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Poincaré recurrence theorem and the strong CP problem

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The existence in the physical QCD vacuum of nonzero gluon condensates, such as $\langle g^2F^2\rangle$, requires dominance of gluon fields with finite mean action density. This naturally allows any real number value for the unit "topological charge" q characterizing the fields approximating the gluon configurations which should dominate the QCD partition function. If q is an irrational number then the critical values of the θ parameter for which CP is spontaneously broken are dense in $\mathbb{R}$, which provides for a mechanism of resolving the strong CP problem simultaneously with a correct implementation of $U_{\rm A}(1)$ symmetry. We present an explicit realization of this mechanism within a QCD motivated domain model. Some model independent arguments are given that suggest the relevance of this mechanism also to genuine QCD.

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I. INTRODUCTION

The functional space $\mathcal{F}_A$ of gluon fields A with finite classical action (in the infinite volume limit),

$$\lim_{V\to\infty} S_V[A] < \infty, \tag{1}$$

can be divided into equivalence classes according to integer values of topological charge

$$\nu = Q[A] = \frac{g^2}{32\pi^2}\int d^4x F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}. \tag{2}$$

Integer values originate from the purely topological properties of the gauge group: the above integral can be rewritten as a surface integral and, since at infinity all fields can be at most pure gauge configurations, the value of that integral is determined by the gauge group alone. According to these equivalence classes, the QCD partition function can be written in the form of a Fourier series,

$$Z(\theta) = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} Z_\nu, \tag{3}$$

where Z_ν is given by a functional integral over the fields belonging to the νth sector of $\mathcal{F}_A$. The representation Eq. (3) is characteristic of non-Abelian gauge groups and appeals to the existence of nonperturbative gluon field configurations, namely, multi-instantons. As a matter of fact, from the very beginning the above construction is based on a quasiclassical principle: the functional integral is assumed to be dominated by fields with minimal classical action, while fields with infinite action are excluded from consideration *ad hoc*. However, as is ultimately required by hadron phenomenology, the physical QCD vacuum is characterized by finite nonzero gluon condensates [1], among which the lowest order condensate can be related to a mean action density times the coupling constant squared [2],

$$\langle g^2 F^a_{\mu\nu}(x)F^a_{\mu\nu}(x)\rangle = 4\lim_{V\to\infty}\langle g^2 S_V\rangle/V \neq 0, \tag{4}$$

which can be provided only by the class of configurations already excluded from Eq. (3), the field configurations with extensive (scaling with the volume V) classical action. One faces the necessity of defining the QCD partition function as an integral over the space $\mathcal{F}_{\mathcal{A}}$ of gluon configurations $\mathcal{A}$ which are allowed to have nonzero classical action density at space-time infinity. For instance, a suitable requirement for $\mathcal{F}_{\mathcal{A}}$ would be

$$\lim_{x^2\to\infty} F^a_{\mu\nu}(x)F^a_{\mu\nu}(x) = \text{const}, \tag{5}$$

with a nonzero constant on the right-hand side (RHS), which can be related to the trace anomaly of the QCD energy-momentum tensor [3,4]. Condition Eq. (5) produces no essential difficulties since the infinity arising from the volume of the system is simply a matter of normalization of the functional integral or, put differently, of normalization of the vacuum energy. The actual values of the gluon condensates and the constant on the RHS of Eq. (5) are defined by the minima of the quantum effective action. This indicates that the properties of the QCD physical vacuum, which are encoded in the various condensates and are responsible for confinement and the mode of realization of chiral symmetry, are due to purely quantum effects. Consequently, the fields to be identified as dominating the QCD functional integral must not have vanishing action density at infinity, otherwise the main effect, the existence of condensates, would be omitted. In order to define the functional integral over fields subject to Eq. (4)

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in an analytical approach, one can represent $\mathcal{A} = B + A$ with the background B from the class of fields dominating the integral and A being small localized fluctuations in this background. The integral over fields B has to be performed exactly, while fluctuations A can be treated perturbatively. The candidate dominating fields $B^a_\mu(x)$ could be required to satisfy another condition for almost all $x \in \mathbb{R}^4$

$$F^a_{\mu\nu}(x)F^a_{\mu\nu}(x) = \text{const}, \tag{6}$$

which, and this is important, neither forbids space-time variation of the strength tensor and the fields $B^a_\mu(x)$ nor devalues the importance of topological (singular) pure gauge field defects of various dimensions distributed throughout the entire Euclidean space-time [4,5].

One can of course look for conditions other than Eqs. (5) and (6) to attempt to specify background fields B such that they would not be assumed to fill space-time almost everywhere and would be described in the spirit of dilute instanton gas or liquid models [6]. However, despite diluteness, any superposition of an infinite number (in the infinite volume) of instantons and anti-instantons represents a field with infinite classical action, which can be different from the class fixed by Eq. (6), but for sure is absent in the space of integration $\mathcal{F}_\mathcal{A}$ subject to Eq. (1).

Equation (4) has an important consequence: in the space of fields $\mathcal{F}_\mathcal{A}$ with extensive classical action the functional $Q[\mathcal{A}]$ can take any real value, rational or irrational, finite or infinite in the limit $V \to \infty$. This signals that the equivalence classes fed into the representation Eq. (3) do not exist in the space $\mathcal{F}_\mathcal{A}$. Nevertheless, the notion of the mean value of the modulus $|Q[\mathcal{A}]|$,

$$Q = \lim_{V\to\infty} \langle |Q_V[\mathcal{A}]| \rangle, \tag{7}$$

is well defined on $\mathcal{F}_\mathcal{A}$ and, as is known from phenomenology and lattice QCD, is a nonzero constant, related to the topological susceptibility χ of the QCD vacuum [7]. In both cases of fields satisfying Eq. (6) and dilute superpositions of instantons, Q can be expressed in terms of a mean action density. There is a crucial difference however between instanton based approximations of the fields with infinite action and backgrounds (as in the domain model for instance [5,8–10]) based on Eq. (6). The literal use of instantons to approximate fields with infinite action assigns integer values of topological charge to the fields from $\mathcal{F}_\mathcal{A}$ and thus transports integer units ν of topological charge from Eq. (3) into QCD with condensates where such integers are irrelevant. Instead, approximations based on Eq. (6) leave the freedom to have any real values for Q for genuine gluon configurations and unit "topological charge" q for the fields approximating them in a particular model of the QCD vacuum.

