Infrared exponent for gluon and ghost propagation in Landau gauge QCD

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In the covariant description of confinement, one expects the ghost correlations to be infrared enhanced. Assuming ghost dominance, the long-range behavior of gluon and ghost correlations in Landau gauge QCD is determined by one exponent κ. The gluon propagator is infrared finite (vanishing) for $\kappa=1/2$ ($\kappa>1/2$) which is still under debate. Here, we study the critical exponent and coupling for the infrared conformal behavior from the asymptotic form of the solutions to the Dyson-Schwinger equations in an ultraviolet finite expansion scheme. The value for κ is directly related to the ghost-gluon vertex. Assuming that it is regular in the infrared, one obtains $\kappa\simeq 0.595$. This value maximizes the critical coupling $\alpha_c(\kappa)$, yielding $\alpha_c^{max}\simeq(4\pi/N_c)0.709 \approx 2.97$ for $N_c=3$. For larger κ the vertex acquires an infrared singularity in the gluon momentum; smaller ones imply infrared singular ghost legs. Variations in α_c remain within 5% from $\kappa=0.5$ to 0.7. Above this range, α_c decreases more rapidly with $\alpha_c\to 0^+$ as $\kappa\to 1^-$ which sets the upper bound on κ.

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I. INTRODUCTION

In gauge theories without Higgs mechanism, particles carrying the global charges of the gauge group cannot strictly be localized. Localized physical states are necessarily neutral in QED and colorless in QCD. The extension to *all* gauge invariant and thus physical states is possible only with a mass gap in the physical world. Then, color-electric charge superselection sectors do not arise in QCD and one concludes confinement.

The necessary conditions for this were formulated more than 20 years ago. In the next section we briefly recall these conditions and how they constrain the infrared behavior of ghost and gluon propagators in the Landau gauge QCD. Based on linear-covariant gauges, their derivation may not fully be divorced from perturbation theory. Their essence is quite generic and summarized in the Kugo-Ojima criterion which should apply in one way or another, whenever some form of Becchi-Rouet-Stora (BRS) cohomology construction does for gauge theories. One way towards a nonperturbative definition of the Landau gauge is provided via stochastic quantization for which the full five-dimensional BRS machinery is in the garage. The time-independent diffusion equation of this formulation is closely related to the Dyson-Schwinger equations (DSEs) in four dimensions as we describe next. Some of the necessary extensions, which have already been implemented in previous DSE studies of infrared exponents for other reasons, imply the Kugo-Ojima criterion. We summarize these studies and how they are confirmed qualitatively in this way, at the end of the Introduction. These various issues related to our study are collected in Secs. I A–I D to supply additional background information.

In Sec. II, we set up the DSE structures relevant for our present study. We summarize the general properties of the ghost-gluon vertex, most importantly its nonrenormalization and ghost-antighost symmetry in the Landau gauge, which will be essential for infrared critical exponents and coupling later. We then present the ultraviolet subtraction procedure with the special care necessary to make sure that it does not artificially affect the infrared. Some confusion arose recently concerning the relation between asymptotic infrared expansions and the renormalization group which we first clarify in Sec. III. We then review the nonperturbative definition of the running coupling that is based on the nonrenormalization of the ghost-gluon vertex in the Landau gauge, and show that in four dimensions it approaches a constant α_c in the infrared whenever this vertex has an asymptotic conformal behavior also. As a by-product of the vertex nonrenormalization, the infrared behavior of both propagators thereby results to be determined by one unique exponent κ in any given dimension. The general machinery to determine the infrared critical exponent and coupling is outlined in Sec. IV. There, we also discuss the results with an additional regularity assumption on the vertex in the infrared, which in four dimensions leads to the values $\kappa\approx 0.595$ and $\alpha_c\equiv\alpha_c^{max}\simeq(4\pi/N_c)0.709 \approx 2.97$ for $N_c=3$. We furthermore discuss the infrared transversality of the vertex and show how this resolves an apparent contradiction with a previous study.

We then discuss more general vertices involving an additional exponent which controls singularities in its external momenta to discuss bounds on α_c and κ. Thereby we will find that values of κ smaller than that for the regular vertex imply infrared divergences in ghost legs, whereas larger ones lead to an infrared divergence of the vertex in the gluon momentum. While the latter can only come together with an infrared vanishing gluon propagator, which will always overcompensate this divergence, the former add to the infrared enhancement of ghost exchanges. In particular, this would have to happen for an infrared finite gluon propagator (with $\kappa=0.5$) as presently favored by lattice simulations.

Our summary and conclusions are given in Sec. V, and we include two appendixes which may provide the interested reader with some more technical details.

A. The Kugo-Ojima confinement criterion

Within the framework of BRS algebra, completeness of the nilpotent BRS-charge Q_B, the generator of the BRS

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transformations, in a state space $\mathcal{V}$ of indefinite metric is assumed. The semidefinite *physical* subspace $\mathcal{V}_{\text{phys}} = \text{Ker}\, Q_B$ is defined on the basis of this algebra by those states which are annihilated by the BRS charge Q_B. Since $Q_B^2 = 0$, this subspace contains the space $\text{Im}\, Q_B$ of so-called daughter states which are images of others, their parent states in $\mathcal{V}$. A physical Hilbert space is then obtained as (the completion of) the covariant space of equivalence classes $\text{Ker}\, Q_B / \text{Im}\, Q_B$, the BRS-cohomology of states in the kernel modulo those in the image of Q_B, which is isomorphic to the space $\mathcal{V}_s$ of BRS singlets. It is easy to see that the image is furthermore contained in the orthogonal complement of the kernel (given completeness they are identical). It follows that states in $\text{Im}\, Q_B$ do not contribute to the inner product in $\mathcal{V}_{\text{phys}}$.

Completeness is thereby important in the proof of positivity for physical states [1,2], because it assures the absence of metric partners of BRS singlets, so-called "singlet pairs." With completeness, all states in $\mathcal{V}$ are either BRS singlets in $\mathcal{V}_s$ or belong to so-called quartets which are metric-partner pairs of BRS-doublets (of parent with daughter states); and this then exhausts all possibilities. The generalization of the Gupta-Bleuler condition on physical states, $Q_B|\psi\rangle = 0$ in $\mathcal{V}_{\text{phys}}$, eliminates half of these metric partners leaving unpaired states of zero norm which do not contribute to any observable. This essentially is the quartet mechanism: Just as in QED, one such quartet, the elementary quartet, is formed by the massless asymptotic states of longitudinal and timelike gluons together with ghosts and antighosts which are thus all unobservable. In contrast to QED, however, one expects the quartet mechanism also to apply to transverse gluon and quark states, as far as they exist asymptotically. A violation of positivity for such states then entails that they have to be unobservable also. Increasing evidence for this has been seen in the transverse gluon correlations over the last years [3].

But that is only one aspect of confinement in this description. In particular, asymptotic transverse gluon and quark states may or may not exist in the indefinite metric space $\mathcal{V}$. If either of them do, and the Kugo-Ojima criterion is realized (see below), they belong to unobservable quartets. Then, the BRS transformations of their asymptotic fields entail that they form these quartets together with ghost-gluon and/or ghost-quark bound states, respectively [2]. It is furthermore crucial for confinement, however, to have a mass gap in transverse gluon correlations. The massless transverse gluon states of perturbation theory must not exist even though they would belong to quartets due to color antiscreening and superconvergence in QCD for less than ten quark flavors [4,5,3].

Confinement depends on the realization of the unfixed global gauge symmetries. The identification of gauge-invariant physical states, which are BRS singlets, with color singlets is possible only if the charge of global gauge transformations is BRS exact *and* unbroken. The sufficient conditions for this are provided by the Kugo-Ojima criterion: Considering the globally conserved current

$$J_\mu^a = \partial_\nu F_{\mu\nu}^a + \{Q_B, D_\mu^{ab}\bar{c}^b\} \quad (\text{with } \partial_\mu J_\mu^a = 0), \tag{1}$$

one realizes that the first of its two terms corresponds to a coboundary with respect to the space-time exterior derivative while the second term is a BRS coboundary. Denoting their charges by G^a and N^a, respectively,

$$Q^a = \int d^3x (\partial_i F_{0i}^a + \{Q_B, D_0^{ab}\bar{c}^b\}) = G^a + N^a. \tag{2}$$

For the first term herein there are only two options, it is either ill-defined due to massless states in the spectrum of $\partial_\nu F_{\mu\nu}^a$, or else it vanishes.

In QED massless photon states contribute to the analogues of both currents in Eq. (1), and both charges on the right-hand side (rhs) in Eq. (2) are separately ill-defined. One can employ an arbitrariness in the definition of the generator of the global gauge transformations (2), however, to multiply the first term by a suitable constant so chosen that these massless contributions cancel. In this way one obtains a well-defined and unbroken global gauge charge which replaces the naive definition in Eq. (2) above [6]. Roughly speaking, there are two independent structures in the globally conserved gauge currents in QED which both contain massless photon contributions. These can be combined to yield one well-defined charge as the generator of global gauge transformations leaving any other combination spontaneously broken, such as the displacement symmetry which led to the identification of the photon with the massless Goldstone boson of its spontaneous breaking [2,7].

If $\partial_\nu F_{\mu\nu}^a$ contains no massless discrete spectrum on the other hand, i.e., if there is no massless particle pole in the Fourier transform of transverse gluon correlations, then $G^a \equiv 0$. In particular, this is the case for channels containing massive vector fields in theories with the Higgs mechanism, and it is expected to be also the case in any color channel for QCD with confinement for which it actually represents one of the two conditions formulated by Kugo and Ojima. In both these situations one first has equally, however,

$$Q^a = N^a = \left\{ Q_B, \int d^3x D_0^{ab}\bar{c}^b \right\}, \tag{3}$$

which is BRS exact. The second of the two conditions for confinement is that this charge be well-defined in the whole of the indefinite metric space $\mathcal{V}$. Together these conditions are sufficient to establish that all BRS-singlet physical states are also color singlets, and that all colored states are thus subject to the quartet mechanism. The second condition thereby provides the essential difference between the Higgs mechanism and confinement. The operator $D_\mu^{ab}\bar{c}^b$ determining the charge N^a will in general contain a *massless* contribution from the elementary quartet due to the asymptotic field $\bar{\gamma}^a(x)$ in the antighost field, $\bar{c}^a \overset{x_0 \to \pm\infty}{\longrightarrow} \bar{\gamma}^a + \cdots$ (in the weak asymptotic limit),

$$D_\mu^{ab}\bar{c}^b \overset{x_0 \to \pm\infty}{\longrightarrow} (\delta^{ab} + u^{ab})\partial_\mu \bar{\gamma}^b(x) + \cdots. \tag{4}$$

Here, the dynamical parameters u^{ab} determine the contribution of the massless asymptotic state to the composite field

$gf^{abc}A_\mu^c\bar{c}^b \xrightarrow{x_0\to\pm\infty} u^{ab}\partial_\mu\bar{\gamma}^b+\cdots$. These parameters can be obtained in the limit $p^2\to 0$ from the Euclidean correlation functions of this composite field, e.g.,

$$\int d^4x e^{ip(x-y)}\langle D_\mu^{ae}c^e(x)gf^{bcd}A_\nu^d(y)\bar{c}^c(y)\rangle$$
$$\equiv\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)u^{ab}(p^2). \tag{5}$$

The theorem by Kugo and Ojima asserts that all $Q^a=N^a$ are well-defined in the whole of $\mathcal{V}$ (and do not suffer from spontaneous breakdown), if and only if

$$u^{ab}\equiv u^{ab}(0)\stackrel{!}{=}-\delta^{ab}. \tag{6}$$

Then, the massless states from the elementary quartet do not contribute to the spectrum of the current in N^a, and the equivalence between physical BRS-singlet states and color singlets is established [1,2,6].

In contrast, if $\det(\mathbb{1}+u)\neq 0$, the global gauge symmetry generated by the charges Q^a in Eq. (2) is spontaneously broken in each channel in which the gauge potential contains an asymptotic massive vector field [1,2]. While this massive vector state results to be a BRS singlet, the massless Goldstone boson states which usually occur in some components of the Higgs field replace the third component of the vector field in the elementary quartet and are thus unphysical. Since the broken charges are BRS exact, this *hidden* symmetry breaking is not directly observable in the physical Hilbert space.

The different scenarios are classified according to the realization of the global gauge symmetry on the whole of the indefinite metric space of covariant gauge theories. If it is unbroken, as in QED and QCD, the first condition is crucial for confinement. Namely, it is then necessary to have a mass gap in the transverse gluon correlations, since otherwise one could in principle have *nonlocal* physical (BRS-singlet and thus gauge-invariant) states with color, just as one has gauge-invariant charged states in QED (e.g., the state of one electron alone in the world with its long-range Coulomb tail). Indeed, with unbroken global gauge invariance, QED and QCD have in common that any gauge invariant localized state must be chargeless/colorless [2]. The question is the extension to nonlocal states as approximated by local ones. In QED this leads to the so-called charge superselection sectors [8], and nonlocal physical states with charge arise. If in QCD, with unbroken global gauge symmetry *and* mass gap, *every* gauge-invariant state can be approximated by gauge-invariant localized ones (which are colorless), one concludes that *every* gauge-invariant (BRS-singlet) state must also be a color singlet.

B. Infrared dominance of ghosts in the Landau gauge

The (second condition in the) Kugo-Ojima confinement criterion, $u=-\mathbb{1}$ leading to well-defined charges N^a, can in Landau gauge be shown by standard arguments employing Dyson-Schwinger equations (DSEs) and Slavnov-Taylor identities (STIs) to be equivalent to an infrared enhanced ghost propagator [6]. In momentum space the nonperturbative ghost propagator of Landau gauge QCD is related to the form factor occurring in the correlations of Eq. (5) as follows:

$$D_G(p)=\frac{-1}{p^2}\frac{1}{1+u(p^2)},\quad \text{with}\quad u^{ab}(p^2)=\delta^{ab}u(p^2). \tag{7}$$

The Kugo-Ojima criterion, $u(0)=-1$, thus entails that the Landau gauge ghost propagator should be more singular than a massless particle pole in the infrared. Indeed, there is quite compelling evidence for this exact infrared enhancement of ghosts in the Landau gauge [9]. For lattice calculations of the Landau gauge ghost propagator, see Refs. [10–12]. The Kugo-Ojima confinement criterion was also tested on the lattice directly [13].

Lattice verifications of the positivity violations for transverse gluon states by now have a long history [14–19]. Numerical extractions of their indefinite spectral density from lattice data are reported in [20]. As mentioned, however, this follows from color antiscreening and superconvergence in QCD already in perturbation theory [4,5], and it is independent of confinement.

Its remaining dynamical aspect resides in the cluster decomposition property of local quantum field theory in this formulation [8,2]. Within the indefinite inner product structure of covariant QCD it can be avoided for colored clusters, only *without mass gap* in the full indefinite space $\mathcal{V}$. In fact, if the cluster decomposition property holds for a gauge-invariant product of two (almost local) fields, it can be shown that both fields are gauge-invariant (BRS-closed) themselves. With mass gap in the physical world, this then eliminates the possibility of scattering a physical asymptotic state into a color singlet consisting of widely separated colored clusters (the "behind-the-moon" problem) [2].

