.

3. Counter examples for Elliott's classification conjecture constructed in [36] and Villadsen algebras of type I [41] satisfy Conjecture 4.5.3 by Theorem 5.2.5. Note that these algebras are unital ASH-algebras of *flat dimension growth*. Thus from [38] have finite radius of comparison.

4. Villadsen algebras of type II [42] are of finite radius comparison (for the same reasons provided in 3) and have unique quasitrace. Therefore these satisfy both the conjectures from Theorem 5.2.5. This means that for each $n \in \mathbb{N}$, we now know that there are unital algebras of stable rank n which satisfy the conjectures.

Remark: For simple C^*-algebras, almost unperforation of $W(A)$ is equivalent to strict comparison (i.e zero radius of comparison) and thus (2) of Theorem 5.2.5 may seem some what redundant when compared to (1). However, in general (without simplicity) it is not clear how the two properties relate to each other.

5.3 Possibility of extending Theorem 5.2.5 to inductive limits

It is natural to attempt to extend the conclusions of parts 1 and 2 of Theorem 5.2.5 to unital inductive limits. More specifically,

Question 5.3.1 *If (A_n, ϕ_n) is an unital inductive system with $LDF(A_n)$ dense in $DF(A_n)$ for each n, is $LDF(A)$ dense in $DF(A)$ for $A = \lim_n A_n$?*

Or more feasibly,

Question 5.3.2 *If (A_n, ϕ_n) is an unital inductive system with either $rc(A_n) < \infty$ for all n or $W(A)$ almost unperofated for all n, is $LDF(A)$ dense in $DF(A)$ for $A = \lim_n A_n$?*

A positive answer to either of these questions would confirm Conjecture 4.5.3 for a much wider class of unital C^*-algebras including ones with infinite radius of comparison. In Proposition 5.3.4, using a classical functional analytic type argument, we prove a result that could be considered an intermediate step in this direction. However, extending Proposition 5.3.4 to a complete answer to the above question remains open. It should be noted that crux of a the matter of providing a positive answer to Questions 5.3.1 or 5.3.2 lies in overcoming the complications that arise form the non-continuity (with respect to inductive limits) of the Cuntz semigroup construction (i.e. $W(A) \ncong \lim_n W(A_n)$). Proposition 5.3.4 does not address this issue.

Suppose B, C are two unital C^*-algebras with non empty quasitrace spaces (for example say B, C are stably finite). Let $\phi : B \to C$ be an unital $*$-homomorphism and let $\phi^\sharp : QT(C) \to QT(B)$ be the induced continuous affine map given by $\phi^\sharp(\tau) = \tau \circ \phi$.

Lemma 5.3.3 *Let $A = \lim_n(A_n, \phi_n)$ where (A_n, ϕ_n) is an inductive system of unital C^*-algebras and unital $*$-homomorphisms. For every $m, k \in \mathbb{N}$, let $\phi_{m,m+k} : A_m \to A_{m+k}$ be the composition $\phi_{m+k-1} \circ \phi_{n+k-2} \circ \ldots. \circ \phi_m$ and let $\phi_{m,\infty} : A_m \to A$ be the induced $*$-homomorphism. Then for any $m \in \mathbb{N}$*

$$\bigcap_{k\in\mathbb{N}} {\phi^\sharp}_{m,m+k}(QT(A_{m+k})) = {\phi^\sharp}_{m,\infty}(QT(A)).$$

Proof If there is an increasing sequence of integers $\{m_k\}$ with $QT(A_{m_k}) = \emptyset, \forall k \in \mathbb{N}$ then $QT(A) = \emptyset$. Thus the conclusion of the Lemma is a triviality.

Let us assume that $QT(A_n) \neq \emptyset$ for every $n \in \mathbb{N}$.

The reverse inclusion is clear. Indeed, if $\rho = {\phi^\sharp}_{m,\infty}(\tau)$ for some $\tau \in QT(A)$ then for each $k \in \mathbb{N}$, $\rho = {\phi^\sharp}_{m,m+k}(\phi_{m+k,\infty}(\tau))$. Hence $\rho \in {\phi^\sharp}_{m,m+k}(QT(A))$, for every $k \in \mathbb{N}$.

To show the forward inclusion, fix $\rho \in \bigcap_{k\in\mathbb{N}} {\phi^\sharp}_{m,m+k}(QT(A_{m+k}))$.

Set $Y_m = \{\rho\}$ and for each $i \neq m$ set,

$$Y_i = \{\phi_{i,m}(\rho)\}, \forall i < m \tag{5.4}$$

$$Y_i = \{\tau \in QT(A_i) : \tau \circ \phi_{m,i} = \rho\}, \forall i > m. \tag{5.5}$$

Let Y be the set,

$$Y = \prod_{i=1}^{\infty} Y_i \subset \prod_{i=1}^{\infty} QT(A_i).$$

Now for each $i \in \mathbb{N}$, let $p_i : Y \to QT(A_i)$ be the restriction of the natural projection map from $\prod_{i=1}^{\infty} QT(A_i)$ onto $QT(A_i)$. From 5.4 and 5.5 we have $p_i = {\phi^\sharp}_{i,j} \circ p_j$ for all $i \leq j$. Therefore, since $QT(A)$ is the inverse limit of $(QT(A_i), {\phi^\sharp}_{i,j})$ [6, Theorem II.4.8.], for every $i \in \mathbb{N}$ there is a continuous map $p : Y \to QT(A)$ such that $p_i = {\phi^\sharp}_{i,\infty} \circ p$ by the universal property of inverse limits.

In particular, there is some $(\tau_n)_{n\in\mathbb{N}} \in Y$ such that $p_m((\tau_n)_{n\in\mathbb{N}}) = {\phi^\sharp}_{m,\infty}(p((\tau_n)_{n\in\mathbb{N}}))$. But by definition of Y, $p_m((\tau_n)_{n\in\mathbb{N}}) = \tau_m = \rho$. Therefore for $\rho_\infty = p((\tau_n)_{n\in\mathbb{N}}) \in QT(A)$ we have $\rho = {\phi^\sharp}_{m,\infty}(\rho_\infty)$ and hence, $\rho \in {\phi^\sharp}_{m,\infty}(QT(A))$.

■

For an unital $*$-homomorphism $\phi : B \to C$, recall that $W(\phi) : W(B) \to W(C)$ induced by ϕ is a morphism of scaled ordered semigroups.

Proposition 5.3.4 *Let (A_n, ϕ_n) be an inductive system of unital C^*-algebras and unital $*$-homomorphisms. Let $A = \lim_{n\to\infty} A_n$ and suppose $LDF(A_n)$ is dense in $DF(A_n)$ for every $n \in \mathbb{N}$. Write $S = \bigcup_{n\in\mathbb{N}} W(\phi_{n,\infty})(W(A_n))$ where $\phi_{n,\infty} : A_n \to A_\infty$ are the natural maps. Then for any $s \in DF(A)$, for every finite subset F of S and each $\epsilon > 0$ there is some $d \in LDF(A)$ such that for every $x \in F$, $|s(x) - d(x)| < \epsilon$.*

Proof For notational convenience let us denote $W(\phi_n), W(\phi_{n,\infty})$ by $\psi_n, \psi_{n,\infty}$ respectively. Clearly S is a subsemigroup of $W(A)$ containing the order unit $\langle 1_A \rangle$ and $(S, \langle 1_A \rangle)$ is a scaled ordered semigroup.

Let $s \in DF(A)$ and fix $\epsilon > 0$. Let F be a finite subset of S. Say $F = \{x_1, x_2, ..., x_t\}$ for some $t \in \mathbb{N}$ and pick some $i_0 \in \mathbb{N}$ so that for every $1 \leq j \leq t$ there is $x_{i_0,j} \in W(A_{i_0})$ with $x_j = \psi_{i_0,\infty}(x_{i_0,j})$. For simplicity let us drop a finite number of A_ns and renumber to take $i_0 = 1$

For all $n \geq 1$ and $1 \leq j \leq t$, let $x_{n+1,j} = \psi_n(x_{n,j}) \in W(A_{n+1})$. For each $n \in \mathbb{N}$, let $s_n = s \circ \psi_{n,\infty} \in DF(A_n)$. Then by density of $LDF(A_n)$ in $DF(A_n)$ for each $n \in \mathbb{N}$ there is $\tau_n \in QT(A_n)$ such that,

$$|d_{\tau_n}(x_{n,j}) - s_n(x_{n,j})| < \frac{\epsilon}{2}, \forall 1 \leq j \leq t. \tag{5.6}$$

Now set $\rho_1 = \tau_1$ and $\rho_n = {\phi^\sharp}_{1,n}(\tau_n)$ for all $n > 1$ where $\phi_{1,n} = \phi_{n-1} \circ \phi_{n-2} \circ .. \circ \phi_1$. Then $(\rho_n)_{n\in\mathbb{N}}$ is a sequence in $QT(A_1)$ and using weak* compactness of $QT(A_1)$, by passing onto a subsequence if required, we assume that $\rho_n \to \rho$ (weak*) for some $\rho \in QT(A)$. But note that $\rho \in {\phi^\sharp}_{1,n}(QT(A_n))$ for all $n > 1$. Hence by 5.3.3, there is some $\tau \in QT(A)$ such that $\rho = {\phi^\sharp}_{1,\infty}(\tau)$.

