FRACTALS AND SUMSETS

by **Qinghe Yin**

Thesis submitted to the University of Adelaide
for the degree of Doctor of Philosophy

Department of Pure Mathematics
The University of Adelaide
South Australia

May 1993

Awarded 1994

CONTENTS

DECLARATION

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and , to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference had been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library being available for photocopying and loan.

Signed: . .... **Date:** 13/5/93

ACKNOWLEDGEMENT

I would like to show my special thanks to my supervisor, Prof. Gavin Brown, for all his help, advice and encouragement throughout my time as a post graduate student in both the University of New South Wales and the University of Adelaide. I would like to thank Prof. C.E. Sutherland, for his help during my period of study at the University of New South Wales. I would also like to thank Dr. Jane Pitman, my co-supervisor in the University of Adelaide, for her help in many aspects.

During my post-graduate study, I was supported by an Overseas Postgraduate Research Scholarship which covered the Overseas Student Charge, and by scholarships granted by the University of New South Wales and the University of Adelaide.

ABSTRACT

This thesis contains four chapters. Chapter 1 consists of the introductory part of the thesis and some mathematical background knowledge.

In Chapter 2, we deal with fractals generated by iterated function systems. We first estimate the Hausdorff dimension and box dimension for Markov attractors of iterated function systems, and fractals with respect to sofic systems. Then we study the structure of this kind of fractal. Finally, we calculate the box dimension of a class of fractal curves which are constructed as a part of Markov attractors of iterated function systems of affine maps.

In Chapter 3 and 4, we study the measure properties of sumsets. Let E, F be Borel subsets of the unit circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. We establish several inequalities of the form

$$m(E+F) \geq \mu(E)^{\alpha}\nu(F)^{\beta}$$

where m is the Lebesgue measure, μ and ν are probability measures on $\mathbf{T}$.

In Chapter 3 we consider μ, ν as probability measures which are uniformly spread on some (digit missing) Cantor sets. For example, when both μ and ν are the probability measure which is uniformly spread on the classical Cantor set C, we have

$$m(E+F) \geq \mu(E)^{\alpha}\mu(F)^{\beta},$$

where α, β fulfil $\alpha+\beta \geq \frac{\log 3}{\log 2}$, $3(\alpha^{-1}+\beta^{-1}) \leq 8$ and $\alpha, \beta \geq \frac{\log 3}{\log 2} - 1$.

In Chapter 4, we deal with coin tossing measures. Let $\mu_p = \mathop{*}\limits_{n=1}^{\infty}(p\delta_0 + (1-p)\delta_{\frac{1}{2^n}})$. Then

$$m(E+F) \geq \mu_{p_1}(E)^2\mu_{p_2}(F)^2$$

when $\max\{p_1, 1-p_1\} + \max\{p_2, 1-p_2\} \leq \frac{3}{2}$.

CHAPTER 1. Introduction and Preliminaries

§1. Introduction

This thesis mainly contains two parts. Chapter 2 is one of the two, in which we deal with the Hausdorff dimension, box dimension and structure of fractals generated by iterated function systems. Another part which contains Chapter 3 and 4, involves measure inequalities of sum sets.

Fractals have been of interest to scientists since the publication of the book[31] of B. Mandelbrot. A fractal by the definition of Mandelbrot is a set whose Hausdorff dimension exceeds its topological dimension. In [2] M. F. Barnsley and S. Demko introduced iterated function systems as a unified way of generating a broad class of fractals. Indeed, many fractals including self-similar sets (such as Cantor middle third set, von Koch curve, Sierpinski gaskets, etc), mixed self-similar sets, self-affine sets, Julia sets, and much more, can be considered as attractors or Markov attractors of iterated function systems. The terminology "self-similar" is used by many authors with different meanings. J.E. Hutchinson[26] gave a precise mathematical definition of self-similar sets and the formula to calculate their Hausdorff dimension. The Hausdorff dimension of self-affine sets was discussed by K.J. Falconer in [18],[20].

Barnsley [3] and Bedford [4] gave the box dimension of a class of self-affine curves. For the general case, [2] gave an estimate of the Hausdorff dimension for attractors of disjoint hyperbolic iterated function systems. D.B. Ellis and M.G. Branton[15] gave an upper bound of the Hausdorff dimension for Markov attractors of disjoint hyperbolic iterated function systems associated with primitive transition matrices and made a natural conjecture concerning the lower bound. C. Bandt[1] constructed fractals by iterated function systems with sofic systems which are more general than attractors and Markov attractors and calculated their Hausdorff dimension in some special cases.

The work which we present in Chapter 2 is mainly based on [15], [1] and [4]. In Chapter 2 we first estimate the Hausdorff dimension and box dimension for Markov attractors of disjoint hyperbolic iterated function systems when the transition matrix M is irreducible.And then we extend the result to more general cases. These results mainly appear in [49]. Afterwards we estimate the Hausdorff dimension of fractals generated with sofic systems under a disjointness condition by similar techniques. This result is also claimed in [49] without proof. Later we discuss the structure of attractors and Markov attractors of iterated function systems. This result appears in [50]. At last we calculate the box dimension of a class of fractal curves which are constructed as a part of Markov attractors of iterated function systems of affine maps.

Let E, F be subsets of a locally compact abelian group X. Of particular interest will be the case when X is the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. Define the sumset of E and F as

$$E + F = \{x + y | x \in E,\ y \in F\}.$$

We can also define the difference set $E - F$ as $E + (-F)$, where $-F = \{-x | x \in F\}$. If both E and F are very "thin"(e.g. in the sense of Haar measure zero), what can we say about the "thickness" of $E + F$? Measure properties of sumsets, especially when X is the real line $\mathbf{R}$ and E, F are Cantor sets on $\mathbf{R}$, are researched and applied by many authors for different purposes(see for examples [8],[10],[25],[29],[30],[32],[34],[36],[37],[38],[40],[42]]). As D.M. Oberlin pointed out "Two of the attributes of a locally compact abelian group are its Haar measure and its addition operation. An aspect of the relation between these is the behavior of the Haar measure of sets which are sums"(see [38]).

In 1946, M. Hall[25] gave a condition under which the sumset of two Cantor sets contains an interval and he used the result to prove that any real number can be expressed as a sum of two numbers in $F(4)$(the set of all the real numbers in the continued fraction of whose fractional part only 1, 2, 3, 4 appear). Measure properties of sumsets are also very important in the research of dynamical systems. For the purpose of studying the structure of the set of all diffeomorphisms on the sphere $\mathbf{S}^2$, S.E. Newhouse in his paper [34] gave a condition, which is similar to the condition of M. Hall, when the intersection of two Cantor sets are not empty, therefore the difference set of them contains an interval. He developed the condition and gave an concept of *thickness* for general Cantor set later([35],[36]). "It is crucial in dynamical systems"(R.F. Williams[47]). In [40] J. Palis and F. Takens gave a limit capacity(upper box dimension) condition when the Lebesgue measure of the difference set of two Cantor sets is zero and used the result to research the structure of bifurcation sets for certain one-parameter families of 2-dimensional diffeomorphism. In [39], Palis raised some questions about difference sets of Cantor sets.

The references directly leading up to the present work are [6],[7],[8],[10]. Now the circle $\mathbf{T}$ is in consideration. Use μ_c to denote the measure on $\mathbf{T}$ which is uniformly spread out on the classical Cantor middle third set C. G. Brown, who is my supervisor, and W. Moran([8]) proved that for any Borel subsets E, F of $\mathbf{T}$, one has

$$m(E+F) \geq \mu_c(E)^\alpha \mu_c(F)^\alpha. \tag{1}$$

where $\alpha = \frac{\log 3}{\log 4}$ and m is the Lebesgue measure. Oberlin([38]) obtained that

$$m(E+F) \geq m(E)^{1-\gamma}\mu_c(F) \tag{2}$$

where $\gamma = \frac{\log 2}{\log 3} = \dim(\mu_c)$. He also noticed the quantitative relation between the exponents of (1) and (2) and the Hausdorff dimensions of the measures involved. For (1) we have

$$\alpha \dim(\mu_c) + \alpha \dim(\mu_c) = 1 = \dim(m).$$

And

$$(1-\gamma)\dim(m) + \dim(\mu_c) = \dim(m)$$

for (2). The proof of (2) is based on the inequality

$$x^\alpha y^\alpha + \max\{x^\alpha(1-y)^\alpha,\ (1-x)^\alpha y^\alpha\} + (1-x)^\alpha(1-y)^\alpha \geq 1 \tag{3}$$

proved by D.R. Woodall([48]), where $0 \leq x,\ y \leq 1$ and $\alpha = \frac{\log 3}{\log 4}$. In [7], Brown proved some analytic inequalities, one of which is

$$1 + x + x^2 \geq (1+x^s)^{1/s}(1+x^t)^{1/t} \tag{4}$$

for $0 \leq x \leq 1$, where s, t are restricted by $\frac{1}{s} + \frac{1}{t} = \frac{\log 3}{\log 2}$ and $3(s+t) \leq 8$. All the others have similar forms. He claimed(without proof) that some of the inequalities he proved can be transferred into measure inequalities of sumsets. In [10] Brown and J.H. Williamson considered the infinite convolution measure

$$\mu_p = \mathop{*}_{n=1}^{\infty}(p\delta_0 + (1-p)\delta_{\frac{1}{2^n}}).$$

The main result of [10] is that there exist an integer n and a $\alpha > 0$ such that for any Borel subsets $E_1,\ E_2, \ldots,\ E_n$, one has

$$m(E_1 + E_2 + \cdots + E_n) \geq \mu_p(E_1)^\alpha \mu_p(E_2)^\alpha \cdots \mu_p(E_n)^\alpha \tag{5}$$

In Chapters three and four we prove several inequalities with the form

$$m(E+F) \geq \mu(E)^{\alpha}\nu(F)^{\beta} \tag{6}$$

where m is the Lebesgue measure, μ and ν are probability measures on $\mathbf{T}$ which are either m or singular to m, and where α, β are numbers satisfying

$$\alpha s(\mu) + \beta s(\nu) \geq 1. \tag{7}$$

Here $s(\cdot)$ is the singular exponent of the given measure.

In Chapter three we consider μ, ν as probability measures which uniformly spread on some (digit missing) Cantor sets. For example, the inequality (1) of Brown and Moran is sharpened as

$$m(E+F) \geq \mu_c(E)^{\alpha}\mu_c(F)^{\beta}$$

where α, β satisfies

$$\begin{cases} \alpha+\beta \geq \dfrac{\log 3}{\log 2} \\ 3(\alpha^{-1}+\beta^{-1}) \leq 8, \qquad \alpha,\ \beta \geq \dfrac{\log 3}{\log 2} - 1. \end{cases}$$

Part of these results appears in [12]. We also point out that a condition like (7) is necessary for an inequality like (6) in general case. In some cases, for example μ_c, we have $s(\mu) = \dim(\mu)$.

In Chapter four, we deal with coin tossing measures. We find that when $\max(p_1, 1-p_1) + \max(p_2, 1-p_2) \leq \frac{3}{2}$ we have

$$m(E+F) \geq \mu_{p_1}(E)^2\mu_{p_2}(F)^2 \tag{8}$$

for any Borel subsets E, F of $\mathbf{T}$. And the condition

$$\max(p_1, 1-p_1) + \max(p_2, 1-p_2) \leq \frac{3}{2}$$

is necessary for (8) in some sense.

The preliminary knowledge which we need appears in the next section.

§2. Preliminaries

Hausdorff Dimension. Let E be a subset of a metric space (X, d). A cover $\mathcal{U} = \{U_i\}$ (countable or finite) of E is called a *δ-cover* if $0 < diam(U_i) \leq \delta$ for each i.

Suppose α is a non-negative number. For any $\delta > 0$ let

$$\mathcal{H}_\delta^\alpha(E) = \inf\{\sum_{U_i \in \mathcal{U}} diam(U_i)^\alpha | \mathcal{U} \text{ is a } \delta\text{-cover of } E\}.$$

$\mathcal{H}_\delta^\alpha(E)$ is increasing as δ is decreasing. Denote

$$\mathcal{H}^\alpha(E) = \lim_{\delta \to 0} \mathcal{H}_\delta^\alpha(E)$$

(may be 0 or ∞). We call $\mathcal{H}^\alpha(E)$ the *α-dimensional Hausdorff measure* of E.

Proposition 1.1. *Suppose that $0 \leq \alpha < \beta$. If $\mathcal{H}^\alpha(E) < \infty$ then*

$$\mathcal{H}^\beta(E) = 0.$$

If $\mathcal{H}^\beta(E) > 0$ then

$$\mathcal{H}^\alpha(E) = \infty.$$

The Hausdorff dimension of E is defined by

$$\dim(E) = \inf\{\alpha | \mathcal{H}^\alpha(E) = 0\} = \sup\{\alpha | \mathcal{H}^\alpha(E) = \infty\}.$$

The "cover" in the above can also be replaced by "open cover" or "closed cover" without changing the value of Hausdorff dimension. By the definition we can get the following result easily(see [27]).

Proposition 1.2. *Let X,Y be two metric space with a map* $\Psi: X \mapsto Y$, *and* δ, c *be positive constants. Then we have*

(a) if $d(\Psi(x), \Psi(y)) \geq cd(x,y)^{\delta}$, *then*

$$\dim(Y) \geq \frac{1}{\delta}\dim(X) \ ;$$

(b) if $\Psi(X) = Y$ *and* $d(\Psi(x), \Psi(y)) \leq cd(x,y)^{\delta}$, *then*

$$\dim(Y) \leq \frac{1}{\delta}\dim(X) \ .$$

Let μ be a probability measure on X. The Hausdorff dimension of μ is defined as

$$\dim(\mu) = \inf\{\dim(E) | \mu(E) = 1\}.$$

Theorem 1.3. *Let* μ *be a measure on [0,1] defined by*

$$\mu = \mathop{*}_{n=1}^{\infty}(p_0\delta_0 + p_1\delta_{\frac{1}{m^n}} + ... + p_{m-1}\delta_{\frac{m-1}{m^n}})$$

where $\mathbf{p} = (p_0, p_1, ..., p_{m-1})$ *is a probability distribution (i.e.* $p_i \geq 0$ *and* $\sum_{i=0}^{m-1} p_i = 1$*). Then*

$$\dim(\mu) = -\frac{1}{\log m}\sum_{i=0}^{m-1} p_i \log p_i$$

(see Billingsley [5]).

Box Dimensions. There are many other definitions of dimension. Box-counting dimension or box dimension is one of the most widely used dimension.

Let E be a bounded non-empty subset of a metric space (X, d) and let $N_{\delta}(E)$ be the smallest number of sets of diameter at most δ which can cover E. The *lower* and *upper box dimensions* of E respectively are defined as

$$\underline{\dim}_B(E) = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}$$
$$\overline{\dim}_B(E) = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}.$$

The upper box dimension is also called *limit capacity* by many authors. If both dimensions are equal we call the common value the *box dimension* of E,

$$\dim_B(E) = \lim_{\delta \to 0} \frac{logN_\delta(E)}{-\log \delta}.$$

The relation of upper and lower box dimensions with Hausdorff dimension is

$$\dim(E) \le \underline{\dim}_B(E) \le \overline{\dim}_B(E).$$

Ergodic Theorem. (see Walters [45]). Let $(X, \mathcal{B}, m)$ be a probability space. We say a map $T : X \mapsto X$ is *measure-preserving*, if T is measurable (i.e. $B \in \mathcal{B} \Rightarrow T^{-1}(B) \in \mathcal{B}$) and $m(T^{-1}(B)) = m(B)$.

A measure-preserving map T is called *ergodic* if the only members B of $\mathcal{B}$ with $T^{-1}(B) = B$ satisfy $m(B) = 0$ or $m(B) = 1$.

Theorem 1.4. *Suppose* $T : (X, \mathcal{B}, m) \mapsto (X, \mathcal{B}, m)$ *is ergodic and* $f \in L^1(X, \mathcal{B}, m)$. *Then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) = \int_X f \, dm \qquad m\text{-a.e.}$$

Non-negative Matrix. Two $n \times n$ matrices are called *equivalent* if there is a permutation that transforms one into the other. An $n \times n$ matrix M is called *reducible* if it is equivalent to a matrix of the form

$$\widetilde{M} = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

where M_{11} and M_{22} are square matrices. Otherwise M is called *irreducible.*

An non-negative square matrix M (all entries of M are non-negative, written as $M \geq 0$) is called *primitive,* if $M^k > 0$ (all entries > 0) for some positive integer k.

Obviously, a primitive matrix is irreducible. For an irreducible non-negative matrix M, we have the following Perron-Frobenius Theorem (see [22]).

Theorem 1.5. *An irreducible non-negative matrix M always has a maximum positive eigenvalue λ. The moduli of all the other eigenvalues do not exceed λ. Moreover there is an eigenvector associated to λ with all positive coordinates.*

We will use $||M||$ to denote the λ in theorem 1.5, and call it the *Perron-Frobenius eigenvalue* of M.

Symbolic Dynamics. Denote

$$\Sigma_n^+ = \prod_{m=1}^{\infty} (1, ..., n),$$

where $n \geq 2$. Then Σ_n^+ is a compact topological space with the product topology of discrete topologies. We may define different metrics with different Hausdorff dimensions on Σ_n^+ which determine the same topology as the product topology. This will play an important role in Chapter 2.

Define a shift map τ on Σ_n^+ by

$$\tau(i_1 i_2 i_3 ...) = (i_2 i_3 ...)$$

and denote the inverse maps of τ as

$$\sigma_i(i_1 i_2 ...) = (i i_1 i_2 ...), \qquad i = 1, 2, ..., N.$$

Let $P = (p_{ij})$ be a stochastic matrix (i.e. $p_{ij} \geq 0$, $\sum_{j=1}^{N} p_{ij} = 1$) and $\mathbf{p} = (p_1, ..., p_N)$ be a probability distribution such that $\mathbf{p}P = \mathbf{p}$. Define a probability measure μ on Σ_n^+ by

$$\mu([i_1 ... i_m]) = p_{i_1} p_{i_1 i_2} ... p_{i_{m-1} i_m}$$

for each cylinder $[i_1 ... i_m]$(i.e. the set of all points of Σ_n^+ whose first n coordinates are $i_1, ..., i_n$). By the Kolmogorov extension theorem μ is a probability measure on $(\Sigma_n^+, \mathcal{B})$ where $\mathcal{B}$ is the Borel algebra of Σ_n^+. If P is irreducible then τ is ergodic on $(\Sigma_n^+, \mathcal{B}, \mu)$(see[5]).

Iterated Function System (i.f.s.). Let X be a compact metric space and $T_i : X \mapsto X$ $(i = 1, ..., n)$ be Borel measurable. Let $C(X)$ denote all continuous real-valued functions on X. $(X; T_1, ..., T_n)$ is called an *iterated function system* (i.f.s.), if there exists a probability distribution $\mathbf{p} = (p_1, ..., p_n)$ such that for any $f \in C(X)$, $\sum_{i=1}^{n} p_i(f \circ T_i)$ is also in $C(X)$ (see Barnsley and Demko [2]).

Clearly when all T_i's are continuous $(X; T_1, ..., T_n)$ is an i.f.s. and for any probability distribution $\mathbf{p}$, $\sum_{i=1}^{n} p_i(f \circ T_i)$ is continuous.

If T_i is a contraction for each i, $(X; T_1, ..., T_n)$ is called a *hyperbolic* iterated function system (h.i.f.s.).

For a h.i.f.s., there exist a non-empty compact subset A which fulfills

$$A = \bigcup_{i=1}^{n} T_i(A)$$

(see Hutchinson [26]). A is called the *attractor* of the h.i.f.s.. There is a continuous map

$$\phi : \Sigma_n^+ \mapsto A$$

defined by

$$\phi(i_1 i_2 ...) = \lim_{m \to \infty} T_{i_1} \cdots T_{i_m}(x), \quad \forall x \in X.$$

The following diagram is commutative

$$\begin{array}{ccc} \Sigma_n^+ & \xrightarrow{\phi} & A \\ \downarrow{\scriptstyle \sigma_i} & & \downarrow{\scriptstyle T_i} \\ \Sigma_n^+ & \xrightarrow{\phi} & A \end{array}$$

for all $i \in \{1, 2, ..., n\}$.

Self-Similarity. When X is a subset of Euclidean space $\mathbf{R}^m$ and T_i is a similitude map(there exist r_i, s.t. $d(T_i(x), T_i(y)) = r_i d(x, y))$ for each i, then the attractor A is called a *self-similar set.* Under the "open set condition" stated below, the Hausdorff dimension of A equals the value of α satisfying

$$\sum_{i=1}^{n} r_i^\alpha = 1.$$

Open Set Condition. We say that the h.i.f.s $(X; T_1, \cdots, T_n)$ satisfies the *open set condition* if there exists a non-empty bounded open set V such that

$$T_i(V) \bigcap T_j(V) = \emptyset, \ i \neq j \ \text{ and } \ T_i(V) \subset V, \ \ i, j = 1, ..., n.$$

CHAPTER 2. Fractals Related to Iterated Function Systems

§1. Introduction

In this chapter we deal with the Hausdorff dimension, box dimension and construction of fractals generated by iterated function systems(i.f.s.). It consists of six sections.

In section 2, we estimate the Hausdorff dimension and box dimension for attractors and Markov attractors of disjoint hyperbolic iterated function systems(h.i.f.s.). The main result of this section appears in Theorem 2.2 and is a generalization of Theorem 2.1 proved by Barnsley and Demko([2]). Theorem 2.1 says that if a disjoint iterated function system $(X; T_1, \cdots, T_n)$ satisfies

$$s_i d(x,\ y) \le d(T_i x,\ T_i y) \le \bar{s}_i d(x,\ y) \quad \forall x, y \in X,\ 1 \le i \le n \tag{1}$$

for some constants $0 < s_i \le \bar{s}_i < 1$, then

$$l \le \dim(A) \le u, \tag{2}$$

where A is the attractor of $(X; T_1, \cdots, T_n)$ and where l and u are given by $\sum_{i=1}^{n} s_i^l = 1$ and $\sum_{i=1}^{n} \bar{s}_i^u = 1$. In Theorem 2.2, we consider the Markov attractor A_M

of the disjoint h.i.f.s. $(X; T_1, \cdots, T_n)$ satisfying (1) with respect to an irreducible transition matrix M. We get

$$l \leq \dim(A_M)) \leq u, \tag{3}$$

where l, u are determined by $||MS^l|| = 1$ and $||M\bar{S}^u|| = 1$. The upper bound part is first proved by Ellis and Branton([15]) for pimitive M (primitivity is a stronger hypothesis than irreducibility). We also get that

$$\overline{\dim}_B(A_M) \leq u. \tag{4}$$

In section 3, we generalize the result of the previous section in two aspects. Firstly, we show that (3) also holds when M is reducible but has a non-zero eigenvalue. Secondly, we prove that when not all T_i's are contractions but $(X; T_1, \cdots, T_n)$ is cyclically contracting with M, $\dim(A_M)$ can also estimated by (3).

In section 4, we use techniques similar to section 2 to estimate Hausdorff dimensions of fractals generated by i.f.s. with sofic systems. The terminology "sofic system" comes from B. Weiss[46]. C. Bandt [1] constructed fractals from i.f.s. with sofic systems. This construction method is more general than the "partial self-similarity" of W.J. Gilbert[24] and the so called "MW method" (see [43])) of R.D. Mauldin and S.C. Williams[33]. [1] calculated the Hausdorff dimension when X is a subset of Euclidean space and T_i's are cyclically contracting similitudes satisfying an open set condition. We assume that X is a compact metric space like in previous sections and T_i's are not similitudes but satisfy (1). Under a disjointness condition we get an estimation of the Hausdorff dimension for this kind of fractal, which is similar to (3) for the Markov attractors. However, a Markov attractor is a special case of a fractal constructed with sofic systems.

In section 5, we discuss the structure of attractors and Markov attractors of hyperbolic iterated function systems. We show that when M is irreducible for a huge amount (in the sense *a.e.* for any ergodic measure) of elements $(j_1, j_2, ...) \in \Sigma_n^+(M^T)$ the Markov attractor A_M can be approached by the set

$$\{T_{j_k} T_{j_{k-1}} \cdots T_{j_1} x;\ k \geq m\}, \quad x \in X.$$

In fact, we will show that

$$A_M = \bigcap_{m=1}^{\infty} \overline{\{T_{j_k} T_{j_{k-1}} \cdots T_{j_1} x;\ k \geq m\}} \qquad \mu\text{-a.e.}$$

for all probability measures μ on $\Sigma_n^+(M^T)$ for which the shift operator is ergodic.

In section 6, we calculate the box dimension for a class of fractal curves. First we construct a class of continuous functions on closed intervals by iterated function systems with affine maps. The graphs of these functions are subsets of Markov attractors of the iterated function systems. Then we calculate the box dimension of the graphs. When the curves appear merely as attractors instead of Markov attractors, Barnsley [3] and Bedford [4] gave the result of box dimension. In general we do not know if the Hausdorff dimension of these curves is equal to box dimension. In the self-affine case, [23] gave a condition under which the Hausdorff dimension and the box dimension are equal.