The necessity for the absence of the massless particle pole in $\partial_\nu F_{\mu\nu}^a$ in the Kugo-Ojima criterion shows that the (unphysical) massless correlations to avoid the cluster decomposition property are *not* the transverse gluon correlations. An infrared suppressed propagator for the transverse gluons in Landau gauge confirms this condition. This holds equally well for the infrared vanishing propagator obtained from DSEs [21,23,22], and conjectured in the studies of the implications of the Gribov horizon [24,25], as for the infrared suppressed but possibly finite ones extracted from improved lattice actions for quite large volumes [26–28].

An infrared finite gluon propagator with qualitative similarities in the transverse components appears to result also in simulations using the Laplacian gauge [29]. Related to the Landau gauge, this gauge fixing was proposed as an alternative for lattice studies in order to avoid Gribov copies [30]. For a perturbative formulation see Ref. [31]. Due to intrinsic nonlocalities, its renormalizability could not be demonstrated so far. Deviations from the Landau gauge condition were observed already at $O(g^2)$ in the bare coupling in Ref. [32]. Moreover, the gluon propagator was seen to develop a large longitudinal component in the nonperturbative regime [29].

In fact, compared to the transverse correlations, it seems to provide the dominant component in the infrared, and it might in the end play a role analogous to that of the infrared enhanced ghost correlations in the Landau gauge. However, the precise relation with Landau gauge still seems somewhat unclear. It is certainly encouraging nevertheless to first of all verify that no massless states contribute to the transverse gluon correlations of the Laplacian gauge either.

C. Nonperturbative Landau gauge

A problem mentioned repeatedly already, which is left in the dark in the description of confinement within the covariant operator formulation presented so far, is the possible influence of Gribov copies [24].

Recently, renewed interest in stochastic quantization arose because it provides ways of gauge fixing in the presence of Gribov copies, at least in principle [33,34]. The relation to Dyson-Schwinger equations is provided by a time-independent version of the diffusion equation in this approach in which gauge-fixing is replaced by a globally restoring drift-force tangent to gauge orbits in order to prevent the probability distribution from drifting off along gauge orbit directions.

In particular, in the limit of the Landau gauge, it is the conservative part of this drift-force, the derivative with respect to transverse gluon-field components of the Faddeev-Popov action, which leads to the standard Dyson-Schwinger equations as clarified by Zwanziger [35]. He furthermore points out that these equations are formally unchanged if Gribov's original suggestion to restrict the Faddeev-Popov measure to what has become known as the interior of the first Gribov horizon is implemented. This is simply because the Faddeev-Popov measure vanishes there, and thus no boundary terms are introduced in the derivation of Dyson-Schwinger equations (DSEs) by this additional restriction. Phrased otherwise, it still provides a measure such that the expectation values of total derivatives with respect to the fields vanish, which is all we need to formally derive the same Dyson-Schwinger equations as those without restriction.

In the stochastic formulation this restriction arises naturally because the probability distribution gets concentrated on the (first) Gribov region as the Landau gauge is approached. Therefore there should be no problem of principle with the existence of Gribov copies in the standard DSEs. However, the distribution of the probability measure among the gauge orbits might be affected by neglecting (the nonconservative) part of the drift force. Ways to overcome this approximation are currently being investigated. Moreover, providing for a correct counting of gauge copies inside the Gribov region, the full stochastic equation will allow comparison with results from Monte Carlo simulations using lattice implementations of the Landau gauge in a much more direct and reliable way. In particular, this should be the case for the lattice analog of the stochastic gauge fixing used in simulations such as those of Refs. [16–18].

Here, we restrict ourselves to the standard Landau gauge DSEs which are best justified nonperturbatively from the stochastic approach to be valid modulo the aforementioned approximation. For their solutions, on the other hand, restricting the support of the Faddeev-Popov measure to the interior of the Gribov region has the effect of additional boundary conditions to select certain solutions from the set of all possible ones which might contain others as well. Consider two invariant functions $Z(k^2)$ and $G(k^2)$ to parametrize the Landau gauge structure,

$$D_{\mu\nu}(k)=\frac{Z(k^2)}{k^2}\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right),\quad D_G(k)=-\frac{G(k^2)}{k^2}, \tag{8}$$

in Euclidean momentum space of gluon and ghost propagator, respectively. Additionally when obtained as DSE solutions, suitable boundary conditions have to be satisfied by these functions. The following infrared bounds were derived by Zwanziger for each of the two as properties of the propagators from the restricted measure.

The observation that the "volume" of configuration space in the infinite-dimensional (thermodynamic) limit is contained in its surface lead to the so-called *horizon condition* which entails that the ghost propagator must be more singular than a massless particle pole in the infrared [35–37],

$$\lim_{k^2\to 0} G^{-1}(k^2)=0. \tag{9}$$

This condition is equivalent to the Kugo-Ojima criterion, $u=-\mathbb{1}$ for well-defined color charges in the Landau gauge, cf., Eqs. (6) and (7) with $G(k^2)=1/[1+u(k^2)]$.

From the proximity of the Gribov horizon in infrared directions Zwanziger furthermore concluded [25] that

$$\lim_{k^2\to 0} Z(k^2)/k^2=0. \tag{10}$$

This removes the massless transverse gluon states of perturbation theory as also required by the Kugo-Ojima criterion. The infrared vanishing of the gluon propagator is a stronger requirement than this, however. It currently remains an open question why this has not been seen in Monte Carlo simulations as yet. An infrared suppression of the gluon propagator itself, rather than $Z(k^2)$, was observed for the Landau gauge in [38] and, more considerably, at large volumes in SU(2) in the three-dimensional case [39–41], as well as in Coulomb gauge [42]. The three-dimensional results are interesting in that the large distance gluon propagator measured there seems incompatible with a massive behavior at low momenta (that was noted also in [18]). At very large volumes, it even becomes negative [40,41]. This is the same qualitative behavior as obtained for the one-dimensional Fourier transform of the DSE results of [22,23] at small values for the remaining momentum components, cf., Fig. 4 of [9] versus Fig. 2 of [40] or Fig. 6 of [41]. Qualitatively, the different dimensionality should not matter much here. On the other hand, the extrapolation of the zero momentum propagator in [41] leads to a finite result which, however, still decreases (slowly) with the volume. This suggests that the physical volumes may still be too small yet and that further study of the volume dependence of the zero momentum gluon propagator might be necessary [43].

D. Infrared exponents in previous studies

Within the standard BRS or Faddeev-Popov formulation, the functions in Eq. (8) have been studied from Dyson-Schwinger equations (DSEs) for the propagators in various truncation schemes of increasing levels of sophistication [3,44]. Typically, the known structures in the three-point vertex functions, most importantly from their Slavnov-Taylor identities and exchange symmetries, have thereby been employed to establish closed systems of nonlinear integral equations that are complete on the level of the gluon, ghost, and quark propagators in Landau gauge. This is possible with systematically neglecting contributions from explicit four-point vertices to the propagator DSEs as well as nontrivial four-point scattering kernels in the constructions of the three-point vertices [3,22]. Employing a one-dimensional approximation, numerical solutions were then obtained in Refs. [22] and [23].

Asymptotic expansion techniques were developed to analytically study the behavior of the solutions in the infrared. The leading infrared behavior was thereby determined by one unique exponent $\kappa \approx 0.92$,

$$Z(k^2) \overset{k^2\to 0}{\sim} \left(\frac{k^2}{\sigma}\right)^{2\kappa} \quad \text{and} \quad G(k^2) \overset{k^2\to 0}{\sim} \left(\frac{\sigma}{k^2}\right)^{\kappa}, \tag{11}$$

with a renormalization group invariant σ, see Sec. III A. The general bounds $0<\kappa<1$ were established in Ref. [22] based on the additional requirement that Z and G have no zeros or poles along the positive real axis, i.e., in the Euclidean domain. Below, we will verify the positivity for the leading infrared behavior of both these functions in the same range, independent of the one-dimensional approximation, and based on some few and quite generic properties of the ghost-gluon vertex alone.

The infrared behavior in Eq. (11) was later confirmed qualitatively by studies of further truncated DSEs. In Ref. [45], a tree-level ghost gluon vertex was used in combination with a one-dimensional approximation which lead to a value of $\kappa \approx 0.77$ for the infrared exponent of ghost and gluon propagation in Landau gauge. Then, in the first infrared asymptotic study of the ghost-gluon system without one-dimensional approximation, the value of $\kappa = 1$ was obtained in Ref. [46]. There is, however, an issue about infrared transversality of the gluon propagator, as we will explain below, which was not addressed in this study. As a result, the correct value for the tree-level vertex is the same as that derived herein for any ghost-gluon vertex with regular infrared limit, $\kappa \approx 0.595$, which was first reported for the tree-level vertex independently in Refs. [47] and [35]. As we furthermore find in our present study, inconsistency arises for $\kappa \to 1$ (from below), and this limit, the upper bound on κ, is therefore excluded.

With $1/2<\kappa$, all these values of the infrared exponent share, however, the same qualitative infrared behavior. The gluon propagator vanishes while the ghost propagator is infrared enhanced. Then, the Kugo-Ojima criterion and the boundary conditions (9) and (10) are both satisfied. The horizon condition seems understandable because the restriction to the Gribov region leads to a positive measure which was implicitly also assumed by requiring solutions without nodes in [22]. Here, depending on the infrared behavior of the ghost-gluon vertex, we will find that this requirement could in principle be maintained also for $\kappa<1/2$. Taken by itself, it only leads to $0<\kappa<1$, and thus to the horizon condition (9). Eventually, with decreasing κ for values smaller than 1/2, infrared singularities in ghost exchanges become too strong for a local field theory description. Around $\kappa=1/2$, however, this argument is just not strong enough, and we cannot turn it into an independent additional argument in favor of Eq. (10) for an infrared vanishing gluon propagator.

II. DYSON-SCHWINGER EQUATIONS

The Dyson-Schwinger equations for the propagators of ghosts and gluons in the pure gauge theory without quarks are schematically represented by the diagrams shown in Fig. 1. With infrared dominance of ghosts, the ghost loop represented by the diagram in the last line of Fig. 1 will provide the dominant contribution to the inverse gluon propagator on the left-hand side in the infrared. In our infrared analysis we will concentrate on this contribution to the (renormalized) gluon DSE which reads in Euclidean momentum space with the notations of [3] (color indices suppressed),

$$D^{-1}_{\mu\nu}(k) = Z_3 D^{0\,-1}_{\mu\nu}(k) - g^2 N_c \tilde{Z}_1 \times \int \frac{d^4q}{(2\pi)^4} G^0_\mu(q,p) D_G(p) D_G(q) G_\nu(p,q) + \cdots, \tag{12}$$

where $p=k+q$, and the contributions from the four remaining gluon loop-diagrams of Fig. 1 were not given explicitly. D^0 is the tree-level propagator, D_G is the ghost propagator, and G_ν is the fully dressed ghost-gluon vertex function with its tree-level counter part denoted by G^0_μ. In the standard linear-covariant gauge the latter is given by the antighost-momentum, $G^0_\mu(q,p) = iq_\mu$. The DSE for the ghost propagator, without truncations at this point, formally reads

$$D_G^{-1}(k) = -\tilde{Z}_3 k^2 + g^2 N_c \tilde{Z}_1 \times \int \frac{d^4q}{(2\pi)^4} i k_\mu D_G(q) G_\nu(q,k) D_{\mu\nu}(k-q). \tag{13}$$

The renormalized propagators, D_G and $D_{\mu\nu}$, and the renormalized coupling g are defined from the respective bare quantities by introducing multiplicative renormalization constants,

$$\tilde{Z}_3 D_G = D_G^{\text{bare}}, \quad Z_3 D_{\mu\nu} = D_{\mu\nu}^{\text{bare}}, \quad Z_g g = g_{\text{bare}}. \tag{14}$$

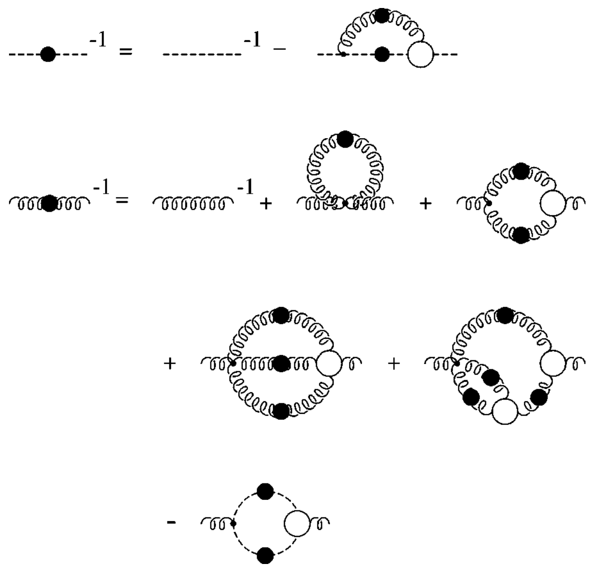

FIG. 1. Dyson-Schwinger equations for the ghost (top) and the gluon (bottom) propagator, diagrammatically.

Furthermore, $\widetilde{Z}_1 = Z_g Z_3^{1/2} \widetilde{Z}_3$ is the ghost-gluon vertex renormalization constant. Before we discuss the properties of the ghost-gluon vertex, essentially the only unknown in Eqs. (12) and (13), we note the following.

If we are allowed to assume that the leading contribution to the inverse gluon propagator in Eq. (12) is completely determined by the ghost loop, this contribution must be transverse in the Landau gauge. In other words, writing

$$D_{\mu\nu}^{-1}(k) = k^2 \delta_{\mu\nu} Z_P^{-1}(k^2) - k_\mu k_\nu Z_R^{-1}(k^2), \tag{15}$$

one should then have $Z_P(k^2) = Z_R(k^2) \equiv Z(k^2)$. Here, in particular, the leading infrared behavior as extracted from the ghost loop alone should not depend on whether we study Z_P or Z_R. With all other contributions subleading, deviations from the transversality of the "vacuum polarization," $Z_P = Z_R$ in the Landau gauge, should also be subleading. We will assess this by studying, in parallel (we sometimes use D dimensions, normally $D=4$ here),

$$Z_P^{-1}(k^2) = \frac{1}{(D-1)k^2} D_{\mu\nu}^{-1}(k) \mathcal{P}_{\mu\nu}(k), \tag{16}$$

$$Z_R^{-1}(k^2) = \frac{1}{(D-1)k^2} D_{\mu\nu}^{-1}(k) \mathcal{R}_{\mu\nu}(k), \tag{17}$$

with

$$\mathcal{P}_{\mu\nu}(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2},$$

$$\mathcal{R}_{\mu\nu}(k) = \delta_{\mu\nu} - D\frac{k_\mu k_\nu}{k^2},$$

respectively. Beyond the leading behavior as dominated by the infrared enhanced propagators within the ghost loop, there will in general be complicated cancellations between longitudinal contributions from various sources to ensure transversality of the gluons in the Landau gauge. These sources can be due to the terms neglected, to truncations of vertices, or to the regularization scheme employed. The tadpole, for example, contributes only to Z_P^{-1}, and so do the quadratic divergences with cutoff regularization. Beyond the leading order one therefore usually employed the $\mathcal{R}$-tensor in the contraction of the gluon DSE in most previous studies since that of Ref. [48].