On the other hand since $\rho_n \to \rho$ in weak*-topology ρ is contained in the norm closure of the convex hull of $\{\rho_n : n \in \mathbb{N}\}$. Thus, choose a finite set of indexes $i_1, i_2, ..., i_k \in \mathbb{N}$ and $0 < t_1, t_2, ..t_k \leq 1$ with $\sum_{p=1}^k t_p = 1$ such that,

$$||\rho - \sum_{p=1}^k t_p \rho_{i_p}|| < \frac{\epsilon}{2}. \tag{5.7}$$

For notational convenience take $\sigma = \sum_{i=1}^k t_j \rho_{i_j}$. Then $\forall 1 \leq j \leq t$,

$$\begin{aligned}
|s_1(x_{1,j}) - d_\sigma(x_{1,j})| &= |\sum_{p=1}^k t_p s_{i_p}(x_{i_p,j}) - \sum_{p=1}^k t_p d_{\tau_{i_p}}(x_{i_p,j})| \\
&\leq \sum_{p=1}^k t_p |s_{i_p}(x_{i_p,j}) - d_{\tau_{i_p}}(x_{i_p,j})| \\
&\leq \sum_{p=1}^k t_p \frac{\epsilon}{2} \quad \text{(by 5.6)} \\
&\leq \frac{\epsilon}{2}
\end{aligned}$$

For every $1 \leq j \leq t$ fix some large enough $r \in \mathbb{N}$ and choose positive contractions $a_{1,j} \in M_r(A_1)_+$ such that $x_{1,j} = \langle a_{1,j} \rangle$. Write $a_{n+1,j} = \phi_n(a_{1,j}), \forall n \in \mathbb{N}$ and $1 \leq j \leq t$.

Note that each $a_{n,j}$ is a positive contraction in $M_r(A_n)$ with $x_{n,j} = \langle a_{n,j} \rangle$. Then $\forall 1 \leq j \leq t$,

$$\begin{aligned} |d_\sigma(x_{1,j}) - d_\rho(x_{1,j})| &= |\lim_{n\to\infty} \sigma((a_{1,j})^{1/n}) - \lim_{n\to\infty} \rho((a_{1,j})^{1/n})| \\ &\leq \limsup_n |\sigma((a_{1,j})^{1/n}) - \rho((a_{1,j})^{1/n})| \\ &\leq \limsup_n ||\sigma - \rho|| ||(a_{1,j})^{1/n})|| \\ &\leq \frac{\epsilon}{2}, \end{aligned}$$

where the last inequality follows from 5.7 and the fact that $a_{1,j}$ is contractive.

Now from 5.8 and 5.8 $\forall 1 \leq j \leq t$,

$$\begin{aligned} |s(x_j) - d_\tau(x_j)| &= |s(\psi_{1,\infty}(x_{1,j})) - d_\tau(\psi_{1,\infty}(x_{1,j}))| \\ &= |s_1(x_{1,j}) - d_\rho(x_{1,j})| \\ &\leq |s_1(x_{1,j}) - d_\sigma(x_{1,j})| - |d_\sigma(x_{1,j}) - d_\rho(x_{1,j})| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

■

6. STABLY APPROXIMATE INTERPOLATION AND CONJECTURE 4.5.4

Interpolation groups is one class of ordered groups for which a good structure theory exists. Our interest of interpolation groups is motivated by their relevance to Conjecture 4.5.4.

From [15, Theorem 10.17] the state space of an interpolation group is a Choquet simplex. Thus, for any unital C^*-algebra A if the ordered group ${K_0}^*(A)$ is an interpolation group then Conjecture 4.5.4 holds for A.

In this chapter we introduce a class of scaled ordered groups that satisfy a *stably approximate* version of interpolation (Definition 6.2.1) and show $S(G, u)$ to be a Choquet simplex for a ordered group G with order unit u that satisfy this weaker notion of interpolation. Our proof is based very much on the ideas of [15]. We simply show that the techniques used in [15] in proving Theorem 6.1.4 for interpolation groups can be adopted to our case after some modifications.

In the final section of the chapter we investigate the possibility of applying Corollary 6.3.8 to confirm Conjecture 4.5.4.

6.1 Interpolation groups

Recall the following definition. Note that we use the terminology and conventions form Chapter 4.

Definition 6.1.1 *Let (G, G_+) be a partially ordered abelian group (Definition 4.3.1) and let $u \in G_+$ be an order unit. (G, G_+) is said to have,* *1. Riesz interpolation property if for all* $a_1, a_2, b_1, b_2 \in G$ *with* $a_i \leq b_j$ *for all* i, j *there is* $c \in G$ *such that,*

$$a_i \leq c \leq b_j, \forall 1 \leq i, j \leq 2.$$

2. Riesz decomposition property if for each $a, b_1, b_2 \in G_+$ *with* $a \leq b_1 + b_2$ *there is* $a_1, a_2 \in G_+$ *such that,*

$$a_1 + a_2 = a \text{ and } a_1 \leq b_1, a_2 \leq b_2.$$

3. Riesz refinement property if for each $a_1, a_2, b_1, b_2 \in G_+$ *with* $a_1 + a_2 = b_1 + b_2$ *there are* $c_{11}, c_{12}, c_{21}, c_{22} \in G_+$ *such that,*

$$a_i = c_{i1} + c_{i2} \text{ and } b_j = b_{1j} + b_{2j}$$

for each $1 \leq i, j \leq 2$

All three properties provided above are equivalent. A proof of this fact is provided in [15, Proposition 2.1].

Proposition 6.1.2 *For any* (G, G_+) *the three properties in Definition 6.1.1 are equivalent.*

Remark: Property (3) of Definition 6.1.1 was first considered by Riesz in [32] for an ordered semigroup with cancellation which is equivalent to a positive cone of an ordered group. Birkhoff introduced the other two properties and proved the equivalence of the three properties in [4]. Note that the first property is defined for any partially ordered set.

Definition 6.1.3 *A partially ordered abelian group* (G, G_+) *which satisfies the Riesz interpolation (equivalently Riesz decomposition or refinement) property is called an interpolation group.*

Our interest on interpolation groups is mainly due to the following theorem.

Theorem 6.1.4 *[18, Theorem I.2.5](also see [15, Theorem 10.17]) If (G, G_+) is an interpolation group with order unit u then its state space $S(G, , u)$ is a (Choquet) simplex.*

All the previous verifications of Conjecture 4.5.4 that we listed in Theorem 4.5.5 utilize this. In other words, in all those cases the group $K_0^*(A)$ is shown to have the Riesz interpolation property.

6.2 Stably approximate interpolation, decomposition and refinement

Definition 6.2.1 *Let (G, G_+) be a partially ordered abelian group with $u \in G_+$ an order unit. We say that (G, G_+, u) has;*

1. Stably approximate interpolation property if for every $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j$ for all i, j and any $\epsilon > 0$ there are $n, k \in \mathbb{N}$ and $c \in G$ with,

$$\frac{k}{n} < \epsilon \text{ and } na_i \leq c \leq nb_j + ku, \forall 1 \leq i, j \leq 2.$$

2. Stably approximate decomposition property if for every $a, b_1, b_2 \in G_+$ with $a \leq b_1 + b_2$ and for any $\epsilon > 0$ there are $k, n \in \mathbb{N}$ with $\frac{k}{n} < \epsilon$ and $a_1, a_2 \in G_+$ such that,

$$\begin{aligned} a_1 + a_2 &= na + ku \\ a_1 &\leq nb_1 + ku \\ a_2 &\leq nb_2 + ku \end{aligned}$$

3. Stably approximate refinement property if for every $a_1, a_2, b_1, b_2 \in G_+$ with $a_1 + a_2 = b_1 + b_2$ and for any $\epsilon > 0$ there are $k, n \in \mathbb{N}$ with $\frac{k}{n} < \epsilon$ and $c_{ij} \in G_+, 1 \leq i, j \leq 2$ such that

$$\begin{aligned} c_{i1} + c_{i2} &= na_i + ku \\ c_{1j} + c_{2j} &= nb_j + ku, \end{aligned}$$

for $1 \leq i, j \leq 2$.

Following the proof of [15, Proposition 2.1] we prove;

Proposition 6.2.2 *Let (G, G_+) be a partially ordered abelian group with $u \in G_+$ an order unit. The three properties listed in Definition 6.2.1 are equivalent.*

Proof Proof of (1) $\implies$ (2);

Let $a, b_1, b_2 \in G_+$ with $a \leq b_1 + b_2$ and fix $\epsilon > 0$. So $0 \leq a$, $0 \leq b_1$ and $a - b_2 \leq b_1$. Since $b_2 \geq 0$, we have $a - b_2 \leq a$.