§2. Markov Attractors

Definition 1. Let $(X;\ T_1, \cdots, T_n)$ be a h.i.f.s. with attractor A. We say $(X;\ T_1, \cdots, T_n)$ is a disjoint h.i.f.s. if $T_i(A) \bigcap T_j(A) = \emptyset$ when $i \neq j$.

The attractor A of a disjoint h.i.f.s. is totally disconnected.

Theorem 2.1. *Let $(X;T_1, \cdots, T_n)$ be a disjoint h.i.f.s.. Let A be the attractor of $(X;T_1, \cdots, T_n)$. Suppose that*

$$s_i d(x,y) \leq d(T_i x, T_i y) \leq \bar{s}_i d(x,y) \qquad \forall x, y \in X,\ 1 \leq i \leq n$$

for some constants $0 < s_i \leq \bar{s}_i < 1$. Then

$$l \leq \dim(A) \leq u,$$

where $\sum_{i=1}^{n} s_i^l = 1$ and $\sum_{i=1}^{n} \bar{s}_i^u = 1$.

This theorem was first established by Barnsley and Demko [2] when X is a compact subset of Euclidean space $\mathbf{R}^m$. Ellis and Branton [15] proved it for general metric spaces. When X is a compact subset of Euclidean space $\mathbf{R}^m$, the upper bound can be replaced by $\min\{u,\ m\}$.

Example 1. Suppose $0 < r_i < 1$, $i = 1, ..., n$. Define a metric d on Σ_n^+ by

$$d(\mathbf{i},\mathbf{j}) = \begin{cases} r_{i_1} \cdots r_{i_k}\ , & i_1 = j_1, \cdots, i_k = j_k; i_{k+1} \neq j_{k+1} \\ 0, & i_1 = j_1, i_2 = j_2, \cdots \\ 1, & i_1 \neq j_1. \end{cases}$$

where $\mathbf{i} = (i_1, i_2, \cdots)$, $\mathbf{j} = (j_1, j_2, \cdots) \in \Sigma_n^+$.

(Σ_n^+, d) is a compact space and we have

$$d(\sigma_i(\mathbf{i}), \sigma_i(\mathbf{j})) = r_i d(\mathbf{i}, \mathbf{j}).$$

for each i. Hence $((\Sigma_n^+, d); \sigma_1, \cdots, \sigma_n)$ is a h.i.f.s.. The attractor of this h.i.f.s. is (Σ_n^+, d) itself. Clearly, $\sigma_i(\Sigma_n^+) \bigcap \sigma_j(\Sigma_n^+) = \emptyset$ when $i \neq j$. This says that $((\Sigma_n^+, d); \sigma_1, \cdots, \sigma_n)$ is a disjoint h.i.f.s.. By theorem 2.1,

$$\dim((\Sigma_n^+, d)) = \alpha$$

where α fulfills $\sum_{i=1}^{n} r_i^\alpha = 1$.

Definition 2. An $n \times n$ matrix M is called a *Markov transition matrix* if all of its entries are 1 or 0. We say a sequence (finite or infinite) $i_1, i_2, \cdots$, where $i_j \in \{1, 2, \cdots, n\}$, is *$M$-admissible*, if

$$M_{i_j i_{j+1}} = 1$$

for all $j = 1, 2, \cdots$.

In the rest of this section, M always stands for a Markov transition matrix.

Denote the set of all the M-admissible element of Σ_n^+ as $\Sigma_n^+(M)$. Then $\Sigma_n^+(M)$ is a closed therefore compact subspace of Σ_n^+.

Definition 3. Let $(X;\ T_1, \cdots, T_n)$ be a h.i.f.s.. Let

$$A_M = \phi(\Sigma_n^+(M)).$$

Then A_M is called the *Markov attractor* of the system associated with M.

Clearly, A_M is a compact subset of the attractor A. The Markov attractor of $((\Sigma_n^+, d); \sigma_1, \cdots, \sigma_n)$ is just $\Sigma_n^+(M)$. Denote

$$J_i = \{\mathbf{i} = (i_1, i_2, \cdots) \in \Sigma_n^+(M) | i_1 = i\}.$$

Then

$$J_i = \bigcup_{M_{ij}=1} \sigma_i(J_j).$$

Let $B_i = A_M \bigcap T_i(A) = \phi(J_i)$. Then

$$B_i = \bigcup_{M_{ij}=1} T_i(B_j) \ .$$

When all T_i's are similitudes, A_M is called a *mixed self-similar set.*

Theorem 2.2. *Assume that $(X; T_1, \cdots, T_n)$ is the same as in theorem 2.1 and M is an irreducible Markov transition matrix, A_M is the Markov attractor associated with M. Then*

$$l \leq \dim(A_M) \leq u,$$

and

$$l \leq \dim(B_i) \leq u \qquad i = 1, \cdots, n,$$

where

$$||MS^l|| = 1 \quad \textit{and} \quad ||M\bar{S}^u|| = 1, \tag{3}$$

and where

$$S^l = \begin{pmatrix} s_1^l & & 0 \\ & \ddots & \\ 0 & & s_n^l \end{pmatrix}, \quad \bar{S} = \begin{pmatrix} \bar{s}_1^u & & 0 \\ & \ddots & \\ 0 & & \bar{s}_n^u \end{pmatrix}.$$

Ellis and Branton [15] proved the upper bound part when M is primitive and left the lowerbound part as a conjecture.

Applying theorem 2.2 to $((\Sigma_n^+, d); \sigma_1, \cdots, \sigma_n)$ we get

$$\dim((\Sigma_n^+(M), d)) = \alpha,$$

where α is determined by $||M\Lambda^\alpha|| = 1$, and where

$$\Lambda^\alpha = \begin{pmatrix} r_1^\alpha & & 0 \\ & \ddots & \\ 0 & & r_n^\alpha \end{pmatrix}.$$

We shall prove this result directly and then use it and proposition 1.2 to prove theorem 2.2.

Proposition 2.3. *Let M be an irreducible Markov transition matrix. Then*

$$\dim((\Sigma_n^+(M), d)) = \alpha,$$

and

$$\dim((J_i, d)) = \alpha, \qquad i = 1, \cdots, n,$$

where α is determined by $||M\Lambda^\alpha|| = 1$.

Proof. Let $\omega = (\omega_1, \cdots, \omega_n)^T$ be a Perron-Frobenius eigenvector of $M\Lambda^\alpha$, such that $\max\{\omega_i\} = 1$. For any cover $\mathcal{U}$ of $\Sigma_n^+(M)$, denote

$$\mathcal{U}(\alpha) = \sum_{U \in \mathcal{U}} diam(U)^\alpha.$$

First we show that $\dim((\Sigma_n^+(M), d)) \leq \alpha$. If we can find a sequence of δ_m-covers of $\Sigma_n^+(M)$ with $\delta_m \downarrow 0$, say $\{\mathcal{U}_m\}$, such that, for some positive constant c,

$$\mathcal{U}_m(\alpha) \leq c < \infty,$$

for each m, then we get the result.

Choose $\delta_m = (\max\{r_i\})^m$ and let

$$\mathcal{U}_m = \{[i_1 \cdots i_m] | (i_1 \cdots i_m) \text{ is } M\text{-admissible}\}.$$

Then $\mathcal{U}_m$ is a δ_m-cover of $\Sigma_n^+(M)$. We show that $\{\mathcal{U}_m\}$ has the required property. We have

$$\begin{aligned}
&\sum_{U\in\mathcal{U}_m} diam(U)^\alpha = \\
=&\sum\{(r_{i_1}\cdots r_{i_m})^\alpha|(i_1\cdots i_m) \text{ is } M\text{-admissible}\}\\
=&\sum_{(i_1\cdots i_m)} r_{i_1}^\alpha(M_{i_1i_2}r_{i_2}^\alpha)\cdots(M_{i_{m-1}i_m}r_{i_m}^\alpha)\\
=&\sum_{i,j} r_i^\alpha(M\Lambda^\alpha)_{ij}^{m-1}\\
\le&(\max\{r_\nu\})^\alpha\sum_i\sum_j(M\Lambda^\alpha)_{ij}^{m-1}\\
\le&(\max\{r_\nu\})^\alpha\cdot\frac{1}{\min\{\omega_\nu\}}\sum_i\sum_j(M\Lambda^\alpha)_{ij}^{m-1}\omega_j\\
=&(\max\{r_\nu\})^\alpha\cdot\frac{1}{\min\{\omega_\nu\}}\sum_i\omega_i.
\end{aligned}$$

The last expression is independent of m.

Now we show that $\dim((\Sigma_n^+(M),d)) \ge \alpha$. We will show that the set $\{\mathcal{U}(\alpha)|\mathcal{U}$ is a cover of $\Sigma_n^+(M)\}$ has a positive lower bound. Then the α-dimensional Hausdorff measure of $(\Sigma_n^+(M),d)$ is positive, therefore $\dim((\Sigma_n^+(M),d)) \ge \alpha$.

Note that any subset of Σ_n^+ can be extended to a cylinder with the same diameter. Since we are interested in the lower bound, and $\Sigma_n^+(M)$ is compact, we may restrict attention to consider the covers consisting of finite cylinders. Suppose $\mathcal{U} = \{C_i|1 \le i \le m\}$ is such a cover. If all C_i's are cylinders formed by strings of the same length, say t, then

$$\begin{aligned}
\mathcal{U}(\alpha) \ge& \sum\{diam([i_1,\cdots,i_t])^\alpha|(i_1\cdots i_t) \text{ is } M\text{-admissible}\}\\
=&\sum\{r_{i_1}^\alpha\cdots r_{i_t}^\alpha|(i_1\cdots i_t) \text{ is } M\text{-admissible}\}\\
=&\sum_{i_1,i_2,\cdots,i_t} r_{i_1}^\alpha(M_{i_1i_2}r_{i_2}^\alpha)\cdots(M_{i_{t-1}i_t}r_{i_t}^\alpha)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} r_i^\alpha \sum_{j=1}^{n} (M\Lambda^\alpha)_{ij}^{t-1} \\
&\geq \sum_{i=1}^{n} r_i^\alpha \sum_{j=1}^{n} (M\Lambda^\alpha)_{ij}^{t-1} \omega_j \\
&= \sum_{i=1}^{n} r_i^\alpha \omega_i .
\end{aligned}
$$

If they don't have the same length, assume the largest one is t, and say, $C_1 = [i_1, \cdots, i_k]$ with $k < t$. Refine the cover by cylinders formed by strings of length t. For C_1 we have

$$
\begin{aligned}
&\sum_{i_{k+1},\cdots,i_t} \{r_{i_1}^\alpha \cdots r_{i_k}^\alpha r_{i_{k+1}}^\alpha \cdots r_{i_t}^\alpha | (i_k i_{k+1} \cdots i_t) \text{ is } M\text{-admissible}\} \\
=& r_{i_1}^\alpha \cdots r_{i_k}^\alpha \sum_{j=1}^{n} (M\Lambda^\alpha)_{i_k j}^{t-k} \\
\leq& r_{i_1}^\alpha \cdots r_{i_k}^\alpha \cdot \frac{1}{\min\{\omega_\nu\}} \sum_{j=1}^{n} (M\Lambda^\alpha)_{i_k j}^{t-k} \omega_j \\
=& r_{i_1}^\alpha \cdots r_{i_k}^\alpha \cdot \frac{\omega_{i_k}}{\min\{\omega_\nu\}} \\
\leq& diam(C_1)^\alpha \cdot \frac{1}{\min\{\omega_\nu\}} .
\end{aligned}
$$

Hence

$$
\begin{aligned}
& diam(C_1)^\alpha \geq \\
\geq& \min\{\omega_\nu\} \cdot \sum_{i_{k+1},\cdots,i_t} \{r_{i_1}^\alpha \cdots r_{i_k}^\alpha r_{i_{k+1}}^\alpha \cdots r_{i_t}^\alpha | (i_k i_{k+1} \cdots i_t) \\
& \qquad \text{is } M\text{-admissible}\}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathcal{U}(\alpha) &= \sum_i diam(C_i)^\alpha \\
&\geq \min\{\omega_\nu\} \cdot \sum \{r_{i_1}^\alpha \cdots r_{i_t}^\alpha | (i_1 \cdots i_t) \text{ is } M\text{-admissible}\} \\
&\geq \min\{\omega_\nu\} \sum_{i=1}^{n} r_i^\alpha \omega_i ,
\end{aligned}
$$

which implies $\inf\{\mathcal{U}(\alpha)|\mathcal{U}$ is a cover of $\Sigma_n^+(M)\} > 0$. Hence $\dim((\Sigma_n^+(M), d)) \geq \alpha$.

As for J_i, since $\Sigma_n^+(M) = \bigcup_{i=1}^{n} J_i$, there exists at least one j such that $\dim((J_j, d)) = \alpha$. Because M is irreducible, for any i there exists a path from i to j. Suppose $(i, i_1, \cdots, i_k, j)$ is such a path. Then the set $\sigma_i\sigma_{i_1}\cdots\sigma_{i_k}(J_j)$ is contained in J_i. Hence

$$\dim((J_i, d)) \geq \dim((\sigma_i\sigma_{i_1}\cdots\sigma_{i_k}(J_j), d)) = \alpha. \quad \blacksquare$$

Theorem 2.2 can be proved by Proposition 2.3 and the following result.

Proposition 2.4. *Define two metrics d_1, d_2 on Σ_n^+ in the same way as in Example 1, where r_i is replaced by s_i and $\bar{s}_i$ respectively, $i = 1, \cdots, n$. Then for any subset E of A one has*

$$\dim(\phi^{-1}(E), d_1) \leq \dim(E) \leq \dim(\phi^{-1}(E), d_2).$$

Proof. By Proposition 1.2, it suffices to show that there exist positive constants c_1, c_2 such that

$$c_1 d_1(\mathbf{i},\mathbf{j}) \leq d(\phi(\mathbf{i}), \phi(\mathbf{j})) \leq c_2 d_2(\mathbf{i},\mathbf{j}),$$

where $\mathbf{i}$, $\mathbf{j}$ are arbitrary elements of Σ_n^+. Suppose that

$$\mathbf{i} = (i_1 \cdots i_k i_{k+1} \cdots), \quad \mathbf{j} = (i_1 \cdots i_k j_{k+1} \cdots)$$

with $i_{k+1} \neq j_{k+1}$. Then

$$\begin{aligned} d(\phi(\mathbf{i}), \phi(\mathbf{j})) &\leq \bar{s}_{i_1} \cdots \bar{s}_{i_k} d(\phi(\tau^k(\mathbf{i})), \phi(\tau^k(\mathbf{j}))) \\ &= d_2(\mathbf{i},\mathbf{j}) d(\phi(\tau^k(\mathbf{i})), \phi(\tau^k(\mathbf{j}))) \\ &\leq d_2(\mathbf{i},\mathbf{j}) diam(X), \end{aligned}$$

where τ is the shift map on Σ_n^+. And

$$\begin{aligned} d(\phi(\mathbf{i}),\phi(\mathbf{j})) &\geq s_{i_1}\cdots s_{i_k} d(\phi(\tau^k(\mathbf{i})),\phi(\tau^k(\mathbf{j}))) \\ &= d_1(\mathbf{i},\mathbf{j}) d(\phi(\tau^k(\mathbf{i})),\phi(\tau^k(\mathbf{j}))) \\ &\geq d_1(\mathbf{i},\mathbf{j}) \cdot \min_{i\neq j}\{d(T_i(A),T_j(A)\}. \end{aligned}$$

By the disjointness condition, $\min_{i\neq j}\{d(T_i(A),T_j(A)\} > 0$. ∎

Remark 1. Clearly, Theorem 2.1 is a special case of Theorem 2.2 when all the entries of M are 1.

Remark 2. The "disjointness" condition plays an important role in Theorem 2.1, Theorem 2.2. From the proof we can see that without the "disjointness" condition the upper bound still remains true. But the lowerbound may not be true, even if the "open set condition" holds.

Example 2(Falconer [20]). Let X be the triangle with vertices (0,0), (1,1), (1,−1), let f_1, f_2 be linear functions restricted on X, which map the three vertices into (0,0), (λ,λ), $(\lambda,0)$ and (0,0), $(\lambda,0)$, $(\lambda,-\lambda)$ respectively, where $0<\lambda<1$. Then $(X;f_1,f_2)$ is a h.i.f.s. which satisfies the "open set condition". For any $x,y\in X$ we have

$$d(f_i(x),f_i(y)) \geq \frac{\lambda}{2} d(x,y), \qquad i=1,2,$$

where the metric d is given by $d((x_1,x_2),(y_1,y_2)) = \max\{|x_1-y_1|,|x_2-y_2|\}$. When $\lambda=\frac{1}{2}$ the l of $(X;f_1,f_2)$ is $\frac{1}{2}$. But the attractor is a single point (0,0).

However, we can slightly generalize the "disjointness" condition of Theorem 2.2. Through the proof we can see that the condition that $(X;T_1,\cdots T_n)$ is a disjoint h.i.f.s. can be replaced by $B_i=\bigcup_{M_{ij}=1} T_i(B_j)$ is a disjoint union for each i. We can also see that the condition

$$s_i d(x,y) \leq d(T_i x, T_i y) \leq \bar{s}_i d(x,y) \tag{4}$$

for all $x,y\in X$ can be relaxed to (4) holds for all $x,y\in A_M$.

Theorem 2.2′. *Suppose that $(X; T_1, \cdots, T_n)$ is a h.i.f.s., and M is an irreducible Markov transition matrix, A_M is the Markov attractor associated with M. If for each i we have*

$$s_i d(x,y) \leq d(T_i x, T_i y) \leq \bar{s}_i d(x,y), \qquad \forall x, y \in A_M$$

and $B_i = \bigcup_{M_{ij}=1} T_i(B_j)$ is a disjoint union, then

$$l \leq \dim(A_M) \leq u,$$

and

$$l \leq \dim(B_i) \leq u, \qquad i = 1, \cdots, N$$

where l and u are given by $||MS^l|| = 1$ and $||M\bar{S}^u|| = 1$.

Further generalization will be given in next section.

Now we consider the upper box dimension for Markov attractors. For a h.i.f.s. $(X; T_1, \cdots, T_n)$ satisfying

$$d(T_i x, T_i y) \leq \bar{s}_i d(x,y) \tag{5}$$

we also have

$$\overline{\dim}_B(A) \leq u \tag{6}$$

where A is the attractor of $(X; T_1, \cdots, T_n)$ and u is given by $\sum_{i=1}^{n} \bar{s}_i^u = 1$ (see [19] p 123). We show that for Markov attractors we have a similar result.

Theorem 2.5. *Let $(X; T_1, \cdots, T_n)$ be a h.i.f.s. satisfying (5). Let A_M be the Markov attractor of $(X; T_1, \cdots, T_n)$ associated with an irreducible Markov transition matrix M. Then*

$$\overline{\dim}_B(A_M) \leq u$$

where u is determined by $||M\bar{S}^u|| = 1$.

We prove Theorem 2.5 in a similar way to the proof of the upper bound part of Theorem 2.2.

Lemma 1. *Under the same assumption as Proposition 2.3 we have*

$$\dim_B((\Sigma_n^+(M), d)) = \dim((\Sigma_n^+(M), d)) = \alpha.$$

Proof. Since we always have $\overline{\dim}_B((\Sigma_n^+(M), d)) \geq \dim((\Sigma_n^+(M), d))$ we need only to show $\overline{\dim}_B((\Sigma_n^+(M), d)) \leq \alpha$. Let $N(\delta)$ be the smallest number of sets of diameter at most δ which can cover $\Sigma_n^+(M)$. By an argument of [19] (p41) it suffices to show that

$$\overline{\lim_{k\to\infty}} \frac{\log N(\gamma^k)}{-k\log\gamma} \leq \alpha$$

for some $0 < \gamma < 1$. We choose $\gamma = \min\{r_i\}$ and cover A_M by M-admissible cylinders of diameters within $(\gamma^{k+1},\ \gamma^k]$. The existence of this kind of cover is due to the choice of γ. Since any two cylinders are either disjoint or one contained in the other, we see that the above cover is disjoint. Denote the cover as $\mathcal{V}_k$. For $[i_1, i_2, \cdots, i_l] \in \mathcal{V}_k$ we have

$$\gamma^{k+1} < r_{i_1} r_{i_2} \cdots r_{i_l} \leq \gamma^k.$$

Hence

$$N(\gamma^k)\gamma^{(k+1)\alpha} < \sum\{(r_{i_1} r_{i_2} \cdots r_{i_l})^\alpha | [i_1, i_2, \cdots, i_l] \in \mathcal{V}_k\}.$$

If we can show that $\sum\{(r_{i_1} r_{i_2} \cdots r_{i_l})^\alpha | [i_1, i_2, \cdots, i_l] \in \mathcal{V}_k\} \leq c$ for some $c > 0$, then we get that

$$\frac{\log N(\gamma^k)}{-k\log\gamma} < \alpha + \frac{\log c - \alpha\log\gamma}{-k\log\gamma}.$$

Therefore $\overline{\lim}_{k\to\infty} \dfrac{\log N(\gamma^k)}{-k\log\gamma} \le \alpha$.

Again we use $\omega = (\omega_1, \cdots, \omega_n)^T$ to denote the Perron-Frobenius eigenvector of $M\Lambda^\alpha$ with $\max\{\omega_i\} = 1$. Suppose the largest length of cylinders in $\mathcal{V}_k$ is t. For $[i_1, i_2, \cdots, i_l] \in \mathcal{V}_k$ $(l < t)$,

$$\begin{aligned}
&\sum\{(r_{i_1} r_{i_2} \cdots r_{i_l} r_{j_{l+1}} \cdots r_{j_t})^\alpha | (j_{l+1}, \cdots, j_t) \\
&\qquad\qquad\qquad \text{is } M\text{-admissible and } M_{i_l j_{l+1}} = 1\} \\
=&(r_{i_1} r_{i_2} \cdots r_{i_l})^\alpha \sum_{j=1}^{n} \{((M\Lambda^\alpha)^{t-l})_{ij} | M_{i_l i} = 1\} \\
>&(r_{i_1} r_{i_2} \cdots r_{i_l})^\alpha \sum_{j=1}^{n} \{((M\Lambda^\alpha)^{t-l})_{ij}\omega_j | M_{i_l i} = 1\} \\
=&(r_{i_1} r_{i_2} \cdots r_{i_l})^\alpha \sum_{M_{i_l i}=1} \omega_i \\
\ge&(r_{i_1} r_{i_2} \cdots r_{i_l})^\alpha \min\{w_j\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&(r_{i_1} r_{i_2} \cdots r_{i_l})^\alpha < \\
<&(\min\{w_j\})^{-1} \sum\{(r_{i_1} r_{i_2} \cdots r_{i_l} r_{j_{l+1}} \cdots r_{j_t})^\alpha | (j_{l+1}, \cdots, j_t) \\
&\qquad\qquad\qquad \text{is } M\text{-admissible and } M_{i_l j_{l+1}} = 1\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum\{(r_{i_1} r_{i_2} \cdots r_{i_l})^\alpha | [i_1, i_2, \cdots, i_l] \in \mathcal{V}_k\} \\
<&(\min\{w_j\})^{-1} \sum\{(r_{i_1} r_{i_2} \cdots r_{i_t})^\alpha | (i_1, i_2, \cdots, i_t) | \text{ is } M\text{-admissible}\} \\
=&(\min\{w_j\})^{-1} \sum_{i,\ j} ((M\Lambda^\alpha)^t)_{ij} \\
\le&(\min\{w_j\})^{-2} \sum_{i,\ j} ((M\Lambda^\alpha)^t)_{ij}\omega_j \\
=&(\min\{w_j\})^{-2} \sum_{i=1}^{n} \omega_i. \qquad \blacksquare
\end{aligned}$$

Lemma 2. *Let X, Y be two metric spaces of finite diameter with a map $\Psi : X \mapsto Y$ which is onto and satisfies*

$$d(\Psi(x), \Psi(y)) \leq cd(x,y)^{\delta}.$$

Then we have

$$\overline{\dim}_B(Y) \leq \frac{1}{\delta}\overline{\dim}_B(X).$$

Lemma 2 is an imitation of Proposition 1.2(b) for upper box-counting dimension.

Proof. For any finite cover $\mathcal{U}$ of X which consist of sets of diameters at most ϵ, $\Psi(\mathcal{U})$ is a finite cover of Y with sets of diameters at most $c\epsilon^{\delta}$. Hence we always have

$$N_{c\epsilon^{\delta}}(Y) \leq N_{\epsilon}(X)$$

which demonstrates the Lemma. ∎

Theorem 2.5 can be proved by Lemma 1, 2 and the fact

$$d(\phi(\mathbf{i}), \phi(\mathbf{j})) \leq d_2(\mathbf{i},\mathbf{j})diam(X),$$

(see the proof of Proposition 2.4).