The tadpole contribution is a momentum-independent constant, so that it will necessarily be subleading as compared to the infrared singular ghost-loop, whenever that singularity is strong enough to lead to an infrared-vanishing gluon propagator, or $Z(k^2) \sim (k^2)^{2\kappa}$ with $\kappa > 1/2$ for $k^2 \to 0$.

The infrared analysis we present below is independent of the regularization and both these reasons in favor of the $\mathcal{R}$-tensor do therefore not apply to our present study. Nevertheless, even with ghost dominance, exact transversality will in general only be obtained by including all different structures possible in the ghost-gluon vertex that can contribute to the leading infrared behavior.

A. The ghost-gluon vertex in the Landau gauge

The ghost-gluon vertex is of particular importance in the analysis of the infrared behavior of the gluon and ghost propagators. We adopt the conventions of Ref. [3]. The arguments of the ghost-gluon vertex denote in the following order the two outgoing momenta for gluon and ghost, and one incoming ghost momentum, cf., Fig. 2,

$$G_\mu^{abc}(k,q,p) = (2\pi)^4 \delta^4(k+q-p) G_\mu^{abc}(q,p), \tag{18}$$

$$G_\mu^{abc}(q,p) = g f^{abc} G_\mu(q,p). \tag{19}$$

Color structures other than the perturbative one assumed here were assessed for the pure Landau gauge theory on the lattice in Ref. [49]. In this study, there was no evidence seen for any significant contribution due to such structures which we will not consider henceforth.

We parametrize the general structure of $G_\mu(q,p)$ which consists of two independent terms by the following form:

$$G_\mu(q,p) = i q_\mu A(k^2;p^2,q^2) + i k_\mu B(k^2;p^2,q^2). \tag{20}$$

One might expect the second structure to be insignificant in the Landau gauge, since it is longitudinal in the gluon momentum k. This is not necessarily the case in Dyson-Schwinger equations, however, since the transversality of the vacuum polarization generally arises from cancellations of different longitudinal contributions as we discussed above.

For later reference, we recall two general properties of this vertex. The implications of these properties are explored below. They both refer to the ghost-gluon vertex in the Landau gauge.

(N) Nonrenormalization, $\widetilde{Z}_1 = Z_g Z_3^{1/2} \widetilde{Z}_3 = 1$ [50], which entails that the vertex reduces to its tree-level form at all symmetric momentum points,

$$G_\mu(q,p)|_{k^2=q^2=p^2} = G^0_\mu(q,p), \tag{21}$$

in a symmetric subtraction scheme. The gauge fields being purely transverse, however, there is a certain freedom left in the definition of the tree-level vertex. *A priori*, any form with $\eta \in [0,1]$,

$$G^0_\mu(q,p) = \eta i q_\mu + \hat{\eta} i p_\mu \quad \text{and} \quad \eta + \hat{\eta} = 1, \tag{22}$$

may be used equally for the Landau gauge. Without further specification, for the functions in Eq. (20) we first have

$$A(x;x,x) = 1 \quad \text{and} \quad B(x;x,x) = \hat{\eta}.$$

The condition on A is η independent. It expresses the essential aspect of nonrenormalization and will be referred to as **(N1)**. The condition on B depends on the ambiguity in defining the Landau gauge, as expected. We call this condition **(N2)**. It reads $B(x;x,x) = 0$ for the transverse limit of the linear-covariant gauge in standard Faddeev-Popov theory, as compared to $B(x;x,x) = 1/2$ for the analogous limit of the ghost-antighost symmetric Curci-Ferrari gauge [51], see also Ref. [3]. These are the two special choices of particular interest, corresponding to $\eta = 1$ and $\eta = \hat{\eta} = 1/2$, respectively. For renormalizability and perturbative aspects of the latter, and for the geometry of the general η gauges, see Refs. [52].

(S) The ghost-antighost conjugation as part of the full Landau gauge symmetry, a semidirect product of $SL(2,\mathbb{R})$ and double BRS invariance, implies [3]

$$A(x;y,z) = A(x;z,y). \qquad \textbf{(S1)}$$

While this holds for all η, again, the B function is more ambiguous. It cannot have definite symmetry properties in general. For the symmetrized Landau gauge, based on the symmetric tree-level vertex with $\eta = \hat{\eta} = 1/2$, the interactions with purely transverse gluons will preserve this exact symmetry of the Landau gauge, however. In the symmetric formulation we therefore expect to have an exactly ghost-antighost symmetric vertex also,

$$G_\mu(q,p) = G_\mu(p,q).$$

Decomposing $B = B_+ + B_-$ with $B_\pm(x;z,y) = \pm B_\pm(x;y,z)$, we then furthermore deduce,

$$2B_+(x;y,z) = A(x;y,z). \qquad \textbf{(S2)}$$

In the fully ghost-antighost symmetric formulation we can thus express the vertex (20) in terms of the functions with definite (anti)symmetry as follows:

$$G_\mu(q,p) = \frac{i q_\mu + i p_\mu}{2} A(k^2;p^2,q^2) + i k_\mu B_-(k^2;p^2,q^2). \tag{23}$$

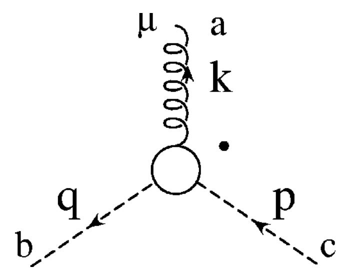

FIG. 2. Conventions for the ghost-gluon vertex $G^{abc}_\mu(k,q,p)$.

The B_- structure is absent at the tree level and it vanishes at all symmetric points. Therefore a symmetric vertex in converse to the logic above also requires $\eta = \hat{\eta} = 1/2$ for the tree-level vertex to be used as the G^0_μ in a symmetric subtraction scheme according to Eq. (21).

At this point, it seems important to stress that, for the infrared exponent of Landau gauge QCD, only the infrared behavior of A is relevant. The critical exponents are of course independent of η. As long as we concentrate on Z_P via Eq. (16) in the gluon DSE (13), the gluon legs of all vertices are transversely contracted. For all internal gluon lines this is automatically true by the transversality of the propagator as in the ghost DSE (13) for example, and for the external lines we just arranged it by hand. Thus the η freedom in the tree-level vertex and the B structure of the full vertex are both irrelevant, as they should be. In an infrared analysis based on this manifestly transverse system we might as well have standard Faddeev-Popov theory in mind with $\eta = 1$, $\hat{\eta} = 0$.

The only place where the η dependence and the B structure do enter is the $\mathcal{R}$-contracted gluon DSE. We therefore introduce the generalized Landau gauge by the above modification of the tree-level vertex here as a purely technical tool to address the transversality issue, i.e., to compare Z_P and Z_R as obtained from Eqs. (16) and (17), respectively. In fact, in order to reconcile ghost dominance with transversality, the result will be that for arbitrary values of η one must essentially have in the infrared (indicated by the superscripts),

$$B^{ir}_+(k^2;p^2,q^2) = \frac{1}{2} A^{ir}(k^2;p^2,q^2), \tag{24}$$

$$B^{ir}_-(k^2;p^2,q^2) = -\frac{p^2 - q^2}{2k^2} A^{ir}(k^2;p^2,q^2). \tag{25}$$

Inserting this into Eq. (20), small rearrangements reveal that the full ghost-gluon vertex therefore has to be transverse in the infrared itself,

$$G^{ir}_\mu(q,p) = \frac{i q_\mu pk - i p_\mu qk}{k^2} A^{ir}(k^2;p^2,q^2). \tag{26}$$

This is in contrast to its perturbative limit where the B structure is suppressed, and it is now also independent of the choice of η. Again, however, the transverse vertex is necessarily symmetric $\propto (i q_\mu + i p_\mu)$ at a symmetric point. In order to extend the subtraction scheme of Eq. (21) nonperturbatively into the infrared, and ensure transversality of the gluon propagator, we thus have to resort to the symmetric choice $\eta=\hat{\eta}=1/2$ in Eq. (22) also [for which Eq. (24) follows trivially with **(S2)**].

B. Truncated Slavnov-Taylor identity

In [22] a Slavnov-Taylor identity of the standard linear-covariant gauge was derived to constrain the ghost-gluon vertex. Since the BRS transformations need some adjustments for other choices such as the ghost-antighost symmetric gauges, as it stands this identity is valid for the case $\eta=1$, $\hat{\eta}=0$ only. A generalization might be worthwhile pursuing. However, considering the transversality of the full vertex in the infrared, this will not provide much additional information to be used in our present study, as we discuss in this section.

Neglecting irreducible ghost-ghost scattering contributions to the Slavnov-Taylor identity (STI) of Ref. [22], and thus maintaining the disconnected contributions to the ghost four-point function only, a truncated Slavnov-Taylor identity is obtained which, in terms of the two structures A, B in the vertex and the ghost propagator, reads

$$\begin{aligned} G(x)\left(\frac{z+y-x}{2}A(y;x,z)-yB(y;x,z)\right) \\ +G(y)\left(\frac{z+x-y}{2}A(x;y,z)-xB(x;y,z)\right) \\ =\frac{zG(x)G(y)}{G(z)}. \end{aligned} \tag{27}$$

Without the symmetry property **(S1)**, a simple solution to Eq. (27) is given by

$$A(x;y,z)=\frac{G(x)}{G(z)},\quad B(x;y,z)\equiv 0. \tag{28}$$

This exact form was used for the ghost-gluon vertex in the study of Wilsonian flow equations for Yang-Mills theory in Ref. [53]. Implementing **(S1)** in addition, the most general solution to Eq. (27) can be written in the form

$$\begin{aligned} A(x;y,z)&=\frac{G(x)}{G(z)}+\frac{G(x)}{G(y)}-1+xf^T(x;y,z),\\ B(x;y,z)&=\frac{G(x)}{G(y)}-1+\frac{x-y+z}{2}f^T(x;y,z). \end{aligned} \tag{29}$$

The undetermined function f^T thereby parametrizes an unknown transverse contribution to the ghost-gluon vertex, $kG^T(q,p)=0$, of the typical type generally remaining unconstrained by the Slavnov-Taylor identities,

$$G^T_\mu(q,p)=(iq_\mu kp-ip_\mu kq)f^T(k^2;p^2,q^2), \tag{30}$$

where $k=p-q$ as before. We obtain, however, from Eq. (29) with **(N1)** and **(S1)**, respectively,

$$f^T(x;x,x)=0\quad\text{and}\quad f^T(x;y,z)=f^T(x;z,y). \tag{31}$$

Since this function is otherwise arbitrary, in particular, in the infrared, the use of solutions (29) somehow seems less appealing for our present study which is concerned about the most general bounds on the infrared exponent that can be derived on the basis of as few and basic assumptions as currently possible.

In the numerical solutions to the coupled system of truncated ghost-gluon DSEs presented in Refs. [23] and [22], the form given in Eq. (29) was used for the ghost-gluon vertex with $f^T\equiv 0$. This solution to the truncated STI (27) (for $\eta=1$) still satisfies **(N1)**, **(N2)**, and **(S1)**. Because it is not purely transverse in the infrared, it should be used in combination with the transversely projected DSE for Z_P from Eq. (16), however, such that only the form of $A(x;y,z)$ in Eq. (29) matters. This causes ultraviolet problems in the numerical studies, see below. If the infrared transversality of the vertex can be maintained on the other hand, by adding suitable transverse terms to a symmetric STI construction to satisfy Eq. (26) above, for example, the $\mathcal{R}$-tensor may be used to contract the gluon DSE via Eq. (17) by which these ultraviolet problems are avoided without doing harm to the infrared structure of the equations.

We believe that this will be the way to proceed with the numerical studies of full solutions to truncated DSEs in the future. In particular, this suggests further developments in the ghost-antighost symmetric formulation.

C. Ultraviolet subtractions and infrared behavior

With the parametrization of the vertex in Eq. (20) we now obtain for the ghost-loop contribution to the gluon DSE (12) the two alternative expressions from the contractions according to Eqs. (16) and (17), respectively,

$$\begin{aligned} \frac{1}{Z_P(k^2)}=Z_3+\frac{g^2N_c}{3}\int\frac{d^4q}{(2\pi)^4}\frac{G(p^2)G(q^2)}{k^2p^2q^2} \\ \times q\mathcal{P}(k)qA(k^2;q^2,p^2)+\cdots, \end{aligned} \tag{32}$$

$$\begin{aligned} \frac{1}{Z_R(k^2)}=Z_3+\frac{g^2N_c}{3}\int\frac{d^4q}{(2\pi)^4}\frac{G(p^2)G(q^2)}{k^2p^2q^2} \\ \times\{\eta(q\mathcal{R}(k)pA(k^2;q^2,p^2)-q\mathcal{R}(k)k\,B(k^2;q^2,p^2)) \\ +\hat{\eta}(p\mathcal{R}(k)pA(k^2;q^2,p^2)-p\mathcal{R}(k)kB(k^2;q^2,p^2))\} \\ +\cdots, \end{aligned} \tag{33}$$

where again $p=k+q$. One can see explicitly here that knowledge of both invariant functions is necessary for an infrared analysis based on the $\mathcal{R}$-tensor, cf., Eq. (33), while only the A structure enters in Eq. (32) obtained with the transverse projector $\mathcal{P}$. We furthermore allowed for the generalized tree-level vertex with $\eta+\hat{\eta}=1$ discussed in **(N)** which does not affect Eq. (32). This makes the equation for Z_P^{-1} particularly well suited for an infrared analysis because then the invariant function $A(k^2;p^2,q^2)$, which parametrizes the essential part of the vertex, is the only unknown remaining in the coupled system with the equally η-independent ghost DSE (13),

$$\frac{1}{G(k^2)}=\tilde{Z}_3-g^2N_c\int\frac{d^4q}{(2\pi)^4}\frac{Z(p^2)G(q^2)}{k^2p^2q^2}\times k\mathcal{P}(p)kA(p^2;k^2,q^2). \tag{34}$$

We used $\tilde{Z}_1=1$ for the Landau gauge in these equations. The ultraviolet divergences of the explicit loop-integrals are compensated by the renormalization constants Z_3, $\tilde{Z}_3$ which we require to be infrared finite. For the ultraviolet subtractions, which are of course k-independent, one then needs to make the following distinctions.