Thus, by (1) of Definition 6.2.1, there are $n, k \in \mathbb{N}$ with $\frac{k}{n} < \epsilon$ and $c_1 \in G$ with,

$$\begin{aligned} 0 &\leq c_1 \\ n(a - b_2) &\leq c_1 \\ c_1 &\leq na + ku \\ c_1 &\leq nb_1 + ku. \end{aligned}$$

Set $c_2 = na + ku - c_1$. Then, $c_1 + c_2 = na + ku$ with $\frac{k}{n} < \epsilon$ and

$$\begin{aligned} c_2 &\leq na + ku - (na - nb_2) \\ &= nb_2 + ku. \end{aligned}$$

Proof of (2) $\implies$ (1):

Suppose $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j$ for all i, j.

Let $\epsilon > 0$.

Set $x = b_2 - a_1$, $y_1 = b_1 - a_1$, and $y_2 = b_2 - a_2$. Then, $x, y_1, y_2 \in G_+$ and

$$x \leq (b_2 - a_1) + (b_1 - a_2) = (b_1 - a_1) + (b_2 - a_2) = y_1 + y_2$$

Therefore, by (2) there are $n, k \in \mathbb{N}$ with $\frac{k}{n} < \epsilon$ and $z_1, z_2 \in G_+$ such that,

$$z_1 + z_2 = nx + ku \tag{6.1}$$

$$z_1 \leq ny_1 + ku \tag{6.2}$$

$$z_2 \leq ny_2 + ku \tag{6.3}$$

Set $c = z_1 + na_1$. So $na_1 \leq c$.

From 6.2,

$$c \leq ny_1 + ku + na_1 = nb_1 + ku.$$

From 6.1,

$$c \leq nx + ku + na_1 = nb_2 + ku.$$

Note that from 6.3,

$$na_2 \leq nb_2 + ku - z_2.$$

Now using 6.1 again,

$$\begin{aligned} na_2 &\leq nb_2 + ku - (nx + ku - z_1) \\ &= na_1 + z_1 \end{aligned}$$

Hence, $na_i \leq c \leq nb_j + ku$ for all i, j.

Proof of (2) $\implies$ (3):

Suppose $a_1, a_2, b_1, b_2 \in G_+$ are such that $a_1 + a_2 = b_1 + b_2$.

Fix $\epsilon > 0$.

Now, $a_1 \leq b_1 + b_2$ and by (2) find $n, k \in \mathbb{N}$ with $\frac{k}{n} < \epsilon$ and $c_{11}, c_{12} \in G_+$ so that,

$$\begin{aligned} c_{11} + c_{12} &= na_1 + ku \\ c_{11} &\leq nb_1 + ku \\ c_{12} &\leq nb_2 + ku \end{aligned}$$

For $1 \leq j \leq 2$, set

$$c_{2j} = nb_j + ku - c_{1j}$$

Then, $c_{1j} + c_{2j} = nb_j + ku$ for $1 \leq j \leq 2$ and,

$$\begin{aligned} c_{21} + c_{22} &= (nb_1 + ku - c_{11}) + (nb_2 + ku - c_{12}) \\ &= nb_1 + nb_2 + 2ku - (na_1 + ku) \\ &= na_2 + ku. \end{aligned}$$

Proof of (3) $\implies$ (2):

Suppose $a, b_1, b_2 \in G_+$ are such that $a \leq b_1 + b_2$.

Fix $\epsilon > 0$.

Write $x_1 = a$ and $x_2 = (b_1 + b_2) - a$. Then $x_1 + x_2 = b_1 + b_2$ and by (3) there are $n, k \in \mathbb{N}$ with $\frac{k}{n} < \epsilon$ and $c_{ij} \in G_+$ for $1 \leq i, j \leq j$ such that

$$\begin{aligned} c_{i1} + c_{i2} &= nx_i + ku, \forall 1 \leq i \leq 2 \\ c_{1j} + c_{2j} &= nb_j + ku, \forall 1 \leq j \leq 2 \end{aligned}$$

Then $c_{11}, c_{12} \in G_+$ satisfy $c_{11} + c_{12} = na + ku$, $c_{1j} \leq nb_j + ku$ and (G, G_+). ■

Definition 6.2.3 *We say that a scaled partially ordered abelian group (G, G_+, u) exhibit stably approximate interpolation if either (hence all) of the three properties defined in Definition 6.2.1 holds for (G, G_+, u).*

Example 6.2.4 *In [28] a scaled pre-ordered group (G, u) is called an asymptotic interpolation group if for every $a_i, b_j \in G$ with $a_i \leq b_j$ for all i, j and for all $\epsilon > 0$, there is some $c, d \in G$ with $||d||_u < \epsilon$ and*

$$a_i \leq c \leq b_j + d, \ \forall 1 \leq i, j \leq 2. \tag{6.4}$$

Here $|| \cdot ||_u$ denotes the order unit norm, i.e. for any $x \in G$,

$$||x||_u = \inf \left\{ \frac{k}{n} : k, n \in \mathbb{N} \ \ \textit{and} \ -ku \leq nx \leq ku \right\}.$$

It is easy to note that such groups satisfy Definition 6.2.1 (1). Indeed, if $c, d \in G$ are as in 6.4, then since $||d||_u < \epsilon$, there are $n, k \in \mathbb{N}$ with $nd \leq ku$ and $\frac{k}{n} < \epsilon$. For such k, n;

$$na_i \leq nc \leq nb_j + nd \leq nb_j + ku, \ \forall 1 \leq i, j \leq 2.$$

Thus all groups with asymptotic interpolation are contained in the class of groups we consider and from [28] (c.f. [43]) this in particular include ordered groups that fail to be interpolation groups.

Example 6.2.5 *[15, Chapter 2] Let $G = \mathbb{Z}^2$ as a group and set $G_+ = \{(a,b) \in G : 2a \geq b \geq 0\}$. Then (G, G_+) is a partially ordered abelian group and $u = (1,1) \in G_+$ is an order unit. Note that (G, G_+) is not an interpolation group. For instance if we set $x_1 = (0,0)$, $x_2 = (0,1)$, $y_1 = (1,1)$ and $y_2 = (1,2)$ then $y_j - x_i \in G_+$ for all i, j but there is no element $c \in G$ with $x_i \leq c \leq y_j$ for all i, j. But (G, G_+, u) has stably approximate interpolation property. To see this let $x_i = (a_i, b_i)$, $y_j = (c_j, d_j) \in G$ where $1 \leq i, j \leq 2$ be such that $x_i \leq y_j$. Then $2(c_j - a_i) \geq (d_j - b_i) \geq 0$ for all i, j. Therefore, for large enough n we may choose $x, y \in \mathbb{Z}$ with*

$$2na_i - nb_i \leq 2x - y \leq 2nc_j - nd_j, \forall i, j,$$

and

$$nb_i \leq y \leq nd_j, \forall i, j.$$

For such x, y we have

$$n(a_i, b_i) \leq (x, y) \leq n(c_j, d_j), \forall i, j.$$

In Proposition 6.2.6 using a simple inductive argument we show that stably approximate refinement property extends to any $2 \times l$ type decompositions. In fact this extends for any $k \times l$ decomposition. However, we will only need to apply it in $2 \times l$ type decompositions and we prove it only for this form.

Proposition 6.2.6 *Suppose (G, G_+, u) exhibits stably approximate interpolation. Suppose $l \in \mathbb{N}, l \geq 2$ and $x + y = z_1 + z_2 + ... + z_l$ for some $x, y, z_1, z_2, ..., z_l \in G_+$. Then there are $n, k_x, k_y, k_i \in \mathbb{N}$ and $x_i, y_i \in G_+$, for $1 \leq i \leq l$ such that $\frac{k_\alpha}{n} < \epsilon$ for each index $\alpha \in \{x, y, 1, 2, ..., l\}$ and*

$$\begin{aligned} x_1 + x_2 + ... + x_l &= nx + k_x u \\ y_1 + y_2 + ... + y_l &= ny + k_y u \\ x_i + y_i &= nz_i + k_i u \end{aligned}$$

where the last equality holds for all $1 \leq i \leq l$.

Proof Case $l = 2$ is the definition of stably approximate refinement property.