It was realized a long time ago (see [11] and, for a recent review, [12]) that the strong CP-problem is a problem of the quasiclassical treatment of the QCD functional integral built into the representation Eq. (3), and that incorporation of the long-range gluon configurations responsible for confinement could potentially remove this problem. However, to date, no way has been found to achieve this and maintain simultaneously the resolution of the $U_\mathrm{A}(1)$ problem traditionally attributed to instantons. Scenarios based on the existence of an additional boson, the axion, have been suggested and have become canonical [12,13].

Nevertheless, as will be discussed below in detail on the basis of a QCD motivated model, and as has been proposed in [4] within the general model independent consideration, the strong CP and $U_\mathrm{A}(1)$ problems have a mechanism for simultaneous resolution within QCD. As is stressed in [4], the order of thermodynamic limit and the limit $\theta \to 0$ are in general not interchangeable and thus independence with respect to θ of the infinite volume QCD partition function does not automatically lead to vanishing topological susceptibility $\chi = \lim_{\theta\to 0}\lim_{V\to\infty} V^{-1} d^2 Z_V(\theta)/d\theta^2$.

The QCD motivated model we shall consider in this article is the domain model [5,8–10]. For completeness and clear identification of the origin of the effect discussed in this article, a comment about the choice of boundary conditions in the domain model is in order. The field B is considered as a background field and is subject to the condition Eq. (6). The system is considered in a large but finite volume V. The total volume is split into a large number of subvolumes—domains. The fluctuation quark fields are defined by the baglike conditions on the domain boundaries. The characteristic size of a domain is much smaller than the characteristic size of the total volume. The thermodynamic limit is understood as the limit when the size of the total volume goes to infinity together with the number of domains while the size of domains stays finite. This construction is sufficient for defining and for practical calculation of the Euclidean functional integral for the partition function. It should be stressed that baglike conditions are very different from the (quasi)periodic conditions typical for formulations based on a compactification of $\mathbb{R}^4$ to the torus. The most important feature for our considerations is that the spectrum λ of the Dirac operator under baglike boundary conditions is asymmetric with respect to $\lambda \to -\lambda$, while (quasi)periodic boundary conditions lead to a symmetric spectrum. In particular, this asymmetry is responsible for the formation of the quark condensate and resolution of the $U_A(1)$ problem in the model as is discussed in detail in [9]. Nevertheless, there is no contradiction between our results and canonical considerations based on toruslike compactifications of $\mathbb{R}^4$ [14]. These are just two different complementary statements of the problem.

Within the domain model the mechanism for resolving the strong CP-problem is realized in the following way. In the presence of nonzero gluon condensates irrational values of Q are permitted and lead to a realization of CP in which the set of critical values of the θ parameter for which the CP-breaking is spontaneous [15] is *dense* in the interval $[-\pi, \pi]$. As a consequence, the infinite volume partition function with infinitesimally small quark masses,

$$Z = \lim_{V\to\infty} Z_V(\theta) = \lim_{V\to\infty} Z_V(0),$$

is independent of θ, and

$$\lim_{V\to\infty}\langle\mathbb{E}\rangle_V^\theta \equiv \lim_{V\to\infty}\langle\mathbb{E}\rangle_V^{\theta=0}, \qquad \lim_{V\to\infty}\langle\mathbb{O}\rangle_{V,\theta} \equiv 0,$$

for any CP-even and CP-odd operators $\mathbb{E}$ and $\mathbb{O}$ respectively, which resolves the problem of CP-violation. Simultaneously, one finds that topological susceptibility in QCD with massive quarks

$$\chi^{\rm QCD} = -\lim_{V\to\infty}\frac{1}{V}\frac{\partial^2}{\partial\theta^2}Z_V(\theta) \neq 0$$

is nonzero, independent of θ, and satisfies the anomalous Ward identity, which indicates a correct implementation of the $U_A(1)$ symmetry. In particular, in the chiral limit the mass of the η' is expressed via the topological susceptibility $\chi^{\rm YM}$ of pure gluodynamics in agreement with the Witten-Veneziano formula.

If Q takes a rational value then only a finite number of such critical points exists in the interval $\theta \in [-\pi, \pi]$, which results in the standard strong CP problem.

To apply a mechanical analogy, an irrational value of Q leads to a picture which is reminiscent of ergodic motion resulting in a dense winding of a torus by a trajectory [16], while motion over closed trajectories can be associated with rational Q.

This article is devoted to an investigation of this mechanism on the basis of the domain model of QCD vacuum [5,8–10]. We omit here all details related to the motivation, formulation and numerous results of the model given in the above papers, apart from those which are specifically needed here. In the next two sections the θ dependence in the model under consideration is elucidated. The Witten-Veneziano mass formula, Gell-Mann–Oakes–Renner relation, the anomalous Ward identity and $\eta\pi\pi$ decay are specifically discussed in Sec. IV. The final section is devoted to explanation of the relation between the mechanism of the strong CP resolution in the domain model and the Poincaré recurrence theorem of ergodic theory which justifies the title of the present paper.