Example 3. Any real number can be expressed as a continued fraction,

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + ...}}}$$

(finite or not)where $a_0 \in \mathbf{Z}$ and $a_i \in \mathbf{Z}^+,\ i \geq 1$. We denote it as $[a_0, a_1, a_2, \cdots]$. Let

$$F(0,n) = \{[0, a_1, a_2, \cdots] | 1 \leq a_i \leq n,\ 1 \leq i < \infty\}.$$

We will estimate the Hausdorff dimensions of $F(0,2)$ by Theorem 2.1. Rogers[41] gave the result $\frac{1}{3} \leq \dim(F(0,2) \leq \frac{2}{3}$.

Let

$$a = \min\{F(0,2)\} = [0,2,1,2,1,\cdots] = \frac{\sqrt{3}-1}{2}$$

and

$$b = \max\{F(0,2)\} = [0,1,2,1,2,\cdots] = \sqrt{3}-1.$$

Define $f_i : [a,b] \mapsto [a,b]$ by

$$f_i([0,a_1,a_2,\cdots]) = [0,i,a_1,a_2,\cdots], \qquad i = 1,\ 2.$$

In fact, we have $f_i(x) = \frac{1}{i+x}$ and $f_i'(x) = -\frac{1}{(i+x)^2}$. Hence

$$\frac{|x-y|}{(i+b)^2} \leq |f_i(x) - f_i(y)| \leq \frac{|x-y|}{(i+a)^2}$$

holds for all $x,\ y \in [a,b]$. Therefore, $([a,b]; f_1, f_2)$ is a h.i.f.s.. Its attractor is $F(0,2)$. Clearly it is disjoint. By theorem 2.1,

$$0.4599\cdots \leq \dim(F(0,2)) \leq 0.6429\cdots,$$

where $0.4599\cdots$ and $0.6429\cdots$ are the solutions of

$$\left(\frac{1}{1+(\sqrt{3}-1)}\right)^{2l} + \left(\frac{1}{2+(\sqrt{3}-1)}\right)^{2l} = 1$$

and

$$\left(\frac{1}{1+\frac{\sqrt{3}-1}{2}}\right)^{2u} + \left(\frac{1}{2+\frac{\sqrt{3}-1}{2}}\right)^{2u} = 1.$$

We can also consider $F(0,2)$ as the attractor of the h.i.f.s. $([a,b]; f_{11}, f_{12}, f_{21}, f_{22})$ where $f_{ij} = f_i \circ f_j$. Since

$$f_{ij}([0,a_1,a_2,\cdots]) = [0,i,j,a_1,a_2,\cdots] \tag{1}$$

i.e.

$$f_{ij}(x) = \frac{1}{i + \frac{1}{j+x}} = \frac{j+x}{ij+1+ix}, \tag{2}$$

we have

$$f'_{ij}(x) = \frac{1}{(ij+1+ix)^2}.$$

Hence

$$\frac{|x-y|}{(ij+1+ib)^2} \leq |f_{ij}(x) - f_{ij}(y)| \leq \frac{|x-y|}{(ij+1+ia)^2}.$$

By Theorem 2.1, we get

$$0.5047\cdots \leq \dim(F(0,2)) \leq 0.5634\cdots,$$

where $l = 0.5047\cdots$ and $u = 0.5634\cdots$ are determined by

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\left(\frac{1}{ij+1+i(\sqrt{3}-1)}\right)^{2l} = 1$$

and

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\left(\frac{1}{ij+1+i(\sqrt{3}-1)/2}\right)^{2u} = 1,$$

Now let E be the subset of $F(0,2)$ defined by

$$E = \{[0, a_1, a_2, \cdots] | a_i = 1 \text{ or } 2, \quad \text{but } (a_i, a_{i+1}, a_{i+2}) \neq (1,2,1), 1 \leq i \leq \infty\}.$$

We show that $\dim(E+E) < 1$ and then $m(E+E) = 0$ where m is the Lebesgue measure. Clearly E is the Markov attractor of $([a,b]; f_{11}, f_{12}, f_{21}, f_{22})$ with respect to

$$M = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Let $\Lambda = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{pmatrix}$. Then the characteristic polynomial of $M\Lambda$ is

$$\begin{aligned} &\det(\lambda I - M\Lambda) \\ =&\lambda^4 - (c_1 + c_4)\lambda^3 - (c_2c_3 + c_2c_4 + c_3c_4)\lambda^2 - c_2c_3c_4\lambda. \end{aligned}$$

Let $c_1 = \sqrt{\bar{s}_{11}} = \frac{1}{2+a} = \frac{3-\sqrt{3}}{3}$, $c_2 = \sqrt{\bar{s}_{12}} = \frac{1}{3+a} = \frac{5-\sqrt{3}}{11}$, $c_3 = \sqrt{\bar{s}_{21}} = \frac{1}{3+2a} = 2 - \sqrt{3}$ and $c_4 = \sqrt{\bar{s}_{22}} = \frac{1}{5+2a} = \frac{4-\sqrt{3}}{13}$. Then

$$\begin{aligned} &\det(\lambda I - M\Lambda) = \\ =&\lambda^4 - \frac{51 - 16\sqrt{3}}{39}\lambda^3 - \frac{313 - 166\sqrt{3}}{143}\lambda^2 - \frac{73 - 41\sqrt{3}}{143}\lambda. \end{aligned}$$

When $\lambda > 0$ we have

$$\det(\lambda I - M\Lambda) > \lambda^4 - 0.60\lambda^3 - 0.18\lambda^2 - 0.02\lambda.$$

Therefore, $|\lambda I - M\Lambda| > 0$ whenever $\lambda \geq 1$, which means that the number u determined by $||M\bar{S}^u|| = 1$ is less than $\frac{1}{2}$ since $||M\bar{S}^u||$ is decreasing with respect to u. Hence we have

$$\dim_H(E) \leq \overline{\dim}_B(E) \leq u < \frac{1}{2}.$$

Therefore

$$\begin{aligned} \dim(E+E) &\leq \dim(E \times E) \\ &\leq \dim(E) + \overline{\dim}_B(E) \quad \text{(see [19] p95)} \\ &< 1. \end{aligned}$$

In fact we can get that

$$0.385... \leq \dim(E) \leq \overline{\dim}_B(E) \leq 0.426...$$

where 0.385... and 0.426... are solutions of $||MS^l|| = 1$ and $||M\bar{S}^u|| = 1$. By the result $\dim(E+E) < 1$ we get that the Markov spectrum below $\sqrt{10}$ which is a subset of $2+(E+E)$ has Lebesgue measure zero(see [14]). The original proof is quite tough.

§3 Generalizations

In this section we will generalize the result of theorem 2.2 in two aspects.

I. M is reducible. Now we assume that M is reducible. We still use $||M||$ to denote the eigenvalue of M whose absolute value is the largest among all the eigenvalues of M. $||M||$ may be 0 for some M. We will show that the result of theorem 2.2 is also true when M is reducible but $||M|| > 0$. First we suppose that

$$M = \begin{pmatrix} M_1 & M_{12} \\ 0 & M_2 \end{pmatrix}$$

where M_1 and M_2 are $m \times m$ and $(n-m) \times (n-m)$ irreducible matrices respectively.

If $M_{12} = 0$, we split $(X; T_1, \cdots, T_n)$ into two h.i.f.s.'s, $(X; T_1, \cdots, T_m)$ and $(X; T_{m+1}, \cdots, T_n)$. Use A_{M_1} and A_{M_2} to denote the Markov attractor of $(X; T_1, \cdots, T_m)$ with respect to M_1 and of $(X; T_{m+1}, \cdots, T_n)$ with respect to M_2. By theorem 2.2 we get

$$l_1 \le \dim(A_{M_1}) \le u_1 \quad \text{and} \quad l_2 \le \dim(A_{M2}) \le u_2$$

where

$$||M_1 S_1^{l_1}|| = ||M_1 \bar{S}_1^{u_1}|| = 1 \quad \text{and} \quad ||M_2 S_2^{l_2}|| = ||M_2 \bar{S}_2^{u_2}|| = 1$$

and

$$S_1^{l_1} = \begin{pmatrix} s_1^{l_1} & & 0 \\ & \ddots & \\ 0 & & s_m^{l_1} \end{pmatrix}, \quad S_2^{l_2} = \begin{pmatrix} s_{m+1}^{l_2} & & 0 \\ & \ddots & \\ 0 & & s_n^{l_2} \end{pmatrix},$$

with similar definitions of $\bar{S}_1^{u_1}$ and $\bar{S}_2^{u_2}$.

Clearly,

$$A_M = A_{M_1} \bigcup A_{M_2} \quad \text{and} \quad l \leq \dim(A_M) \leq u$$

where

$$l = \max\{l_1, l_2\} \ , \quad u = \max\{u_1, u_2\}.$$

And we have

$$\|MS^l\| = \max\{\|M_1 S_1^l\}\|, \ \|M_2 S_2^l\|\} = 1,$$

and

$$\|M\bar{S}^{\bar{u}}\| = \max\{\|M_1 \bar{S}_1^u\|, \ \|M_2 \bar{S}_2^u\|\} = 1 \ .$$

If $M_{12} \neq 0$, the M-admissible sequence related to M_{12} is

$$i_1, i_2, \cdots, i_k, j_{k+1}, j_{k+2}, \cdots$$

where $(i_1, \cdots, i_k)$ and $(j_{k+1}, j_{k+2}, \cdots)$ are M_1 and M_2 admissible respectively with $M_{i_k j_{k+1}} = 1$. Since the set

$$\{(i_1, i_2, \cdots, i_k) \mid (i_1, i_2, \cdots, i_k) \ \text{is } M\text{-admissible}, k = 1, 2, \cdots\}$$

is countable, we may denote the related mappings by $T(1), T(2), \cdots$. Using $A_{M_{12}}$ to denote the set of all the M-attractive points related with M_{12} , we have

$$A_{M_{12}} = \bigcup_{i=1}^{\infty} T(i)(A_{M_2}) \ .$$

Hence

$$\dim(A_M) = \max\{\dim(A_{M_1}) \ , \ \dim(A_{M_2})\}$$

and

$$l \leq \dim(A_M) \leq u \ .$$

Certainly

$$\|MS^l\| = 1 \quad \text{and} \quad \|M\bar{S}^u\| = 1 \ .$$

In general, a Markov transition matrix M with $\|M\| > 0$, is equivalent to

$$\widetilde{M} = \begin{pmatrix} M_1 & & & * \\ & \ddots & & \\ & & M_{k-1} & \\ 0 & & & M_k \end{pmatrix}$$

where $M_1, \cdots, M_{k-1}$ are irreducible, and M_k irreducible or has the form

$$M_k = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

Without losing generality, we can assume

$$M = \begin{pmatrix} M_1 & & & * \\ & \ddots & & \\ & & M_{k-1} & \\ 0 & & & M_k \end{pmatrix}$$

If M_k is irreducible, from the above we know that

$$\dim(A_M) = \max\{ \dim(A_{M_1}), ..., \dim(A_{M_k}) \} \ .$$

Hence

$$l \leq \dim(A_M) \leq u$$

where

$$l = \max\{l_1, \cdots, l_k\} \quad \text{and} \quad u = \max\{u_1, \cdots, u_k\}$$

and l_i, u_i have the same meaning as in the case $k = 2$.

In the case $M_k = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$, there is no M_k-admissible element in $\Sigma_n^+(M)$, so $A_{M_k} = \emptyset$. Since $\|M\| > 0$, we must have $k > 1$. Again we have

$$\dim(A_M) = \max\{\dim(A_{M_1}), \cdots, \dim(A_{M_{k-1}}), \dim(A_{M_k})\}$$

and

$$l \leq \dim(A_M) \leq u$$

where

$$l = \max\{l_1, \cdots, l_{k-1}\} \quad \text{and} \quad u = \max\{u_1, \cdots, u_{k-1}\} \ .$$

In both cases we can easily see

$$\|MS^l\| = 1 \qquad \text{and} \qquad \|M\bar{S}^u\| = 1 \ .$$

Now we have proved

Theorem 2.6. *Assume that $(X; T_1, \cdots, T_n)$ is a hyperbolic iterated function system, and M is a Markov transition matrix with at least one non-zero eigenvalue and that $B_i = \bigcup_{M_{ij}=1} T_i(B_j)$ is a disjoint union. Then we have*

$$l \leq \dim(A_M) \leq u$$

where

$$||MS^l|| = 1 \qquad \text{and} \qquad ||M\bar{S}^u|| = 1.$$

As for the box dimension, we also have $\overline{\dim}_B(A_M) \leq u$. But we can not get this result from the above discussion, since for a countable union $E = \bigcup_{i=1}^{\infty} E_i$, we do not have $\overline{\dim}_B(E) = \sup_i\{\overline{\dim}_B(E_i)\}$ in general. We need some further calculation.

Again we assume $M = \begin{pmatrix} M_1 & M_{12} \\ 0 & M_2 \end{pmatrix}$ and M_1, M_2 are irreducible. We calculate that

$$\overline{\dim}_B((\Sigma_n^+(M), d_2)) \leq u.$$

We have $\Sigma_n^+(M) = \Sigma_n^+(M_1) \bigcup \Sigma_n^+(M_2) \bigcup \Sigma_n^+(M_{12})$, where

$$\Sigma_n^+(M_{12}) = \bigcup \{(i_1, ..., i_k, j_1, j_2, ...) | (i_1, ..., i_k) \text{ is } M\text{-admissible}, \\ (j_1, j_2, ...) \in \Sigma_n^+(M_2) \text{ and } M_{i_k j_1} = 1\}.$$

Let $\delta = \min\{\bar{s}_i\}$. We calculate that how many cylinders of diameter at most δ^m we need to cover $\Sigma_n^+(M_{12})$. We use $N_1(\delta^m)$, $N_2(\delta^m)$ and $N_{12}(\delta^m)$ to denote the least number of cylinders of diameter at most δ^m are used to cover $\Sigma_n^+(M_1)$, $\Sigma_n^+(M_2)$ and $\Sigma_n^+(M_{12})$.

Since $\overline{\dim}_B((\Sigma_n^+(M_1), d_2)) \leq u$ and $\overline{\dim}_B((\Sigma_n^+(M_)), d_2)) \leq u$, For given $\epsilon > 0$ there exists an N, whenever $m \geq N$ we have

$$\frac{\log N_i(\delta^m)}{-\log \delta^m} < u + \epsilon, \quad i = 1, 2.$$

Therefore there exists a $c > 0$ such that $N_i(\delta^k) \leq c\delta^{-k(u+\epsilon)}$ for each k.

We decompose $\Sigma_n^+(M_{12})$ in the following way: Let

$$J_k = \{(i_1, ..., i_t, j_1, j_2, ...) \in \Sigma_n^+(M_{12}) | \delta^{k+1} < \bar{s}_{i_1} ... \bar{s}_{i_t} \leq \delta^k\}.$$

Then $\Sigma_n^+(M_{12}) = \bigcup_{k=0}^{\infty} J_k$. Since $(i_1, ..., i_t)$ is M-admissible, there are at most $c_1 N_1(\delta^{k+1})$ different such elements for each k, where c_1 is a constant not exceeding $\max\{i \mid \frac{\min\{\bar{s}_j\}}{\max\{\bar{s}_j\}^i} \leq 1\} + 1$. Therefor for $k \leq m$ we can use at most

$c_1 N_1(\delta^{k+1}) \cdot N_2(\delta^{m-k+1}) \leq c_2 \delta^{-(m+2)(u+\epsilon)}$ cylinders of diameter at most δ^m to cover J_k, where $c_2 = c_1 c^2$. Obviously, $\bigcup_{k=m+1}^{\infty} J_k$ can be covered by at most $N_1(\delta^m)$ such cylinders. Hence $\Sigma_n^+(M_{12})$ can be covered by $(m+1)c_2\delta^{-(m+2)(u+\epsilon)}$ such cylinders. Therefore

$$\dim_B((M_{12}, d_2)) \leq u + \epsilon.$$

By the arbitrariness of ϵ we get $\dim_B((M_{12}, d_2)) \leq u$. Hence under the same assumption of Theorem 2.6, we can get that

$$\dim_B((M_{12}, d_2)) \leq u.$$

By lemma 2 of §2 we get

Theorem 2.7. *Under the same assumption of Theorem 2.6, we have*

$$\dim_B(A_M) \leq u$$

where u is the same as in Theorem 2.6.

II. Not all T_i's need be contractions. In this paragraph we generalize theorem 2.2 to the case $(X;\ T_1, \cdots, T_n)$ is cyclically contracting with respect to M.

Definition 4. A Markov transition matrix M is given. A *path* from i_1 to i_k is a finite M-admissible sequence $i_1, i_2, \cdots i_k$. A *cycle* is a path with $M_{i_k i_1} = 1$. By *elementary* path or *elementary* cycle we mean a path or a cycle for which $i_s \neq i_t$, when $s \neq t$.

Definition 5. Let $(X;\ T_1, \cdots, T_n)$ be an i.f.s., where T_i's are Lipschitz maps with $Lip(T_i) = r_i$(where $Lip(T) = \inf\{L | d(Tx, Ty) \leq Ld(x, y), \forall x, y \in X\}$), and M be a Markov transition matrix. $(X;\ T_1, \cdots, T_n)$ is called *cyclically contracting* with respect to M if for any elementary cycle $i_1, \cdots, i_k$ we have $r_{i_1} r_{i_2} \cdots r_{i_k} < 1$.

Theorem 2.8. (Feiste[21]) *Let $(X; T_1, \cdots, T_n)$ be an i.f.s., where T_i's are Lipschitz maps. If $(X; T_1, \cdots, T_n)$ is cyclically contracting with respect to an irreducible Markov transition matrix M then there is a unique N-tuple $B = (B_1, \cdots, B_n)$ of compact subsets $B_i \subset X$ with*

$$B_i = \bigcup_{M_{ij}=1} T_i(B_j)$$

for all $i \in \{1, \cdots, n\}$.

Let $A_M = \bigcup_{i=1}^{n} B_i$. We also call it Markov attractor of $(X; T_1, \cdots, T_n)$, though the attractor A may not exist in this situation. If $B_i \bigcap B_j = \emptyset$ we also call $(X; T_1, \cdots, T_n)$ disjoint iterated function system.

Now we generalize Theorem 2.2 to the case $(X; T_1, \cdots, T_n)$ is cyclically contracting.

Suppose that M is an irreducible Markov transition matrix, $\{r_1, r_2, \cdots, r_n\}$ is a group of positive constants such that for any elementary cycle $i_1, i_2, \cdots, i_k$ we have $r_{i_1} r_{i_2} \cdots r_{i_k} < 1$, and α a constant satisfying $||M\Lambda^\alpha|| = 1$ where

$$\Lambda^\alpha = \begin{pmatrix} r_1^\alpha & & 0 \\ & \ddots & \\ 0 & & r_n^\alpha \end{pmatrix}.$$

Let $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be the Perron-Frobenius eigenvector with $\max\{\omega_i\} = 1$ Like Example 1 of §2, we define a metric d on $\Sigma_n^+(M)$ by

$$d(\mathbf{i}, \mathbf{j}) = \begin{cases} (r_{i_1} \cdots r_{i_k})^\alpha \omega_{i_k}, & i_1 = j_1, \cdots, i_k = j_k;\ i_{k+1} \neq j_{k+1} \\ \max\{r_i\}, & i_1 \neq j_1, \end{cases}$$

where $\mathbf{i} = (i_1, i_2, \cdots)$, $\mathbf{j} = (j_1, j_2, \cdots)$ are M-admissible elements of Σ_n^+. We need to check the triangle inequality.

Suppose that $\mathbf{i} = (i_1, \cdots, i_k, i_{k+1}, \cdots)$, $\mathbf{j} = (i_1, \cdots, i_k, j_{k+1}, \cdots)$ and $\mathbf{t} = (i_1, \cdots, i_r, t_{r+1}, \cdots)$ are elements of $\Sigma_n^+(M)$ with $i_{r+1} \neq t_{r+1}, i_{k+1} \neq j_{k+1}$ and $r < k$. We have

$$\begin{aligned} d(\mathbf{i}, \mathbf{t}) &= (r_{i_1} \cdots r_{i_r})^l v_{i_r} \\ &= (r_{i_1} \cdots r_{i_r})_l \sum_{\nu=1}^{n} (M\Lambda^\alpha)_{i_r \nu}^{k-r} v_\nu \\ &\geq (r_{i_1} \cdots r_{i_k})^\alpha \omega_{i_k} \\ &= d(\mathbf{i}, \mathbf{j}). \end{aligned}$$

Hence $d(\mathbf{i},\mathbf{j}) \leq d(\mathbf{i},\mathbf{t}) + d(\mathbf{t},\mathbf{j})$. It is shown that d is really a metric on $\Sigma_n^+(M)$. But this time d is not a metric on Σ_n^+. In the same way as section 2 we can also prove

Proposition 2.9. $\dim((\Sigma_n^+(M), d)) = \dim_B((\Sigma_n^+(M), d)) = 1.$

By Proposition 2.9, Proposition 1.2 ($\delta = l^{-1}$ for first part, $\delta = u^{-1}$ for second) and Lemma 2 of §2 we get

Theorem 2.10. *Let M be an irreducible Markov transition matrix, and $\{X; T_1, \cdots, T_n\}$ be an iterated function system cyclically contracting with M satisfying*

$$s_i d(x,y) \leq d(T_i x, T_i y) \leq \bar{s}_i d(x,y).$$

Suppose that $B_i = \bigcup_{j=1}^{n} T_i(B_j)$ is a disjoint union for each i. Then

$$l \leq \dim(A_M) \leq \overline{\dim}_B(A_M) \leq u,$$

and

$$l \leq \dim(B_i) \leq \overline{\dim}_B(B_i) \leq u, \qquad i = 1, \cdots, n,$$

where $\|MS^l\| = 1$ and $\|M\bar{S}^u\| = 1$.

In the same way as part I, we can generalize Theorem 2.10 to the case $\|M\| > 0$ (needn't be irreducible). At last we get

Theorem 2.11. *Suppose that M is a Markov transition matrix with at least one non-zero eigenvalue, and $(X; T_1, \cdots, T_n)$ is an iterated function system cyclically contracting with M satisfying*

$$s_i d(x,y) \le d(T_i x, T_i y) \le \bar{s}_i d(x,y),$$

and that $B_i = \bigcup_{i=1}^{n} T_i(B_j)$ is a disjoint union for each i. Then we have

$$l \le \dim(A_M) \le \overline{\dim}_B(A_M) \le u$$

where $\|MS^l\| = 1$ and $\|M\bar{S}^u\| = 1$.

§4 Sofic Systems

Definition 6. Let Q_k be a nonempty subset of $\{1,\cdots,n\}) \times \{1,\cdots,m\}$, for $k = 1,\cdots,m$. The subset of Σ_n^+

$$F = \{\mathbf{i} = (i_1, i_2, \cdots) \in \Sigma_n^+ | \text{there are } k_0, k_1, \cdots \in \{1,\cdots,m\} \text{ with } (i_\nu, k_\nu) \in Q_{k_{\nu-1}} \}$$

is called a *sofic system.*

We say a finite sequence $(i_1, i_2, \cdots, i_s)$ is *F-admissible*, if there is an element of F which begins with $(i_1, i_2, \cdots, i_s)$.

Let $(X; T_1, \cdots, T_n)$ be an i.f.s., where T_i is a Lipschitz map with $Lip(T_i) = r_i$ for each i. If for any cycle $(i_1, i_2, \cdots, i_l, i_1)$ which appears in an element of F, one has

$$r_1 \cdot r_2 \cdots r_l < 1$$

we say that $(X; T_1, \cdots, T_n)$ is *cyclically contracting* with F.

Bandt[1] showed that if $(X; T_1, \cdots, T_n)$ is cyclically contracting with F, then there exist non-empty compact subsets C_1, C_2, ..., C_m such that

$$C_k = \bigcup \{T_i(C_\nu) | (i, \nu) \in Q_k\}. \tag{1}$$

Let $C = \bigcup_{k=1}^{m} C_k$, and $F_k = \{\mathbf{i} = (i_1, i_2, \cdots) \in F | k_0 = k\}, i = 1, ..., m$. If $(X; T_1, \cdots, T_n)$ is a h.i.f.s., then we have

$$\phi(F) = C, \quad \text{and } \phi(F_k) = C_k, \quad k = 1, ..., m, \tag{2}$$

where $\phi: \Sigma_n^+ \mapsto A$ is defined in §1.2. If $(X; T_1, \cdots, T_n)$ is not a h.i.f.s., we can define $\phi: F \mapsto C$ in the same way and we also have (2).

When $m = 1$, we have $C = A$. For $m = n$ and $Q_i = \{(i,j),\ M_{ij} = 1\}$, where $M = (M_{ij})_{n\times n}$ is a Markov transition matrix, we get $C_i = B_i$, for $i = 1, ..., n$.

Now suppose that $(X; T_1, ..., T_n)$ is cyclically contracting with F. Denote $\mathbf{r} = (r_1, ..., r_n)$, where $r_i = Lip(T_i)$. Construct a matrix $M(\mathbf{r}, \alpha)$ by letting

$$M_{ij} = \sum \{r_k^\alpha | (k,j) \in Q_i\}. \tag{3}$$

In order to use the Perron-Frobenius theorem, we assume that $||M(r,\alpha)||$ is irreducible. Then there exists a unique value of α, such that

$$||M(\mathbf{r}, \alpha)|| = 1, \tag{4}$$

where $||M||$ is the Perron-Frobenius eigenvalue of M.