Assume that the statement holds for some $l \in \mathbb{N}$ with $l \geq 2$. Suppose $x, y \in G_+$ and $x + y = z_1 + z_2 + ... + z_{l+1}$, where $z_i \in G_+$. By assumption there are $\tilde{n}, \tilde{k}_\alpha \in \mathbb{N}$, $\alpha = x, y, 1, 2, ..., l$ with $\frac{\tilde{k}_\alpha}{\tilde{n}} < \frac{\epsilon}{2}$ and $\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, ..., \tilde{x}_{l-1}, \tilde{y}_{l-1}, X_l, Y_l \in G_+$ such that,

$$\sum_{i=1}^{l-1} \tilde{x}_i + X_l = \tilde{n}x + \tilde{k}_x u \tag{6.5}$$

$$\sum_{i=1}^{l-1} \tilde{y}_i + Y_l = \tilde{n}y + \tilde{k}_y u \tag{6.6}$$

$$\tilde{x}_i + \tilde{y}_i = \tilde{n}z_i + \tilde{k}_i u, \forall 1 \leq i \leq l-1 \tag{6.7}$$

$$X_l + Y_l = \tilde{n}(z_l + z_{l+1}) + \tilde{k}_l u \tag{6.8}$$

From 6.8, $X_l + Y_l = \tilde{n}z_l + (\tilde{n}z_{l+1} + \tilde{k}_l u)$ and hence by stably approximate refinement property, there are $N, K \in \mathbb{N}$ with $\frac{K}{N} < \frac{\epsilon}{2}$ and $\tilde{x}_l, \tilde{y}_l, \tilde{x}_{l+1}, \tilde{y}_{l+1} \in G_+$ such that,

$$\tilde{x}_l + \tilde{x}_{l+1} = NX_l + Ku \tag{6.9}$$

$$\tilde{y}_l + \tilde{y}_{l+1} = NY_l + Ku \tag{6.10}$$

$$\tilde{x}_l + \tilde{y}_l = N\tilde{n}z_l + Ku \tag{6.11}$$

$$\tilde{x}_{l+1} + \tilde{y}_{l+1} = N(\tilde{n}z_{l+1} + \tilde{k}_l u) + Ku \tag{6.12}$$

Now let $n = N\tilde{n}$.

For $\alpha = x, y$ set $k_\alpha = N\tilde{k}_\alpha + K$, and for $i = 1, 2, ..., l-1$, set $k_i = N\tilde{k}_i$.

Let $k_l = K$ and $k_{l+1} = N\tilde{k_{l+1}} + K$.

Clearly $\frac{k_\alpha}{n} < \epsilon, \forall \alpha = x, y, 1, 2, .., l, l+1$.

For all $1 \leq i \leq l-1$ let,

$$x_i = N\tilde{x}_i \text{ and } y_i = N\tilde{y}_i$$

For $i = l, l+1$ set

$$x_i = \tilde{x}_i \text{ and } y_i = \tilde{y}_i.$$

Then, from 6.5 and 6.9,

$$\begin{aligned}\sum_{i=1}^{l+1} x_i &= \sum_{i=1}^{l-1} N\tilde{x}_i + (\tilde{x}_l + \tilde{x}_{l+1}) \\ &= \sum_{i=1}^{l-1} N\tilde{x}_i + (NX_l + Ku) \\ &= N(\tilde{n}x + \tilde{k}_x u) + Ku \\ &= nx + k_x u\end{aligned}$$

By a similar argument,

$$\sum_{i=1}^{l+1} y_i = ny + k_y u$$

From 6.7, for all $1 \leq i \leq l-1$,

$$x_i + y_i = N\tilde{x}_i + N\tilde{y}_i = N(\tilde{n}z_i + \tilde{k}_i u) = nz_i + k_i u$$

From 6.11,

$$x_l + y_l = N\tilde{x}_l + N\tilde{y}_l = \tilde{n}z_l + Ku = nz_l + k_l u.$$

From 6.12

$$x_{l+1} + y_{l+1} = \tilde{x}_{l+1} + \tilde{y}_{l+1} = N(\tilde{n}z_{l+1} + \tilde{k}_l u) + Ku = nz_{l+1} + k_{l+1} u.$$

■

6.3 Positive homomorphisms on groups with stably approximate interpolation

In this section we show that Theorem 6.1.4 generalizes to scaled partially ordered groups which exhibit stably approximate interpolation. To present the proof of this fact we require few more definitions and results concerning ordered groups. For the sake of completeness and the convenience of the reader we recall these here.

Recall that a partially ordered set X is said to be *lattice* ordered if every finite subset S of X has a supremum and an infimum in X.

Definition 6.3.1 *A partially ordered abelian group* (G, G_+) *is called a* lattice ordered group *if* G *is lattice ordered as a partially ordered set.*

Proposition 6.3.2 *[15, Proposition 1.5] For a partially ordered abelian group* (G, G_+) *the following are equivalent.*

1. G *is lattice ordered.*

2. (G, G_+) *is directed and every pair of elements in the partially ordered set* G_+ *has an infimum in* G_+*.*

3. (G, G_+) *is directed and every pair of elements in the partially ordered set* G_+ *has a supremum in* G_+*.*

A partially ordered abelian group G is said to be *Dedekind complete* if every non empty subset of G which is bounded above in G has a supremum in G.

Remarks:

1. (G, G_+) is Dedekind complete iff every non empty subset of G which is bounded below has an infimum in G.

2. Dedekind complete ordered group (G, G_+) is lattice ordered iff (G, G_+) is directed.

Definition 6.3.3 *A* positive homomorphism *on a partially ordered abelian group* (G, G_+) *is an additive map* $f : G \to \mathbb{R}$ *such that* $f(G_+) \subset \mathbb{R}^+$*. We will use* $Hom_+(G, G_+)$ *(or simply* $Hom_+(G)$ *when* G_+ *is clear) to denote the set of all positive homomorphisms on the ordered group* (G, G_+)*.*

Remark: For a non zero (G, G_+, u), $f : G \to \mathbb{R}$ is a positive homomorphism iff $f = \alpha s$ for some $s \in S(G, G_+, u)$ and $\alpha \in \mathbb{R}_+$.

Given an ordered group (G, G_+) let,

$$\Delta \mathrm{Hom}_+(G) = \{f - g \,:\, f, g \in \mathrm{Hom}_+(G)\}\,.$$

With pointwise operations $\Delta\mathrm{Hom}_+(G)$ is a group as well as a real vector space. Moreover, the pair $(\Delta\mathrm{Hom}_+(G), \mathrm{Hom}_+(G))$ form a directed ordered group as well as a *partially ordered real vector space*

Recall that a *simplex* is a convex subset K in a real vector space E where K is affinely homeomorphic to a base for a lattice cone in some real vector space. *Choquet* simplex is simply a compact simplex.

From the remark following Definition 6.3.3 it is not hard to see that $S(G,u)$ form a base for the cone $\mathrm{Hom}_+(G)$ for any ordered group G with order unit u. Thus, crux of the work of [15] in proving Theorem 6.1.4 lies in establishing the fact that $\mathrm{Hom}_+(G)$ is a lattice cone for a scaled interpolation group G. In fact [15] confirms a stronger result. Namely, [15, Corollary 2.28] show that $(\Delta\mathrm{Hom}_+(G), \mathrm{Hom}_+(G))$ is Dedekind complete for any directed interpolation group G. Note that whenever an ordered group G has an order unit then G is directed. Hence, [15, Corollary 2.28] in particular applies to any scaled ordered group G (with interpolation). Thus, for such G, $(\Delta\mathrm{Hom}_+(G), \mathrm{Hom}_+(G))$ is a directed Dedekind complete group and hence one get $\mathrm{Hom}_+(G)$ to be lattice cone.

In what proceeds we will make suitable modifications to techniques used in [15] to prove that the conclusion of Theorem 6.1.4 holds for $(\Delta\mathrm{Hom}_+(G), \mathrm{Hom}_+(G))$ when G is a scaled ordered group that exhibits stably approximate interpolation. In Theorem 6.3.7 we confirm a weaker version of [15, Corollary 2.28] for groups with stably approximate interpolation and it is easily seen that this still imply $S(G,u)$ to be a Choquet simplex (Corollary 6.3.8).

Definition 6.3.4 *Given an ordered group (G, G_+), a map $d : G_+ \to \mathbb{R}$ is called a subadditive map if $d(0) = 0$ and for all $a, b \in G_+$, $d(a+b) \le d(a) + d(b)$.*

For a subadditive map d on (G, G_+) and $\epsilon > 0$ let,

$$D_\epsilon(x) = \left\{ \frac{\sum_{i=1}^{l} d(x_i)}{n} : n,k,l \in \mathbb{N} \text{ with } \frac{k}{n} < \epsilon \text{ and } \forall 1 \le i \le l, x_i \in G_+, \sum_{i=1}^{l} x_i = nx + ku \right\}$$

for each $x \in G_+$.

Following simple observation on the sets $D_\epsilon(x)$ proves crucial in overcoming the technical difficulties in extending the arguments of [15] to groups with stably approximate interpolation.

Lemma 6.3.5 *Let $d : G_+ \to \mathbb{R}$ be subadditive and let $D_\epsilon(x)$ be defined as above. For a fixed $x \in G_+$ if there is some $\epsilon_0 > 0$ such that $D_{\epsilon_0}(x)$ is bounded above, then $\lim_{\epsilon \to 0} D_\epsilon(x)$ exists finitely. Furthermore this limit is equal to $\inf_{\epsilon>0} \sup D_\epsilon(x)$ and is not less than $d(x)$.*

Proof Clearly for each $x \in G_+$ and for all $\epsilon > 0$, we have $d(x) \in D_\epsilon(x)$. Thus, for each x, $d(x) \leq \sup_{\epsilon>0} D(x)$.