II. THETA DEPENDENCE IN THE DOMAIN MODEL

The model is given in terms of the following partition function for $N \to \infty$ domains of radius R:

$$\begin{aligned}\mathcal{Z} = \mathcal{N}\lim_{V,N\to\infty}\int_{\Omega_{\alpha,\vec\beta}} d\alpha d\vec\beta \prod_{i=1}^N \int_\Sigma d\sigma_i \int_{\mathcal{F}_\psi^i}\mathcal{D}\psi^{(i)}\mathcal{D}\bar\psi^{(i)} \\ \times \int_{\mathcal{F}_A^i}\mathcal{D}A^i\delta[D(\breve B^{(i)})A^{(i)}]\Delta_{\rm FP}[\breve B^{(i)}, A^{(i)}] \\ \times e^{-S_{V_i}^{\rm QCD}[A^{(i)}+B^{(i)},\psi^{(i)},\bar\psi^{(i)}]-i\theta Q_{V_i}[A^{(i)}+B^{(i)}]},\end{aligned} \tag{8}$$

where the functional spaces of integration $\mathcal{F}_A^i$ and $\mathcal{F}_\psi^i$ are specified by the boundary conditions $(x-z_i)^2 = R^2$

$$\begin{aligned}\breve n_i A^{(i)}(x) = 0, \qquad i\not\eta_i(x)e^{i(\alpha+\beta^a\lambda^a/2)\gamma_5}\psi^{(i)}(x) = \psi^{(i)}(x), \\ \bar\psi^{(i)}e^{i(\alpha+\beta^a\lambda^a/2)\gamma_5}i\not\eta_i(x) = -\bar\psi^{(i)}(x).\end{aligned} \tag{9}$$

Here $\breve n_i = n_i^a t^a$ with the generators t^a of $SU_{\mathbf{c}}(3)$ in the adjoint representation, the α and β^a are random chiral angles associated with the chiral symmetry violating boundary condition Eq. (9) in the presence of N_f ($a = 1, \ldots, N_f^2-1$) quark flavors. The λ^a are $SU(N_f)$ flavor generators normalized such that ${\rm Tr}(\lambda^a\lambda^b) = 2\delta^{ab}$. Averaging over the angles α, β^a is included in the partition function. The thermodynamic limit assumes $V, N\to\infty$ but with the density $v^{-1} = N/V$ taken fixed and finite. The partition function is formulated in a background field gauge with respect to the domain mean field, which is approximated inside and on the boundaries of the domains by a covariantly constant (anti)self-dual gluon field with the field-strength tensor of the form

$$F_{\mu\nu}^a(x) = \sum_{j=1}^N n_j^a B_{\mu\nu}^{(j)}\vartheta(1-(x-z_j)^2/R^2),$$

with $B_{\mu\nu}^{(j)}B_{\mu\rho}^{(j)} = B^2\delta_{\nu\rho}$. Here z_j^μ are the positions of the centers of domains in Euclidean space. The measure of integration $d\sigma_i$ over parameters characterizing domains can be found in any of the papers [5,8–10]. The background field in Eq. (10) is designed to satisfy Eq. (6).

In a sense, this model represents a "step-function" approximation to a class of fields with infinite action which are assumed to dominate the QCD partition function. The building blocks of this approximation are domains with the same mean size R and mean action density B^2, whose values are determined from the phenomenological string tension in [5]. Moreover, as the field in domains is taken to be (anti)self-dual, a mean absolute value q of the integral of the topological charge density over the domain volume $v = \pi^2R^4/2$ is attributed to a domain, and is expressed through the domain radius and field strength as $q = B^2R^4/16$. This quantity can take any real values but for B, R fixed as in [5] $q \approx 0.15\ldots$. It should be noted also that domain boundaries are assumed to be populated by pure gauge singularities which look like topological defects (instantons, monopoles, vortices and domain walls) but only locally; globally no topologically conserved number can be associated with them. The role of these defects in the model is to generate boundary conditions for the fluctuations of gluon and quark fields A and ψ inside domains [5]. In particular, at the location of pure gauge singularities ∂V_i color quark currents have to satisfy the condition

$$\bar{\psi}(x)\not{\eta}_i(x)t^a\psi(x) = 0, \qquad x \in \partial V_i.$$

with η_i^μ being a vector normal to the surface ∂V_i of the ith domain where the singularities are located. This leads to the condition Eq. (9) with arbitrary parameters α, β^a. Simultaneously it is natural to require that colorless quark currents like

$$\bar{\psi}(x)\lambda^a\psi(x),\ \bar{\psi}(x)\gamma_5\lambda^a\psi(x),\ \bar{\psi}(x)\not{\eta}_i(x)\gamma_5\lambda^a\psi(x), \text{etc.}$$

are continuous at ∂V_i. Such continuity conditions Eq. (10) fix the parameters α, β^a to be the same at all singular surfaces.

Apart from chiral angles and neglecting gluon fluctuations A in lowest order of the perturbation expansion, the integrations in the partition function Eq. (8) give [9]

$$Z_V(\theta) = \mathcal{N}\int_{\Omega_{\alpha\beta}} d\alpha d\vec{\beta}\exp\{-V\mathcal{F}(\alpha,\beta,\theta)\},$$

with the free energy density as a function of chiral angles in the presence of infinitesimally small quark masses m_i

$$\mathcal{F} = -\frac{1}{v}\ln[\cos q(W_{N_f} - \theta)] - \aleph\sum_{i=1}^{N_f} m_i\cos\Phi_i, \tag{10}$$

and where

$$W_{N_f} = \sum_{i=1}^{N_f}\arctan(\tan\Phi_i), \qquad \Phi_i = \alpha + B_i.$$

The functions B_i for various numbers of flavors are

$$B_1 = 0 \quad \text{for } N_f = 1, \tag{11}$$

$$B_1 = \frac{|\vec{\beta}|}{2}, \qquad B_2 = -\frac{|\vec{\beta}|}{2}, \qquad \text{for } N_f = 2, \tag{12}$$

and, rather than give explicit representations for $N_f = 3$, we give several identities useful for later purposes:

$$\sum_{i=1}^{3} B_i = 0 \tag{13}$$

$$\sum_{i=1}^{3} B_i^2 = \frac{1}{2}\sum_{a=1}^{8}(\beta^a)^2 \tag{14}$$

$$\sum_{i=1}^{3} B_i^3 = \frac{3}{4\sqrt{3}}\sum_{a=1}^{3}(\beta^a)^2\beta^8 + \cdots, \tag{15}$$

where the dots denote other cubic terms but involving $a = 4, \ldots, 8$. Equations (12)–(15) come from the diagonalization of the fermionic boundary conditions Eq. (9) in the flavor space. The quantity $\aleph$ has been explicitly computed in [9] and appears in the domain model as a function of the field strength B and domain radius R,