When X is a subset of Euclidean space R^q, T_i's are cyclically contracting similitudes satisfying a kind of open set condition, Bandt proved that

$$\dim(C_i) = \alpha, \quad i = 1, ..., n,$$

where α is determined by (4).

Now we want to estimate the upper and lower bound of the Hausdorff dimension of C_k's in general metric spaces with T_i's being not similitudes but satisfying

$$s_i d(x,y) \le d(T_i(x), T_i(y)) \le \bar{s}_i d(x,y).$$

Theorem 2.12. *Suppose that $\{X; T_1, ..., T_n\}$ is cyclically contracting with F and satisfies that*

$$s_i d(x,y) \le d(T_i(x), T_i(y)) \le \bar{s}_i d(x,y),$$

and that the matrix defined by (3) is irreducible. Suppose that for each k

$$C_k = \bigcup_{(i,\nu)\in Q_k} T_i(C_\nu)$$

is a disjoint union. Then

$$l \leq \dim(C) \leq \overline{\dim}_B(C) \leq u,$$

and

$$l \leq \dim(C_k) \leq \overline{\dim}_B(C_k) \leq u, \qquad k = 1, \cdots m,$$

where l and u are determined by

$$||M(\mathbf{s}, l)|| = 1 \text{ and } ||M(\bar{\mathbf{s}}, u)|| = 1$$

respectively.

Proof. First we assume that $(X; T_1, ..., T_n)$ is a h.i.f.s.. Like in §2, we work on Σ_n^+. Let $r_i \in (0, 1)$, $i = 1, ..., n$, and d be the metric defined in Example 1. Then we have

Lemma 1. *Suppose that for each i*

$$F_k = \bigcup_{(i,\nu)\in Q_k} \sigma_i(F_\nu)$$

is a disjoint union. Then

$$\dim(F_k, d) = \dim_B(F_k, d) = \alpha,$$

where α is determined by (4).

Proof of Lemma 1. Let $\omega = (\omega_1, ..., \omega_m)^T$ be the Perron-Frobenius eigenvector of $M(\mathbf{r}, \alpha)$ with $\max\{\omega_k\} = 1$. First we show that $\overline{\dim}_B(F_k) \leq \alpha$. As in the proof of Lemma 1 of §2, let $\gamma = \min\{r_i\}$. We calculate the the number of F-admissible cylinders of diameter at most γ^r being needed to cover F_k. Let

$$\mathcal{V}_r = \{[i_1, \cdots, i_s] | (i_1, \cdots, i_s) \text{ is } F_k\text{-admissibale},$$
$$r_{i_1} \cdots r_{i_{s-1}} > \gamma^r, \text{ and } r_{i_1} \cdots r_{i_s} \leq \gamma^r\}.$$

Then $\mathcal{V}_r$ is a disjoint cover of F_k with cylinders of diameter at most γ^r. Suppose the largest length of cylinders in $\mathcal{V}_r$ is t. For $[i_1, \cdots, i_s] \in \mathcal{V}_r$ with $s < t$, we have

$$(r_{i_1} \cdots r_{i_s})^\alpha \leq \frac{1}{\min\{\omega_j\}} (r_{i_1} \cdots r_{i_s})^\alpha \sum_{k_s} \omega_{k_s}$$

where the index k_s is determined by

$$k_0, k_1, \cdots, k_s \in \{1, \cdots, m\}, \ (i_\nu, k_\nu) \in Q_{k_{\nu-1}}, \ k_0 = k.$$

Therefore

$$\begin{aligned}
&(r_{i_1} \cdots r_{i_s})^\alpha \leq \\
\leq & \frac{1}{\min\{\omega_j\}} (r_{i_1} \cdots r_{i_s})^\alpha \sum_{k_s} \sum_j [(M(\mathbf{r}, \alpha))^{t-s}]_{k_s j} \omega_j \\
\leq & \frac{1}{\min\{\omega_j\}} (r_{i_1} \cdots r_{i_s})^\alpha \sum_{k_s} \sum_j [(M(\mathbf{r}, \alpha))^{t-s}]_{k_s j} \\
= & \frac{1}{\min\{\omega_j\}} (r_{i_1} \cdots r_{i_s})^\alpha \sum_{i_{s+1}, \cdots, i_t} \{r_{i_{s+1}}^\alpha \cdots r_{i_t}^\alpha | (i_1, \cdots, i_s, i_{s+1} \cdots, i_t) \\
& \qquad \text{is } F_k\text{-admissible}\}
\end{aligned}$$

Use N_r to denote the number of cylinders in $\mathcal{V}_r$. Then

$$\begin{aligned}
N_r \gamma^{(r+1)\alpha} < & \sum_{[i_1, \cdots, i_s] \in \mathcal{V}_r} (r_{i_1} \cdots r_{i_s})^\alpha \\
\leq & \frac{1}{\min\{\omega_j\}} \sum_{i_1, \cdots, i_t} \{(r_{i_1} \cdots r_{i_t})^\alpha | (i_1, \cdots, i_t) \text{is } F_k\text{-admissible}\} \\
= & \frac{1}{\min\{\omega_j\}} \sum_j [(M(\mathbf{r}, \alpha))^t]_{kj}
\end{aligned}$$

$$\leq \frac{1}{(\min\{\omega_j\})^2} \sum_j [(M(\mathbf{r},\alpha))^t]_{kj}\omega_j$$
$$= \frac{\omega_k}{(\min\{\omega_j\})^2}$$
$$\leq \frac{1}{(\min\{\omega_j\})^2}.$$

Hence we get $\overline{\dim}_B(F_k) \leq \alpha$.

Next we show that $\dim(F_k, d) \geq \alpha$. Suppose $\mathcal{U} = \{U_i\}$ is any open cover of F_k. We show that

$$\inf\{\mathcal{U}(\alpha)\} > 0,$$

the infimum is for all open covers. By the same reason as in the proof of Theorem 2.2, we need only consider covers consisting of finite many cylinders. Suppose $\mathcal{U} = \{D_i\}$ is such a cover.

If all the D_i's are formed by strings of the same length, say t, then

$$\sum diam(D_\nu)^\alpha \geq$$
$$\geq \sum \{r_{i_1}^\alpha \cdots r_{i_t}^\alpha | \text{there exist } k_1, k_2, ..., k_t$$
$$\text{s.t. } (i_\nu, k_\nu) \in Q_{k_{\nu-1}}, k_0 = k\}$$
$$\geq \min\{\omega_\nu\} > 0.$$

If they don't have the same length, assume the largest one is t, and say, $D_1 = [i_1, ..., i_s]$ with $s < t$. Refine the cover by cylinders formed by strings of length of t. For D_1 we have

$$\sum_{(i_{s+1},\ldots,i_t)} \{r_{i_1}^\alpha \cdots r_{i_s}^\alpha r_{i_{s+1}}^\alpha \cdots r_{i_t}^\alpha | \text{there exist } k_{s+1}, ..., k_t$$
$$\text{s.t.} (i_\nu, k_\nu) \in Q_{k_{\nu-1}}, k_0 = k\}$$
$$\leq r_{i_1}^\alpha \cdots r_{i_s}^\alpha \cdot \frac{1}{\min\{\omega_\nu\}} \sum_{i,j} (M(\mathbf{r},\alpha))_{ij}^{t-s}\omega_j$$
$$\leq r_{i_1}^\alpha \cdots r_{i_s}^\alpha \cdot \frac{m}{\min\{\omega_\nu\}}$$
$$= c_1 diam(D_1)^\alpha.$$

Hence

$$diam(D_1)^{\alpha} \geq$$

$$\geq \sum_{(i_{s+1},\ldots,i_t)} \{r_{i_1}^{\alpha} \cdots r_{i_s}^{\alpha} r_{i_{s+1}}^{\alpha} \cdots r_{i_t}^{\alpha} | \text{there exist } k_1, k_2, \ldots, k_t$$

$$\text{s.t. } (i_\nu, k_\nu) \in Q_{k_{\nu-1}}, k_0 = k\}.$$

Therefore

$$\sum diam(D_\nu)^{\alpha} \geq c_1^{-1} \min\{\omega_k\} = c > 0,$$

which implies $\inf\{\mathcal{U}(\alpha)\} \geq c > 0$. The lemma is proved.

Lemma 2. *The condition* $C_k = \bigcup_{(i,\nu)\in Q_k} T_i(C_\nu)$ *being a disjoint union for each* k *implies that* ϕ *is invertible on each* F_k.

Let $\mathbf{i} = (i_1 i_2 \ldots i_m i_{m+1} \ldots)$ and $\mathbf{j} = (i_1 i_2 \ldots i_m j_{m+1} \ldots)$ are two different elments of F_k with $i_{m+1} \neq j_{m+1}$. We show that $\phi(\mathbf{i}) \neq \phi(\mathbf{j})$. Since $\mathbf{i}$, $\mathbf{j} \in F_k$, there exist $k_1, \ldots, k_m, k_{m+1}$, and $k'_1, \ldots, k'_m, k'_{m+1}$, such that $(i_\nu, k_\nu) \in Q_{k_{\nu-1}}$, $(i_\nu, k'_\nu) \in Q_{k'_{\nu-1}}$ for $\nu = 1, \ldots, m$, where $k_0 = k'_0 = k$, and $(i_{m+1}, k_{m+1}) \in Q_{k_m}$, $(j_{m+1}, k'_{m+1}) \in Q_{k'_m}$. Let

$$\mu = \min\{m+1, \ \nu | k_\nu \neq k'_\nu\}.$$

If $\mu = m+1$, let $x = \phi(i_{m+1} \ldots)$ and $y = \phi(j_{m+1} \ldots)$. Then

$$x \in T_{i_{m+1}}(C_{k_{m+1}}) \subset C_{k_m}, \quad \text{and} \quad y \in T_{j_{m+1}}(C_{k'_{m+1}}) \subset C_{k_m}.$$

Hence $x \neq y$, and hence

$$\phi(\mathbf{i}) = T_{i_1} \cdots T_{i_m}(x) \neq T_{i_1} \cdots T_{i_m}(y) = \phi(\mathbf{j}).$$

If $\mu < m+1$, let $x = \phi(i_\mu \ldots i_m i_{m+1} \ldots)$ and $y = \phi(i_\mu \ldots i_m j_{m+1} \ldots)$. Then

$$x \in T_{i_\mu}(C_{k_\mu}) \subset C_{k_{\mu-1}}, \quad \text{and} \quad y \in T_{i_\mu}(C_{k'_\mu}) \subset C_{k_{\mu-1}}.$$

Therefore $x \neq y$, and $\phi(\mathbf{i}) = T_{i_1} \cdots T_{i_{\mu-1}}(x) \neq T_{i_1} \cdots T_{i_{\mu-1}}(y) = \phi(\mathbf{j})$.

Now define two metrics d_1, d_2 on Σ_n^+ by using $\mathbf{s}$ and $\bar{\mathbf{s}}$ instead of $\mathbf{r}$ respectively. Then

Lemma 3. *There exist constants c_1, $c_2 > 0$ such that for any $\mathbf{i}$, $\mathbf{j} \in F_k$ we have*

$$c_1 d_1(\mathbf{i},\mathbf{j}) \leq d(\phi(\mathbf{i}), \phi(\mathbf{j})) \leq c_2 d_2(\mathbf{i},\mathbf{j}).$$

Let $\mathbf{i} = (i_1 i_2 ... i_n i_{n+1} ...)$ and $\mathbf{j} = (i_1 i_2 ... i_n j_{n+1} ...)$ are two different elements of F_k with $i_{n+1} \neq j_{n+1}$. And $k_1, ..., k_n, k_{n+1}$, $k'_1, ..., k'_n, k'_{n+1}$, μ have the same meaning as in the proof of lemma 2. Clearly, we have

$$d(\phi(\mathbf{i}), \phi(\mathbf{j})) \leq diam(X) d_2(\mathbf{i},\mathbf{j}).$$

Let

$$c_1 = \min\{d(T_i(C_\nu), T_r(C_t)) | (i,\nu), (r,t) \in Q_k, \quad (i,\nu) \neq (r,t)\}.$$

Then $c_1 > 0$. If $\mu = m+1$, it is easy to see that $d(\phi(\mathbf{i}), \phi(\mathbf{j}) \geq c_1 d_1(\mathbf{i},\mathbf{j})$. If $\mu \leq m$, then

$$d_1(\phi(\mathbf{i}), \phi(\mathbf{j})) \geq c_1 s^l_{i_1} \cdots s^l_{i_{\mu-1}}$$
$$\geq c_1 s^l_{i_1} \cdots s^l_{i_m} = c_1 d_1(\mathbf{i}\ ,\mathbf{j}).$$

Lemma 1 to 3 give the proof of theorem 2.12 when $(X; T_1, \cdots, T_2)$ is a h.i.f.s.. If $(X; T_1, \cdots, T_2)$ is not a h.i.f.s., like what we did in §2, we define a metric d' on F. For two different elements $\mathbf{i}, \mathbf{j} \in F$ as in the above, denote

$$\theta(\mathbf{i},\mathbf{j}) = \begin{cases} 1, & m = 0,1 \\ \max\{\omega_{k_m} | k_1, ..., k_m \in \{1, ..., m\}, & \\ \qquad (i_\nu, k_\nu) \in Q_{i_{\nu-1}}, \nu = 2, ...m\}, & m \geq 2. \end{cases}$$

Define

$$d'(\mathbf{i},\mathbf{j}) = \begin{cases} \theta(\mathbf{i},\mathbf{j}) \max\{r_\nu^l\}, & m = 0 \\ (r_{i_1}...r_{i_m})^l \theta(\mathbf{i},\mathbf{j}), & m \geq 1 \\ 0, & \mathbf{i} = \mathbf{j}. \end{cases}$$

We have to check the triangle inequality. For $\mathbf{t} = (i_1 i_2 ... i_s t_{s+1} ...)$, if $s \geq m$ or $m = 1$, $d'(\mathbf{i},\mathbf{j}) \leq d'(\mathbf{i},\mathbf{t}) + d'(\mathbf{j},\mathbf{t})$ is a trivial result. Suppose $s < m$ and $m \geq 2$. Assume that $k_1, ...k_m$ fulfil $(i_\nu, k_\nu) \in Q_{k_{\nu-1}}$ for $\nu = 2, ..., m$, and $\theta(\mathbf{i},\mathbf{j}) = \omega_{k_m}$. Then

$$\begin{aligned} d'(\mathbf{i},\mathbf{t}) &= (r_{i_1}...r_{i_r})^\alpha \theta(\mathbf{i},\mathbf{t}) \\ &\geq (r_{i_1}...r_{i_s})^\alpha \omega_{k_s} \\ &= (r_{i_1}...r_{i_s})^\alpha \sum_{\nu=1}^{m} [M(\mathbf{r}, l)^{m-r}]_{k_s \nu} \omega_\nu \\ &\geq (r_{i_1}...r_{i_s} r_{k_{s+1}} ... r_{s_m})^\alpha \omega_{k_m} \\ &= d'(\mathbf{i},\mathbf{j}). \end{aligned}$$

Like in the above, we can prove the following results.

Lemma 4. $\dim(F_k, d') = \dim_B(F_k, d') = 1.$

Lemma 5. *There exist* $c_1,\ c_2 > 0$ *such that*

$$c_1 d_1'((\mathbf{i},\mathbf{j})^{1/l} \leq d(\phi(\mathbf{i}), \phi(\mathbf{j})) \leq c_2 d_2'(\mathbf{i},\mathbf{j})^{1/u}$$

By lemma 2, lemma 4, lemma 1 and proposition 1.2, we can get the expected result. The proof is completed. ∎

Like what we did in §3, we can also generalize Theorem 2.12 to the case when the matrix M is reducible.

Theorem 2.13. *Suppose that* $\{X; T_1, ..., T_n\}$ *is cyclically contracting with* F *and satisfies that*

$$s_i d(x,y) \le d(T_i(x), T_i(y)) \le \bar{s}_i d(x,y).$$

Suppose that for each k

$$C_k = \bigcup_{(i,\nu)\in Q_k} f_k(C_\nu)$$

is a disjoint union. Then

$$l \le \dim(C) \le \overline{\dim}_B(C) \le u,$$

where l *and* u *are determined by*

$$||M(\mathrm{s}, l)|| = 1 \text{ and } ||M(\bar{\mathrm{s}}, u)|| = 1$$

respectively, and where $||M||$ *denotes the eigenvalue of* M *with the largest modulus.*

§5. Construction of Attractors

Let $(X; T_1, \cdots T_n)$ be a h.i.f.s., A be the attractor. We know that for any $a \in A$, there is an $\mathbf{i} = (i_1, i_2, \cdots) \in \Sigma_n^+$ such that

$$a = \lim_{m \to \infty} T_{i_1} \cdots T_{i_m}(x), \quad \forall x \in X. \tag{1}$$

For using a computer to construct fractals, (1) is not efficient, since the expression of (1) is not inheritable. Hence we consider the converse order:

$$T_{i_n} T_{i_{n-1}} \cdots T_{i_1} x \ . \tag{2}$$

This sequence does not converge for most of the $\mathbf{i}$'s in Σ_n^+. Elton proved an ergodic theorem related to (2) for an i.f.s. which is contractive "on the average". We might hope that we can use (2) to draw the pictures of the attractor and Markov attractors of h.i.f.s..

For $\mathbf{j} = (j_1, j_2, ...) \in \Sigma_n^+$, let

$$A_m(\mathbf{j}, x) = \overline{\{T_{j_k} T_{j_{k-1}} \cdots T_{j_1} x;\ k \geq m\}}, \ x \in X$$

and

$$A(\mathbf{j}, x) = \bigcap_{m=1}^{\infty} A_m(\mathbf{j}, x) \ .$$

C.E. Sutherland ([44]) conjectured that for almost all $\mathbf{j} \in \Sigma_n^+$, we have

$$A(\mathbf{j}, x) = A \ .$$

We will prove the conjecture is true. However, we will give a more general result on Markov attractors of h.i.f.s..

First we claim that $A(\mathbf{j}, x)$ is independent of x.

Lemma 1. *Suppose that $(X;T_1,...,T_N)$ is a h.i.f.s., Then we have*

$$A(\mathbf{j},x)=A(\mathbf{j},y),\quad \forall x,y\in X.$$

Proof. By definition

$$A(\mathbf{j},x)=\bigcap_{m=1}^{\infty}A_m(\mathbf{j},x)=\bigcap_{m=1}^{\infty}\overline{\{T_{j_k}T_{j_{k-1}}\cdots T_{j_1}x;\ k\geq m\}}\ .$$

It's easy to see that $a\in A(\mathbf{j},x)$ if and only if there exists an increasing integer sequence $\{l_n\}$ such that

$$a=\lim_{n\to\infty}T_{j_{l_n}}T_{j_{l_n-1}}\cdots T_{j_{l_{n-1}}}\cdots T_{j_{l_1}}\cdots T_{j_1}x\ .$$

For $y\neq x$, we have

$$d(T_{j_{l_n}}T_{j_{l_n-1}}\cdots T_{j_1}x,\ T_{j_{l_n}}T_{j_{l_n-1}}\cdots T_{j_1}y)\leq r^{l_n}d(x,\ y)\to 0\ .$$

Hence

$$a=\lim_{n\to\infty}T_{j_{l_n}}T_{j_{l_n-1}}\cdots T_{j_{l_{n-1}}}\cdots T_{j_{l_1}}\cdots T_{j_1}y\ .$$

which implies $A(\mathbf{j},x)=A(\mathbf{j},y)$ $(=A(\mathbf{j})\ )$. Lemma 1 has been proved. ∎

In the following we will use $A(\mathbf{j})$ instead of $A(\mathbf{j},x)$. One of our main results is

Theorem 2.14. *Suppose that $(X;T_1,...,T_N)$ is a h.i.f.s., M is a $N\times N$ Markov transition matrix. If $\mathbf{j}\in\Sigma_n^+(M^T)$ contains all finite M^T-admissible sequences, then*

$$A(\mathbf{j})=A_M\ .$$

When all the entries of M are 1, we have

Corollary. *Suppose that* $(X;T_1,...,T_N)$ *is a h.i.f.s.. If* $\mathbf{j} \in \Sigma_n^+(M^T)$ *contains all finite strings of* $\{1,2,...,N\}$, *then*

$$A(\mathbf{j}) = A \ .$$

Proof. Assume $\mathbf{j} \in \Sigma_n^+(M^T)$ as in Theorem 1. At first we show that $A_M \subset A(\mathbf{j})$.

For any $a \in A_M$, there exists an $\mathbf{i} \in \Sigma_n^+(M)$ such that

$$a = \lim_{n \to \infty} T_{i_1}T_{i_2}...T_{i_n}x, \quad \forall x \in X.$$

For fixed n, $(i_n, i_{n-1}, ..., i_2, i_1)$ is M^T-admissible, hence appears in $\mathbf{j}$. Suppose $(j_{l_n-n+1}, j_{l_n-n+2}, ..., j_{l_n-1}, j_{l_n})$ is the first time that $(i_n, i_{n-1}, ..., i_2, i_1)$ appears in $\mathbf{j}$. Denote

$$x_n = T_{j_{l_n-n}}T_{j_{l_n-n-1}} \cdots T_{j_1}x \ .$$

Then

$$\begin{aligned}
&d(T_{i_1}T_{i_2}\cdots T_{i_n}x, T_{j_{l_n}}T_{j_{l_n-1}}\cdots T_{j_{l_n-n+1}}T_{j_{l_n-n}}\cdots T_{j_1}x) \\
=&d(T_{i_1}T_{i_2}\cdots T_{i_n}x, T_{j_{l_n}}T_{j_{l_n-1}}\cdots T_{j_{l_n-n+1}}x_n) \\
=&d(T_{i_1}T_{i_2}\cdots T_{i_n}x, T_{i_1}T_{i_2}\cdots T_{i_n}x_n) \\
\le& r^n d(x,x_n) \le r^n \mathrm{diam}(X) \to 0,
\end{aligned}$$

since X is compact. Hence

$$a = \lim_{n \to \infty} T_{j_{l_n}}T_{j_{l_n-1}}\cdots T_{j_1}x,$$

for an increasing integer sequence $\{l_n\}$. So $a \in A(\mathbf{j})$ which implies $A_M \subset A(\mathbf{j})$.

Now we show that $A(\mathbf{j}) \subset A_M$. For $a \in A(\mathbf{j})$, there exists an increasing integer sequence $\{l_n\}$, s.t.

$$a = \lim_{n \to \infty} T_{j_{l_n}}T_{j_{l_n-1}}\cdots T_{j_{l_{n-1}}}\cdots T_{j_{l_1}}\cdots T_{j_1}x \ .$$

Since $\mathbf{j} \in \Sigma_n^+(M^T)$, then $(j_{l_1}, ... j_1)$ is M-admissible. Hence it can be extended into an element of $\Sigma_n^+(M)$, denote as $\mathbf{j}^{(1)}$, which begins with $(j_{l_n}, ..., j_1)$. Similarly, we can get $\mathbf{j}^{(2)}$, $\mathbf{j}^{(3)}$, ..., $\mathbf{j}^{(n)}$,... . Now we ge t a sequence $\{\mathbf{j}^{(n)}\}$ of $\Sigma_n^+(M)$. Since $\Sigma_n^+(M)$ is a compact subspace of the metric space Σ_n^+, there exists a subsequence $\{\mathbf{j}^{(n_k)}\}$ which converges to an element $\mathbf{i} \in \Sigma_n^+(M)$. It's easy to see

$$\begin{aligned} a &= \lim_{n \to \infty} T_{j_{l_n}} \cdots T_{j_1} x \\ &= \lim_{n \to \infty} T_{i_1} \cdots T_{i_n} x \ \in A_M \ , \end{aligned}$$

hence $A(\mathbf{j}) \subset A_M$. ∎

Let $J_M = \{\mathbf{j} \in \Sigma_n^+(M^T);\ \mathbf{j}$ contains every finite M^T-admissible sequence$\}$. We will see that J_M is quite "large" in the sense that for many probability measures which are concentrated on $\Sigma_n^+(M^T)$, J_M has measure 1. We have

Theorem 2.15. *Suppose that μ_M is a probability measure on $(\Sigma_n^+, \mathcal{B})$, such that* $\operatorname{supp}(\mu_M) = \Sigma_n^+(M^T)$ *and the shift operator τ is ergodic on $(\Sigma_n^+,\ \mathcal{B},\ \mu_M)$, where $\mathcal{B}$ is the Borel algebra of (Σ_n^+, d). Then*

$$\mu_M(J_M) = 1 \ .$$

When all the elements of M are 1, we use J instead of J_M and μ instead of μ_M.