If $0 < \epsilon < \epsilon_0$, then $D_\epsilon(x) \subset D_{\epsilon_0}(x)$ and hence for all $\epsilon < \epsilon_0$,

$$\sup D_\epsilon(x) \leq \sup D_{\epsilon_0}(x) < \infty.$$

Hence we clearly have, $d(x) \leq \inf_{\epsilon>0} \sup D_\epsilon(x) = \lim_{\epsilon\to 0} D_\epsilon(x) < \infty$. ■

Lemma 6.3.6 *Let (G, G_+, u) be a scaled partially ordered abelian group which exhibits stably approximate interpolation. Let $d : G_+ \to \mathbb{R}$ be subadditive with $d(ma) = md(a)$ for each $a \in G_+$, $m \in \mathbb{N}$. Furthermore assume that there is $M_u \geq 0$ such that*

$$d(x) \leq \frac{d(nx + ku) + kM_u}{n}$$

for all $x \in G_+$ and $k, n \in \mathbb{N}$. Suppose for all $x \in G_+$ there is some $\delta_x > 0$ such that the set $D_{\delta_x}(x)$ is bounded above. Then the function $f : G_+ \to \mathbb{R}$ defined by

$$f(x) = \lim_{\epsilon\to 0} \sup D_\epsilon(x) = \inf_{\epsilon>0} \sup D_\epsilon(x)$$

is an additive homomorphism and f uniquely extends to a group homomorphism on G.

Proof Since u is an order unit, (G, G_+) is directed. Thus, any additive map $h : G_+ \to [0, \infty)$ extends to a group homomorphism on G. Hence, the final conclusion is immediate once we show f to be additive.

From Lemma 6.3.5 the function f is well defined and $f(x) \geq d(x), \forall x \in G_+$.

We show that f is additive.

Fix $x, y \in G_+$ and $\eta > 0$.

By definition of f choose $\epsilon_0 > 0$ with $\epsilon_0 M_u < \eta$ such that the following hold for each $z \in \{x, y, x+y\}$ and all $\epsilon \leq \epsilon_0$;

$$f(z) \leq \sup D_\epsilon(z) \leq f(z) + \frac{\eta}{6}. \tag{6.13}$$

Choose $n_x, k_x, n_y, k_y \in \mathbb{N}$ with $\frac{k_x}{n_x}, \frac{k_y}{n_y} < \frac{\epsilon_0}{2}$ and $x_1, x_2, ..., x_s, y_1, y_2, ..., y_t \in G_+$ such that;

$$x_1 + x_2 + ... + x_s = n_x x + k_x u \tag{6.14}$$

$$y_1 + y_2 + ... + y_s = n_y y + k_y u \tag{6.15}$$

and

$$\sup D_{\frac{\epsilon_0}{2}}(x) - \frac{\eta}{6} \leq \frac{d(x_1) + d(x_2) + ... + d(x_s)}{n_x} \tag{6.16}$$

$$\sup D_{\frac{\epsilon_0}{2}}(y) - \frac{\eta}{6} \leq \frac{d(y_1) + d(y_2) + ... + d(y_t)}{n_y}. \tag{6.17}$$

Set $n = n_x n_y$ and $k = n_y k_x + n_x k_y$.

Then, $\frac{k}{n} < \epsilon_0$ and from 6.14 and 6.15,

$$\sum_{i=1}^{s} n_y x_i + \sum_{j=1}^{t} n_x y_j = n(x+y) + ku.$$

Therefore,

$$\begin{aligned}
\sup D_{\epsilon_0}(x+y) &\geq \frac{\sum_{i=1}^{s} d(n_y x_i) + \sum_{j=1}^{t} d(n_x y_j)}{n} \\
&= \frac{\sum_{i=1}^{s} n_y d(x_i)}{n} + \frac{\sum_{j=1}^{t} n_x d(y_j)}{n} \\
&= \frac{\sum_{i=1}^{s} d(x_i)}{n_x} + \frac{\sum_{j=1}^{t} d(y_j).}{n_y}.
\end{aligned}$$

Hence, from 6.16 and 6.17,

$$\sup D_{\epsilon_0}(x+y) \geq \sup D_{\frac{\epsilon_0}{2}}(x) + \sup D_{\frac{\epsilon_0}{2}}(y) - \frac{\eta}{3}.$$

From 6.13 with $z = x, y$ we get,

$$\sup D_{\epsilon_0}(x+y) \geq f(x) + f(y) - \frac{2\eta}{3}.$$

Using 6.13 again (now with $z = x + y$),

$$f(x+y) + \frac{\eta}{6} \geq \sup D_{\epsilon_0}(x+y) \geq f(x) + f(y) - \frac{2\eta}{3}.$$

Thus as $\eta > 0$ is arbitrary we have

$$f(x+y) \geq f(x) + f(y).$$

Now to show the reverse inequality choose $n_0, k_0 \in \mathbb{N}$ with $\frac{k_0}{n_0} < \frac{\epsilon_0}{2}$ and some $z_1, z_2, ..., z_l$ such that

$$n_0(x+y) + k_0 u = z_1 + z_2 + ... + z_l \tag{6.18}$$

and

$$\sup D_{\frac{\epsilon_0}{2}}(x+y) - \frac{\eta}{6} \leq \frac{d(z_1) + d(z_2) + ... + d(z_l)}{n_0}. \tag{6.19}$$

Apply Theorem 6.2.6 to equation 6.18 with $x = n_0 x$ and $y = n_0 y + k_0 u$ to choose $n, k_x, k_y, k_i \in \mathbb{N}$, $x_i, y_i \in G_+$, for $1 \leq i \leq l$ such that $\frac{k_\alpha}{n} < \epsilon_0$ for each index $\alpha \in \{x, y, 1, 2, ..., l\}$ and

$$x_1 + x_2 + ... + x_l = nn_0 x + k_x u \tag{6.20}$$

$$y_1 + y_2 + ... + y_l = nn_0 y + (nk_0 + k_y)u \tag{6.21}$$

$$x_i + y_i = nz_i + k_i u. \tag{6.22}$$

By using the above three equations and that u is an order unit, $\sum_{i=1}^{l} k_i = k_x + k_y$. From 6.22 and assumptions on d,

$$\begin{aligned}
\sum_{i=1}^{l} d(z_i) &\leq \sum_{i=1}^{l} \frac{d(nz_i + k_i u) + k_i M_u}{n} \\
&= \sum_{i=1}^{l} \frac{d(x_i + y_i)}{n} + \frac{M_u}{n} \sum_{i=1}^{l} k_i \\
&\leq \frac{\sum_{i=1}^{l} d(x_i)}{n} + \frac{\sum_{j=1}^{l} d(y_j).}{n} + \frac{M_u(k_x + k_y)}{n} \\
&\leq \frac{\sum_{i=1}^{l} d(x_i)}{n} + \frac{\sum_{j=1}^{l} d(y_j)}{n} + M_u \epsilon_0
\end{aligned}$$

Thus from 6.19,

$$\sup D_{\frac{\epsilon_0}{2}}(x+y) - \frac{\eta}{6} \quad \leq \quad \frac{\sum_{i=1}^{l} d(x_i)}{nn_0} + \frac{\sum_{j=1}^{l} d(y_j)}{nn_0} + \frac{M_u \epsilon_0}{n_0} \tag{6.23}$$

Since $\frac{k_x}{nn_0} < \frac{\epsilon_0}{2}$, from 6.20 it follows that,

$$\frac{\sum_{i=1}^{l} d(x_i)}{nn_0} \leq \sup D_{\epsilon_0}(x).$$

Similarly as $\frac{nk_0+k_y}{n} < \epsilon_0$, by 6.21,

$$\frac{\sum_{i=1}^{l} d(y_i)}{nn_0} \leq \sup D_{\epsilon_0}(y).$$

Combining the last two inequalities with 6.23,

$$\sup D_{\frac{\epsilon_0}{2}}(x+y) - \frac{\eta}{6} \quad \leq \quad \sup D_{\epsilon_0}(x) + \sup D_{\epsilon_0}(y) + \frac{M_u \epsilon_0}{n_0}.$$

Therefore by definition of f,

$$f(x+y) - \frac{\eta}{6} \leq \sup D_{\epsilon_0}(x) + \sup D_{\epsilon_0}(y) + \frac{M_u \epsilon_0}{n_0}.$$

By applying 6.13 with $z = x, y$,

$$f(x+y) - \frac{\eta}{6} \leq f(x) + f(y) + \frac{\eta}{3} + \frac{M_u \epsilon_0}{n_0}.$$

Recall that by the choice of ϵ_0, $M_u \epsilon_0 < \frac{\eta}{3}$. Thus,

$$f(x+y) \leq f(x) + f(y) + \eta.$$

Since $\eta > 0$ is arbitrary this completes the proof. ■

Theorem 6.3.7 *Let (G, G_+) be a directed partially ordered abelian group with order unit u. Suppose that (G, G_+, u) exhibits stably approximate interpolation. Let $S \subset \Delta Hom_+(G)$ be non-empty and bounded above in $(\Delta Hom_+(G), Hom_+(G))$ with the set $\{f(u) : f \in S\}$ bounded below. Then S has a supremum in $(\Delta Hom_+(G), Hom_+(G))$.*

Proof Let $H = \Delta Hom_+(G)$ and $H_+ = Hom_+(G)$. Let us use $\leq^+$ to denote the partial order induced by H_+. That is for all $f, g \in H$,

$$f \leq^+ g \iff g - f \in H_+.$$

Take $S = \{f_i : i \in I\} \subset H$ and suppose that $g \in H$ is an upper bound for S. Let $M_u > 0$ be such that $|f_i(u)| \leq M_u$ for all $i \in I$.