(i) In DSEs for propagators of massless or infrared enhanced degrees of freedom we can perform the limit $k^2\to 0$. In the present case we expect this for the ghosts, i.e., the left-hand side (lhs) of Eq. (34) will approach a finite constant $C_G\equiv\lim_{x\to 0}G^{-1}(x)<\infty$ which can be zero, however. In such a case, the renormalization constant can easily be eliminated, e.g., here we then have

$$\tilde{Z}_3=g^2\mu^{4-D}N_c\frac{D-1}{D}\int\frac{d^Dq}{(2\pi)^D}\times\frac{1}{q^4}Z(q^2)G(q^2)A(q^2;0,q^2)+C_G. \tag{35}$$

We adopted the conventions of dimensional regularization with $D\to 4^-$ here, but others may be employed as desired. If we assume regularity of A in the origin, we can conclude that $A(x;0,x)\to 1$ for $x\to 0$ from **(N1)**. Generally, if $A(x;0,x)$ does not vanish for $x\to 0$, then Eq. (35) tells us that $Z(q^2)G(q^2)\to 0$ for $q^2\to 0$ in the infrared. Otherwise, $\tilde{Z}_3$ would be infrared divergent for $D\leqslant 4$. On the other hand, $g^2Z(q^2)G^2(q^2)$ is more and more recognized to be a good candidate for a nonperturbative definition of the running coupling in the Landau gauge, see Sec. III A. However, this definition can only be reasonable if we are able to arrange matters such that it does not vanish in the infrared (in fact, the running coupling must be monotonic to avoid a double valued β-function with spurious zeros).

So that with $Z(q^2)G^2(q^2)\not\to 0$ and $Z(q^2)G(q^2)\to 0$ we must have infrared enhancement of ghosts. In particular, $C_G=0$ in Eq. (35) and no such constant is then possible in a nonperturbative renormalization scheme.

We can also reverse the logic here, and regard $C_G=0$ as an additional boundary condition on a set of possible solutions to the DSE (34). This then implements Zwanziger's horizon condition (9) to select the solution for the restricted Faddeev-Popov weight that vanishes outside the Gribov horizon [35], and that at the same time provides a positive definite $G(k^2)>0$ [22]. Once this selection is made, however, it then follows that $Z(q^2)G^2(q^2)\to$const for $q^2\to 0$ (in $D=4$ dimensions) which is a consequence of the nonrenormalization of the vertex **(N1)**, as we will show in Sec. III C.

So let us adopt $C_G=0$ and concentrate on the possible solutions with infrared enhanced ghosts from now on.

With Eq. (35) in Eq. (34) we explicitly remove the ultraviolet divergence and obtain a manifestly finite equation to study the infrared behavior of G in $D=4$ dimensions for a given form of A in the infrared,

$$\frac{1}{G(k^2)}=g^2N_c\int\frac{d^4q}{(2\pi)^4}\left\{\frac{3}{4}\frac{Z(q^2)G(q^2)}{q^4}A(q^2;0,q^2)-k\mathcal{P}(p)k\frac{Z(p^2)G(q^2)}{k^2p^2q^2}A(p^2;k^2,q^2)\right\}. \tag{36}$$

While the ultraviolet subtraction is rather simple with $k^2\to 0$ herein, without this subtraction, a naive infrared analysis will be aggravated by the ultraviolet divergences. Thus the safe order of formal steps is to perform the ultraviolet subtraction before the infrared analysis in this case. The opposite order applies for the gluon DSE.

(ii) In DSEs for propagators of massive degrees of freedom or even infrared-vanishing correlations, the explicit ultraviolet subtraction is subleading in the infrared, and it cannot simply be extracted from the limit $k^2\to 0$. This should certainly be the case for the transverse gluon correlations. The least we expect as a necessary condition for confinement is the mass gap. The horizon condition implies an even stronger infrared singularity, as mentioned in the Introduction. In either case we have $Z^{-1}(k^2)\to\infty$ for $k^2\to 0$ for the lhs in DSEs such as Eq. (32) or (33). The advantage is that the coefficients of the divergent terms in an asymptotic infrared expansion can be extracted without bothering with ultraviolet subtractions in the first place, since ultraviolet divergences will only occur at the subleading constant level in this expansion. To be specific, inside the ghost loop, the infrared enhanced asymptotic forms of ghost propagators (together the leading behavior of the ghost-gluon vertex) will converge in the ultraviolet. Here the problem rather is to extract the necessary ultraviolet subtraction without introducing by hand spurious infrared divergences. This problem had already been dealt with in Refs. [23] and [22]. What generally needs to be done in such a case is to reverse the order of (i) above and to isolate the infrared divergent contributions to the gluon DSE on both sides prior to the ultraviolet subtraction, in order to define an infrared finite renormalization constant Z_3.

Assume the coefficients in the infrared divergent terms of the asymptotic expansions for the ghost propagator and the vertex are known as well as the corresponding asymptotic forms denoted by $G^{ir}(q^2)$ and $A^{ir}(k^2;p^2,q^2)$. We can then, e.g., in Eq. (32) subtract on both sides an ultraviolet finite contribution of the form,

$$\frac{1}{Z_P^{ir}(k^2)}\equiv\frac{g^2N_c}{3}\int\frac{d^4q}{(2\pi)^4}\frac{q\mathcal{P}(k)q}{k^2p^2q^2}\times G^{ir}(p^2)G^{ir}(q^2)A^{ir}(k^2;q^2,p^2), \tag{37}$$

which we will actually use to determine $Z_P^{ir}(k^2)$ below, or an analogous expression for $Z_R^{ir}(k^2)$ from Eq. (33). In both cases, this infrared subtraction in the gluon DSE has to be performed up to the necessary order such that

$$\frac{1}{Z(k^2)} - \frac{1}{Z^{ir}(k^2)} \to C_A = \text{const} \quad \text{for} \quad k^2 \to 0. \tag{38}$$

On the other hand, the k-independent constant contribution to the rhs of the DSE is the one that contains the (overall) logarithmic divergence which is absorbed in the renormalization constant Z_3. To extract that from the infrared subtracted gluon DSE, analogous to Eq. (35) from the ghost DSE in (i) above, we first rewrite the gluon DSE (32), adding a zero via the definition in Eq. (37),

$$\begin{aligned} Z_3 = {} & \frac{1}{Z_P(k^2)} - \frac{1}{Z_P^{ir}(k^2)} - \frac{g^2\mu^{4-D}N_c}{D-1}\int \frac{d^Dq}{(2\pi)^D} \\ & \times \frac{q\mathcal{P}(k)q}{k^2p^2q^2}\big(G(p^2)G(q^2)A(k^2;q^2,p^2) \\ & - G^{ir}(p^2)G^{ir}(q^2)A^{ir}(k^2;q^2,p^2)\big) + \cdots . \end{aligned} \tag{39}$$

A further technical complication arises from the additional power of q^2/k^2 which prevents us from taking the naive limit $k^2 \to 0$ herein. This problem is due to contributions which superficially contain quadratic ultraviolet divergences. As also the tadpole, for example, these are of the order $\sim 1/k^2$ in the DSE for Z_P^{-1}. If $\kappa > 1/2$ they occur at a subleading order in the infrared expansion. Then, this problem is irrelevant for our present study which is concerned about the leading order only. Moreover, the sum of all quadratically divergent contributions must vanish in the gluon DSE from gauge invariance. This cannot be seen from the ghost loop alone, however. All such contributions would have to be maintained to see this cancellation explicitly before one can let $k^2 \to 0$ in Eq. (39) to extract the logarithmically divergent contribution that defines the ultraviolet renormalization constant Z_3. In particular, this is necessary when the transversely projected gluon DSE for Z_P is used beyond the infrared analysis presented here. For the $\mathcal{R}$ tensor, leading to Z_R via Eq. (17) which is free from quadratically divergent contributions [48], and with the generalized tree-level vertex $G_\mu = G_\mu^0$, to give an example without such complication, one readily verifies that

$$\begin{aligned} Z_3 = {} & -\left(\frac{2(D-2)}{D(D+2)} - \eta\hat{\eta}\right) g^2\mu^{4-D}N_c \int \frac{d^Dq}{(2\pi)^D} \\ & \times \frac{1}{q^4}\{G^2(q^2) - [G^{ir}(q^2)]^2\} + \cdots , \end{aligned} \tag{40}$$

from $k^2 \to 0$ in Eq. (33) with the analogue of (37) for the $\mathcal{R}$ tensor. It is infrared finite by construction, and it gives the correct perturbative ghost-loop contribution to the gluon renormalization constant Z_3, ultraviolet divergent for $D \to 4$. Contrary to the ghost DSE renormalization in Eq. (35), an additive constant is possible in Eq. (40) and is included in the terms not given explicitly here. It can be used to adjust the constant C_A in the gluon DSE which is subleading in the infrared, cf., Eq. (38).

To summarize, it should be possible to impose renormalization conditions on the full propagators to equal the tree level ones at an arbitrary (space-like) subtraction point $k^2 = \mu^2 > 0$. To do this, we have two independent conditions at our disposal which fix the physically insignificant overall factors in each of the two propagators. For massless tree-level propagators, we cannot without loss of generality extend this subtraction scheme to include $k^2 = \mu^2 = 0$, however, since that forced the full propagators to have the same massless single-particle singularity. Here, the necessity of a mass gap in the transverse gluon correlations entails that $1/Z(k^2)$ is infrared divergent, and it is thus impossible to fix a multiplicative factor by a subtraction at $\mu = 0$ requiring that $1/Z(0)$ be unity (or any other finite value). We can, however, fix this factor by assigning the infrared subtracted $1/Z(k^2) - 1/Z^{ir}(k^2)$ at $k^2 = 0$ a certain value C_A.

That sets one of the two conditions available. Of course, the same argument applies to the ghost propagator at $\mu = 0$. In this case, it is because both the Kugo-Ojima criterion and the horizon condition tell us that the full ghost propagator should not have the singularity structure of the free-massless tree-level one. In particular, with $1/G(k^2) \to 0$ we cannot fix the overall factor by subtracting G^{-1} at zero. As mentioned above, this case is different in that a nonvanishing constant contribution to the ghost DSE would be infrared dominant and cannot occur together with the infrared enhanced ghost correlations. To fix the multiplicative factor in the ghost propagator implicitly, we use the second of the two independent renormalization conditions on the product of both propagators,

$$G^2(\mu^2)Z(\mu^2) \overset{!}{=} 1, \tag{41}$$

which can be used to define a nonperturbative running coupling in the Landau gauge as we discuss next.

III. RENORMALIZATION INDEPENDENT INFRARED ANALYSIS

A. Infrared expansion and renormalization group

Herein, we adopt the nonperturbative renormalization scheme introduced in Refs. [23] and [22]. To review this scheme briefly, recall that the formal solutions to the renormalization group equations for the gluon and the ghost propagator, e.g., for the latter this is Eq. (A1) in Appendix A, can be written in the general forms,

$$Z(k^2) = \exp\left\{-2\int_g^{\bar{g}(t_k,g)} dl \frac{\gamma_A(l)}{\beta(l)}\right\} f_A[\bar{g}(t_k,g)], \tag{42}$$

$$G(k^2) = \exp\left\{-2\int_g^{\bar{g}(t_k,g)} dl \frac{\gamma_G(l)}{\beta(l)}\right\} f_G[\bar{g}(t_k,g)], \tag{43}$$

respectively. Here, $t_k = (\ln k^2/\mu^2)/2$, and $\overline{g}(t,g)$ is the running coupling, the solution of $d/dt\overline{g}(t,g) = \beta(\overline{g})$ with $\overline{g}(0,g) = g$ and the Callan-Symanzik β function, perturbatively, $\beta(g) = -\beta_0 g^3 + \mathcal{O}(g^5)$. The exponential factors are the multiplicative constants for finite renormalization group transformations $(\mu \rightarrow \mu')$,

$$\mathcal{Z}_3(\mu',\mu) = \exp\left\{ -2\int_g^{g'} dl \frac{\gamma_A(l)}{\beta(l)} \right\}, \tag{44}$$

$$\widetilde{\mathcal{Z}}_3(\mu',\mu) = \exp\left\{ -2\int_g^{g'} dl \frac{\gamma_G(l)}{\beta(l)} \right\}, \tag{45}$$

with $\mu'^2 = k^2$ in Eqs. (42) and (43), where $\gamma_A(g)$ and $\gamma_G(g)$ are the anomalous dimensions of the gluon and the ghost fields, respectively.

The structure of Eqs. (42) and (43) is summarized as follows. The momentum dependence of the propagator functions $Z(k^2)$ and $G(k^2)$ is completely determined by the running coupling evaluated at k^2, which is renormalization group invariant, i.e., μ independent, since $(d/d\ln\mu)\overline{g}(t_k,g) = [\mu\partial/\partial\mu + \beta(g)\partial/\partial g]\overline{g}(t_k,g) = 0$. We can therefore parametrize this momentum dependence by a function of the ratio of k^2 over a renormalization group invariant, dynamically generated momentum scale $\sigma \propto \Lambda^2_{\mathrm{QCD}}$.

The μ dependence of the propagators, on the other hand, is then given only by the $g \equiv g(\mu)$ of the lower bound in the exponential renormalization factors. We can therefore always separate these two dependences, that on (g,μ) versus that on k^2/σ, in Eqs. (42) and (43) into multiplicative factors by conveniently choosing a g_0 such that $\mu^2 = \sigma$ at $g = g_0$,

$$\sigma = \mu^2 \exp\left\{ -2\int_{g_0}^{g} \frac{dl}{\beta(l)} \right\}, \tag{46}$$

which, at the same time, defines σ to be a renormalization group invariant momentum scale as promised.

Via this factorization of the (g,μ) dependence, we can now make the *Ansatz* that the propagator functions Z and G have asymptotic infrared expansions to some order N in terms of k^2/σ involving renormalization group invariant exponents and coefficients,

$$Z(k^2) \rightarrow \exp\left\{ -2\int_g^{g_0} dl \frac{\gamma_A(l)}{\beta(l)} \right\} \sum_n^N e_n \left(\frac{k^2}{\sigma} \right)^{\epsilon_n}, \tag{47}$$

$$G(k^2) \rightarrow \exp\left\{ -2\int_g^{g_0} dl \frac{\gamma_G(l)}{\beta(l)} \right\} \sum_n^N d_n \left(\frac{k^2}{\sigma} \right)^{\delta_n}, \tag{48}$$

for $k^2/\sigma \rightarrow 0$. Here we use a notation similar to that introduced in [54]. We note, however, that our expansion involves the RG invariant scale σ while the renormalization scale μ was used in [54]. The difference can be absorbed in a redefinition of the coefficients e_n, d_n as we explain in Appendix A. Most importantly, this implies that our coefficients e_n, d_n are also (g,μ) independent.