Corollary. *Suppose that μ is a probability measure on $(\Sigma_n^+, \mathcal{B})$, such that* $\operatorname{supp}(\mu) = \Sigma_n^+$ *and τ is ergodic on $(\Sigma_n^+,\ \mathcal{B},\ \mu)$. Then*

$$\mu(J) = 1 \ .$$

Proof. We show that $\mu_M(J_M^c) = 0$. Let

$$J^c(i_1, i_2, ..., i_n) = \{\mathbf{j} \in \Sigma_n^+;\ (i_1, i_2, ..., i_n) \text{ doesn't appear in } \mathbf{j}\} \ .$$

Then

$$J_M^c = \left(\bigcup{}' J^c(i_1, i_2, ..., i_n)\right) \bigcup \Sigma_n^+(M^T)^c ,$$

where $\bigcup{}'$ is the union for all M^T-admissible $(i_1, i_2, ..., i_n)$. We have

$$\mu_M(\Sigma_n^+(M^T)^c) = 0 ,$$

and all the finite M^T-admissible sequences consist of a countable set. Hence it's enough to show that

$$\mu_M\left(J^c(i_1, i_2, ..., i_n)\right) = 0$$

where $(i_1, i_2, ..., i_n)$ is M^T-admissible. Since the shift operator τ is ergodic on $(\Sigma_n^+, \mathcal{B}, \mu_M)$, we have

$$\lim_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} f(\tau^k \mathbf{i}) = \int_{\Sigma_n^+} f(\mathbf{j}) d\mu_M \quad \mu_M - \text{a.e.} \tag{1}$$

for any integrable function f on Σ_n^+. For a finite M^T-admissible sequence $(i_1, i_2, ..., i_n)$, let

$$f = \chi_{[i_1, i_2, ... i_n]}$$

be the characteristic function of $[i_1, i_2, ... i_n]$. Then

$$f(\tau^k(\mathbf{j}) = 0, \quad \forall k, \ \forall \mathbf{j} \in J^c(i_1, i_2, ... i_n) .$$

Hence

$$\frac{1}{m} \sum_{k=0}^{m-1} f(\tau^k(\mathbf{j})) = 0, \quad \forall m, \ \forall \mathbf{j} \in J^c(i_1, i_2, ..., i_n) .$$

But

$$\begin{aligned} \int_{\Sigma_n^+} f(\mathbf{j}) d\mu_M &= \int_{[i_1, i_2, ..., i_n]} 1 d\mu_M \\ &= \mu_M([i_1, i_2, ..., i_n]) \\ &= p_{i_1} p_{i_1 i_2} ... p_{i_{n-1} i_n} \\ &> 0 , \end{aligned}$$

since $(i_1, i_2, ..., i_n)$ is M^T-admissible. Hence we get

$$\mu_M(J^c(i_1, i_2, ..., i_n)) = 0 .$$

The proof is completed. ∎

Theorem 2.14 and Theorem 2.15 tell us that for "almost all" $\mathbf{j} \in \Sigma_n^+(M^T)$ (Σ_n^+) all the points produced in (2) form a set which is dense in A_M (A). Hence we can use (2) to construct the Markov attractor A_M (attractor A) of the h.i.f.s.

There are many probability measures which are concentrated on $\Sigma_n^+(M)$ and make τ ergodic. In fact, suppose that $P = (p_{ij})$ is a stochastic matrix $(p_{ij} \geq 0, \sum_{j=1}^N p_{ij} = 1)$ such that

$$p_{ij} \begin{cases} > 0, & \text{if } M_{ij} = 1 \\ = 0, & \text{if } M_{ij} = 0. \end{cases}$$

Then P is irreducible, since M is irreducible. Let $\mathbf{p} = (p_1, p_2, ..., p_N)$ be the left eigenvector of P corresponding to it's Perron-Frobenius eigenvalue ($= 1$), such that $p_1 + p_2 + ... + p_N = 1$. Then we have $\mathbf{p}P = \mathbf{p}$. Define a measure ν on $(\Sigma_n^+, \mathcal{B})$ by

$$\nu([i_1, 1_2, ..., i_n]) = p_{i_1} p_{i_1 i_2} p_{i_2 i_3} ... p_{i_{n-1} i_n} ,$$

where $[i_1, i_2, ..., i_n]$ is the cylinder set of Σ_n^+. Clearly, $\text{supp}(\nu) = \Sigma_n^+(M^T)$. And σ is ergodic on $(\Sigma_n^+, \mathcal{B}, \mu_M)$. Hence for all $x \in X$ we have

$$A(\mathbf{j}, x) = A_M \qquad \nu\text{-a.e.}$$

Now we give two particular examples of the measure defined above.

1. Let $p_{ij} = m_{ji}/(m_{1i} + m_{2i} + ... + m_{Ni})$. Then we get an irreducible stochastic matrix P. We use $\mu_M^{(1)}$ to denote the related measure. Then for all $x \in X$,

$$A(\mathbf{j}, x) = A_M \qquad \mu_M^{(1)} - \text{a.e.}$$

When all the entries of M are 1, we use μ instead of $\mu_M^{(1)}$. It's easy to see that μ is a probability measure generated by the distribution of $\{1, 2, ..., N\}$ with each $i \in \{1, 2, ..., N\}$ having the same probability $1/N$.

2. Suppose λ is the Perron-Frobenius eigenvalue of M, $\mathbf{u} = (u_1, u_2, ..., u_N)^t$ and $\mathbf{v} = (v_1, v_2, ..., v_N)^t$ are left and right P-F eigenvectors of M respectively, such that $\mathbf{u}^t\mathbf{v} = 1$. Let

$$\mu_M^{(2)}([i_1, i_2, ..., i_n]) = \lambda^{n-1} v_{i_1} m_{i_2 i_1} m_{i_3 i_2} ... m_{i_n i_{n-1}} u_{i_n} \ .$$

$\mu_M^{(2)}$ can also be extended to be probability measure on $(\Sigma_n^+, \mathcal{B})$. Again we have $\text{supp}(\mu_M^{(2)}) = \Sigma_n^+(M^T)$. When all the entries of M are 1, we get the same measure μ as in the first example. The measure $\mu_M^{(2)}$ takes the same value on all M^T-admissible cylinder sets which have the same length and have the same beginning and ending numbers.

Now we define $\mu_M^{(2)}$ in another way. Let $p_{ij} = m_{ji}u_j/\lambda u_i$. Then $p_{ij} \geq 0$ and $\sum_{j=1}^{N} p_{ij} = 1$. Hence $P = (p_{ij})$ is an irreducible stochastic matrix and $\mu_M^{(2)}$ is the related measure. So $(\Sigma_n^+, \mathcal{B}, \mu_M^{(2)}, \tau)$ is ergodic. By Theorem 1 and 2,

$$A(\mathbf{j}, x) = A_M \qquad \mu_M^{(2)} - \text{a.e.}$$

§6. Box dimension of fractal curves

In this section, the metric space in consideration is a rectangular subset $I_1 \times I_2$ of $\mathbf{R}^2$. Without loss of generality, we let $I_1 = I_2 = [0,1]$ and use J to denote $[0,1] \times [0,1]$. Now we construct a class of curves which are graphs of real continuous functions with the domain $[0,1]$.

For $i = 1, 2, \cdots, k$, define $T_{ij} : J \mapsto J$ by

$$T_{ij}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ b_{ij} & c_{ij} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_i \\ y_{ij} \end{pmatrix}, \; j = 1, 2, ..., l_i.$$

For $n = l_1 + l_2 + \cdots + l_k$, define an $n \times n$ matrix M in the following way: We use $M_{(ij)(uv)}$ to denote the $(l_0 + \cdots + l_{i-1} + j,\ l_0 + \cdots + l_{u-1} + v)$ element of M, this uses the convention that $l_0 = 0$. First we let $M_{(ij)(iv)} = \delta_{jv}$. Further more, for each (ij) and each u we define $M_{(ij)(uv)} = 1$ for exactly one $v \in \{1, 2, \cdots, l_u\}$, and $M_{(ij)(uv)} = 0$ for all other cases. Suppose that T_{ij}'s are contractive maps and satisfy the following conditions:

1. $a_i > 0$ and $a_1 + a_2 + \cdots + a_k = 1$, $x_1 = 0$ and $x_{i+1} = a_1 + \cdots + a_i$, $i = 1, 2, ..., k-1$;

2. let $(0, y_j)$ be the fixed point of T_{1j}, $j = 1, 2, ..., l_1$ and $(1, y'_j)$ be the fixed point of T_{kj}, $j = 1, 2, ..., l_k$. We assume that there exists a $y_0 \in [0,1]$ such that $P_2 T_{uv}\begin{pmatrix} 0 \\ y_j \end{pmatrix} = y_0$ if $M_{(uv)(1j)} = 1$, $u \neq 1$ and $P_2 T_{uv}\begin{pmatrix} 1 \\ y'_j \end{pmatrix} = y_0$ if $M_{(uv)(kj)} = 1$, $u \neq k$, where P_2 is the projective map to the second coordinate.

Then we get a hyperbolic iterated function system.

Let $G = \bigcup_{i=1}^{k} B_{i1}$, where $B_{ij} = \bigcup_{M_{(ij)(uv)}=1} T_{ij}(B_{uv})$ (cf. p17). Then we have

Theorem 2.16. *G is the graph of a continuous function $\phi : [0,1] \mapsto \mathbf{R}$.*

Proof. For each sequence $i_1, i_2, \cdots$, by the definition of M, there exists exactly one sequence $(i_1 1), (i_2 j_2), \cdots$ such that $M_{(i_t j_t)(i_{t+1} j_{t+1})} = 1$. If the elements of the sequence $i_1, i_2, \cdots$ are not all 1 or k except finitely many, then there exists exactly one point (x, y) such that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lim_{m \to \infty} T_{i_1 1} T_{i_2 j_2} \cdots T_{i_m j_m} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Define $\phi(x) = y$.

For the sequence $i_1 i_2 \cdots i_m 111 \cdots$ and $i_1 i_2 \cdots i_m - 1kkk \cdots$ let

$$\begin{pmatrix} x \\ y \end{pmatrix} = T_{i_1 1} \cdots T_{i_m j_m} \begin{pmatrix} 0 \\ y_j \end{pmatrix}, \quad \text{where } M_{(i_m j_m)(1j)} = 1$$

and

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T_{i_1 1} \cdots T_{i_m - 1 j'_m} \begin{pmatrix} 1 \\ y'_{j'} \end{pmatrix}, \quad \text{where } M_{(i_m - 1 j'_m)(kj')} = 1.$$

By condition 1 and 2, we can easily see that $T_{i_m j_m} \begin{pmatrix} 0 \\ y_j \end{pmatrix} = T_{i_m - 1 j'_m} \begin{pmatrix} 1 \\ y'_{j'} \end{pmatrix}$. Hence $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$. Again, we define $\phi(x) = y$. Therefore the function $\phi : [0,1] \mapsto \mathbf{R}$ is well defined.

Next we show that ϕ is continuous by showing that G is a continuous image of $[0,1]$.

For $x = \sum_{m=1}^{\infty} \frac{i_m - 1}{k^m}$, $i_m \in \{1, 2, \cdots, k\}$, define $\pi : [0,1] \mapsto G$

$$\pi(x) = \lim_{m \to \infty} T_{i_1 1} T_{i_2 j_2} \cdots T_{i_m j_m} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where j_t determined by $M_{(i_{t-1}j_{t-1})(i_tj_t)} = 1$. We show that π is continuous. Let $\alpha = \max_{i,j}\{\mathrm{Lip}(T_{ij})\}$. Given $\epsilon > 0$, choose N large enough that $\alpha^N < \epsilon/2\sqrt{2}$. Let $\delta = k^{-N+1}$. For $x = \sum_{m=1}^{\infty} \frac{i_m - 1}{k^m}$ and $x' = \sum_{m=1}^{\infty} \frac{u_m - 1}{k^m}$, if $|x - x'| < \delta$ we must have $i_1 = u_1,\ i_2 = u_2, \cdots, i_N = u_N$ or $i_1 = u_1,\ i_2 = u_2, \cdots, i_{l-1} = u_{l-1},\ i_l = u_l + 1$ and $i_{l+1} = \cdots = i_N = 1,\ u_{l+1} = \cdots = u_N = k$. In the first case, it is easy to see $|\pi(x) - \pi(x')| < \epsilon$. In the second case, we have

$$\pi(x) = \lim_{m \to \infty} T_{i_1 1} T_{i_2 j_2} \cdots T_{i_l j_l} (T_{1j})^{N-l} T_{i_{N+1} j_{N+1}} \cdots T_{i_m j_m} \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and

$$\pi(x') = \lim_{m \to \infty} T_{i_1 1} T_{i_2 j_2} \cdots T_{i_l - 1, v_l} (T_{kv})^{N-l} T_{u_{N+1} v_{N+1}} \cdots T_{u_m v_m} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As in the above, let $(0, y_j)$ be the fixed point of T_{1j} and $(1, y'_v)$ the fixed point of T_{kv}. By the second assumption we know that

$$T_{i_l j_l} \begin{pmatrix} 0 \\ y_j \end{pmatrix} = T_{i_l - 1 v_l} \begin{pmatrix} 1 \\ y'_v \end{pmatrix}.$$

Let $E = T_{i_l j_l}(T_{1j})^{N-l}(J) \bigcup T_{i_l - 1 v_l} T_{i_l - 1, v_l}(T_{kv})^{N-l}(J)$. Then $\mathrm{diam}(E) \le 2\alpha^{N-l+1}\sqrt{2}$, since the two parts of the union have a common point. Therefore

$$\begin{aligned} |\pi(x) - \pi(x')| &\le \mathrm{diam}(T_{i_1 1} T_{i_2 j_2} \cdots T_{i_{l-1} j_{l-1}}(E)) \\ &\le \alpha^{l-1} \mathrm{diam}(E) < \epsilon. \qquad \blacksquare \end{aligned}$$

Example. Let $k > 2$. Define $T_{ij} : J \mapsto J$ $(i = 1, 2, ..., k;\ j = 1, 2)$ as follows:

$$T_{i1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{i-1}{k} \\ 1 - \alpha \end{pmatrix},$$

$$T_{i2} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & 1 - \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{i-1}{k} \\ 0 \end{pmatrix},$$

where $\min\{\alpha, 1-\alpha\} > \frac{1}{k}$. Let M be defined by

$$M_{(ij)(uv)} = \begin{cases} 1 & \text{if } (ij) = (uv) \text{ or } i \neq u,\ j \neq v \\ 0 & \text{otherwise.} \end{cases}$$

When $k = 3$

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that (1) and (2) are satisfied. The continuous function, denoted as $f_{k,\alpha}$, can be defined in the following way: for $x = \sum_{m=1}^{\infty} \frac{x_m}{k^m}$, $x_m \in \{0, 1, ..., k-1\}$, let

$$f_{k,\alpha}(x) = \sum_{m=1}^{\infty} \alpha^{l_m}(1-\alpha)^{m-l_m} u_m$$

where $u_1 = 1$ and

$$u_{m+1} = \begin{cases} u_m & \text{if } x_{m+1} = x_m \\ 1 - u_m & \text{otherwise} \end{cases}$$

and $l_m = u_1 + u_2 + \cdots + u_m - 1$. When $\alpha = \frac{1}{2}$ we call this function the Bush function, because it was first defined by K.A. Bush[13] as an example of a continuous nowhere differentiable function.

Next we calculate the box dimension of G under certain conditions. We first establish a more general result.

Let $(J; T_1, T_2, \cdots, T_n)$ be a h.i.f.s. where $T_i : J \mapsto J$ is defined by

$$T_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ b_i & c_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

with $0 < |a_i| < |c_i|$. Let M be an $n \times n$ irreducible Markov transition matrix. Let A_M be the Markov attractor of the h.i.f.s. associated with M. Let s be the number such that

$$||M \begin{pmatrix} |c_1||a_1|^{s-1} & & 0 \\ & \ddots & \\ 0 & & |c_n||a_n|^{s-1} \end{pmatrix}|| = 1. \qquad (1)$$

Then we have

Proposition 2.17. $\dim_B(A_M) \leq s$.

Proof. Like in section 2, we use $\Sigma_n^+(M)$ to denote the set of all M-admissible elements of Σ_n^+. Let

$$M(s) = M \begin{pmatrix} |c_1||a_1|^{s-1} & & 0 \\ & \ddots & \\ 0 & & |c_n||a_n|^{s-1} \end{pmatrix}.$$

By the Perron-Frobenius Theorem, there exists a vector $\mathbf{p} = (p_1, p_2, \cdots, p_n)^T$ with $p_i > 0$ such that

$$M(s)\mathbf{p} = \mathbf{p}.$$

We assume that $\sum_{i=1}^{n} p_i = 1$. Define a probability measure on Σ_n^+ by letting

$$\mu([i]) = p_i,$$

$$\mu([ij]) = M(s)_{ij} p_j,$$

$$\cdots\ \cdots$$

$$\mu([i_1 i_2 \cdots i_k]) = M(s)_{i_1 i_2} M(s)_{i_2 i_3} \cdots M(s)_{i_{k-1} i_k} p_{i_k},$$

where $[i_1 i_2 \cdots i_k]$ is the cylinder set which contains all elements beginning with $i_1 i_2 \cdots i_k$. Clearly, the support of μ is $\Sigma_n^+(M)$. Let $a = \min\{|a_1|, |a_2|, \cdots, |a_n|\}$.

Given $\delta > 0$, suppose $a^m \geq \delta > a^{m+1}$. For each $x \in A_M$, there exist $i_1, i_2, \cdots, i_l$ with $\delta > |a_{i_1} a_{i_2} \cdots a_{i_l}| > a^{m+2}$ and $M_{i_j i_{j+1}} = 1$ such that $x \in T_{i_1} \cdots T_{i_l} J = J_{i_1 \cdots i_l}$. Let

$$W = \{[i_1 \cdots i_l] | l \text{ is the first number}$$
$$\text{s.t. } \delta > |a_{i_1} a_{i_2} \cdots a_{i_l}| > a^{m+2}, M_{i_j i_{j+1}} = 1\}.$$

It is easy to see that if $[i_1 \cdots i_l]$, $[j_1 \cdots j_t] \in W$ and $[i_1 \cdots i_l] \neq [j_1 \cdots j_t]$, then $[i_1 \cdots i_l] \bigcap [j_1 \cdots j_t] = \emptyset$. Therefore W is a disjoint cover of $\Sigma_n^+(M)$.

Now we calculate how many δ-squares (square of side length δ) are needed to cover A_M. In the case where all b_i are zero, The height and width of $J_{i_1 \cdots i_l}$ (denoted as $|J_{i_1 \cdots i_l}|_H$ and $|J_{i_1 \cdots i_l}|_W$) are $|c_{i_1} \cdots c_{i_l}|$ and $|a_{i_1} \cdots a_{i_l}| < \delta$ respectively. Hence, at most

$$[\frac{|c_{i_1} \cdots c_{i_l}|}{|a_{i_1} \cdots a_{i_l}|}] + 1$$

δ-squares are needed to cover $[i_1 \cdots i_l]$.

$$\begin{aligned}
&\sum_{[i_1 \cdots i_l] \in W} \left([\frac{|c_{i_1} \cdots c_{i_l}|}{|a_{i_1} \cdots a_{i_l}|}] + 1 \right) \\
\leq& 2 \sum_{[i_1 \cdots i_l] \in W} |\frac{c_{i_1} \cdots c_{i_l}}{a_{i_1} \cdots a_{i_l}}| \\
\leq& 2 \sum_{[i_1 \cdots i_l] \in W} |\frac{c_{i_1} \cdots c_{i_l}}{a_{i_1} \cdots a_{i_l}}| (\frac{|a_{i_1} \cdots a_{i_l}|}{a^2})^s \delta^{-s} \\
=& \frac{2}{a^{2s}} \cdot \delta^{-s} \sum_{[i_1 \cdots i_l] \in W} |c_{i_1} \cdots c_{i_l}| |a_{i_1} \cdots a_{i_l}|^{s-1} \\
\leq& \frac{2}{a^{2s}} \cdot \frac{1}{\min_j \{p_j\}} \cdot \delta^{-s} \sum_{[i_1 \cdots i_l] \in W} |c_{i_1} \cdots c_{i_l}| |a_{i_1} \cdots a_{i_l}|^{s-1} p_{i_l} \\
=& \frac{2}{a^{2s}} \cdot \frac{1}{\min_j \{p_j\}} \cdot \delta^{-s} \sum_{[i_1 \cdots i_l] \in W} M(s)_{i_1 i_2} \cdots M(s)_{i_{l-1} i_l} p_{i_l} \\
=& \frac{2}{a^{2s}} \cdot \frac{1}{\min_j \{p_j\}} \cdot \delta^{-s} \sum_{[i_1 \cdots i_l] \in W} \mu([i_1 \cdots i_l]) \\
=& \frac{2}{a^{2s}} \cdot \frac{1}{\min_j \{p_j\}} \cdot \delta^{-s}.
\end{aligned}$$

Therefore, for any $\delta > 0$, at most $\frac{2}{a^{2s}} \cdot \frac{1}{\min_j\{p_j\}} \cdot \delta^{-s}$ are needed to cover A_M. Hence $\dim_B(A_M) \leq s$.

When not all $b_i = 0$, we can get the result by showing that

$$|J_{i_1 \cdots i_m}|_H \leq B|c_{i_1} \cdots c_{i_m}|$$

for some positive constant B. We will do that in the following Lemma. ∎

Lemma. *We have that*

$$|c_{i_1} \cdots c_{i_m}| \leq |J_{i_1 \cdots i_m}|_H \leq B|c_{i_1} \cdots c_{i_m}|,$$

where $B \geq 1$ is a positive constant.

Proof. First we show that $|J_{i_1 \cdots i_m}|_H \leq B|c_{i_1} \cdots c_{i_m}|$. When $m = 1$, we have

$$|J_{i_1}|_H \leq |c_{i_1}| + |b_{i_1}|.$$

Let $c = \max\{\frac{|b_i|}{|c_i|}\}$. Then

$$|J_{i_1}|_H \leq (1 + c)|c_{i_1}|.$$

Assume that

$$|J_{i_2 \cdots i_{m+1}}|_H \leq B_m|c_{i_2} \cdots c_{im+1}|.$$

Then

$$\begin{aligned}
&|J_{i_1 i_2 \cdots i_{m+1}}|_H \\
=&|T_{i_1} J_{i_2 \cdots i_{m+1}}|_H \\
\leq&|c_{i_1}||J_{i_2 \cdots i_{m+1}}|_H + |b_{i_1}||J_{i_2 \cdots i_{m+1}}|_W \\
=&|c_{i_1}||J_{i_2 \cdots i_{m+1}}|_H + |b_{i_1}||a_{i_2} \cdots a_{i_{m+1}}| \\
\leq&B_m|c_{i_1} \cdots c_{i_{m+1}}| + c|\frac{a_{i_2} \cdots a_{i_{m+1}}}{c_{i_2} \cdots c_{i_{m+1}}}||c_{i_1} c_{i_2} \cdots c_{i_{m+1}}| \\
\leq&(B_m + cd^m)|c_{i_1} \cdots c_{i_{m+1}}|,
\end{aligned}$$

where $d = \max_j\{\frac{|a_j|}{|c_j|}\}$. Hence we can choose $B_{m+1} = B_m + cd^m$. Notice that $B_1 = 1 + c$. Therefore,

$$B_m = 1 + \sum_{i=0}^{m-1} cd^k < 1 + \frac{c}{1-d}.$$

Hence the number $1 + \frac{c}{1-d}$ can be chosen as B.

As for the other part, we can see easily that the second coordinate of $T_{i_1} \cdots T_{i_m} \begin{pmatrix} x \\ y \end{pmatrix}$ is $c_{i_1} \cdots c_{i_m} y + \alpha x + \beta$ where α and β are independent of x, y. Therefore

$$|J_{i_1 \ldots i_m}|_H \geq |c_{i_1} \cdots c_{i_m}|. \qquad \blacksquare$$

Now we assume that the h.i.f.s. satisfies the following condition:

3. open set condition. If $M_{ij} = 1$ and $i \neq j$, then $T_i(J^\circ) \bigcap T_j(J^\circ) = \emptyset$, where J° is the interior of J.

4. for any $i_1 i_2 \cdots i_m$ with $M_{i_j i_{j+1}} = 1$, let

$$y_{i_1 i_2 \cdots i_m} = \inf\{y;\ (x, y) \in J_{i_1 i_2 \cdots i_m} \bigcap A_M \text{ for some } x\}$$

and

$$y^*_{i_1 i_2 \cdots i_m} = \sup\{y;\ (x, y) \in J_{i_1 i_2 \cdots i_m} \bigcap A_M \text{ for some } x\}.$$

We assume that there exist $A > 0$ such that

$$y^*_{i_1 i_2 \cdots i_m} - y_{i_1 i_2 \cdots i_m} \geq A |c_{i_1} c_{i_2} \cdots c_{i_m}|.$$

5. for any $i_1 i_2 \cdots i_m$ with $M_{i_j i_{j+1}} = 1$

$$P_2\big(J_{i_1 i_2 \cdots i_m} \bigcap A_M\big) = [y_{i_1 i_2 \cdots i_m}, y^*_{i_1 i_2 \cdots i_m}],$$

is an interval.