Define $d : G_+ \to [0, \infty)$ by,

$$d(x) = \sup_{i \in I} f_i(x), \forall x \in G_+.$$

Since $f_i(x) \leq g(x), \forall i \in I$ and for all $x \in G_+$, d is well defined and $d(0) = 0$. Since each f_i is additive, d is subadditive with $d(mx) = mx$, $\forall x \in G_+$ and $\forall m \in \mathbb{N}$. Furthermore for any $x \in G_+$ and $k, n \in \mathbb{N}$ and for a fixed $i_o \in I$ we have,

$$\begin{aligned} f_{i_0}(x) &= \frac{f_{i_0}(nx + ku)}{n} - \frac{kf_{i_0}(u)}{n} \\ &\leq \frac{f_{i_0}(nx + ku)}{n} + \frac{kM_u}{n} \\ &\leq \frac{\sup_{i \in I} f_i(nx + ku)}{n} + \frac{kM_u}{n} \\ &= \frac{d(nx + ku) + kM_u}{n} \end{aligned}$$

Thus,

$$d(x) = \sup_{i_0 \in I} f_{i_0}(x) \leq \frac{d(nx + ku) + kM_u}{n}$$

For any $\epsilon > 0$ and $x \in G_+$, let $D_\epsilon(x)$ be as before.

For any decomposition $\sum_{i=1}^{l} x_i = nx + ku$ with $\frac{k}{n} < \epsilon$ and $x_1, x_2, ..., x_l \in G_+$,

$$\begin{aligned} \frac{d(x_1) + d(x_2) + ... + d(x_l)}{n} &\leq \frac{g(x_1) + g(x_2) + ... + g(x_l)}{n} \\ &= \frac{g(x_1 + x_2 + ... + x_l)}{n} \\ &= \frac{g(nx + ku)}{n} \\ &= g(x) + \frac{kg(u)}{n} \\ &\leq g(x) + \epsilon|g(u)|. \end{aligned} \tag{6.24}$$

Hence, for every $x \in G_+$ and $\epsilon > 0$, $D_\epsilon(x)$ is bounded above by $g(x) + \epsilon|g(u)|$.

Thus, from Lemma 6.3.6 the function $f : G_+ \to [0, \infty)$ defined by,

$$f(x) = \lim_{\epsilon \to 0} \sup D_\epsilon(x) = \inf_{\epsilon > 0} \sup D_\epsilon(x), \forall x \in G_+$$

is a well defined additive homomorphism on G_+ and f extends uniquely to a group homomorphism on G. We now verify that f is in H and is in fact the supremum of the set S in H.

Fix $i_0 \in I$,$\epsilon > 0$ and $x \in G_+$.

Suppose $\sum_{i=1}^{l} x_i = nx + ku$ with $\frac{k}{n} < \epsilon$ and $x_1, x_2, .., .x_l \in G_+$.

Then,

$$\begin{aligned}
\frac{d(x_1) + d(x_2) + ... + d(x_l)}{n} &\geq \frac{f_{i_0}(x_1) + f_{i_0}(x_2) + ... + f_{i_0}(x_l)}{n} \\
&= \frac{f_{i_0}(nx + ku)}{n} \\
&= f_{i_0}(x) + \frac{kf_{i_0}(u)}{n} \\
&\geq f_{i_0}(x) - \epsilon|f_{i_0}(u)|.
\end{aligned}$$

Thus, for each $i_0 \in I$, $x \in G_+$ and for any $\epsilon > 0$,

$$\sup D_\epsilon(x) \geq f_{i_0}(x) - \epsilon|f_{i_0}(u)|.$$

Hence, for each $i_0 \in I$, $x \in G_+$,

$$f(x) = \lim_{\epsilon \to 0} \sup D_\epsilon(x) \geq f_{i_0}(x).$$

Then, for each $i_0 \in I$, $f - f_{i_0} \in H_+$,as f and f_i are group homomorphisms. In particular, fixed i_0 we have $f = [(f - f_{i_0}) + f_{i_0}] \in H$ and $f_i \leq^+ f, \forall i \in I$.

Now suppose $h \in H$ is any upper bound for S.

Then as we have shown in 6.24,

$$\sup D_\epsilon(x) \leq h(x) + \epsilon|h(u)|, \forall \epsilon > 0, \forall x \in G_+.$$

Thus,

$$f(x) = \lim_{\epsilon \to 0} \sup D_\epsilon(x) \leq h(x), \forall x \in G_+.$$

Hence, $f \leq^+ h$ and f is the supremum of S in H. ■

Corollary 6.3.8 *Let (G, G_+, u) be a scaled partially ordered group that exhibits stably approximate interpolation. Then $S(G, u)$ is a Choquet simplex.*

Proof We need to show that $(\Delta\mathrm{Hom}_+(G), \mathrm{Hom}_+(G))$ is lattice ordered. Suppose $S = \{f_i : 1 \leq i \leq k\}$ is a finite set in $\Delta\mathrm{Hom}_+(G)$. Choose $h_i, g_i \in \mathrm{Hom}_+(G)$ such that $f_i = h_i - g_i$ for all $1 \leq i \leq k$. Then, S is bounded above by $h_1 + h_2 + ... + h_k$. Clearly $\{f_i(u) : 1 \leq i \leq k\}$ is bounded. Thus, S has a supremum by Theorem 6.3.7. ■

6.4 Applying results of 6.3 to answer Conjecture 4.5.4

Note that one could define the notion of stably approximate interpolation for ordered semigroups as well.

Definition 6.4.1 *Let $(W, \leq, u)$ be a scaled partially ordered abelian semigroup. We say that $(W, \leq, u)$ has stably approximate interpolation property if for any given $a_1, a_2, b_1, b_2 \in W$ with $a_i \leq b_j$ for all i, j and for all $\epsilon > 0$ there are $k, n \in \mathbb{N}$ and $c, d \in W$ such that $\frac{k}{n} < \epsilon$ and*

$$a_i + d \leq c \leq b_j + d + ku, \forall 1 \leq i, j \leq 2.$$

Remarks:

1. If W is a group then Definition 6.4.1 agrees with Definition 6.2.1 (1).

2. One could define the analogues of Definition 6.2.1 (2) and (3) for ordered abelian semigroups W. However, unless W has cancellation (in which case W is the positive cone of a partially ordered group) there properties are not equivalent in W.

Given an ordered semigroup $(W, \leq)$, let $G(W)$ denote the Grothendieck group of W and let $\gamma : W \to G(W)$ be the natural additive map. So, $G(W) = \{\gamma(b) - \gamma(a) : a, b \in W\}$. Let us set $G(W)_{++} = \{\gamma(b) - \gamma(a) : a, b \in W, a \leq b\}$. It is easily seen that $(G(W), G(W)_{++})$ form a partially ordered abelian group and if u is an order unit for W then $\gamma(u)$ is an order unit for $G(W)$. Note that the construction of the group ${K_0}^*(A)$ for a C^*-algebra is just a special case of this with $W = W(A)$.

Lemma 6.4.2 *[29, c.f Lemma 4.2] Let $(W, \leq, u)$ be a scaled partially ordered abelian semigroup that has stably approximate interpolation property. Then the scaled ordered group $(G(W), G(W)_{++}, \gamma(u))$ has stably approximate interpolation property.*

Proof Let $g_i, h_j \in G(W)$ be such that $g_i \leq h_j, \forall i, j$. Fix $\epsilon > 0$.

Note that one may select $a_i, b_j, z \in W$ where $1 \leq i, j \leq 2$ such that,

$$g_i = \gamma(a_i) - \gamma(z) \quad \text{and} \quad h_j = \gamma(b_j) - \gamma(z), \forall i, j \tag{6.25}$$

Indeed $g_i = \gamma(x_i) - \gamma(v_i)$ and $h_j = \gamma(y_j) - \gamma(w_j)$ for some $x_i, y_j, v_i, w_j \in W$, $1 \leq i, j \leq 2$. We simply set $z = v_1 + v_2 + w_1 + w_2$, $a_1 = x_1 + v_2 + w_1 + w_2$, $a_2 = x_2 + v_1 + w_1 + w_2$, $b_1 = y_1 + v_1 + v_2 + w_2$ and $b_2 = y_2 + v_1 + v_2 + w_1$.