$$\aleph = \frac{1}{\pi^2R^3}\,\mathrm{Tr}\sum_{k=1}^{\infty}\frac{k}{k+1}\Big[M(1, k+2, z) - \frac{z}{k+2}M(1, k+3, -z) - 1\Big],$$

where M is the confluent hypergeometric function, $z = \hat{n}BR^2/2$ and the color Tr denotes summation over elements of diagonal color matrix $\hat{n}$. With B, R as determined in [5] $\aleph = (237.8\ \mathrm{MeV})^3$. It determines the value of the quark condensate

$$\langle\bar{\psi}\psi\rangle = -\aleph$$

and arises from the asymmetry of the Dirac operator spectrum [17] with the boundary conditions Eq. (9). Because of Eq. (13) (which applies for any N_f), we have

$$W_{N_f} = N_f\alpha(\mathrm{mod}\,\pi).$$

Thus, the anomalous part of free energy density, that involving the topological charge q, is independent of the β^a which manifests the Abelian property of the anomaly. The significance of Eq. (10) is that it is derived directly from the domain *ansatz* for the QCD vacuum but also corresponds to the zero momentum limit of the effective chiral Lagrangian of [7] up to the somewhat different encoding of the anomaly. A task in the following is then verifying that this encoding of the symmetries of low energy phenomenology reproduces the expected phenomena of spontaneous chiral symmetry breaking and resolution of the $U_A(1)$ problem but, more importantly, that it also solves the strong CP problem within this QCD vacuum approach.

In the thermodynamic limit $V \to \infty$ the minima of $\mathcal{F}$ define the vacua of the system under consideration. For massless quarks $m_i \equiv 0$ the anomalous term gives a discrete set of degenerate minima at

$$\alpha_{kl}(\theta) = \frac{\theta}{N_f} + \frac{2\pi l}{qN_f} + \frac{\pi k}{N_f} \qquad (k \in Z, l \in Z). \tag{16}$$

Thus, due to the anomaly the continuous $U_A(1)$ is reduced to a discrete group of chiral transformations connecting the minima Eq. (16) with each other. In the absence of additional sources violating chiral symmetries like quark masses, this discrete symmetry is sufficient to provide for zero mean values of all chirally noninvariant operators.

III. IRRATIONAL VALUE OF PARAMETER *q*

There is a crucial difference between rational and irrational values of q. For q rational there is a *finite* number of distinct minima and all others are 2π-equivalent to one of these minima. In other words, there is a periodic structure which allows one to split the set Eq. (16) into a finite number of equivalence classes.

If q is an irrational number then all minima in the interval $\alpha \in [-\infty, \infty]$ are distinct: there are no minima which are 2π-equivalent to each other. The distance between a pair of minima is

$$\Delta(k_1, l_1|k_2, l_2) = 2\pi m(k_1, l_1|k_2, l_2) + \delta(k_1, l_1|k_2, l_2)$$

with $m(k_1, l_1|k_2, l_2)$ some integer, and $\delta(k_1, l_1|k_2, l_2)$ a number in the interval $(0, 2\pi)$ which is different for any pair $(k_1, l_1|k_2, l_2)$, $l_1 \neq l_2$. Unlike the case of rational q where only several numbers δ arise, the set of all $\delta(k_1, l_1|k_2, l_2)$ is dense in the interval $(0, 2\pi)$ for irrational q. The periodicity, typical for a rational q, is lost for irrational values of q and the set of all minima cannot be split into 2π-equivalence classes. However, these discrete minima of the free energy are separated by infinite energy barriers in the infinite volume limit, and no flavor singlet Goldstone modes are expected.

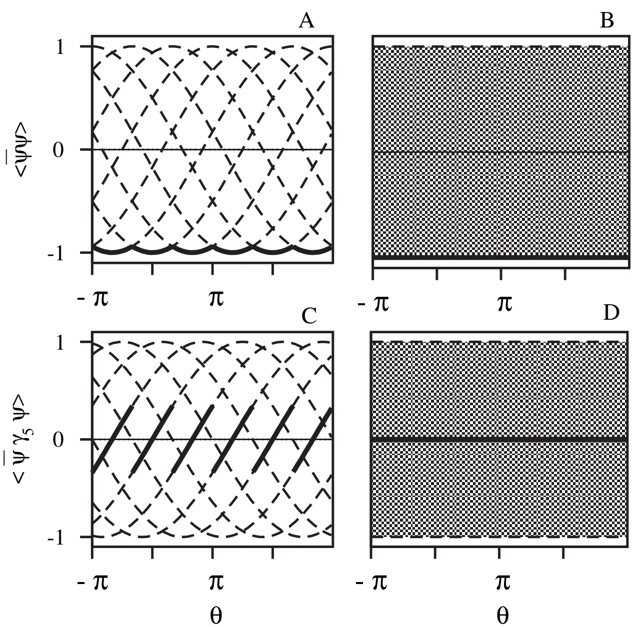

FIG. 1. The scalar (A and B) and pseudoscalar (C and D) quark condensates as functions of θ for $N_f = 3$ in units of $\aleph$. The plots A and C are for rational $q = 0.15$, while B and D correspond to any irrational q, for instance to $q = \frac{3}{2.02\pi^2} = 0.150\,47\ldots$ which is numerically only slightly different from 0.15. The dashed lines in A and C correspond to discrete minima of the free energy density which are degenerate for $m \equiv 0$. The solid bold lines denote the minimum which is chosen by an infinitesimally small mass term for a given θ. Points on the solid line in A, where two dashed lines cross each other, correspond to critical values of θ, at which CP is broken spontaneously. This is signalled by the discontinuity in the pseudoscalar condensate in C. For irrational q, as illustrated in B and D, the dashed lines densely cover the strip between 1 and -1, and the set of critical values of θ turns out to be dense in $\mathbb{R}$. As will be discussed in the last section, an analogy can be seen between A (C) and motion on a torus over a closed trajectory, while a manifestation of the Poincaré recurrence theorem resulting in dense winding of a torus (for example, see [16]) can be recognized in B (D).