Theorem 2.18. *Suppose the h.i.f.s. satisfies the above conditions. Then*

$$\dim_B(A_M) = s.$$

Proof. We only need to show that $\dim_B(A_M) \geq s$. Given $0 < \delta < 1$, assume that $a^m \geq \delta > a^{m+1}$, where $a = \min_i\{|a_i|\}$. For any $x \in A_M$ there exist $i_1, \cdots i_l$ with $M_{i_j i_{j+1}} = 1$ and $\delta \leq a_{i_1} \cdots |a_{i_l} < a^{m-2}|$ such that $x \in J_{i_1 \cdots i_l}$. Because of the assumptions 4 and 5, there are at least $[\frac{A|c_{i_1} \cdots c_{i_l}|}{\delta}]$ δ-squares which intersect with $J_{i_1 \cdots i_l} \bigcap A_M$. From the fact that $|J_{i_1 \cdots i_l}|_H \geq |c_{i_1} \cdots c_{i_m}| > |J_{i_1 \cdots i_l}|_W = |a_{i_1} \cdots a_{i_l}| > \delta$ and the open set condition, each δ-square intersects at most 4 such sets. We again use W to denote all the cylinders $[i_1 \cdots i_l]$ mentioned above. Again W is a disjoint cover of $\Sigma_n^+(M)$. The following calculation gives us an estimate for the lower bound of the number of δ-squares that are needed to cover A_M.

$$\begin{aligned}
&\sum_{[i_1 \cdots i_l] \in W} \frac{1}{4}[\frac{|c_{i_1} \cdots c_{i_l}|}{\delta}] \\
\geq & \frac{1}{8} \sum_{[i_1 \cdots i_l] \in W} \frac{|c_{i_1} \cdots c_{i_l}|}{\delta} \\
\geq & \frac{1}{8} \sum_{[i_1 \cdots i_l] \in W} \frac{|c_{i_1} \cdots c_{i_l}|}{a^m} \\
\geq & \frac{1}{8} \sum_{[i_1 \cdots i_l] \in W} |\frac{c_{i_1} \cdots c_{i_l}}{a_{i_1} \cdots a_{i_l}}| \cdot \frac{1}{a^2} \\
= & \frac{1}{8a^2} \sum_{[i_1 \cdots i_l] \in W} |c_{i_1} \cdots c_{i_l}| \cdot |a_{i_1}|^{s-1} \cdots |a_{i_l}|^{s-1} \cdot |a_{i_1} \cdots a_{i_l}|^{-s} \\
\geq & \frac{1}{8a^2} \sum_{[i_1 \cdots i_l] \in W} \mu([i_1 \cdots i_l]) \cdot (\frac{\delta}{a^2})^{-s} \\
= & \frac{1}{8} \cdot a^{2(s-1)} \cdot \delta^{-s}.
\end{aligned}$$

Therefore, $\dim(A_M) \geq s$. ∎

In the proofs of Proposition 2.17 and Theorem 2.18 we can use $B_i = \bigcup_{M_{ij}=1} T_i(B_j)$ to replace A_M and get the same result.

Corollary. *Under the same assumption of Theorem 3, we have*

$$\dim_B(B_i) = s, \quad i = 1, ..., n.$$

Now we come back to the curve G defined in the above. Since G is a curve, condition 5 is satisfied.

Theorem 2.19. *Assume that $|c_{ij}| > a_i$ and the h.i.f.s. in Theorem 2.16 satisfies condition 3 and 4. Suppose that the matrix M is irreducible. Then*

$$\dim_B(G) = s.$$

Example(continued). By Theorem 2.19 we can calculate the box dimension for the graph of the of the function $f_{k,\alpha}$ defined in Example 1. Use $G_{k,\alpha}$ to denote the graph of the function $f_{k,\alpha}$. First we check that $G_{k,\alpha}$ satisfies condition 4. Given $i_1, i_2, ..., i_m$,

$$P_1(J_{(i_1 1)(i_2 j_2)\cdots(i_m j_m)}) = [a, b]$$

where $a = \sum_{j=1}^{m} \frac{i_j - 1}{k^j}$ and $b = a + \frac{1}{k^m}$. By the definition of $f_{k,\alpha}$, for any $x \in (a, b)$ we have the same u_j, $(j = 1, 2, ..., m)$ in the expression of $f_{k,\alpha}(x)$. Suppose $u_m = 0$. Let $x_1 = a + (i_m - 1) \sum_{j=m+1}^{\infty} \frac{1}{k^j}$ and $x_2 = a + \sum_{j=m+1}^{\infty} \frac{l}{k^j}$, where $0 \le l \le k - 1$ and $l \ne i_m - 1$. Then

$$\min_{x \in [a,b]} f_{k,\alpha}(x) = f_{k,\alpha}(x_1) = \sum_{j=1}^{m} \alpha^{l_j}(1-\alpha)^{j-l_j} u_j$$

and

$$\max_{x \in [a,b]} f_{k,\alpha}(x) = f_{k,\alpha}(x_2)$$
$$= \sum_{j=1}^{m} \alpha^{l_j}(1-\alpha)^{j-l_j} u_j + \alpha^{l_m}(1-\alpha)^{m-l_m} \sum_{j=1}^{\infty} \alpha^j.$$

Hence we can choose $A = B = \frac{1}{1-\alpha}$ in condition 4. If $u_m = 1$ we can get the same result.

Let $\Lambda = \mathrm{diag}(a, b, a, b, \cdots, a, b)$ be a $2k \times 2k$ diagonal matrix whose diagonal elements are a and b alternatively with $a, b > 0$. We have

$$||M\Lambda|| = \frac{a + b + \sqrt{(a-b)^2 + 4(k-1)^2 ab}}{2}.$$

Choose $a = \alpha(\frac{1}{k})^{s-1}$, $b = (1-\alpha)(\frac{1}{k})^{s-1}$ and let $||M\Lambda|| = 1$. We get

$$\dim_B(G_{k,\alpha}) = s = 1 + \frac{\log(1 + \sqrt{(2\alpha-1)^2 + 4(k-1)^2\alpha(1-\alpha)} - \log 2}{\log k}.$$

When $\alpha = \frac{1}{2}$, using G_k to denote the graph of the Bush function, we have

$$\dim_B(G_k) = 2 - \frac{\log 2}{\log k}.$$

CHAPTER 3. Measure Inequalities of Sumsets, I

§1 Introduction.

In this and the next chapter we establish some measure inequalities for sumsets. In this chapter we treat with Cantor-type measures. For convenience, we will use (a, b) (or (a, b, c)) to denote $\max\{a, b\}$(or $\max\{a, b, c\}$) in both this and next chapter, since it will appear very frequently.

The are five sections in this chapter. Our main results will appear in section 2 and be proved in section 3. We establish several inequalities with the form

$$m(E + F) \geq \mu(E)^{\alpha}\nu(F)^{\beta} \tag{1}$$

where μ, ν are Lebesgue measure or probability measures uniformly spreading on sets of numbers in whose base 3 or 4 expansion one or two digits do not appear. We find that for this kind of measures if (1) holds for all Borel subsets E, F we must have

$$\alpha s(\mu) + \beta s(\nu) \geq s(m) \geq 1 \tag{2}$$

where $s(\cdot)$ is the singular exponent of the measure(defined below). For measures discussed here we have $s(\cdot) = \dim(\cdot)$. Hence (2) can be written as

$$\alpha \dim(\mu) + \beta \dim(\nu) \geq \dim(m) = 1$$

as Oberlin [38] noticed.

In order to prove one of our main result, we need the inequality

$$1 + x + x^2 + x^3 \geq (1 + x^s)^{\frac{1}{s}} (1 + x^t + x^{2t})^{\frac{1}{t}}, \qquad 0 \leq x \leq 1$$

Where $s,\ t \geq 1$ satisfying $\begin{cases} 3s + 8t \leq 15 \\ \frac{1}{2}\frac{1}{s} + \frac{\log 3}{\log 4}\frac{1}{t} = 1. \end{cases}$ The proof of this inequality which is quite long appears in section 4.

In section 5 we establish an inequality for sumset of three sets.

§2. Main Results.

To begin with this section, we define the measures involved. We always use m to stand for the Lebesgue measure on $\mathbf{T}$. Use μ_c to denote the probability measure which uniformly spreads on the Cantor middle third set, i.e.

$$\mu_c = \mathop{*}_{n=1}^{\infty}(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_{\frac{2}{3^n}}).$$

Let

$$\nu_1 = \mathop{*}_{n=1}^{\infty}(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_{\frac{1}{4^n}})$$

and

$$\nu_2 = \mathop{*}_{n=1}^{\infty}(\frac{1}{3}\delta_0 + \frac{1}{3}\delta_{\frac{1}{4^n}} + \frac{1}{3}\delta_{\frac{2}{4^n}}).$$

Before stating the main results, we look at two examples.

Example 1. It is obvious that for any Borel subsets E, F, we have

$$m(E+F) \geq m(E)^{\alpha} m(F)^{\beta}, \tag{1}$$

where $0 \leq \alpha,\ \beta \leq 1$ and $\alpha + \beta = 1$.

Example 2. Let

$$\nu_3 = \mathop{*}_{n=1}^{\infty}(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_{\frac{2}{4^n}}).$$

We have $m = \nu_1 * \nu_3$. If E is a subset of $supp(\nu_1)$ and F a subset of $supp(\nu_3)$, then

$$m(E+F) = \nu_1(E)\nu_3(F).$$

Hence for any Borel subsets E, F we have

$$m(E+F) \geq \nu_1(E)\nu_3(F). \tag{2}$$

Notice that (1) and (2) have the same form

$$m(E+F) \geq \mu_1(E)^\alpha \mu_2(F)^\beta, \tag{3}$$

where $0 \leq \alpha,\ \beta \leq 1$ and satisfying

$$\alpha \dim(\mu_1) + \beta \dim(\mu_2) = \dim(m). \tag{4}$$

Further more, (1) and (2) can be generalized as

$$m(E+F) \geq m(E)^\alpha m(F)^\beta; \qquad \alpha, \beta \geq 0,\ \alpha + b \geq 1 \tag{1'}$$

and

$$m(E+F) \geq \nu_1(E)^\alpha \nu_3(F)^\beta; \qquad \alpha, \beta \geq 1. \tag{2'}$$

We will build such kind of inequalities for m, μ_c, ν_1 and ν_2. Our main result is

Theorem 3.1. *Let m be the Lebesgue measure on $\mathbf{T}$, μ_c, ν_1 and ν_2 be defined as the above. Then for any Borel subsets E, F of $\mathbf{T}$, we have*

$$m(E+F) \geq \mu_c(E)^\alpha \mu_c(F)^\beta, \qquad \begin{cases} \alpha + \beta \geq \dfrac{\log 3}{\log 2} \\ 3(\alpha^{-1} + \beta^{-1}) \leq 8, \quad \alpha,\ \beta \geq \dfrac{\log 3}{\log 2} - 1, \end{cases} \tag{5}$$

$$m(E+F) \geq m(E)^\alpha \mu_c(F)^\beta, \qquad \begin{cases} \alpha + \dfrac{\log 2}{\log 3}\beta \geq 1 \\ \alpha \geq 1 - \dfrac{\log 2}{\log 3}, \end{cases} \tag{6}$$

$$m(E+F) \geq m(E)^\alpha \nu_1(F)^\beta, \qquad \begin{cases} \alpha + \dfrac{1}{2}\beta \geq 1 \\ \alpha \geq \dfrac{1}{2}, \end{cases} \tag{7}$$

$$m(E+F) \geq \nu_1(E)^\alpha \nu_2(F)^\beta, \qquad \begin{cases} \dfrac{1}{2}\alpha + \dfrac{\log 3}{\log 4}\beta \geq 1 \\ 3\alpha^{-1} + 8\beta^{-1} \leq 15, \quad \alpha \geq \alpha_0, \quad \beta \geq \beta_0, \end{cases} \tag{8}$$

where $\alpha_0 = \min\{\alpha_1, \alpha_2\}$, $\beta_0 = \min\{\beta_1, \beta_2\}$ *and* (α_1, β_1), (α_2, β_2) *are solutions of* $\begin{cases} \frac{1}{2}\alpha + \frac{\log 3}{\log 4}\beta = 1 \\ 3\alpha^{-1} + 8\beta^{-1} = 15 \end{cases}$.

We also wish to establish analogous inequalities for which the measures in the right hand are both ν_2, or ν_2 and m. But the method we will use to prove Theorem 3.1 does not work in that situation. However, we have the following partial results for the first case.

Theorem 3.2. *Let* ν_2 *be the same as the above. Then for any Borel subsets* E, F *of* $\mathbf{T}$, *we have*

$$m(E+F) \geq \nu_2(E)^\alpha \nu_2(F)^\beta, \qquad \alpha, \beta \geq 1. \tag{9}$$

If E *is a Borel subset such that*

$$E \bigcap supp(\nu_2) = \{c_k + \sum_{i=k+1}^{\infty} \frac{\epsilon_i}{4^i}, \quad \epsilon_i = 0, 1, 2\}$$

where $c_k = \sum_{i=1}^{k} \frac{\epsilon_i}{4^i}$ *(*ϵ_i *=0, 1 or 2) is a constant, then for any Borel subset* F *we have*

$$m(E+F) \geq \nu_2(E)^\alpha \nu_2(F)^\beta, \qquad \begin{cases} \alpha + \beta \geq \dfrac{\log 4}{\log 3} \\ \alpha, \ \beta \geq \dfrac{\log 4}{\log 3} - 1. \end{cases} \tag{10}$$

The measure inequalities for sumsets with the form (4) under conditions like (3) related to the Hausdorff dimensions of the measures involved are not only when the left hand side is the Lebesgue measure. For ν_1 and ν_2 defined in the above, we have

Proposition 3.3. *For any Borel subsets* E, F *of* $\mathbf{T}$ *one has*

$$\nu_2(E+F) \geq \nu_1(E)^\alpha \nu_1(F)^\beta, \qquad \begin{cases} \alpha + \beta \geq \dfrac{\log 3}{\log 2} \\ 3(\alpha^{-1} + \beta^{-1}) \leq 8, \quad \alpha, \ \beta \geq \dfrac{\log 3}{\log 2} - 1. \end{cases} \tag{11}$$

The first condition of (11) is equivalent to

$$\alpha \dim(\nu_1) + \beta \dim(\nu_1) = \dim(\nu_2).$$

The above results will be proved in following sections. The following examples tell us that we can not build an measure inequality (3) for all probability measures μ_1, μ_2 under the condition (4).

Example 3. We have $\dim(\nu_1) = \frac{1}{2}$. But

$$supp(\nu_1) + supp(\nu_1) = \{\sum_{n=1}^{\infty} \frac{\epsilon_n}{3^n} | \epsilon_n = 0, 1\},$$

with Lebesgue measure 0.

Example 4. Let

$$\mu_{\frac{3}{4}} = \mathop{*}_{n=1}^{\infty} (\frac{3}{4}\delta_0 + \frac{1}{4}\delta_{\frac{1}{2^n}}).$$

Then

$$\dim(\mu_{\frac{3}{4}}) = \frac{-\frac{3}{4}\log\frac{3}{4} - \frac{1}{4}\log\frac{1}{4}}{\log 2} = 0.81\cdots.$$

Let $E = F = [0, \frac{1}{2^k}]$. Then $E + F = [0, \frac{1}{2^{k-1}}]$. But for α, β with $\alpha + \beta = \frac{1}{\dim(\mu_{\frac{3}{4}})}$,

$$\mu_{\frac{3}{4}}(E)^{\alpha}\mu_{\frac{3}{4}}(F)^{\beta} > \mu_{\frac{3}{4}}(E)\mu_{\frac{3}{4}}(F) = \left(\frac{3}{4}\right)^{2k} > \frac{1}{2^{k-1}} = m(E + F)$$

when k is large enough. Measures defined in this way will be discussed in the next chapter.

The next example shows that for some probability measures, though (4) does not hold, we still have (3).

Example 5. Let $\{s_j\}$ be a sequence of numbers decreasing to 0 and let $0 = m_0 < m_1 < m_2 < \cdots$ be a sequence of integers increasing rapidly enough to ensure that

$$\begin{cases} (m_1 - m_0 + (m_3 - m_2) + \cdots + (m_{2j-1} - m_{2j-2}) \le s_j m_{2j} \\ (m_2 - m_1) + (m_4 - m_3) + \cdots + (m_{2j} - m_{2j-1}) \le s_j m_{2j+1}. \end{cases} \tag{12}$$

Let $A = \left\{ \sum_{j=0}^{\infty} \sum_{n=m_{2j}+1}^{m_{2j+1}} \frac{\epsilon_n}{2^n} | \epsilon_n = 0, 1 \right\}$ and $B = \left\{ \sum_{j=1}^{\infty} \sum_{n=m_{2j-1}+1}^{m_{2j}} \frac{\epsilon_n}{2^n} | \epsilon_n = 0, 1 \right\}$. Then $\dim(A) = \dim(B) = 0$(see Falconer[17] p73). Define

$$\mu_1 = \mathop{*}_{j=0}^{\infty} \mathop{*}_{n=m_{2j}+1}^{m_{2j+1}} (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_{\frac{1}{2^n}})$$

and

$$\mu_2 = \mathop{*}_{j=1}^{\infty} \mathop{*}_{n=m_{2j-1}+1}^{m_{2j}} (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_{\frac{1}{2^n}}).$$

We have $supp(\mu_1) = A$ and $supp(\mu_2) = B$. Hence $\dim(\mu_1) = \dim(\mu_2) = 0$. It is easy to see that $\mu_1 * \mu_2 = m$. Therefore for any Borel subsets E, F we have

$$m(E + F) \ge \mu_1(E)\mu_2(F).$$

Now we define a quantity to characterize the singularity of regular probability measures on $\mathbf{T}$.

Definition. Let μ be a regular probability measure on $\mathbf{T}$. For any $x \in \mathbf{T}$ define

$$s(\mu, x) = \liminf_{\delta \to 0} \frac{\log \mu(x - \delta, x + \delta)}{\log(2\delta)},$$

and call it the *local scaling exponent* of μ. Let

$$s(\mu) = \inf_{x \in \mathbf{T}} \{s(\mu, x)\}.$$

We call $s(\mu)$ the *singular exponent* of μ.

If μ is a discrete measure we have $s(\mu) = 0$. For Lebesgue measure m, $s(m) = 1$. For μ_1, μ_2 in example 3, $s(\mu_1) = s(\mu_2) = 0$. If $s(\mu, x) = s(\mu)$ for μ-a.e. it happens that $s(\mu) = \dim(\mu)$(see Falconer [19] p61). The Lebesgue measure m, and μ_c, $\nu_i (i = 1, 2, 3)$ defined above are of this kind. In general we have $s(\mu) \le \dim(\mu)$. For the measure $\mu_{\frac{3}{4}}$ defined in Example 4, $s(\mu_{\frac{3}{4}}) = \frac{\log 4 - \log 3}{\log 2} = 2 - \frac{\log 3}{\log 2}$ which is less than $\dim(\mu_{\frac{3}{4}}) = 2 - \frac{3}{4}\frac{\log 3}{\log 2}$.

Theorem 3.4. *Let μ, ν be regular probability measures on $\mathbf{T}$. If the the inequality*

$$\nu(E+F) \geq \mu(E)^{\alpha}\mu(F)^{\beta} \tag{13}$$

holds for any Borel subset E, F, then we have

$$(\alpha+\beta)s(\mu) \geq s(\nu). \tag{14}$$

Proof. Suppose that $(\alpha+\beta)s(\mu) = s(\nu) - 2\epsilon$ where $\epsilon > 0$. By definition there exist $x \in \mathbf{T}$ and $\delta_n \downarrow 0$, such that

$$\frac{\log \mu(x-\delta_n, x+\delta_n)}{\log |2\delta_n|} < s(\mu) + \epsilon/(\alpha+\beta). \tag{15}$$

Hence

$$\begin{aligned} \mu(x-\delta_n, x+\delta_n)^{\alpha+\beta} &> |2\delta_n|^{(s(\mu)+\epsilon/(\alpha+\beta))(\alpha+\beta)} \\ &= |2\delta_n|^{s(\nu)-\epsilon}. \end{aligned}$$

But $\nu(x-2\delta_n, x+2\delta_n) \leq |4\delta_n|^{s(\nu)} = 2^{s(\nu)}|2\delta_n|^{s(\nu)}$. When n large enough we have $|2\delta_n|^{-\epsilon} > 2^{s(\nu)}$. Therefore for $E = F = [x-\delta_n, x+\delta_n]$ we have $\mu(E)^{\alpha}\mu(F)^{\beta} > \nu(E+F)$. ∎

By Theorem 3.4 we know that the restriction on α, β related to Hausdorff dimension of measures is a necessary condition for (5) and (11). If there are two different measures μ_1, μ_2 in the right hand side of (13), Example 5 shows that we may not have a result analogous to (14) in general. However, by the proof of Theorem 3.4 we can immediately obtain the following corollary.

Corollary. *Let μ_1, μ_2 and ν be regular probability measures on $\mathbf{T}$. Suppose that there exists a sequence $\{\delta_n\} \downarrow 0$ such that $s(\mu_1) = \lim_{n\to\infty} \frac{\log \mu_1(x_1-\delta_n, x_1+\delta_n)}{\log|2\delta_n|}$ and $s(\mu_2) = \lim_{n\to\infty} \frac{\log \mu_2(x_2-\delta_n, x_2+\delta_n)}{\log|2\delta_n|}$ for some x_1, $x_2 \in \mathbf{T}$. If the inequality*

$$\nu(E+F) \geq \mu_1(E)^{\alpha}\mu_2(F)^{\beta}$$

holds for any Borel subset E, F, then we have

$$\alpha s(\mu_1) + \beta s(\mu_2) \geq s(\nu).$$

The inequalities (6), (7), (8) and (10) are of this type.

Here we also point out that a restriction like α, $\beta \geq \frac{\log 3}{\log 2} - 1$ for (5) (except (8)) is also necessary. To see this, let $\alpha = 2$, β be any number less than $\frac{\log 3}{\log 2} - 1$, say $\beta = \frac{\log 3}{\log 2} - 1 - \epsilon$ $(\epsilon > 0)$. Let $E = C$ the classical Cantor set, and

$$F = \{\sum_{n=k}^{\infty} \frac{\epsilon_n}{3^n} | \epsilon_n = 0, 2\}.$$

Since $F + F \subset [0, \frac{4}{3^k}]$, we get

$$m(E+F) \leq \frac{2^{k+2}}{3^k}.$$

But

$$\begin{aligned}\mu_c(E)^\alpha \mu_c(F)^\beta &= \left(\frac{1}{2^k}\right)^{\frac{\log 3}{\log 2} - 1 - \epsilon} \\ &= \frac{2^{k(1+\epsilon)}}{3^k} > m(E+F)\end{aligned}$$

when k is large enough. As for the restriction $3(\alpha^{-1} + \beta^{-1}) \leq 8$ for (5) and (11), $3\alpha^{-1} + 8\beta^{-1} \leq 15$ for (8), they are required by the the method we will use. We do not know whether they are necessary or not.

§3. Reduction Process

In this section, we reduce the measure inequalities into single variable inequalities. In [7] Brown proved

Theorem 3.5. *Suppose that $s,\ t \geq 1$. Then for all $0 \leq x \leq 1$ we have*

$$1 + x + x^2 \geq (1 + x^s)^{1/s}(1 + x^t)^{1/t}, \qquad s^{-1} + t^{-1} = \frac{\log 3}{\log 2} \tag{1}$$

if and only if $3(s + t) \leq 8$;

$$1 + x + x^2 \geq (1 + x^s)^{1/s}(1 + x^t + x^{2t})^{1/t}, \qquad \frac{\log 2}{\log 3}s^{-1} + t^{-1} = 1; \tag{2}$$

and

$$1 + x + x^2 + x^3 \geq (1 + x^s)^{1/s}(1 + x^t + x^{2t} + x^{3t})^{1/t}, \tag{3}$$

whenever $\frac{1}{2}s^{-1} + t^{1} = 1$.

In [12] we use (3) to prove the measure inequality

$$m(E + F) \geq \nu_1(E)^\alpha m(F)^\beta$$

where $\frac{1}{2}\alpha + \beta = 1,\ 0 \leq \alpha,\ \beta \leq 1$. It is an immediate result that the inequality holds when $\frac{1}{2}\alpha + \beta \geq 1,\ \beta \geq \frac{1}{2}$. All the other inequalities can be proved similarly. In the following we will give the proof of (5) in Theorem 3.1(denote as (3.1)–(5)) in detail. Since a Lemma of Brown and Shepp(Lemma 4 of [9]) we be used in this and following sections, we copy it below:

Proposition 3.6. (Brown & Shepp) *Suppose that, for* $i = 0,\ 1,\ 2,\ s_i \geq 1,\ t_i \geq 1$ *and* $as_i^{-1} + bt_i^{-1} = 1$, *for some constants* $a,\ b$ *with* $\frac{a}{b} = \frac{\log n}{\log m}$ *then*

$$S_{s_0}(x)S_{t_0}(y) \leq \max_{i=1,2}\{S_{s_i}(x)S_{t_i}(y)\}$$

Where

$$S_s(x) = \left(\sum_{i=1}^{m} x_i^s\right)^{1/s}, \qquad S_t(y) = \left(\sum_{i=1}^{n} y_i^t\right)^{1/t}$$

and $x_i,\ y_i \geq 0$.