Then, for all $1 \leq i, j \leq$ there are $s_{ij}, t_{ji} \in W$ such that

$$s_{ij} \leq t_{ij}, \forall 1 \leq i, j \leq 2 \tag{6.26}$$

and

$$\gamma(b_j) - \gamma(a_i) = \gamma(t_{ij}) - \gamma(s_{ij}), \forall 1 \leq i, j \leq 2.$$

From the final equation, for appropriately chosen $w \in W$,

$$b_j + s_{ij} + w = a_i + t_{ij} + w, \forall 1 \leq i, j \leq 2 \tag{6.27}$$

Now form 6.26 and 6.27, for all $1 \leq i, j \leq 2$,

$$b_j + t_{ij} + w \geq a_i + t_{ij} + w$$

Hence, for $t = w + t_{11} + t_{12} + t_{21} + t_{22}$,

$$b_j + t \geq a_i + t, \forall 1 \leq i, j \leq 2 \tag{6.28}$$

By 6.28 and stably approximate interpolation property of W, there is $n, k \in \mathbb{N}$ with $\frac{k}{n} < \epsilon$ and $c, d \in W$ such that,

$$na_i + d \leq c \leq nb_i + d + ku, \forall 1 \leq i, j \leq 2.$$

Hence,

$$n\gamma(a_i) \leq \gamma(c) \leq n\gamma(b_j) + k\gamma(u),\ \forall 1 \leq i, j \leq 2.$$

Thus by 6.25, for $C = \gamma(c) - nz$,

$$ng_i \leq C \leq nh_j + k\gamma(u), \forall 1 \leq i, j \leq 2.$$

■

From Lemma 6.4.2 and Corollary 6.3.8 we get;

Corollary 6.4.3 *Let A be any untial C^*-algebra. If $(W(A), \langle 1_A \rangle)$ has stably approximate interpolation property then $DF(A)$ is a Choquet simplex.*

Note that (see [3, Lemma 2.20] for example) since A is stably finite for any two projections $p, q \in M_\infty(A)$ one has $p \sim q$ in the Cuntz sense if and only if $p \sim q$ in the Murray-von Neumann sense. For any A, saying $p \preccurlyeq q$ in Murray-von Neumann sense is equivalent to saying $p \preccurlyeq q$ in the Cuntz sense. In proving the following Corollary we will use these observations to identify the ordered semigroup $V(A)$ as an (ordered) subsemigroup of $W(A)$.

Corollary 6.4.4 *Let A be a unital stably finite C^*-algebra with real rank zero. If $rc(A) < \infty$ then $DF(A)$ is a Choquet simplex.*

Proof From Corollary 6.4.3 we only have to show that $W(A)$ has stably approximate interpolation.

Since A is of real rank zero, for every $a \in M_\infty(A)_+$ and $\epsilon > 0$ there is some $b \in M_\infty(A)_+$ with $|\sigma(b)| < \infty$ and

$$||a - b|| < \epsilon.$$

Thus, it follows that

$$\langle a \rangle = \sup_{n \in \mathbb{N}} \langle p_n \rangle$$

for a sequence of projections $p_n \in M_\infty(A)$ with,

$$\langle (a - \frac{1}{2^n})_+ \rangle \leq \langle p_n \rangle \leq \langle (a - \frac{1}{2^{n+1}})_+ \rangle \leq \langle p_{n+1} \rangle, \, \forall n \in \mathbb{N}.$$

.

Suppose that $a_i, b_j \in M_\infty(A)_+$ are such that,

$$\langle a_i \rangle \leq \langle b_j \rangle, \, \forall 1 \leq i, j \leq 2. \tag{6.29}$$

Choose sequences of ${p_n}^{(i)}, {q_n}^{(j)} \in \mathcal{P}(M_\infty(A))$ such that for each $1 \leq i, j \leq 2$,

$$\langle a_i \rangle = \sup_{n \in \mathbb{N}} \langle {p_n}^{(i)} \rangle \text{ and } \langle b_j \rangle = \sup_{n \in \mathbb{N}} \langle {q_n}^{(j)} \rangle, \tag{6.30}$$

with

$$\langle (a_i - \frac{1}{2^n}) \rangle \leq \langle {p_n}^{(i)} \rangle \leq \langle (a_i - \frac{1}{2^{n+1}}) \rangle \leq \langle {p_{n+1}}^{(i)} \rangle, \forall n \in \mathbb{N} \tag{6.31}$$

and

$$\langle (b_j - \frac{1}{2^n}) \rangle \leq \langle {q_n}^{(j)} \rangle \leq \langle (b_j - \frac{1}{2^{n+1}}) \rangle \leq \langle {q_{n+1}}^{(j)} \rangle, \forall n \in \mathbb{N} \tag{6.32}$$

Now since $\langle a_i \rangle \leq \langle b_j \rangle$ for all i, j, give any $n \in \mathbb{N}$ from 6.30 and 6.32 we may select $m_n \in \mathbb{N}$ such that,

$$\langle {p_n}^{(i)} \rangle \leq \langle {q_{m_n}}^{(j)} \rangle, \forall 1 \leq i, j \leq 2. \tag{6.33}$$

Furthermore we may choose $(m_n)_{n \in N}$ such that $m_n < m_{n+1}$.

Since A is stably finite, from the paragraph that proceeded the Corollary we may assume the analogue of 6.33 for the respective Murray-von Neumann classes to hold in $V(A)$.

Note that, since A is of real rank zero, $W(A)$ has the Riesz refinement property by [2, Propostion 1.2 and Theorem 7.2] (or see [46]). Hence, from [28, Proposition 2.2] for any fixed $\epsilon > 0$, there is $r_{1,\epsilon}, c_{1,\epsilon} \in \mathcal{P}(A_\infty)$ with $||[r_{1,\epsilon}]||_u < \frac{\epsilon}{2}$,

$$[{p_1}^{(i)}] \leq [c_{1,\epsilon}] \leq [{q_{m_1}}^{(j)}] + [r_{1,\epsilon}], \forall 1 \leq i, j \leq 2.$$

Using the same reasoning ([28, Propostion 2.2] and [46]) with an inductive argument we may proceed to construct sequences $(r_{,\epsilon})_{n\in\mathbb{N}}$ and $(c_{n,\epsilon})_{n\in\mathbb{N}}$ such that

$$[{p_n}^{(i)}] \leq [c_{n,\epsilon}] \leq [{q_{m_n}}^{(j)}] + [r_{n,\epsilon}]\,, \forall 1 \leq i,j \leq 2 \tag{6.34}$$

with $||r_{n,\epsilon}||_{\langle 1_A\rangle} < \frac{\epsilon}{2^n}$ and $[c_{n-1,\epsilon}] \leq [c_{n,\epsilon}]$ for every $n \geq 2$.

From hereon we will write $||\cdot||$ to denote $||\cdot||_{\langle 1_A\rangle}$

Let $r_\epsilon = \sum_{i=1}^{\infty} \frac{r_{i,\epsilon}}{2^i} \in A \otimes \mathcal{K}$ where $\mathcal{K}$ denotes the compact operators on some separable Hilbert space.

Then, in $Cu(A) = W(A \otimes \mathcal{K})$,

$$\langle r_{n,\epsilon}\rangle \leq \langle r_\epsilon\rangle\,, \forall n \in \mathbb{N}\,. \tag{6.35}$$

By identifying the ordered semigroup $V(A)$ as a (ordered) subsemigroup in $W(A)$ again, we note that $(\langle c_{n,\epsilon}\rangle)_{n\in\mathbb{N}}$ is an increasing sequence in $W(A) \subset Cu(A)$. Therefore, since $Cu(A)$ is closed under suprema of upward directed sets, there is some $\langle c_\epsilon\rangle \in Cu(A)$ such that

$$\langle c_\epsilon\rangle = \sup_{n\in\mathbb{N}} \langle c_{n,\epsilon}\rangle. \tag{6.36}$$

Now from 6.33, 6.35, 6.36 and using the identifications $V(A) \subset W(A) \subset Cu(A)$ once more,

$$\langle {p_n}^{(i)}\rangle \leq \langle c_\epsilon\rangle \leq \langle {q_{m_n}}^{(j)}\rangle + \langle r_\epsilon\rangle\,, \forall 1 \leq i,j \leq 2, \forall n \in \mathbb{N}$$

Hence, from 6.30,

$$\langle a_i\rangle \leq \langle c_\epsilon\rangle \leq \langle b_j\rangle + \langle r_\epsilon\rangle\,, \forall 1 \leq i,j \leq 2.$$

Note that for every $\tau \in QT(A)$,

$$d_\tau(\langle r_\epsilon\rangle) \leq \sum_{i=1}^{\infty} d_\tau(\langle r_{i,\epsilon}\rangle) \leq \sum_{i=1}^{\infty} ||[r_{i,\epsilon}]|| \leq \epsilon. \tag{6.37}$$

Hence, since radius of comparison of A is assumed to be of finite, if $k \in \mathbb{N}$ and $k > rc(A) + \epsilon$ then $\langle r_\epsilon\rangle \leq k\langle 1_A\rangle$. From [7], $rc(A) < \infty$ also implies $W(A)$ to be hereditary. Combining these we conclude that $\langle r_\epsilon\rangle$ and $\langle c_\epsilon\rangle$ are elements in $W(A)$.