For any real q this discrete symmetry and continuous nonsinglet flavor chiral symmetry are spontaneously broken, which becomes manifest if the infinitesimal masses are switched on. The term linear in masses in Eq. (10) selects a minimum for which $\alpha_{kl}(\theta)$ and the flavor nonsinglet angles β^a maximize $\sum_i^{N_f} m_i \cos(\Phi_i)$. There are specific values of the θ which are critical in the sense that two different minima $\alpha_{kl}(\theta_c)$ and $\alpha_{k'l'}(\theta_c)$ are degenerate in the presence of a mass term; these degenerate minima are CP-conjugates. The so-called Dashen phenomenon occurs [15]: CP is broken spontaneously for $\theta = \theta_c$ while for noncritical values CP is explicitly broken. For rational q there is only a finite number of θ_c in the interval $(-\pi, \pi]$, which includes [7,18] the value $\theta = \pi$. Irrational values of q lead to a drastically different picture: the set of critical values θ_c is dense in $\mathbb{R}$. In other words, any real value of θ is either critical or is a limit of a Cauchy sequence of critical values. CP is broken spontaneously in any arbitrarily small vicinity of any real value of θ, in particular, in the vicinity of $\theta = 0$. Understood in the sense of limiting values, the pseudoscalar condensate vanishes in both CP-conjugate vacua. The qualitative (and, in a sense, dramatic) difference in the consequences of q being rational or irrational is illustrated in Fig. 1.

Note, that at $\theta = 0$ CP is just an exact symmetry, it is not broken at all (neither spontaneously, nor explicitly). The point $\theta = 0$ is a limiting point of a set of critical points but it does not belong to this set: $\theta = 0$ itself is not critical. Thus there is no contradiction with the Vafa-Witten theorem [19].

IV. MASS RELATIONS AND ANOMALOUS WARD IDENTITY

Flavor $SU_{\mathrm{L}}(N_f) \times SU_{\mathrm{R}}(N_f)$ is broken spontaneously and the appropriate number of corresponding Goldstone modes are expected, while $U_A(1)$ is realized in a more complex non-Goldstone form for any θ, as shown in [10]. To make this manifest at the level of the free energy Eq. (10), one can consider small variations of the variables α and β^a in the vicinity of the minimizing configurations $\{\alpha = \alpha_{\min}(\theta), B_1 = B_2 = B_3 = 0\}$. To this end it is sufficient to consider variations $\delta\alpha$, $\delta\beta^a$. We organize the expansion as follows:

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_\beta + \mathcal{F}_\alpha. \tag{17}$$

The leading term is $\mathcal{F}_0 = -N_f \aleph m \cos[\alpha_{kl}(\theta)]$ which we have already identified in [10] as generating the quark condensate per flavor including its θ dependence, $\langle \bar{q}_n q_n \rangle = -\aleph \cos[\alpha_{kl}(\theta)]$. Fixing now $N_f = 3$ we next consider the terms involving fluctuations in $\delta\beta^a$:

$$\mathcal{F}_\beta = \frac{1}{4}\aleph m \cos[\alpha_{kl}(\theta)]\sum_{a=1}^{8}(\delta\beta^a)^2 - \frac{1}{8\sqrt{3}}\aleph m \sin[\alpha_{kl}(\theta)] \times \sum_{a=1}^{3}(\delta\beta^a)^2\delta\beta^8, \tag{18}$$

where we have shown all quadratic terms and one of the cubic terms. Equations (13)–(15) are used to derive this result. Let us now identify the flavor structure of the fermionic boundary condition Eq. (9) with that used in effective chiral theories [7] via

$$i(\beta^a\lambda^a/2 + \alpha) \equiv i\frac{\sqrt{2}}{F_\pi}\left(\pi^a\tau^a + \frac{1}{\sqrt{N_f}}\eta_0\right),$$

where π^a and η_0 are, respectively, identified as the non-singlet and singlet mesons and the τ^a are normalized such that $\mathrm{Tr}(\tau^a\tau^a) = \delta^{ab}$. The pion-decay constant is $F_\pi = 93$ MeV. Thus we identify, for example, for the neutral flavorless mesons, the fluctuations as

$$\delta\beta^3 = \frac{2\pi_0}{F_\pi}, \qquad \delta\beta^8 = \frac{2\pi_8}{F_\pi}, \qquad \delta\alpha = \sqrt{\frac{2}{N_f}}\frac{\eta_0}{F_\pi}. \tag{19}$$

Certainly what follows is not more than an illustration of what one can expect for the meson spectrum and interactions: a more solid investigation requires the construction of the effective meson action and calculation of the relativistic bound state spectrum in the spirit of [9]. Nevertheless, if we identify the coefficients of the terms in Eq. (18) which are quadratic in the variations as "meson" mass terms then we obtain

$$F_\pi^2 m_{\pi/\eta}^2 = 2m\aleph.$$

This takes into account that for irrational q one has $\cos[\alpha_{kl}(\theta)] = 1$ and $\sin[\alpha_{kl}(\theta)] = 0$ for any θ (see the bold solid lines in plots B and D in Fig. 1). Using our identification of $\aleph$ with the quark condensate, we thus recover the Gell-Mann–Oakes–Renner relation for masses of pions ($a = 1, 2, 3$) and the eta-meson ($a = 8$).