Proof of (3.1)–(5). Since if (3.1)–(5) hold for a pair of $(\alpha,\ \beta)$ then it holds for any pair $(\alpha',\ \beta')$ where $\alpha' \geq \alpha$, $\beta' \geq \beta$, since if $\alpha > 1$ we always have $1 + \alpha > \frac{\log 3}{\log 2}$ and $3(\alpha^{-1} + 1) < 8$, we can assume that $\beta \leq 1$. By the regularity of m and μ_c it is enough to prove the inequality for closed E and F. Furthermore we may assume E, F are closed subsets of the Cantor set C. Let

$$A_n = \left\{\sum_{k=1}^{n} \frac{\epsilon_k}{3^k} | \text{there exist } x \in E, \text{ s.t. } x = \sum_{k=1}^{\infty} \frac{\epsilon_k}{3^k},\ \epsilon = 0 \text{ or } 2\right\},$$

$$B_n = \left\{\sum_{k=1}^{n} \frac{\epsilon_k}{3^k} | \text{there exist } x \in F, \text{ s.t. } x = \sum_{k=1}^{\infty} \frac{\epsilon_k}{3^k},\ \epsilon = 0 \text{ or } 2\right\},$$

and $E_n = A_n + [0, \frac{1}{3^n}]$, $F_n = B_n + [0, \frac{1}{3^n}]$. Then

$$E = \bigcap_{n=1}^{\infty} E_n, \qquad F = \bigcap_{n=1}^{\infty} F_n$$

and

$$E + F = \bigcap_{n=1}^{\infty} (E_n + F_n).$$

Clearly $m(E_n + F_n) \geq |A_n + B_n| \cdot \frac{1}{3^n}$, where we use $|\cdot|$ to denote the number of elements of the corresponding set. Therefore,

$$m(E + F) = \lim_{n\to\infty} m(E_n + F_n) \geq \lim_{n\to\infty} \frac{|A_n + B_n|}{3^n}.$$

Since $\mu_c(E) = \lim_{n\to\infty} \mu_c(E_n) = \lim_{n\to\infty} \frac{|A_n|}{2^n}$ and $\mu_c(F) = \lim_{n\to\infty} \frac{|B_n|}{2^n}$. It suffices to show

$$|A_n + B_n| \geq \left(\frac{3}{2^{\alpha+\beta}}\right)^n |A_n|^\alpha |B_n|^\beta.$$

We do this inductively.

As Brown and Moran did in [8], let

$$A_{n+1}^i = \{\sum_{k=1}^{n+1} \frac{\epsilon_k}{3^k} \in A_{n+1} | \epsilon_{n+1} = i\},$$

$$B_{n+1}^i = \{\sum_{k=1}^{n+1} \frac{\epsilon_k}{3^k} \in B_{n+1} | \epsilon_{n+1} = i\},$$

for $i = 0,\ 2$. By inductive assumption

$$\begin{aligned}|A_{n+1} + B_{n+1}| &\geq |A_{n+1}^0 + B_{n+1}^0| + (|A_{n+1}^0 + B_{n+1}^2|, |A_{n+1}^2 + B_{n+1}^0|) + \\ &\quad + |A_{n+1}^2 + B_{n+1}^2| \\ &\geq [|A_{n+1}^0|^\alpha |B_{n+1}^0|^\beta + (|A_{n+1}^0|^\alpha |B_{n+1}^2|^\beta, |A_{n+1}^2|^\alpha |B_{n+1}^0|^\beta) + \\ &\quad + |A_{n+1}^2|^\alpha |B_{n+1}^2|^\beta] \cdot \left(\frac{3}{2^{\alpha+\beta}}\right)^n\end{aligned}$$

We hope that the last expression is not less than $\left(\frac{3}{2^{\alpha+\beta}}\right)^{n+1} |A_{n+1}|^\alpha |B_{n+1}|^\beta$. It is equivalent to

$$a^\alpha b^\beta + (a^\alpha(1-b)^\beta, (1-a)^\alpha b^\beta) + (1-a)^\alpha(1-b)^\beta \geq \frac{3}{2^{\alpha+\beta}}, \qquad (4)$$

for all $a,\ b \in (0,\ 1)$. We have

Proposition 3.7. *The inequality (4) is true if and only if*

$$1 + x + x^2 \geq \frac{3}{2^{\alpha+\beta}}(1 + x^{\alpha^{-1}})^\alpha (1 + x^{\beta^{-1}})^\beta, \qquad 0 \leq x \leq 1. \qquad (5)$$

Proof. Assume that (4) is true. Let $a,\ b \geq \frac{1}{2}$. Multiplying both sides by $\frac{1}{a^\alpha b^\beta}$, and denoting $\left(\frac{1-a}{a}\right)^\alpha$ as x, $\left(\frac{1-b}{b}\right)^\beta$ as y, we get

$$1 + (x,\ y) + xy \geq \frac{3}{2^{\alpha+\beta}}(1 + x^{\alpha^{-1}})^\alpha (1 + y^{\beta^{-1}})^\beta, \qquad 0 \leq x,\ y \leq 1. \qquad (6)$$

When $x = y$ (6) becomes (5).

Now supposing (5) holds we prove (4) is true.

If $a = 0$ or 1 (4) becomes

$$b^\beta + (1-b)^\beta \geq \frac{3}{2^{\alpha+\beta}}. \tag{7}$$

We need only check that (7) hold for $b = 0, \frac{1}{2}, 1$. When $b = 0$ or 1, we get $1 \geq \frac{3}{2^{\alpha+\beta}}$ which is true since $\alpha + \beta \geq \frac{\log 3}{\log 2}$. For $b = \frac{1}{2}$ we get the true statement $2^{1-\beta} \geq \frac{3}{2^{\alpha+\beta}}$(since $\alpha \geq \frac{\log 3}{\log 2} - 1$).

Now we assume that $a > \frac{1}{2}$, $b > 0$. Like in the above, multiplying both sides of (4) by $\frac{1}{a^\alpha b^\beta}$, letting $x = \left(\frac{1-a}{a}\right)^\alpha$ and $y = \left(\frac{1-b}{b}\right)^\beta$, then (4) becomes

$$1 + (x,\ y) + xy \geq \frac{3}{2^{\alpha+\beta}}(1 + x^{\alpha^{-1}})^\alpha(1 + y^{\beta^{-1}})^\beta, \qquad 0 \leq x \leq 1,\ y \geq 0. \tag{8}$$

Consider $1 + (x,\ y) + xy - \frac{3}{2^{\alpha+\beta}}(1 + x^{\alpha^{-1}})^\alpha(1 + y^{\beta^{-1}})^\beta$ as a function of y. Then its second order derivative is not positive except when $y = x$ where the derivative does not exist. Hence, by the concaveness of the function, we need only check (8) holds for $y = 0$, x and ∞.

When $y = 0$ we get

$$1 + x \geq \frac{3}{2^{\alpha+\beta}}(1 + x^{\alpha^{-1}})^\alpha. \tag{9}$$

If $\alpha \leq 1$, (9) is true obviously, because $\frac{3}{2^{\alpha+\beta}} \leq 1$ and $(1 + x^{\alpha^{-1}})^\alpha \leq 1 + x$. Since $\beta \geq \frac{\log 3}{\log 2} - 1$, we have $\frac{3}{2^{\alpha+\beta}} \leq 2^{1-\alpha}$. When $\alpha > 1$ we show that

$$1 + x \geq 2^{1-\alpha}(1 + x^{\alpha^{-1}})^\alpha, \tag{10}$$

which is equivalent to

$$\frac{1 + z^\alpha}{2} \geq \left(\frac{1+z}{2}\right)^\alpha, \qquad 0 < z < 1. \tag{11}$$

Let $f(z) = \frac{1+z^\alpha}{2} - \left(\frac{1+z}{2}\right)^\alpha$. Then

$$f(0) = \frac{1}{2} - \frac{1}{2^\alpha} > 0, \qquad f(1) = 0$$

and

$$f'(z) = \frac{\alpha}{2} z^{\alpha-1} - \frac{\alpha}{2}\left(\frac{1+z}{2}\right)^{\alpha-1} \leq 0.$$

Hence, $f(z) > 0$ for all $z \in (0,\ 1)$. Then (11) and then (10) are true.

When $y = x$ we get (5) which is true by assumption. When $y = \infty$ we get (9) again. ∎

Now we have to show that (5) really holds when α and β satisfy

$$\begin{cases} \alpha + \beta \geq \dfrac{\log 3}{\log 2} \\ 3(\alpha^{-1} + \beta^{-1}) \leq 8, \quad \alpha,\ \beta \geq \dfrac{\log 3}{\log 2} - 1. \end{cases}$$

Proposition 3.8. *For $0 \leq x \leq 1$ we have*

$$1 + x + x^2 \geq \frac{3}{2^{\alpha+\beta}}(1 + x^{\alpha^{-1}})^\alpha (1 + x^{\beta^{-1}})^\beta, \qquad 0 \leq x \leq 1, \tag{5}$$

where α, β satisfy $\begin{cases} \alpha + \beta \geq \dfrac{\log 3}{\log 2} \\ 3(\alpha^{-1} + \beta^{-1}) \leq 8, \quad \alpha,\ \beta \geq \dfrac{\log 3}{\log 2} - 1. \end{cases}$

Proof. By Proposition 3.6, we need only consider $3(\alpha^{-1} + \beta^{-1}) = 8$. Let $u = \alpha + \beta$ and consider α and β as functions of u determined by

$$\begin{cases} \alpha + \beta = u \\ 3(a^{-1} + \beta^{-1}) = 8. \end{cases} \tag{12}$$

Let

$$f(u, x) = \frac{3}{2^u}(1 + x^{\alpha^{-1}})^\alpha (1 + x^{\beta^{-1}})^\beta.$$

When $u = \frac{\log 3}{\log 2}$, we have $f(u,x) \le 1 + x + x^2$ by (1). We will prove the proposition by showing that $f(u,x)$ is decreasing with respect to u. We may assume that $\alpha > \beta$, since when $\alpha = \beta = \frac{3}{4}$ we have $u = \frac{3}{2} < \frac{\log 3}{\log 2}$. For the convenience of calculation we consider the logarithm of $f(u,x)$. Let

$$g(u,x) = \ln f(u) = \ln 3 - u \ln 2 + \alpha \ln(1 + x^{\alpha^{-1}}) + \beta \ln(1 + x^{\beta^{-1}}).$$

Then

$$g'_u(u,x) = -\ln 2 + \left[\ln(1 + x^{\alpha^{-1}}) - \frac{x^{\alpha^{-1}} \ln x}{\alpha(1 + x^{\alpha^{-1}})}\right] \frac{d\alpha}{du} + \left[\ln(1 + x^{\beta^{-1}}) - \frac{x^{\beta^{-1}} \ln x}{\beta(1 + x^{\beta^{-1}})}\right] \frac{d\beta}{du}.$$

From (12) we get $\frac{d\alpha}{du} = \frac{\alpha^2}{\alpha^2 - \beta^2}$, $\frac{d\beta}{du} = \frac{-\beta^2}{\alpha^2 - \beta^2}$. Therefore,

$$g'_u(u,x) = -\ln 2 + \frac{1}{\alpha^2 - \beta^2} \left[\alpha\left(\alpha \ln(1 + x^{\alpha^{-1}}) - \frac{x^{\alpha^{-1}} \ln x}{1 + x^{\alpha^{-1}}}\right) - \beta\left(\beta \ln(1 + x^{\beta^{-1}}) - \frac{x^{\beta^{-1}} \ln x}{1 + x^{\beta^{-1}}}\right)\right].$$

Let

$$h(u,x) = \alpha\left(\alpha \ln(1 + x^{\alpha^{-1}}) - \frac{x^{\alpha^{-1}} \ln x}{1 + x^{\alpha^{-1}}}\right) - \beta\left(\beta \ln(1 + x^{\beta^{-1}}) - \frac{x^{\beta^{-1}} \ln x}{1 + x^{\beta^{-1}}}\right).$$

Then

$$\begin{aligned} h'_x(u,x) &= \left(\frac{\alpha x^{\alpha^{-1}-1}}{1 + x^{\alpha^{-1}}} - \frac{x^{\alpha^{-1}-1} \ln x + \alpha x^{\alpha^{-1}-1}}{1 + x^{\alpha^{-1}}} - \frac{x^{2\alpha^{-1}-1} \ln x}{(1 + x^{\alpha^{-1}})^2}\right) \\ &\quad - \left(\frac{\beta x^{\beta^{-1}-1}}{1 + x^{\beta^{-1}}} - \frac{x^{\beta^{-1}-1} \ln x + \beta x^{\beta^{-1}-1}}{1 + x^{\beta^{-1}}} - \frac{x^{2\beta^{-1}-1} \ln x}{(1 + x^{\beta^{-1}})^2}\right) \\ &= x^2 \ln x \cdot \left(-\frac{x^{\alpha^{-1}}}{(1 + x^{\alpha^{-1}})^2} + \frac{x^{\beta^{-1}}}{(1 + x^{\alpha^{-1}})^2}\right) > 0 \end{aligned}$$

since the function $\frac{y}{(1+y)^2}$ is increasing when y is increasing in (0,1), and $x^{\alpha^{-1}} > x^{\beta^{-1}}$. Hence

$$g'_u(u,x) \le -\ln 2 + \frac{1}{\alpha^2 - \beta^2} h(u,1) = 0,$$

which completes the proof.

Now we have proved (3.1)–(5). Through the proof we can see that μ_c can be replaced by the the following measure

$$\mu = \mathop{*}_{n=1}^{\infty}(\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{\frac{1}{3^n}})$$

which is carried on the set $C_1 = \frac{1}{2}C$, where C is the Cantor middle third set. In fact we need only replace 2 by 1 in the definition of A_n and B_n.

Proposition 3.3 can also be proved in the same way. For the proof of (3.1)–(6), (7), (8) we need

Proposition 3.9. *Suppose that* $a_i,\ b_i \geq 0$ *and* $\sum a_1 = 1, \sum b_i = 1$ *where the sum is for all the* i *appearing in each inequality. We have the following inequalities:*

$$(a_0^\alpha b_0^\beta,\ a_2^\alpha b_1^\beta) + (a_0^\alpha b_1^\beta,\ a_1^\alpha b_0^\beta) + (a_1^\alpha b_1^\beta,\ a_2^\alpha b_0^\beta) \geq 1, \qquad \alpha + \frac{\log 2}{\log 3}\beta = 1,\ 0 \leq \alpha, \beta \leq 1; \tag{13}$$

$$(a_0^\alpha b_0^\beta,\ a_3^\alpha b_1^\beta) + (a_0^\alpha b_1^\beta,\ a_1^\alpha b_0^\beta) + (a_1^\alpha b_1^\beta,\ a_2^\alpha b_0^\beta) + (a_2^\alpha b_1^\beta,\ a_3^\alpha b_0^\beta) \geq 1, \qquad \alpha + \frac{1}{2}\beta = 1,\ 0 \leq \alpha, \beta \leq 1; \tag{14}$$

$$a_0^\alpha b_0^\beta + (a_0^\alpha b_1^\beta,\ a_1^\alpha b_0^\beta) + (a_1^\alpha b_1^\beta,\ a_2^\alpha b_0^\beta) + a_2^\alpha b_1^\beta \geq 1, \qquad \frac{1}{2}\alpha + \frac{\log 3}{\log 4}\beta = 1,\ 3\alpha^{-1} + 8\beta^{-1} \leq 15. \tag{15}$$

(13) and (14) can be proved by (2) and (3) of Theorem 3.5 respectively. The proof of (15) needs the following theorem.

Theorem 3.10. *The inequality*

$$1 + x + x^2 + x^3 \geq (1 + x^s)^{\frac{1}{s}}(1 + x^t + x^{2t})^{\frac{1}{t}} \tag{16}$$

holds for $0 \le x \le 1$, *if and only if*

$$3s + 8t \le 15, \tag{17}$$

where $s,\ t \ge 1$ *satisfy*

$$\frac{1}{2}\frac{1}{s} + \frac{\log 3}{\log 4}\frac{1}{t} = 1. \tag{18}$$

The proof of Theorem 3.10, which is quite long, will be given in next section.

For the proof of (15) we also need

$$1 + x + x^2 + x^3 \ge (1 + x^s)^{\frac{1}{s}}(1 + x^t)^{\frac{1}{t}} \tag{19}$$

where $s,\ t$ are the same as in Theorem 3.10. In fact we have (19) for $s,\ t$ satisfying $s,\ t \ge 1$ and $\frac{1}{s} + \frac{1}{t} \le 1.5$ (see Kemp [28]). From (18) we get

$$\frac{1}{s} + \frac{1}{t} = 2 - (1 + \frac{\log 3}{\log 2})\frac{1}{t} < 1.5,$$

since $t < 2$ by (17).

We wish that the following inequality were true:

$$(a_0^\alpha b_0^\beta,\ a_2^\alpha b_2^\beta) + (a_0^\alpha b_1^\beta,\ a_1^\alpha b_0^\beta) + (a_0^\alpha b_2^\beta,\ a_1^\alpha b_1^\beta,\ a_2^\alpha b_0^\beta) + (a_1^\alpha b_2^\beta,\ a_2^\alpha b_1^\beta) \ge 1 \tag{20}$$

where $0 \le \alpha,\ \beta \le 1,\ \alpha + \beta = \frac{\log 4}{\log 3}$ and $a_i,\ b_i \ge 0,\ \sum a_i = \sum b_i = 1$. Unfortunately, (20) does not hold in general. Let $a_0 = a_2 = b_0 = b_2 = \frac{1}{2}$ and $a_1 = b_1 = 0$. Then the left hand side equals $2^{1-\frac{\log 4}{\log 3}} < 1$. However, (20) is true when $a_0 = a_1 = a_2 = \frac{1}{3}$, i.e.

$$(b_0^\beta,\ b_2^\beta) + (b_1^\beta,\ b_0^\beta) + (b_2^\beta,\ b_1^\beta,\ b_0^\beta) + (b_2^\beta,\ b_1^\beta) \ge 3^\alpha, \tag{21}$$

by which we can prove (3.2)–(10).

Since the left hand side of (21) is symmetric, we may assume $b_0 \geq b_1,\ b_2$. Then (21) becomes

$$3b_0^\beta + (b_1^\beta,\ b_2^\beta) \geq 3^\alpha. \tag{22}$$

The worst case of (22) is $b_1 = b_2$. therefore, we need to show

$$3b_0^\beta + \beta_1^\beta \geq 3^\alpha, \qquad b_0 \geq b_1 \geq 0,\ b_0 + 2b_1 = 1. \tag{23}$$

Let $\left(\frac{b_1}{b_0}\right)^\beta = x$. Then (22) is equivalent to

$$3 + x \geq 3^\alpha(1 + 2x^{\beta^{-1}})^\beta, \qquad 0 \leq x \leq 1. \tag{24}$$

Let $f(x) = 3 + x - 3^\alpha(1 + 2x^{\beta^{-1}})^\beta$. We have

$$f(0) = 3 - 3^\alpha \geq 0, \qquad f(1) = 0$$

and

$$f''(x) = -2 \cdot 3^\alpha(\beta^{-1} - 1)(1 + 2x^{\beta^{-1}})^{\beta-2} \cdot x^{\beta^{-1}-2} \geq 0.$$

Hence $f(x) \geq 0$ for all $0 \leq x \leq 1$.

Suppose E, F are the same as in (3.2)–(10). Let

$$E_n = c_k + \{ \sum_{i=k+1}^{n} \frac{\epsilon_i}{4^i}, \quad \epsilon_i = 0, 1, 2\}$$

where $c_k = \sum_{i=1}^{k} \frac{\epsilon_i}{4^i}$ (ϵ_i =0, 1 or 2) is a constant, and

$$F_n = \{\sum_{i=1}^{n} \frac{\epsilon_i}{4^i} | \text{there exists} \sum_{i=1}^{\infty} \frac{\epsilon_i}{4^i} \in F,\ \epsilon_i = 0, 1 \text{ or } 2\}.$$

Like before, by (21) we can show that

$$|E_n + F_n| \geq |E_n|^\alpha |F_n|^\beta, \tag{25}$$

which derives (3.2)–(10). It is enough to check (25) is true when $n = k$, the first step of induction, which is a trivial result.

As for (3.2)–(9), it can be proved by

$$(a_0b_0,\ a_2b_2) + (a_0b_1,\ a_1b_0) + (a_0b_2,\ a_1b_1,\ a_2b_0) + (a_1b_2,\ a_2b_1) \geq \frac{4}{9}, \qquad (26)$$

where $a_i,\ b_i \geq 0$ and $\sum a_i = \sum b_i = 1$. In the following we show (26) is true.

We have

$$\text{LHS of (26)} \geq a_0(b_0 + b_1 + b_2) + (a_1b_2,\ a_2b_1) = a_0 + (a_1b_2,\ a_2b_1) \qquad (27)$$

Hence (26) holds whenever $a_0 \geq \frac{4}{9}$. Similarly, (26) holds if there exists an a_i or $b_i \geq \frac{4}{9}$. Therefore we may restrict a_i's and b_i's to lie in the interval $[\frac{1}{9},\ \frac{4}{9}]$(denote as $[c_1,\ d_1]$). It is easy to see that under this restriction we have $(a_1b_2,\ a_2b_1) \geq c_1d_1$. From (27) we know that (26) is true whenever $a_0 \geq \frac{4}{9} - c_1d_1$. Furthermore, (26) is true if a_i's or b_i's lie outside the interval $[\frac{1}{9} + 2c_1d_1,\ \frac{4}{9} - c_1d_1]$(denote as $[c_2,\ d_2]$). Inductively, we get that (26) holds if a_i's or b_i's lie outside the interval $[c_n,\ d_n]$, where $d_n = \frac{4}{9} - c_{n-1}d_{n-1},\ c_n = 1 - 2d_n$.

Now we show that $d_n \downarrow \frac{1}{3}$ and therefore $c_n \uparrow \frac{1}{3}$. In fact we have

$$d_n = \frac{4}{9} - (1 - 2d_{n-1})d_{n-1}. \qquad (28)$$

Since the function $(1 - 2x)x$ is decreasing when $x > \frac{1}{4}$, and $d_n \geq \frac{4}{9} - \max\{(1 - 2x)x\} = \frac{4}{9} - \frac{1}{8} > \frac{1}{4}$, we know that d_n is decreasing as n is increasing. Hence d_n is decreasing to the limit value $\frac{1}{3}$ which is a root of $x = \frac{4}{9} - (1 - 2x)x$.

The proof is completed by the fact that when $a_i = b_i = \frac{1}{3}(i = 0, 1, 2)$ (24) is an equality.

§4. A Proof of An Inequality

In this section we give a proof of Theorem 3.10. For the convenience of statement, we copy Theorem 3.10 below.

Theorem 3.10. *The inequality*

$$1 + x + x^2 + x^3 \geq (1 + x^s)^{\frac{1}{s}}(1 + x^t + x^{2t})^{\frac{1}{t}} \tag{1}$$

holds for $0 \leq x \leq 1$, *if and only if*

$$3s + 8t \leq 15, \tag{2}$$

where $s,\ t \geq 1$ *satisfy*

$$\frac{1}{2}\frac{1}{s} + \frac{\log 3}{\log 4}\frac{1}{t} = 1. \tag{3}$$

Proof. First we show that the condition $3s + 8t \leq 15$ is necessary. Let $x = 1 - y$. Considering the second order Taylor expansion with respect to y of both sides of (1), we get

$$1 + x + x^2 + x^3 = 4 - 6y + 4y^2 + o(y^2), \tag{4}$$

and

$$\begin{aligned} &(1 + x^s)^{\frac{1}{s}}(1 + x^t + x^{2t})^{\frac{1}{t}} \\ &= [2^{\frac{1}{s}} - 2^{\frac{1}{s}-1}y + 2^{\frac{1}{s}-3}(s-1)y^2 + o(y^2)][3^{\frac{1}{t}} - 3^{\frac{1}{t}}y + 3^{\frac{1}{t}-1}ty^2 + o(y^2)] \\ &= 4 - 6y + \frac{3s + 8t + 9}{6}y^2 + o(y^2) \quad (\text{by } (3),\ 2^{\frac{1}{s}}3^{\frac{1}{t}} = 4). \end{aligned} \tag{5}$$

Comparing (4) and (5) we know that when $3s + 8t > 15$, we can choose x close to 1 (i.e. y close to 0) which makes (1) false.