The nonrenormalization **(N)** of the ghost-gluon vertex in the infrared, cf., $\widetilde{Z}_1 = Z_g Z_3^{1/2} \widetilde{Z}_3 = 1$ which was derived, in particular, for a symmetric subtraction scheme $k^2 = p^2 = q^2 = \mu^2$ with $\mu^2 \rightarrow 0$ [50], now entails for the renormalization factors in Eqs. (44) and (45) that

$$\mathcal{Z}_3^{1/2}(\mu',\mu)\widetilde{\mathcal{Z}}_3(\mu',\mu) = \frac{g'}{g} \equiv \frac{\overline{g}(t,g)}{g}, \quad t = \ln\frac{\mu'}{\mu}, \tag{49}$$

in Landau the gauge, which is equivalent to

$$2\gamma_G(g) + \gamma_A(g) = -\frac{1}{g}\beta(g). \tag{50}$$

This is, in fact, what allows one to define a nonperturbative running coupling as introduced in Refs. [23] and [22] by

$$g^2 Z(\mu'^2) G^2(\mu'^2) \overset{!}{=} g'^2 = \overline{g}^2[\ln(\mu'/\mu), g]. \tag{51}$$

It reduces to the unique perturbative definition for large μ, μ', is renormalization group invariant, dimensionless, and thus as good as any nonperturbative definition can be. The fact that no constant of proportionality is involved in Eq. (51) implies a specific renormalization condition. It corresponds to requiring the condition on the propagators,

$$Z(\mu^2) = f_A(g), \quad G(\mu^2) = f_G(g) \quad \text{with} \quad f_G^2 f_A = 1, \tag{52}$$

which is incomplete, of course, to fix both their values separately at an arbitrary $k^2 = \mu^2$. The perturbative limits are, however, $f_{A,G} \rightarrow 1, g \rightarrow 0$, corresponding to the perturbative momentum subtraction scheme,

$$Z(\mu^2) = 1 \quad \text{and} \quad G(\mu^2) = 1 \tag{53}$$

for an asymptotically large subtraction point $k^2 = \mu^2$.

With this so-defined running coupling, by Eq. (51), the existence of an infrared fixed point, $\overline{g}(t,g) \rightarrow g_c$ finite for $t \rightarrow -\infty$, then follows in the Landau gauge to be in one-to-one correspondence with the scaling law for the leading infrared exponents of gluon and ghost propagation in the form (for $D = 4$),

$$\epsilon_0 + 2\delta_0 = 0. \tag{54}$$

To make this explicit, consider the leading infrared behavior from Eqs. (47) and (48), with the exponential factors therein expressed by Eqs. (44) and (45), for $k^2 \rightarrow 0$,

$$Z(k^2) \rightarrow \mathcal{Z}_3(\sqrt{\sigma},\mu) e_0 \left(\frac{k^2}{\sigma} \right)^{\epsilon_0}, \tag{55}$$

$$G(k^2) \rightarrow \widetilde{\mathcal{Z}}_3(\sqrt{\sigma},\mu) d_0 \left(\frac{k^2}{\sigma} \right)^{\delta_0}, \tag{56}$$

which, from Eq. (49), entails that the infrared behavior of the running coupling $\alpha(k^2) \equiv \overline{g}^2(t_k,g)/(4\pi)$ is given by

$$\alpha(k^2) \equiv \frac{g^2}{4\pi} Z(k^2) G^2(k^2) \to \frac{g_0^2 e_0 d_0^2}{4\pi} \left(\frac{k^2}{\sigma}\right)^{\epsilon_0 + 2\delta_0}. \quad (57)$$

We therefore introduce $\alpha_c = g_0^2 e_0 d_0^2/(4\pi)$ in the following. It represents the infrared fixed point, $\alpha(k^2) \to \alpha_c$ for $k^2 \to 0$, which occurs exactly if Eq. (54) holds.

The infrared scaling behavior in Eq. (54) was first observed in the solutions to truncated DSEs in Refs. [22] and [23]. It was recently derived from the ghost DSE in Ref. [54]. Therein, an additional assumption on the vertex was used which is a bit *ad hoc* and which is actually not necessary. In Sec. III C below, we therefore present an alternative derivation of the infrared behavior (54), from Ref. [47], which is based on the nonrenormalization, condition **(N1)** for the vertex alone.

B. Vertex *Ansatz*

All we need in our infrared analysis is an *Ansatz* for the invariant function A which parametrizes the relevant structure of the ghost-gluon vertex in the Landau gauge. Anticipating a conformal behavior in the infrared also for the vertex, we first write,

$$A^{ir}(k^2;p^2,q^2) = \left(\frac{k^2}{\sigma}\right)^n \left(\frac{p^2}{\sigma}\right)^m \left(\frac{q^2}{\sigma}\right)^l. \quad (58)$$

The nonrenormalization condition **(N1)** for the Landau-gauge vertex then leads to the constraint,

$$l + m + n = 0, \quad (59)$$

which will be implemented in our analyses throughout. We will later also allow sums of terms of this form in order to explore versions of this *Ansatz* which are symmetrized with respect to the ghost legs.

The scaling law for the infrared propagators in Eq. (54) then follows for such a sum of terms (58) via Eq. (59) only from **(N1)**, as we shall show in the next section.

If we require, in addition, that $A(k^2;p^2,q^2)$ remain finite when one of the ghost momenta vanishes, one of the terms of the form (58) must exist in the sum with either

$$m = 0, \quad l = -n, \quad \text{or} \quad l = 0, \quad m = -n. \quad (60)$$

All other possible terms must then vanish and thus have $m > 0$ or $l > 0$, respectively; and if the finite contribution to A in that limit was to be in itself symmetric with respect to the two ghost momenta, one would only be left with $A^{ir} \equiv 1$ as in the tree-level vertex, since then $l = m = 0$. In a ghost-antighost symmetric sum of two terms on the other hand, we need one of each kind together with $n < 0$ to avoid infrared divergent ghost legs.

After the infrared scaling (54) and the general formulas for the infrared contributions of terms of the genuine form Eqs. (58) and (59) will be derived, we will assume relations as in Eq. (60), in addition. The joint infrared exponent κ for ghosts and gluons then is a function of a single critical exponent n which is left as an open parameter in their vertex. To exemplify its influence, we will explicitly calculate this function for the following three *Ansätze*:

$$\text{(i)} \quad A^{ir}(k^2;p^2,q^2) = \left(\frac{k^2}{q^2}\right)^n, \quad (61)$$

$$\text{(ii)} \quad A^{ir}(k^2;p^2,q^2) = \frac{1}{2}\left[\left(\frac{k^2}{q^2}\right)^n + \left(\frac{k^2}{p^2}\right)^n\right], \quad (62)$$

$$\text{(iii)} \quad A^{ir}(k^2;p^2,q^2) = \left(\frac{k^2}{q^2}\right)^n + \left(\frac{k^2}{p^2}\right)^n - 1. \quad (63)$$

In the form (i) of Eq. (61) the *Ansatz* does not bother about ghost-antighost symmetry. For $n = \delta_0 \equiv -\kappa$, this form contains the infrared behavior of the nonsymmetric solution (28) to the truncated STI (27) for the vertex of Ref. [53] as a special case. In both symmetrized versions (ii) and (iii) one furthermore has $n \leqslant 0$ if infrared divergences associated with the ghost legs are to be avoided in the ghost-gluon vertex function. Version (ii) in Eq. (62) yields a finite and constant $A(q^2;0,q^2) = 1/2$, while (iii) is an example for the possibility that $A(q^2;0,q^2) = 0$. Version (iii) in Eq. (63) for $n = \delta_0 \equiv -\kappa$ includes the behavior of the symmetric solution (29) to the STI (27) with $f^T \equiv 0$ as obtained in Ref. [22]. All three versions satisfy **(N1)**, of course, and they all reduce to the tree-level vertex at $n = 0$.

C. Unique infrared exponent from ghost DSE

With the general form of our *Ansatz* (58) for the relevant part of the ghost-gluon vertex, we can now extract the leading contributions in the infrared on both sides of the ghost DSE (36). Here, with Z^{ir} and G^{ir} denoting the leading infrared behavior of the propagators as given in Eqs. (55) and (56), respectively, we first note that

$$Z^{ir}(p^2) G^{ir}(q^2) = \frac{g_0^2}{g^2} \frac{e_0 d_0}{\tilde{\tilde{Z}}_3(\sqrt{\sigma},\mu)} \left(\frac{p^2}{\sigma}\right)^{\epsilon_0} \left(\frac{q^2}{\sigma}\right)^{\delta_0}, \quad (64)$$

where use has been made also of Eq. (49) for $\mu' = \sqrt{\sigma}$, $g' = g_0$. The lhs of the ghost DSE (36) approaches, for $k^2 \to 0$,

$$G^{-1}(k^2) \to \tilde{\tilde{Z}}_3^{-1}(\sqrt{\sigma},\mu) d_0^{-1} \left(\frac{k^2}{\sigma}\right)^{-\delta_0}. \quad (65)$$

To obtain the leading behavior at small k^2/σ of the rhs in Eq. (36), we replace the undetermined functions in the integrand of Eq. (36) by the from given in Eq. (64) and the *Ansatz* (58) for the leading infrared behavior of the vertex. The difference in the integrand between the full functions, Z, G, and A, and their asymptotic infrared forms, Z^{ir}, G^{ir}, and A^{ir}, is subleading and it produces, upon integration, terms that are also subleading in an expansion of the rhs of the DSE. This procedure is not restricted to the leading infrared behavior. It can straightforwardly be generalized for an infrared expansion to a given order, as long as all integrals in this expansion remain finite. This is true for the leading infrared forms as discussed above. When these are inserted, the integral in Eq.

(36) converges in $D=4$ dimensions. The role of the first term in Eq. (36), however, is to just guarantee convergence of all necessary integrals at exactly $D=4$. It is certainly necessary for the subtraction of the ultraviolet divergences when the full functions with their logarithmic momentum dependences are inserted. Here we only need this term if we insist on a calculation in $D=4$ involving convergent integrals in every step. We can, however, obtain the same result by analytic continuation of integrals performed in D dimensions. Though this is of course precisely along the rules of dimensional regularization, this notion might be misleading here, since nothing is there to be regularized in the first place. It is adopted here for convenience only. Upon insertion of the leading infrared forms (64) and (58), the first term in Eq. (36), in D dimensions, then yields a contribution to the ghost DSE proportional to

$$\left(\frac{k^2}{\sigma}\right)^m \int \frac{d^D q}{(2\pi)^D} \left(\frac{q^2}{\sigma}\right)^{\epsilon_0+\delta_0+l+n-2} \tag{66}$$

which vanishes for any D in the analytic definition.

Therefore with Eqs. (64) and (58) in the integral for the relevant part of the rhs in D dimensions, and with Eq. (65) for the lhs, the leading contributions to both sides of the ghost DSE (36) for $k^2 \to 0$ are readily extracted to yield,

$$\begin{aligned}\left(\frac{k^2}{\sigma}\right)^{-\delta_0} =& -N_c 4\pi\alpha_c \left(\frac{k^2}{\sigma}\right)^{\epsilon_0+\delta_0+D/2-2+l+m+n} \\ &\times \int \frac{d^D q}{(2\pi)^D} \left(\frac{1}{q^2}\right)^{D/2} \frac{k\mathcal{P}(p)k}{k^2} \left(\frac{q^2}{k^2}\right)^{D/2-1+\delta_0+l} \\ &\times \left(\frac{k^2}{p^2}\right)^{1-\epsilon_0-n}.\end{aligned} \tag{67}$$

Here, $p=k\pm q$ again, and we have $4\pi\alpha_c = g_0^2 e_0 d_0^2$ as introduced in Eq. (57) of Sec. III A; and just as for the usual replacement $g\to g\mu^{2-D/2}$, the dimension of the coupling for $D\neq 4$ has been taken care of in Eq. (67) by replacing $g_0^2 \to g_0^2\sigma^{2-D/2}$. The resulting explicit exponent of the scale can be combined with the corresponding exponent of an extra external momentum factor into $(k^2/\sigma)^{D/2-2}$ which was added to the total exponent of this ratio on the rhs of Eq. (67). The dimensionless integral therein is written in a form ready to apply the integration formula (B1) of Appendix B. The result is k independent as we will see explicitly in the next section. Therefore from the nonrenormalization of the vertex, condition **(N1)** which implies $l+m+n=0$, Eq. (67) for the leading infrared contributions to both sides of the ghost DSE entails that

$$\epsilon_0 + 2\delta_0 = 2 - D/2. \tag{68}$$

Thus the infrared behavior of the running coupling from Eq. (57) equivalently follows to be of the form,

$$\alpha(k^2) \to \alpha_c \left(\frac{k^2}{\sigma}\right)^{2-D/2}, \quad \text{for} \quad k^2 \to 0, \tag{69}$$

which is infrared finite for $D=4$. From now on, we therefore parametrize the leading infrared exponents of ghost and gluon propagation by the joint exponent κ again, in D dimensions,

$$\delta_0 = -\kappa, \quad \epsilon_0 = 2\kappa + 2 - D/2. \tag{70}$$

Infrared enhancement of ghosts follows for exponents $0<\kappa<(D-2)/2$ with the upper bound from the temperedness of local fields. In addition, $(D-2)/4\leqslant\kappa$ for the mass gap in transverse gluon correlations. Together, this then establishes the Kugo-Ojima confinement criterion in the Landau gauge. Zwanziger's horizon condition furthermore excludes equality at the lower bound. We thus expect solutions for

$$(D-2)/4 < \kappa < (D-2)/2. \tag{71}$$

Once this exponent is determined, depending on the infrared exponents of the ghost-gluon vertex, from Eq. (67) we furthermore obtain

$$\alpha_c = \frac{2^{D-2}\pi^{D/2-1}}{N_c I_G^{(D)}(\kappa,l,n)}, \tag{72}$$

where $I_G^{(D)}(\kappa,l,n)$ is a ratio of Γ functions proportional to the integral in Eq. (67) which we determine next.