More significantly for the strong *CP*-problem we observe that in higher order terms in the decomposition those which are *CP*-even, namely, involving even powers of the fluctuations, come with a coefficient $\cos[\alpha_{kl}(\theta)] = 1$ for the allowed vacua. All *CP*-odd terms, those with odd powers of the fluctuations, are proportional to $\sin[\alpha_{kl}(\theta)]$ which exactly vanishes. *CP* is thus unbroken. For example, the term which would correspond to an $\eta\pi\pi$ interaction is the cubic term in Eq. (18). Using the identifications Eq. (19), this is

$$-\sin[\alpha_{kl}(\theta)]\frac{m_\pi^2}{2\sqrt{3}F_\pi}\pi_8(\pi_0)^2. \tag{20}$$

For q irrational this term exactly vanishes: the decay $\eta \to \pi\pi$ is exactly suppressed. For q rational, we can expand Eq. (20) in the vicinity of $\theta = 0$ using Eq. (16) and obtain $-[\theta/(6\sqrt{3})](m_\pi^2/F_\pi)\pi_8(\pi_0)^2$ which is the analogue of Eq. (8) in [20] (and similar results elsewhere) for three degenerate quarks. In the context of approaches such as [20], there is an *a priori* assumption that one can expand in small θ: the θ dependence is first chirally rotated from the topological charge density term into the quark mass terms and then an expansion performed for small quark mass. Using the small upper bound of the neutron dipole moment [21] such considerations lead to the expectation of an unusually small value for θ, leading to the familiar fine-tuning problem. However, in the context of irrational topological charge in the domain model, as a particular study of the class of fields specified by the condition Eq. (6) where the θ dependence can be handled exactly, we see that such an expansion misses the possibility of the cancellation of *CP*-conjugate vacua in *CP*-odd quantities. The absence of *CP*-violating hadronic processes such as $\eta \to \pi\pi$ (or, if baryons could be included in these considerations, a neutron dipole moment) is not then indicative of vanishing θ; physical observables are independent of θ, as in Fig. 1 (B,D), and still *CP* symmetry is exact in QCD with condensates, confinement and the correct realization of chiral symmetry, even in the presence of the θ term.

Finally we turn to the fluctuations in the singlet directions α. Consider then the integral

$$Z_V(\theta) = \mathcal{N}\int_{-\infty}^{+\infty} d\alpha e^{-V\mathcal{F}(\alpha,\vec{\beta}_{\min},\theta)}$$

with $V = vN$ and finite $N \gg 1$. The integral can be computed by means of the saddle-point approximation. The set of minima of the free energy density is obtained from the solution of equation

$$\begin{aligned}\mathcal{F}'_\alpha(\alpha, \vec{\beta}_{\min}, \theta) &= -v^{-1}\tan(qN_f\alpha - 2qk\pi - q\theta - 2l\pi) \\ &\quad - N_f m\aleph\sin(\alpha) \\ &= 0.\end{aligned}$$

The free energy density and its second derivative at the minima α_{kl} is

$$\begin{aligned}\mathcal{F} &= -\aleph m N_f\cos[\alpha_{kl}(\theta)] \\ \mathcal{F}''_\alpha &= \chi^{\mathrm{YM}}N_f^2 + \aleph m N_f\cos[\alpha_{kl}(\theta)],\end{aligned}$$

where $\chi^{\mathrm{YM}} = q^2/v = B^4R^4/128\pi^2$ is the topological susceptibility in the absence of quarks which in the domain model was evaluated in [5] to be approximately $(197\ \mathrm{MeV})^4$. With the identification of $\delta\alpha$ of Eq. (19) we thus extract the Witten-Veneziano mass formula for the eta-prime meson

$$m_{\eta'}^2F_\pi^2 = 2N_f\chi^{\mathrm{YM}} + m_\pi^2F_\pi^2 \tag{21}$$

in the physical vacuum. Continuing with the evaluation of the integral in the standard way we arrive at

$$Z_V(\theta) = \lim_{L\to\infty} \mathcal{N}_{L,N} \sum_{l=-L}^{L} \sum_{k=1}^{N_f-1} \times \frac{e^{NN_f vm\aleph \cos[\alpha_{kl}(\theta)]}}{[\chi^{\rm YM} N_f^2 + \aleph m N_f \cos[\alpha_{kl}(\theta)]]^{1/2}} \times [1 + \mathcal{O}(m)][1 + \mathcal{O}(1/N)], \tag{22}$$

with $Z_V(0) = 1$. The sum above runs over all local minima of the free energy, enumerated by k, l.

Recall now that the topological susceptibility

$$\chi \equiv \left(\frac{g^2}{32\pi^2}\right)^2 \int d^4x \langle F\cdot \tilde{F}(x) F\cdot \tilde{F}(0)\rangle = -\lim_{V\to\infty}\lim_{\theta\to 0} \frac{1}{Z_V(0)} \frac{1}{V} \frac{\partial^2}{\partial\theta^2} Z_V(\theta). \tag{23}$$

It can thus be extracted from the partition function by the double variation with respect to θ. It is important to note here that this relies on changing the order of taking the derivatives with respect to θ and functional integration, which is well defined only for a certain regularized form of the functional integral. In particular, the total volume V must be large but finite. In our context it means that the differentiation of Eq. (22) with respect to θ must be performed before taking the thermodynamic limit $N \to \infty$. When V is large but finite $Z_V(\theta)$ depends on theta. After the second derivative over θ is taken the limit $N\to\infty$ chooses the global minimum, i.e. l, k corresponding to the global minimum. The key point here is that for irrational q, $\forall \epsilon > 0$ and $\forall\theta$ there exist integers (l, k) [specifying the vacuum $\alpha_{kl}(\theta)$] such that $1 - \cos(\theta/N_f + 2\pi l/qN_f) < \epsilon$. The existence of such a minimum eliminates dependence of observables on θ. Moreover, taking the limit $\theta\to 0$ is unnecessary in Eq. (23): as defined by Eq. (23) but with $\theta \to 0$ omitted, the CP-even quantity χ does not depend on θ. We repeat though that the most important point here is that differentiation with respect to θ and the thermodynamic limit are not interchangeable; but deriving with respect to θ first is precisely how χ is obtained through Eq. (23).

Now let us use this to extract the topological susceptibility in the presence of light quarks. The aim, on the one hand, is to check consistency with the anomalous Ward identity of [22], and on the other to confirm the suppression of the susceptibility by quark loops, as proposed by [23] to fulfill the identities. According to Eq. (23) the result is

$$\chi^{\rm QCD} = \frac{m\aleph}{N_f} + \mathcal{O}(m^2) \tag{24}$$

and consistent (in Euclidean space) with the anomalous Ward identity of [22]:

$$N_f^2 \chi^{\rm QCD} = N_f m_\pi^2 F_\pi^2 + \mathcal{O}(m_\pi^4). \tag{25}$$

V. DISCUSSION

A. The Poincaré recurrence theorem

It is instructive to compare some aspects of the pictures arising for rational (in particular, integer) *versus* irrational values of q. The fundamental difference between these two cases is clearly seen in Fig. 1 but still requires precise formulation. As a matter of fact, plots (A,B) represent all inequivalent curves $-\cos[\alpha_{kl}(\theta)]$ while (C,D) show all inequivalent curves $\sin[\alpha_{kl}(\theta)]$.