Furthermore from Theorem 5.2.5 (1), $LDF(A)$ is dense in $DF(A)$ and therefore by 6.37,

$$||\langle r_\epsilon\rangle|| = \sup_{s\in DF(A))} s(\langle r_\epsilon\rangle) \leq \epsilon.$$

Therefore, we may choose some $k, n \in \mathbb{N}$ with $\frac{k}{n} < \epsilon$ such that;

$$n\langle a_i\rangle \leq n\langle c_\epsilon\rangle \leq n\langle b_j\rangle + n\langle r_\epsilon\rangle \leq n\langle b_j\rangle + k\langle 1_A\rangle\,, \forall 1 \leq i, j \leq 2.$$

As $\langle c_\epsilon\rangle \in W(A)$ this completes the proof.

■

Question 6.4.5 *Are there unital C^*-algebras A for which $W(A)$ fail to satisfy the asymptotic interpolation property in the sense of [28] but exhibit stably approximate interpolation?*

Question 6.4.6 *Do the properties of state spaces $S(G, u)$ shown to hold in [28] for an asymptotic interpolation group G still hold if G is assumed to have stably approximate interpolation ?*

Question 6.4.7 *What kind of closure properties hold for the class of ordered groups with stably approximate interpolation?*

REFERENCES

REFERENCES

[1] R. Antoine, J. Bosa, F. Perera, and H. Petzka, *Geometric structure of dimension functions of certain continuous fields*, *J. Funct. Anal.*, 266 (4) : 2403-2423, 2014.

[2] P. Ara, K. R. Goodearl, K.C. O'Meara and E. Padro, *Separative cancellation for projective modules over exchange rings*, *Israel J. Math.* 105 (1): 105-137, 1998

[3] P. Ara, F. Perera and A. Toms, *K-theory for operator algebras. Classification of C^*-algebras. Aspects of operator algebras and applications, 1-71*, *Contemp. Math. 534*. Amer. Math. Soc., Providence, RI, 2011.

[4] G. Birkhoff, Lattice-ordered groups. *Ann. of Math.*, 43 (2): 298-331, 1942.

[5] B. Blackadar, *Comparison theory for simple C^*-algebras*, in Operator Algebras and Applications, eds. D. E. Evans and M. Takesaki, London Math. Sot. Lecture Notes Ser., Vol. 135, Cambridge Univ. Press, London/New York, 1989

[6] B. Blackadar and D. Handelman, Dimension functions and traces on C^*-algebras. *J. Funct. Anal.*, 45: 297-340, 1982.

[7] B. Blackadar, L. Robert, A. P. Tikuisis, A. S. Toms, W. Winter, An algebraic approach to the radius of comparison. *Trans. Am. Math. Soc.*, 364: 3657-3674, 2012.

[8] B. Blackadar and M. Rørdam, Extending states on preordered semigroups and the existence of quasitraces on C^*-algebras. *Journal of Algebra*, 152: 240-47, 1992.

[9] N.P. Brown and F. Perera and A.S. Toms, The Cuntz semigroup, the Elliott conjecture, and dimension functions on C^*-algebras. *J. Reine Angew. Math*, 621:191-211, 2008.

[10] N.P. Brown and A.S. Toms, *There applications of the Cuntz semigroup Int. Math Res. Not.*, doi: 10.1093/imrnirnm068, 2007.

[11] J. Cuntz, *The structure of multiplication and addition in single C^*-algebras* *Math. Scand.*, 40: 215-233, 1977.

[12] J. Cuntz, *Dimension functions on simple C^*-algebras* *Math. Ann.*, 233: 145 -153, 1978.

[13] M. Dadarlat and A.S. Toms, *Ranks of operators in simple C^*-algebras* *J.Funct.Anal.*, 259: 1209-1229, 2010.

[14] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton University Press, Princeton, New Jersey, 1952.

[15] K. R. Goodearl, *Partially ordered abelian groups with interpolation*, *Math. Surveys and Monographs* 20, Amer. Math. Soc., Providence, 1986.

[16] K. R. Goodearl, *Riesz decomposition in inductive limit C^*-algebras*, *Rocky Mountain J. Math.*, 24: 1405-1430, 1994.

[17] K. R. Goodearl and D. Handelman, *Rank functions and K_0 of regular rings*, *J. Pure and appl. Algebra*, 7: 195-216, 1976.

[18] K. R. Goodearl, D. Handelman and J. W. Lawrence *Affine representations of Grothendieck groups and applications to Rickart C^*-algebra and $\aleph_0$-continuous regular rings*, *Mem, Amer. Math. Soc.*, 234, 1980.

[19] U. Haagerup, *Quasitraces on exact C^*-algebras are traces*, *Royal Society of Canada*, 36(02-03):67–92, 2014.

[20] D. Handelman, *Homomorphisms of C^*-algebras to finite AW^*-algebras,*, *Michigan Math. J.*, 28: 229-240, 1981.

[21] D. Husemoller, *Fiber Bundles*, Third Ed. *Graduate Texts in Mathematics; 20.* Springer-Verlag New York, Inc. New York, 1994.

[22] E. Kirchberg and M. Rrdam, *Non-simple purely infinite C^* algebras*, Second Ed., *Amer. J. Math*, 122: 637-666, 2000.

[23] F. Larsen and N. J. Lausten and M. Rørdam, *An introduction to K-Theory for -C^*-algebras.* Cambridge University press, Cambridge, 2000.

[24] James R. Munkres, *Topology*, Second Ed., NJ Prentice Hall, Inc., Upper Saddle River, 2000.

[25] Gerard J. Murphy, *C^*-algebras and Operator Theory*, Academic Press, Boston, 1990.

[26] F. J. Murray and J. von Neumann, *On rings of operators*, *Ann. of Math.*, 37: 116-229, 1936.

[27] K. Nagami, *Dimension Theory*, Academic Press, New York, 1970.

[28] E. Pardo, *Metric completions of ordered groups and K_0 of exchange rings*, *Trans. Am. Math. Soc.*, 350 (3): 913-933, 1998.

[29] F. Perera, *The structure of positive elements for C^*-algebras with real rank zero*, *Internat. J. Math*, 8: 383-405, 1997.

[30] F. Perera and A. S. Toms, *Recasting the Elliott conjecture*, *Math. Ann.*, 338 (03): 669702, 2007.

[31] N. C. Phillips, *Recursive subhomogeneous algebras*, *Trans. Am. Math. Soc.*, 359 (10): 4595-4623, 2007.

[32] F. Riesz, *Sur quelques notions fondamentales dans la theorie generale des operations lineaires* *Ann. of Math.*, 41:174-206, 1940.

[33] M. Rørdam, *On the structure of simple C^*-algebras tensored with a UHF-algebra, II* *J. Funct. Anal.*, 107:255-269, 1992.

[34] R. G. Swan, *Vector bundles and projective modules*, *Trans. Amer. Math. Soc.*, 105: 264-277, 1962

[35] J.-P. Serre, *Faisceaux algdbriques cohe'rents*, *Ann. of Math.*, 6(2): 197-278. , 1955.

[36] A. Toms, *On the classification problem for nuclear C^*-algebras*, *Ann. of Math*, 167(2):10591074, 2008.

[37] A. Toms, *K-theoretic rigidity and slow dimension growth*, *Invent. Math.*, 183(2):225-244, 2008.

[38] A. S. Toms, *Stability in the Cuntz semigroup of a commutative C^*-algebra*, *Proc. Lond. Math. Soc. (3)*, 96(1): 1-25, 2008.

[39] A. S. Toms, *Comparison theory and smooth minimal C^*-dynamics*, *Comm. Math. Phys*, 96(2): 401-433, 2009.

[40] A. S. Toms, *Flat dimension growth for C^*-algebras*, *C-algebras, J. Funct. Anal.*, 238: 678708, 2006.

[41] J. Villadsen *Simple C^*-algebras with perforation*, *J. Funct. Anal.*, 154:110-116, 1998.

[42] J. Villadsen *On the stable rank of simple C^*-algebras*, *J. Amer. Math. Soc.*, 12:1091-1102, 1999.

[43] F. Wehrung *Tensor products of structures with interpolation*, *Pacific J. Math*, 176: 267-285, 1996.

[44] W. Winter, *Nuclear dimension and $\mathcal{Z}$-stability of pure C^*-algebras*, *Invent. Math.*, 187:259342, 2012.

[45] G. W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, New York, 1978.

[46] S. Zhang *Matricial structure and homotopy type of simple C^*-algebras of real rank zero*, *J.Operator Theory*, 26: 283-312, 1991.

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