IV. INFRARED EXPONENT FOR GHOST-GLUON SYSTEM, RESULTS

The same procedure that led to Eq. (67) in the ghost case can be applied to the gluon DSE. In this case, we simply insert the leading infrared behavior of the propagators from Eqs. (55) and (56) and the vertex (58) directly into Eq. (37). In D dimensions, this then analogously leads to

$$\left(\frac{k^2}{\sigma}\right)^{-\epsilon_0} = N_c 4\pi\alpha_c \left(\frac{k^2}{\sigma}\right)^{2\delta_0+D/2-2} \frac{I_{Z_P}^{(D)}(\kappa,l,n)}{2^D\pi^{D/2}}, \tag{73}$$

where we used $l+m+n=0$, and again, we conclude the relation for the exponents in Eq. (68). A comparison of Eqs. (72) and (73) furthermore tells us that

$$I_G^{(D)}(\kappa,l,n) \overset{!}{=} I_{Z_P}^{(D)}(\kappa,l,n), \tag{74}$$

which determines the values allowed for the exponents n, l, and κ. The two dimensionless integrals in the infrared expansions of the DSEs, $I_G^{(D)}$ for the ghost and $I_{Z_P}^{(D)}$ for the transverse ghost-loop contribution to the gluon propagator, are explicitly given by

$$\frac{I_G^{(D)}(\kappa,l,n)}{2^D\pi^{D/2}}=-\int\frac{d^Dq}{(2\pi)^D}\left(\frac{1}{q^2}\right)^{D/2}\frac{k\mathcal{P}(p)k}{k^2}\times\left(\frac{q^2}{k^2}\right)^{D/2-1-\kappa+l}\left(\frac{k^2}{p^2}\right)^{D/2-1-2\kappa-n}, \quad (75)$$

$$\frac{I_{Z_P}^{(D)}(\kappa,l,n)}{2^D\pi^{D/2}}=\int\frac{d^Dq}{(2\pi)^D}\left(\frac{1}{q^2}\right)^{D/2}\frac{q\mathcal{P}(k)q}{(D-1)k^2}\times\left(\frac{q^2}{k^2}\right)^{D/2-1-\kappa-n-l}\left(\frac{k^2}{p^2}\right)^{1+\kappa-l}. \quad (76)$$

To compute these, we first note that

$$\frac{k\mathcal{P}(p)k}{k^2}=\frac{1}{2}\left(1+\frac{q^2}{p^2}+\frac{q^2}{k^2}\right)-\frac{1}{4}\left(\frac{k^2}{p^2}+\frac{p^2}{k^2}+\frac{q^4}{k^2p^2}\right),$$

$$\frac{q\mathcal{P}(k)q}{k^2}=\frac{1}{2}\left(\frac{q^2}{k^2}+\frac{p^2}{k^2}+\frac{p^2q^2}{k^4}\right)-\frac{1}{4}\left(1+\frac{q^4}{k^4}+\frac{p^4}{k^4}\right).$$

With these in Eqs. (75) and (76), respectively, it is now straightforward to apply the formula in Eq. (B1) repeatedly with suitable substitutions for the exponents α and β. For each of the two integrals this leads to a sum of six ratios of Γ functions with arguments that differ by integer values corresponding to the above six terms with different powers of the momenta in each of the two integrals. After some tedious applications of the Γ's functional identity, these six ratios can be combined into a single one, to the effect that,

$$I_G^{(D)}(\kappa,l,n)=-\frac{D-1}{2}\frac{\Gamma\left(\frac{D}{2}-\kappa+l\right)\Gamma(1+2\kappa+n)\Gamma(-\kappa-l-n)}{\Gamma\left(\frac{D}{2}-2\kappa-n\right)\Gamma(1+\kappa-l)\Gamma\left(\frac{D}{2}+1+\kappa+l+n\right)}, \quad (77)$$

$$I_{Z_P}^{(D)}(\kappa,l,n)=\frac{1}{2}\frac{\Gamma\left(\frac{D}{2}-\kappa+l\right)\Gamma\left(1-\frac{D}{2}+2\kappa+n\right)\Gamma\left(\frac{D}{2}-\kappa-l-n\right)}{\Gamma(D-2\kappa-n)\Gamma(1+\kappa-l)\Gamma(1+\kappa+l+n)}. \quad (78)$$

Though these quite general results might look complicated at first, in fact, they are surprisingly simple. To appreciate this, consider the following special case first.

A. Tree-level ghost-gluon vertex

In this section we concentrate on the results in the event that the ghost-gluon vertex reduces to its tree-level form in the infrared, $G_\mu^{ir}=G_\mu^0$. This corresponds to $A^{ir}=1$ and, for Z_R, the generalized form with $B^{ir}=\hat{\eta}$. We thus set $l=n=0$ in the general result of Eqs. (77) and (78), and omit the arguments l and n here. In $D=4$ dimensions, this amounts to

$$I_G^{(4)}(\kappa)=-\frac{3}{2}\frac{\Gamma^2(-\kappa)\Gamma(2\kappa-1)}{\Gamma(-2\kappa)\Gamma(\kappa-1)\Gamma(\kappa+3)}, \quad (79)$$

$$I_{Z_P}^{(4)}(\kappa)=\frac{1}{2}\frac{\Gamma^2(-\kappa)\Gamma(2\kappa-1)}{\Gamma^2(\kappa-1)\Gamma(4-2\kappa)}. \quad (80)$$

The result in Eq. (79) then agrees with both versions given for the tree-level vertex in Ref. [46]. There, another method was employed leading to a rather complicated form which consists of sums of confluent hypergeometric functions and which is quite difficult to simplify any further. We explain this method and the connection to ours in Appendix B, cf., Eq. (B1) versus Eq. (B7), in particular. The two different forms for the result in Ref. [46] thereby arose from either choosing the internal gluon or the ghost momentum as the integration variable in the loop. Here, both give the same in the first place. The symmetry $p^2\leftrightarrow q^2$ in the integrand on the lhs of Eq. (B1) is manifest on the rhs by its explicit invariance under $\alpha\to\alpha'=D/2-\beta$ together with $\beta\to\beta'=D/2-\alpha$.

The self-consistency condition (74), for both DSEs to yield the same value of the constant α_c, is also implemented quite easily in Eqs. (79) and (80). In this case, for the tree-level vertex, one derives the condition,

$$12(3-2\kappa)(2\kappa-1)=(\kappa+2)(\kappa+1). \quad (81)$$

This is quadratic in κ, and the two possibilities are

$$\kappa=\frac{1}{98}(93\mp\sqrt{1201})\approx\{\underline{0.59535},1.3026\}, \quad (82)$$

with one unique root in $0<\kappa_1<1$ which we have underlined in Eq. (82). This result was first obtained independently in Refs. [47] and [35]. The corresponding value of α_c is given by

$$\alpha_c=\frac{4\pi}{N_cI_G^{(4)}(\kappa_1)}\approx 2.9717 \quad \text{for} \quad N_c=3, \quad (83)$$

with $I_G^{(4)}(\kappa_1)=I_{Z_P}^{(4)}(\kappa_1)\approx 1.4096$. It is smaller by this last factor than the value of $\alpha_c=4\pi/N_c$ derived from $I_G^{(4)}(\kappa)=1$ for $\kappa=1$ with the tree-level vertex (and also $D=4$) in Ref. [46]. This disagreement is due to the difference between the transverse ghost-loop contribution Z_P employed here, and the $\mathcal{R}$-contracted Z_R in Ref. [46]. Our calculation for Z_R, see below, with $A=1$ and $B=\hat{\eta}$ for the generalized tree-level vertex leads to

$$I_{Z_R}^{(4)}(\kappa)=(4\kappa-2)\left(1-2\eta\hat{\eta}\frac{(2\kappa-3)}{(\kappa-1)}\right)I_{Z_P}^{(4)}(\kappa). \tag{84}$$

For $\eta=1$ or 0 this is equivalent to the sum of 12 confluent hypergeometric functions given as the result for the same integral in Ref. [46]. The $D=4$ results for the tree-level ghost-gluon vertex are summarized in Fig. 3. In standard Faddeev-Popov theory $\eta=1$, and transversality of the ghost loop, $I_{Z_R}^{(4)}=I_{Z_P}^{(4)}$, required $\kappa=3/4$. In order to tune Z_R for transversality at the self-consistent value of the tree-level exponent κ_1, on the other hand, we would need $\eta\approx 1.16$ or -0.16 which appear to be rather unnatural. It is not possible in the ghost-antighost symmetric formulation.

Another important difference between $I_{Z_P}^{(4)}$ and $I_{Z_R}^{(4)}$ for the tree-level vertex, as seen in Fig. 3 and Eq. (84), is the observation that $I_{Z_P}^{(4)}(\kappa)$ has a pole at $\kappa=1/2$. The gluon propagator then necessarily vanishes in the infrared: If it was to approach a constant, one had to have $\kappa=1/2$. In this case, however, its constant limit was proportional to $1/I_{Z_P}^{(4)}(\kappa)$ which vanishes for $\kappa\to 1/2$. One thus obtains the strict lower bound $1/2<\kappa$ for the tree-level case. Zwanziger's horizon condition is then satisfied. The apparently infrared finite extrapolations from lattice calculations are an open question still, however.

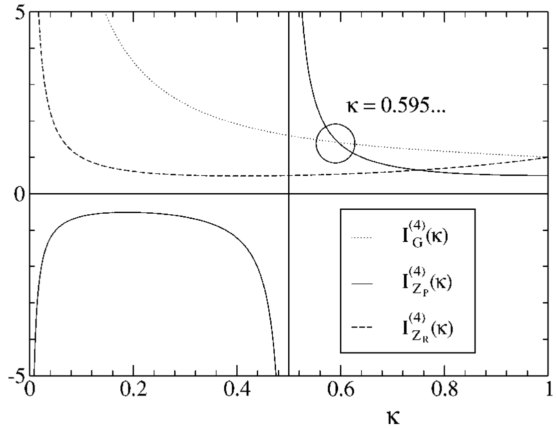

FIG. 3. The infrared integrals $I_G^{(D)}$, $I_{Z_P}^{(D)}$, and $I_{Z_R}^{(D)}$ in $D=4$ dimensions with the tree-level ghost-gluon vertex, cf., Eqs. (79) and (80) and the $\eta=1$ case from Eq. (84), respectively. The corresponding value for the exponent $\kappa_1\approx 0.595$ is obtained from the intersection point of I_G and I_{Z_P} as marked by the circle.

B. Infrared transversality

This completes our presentation for the tree-level vertex. Before we discuss more general possibilities, in particular, the cases (i)–(iii) in Sec. III B, we study the constraints from transversality of the gluon propagator on the vertex in the infrared in this section.

For the $\mathcal{R}$-contracted infrared contribution of the ghost-loop in the gluon DSE, Eq. (33), we must specify a form for the second, the longitudinal B structure of the vertex, in addition. The fact that the leading infrared behavior of the propagators should not depend on the choice of studying Z_P or Z_R can be used to construct an infrared form of $B(k^2,p^2,q^2)$ analogous A^{ir} in Eq. (58).

First, we express the integrand in Eq. (33) for Z_R in terms of that for Z_P in Eq. (32) plus a correction term which, for a given *Ansatz* A^{ir}, is required to vanish (at least upon integration). For the four terms in the curly brackets in Eq. (33), this leads to

$$\begin{aligned}\{\cdots\}=&\,q\mathcal{P}(k)qA(k^2;q^2,p^2)-(D-1)\\&\times(\hat{\eta}k^2+qk)\left[\left(1+\frac{qk}{k^2}\right)A(k^2;q^2,p^2)\right.\\&\left.-B(k^2;q^2,p^2)\right].\end{aligned} \tag{85}$$

The first term on the rhs herein reproduces the result for $I_{Z_P}^{(D)}(\kappa,l,n)$ from Eq. (76). With tree-level settings $A=1$, $B=\hat{\eta}$ one readily verifies Eq. (84).

The difference between the infrared integrals, $\Delta^{(D)}\equiv I_{Z_P}^{(D)}-I_{Z_R}^{(D)}$, can then be written,

$$\begin{aligned}\frac{\Delta^{(D)}}{2^D\pi^{D/2}}=&\int\frac{d^Dq}{(2\pi)^D}\left(\frac{1}{q^2}\right)^{D/2}\left(\frac{q^2}{k^2}\right)^{D/2-1-\kappa}\left(\frac{k^2}{p^2}\right)^{1+\kappa}\\&\times\frac{1}{4}\left(\hat{\eta}-\eta+\frac{p^2}{k^2}-\frac{q^2}{k^2}\right)\\&\times\left[\left(1+\frac{p^2}{k^2}-\frac{q^2}{k^2}\right)A^{ir}-2B^{ir}\right].\end{aligned} \tag{86}$$

The order of the arguments in A^{ir} and B^{ir} is the same here as that in Eq. (85) above. The ghost-loop integration projects onto terms that are overall symmetric in $p^2\leftrightarrow q^2$. The antisymmetric ones vanish upon integration. We use the symmetry **(S1)** of A and the decomposition $B=B_++B_-$ into (anti)symmetric parts $B_\pm(x;y,z)=\pm B_\pm(x;z,y)$ for B again, and sort out the symmetric part of the integrand in Eq. (86). It vanishes if

$$(\hat{\eta}-\eta)A^{ir}(k^2;q^2,p^2)=(\hat{\eta}-\eta)2B_+^{ir}(k^2;q^2,p^2),$$

and

$$\frac{p^2-q^2}{2k^2}A^{ir}(k^2;q^2,p^2)=B_-^{ir}(k^2;q^2,p^2). \tag{87}$$

The first condition shows that $2B_+^{ir}=A^{ir}$ for $\hat{\eta}\neq\eta$. In the symmetric case, $\hat{\eta}=\eta=1/2$, no such restriction is implied here, but the same relation is then given by **(S2)**, see Sec. II A above. Therefore Eqs. (24) and (25) of Sec. II A follow as sufficient conditions for infrared transversality independent of η. Without turning them into necessary conditions this is a rather trivial result. As we showed in Sec. II A, the conditions in Eqs. (24) and (25) imply that the vertex itself is transverse in the infrared. This is always a possibility to warrant infrared transversality, however. The necessary condition is $\Delta^{(D)}=0$.

The point here is to demonstrate that, apart from possible accidental cancellations, e.g., with tuning $\eta=1.16$ for the tree-level case of the last section, for general η, there are really no possibilities left other than infrared transversality of the vertex itself. In particular, we should be allowed to choose η such as $\eta=1$ for the standard linear-covariant or $\eta=1/2$ for the ghost-antighost symmetric case. We might then further say that we are not interested in contributions to the vertex which themselves vanish upon integration in Eq. (86) for $\Delta^{(D)}$. Up to such irrelevant contributions, which neither contribute to the gluon nor the ghost DSE, Eqs. (24) and (25) and thus the transversality of the vertex in the infrared are also necessary conditions for the infrared transversality of the gluon correlations in the Landau gauge independent of η.

C. Bounds on the infrared exponent

We now go back to the general results given in Eqs. (77) and (78). With these results for the necessary infrared integrals it requires little effort to explore infrared forms $A^{ir}(k^2;p^2,q^2)$ other than $A^{ir}=\text{const}$ for the A structure in the ghost-gluon vertex. First, in four dimensions, the self-consistency condition in Eq. (74) for these integrals leads to

$$\begin{aligned}(l+n+\kappa-1)(l+n+\kappa)(l+n+\kappa+1)(l+n+\kappa+2)\\ \overset{!}{=}-3(n+2\kappa)(n+2\kappa-1)(n+2\kappa-2)\\ \times(n+2\kappa-3).\end{aligned} \tag{88}$$

This then corresponds to the *Ansatz* for the vertex as given in Eqs. (58) and (59) with only condition **(N1)** being implemented at this stage. It is the starting point for the discussion of the three special cases introduced in Eqs. (61)–(63) of Sec. III B.

Case (i). Here, we simply need to set $n+l=0$. We then obtain form (88) for the possible solutions to Eq. (74) which here reads,

$$I_G^{(4)}(\kappa,-n,n)\overset{!}{=}I_{Z_P}^{(4)}(\kappa,-n,n), \tag{89}$$

the quartic equation to, e.g., determine $n(\kappa)$,

$$(\kappa-1)\kappa(\kappa+1)(\kappa+2)=-3(n+2\kappa)\cdots(n+2\kappa-3). \tag{90}$$

The four real roots to this equation are shown for κ in $[0,1]$ in Fig. 4 on the left.

The top branch, with $1\leqslant n\leqslant 3$, leads to negative $I_G^{(4)}$ and $I_{Z_P}^{(4)}$. Both propagators, $Z(k^2)$ and $G(k^2)$, then necessarily had zeros at some finite k^2, and it resulted that $\alpha_c\leqslant 0$ (equality at the bounds $n=3,1$ for $\kappa=0,1$). We therefore rule out this solution as being unphysical.

Also, we are particularly interested in solutions with at most weak singularities in vertex functions and require therefore $|n|<1$. This is the case for the next branch with $0<n<2$ provided $0.4222<\kappa$. This branch will no longer exist after symmetrization with respect to ghost-antighost momenta [see case (ii) below], however.