First of all, for rational q there is a finite number $n_{\rm c}$ of critical points ($\{\theta_{\rm c}^i\}, i = 1, \dots, n_{\rm c}$) in the interval $-\pi < \theta \leq \pi$ and a countable set of critical points ($\{\theta_{\rm c}^i + 2\pi j\}, i = 1, \dots, n_{\rm c}, j \in \mathbb{Z}$) in $\mathbb{R}$, distributed $(2\pi/n_c)$-periodically in $\mathbb{R}$. In particular, $\theta = 0$ is not critical while $\theta = \pi$ is critical for any rational q. The number of critical values $n_{\rm c}$ is determined by the integer numerator q_1 of a rational number $N_f q = q_1/q_2$. For irrational q the interval $-\pi < \theta \leq \pi$ already contains a countable set of critical points ($\{\theta_{\rm c}^i\}, i = 1, \dots, \infty$) which is dense in this interval. Thus $\forall\theta \in \mathbb{R}$ we have either a critical point or a limiting point of a sequence of critical points. In particular, $\theta = 0$ is such a limiting point for any irrational q.

Furthermore, for rational q and $\forall\theta$, the set $\{C_j(\theta)\}$ ($j = 1, \dots, n_c$) contains a finite number of points. Here $C_j(\theta) = \cos[\alpha_j(\theta)]$. The representation for $Z_V(\theta)$ is analogous to Eq. (22) but contains a finite sum over $j = 1, \dots, n_c$, only one term of which dominates the sum $\forall\theta \neq \theta_{\rm c}$; the rest of the terms are exponentially suppressed at large volume. Such a representation indicates that $Z_V(\theta)$ is differentiable to infinite order $\forall\theta \neq \theta_{\rm c}$, which, in particular, means that CP-even observables can be computed for small θ as Taylor series in θ. At critical θ the first derivative of $Z_V(\theta)$ is discontinuous.

Irrational values of q change this behavior radically: $C = 1$ is the limiting point of a sequence $\{C_j(\theta)\}$ ($j = 1, \dots, \infty$) $\forall\theta$. This means by definition that $\forall\varepsilon > 0$ and $\forall\theta\exists J_\varepsilon(\theta)$ such that $1 - C_j(\theta) < \varepsilon$, $\forall j > J_\varepsilon(\theta)$. Simultaneously $|\mathcal{S}_j(\theta)| < \varepsilon$, $\forall j > J_\varepsilon(\theta)$. Here $\mathcal{S}_j(\theta) = \sin[\alpha_j(\theta)]$. The dependence of J_ε on θ is discontinuous $\forall\theta$ and thus the sum in Eq. (22) in the infinite volume limit represents the function $Z_V(\theta)$ which, strictly speaking, is not differentiable $\forall\theta$. However, for finite volume, each term in the series is a differentiable function and the sum, regularized by $L < \infty$ and $N < \infty$, can be differentiated term by term, with a subsequent taking of the limits $V, L \to \infty$ which are well defined. In this sense Eq. (22) gives a prescription for computing derivatives with respect to θ.

To underscore that the character of the θ dependence realized in the domain model does not represent anything unusual or "exotic" it is useful to map the above considerations to some textbook examples of application of the Poincaré recurrence theorem—one of the cornerstones of ergodic theory (for example, see [16]).

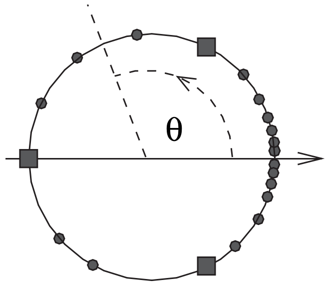

FIG. 2. Values of $\theta_l = g^l\theta$ as defined in Eq. (26): boxes occur for rational $q = q_1/q_2$ with $q_1 N_f = 3$, while circles represent some of the points densely distributed, in particular, in the neighborhood of $\theta = 0$ for irrational q.

The first straightforward example is just the present problem slightly rephrased. Consider the group of rotations by angle β acting on a point at polar angle θ on a circle. The lth element of this group corresponds to the angle

$$\theta_l = g^l\theta = \theta + \beta l (\mathrm{mod}\, 2\pi), \qquad \beta = 2\pi b.$$

According to the Poincaré recurrence theorem, for any rational b there exists a number l_b such that $\theta_{l_b} = \theta$. The points represented in Fig. 2 by the boxes correspond to the elements of the group. For any irrational b the set $\{\theta_l, l = 0\ldots\infty\}$ is dense on the circle everywhere, particularly in the vicinity of $\theta = 0$ as is illustrated by the filled circles in Fig. 2. Identifying the θ-parameter of QCD with the polar angle in Fig. 2 and the parameter b with the "topological charge" q by $b = 1/N_f q$ establishes the desired map.

Another simple example of the application of the Poincaré theorem is that of uniform motion on a torus,

$$\dot{\theta} = b_\theta, \qquad \dot{\phi} = b_\phi,$$

where θ and ϕ are the latitude and longitude of a point P on the torus as is shown in Fig. 3. If the ratio of velocities $b = b_\theta/b_\phi$ is a rational number then the point moves along a closed trajectory which can be characterized by an integer winding number. In the case of irrational b the trajectory is dense on the torus. If we take for simplicity $b_\phi = 1$, identify $b_\theta = 1/qN_f$ and replace the discrete parameter $2\pi l$ by a continuous variable t, then the plot A in Fig. 1 would correspond to motion on the torus over a closed trajectory with a winding number equal to the number of critical points n_c. An irrational q leads to a correspondence between the curves densely covering the strip in plot B to the motion of a point on a torus over the trajectory densely covering the torus.

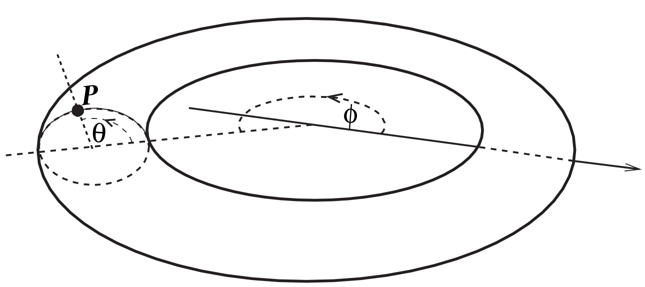

FIG. 3. Coordinates on a torus, as discussed in the text.

If we were to take the liberty of extrapolating the QCD motivated domain model considerations of this paper to genuine QCD, then we would conclude that for the class of background fields with $Q[\mathcal{A}]$ taking rational values the $U_A(1)$ problem can be solved but at the cost of simultaneously inviting the strong CP problem. On the other hand, irrational values of $Q[\mathcal{A}]$ are as natural for QCD with nonzero gluon condensates as rational (in particular, integer) values but providing clear signatures of resolving the $U_A(1)$ problem without creating the CP problem in strong interactions.

B. Model independent consideration

In this final part of the discussion, we give arguments as to how the mechanism for resolving the strong CP-problem can be identified in QCD in a model independent way. We start with the anomalous Ward identities of, for example, [18]. These identities are model independent and arise from the symmetries of QCD and basic properties of the pseudoscalar meson spectrum. A compact summary of this approach is given in Appendix B.1 of [10]. Without repeating those derivations here, a key result obtained is a differential equation for the θ dependence of the matrix of condensates

$$V_{ij} \propto \langle 0|\bar{q}_i q_j|0\rangle \tag{26}$$

whose phases are parametrized by angles ϕ_i. The true vacua are obtained by finding local minima with respect to perturbation of these angles by small ω_i.

From this one derives that a CP-even symmetry breaking term is given by

$$\langle \epsilon H'_{\mathrm{even}}\rangle = 2m\sum_i \cos\phi_i \tag{27}$$

and a CP-odd symmetry breaking term is given by

$$\langle \epsilon H'_{\mathrm{odd}}\rangle = 2m\sum_i \sin\phi_i. \tag{28}$$

One has the following model independent requirements then:

No $U(1)$ Goldstone boson, summarized in the equation

$$\frac{\partial}{\partial\theta}\sum_i \phi_i = 1. \tag{29}$$

Stationarity condition with respect to chiral perturbations, reflected by the equation

$$\sin\phi_i = \text{independent of } i. \tag{30}$$

Minimum with respect to chiral perturbations,

$$\cos\phi_i > 0. \tag{31}$$

CP-conservation,

$$\sum_i \sin\phi_i = 0, \tag{32}$$

which might hold for only one value of θ (e.g. $\theta = 0$) or, ideally, may somehow hold for all θ.

2π-periodicity. This is summarized in the equation

$$\sum_i \cos\phi_i|_\theta = \sum_i \cos\phi_i|_{\theta+2n\pi} \tag{33}$$

but cosine could just as well be replaced here by sine. This condition emerges from previous conditions but only in the presence [18] of the Dashen phenomenon at least at $\theta = \pi$.

Each of these conditions is realized in the domain model [10] and has appeared in previous sections of this paper. In the following we shall only derive results from these equations.

From Eq. (29) we extract the result

$$\sum_i \phi_i = \theta + c$$

for some θ-independent constant c. We can give the individual angles as

$$\phi_i = (\theta + c_i)/N, \qquad N^{-1}\sum_i c_i = c.$$

From Eq. (30) we have in turn that

$$\sin[(\theta + c_1)/N] = \sin[(\theta + c_2)/N] = \cdots = \sin[(\theta + c_N)/N],$$

thus

$$c_i = c \equiv \kappa\pi$$

is independent of i up to shifts of $2k\pi$. All the allowed solutions to Eqs. (29)–(31) are classified by

$$\phi_1 = \cdots = \phi_N = \varphi_k \equiv \varphi_0 + 2\pi k$$

with the restriction $\varphi_0 = (\kappa\pi + \theta)/N \in]-\pi/2, \pi/2[$, providing for positivity of the cosine. At this point κ is a real number.

Let us first implement CP-invariance at $\theta = 0$ just keeping κ arbitrary. Since ϕ_i are independent of i it suffices that $\sin\phi_i = 0$. Thus

$$\sin\kappa\pi/N = 0$$

which leads to $\kappa = Nm$ with m integer. Thus either κ can be absorbed in k or should be set to zero. The latter choice leads to the statement in [18] that CP-invariance at $\theta = 0$ fixes unambiguously the integration constant from Eq. (29).

Alternately let us characterize κ before seeking to impose CP-invariance at $\theta = 0$. Note to this end that any real number can be written as a product of an integer and some number between zero and one,

$$\kappa = 2l\xi, \qquad l \in \mathbb{Z}, \qquad \xi \in [0, 1/2]. \tag{34}$$

This opens the possibility that different values of l are appropriate for different ranges of θ subject to the above conditions. Thus far for CP-invariance at $\theta = 0$ we must have $l = 0$. But for $\theta \neq 0$ nonzero values of l satisfying condition

$$2l\xi\pi + \theta = 0 (\bmod 2\pi)$$

are now available due to the discrete nature of this label.

We are now ready to map the model independent approach identically to the framework that emerged from the domain model via the presence of the parameter ξ, which can have irrational values. In view of Eq. (16) it is tempting to make the identification

$$\xi \equiv 1/q.$$

The conclusion we draw is that the spectrum of solutions to the theta dependence of the QCD Ward identities is rich enough to exhibit the behavior seen in and originally derived in an explicit QCD motivated model, the domain model, which solves the strong CP-problem. However, the *interpretation* of the parameter ξ is possible at this stage only within the model, namely, as the mean value of a fraction of the topological charge related to the space-time regions where the dominating gluon configurations in the QCD vacuum can be considered as homogeneous, or succinctly—the mean topological charge *per* domain.

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