For the bottom branch with $0\leqslant n\leqslant -2$, the restriction to weakly singular vertices, $-1<n$ leads to $\kappa<0.4767$. With this solution, it is therefore impossible to obtain the mass gap in transverse gluon correlations, for which $0.5\leqslant\kappa$. Furthermore, it does not survive the symmetrization in (ii) either.

For the branch that includes the tree-level result, that with $n(\kappa_1)=0$, the critical coupling from Eq. (72), with $D=4$ and $N_c=3$, is shown as a function of κ in Fig. 5 (dashed line). Its maximum $\alpha_c^{max}\approx 2.9798$ occurs at $\kappa\approx 0.6174$. It is thus slightly larger than the tree-level value $\alpha_c(\kappa_1)\approx 2.9717$ given in Eq. (83).

Case (ii). This case corresponds to a sum of two terms, one with $m=0$ and $l=-n$ as in case (i) above, and the other with $l=0$ and $m=-n$. Since the ghost-loop contribution I_{Z_P} is manifestly symmetric in $p^2\leftrightarrow q^2$, this symmetrization only affects the infrared contribution I_G to the ghost DSE, and we can therefore write

$$I_G(\kappa,n)\equiv\frac{1}{2}(I_G^{(4)}(\kappa,-n,n)+I_G^{(4)}(\kappa,0,n)), \tag{91}$$

$$I_{Z_P}(\kappa,n)\equiv I_{Z_P}^{(4)}(\kappa,-n,n). \tag{92}$$

The self-consistency condition $I_G(\kappa,n)=I_{Z_P}(\kappa,n)$ can now be used to obtain

$$\begin{aligned}\frac{(\kappa-1)\cdots(\kappa+2)(n+\kappa-1)\cdots(n+\kappa+2)}{(\kappa-1)\cdots(\kappa+2)+(n+\kappa-1)\cdots(n+\kappa+2)}\\ \overset{!}{=}-\frac{3}{2}(n+2\kappa)(n+2\kappa-1)(n+2\kappa-2)\\ \times(n+2\kappa-3).\end{aligned} \tag{93}$$

This equation has eight roots $n(\kappa)$ which, in general, come in complex conjugate pairs. The real roots for κ in $[0,1]$ are shown in Fig. 4 on the right.

One discovers that two out of the originally four branches, from case (i) above, are almost unaffected by the symmetrization employed here: These are the unphysical one at the top, with $1\leqslant n\leqslant 3$ and much the same $\alpha_c\leqslant 0$, and the one connected to the tree-level result, with $n(\kappa_1)=0$ for κ_1

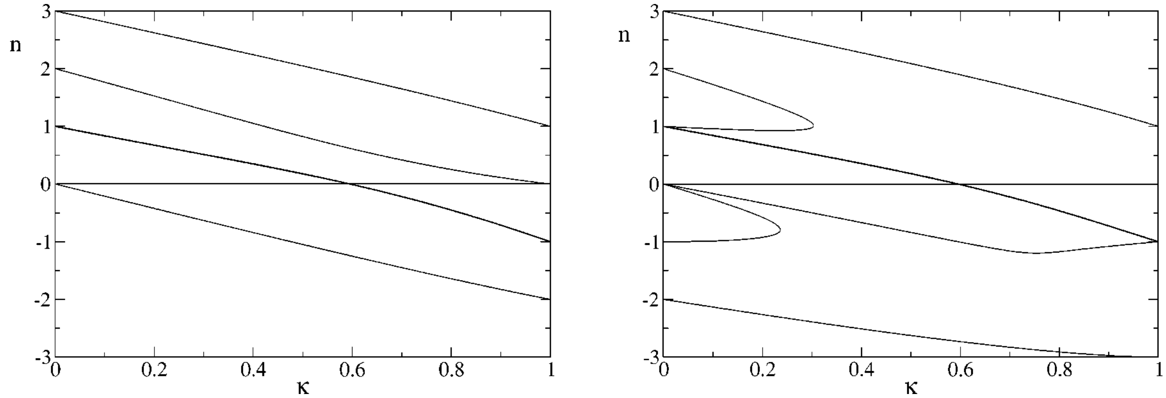

FIG. 4. Roots without symmetrization (left), real roots with additional symmetrization (right).

$=(93-\sqrt{1201})/98$. Superimposing both results for the latter, with and without the symmetrization, the corresponding solutions $n(\kappa)$ turn out to be almost indistinguishable in the whole range $0<\kappa<1$ on the scales of our plots. A little bit more appreciable, but not very significant still, are the differences in the corresponding values for α_c as compared to each other in Fig. 5. For the symmetric solution (solid line) the maximum occurs at the value κ_1 for the tree-level vertex, $\alpha_c^{max}=\alpha_c(\kappa_1)$, at which both solutions intersect. As explained in Sec. III B, if we furthermore require the vertex-structure A to have no infrared divergences associated with the ghost momenta, we must have $n\leqslant 0$ for the ghost-antighost symmetric vertex, in addition. Therefore we can find physically acceptable solutions in the range

$$\kappa_1\leqslant\kappa<1 \quad \text{and} \quad 0<\alpha_c(\kappa)\leqslant\alpha_c(\kappa_1), \tag{94}$$

correspondingly, with the values $\kappa_1\approx 0.59535$ and $\alpha_c(\kappa_1)\approx 2.9717$ for the bounds as obtained, respectively, from Eqs. (82) and (83) with the tree-level or regular vertex.

For completeness we mention that the new branch for the symmetrized vertex (ii) with $-1.2<n<0$, with possibly interesting solutions $-1\leqslant n$ for $\kappa\leqslant\kappa_1$, leads to $\alpha_c\leqslant 0$ throughout, and this is also the case for the bottom branch with $n<-2$.

In the range of particular interest, $1/2\leqslant\kappa<1$, we are thus left with the branch of solutions including $n(\kappa_1)=0$ as the only one with $\alpha_c>0$ after the symmetrization in (ii). At the same time, it seems quite encouraging that this branch, the only physically relevant one, is practically unaffected by the symmetrization.

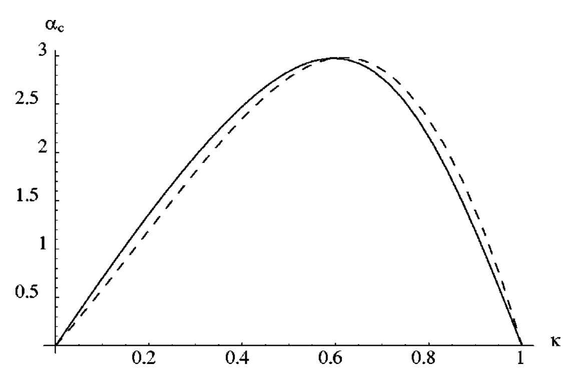

FIG. 5. The critical value of the running coupling over the infrared exponent κ for case (i) (dashed) and case (ii) (solid).

Case (iii). In this example, the infrared behavior of the ghost-gluon vertex as given in Eq. (63) is such that it again satisfies the conditions **(N1)**, from nonrenormalization, and **(S1)**, from ghost-antighost symmetry. In the limit where one ghost momentum vanishes, one now has $A(q^2;q^2,0)=0$, however. We will see that this has no dramatic consequences either on the physically interesting solutions found in the range (94) above.

Here, by the same arguments as in case (ii), we can now express the leading infrared integrals in both DSEs,

$$I_G(\kappa,n)\equiv I_G^{(4)}(\kappa,-n,n)+I_G^{(4)}(\kappa,0,n)-I_G^{(4)}(\kappa,0,0),$$

$$I_{Z_P}(\kappa,n)\equiv 2I_{Z_P}^{(4)}(\kappa,-n,n)-I_{Z_P}^{(4)}(\kappa,0,0). \tag{95}$$

Due to the $(l=n=0)$ contributions herein, which arise from the tree-level term (with negative sign) in Eq. (63), it is generally no longer possible to derive the solutions to $I_G(\kappa,n)=I_{Z_P}(\kappa,n)$ as the roots of simple polynomials. Searching the physically interesting range of parameters numerically, starting from the known solution for $n=0,\ \ \kappa=\kappa_1$, we obtain the dashed curve for $n(\kappa)$ as compared to the corresponding branch for case (ii) in Fig. 6. Again requiring $n\leqslant 0$ to avoid infrared divergent ghost legs, we find that the solutions in the two cases are remarkably close to each other with $-\kappa<n\leqslant 0$ for $\kappa_1\leqslant\kappa<1$, and again we find that $n\to-\kappa$ in the limit $\kappa\to 1$ in which $\alpha_c\to 0$ in all three cases, however.

One might think that another solution with $n=0$ exists for $\kappa=1/2$ and the case (iii) vertex. Since that could have important implications, we note here that this is actually not the case. We know that I_{Z_P} in Eq. (95) reduces to the form for the tree-level vertex at $n=0$, which does not lead to a solution in four dimensions, cf., Fig. 3. The fact that the dashed line in Fig. 6 appears to approach $n\to 0^+$ for $\kappa\to(1/2)^+$ is explained as follows. For sufficiently small $n=\epsilon$, we find from Eq. (95) that

$$I_{Z_P}(\kappa,\epsilon)\sim\left(\frac{1}{\kappa-(1-\epsilon)/2}-\frac{1}{2}\,\frac{1}{\kappa-1/2}\right), \quad \text{for} \quad \kappa\to\frac{1}{2},$$

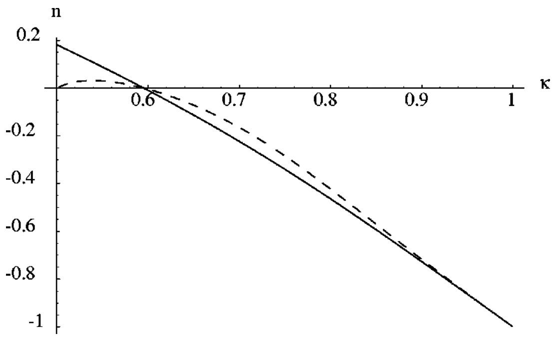

FIG. 6. Solution $n(\kappa)$ for case (iii) (dashed) compared to the tree-level branch of case (ii) (solid) in the range $1/2<\kappa<1$, both with $n(\kappa_1)=0$ at $\kappa_1=(93-\sqrt{1201})/98$.

is dominated by two nearby poles with opposite signs. Therefore for arbitrarily small but finite $\epsilon>0$ the pole at $\kappa=1/2$ in the contribution from the last term, the negative of that in the tree-level vertex case, will always lead to an intersection just above $\kappa=1/2$ with $I_G(\kappa,0)$ which approaches a constant (corresponding to the value $4\pi/[3I_G(0.5,0)]\approx 2.62$ in Fig. 8). For $n=\epsilon=0$ on the other hand, both poles coincide and their residues sum up to that of the tree-level case which is now positive, and thus, the intersection point then disappears. This is confirmed also numerically and demonstrated in Fig. 7.

Of course, if we relax the condition $n\leqslant 0$, we can have self-consistent solutions also for infrared exponents $\kappa<\kappa_1$, including those for $\kappa=1/2$ in cases (i) and (ii). At the same time, this leads to a singularity in $A(k^2;p^2,q^2)$ as $q^2\to 0$.

Negative n on the other hand leads to an infrared divergence associated with the gluon leg. However, as long as $-n<\kappa$ this is overcompensated by the gluon propagator attached to that leg because $Z(k^2)\sim(k^2)^{2\kappa}$. For $n=-\kappa$ an effective massless particle pole would be left in a gluon exchange between two vertices, $G_\mu D_{\mu\nu}(k)G_\nu\sim 1/k^2$. In all solutions we report here, $n\to-\kappa$ for $\alpha_c\to 0$ (both from above). Therefore this limit cannot be reached, since then, at least one of the leading infrared coefficients d_0 or e_0 in the propagators vanishes which contradicts the assumptions, cf., Eqs. (55)–(57).

No such compensation occurs for divergences associated with ghost legs. The ghost correlations are themselves infra-

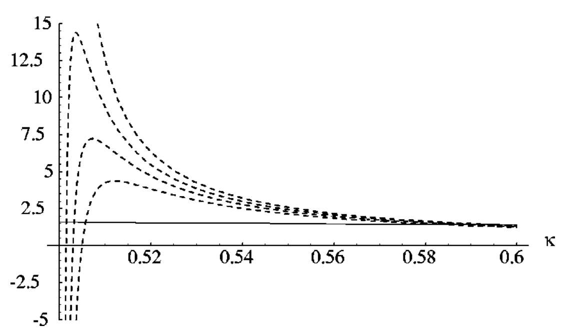

FIG. 7. $I_{Z_p}(\kappa,n)$ (dashed) for several small values $n=\{0.01,0.006,0.003\}$ and $n=0$ over the infrared exponent κ in case (iii). The intersection with $I_G(\kappa,0)$ (solid) near $\kappa=1/2$ disappears for $n=0$.

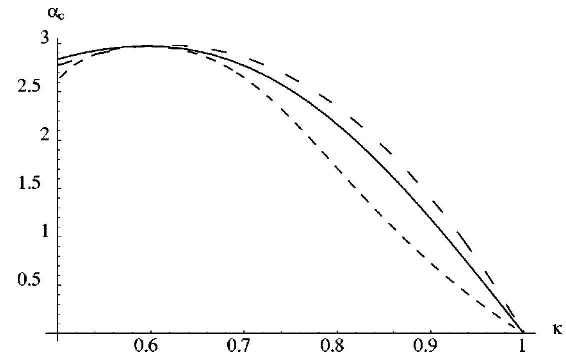

FIG. 8. The value of α_c in the range $1/2<\kappa<1$ for all three cases, (i) (long dashed), (ii) (solid), and (iii) (short dashed).

red enhanced. This infrared enhancement will persist for the ghost correlations between their vertices, if $0<\kappa-m-l<1$. Above the upper bound, the infrared divergences become too severe for the description in terms of local fields. For the cases with the ghost-antighost symmetry **(S1)** of Landau gauge, we obtain from this restriction the upper bound $n<(1-\kappa)/2\leqslant 0.25$ for $\kappa\leqslant 1/2$. This, however, leaves just enough room that it alone does not rule out the solutions with positive n found for $1/2\leqslant\kappa\leqslant\kappa_1$ in case (ii), as seen in Fig. 6.

V. SUMMARY AND CONCLUSIONS

The Dyson-Schwinger equations of standard Faddeev-Popov theory in the Landau gauge, when supplemented by additional boundary conditions, can be derived as an approximation to the time-independent diffusion equation of stochastic quantization which is valid nonperturbatively [35]. The nonconservative part of the drift force that is neglected in this approximation cannot be described by local interactions. The effect of this part will have to be investigated in the future. It may well be responsible for the kind of ''Gribov noise'' observed in lattice calculations.

Here, we studied a slightly more general definition of the Landau gauge as a limit of a wider class including nonlinear covariant gauges [52]. This limit is controlled by an additional free parameter η in the tree-level vertex (with $\eta=1$ in Faddeev-Popov theory). In particular, we find that nonrenormalization of the vertex in a symmetric subtraction scheme and infrared transversality of the gluon propagator in Landau gauge can only go together with the manifestly ghost-antighost symmetric choice $\eta=1/2$. In the light of the recent progress connecting the linear-covariant gauge with time-independent stochastic quantization, the ghost-antighost symmetric Curci-Ferrari gauges might therefore also deserve to be reconsidered for similar connections.

Optimistically assuming that perfect sense can be made of Dyson-Schwinger equations nonperturbatively some day, we studied the infrared critical exponent and coupling for gluon and ghost propagation in the Landau gauge in quite some generality. We gave two reasons for assuming ghost dominance, the Kugo-Ojima criterion for confinement and the horizon condition to restrict the measure to the first Gribov region, and implemented this as a boundary condition in our infrared asymptotic discussion of the DSE solutions. Central to an understanding of the infrared exponents for gluon and ghost propagation in Landau gauge then is a knowledge of their vertex. Assuming it has a regular infrared limit, we obtain $\kappa \simeq 0.595$. For the ghost-antighost symmetric vertices, this value maximizes the critical coupling $\alpha_c(\kappa)$, yielding $\alpha_c^{max} \approx 2.97$, as summarized once more for κ between 0.5 and 1 in Fig. 8. For larger κ the vertex acquires an infrared singularity in the gluon momentum, smaller ones imply infrared singular ghost legs.

Quite encouragingly, numerical solutions to truncated Dyson-Schwinger equations have recently been obtained with the infrared behavior of the regular vertex, as derived here, for the whole momentum range up to the perturbative ultraviolet regime and without one-dimensional approximation in Ref. [55].

An important detail in all our considerations is the nonrenormalization of the ghost-gluon vertex in the Landau gauge. Derived from standard Slavnov-Taylor identities it is one of the arguments that hold at all orders of perturbation theory. That it is also true nonperturbatively, however, is another additional assumption. It is therefore quite important and interesting that this has been assessed and verified within the numerical errors in calculations using the Landau gauge on the lattice [56]. Calculating both propagators simultaneously, this study furthermore appears to confirm a unique exponent for the combined infrared behavior of gluons and ghosts for the first time in a lattice calculation implementing the Landau gauge condition. Also, for SU(2) this study reports preliminary values of α_c that are fully consistent with the results obtained here somewhere near the maximum $\alpha_c^{max} \simeq (4\pi/N_c) 0.71 \approx 4.5$ for $N_c = 2$. It will be very interesting to see the final errors, so that we will be in the fortunate position to restrict further the range of both α_c and κ. At the moment, the combined evidence seems to indicate that the result will be somewhere in the range around $\kappa = 0.5$ and the maximum near $\kappa = 0.6$. Unfortunately, with this conclusion the question about an infrared vanishing versus finite gluon propagator must therefore remain open, for the time being.

Note added. P. Petreczky kindly reminded us of the lattice Landau gauge results for the three-dimensional gluon propagator of Refs. [40] and [41]. We gratefully acknowledge communications with him on their results. From communications with A. Davydychev, we learned that we might have inadvertently given the impression to consider formula (B1) as new in any sense. This is not at all the case. The 2-line derivation in Eqs. (B2) and (B3) below is given for the convenience to the reader. He furthermore points out that relation (B13) follows from Eq. (11) listed on p. 534 in the tables of Ref. [70]. We gratefully acknowledge this information.

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APPENDIX A: GHOST RG EQUATION REEXAMINED

We repeat the renormalization group (RG) analysis of Ref. [54] for the ghost propagator with some minor corrections. These corrections do not affect the main conclusions of Ref. [54] as far as we can judge. The correct versions, in particular, of Eqs. (8)–(11) in Ref. [54], are necessary, however, in order to establish the equivalence of their asymptotic infrared expansion and the one adopted via Eq. (48) herein, which is a minor variation of the expansion techniques developed previously [57–62,22].

First, recall the RG equation for the ghost propagator $G(k^2) \equiv G(k^2,\mu^2)$, in this form also given in Eq. (6) of [54],

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - 2\gamma_G(g)\right) G(k^2,\mu^2) = 0. \tag{A1}$$

The formal solution to this equation is given by Eq. (43). With the infrared *Ansatz* in the form of Eq. (5) in [54],

$$G(k^2,\mu^2) \simeq \sum_n^N d'_n(g) \left(\frac{k^2}{\mu^2}\right)^{\delta_n}, \tag{A2}$$

here denoting the coefficients of [54] as the primed ones, d'_n, to distinguish from those in Eq. (48), we first obtain

$$\beta(g)\left(\frac{\partial d'_n}{\partial g} + d'_n \ln(k^2/\mu^2) \frac{\partial \delta_n}{\partial g}\right) - 2 d'_n(\delta_n + \gamma_G(g)) = 0, \tag{A3}$$

at variance with Eq. (8) of Ref. [54] in two minor ways [by the factor of 2 and the sign of the $\gamma_G(g)$ term]. Nevertheless, with their conclusion that therefore $\partial \delta_n / \partial g = 0$, we find for the coefficients

$$\frac{\partial d'_n}{\partial g} = \frac{2(\delta_n + \gamma_G(g))}{\beta(g)} d'_n, \tag{A4}$$

the general solution of which takes the form

$$d'_n(g) \propto \exp \int^g \frac{2(\delta_n + \gamma_G(l))}{\beta(l)} dl. \tag{A5}$$

This, however, is incompatible with Eq. (9) of Ref. [54],

$$d'_n(g) = \text{const}\, g^{-2(\delta_n + \gamma_G)/2\gamma_G + \gamma_A}. \tag{A6}$$

In particular, since we just noted that the exponents δ_n are g independent, they are either zero or one would need $\beta(g) = -\text{const} \times g$ to obtain Eq. (A6) from Eq. (A5). By virtue of Eq. (50), $\beta(g) = -g(2\gamma_G + \gamma_A) \sim -g$, this would imply that $2\gamma_G + \gamma_A$ is g independent. If we would then conclude in addition that both γ_G and γ_A are g independent, only then we would obtain Eq. (A6) from Eq. (A5).

Note that such a behavior, $\beta(g) \sim -g$, though in principle possible in the infrared, would not lead to a fixed point and thus contradict the other results of [54] as we discussed in Sec. III A above. Not restricted to such a specific behavior, here we go back to the general form of the d'_n in Eq. (A5). First, remember that $g = g_0$ for $\mu^2 = \sigma$. In this case, the exponential factor in our expansion (48) becomes unity, and Eqs. (48) and (A2) agree. Thus $d_n = d_n'(g_0)$, and we can split the solution to Eq. (A4) for d_n' with this initial condition into factors as follows:

$$d_n'(g) = d_n \exp\left\{ 2\,\delta_n \int_{g_0}^{g} \frac{dl}{\beta(l)} \right\} \exp\left\{ 2 \int_{g_0}^{g} \frac{\gamma_G(l)}{\beta(l)}\, dl \right\}. \quad \text{(A7)}$$

The first exponential factor herein, with Eq. (46), is equal to $(\mu^2/\sigma)^{\delta_n}$, and can be used to replace $\mu^2 \to \sigma$ in Eq. (A2). The last exponential in Eq. (A7) is the same overall factor determining the (g,μ) dependence of the ghost propagator as in Eq. (48). Substituting Eq. (A7) into the expansion (A2) of Ref. [54], one obtains,

$$G(k^2,\mu^2) \simeq \exp\left\{ 2 \int_{g_0}^{g} \frac{\gamma_G(l)}{\beta(l)}\, dl \right\} \sum_n^N d_n \left(\frac{k^2}{\sigma} \right)^{\delta_n}, \quad \text{(A8)}$$

which agrees with the RG invariant expansion of Eq. (48).

APPENDIX B: TWO WAYS TO DO THE *D*-DIMENSIONAL INTEGRALS

The basic formula we employ for the infrared analysis in Sec. III involves D-dimensional integrals of the following form which converge for $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta-\alpha) > 0$, $\mathrm{Re}(\beta) < D/2$:

$$\int \frac{d^Dq}{(2\pi)^D} \left(\frac{1}{q^2} \right)^{D/2} \left(\frac{q^2}{k^2} \right)^{\alpha} \left(\frac{k^2}{p^2} \right)^{\beta} = \frac{1}{2^D \pi^{D/2}} \frac{\Gamma(\alpha)\Gamma(D/2-\beta)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(D/2-\alpha)\Gamma(D/2+\alpha-\beta)}, \quad \text{(B1)}$$

where $p = k \pm q$ and an explicit factor $(k^2)^{\beta-\alpha}$ was introduced to render the integral dimensionless. This is a textbook formula, cf., Eq. (2.5.178) in [63]. For a simple derivation with the general exponents [64] one observes that the lhs of Eq. (B1) is a convolution integral which reduces to an ordinary product upon Fourier transformation. Using

$$\frac{1}{(q^2)^{\gamma}} = \frac{\Gamma(D/2-\gamma)}{4^{\gamma} \pi^{D/2} \Gamma(\gamma)} \int d^Dx\, (x^2)^{\gamma - D/2} e^{-iqx} \quad \text{(B2)}$$

for the two factors in the convolution (with the power γ given by $D/2-\alpha$ and β, respectively), one thus obtains

$$\int \frac{d^Dq}{(2\pi)^D} \left(\frac{1}{q^2} \right)^{D/2} \left(\frac{q^2}{k^2} \right)^{\alpha} \left(\frac{k^2}{p^2} \right)^{\beta} = \frac{4^{\alpha-\beta}}{(2\pi)^D} \frac{\Gamma(\alpha)\Gamma(D/2-\beta)}{\Gamma(\beta)\Gamma(D/2-\alpha)} \int \frac{d^Dx}{(x^2)^{D/2}} (k^2x^2)^{\beta-\alpha} e^{-ikx}. \quad \text{(B3)}$$

Equation (B1) then follows from a further application of the Fourier transform (B2) herein, now with $\gamma = \beta - \alpha$.

Alternatively, we can do the integral in Eq. (B1) which we denote henceforth by $f_D(\alpha,\beta)$ in a straightforward though less elegant way by first performing all but one of the angles of the polar coordinates in D-dimensional momentum space,

$$f_D(\alpha,\beta) = \frac{K(D)}{(2\pi)^D} \frac{D}{2} \int \frac{dq^2}{q^2} \left(\frac{q^2}{k^2} \right)^{\alpha} \frac{1}{B[(D-1)/2,1/2]} \times \int_{-1}^{1} dz\, (1-z^2)^{(D-3)/2} \left(\frac{k^2}{p^2} \right)^{\beta}, \quad \text{(B4)}$$

where $K(D) = 2\pi^{D/2}/[D\Gamma(D/2)]$ is the volume of the D-dimensional unit ball, $p^2 = q^2 + k^2 - 2kqz$, and the Euler beta function is given by

$$B[(D-1)/2,1/2] = \int_0^1 dt\, t^{-1/2} (1-t)^{(D-3)/2} = \int_{-1}^{1} dz\, (1-z^2)^{(D-3)/2}. \quad \text{(B5)}$$

For the azimuthal integration in Eq. (B4), the formula 2 in **3.665** of Ref. [65] can be used to obtain,

$$\begin{aligned} f_D(\alpha,\beta) &= \frac{K(D)}{(2\pi)^D} \frac{D}{2} \left[\int_0^{k^2} \frac{dq^2}{q^2} \left(\frac{q^2}{k^2} \right)^{\alpha} {}_2F_1\left(\beta, \beta - \frac{D}{2} + 1; \frac{D}{2}; \frac{q^2}{k^2} \right) + \int_{k^2}^{\infty} \frac{dq^2}{q^2} \left(\frac{q^2}{k^2} \right)^{\alpha-\beta} {}_2F_1\left(\beta, \beta - \frac{D}{2} + 1; \frac{D}{2}; \frac{k^2}{q^2} \right) \right] \\ &= \frac{1}{2^D \pi^{D/2} \Gamma(D/2)} \left[\int_0^1 \frac{dx}{x} (x^{\alpha} + x^{\beta-\alpha})\, {}_2F_1\left(\beta, \beta - \frac{D}{2} + 1; \frac{D}{2}; x \right) \right] \end{aligned} \quad \text{(B6)}$$

$$= \frac{1}{2^D \pi^{D/2} \Gamma(D/2)} \left[\frac{1}{\alpha}\, {}_3F_2\left(\beta, \beta - \frac{D}{2} + 1, \alpha; \frac{D}{2}, \alpha+1; 1 \right) + \frac{1}{\beta-\alpha}\, {}_3F_2\left(\beta, \beta - \frac{D}{2} + 1, \beta-\alpha; \frac{D}{2}, \beta-\alpha+1; 1 \right) \right], \quad \text{(B7)}$$

where for the last step the following integration formula for the generalized hypergeometric functions led to the final result in Eq. (B7):

$$\int_0^1 dx x^{\gamma} {}_pF_q(\mathbf{a};\mathbf{b};x) = \frac{1}{\gamma+1} {}_{p+1}F_{q+1} \times(\{\mathbf{a},\gamma+1\};\{\mathbf{b},\gamma+2\};1), \tag{B8}$$

which is most easily derived for $\gamma>-1$ (see, e.g., the appendixes of Refs. [46] and [47]) from the power series expansion for the generalized hypergeometric functions,

$${}_pF_q(\mathbf{a};\mathbf{b};z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{B9}$$

by noting the relation,

$$\frac{1}{\gamma+n+1} = \frac{1}{\gamma+1} \frac{(\gamma+1)_n}{(\gamma+2)_n}, \tag{B10}$$

for the Pochhammer symbols,

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{B11}$$

For other general properties and a variety of relations amongst the different hypergeometric functions, e.g., see Refs. [66–69]. A well-known one, for example, expresses the Gauss series as a ratio of gamma functions,

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{B12}$$

Many more relations of this kind, including less known ones, are listed in the tables on hypergeometric functions of Ref. [70]. We originally thought it might be interesting to note that the two ways to calculate $f_D(\alpha,\beta)$ leading to the rhs of Eqs. (B1) and (B7), respectively, allow one to devise some of these additional relations. For example, by simple renamings, a comparison of the rhs in Eqs. (B1) and (B7) yields,

$$\begin{aligned} &\frac{1}{a} {}_3F_2(a,a+b,a+b-c+1;c,a+1;1) \\ &\quad + \frac{1}{b} {}_3F_2(b,a+b,a+b-c+1;c,b+1;1) \\ &\quad = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \\ &\quad = B(a,b) \, {}_2F_1(a,b;c;1). \end{aligned} \tag{B13}$$

This formula follows with replacing $b\to a+b$, $1-c\to c-a-b$, and $d\to c$ upon rearrangement from Eq. (11) on p. 534 in [70], and our presentation here seems obsolete now [71]. At least, the equivalence of the results from the infrared analysis of DSEs in Ref. [46], to the expressions in Eqs. (79) and (84) with $\eta=0$ for the tree-level vertex case of Sec. IV A is explicitly established in this way. The second procedure to calculate integrals such as $f_D(\alpha,\beta)$ was thereby used in Ref. [46]. Each of the results therein are readily expressed in terms of one simple ratio of gamma functions when using the relations presented in this appendix. Though equivalent to Eq. (B7) of course, use of Eq. (B1) thereby is far more convenient for all practical purposes.

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