Now we prove (1) under (2) and (3). By Lemma 3.6, we need only to prove (1) under

$$\frac{1}{2}\frac{1}{s} + \frac{\log 3}{\log 4}\frac{1}{t} = 1, \quad 3s + 8t = 15. \tag{6}$$

From (6) we get

$$s = 5 - \frac{8}{3}t, \quad t_1 = 1.012670... \text{ and } t_2 = 1.467310...$$

Let

$$F(x) = \log(1 + x + x^2 + x^3) - \frac{1}{s}\log(1 + x^s) - \frac{1}{t}\log(1 + x^t + x^{2t}).$$

We need to show that $F(x) \geq 0$ for $0 \leq x \leq 1$ when $t = t_1,\ t_2$. We have

$$F(0) = 0, \quad F(1) = 0. \tag{7}$$

Consider the first derivative of $F(x)$:

$$F'(x) = \frac{1 + 2x + 3x^2}{1 + x + x^2 + x^3} - \frac{x^{s-1}}{1 + x^s} - \frac{x^{t-1} + 2x^{2t-1}}{1 + x^t + x^{2t}},$$

which is a positive multiple of the following function (denote $u = t/3$, then $s = 5 - 8u$):

$$\begin{aligned} G(x) &= (1 + 2x + 3x^2)(1 + x^s)(1 + x^t + x^{2t}) - (1 + x + x^2 + x^3)x^{s-1}(1 + x^t + x^{2t}) \\ &\quad - (1 + x + x^2 + x^3)(1 + x^s)(x^{t-1} + 2x^{2t-1}) \\ &= 1 + 2x + 3x^2 + x^{6-8u} + 2x^{7-8u} + x^{3u+1} + 2x^{3u+2} - x^{5-5u} + x^{7-5u} - x^{6u} \\ &\quad + x^{6u+2} + 2x^{5-2u} - x^{4-8u} - 2x^{4-5u} - 3x^{4-2u} - x^{6-2u} - x^{3u-1} - 2x^{6u-1}, \end{aligned}$$

and we have

$$G(0) = 1, \quad G(1) = 0. \tag{8}$$

If we can prove $G(x)$ has exactly one root in$(0,1)$, then by (7) and (8) we can get the expected result.

I. $\mathbf{t_1 = 1.0126705730315177...}$

Define $H(x) = x^{-1}G(x)$. Then

$$H(0) = +\infty, \quad H(1) = 0. \tag{9}$$

We show that $H(x)$ has exactly one root in $(0,1)$. Write $H(x)$ in power increasing order:

$$\begin{aligned} H(x) =& x^{-1} - x^{3u-2} + 2 - 2x^{6u-2} - x^{3-8u} + 3x + x^{3u} - x^{6u-1} - 2x^{3-5u} + 2x^{1+3u} \\ &+ x^{5-8u} - x^{4-5u} - 3x^{3-2u} + x^{1+6u} + 2x6 - 8u - 2x^{4-2u} + x^{6-5u} - x^{5-2u}. \end{aligned}$$

Then

$$\begin{aligned} H'(x) =& - x^{-2} - (3u-2)x^{3u-3} - 2(6u-2)x^{6u-3} - (3-8u)x^{2-8u} + 3 + 3ux^{3u-1} \\ &- (6u-1)x^{6u-2} - 2(3-5u)x^{2-5u} + 2(1+3u)x^{3u} + (5-8u)x^{4-8u} \\ &- (4-5u)x^{3-5u} - 3(3-2u)x^{2-2u} + (1+6u)x^{6u} + 2(6-8u)x^{5-8u} \\ &- 2(4-2u)x^{3-2u} + (6-5u)x^{5-5u} - (5-2u)x^{4-2u}, \end{aligned}$$

and

$$H'(0) = -\infty, \quad H'(1) = 0. \tag{10}$$

Furthermore, we have

$$H''(0) = +\infty, \quad H''(1) = 0 \tag{11}$$

and

$$H^{(4)}(x) = 24x^{-5} - (2-3u)(3-3u)(4-3u)(5-3u)x^{3u-6}$$
$$+ 2(6u-2)(3-6u)(4-6u)(5-6u)x^{6u-6} + (3-8u)(8u-2)(8u-1)8ux^{-1-8u}$$
$$+ 3u(3u-1)(2-3u)(3-3u)x^{3u-4} - (6u-1)(6u-2)(3-6u)(4-6u)x^{6u-5}$$
$$- 2(3-5u)(2-5u)(5u-1)5ux^{-1-5u} - 6u(1+3u)(3u-1)(2-3u)x^{3u-3}$$
$$- (5-8u)(4-8u)(3-8u)(8u-2)x^{1-8u} + (4-5u)(3-5u)(2-5u)(5u-1)x^{-5u}$$
$$+ 3(3-2u)(2-2u)(1-2u)2ux^{-1-2u} + (1+6u)6u(6u-1)(6u-2)x^{6u-3}$$
$$+ 2(6-8u)(5-8u)(4-8u)(3-8u)x^{2-8u} - 2(4-2u)(3-2u)(2-2u)(1-2u)x^{-2u}$$
$$+ (6-5u)(5-5u)(4-5u)(3-5u)x^{2-5u} - (5-2u)(4-2u)(3-2u)(2-2u)x^{1-2u}.$$

If we know that $H^{(4)}(x) > 0$, for $0 < x < 1$, then $H''(x)$ is convex in $(0,1)$. By (11) we know that $H''(x)$ has at most one root in $(0,1)$. If $H''(x)$ has no root in $(0,1)$ then $H''(x) > 0$ for $x \in (0,1)$. Therefore $H'(x) < 0$ for $x \in (0,1)$. Because of (9), we get $H(x) > 0$, which implies $G(x) > 0$, and then $F'(x) > 0$ for $0 < x < 1$. It contradicts with (7). Hence $H''(x)$ has exactly one root in$(0,1)$. At this stage, we can see that $G(x)$ has exactly one root in (0,1) easily.

Now we show that $H''(x) > 0$ for $x \in (0,1)$. Denote the absolute value of the coefficient of the i-th term of $H^{(4)}(x)$ as $a_i(u)$ and the exponent as α_i. Then we can write $H^{(4)}(x)$ as

$$\begin{aligned}H^{(4)}(x) =& a_1(u)x^{\alpha_1} - a_2(u)x^{\alpha_2}\\ &+ a_3(u)x^{\alpha_3} + a_4(u)x^{\alpha_4} + a_5(u)x^{\alpha_5}\\ &- a_6(u)x^{\alpha_6} - a_7(u)x^{\alpha_7} - a_8(u)x^{\alpha_8} - a_9(u)x^{\alpha_9}\\ &+ a_{10}(u)x^{\alpha_{10}} + a_{11}(u)x^{\alpha_{11}} + a_{12}(u)x^{\alpha_{12}} + a_{13}(u)x^{\alpha 13}\\ &- a_{14}(u)x^{\alpha_{14}} + a_{15}(u)x^{\alpha_{15}} - a_{16}(u)x^{\alpha_{16}}\end{aligned}$$

Since $\alpha_1 < \alpha_2 < ... < \alpha_{16}$ and $0 < x < 1$, we need only to compare the coefficients. Consider u as a variable and define

$$f_1(u) = a_1(u) - a_2(u),$$

$$f_2(u) = a_3(u) + a_4(u) + a_5(u) - a_6(u) - a_7(u) - a_8(u) - a_9(u),$$

$$f_3(u) = a_{10}(u) + a_{11}(u) + a_{12}(u) + a_{13}(u) - a_{14}(u),$$

$$f_4(u) = a_{15}(u) - a_{16}(u).$$

We show that when $u = t_1/3 = 0.33755...$, we have

$$\begin{cases} f_1(u) > 0, \\ f_1(u) + f_2(u) > 0, \\ f_3(u) > 0, \\ f_3(u) + f_4(u) > 0. \end{cases}$$

In fact, $f_1(u) = -96 + 462u + 639u^2 - 378u^3 + 81u^4$ and

$$f_1'(u) = 462 - 1278u + 1134u^2 - 324u^3,$$

$$f_1''(u) = -1278 + 2286u - 984u^2 < 0 \quad (\text{since } B^2 - 4AC < 0).$$

Hence for $0 \leq u \leq 0.4$ we have

$$f_1'(u) \geq f_1'(0.4) = 75.504 > 0.$$

So for $u = t_1/3 = 0.337556...$, we have

$$f_1(u) \geq f_1(0.33755) \geq 0.626.$$

Similarly, for $f_2(u)$ we have

$$f_2(u) = -144 + 1018u - 3001u^2 + 5342u^3 - 4895u^4,$$

$$f_2'(u) = 1018 - 6002u + 16026u^2 - 19580u^3,$$

$$f_2''(u) = -6002 + 32052u - 58740u^2 < 0 \quad (B^2 - 4AC < 0),$$

and $f_2'(0.35) = 40.9925 > 0$. Therefore for that particular value of u, we have

$$f_2'(u) \geq f_2'(0.33755) \geq -0.401.$$

Hence

$$f_1(u) + f_2(u) \geq 0.626 - 0.401 = 0.225 > 0.$$

In the same way, we can get

$$f_3(u) \geq f_3(0.338) \geq 2.02.$$

Lastly,

$$f_4(u) = 240 - 1402u + 2691u^2 - 2138u^3 + 609u^4,$$

$$f_4'(u) = -1402 + 5328u - 6414u^2 + 2436u^3,$$

$$f_4''(u) = 5382 - 12828u + 7308u^2.$$

This time we have $f_4''(0.3) > 0$, $f_4''(0.5) > 0$ and $f_4''(1) < 0$. Therefore $f_4''(u) > 0$ for $0.3 \leq u \leq 0.5$. By the fact that $f_4''(0.3) = -298.8879$ and $f_4''(0.5) = -10$, we get that

$$f_4'(u) < 0, \quad u \in (0.3, 0.5).$$

Hence for the particular u, we have

$$f_4(u) \geq f_4(0.34) \geq -1.5.$$

At last we get

$$f_3(u) + f_4(u) \geq 2.02 - 1.5 = 0.52 > 0.$$

II. $\mathbf{t_2 = 1.467310677329...}$

It is similar to last the paragraph, but more complicated. This time we define

$$H(x) = x^{-\frac{1}{2}} G(x)$$

and write it in power increasing order

$$
\begin{aligned}
H(x) =& x^{-\frac{1}{2}} - x^{\frac{7}{2}-8u} - x^{3u-\frac{3}{2}} + 2x^{\frac{1}{2}} - 2x^{\frac{7}{2}-5u} - 2x^{6u-\frac{3}{2}} \\
&+ 3x^{\frac{3}{2}} + x^{\frac{11}{2}-8u} + x^{3u-\frac{1}{2}} - x^{\frac{9}{2}-5u} - x^{6u-\frac{1}{2}} - 3x^{\frac{7}{2}-2u} \\
&+ 2x^{\frac{13}{2}-8u} + 2x^{\frac{3}{2}+3u} - 2x^{\frac{9}{2}-2u} + x^{\frac{13}{2}-5u} + x^{\frac{3}{2}+6u} - x^{\frac{11}{2}-2u}.
\end{aligned}
$$

Like in the last paragraph, we have

$$
\begin{cases}
H(0) = +\infty, \quad H(1) = 0; \\
H'(0) = -\infty, \quad H'(1) = 0; \\
H''(0) = +\infty, \quad H''(1) = 0.
\end{cases}
$$

We will show that $H^{(4)}(x) > 0$. Unlike the last paragraph, we can not prove it directly. We need to consider $H^{(6)}(x)$.Check that

$$
\begin{aligned}
&H^{(4)}(0) = +\infty, \quad H^{(4)}(1) = 6 > 0, \\
&H^{(5)}(0) = -\infty, \quad H^{(5)}(1) = -6 < 0.
\end{aligned}
$$

By the same method used in the last paragraph, we will show that

$$
H^{(6)}(x) > 0,
$$

which completes the proof. Now

$$
\begin{aligned}
&H^{(6)}(x) = \frac{10395}{64}x^{-\frac{13}{2}} \\
&-(8u-\frac{7}{2})(8u-\frac{5}{2})(8u-\frac{3}{2})(8u-\frac{1}{2})(8u+\frac{1}{2})(8u+\frac{3}{2})x^{-\frac{5}{2}-8u} \\
&-(\frac{3}{2}-3u)(\frac{5}{2}-3u)(\frac{7}{2}-3u)(\frac{9}{2}-3u)(\frac{11}{2}-3u)(\frac{13}{2}-3u)x^{3u-\frac{15}{2}} \\
&-\frac{945}{32}x^{-\frac{11}{2}}-2(\frac{7}{2}-5u)(\frac{5}{2}-5u)(5u-\frac{3}{2})((5u-\frac{1}{2})(5u+\frac{1}{2})(5u+\frac{3}{2})x^{-\frac{5}{2}-5u} \\
&-2(6u-\frac{3}{2})(6u-\frac{5}{2})(\frac{7}{2}-6u)(\frac{9}{2}-6u)(\frac{11}{2}-6u)(\frac{13}{2}-6u)x^{6u-\frac{15}{2}} \\
&+\frac{945}{64}x^{-\frac{9}{2}}+(\frac{11}{2}-8u)(\frac{9}{2}-8u)(8u-\frac{7}{2})(8u-\frac{5}{2})(8u-\frac{3}{2})(8u-\frac{1}{2})x^{-\frac{1}{2}-8u} \\
&+(3u+\frac{1}{2})(3u-\frac{1}{2})(\frac{3}{2}-3u)(\frac{5}{2}-3u)(\frac{7}{2}-3u)(\frac{9}{2}-3u)x^{3u-\frac{11}{2}} \\
&+(\frac{9}{2}-5u)(\frac{7}{2}-5u)(\frac{5}{2}-5u)(5u-\frac{3}{2})(5u-\frac{1}{2})(5u+\frac{1}{2})x^{-\frac{3}{2}-5u} \\
&+(6u-\frac{1}{2})(6u-\frac{3}{2})(6u-\frac{5}{2})(\frac{7}{2}-6u)(\frac{9}{2}-6u)(\frac{11}{2}-6u)x^{6u-\frac{13}{2}} \\
&+3(\frac{7}{2}-2u)(\frac{5}{2}-2u)(\frac{3}{2}-2u)(2u-\frac{1}{2})(2u+\frac{1}{2})(2u+\frac{3}{2})x^{\frac{5}{2}-2u} \\
&-2(\frac{13}{2}-8u)(\frac{11}{2}-8u)(\frac{9}{2}-8u)(8u-\frac{7}{2})(8u-\frac{5}{2})(8u-\frac{3}{2})x^{\frac{1}{2}-8u} \\
&-2(\frac{3}{2}+3u)(\frac{1}{2}+3u)(3u-\frac{1}{2})(\frac{3}{2}-3u)(\frac{5}{2}-3u)(\frac{7}{2}-3u)x^{3u-\frac{9}{2}} \\
&-2(\frac{9}{2}-2u)(\frac{7}{2}-2u)(\frac{5}{2}-2u)(\frac{3}{2}-2u)(2u-\frac{1}{2})(2u+\frac{1}{2})x^{-\frac{3}{2}-2u} \\
&-(\frac{13}{2}-5u)(\frac{11}{2}-5u)(\frac{9}{2}-5u)(\frac{7}{2}-5u)(\frac{5}{2}-5u)(5u-\frac{3}{2})x^{\frac{1}{2}-5u} \\
&-(\frac{3}{2}+6u)(\frac{1}{2}+6u)(6u-\frac{1}{2})(6u-\frac{3}{2})(6u-\frac{5}{2})(\frac{7}{2}-6u)x^{6u-\frac{9}{2}} \\
&+(\frac{11}{2}-2u)(\frac{9}{2}-2u)(\frac{7}{2}-2u)(\frac{5}{2}-2u)(\frac{3}{2}-2u)(2u-\frac{1}{2})x^{-\frac{1}{2}-2u}.
\end{aligned}
$$

Let

$$
\begin{aligned}
f_1(u) &= a_1(u) - \sum_{i=2}^{6} a_i(u) \\
&= -6216.328125 + 60563.25u - 241881.5625u^2 + 511830u^3 \\
&\quad - 651543.75u^4 + 613188u^5 - 387435u^6,
\end{aligned}
$$

and

$$f_2(u) = \sum_{i=7}^{12} a_i(u) - \sum_{i=13}^{17} a_i(u)$$
$$= 6378.75 - 86083.5u + 480716.3125u^2 - 1424670u^3$$
$$+ 2371723.75u^4 - 2111364u^5 + 788299u^6,$$

where $a_i(u)$ (> 0) is the absolute value of the coefficient of the i-th term of $H^{(6)}(x)$.

We show that

$$\begin{cases} f_1(u) > 0 \\ f_2(u) > 0, \end{cases}$$

for $u = t_2/3 = 0.489103559...$, which implies $H^{(6)}(x) > 0$ for $0 < x < 1$. In fact, for positive terms of f_1 and f_2 using 0.489103, which is slightly less than $u = t_2/3$, and for negative terms using 0.489104, which is slightly larger, in place of u, we get

$$\begin{cases} f_1(u) \geq 0.79, \\ f_2(u) \geq 0.54. \end{cases}$$

The proof is completed.

§5. Sumset of three sets.

In this section we consider the Lebesgue measure of sumset of 3 subsets with positive measure of ν_1 defined in §2. Suppose E, F, G are Borel subsets of $\mathbf{T}$. We hope that there exist $\alpha,\ \beta,\ \gamma > 0$ with $(\alpha+\beta+\gamma)\dim(\nu_1) = \dim(m)$ or simply $\alpha+\beta+\gamma = 2$ such that

$$m(E+F+G) \geq \nu_1(E)^\alpha \nu_1(F)^\beta \nu_1(G)^\gamma. \tag{1}$$

Indeed by Proposition 3.3 and (8) of Theorem 3.1, we know that there exist $\beta',\ \gamma' > 0$ with $\beta'+\gamma' = \dfrac{\log 3}{\log 2}$ such that

$$\nu_2(F+G) \geq \nu_1(F)^{\beta'} \nu_1(G)^{\gamma'},$$

and $\alpha,\ u > 0$ with $\frac{1}{2}\alpha + \dfrac{\log 3}{\log 4}u = 1$ such that

$$m(E+F+G) \geq \nu_1(E)^\alpha \nu_2(F+G)^u.$$

Let $\beta = \beta' u$ and $\gamma = \gamma' u$. Then we have $\alpha+\beta+\gamma = 2$ and get (1). Inaddition, we have $\alpha^{-1}+\beta^{-1}+\gamma^{-1} \leq 5$.

Denote the solutions of $\begin{cases} \dfrac{1}{2}\alpha + \dfrac{\log 3}{\log 4}\beta = 1 \\ 3\alpha^{-1} + 8\beta^{-1} = 15 \end{cases}$ as

$$\begin{cases} \alpha_1 = 0.434868\cdots \\ \beta_1 = 0.987487\cdots \end{cases} \quad \text{and} \quad \begin{cases} \alpha_2 = 0.919818\cdots \\ \beta_2 = 0.681518\cdots \end{cases}$$

and the solutions of $\begin{cases} \alpha+\beta = \dfrac{\log 3}{\log 2} \\ 3(\alpha^{-1}+\beta^{-1}) = 8 \end{cases}$ as

$$\begin{cases} \alpha_1' = \beta_2' = 0.975963\cdots \\ \alpha_2' = \beta_1' = 0.608999\cdots \end{cases}$$

Let $a = \alpha_2'\beta_2 = 0.415044\cdots$, $b = \alpha_2 = 0.919818\cdots$, $c = \alpha_1 = 0.434868\cdots$ and $d = \alpha_1'\beta_1 = 0.963751\cdots$. Then we have

Theorem 3.11. *For any Borel subsets E, F, G of $\mathbf{T}$ one has*

$$m(E+F+G) \geq \nu_1(E)^\alpha \nu_1(F)^\beta \nu_1(G)^\gamma \tag{1}$$

where α, β, γ satisfy

$$\alpha+\beta+\gamma = 2 \tag{2}$$

and are located in any one of the three intervals $[a,\ b]$, $[c,\ d]$ and $[0.5,\ 1]$.

When α, β, γ are located in $[a,\ b]$ or $[c,\ d]$ Theorem 3.11 can be proved directly by Proposition 3.3 and (8) of Theorem 3.1. For the interval $[0,\ 1]$ the proof is based on a particular one variable inequality.

Theorem 3.12. *Suppose that $r,\ s,\ t \geq 1$ with $\frac{1}{r}+\frac{1}{s}+\frac{1}{t} = 2$ and $1 \leq r,\ s,\ t \leq 2$. Then for all $0 \leq x \leq 1$ we have*

$$1+x+x^2+x^3 \geq (1+x^r)^{\frac{1}{r}}(1+x^s)^{\frac{1}{s}}(1+x^t)^{\frac{1}{t}}. \tag{3}$$

Proof. First we claim that we need only to prove (3) for $r=1$, $s=t=2$. In fact, we may assume that $t \geq r,\ s$. If $t=2$, then $1 \leq r,\ s \leq 2$ and $\frac{1}{r}+\frac{1}{s} = 1.5$. By Proposition 3.6,

$$(1+x^r)^{1/r}(1+x^s)^{1/s} \leq (1+x)(1+x^2)^{1/2},$$

Then

$$(1+x^r)^{1/r}(1+x^s)^{1/s}(1+x^t)^{1/t} \leq (1+x)(1+x^2)^{1/2}(1+x^2)^{1/2}.$$

If $t < 2$, suppose $s \geq r$, choose s_0 such that $\frac{1}{2} + \frac{1}{s_0} + \frac{1}{r} = 2$. It must be that $s_0 \geq 1$, and we have

$$(1 + x^r)^{1/r}(1 + x^s)^{1/s}(1 + x^t)^{1/t} \leq (1 + x^r)^{1/r}(1 + x^{s_0})^{1/s_0}(1 + x^2)^{1/2}.$$

Hence

$$\begin{aligned}(1 + x^r)^{1/r}(1 + x^s)^{1/s}(1 + x^t)^{1/t} &\leq (1 + x)(1 + x^2)^{1/2}(1 + x^2)^{1/2} \\ &= 1 + x + x^2 + x^3 \qquad \blacksquare\end{aligned}$$

Proof of Theorem 3.11. Suppose that $\nu_1(E) \geq \nu_1(F) \geq \nu_1(G)$. By (8) of Theorem 3.1, we have

$$m(E + F + G) \geq \nu_1(E)^b \nu_2(F + G)^{\beta_2},$$

for $b = \alpha_2$, β_2 defined before. And by Proposition 3.3,

$$\nu_2(F + G) \geq \nu_1(F)^{\alpha_1'} \nu_1(G)^{\beta_1'}.$$

Hence, for $\alpha_0 = b = 0.919\cdots$, $\beta_0 = \beta_2\alpha_1' = 0.665\cdots$ and $\gamma = a = \beta_2\beta_1' = 0.415\cdots$, we have

$$m(E + F + G) \geq \nu_1(E)^{\alpha_0} \nu_1(F)^{\beta_0} \nu_1(G)^{\gamma_0}.$$

If $\alpha,\ \beta,\ \gamma \in [a,\ b]$ with $\alpha+\beta+\gamma = 2$, we must have $\alpha \leq \alpha_0$ and $\alpha+\beta \leq \alpha_0+\beta_0$(since $\gamma \geq \gamma_0 = a$). Therefore

$$\begin{aligned}m(E + F + G) &\geq \nu_1(E)^{\alpha_0} \nu_1(F)^{\beta_0} \nu_1(G)^{\gamma_0} \\ &\geq \nu_1(E)^{\alpha} \nu_1(F)^{\beta} \nu_1(G)^{\gamma}.\end{aligned}$$

Similarly, we can show that whenever $\alpha,\ \beta,\ \gamma \in [c,\ d]$ with $\alpha + \beta + \gamma = 2$, we have

$$m(E + F + G) \geq \nu_1(E)^{\alpha} \nu_1(F)^{\beta} \nu_1(G)^{\gamma}.$$

For the case $\alpha,\ \beta,\ \gamma \in [0,\ 1]$, we need only to consider $\alpha = 1,\ \beta = \gamma = \frac{1}{2}$ by the same reason as above. Like in §3, let

$$E_n = \Big\{\sum_{k=1}^{n} \frac{\epsilon_k}{4^k} \mid \text{there exist } \sum_{k=1}^{\infty} \frac{\epsilon_k}{4^k} \in E,\ \epsilon_i = 0 \text{ or } 1\Big\},$$

and

$$E_n^i = \Big\{\sum_{k=1}^{n} \frac{\epsilon_k}{4^k} \in E_n,\ \epsilon_n = i\Big\}$$

for $i = 0,\ 1$. And define $F_n,\ F_n^i$ and $G_n,\ G_n^i$ in the same way. Then

$$\begin{aligned}
&|E_n + F_n + G_n| \geq \qquad (4)\\
\geq &|E_n^0 + F_n^0 + G_n^0| + \Big(|E_n^1 + F_n^0 + G_n^0|,\ |E_n^0 + F_n^1 + G_n^0|,\ |E_n^0 + F_n^0 + G_n^1|\Big)\\
&+ \Big(|E_n^1 + F_n^1 + G_n^0|,\ |E_n^1 + F_n^0 + G_n^1|,\ |E_n^0 + F_n^1 + G_n^1|\Big) + |E_n^1 + F_n^1 + G_n^1|\\
\geq &|E_n^0||F_n^0|^{\frac{1}{2}}|G_n^0|^{\frac{1}{2}} + \Big(|E_n^1||F_n^0|^{\frac{1}{2}}|G_n^0|^{\frac{1}{2}},\ |E_n^0||F_n^1|^{1/2}|G_n^0|^{\frac{1}{2}},\ |E_n^0||F_n^0|^{\frac{1}{2}}|G_n^1|^{\frac{1}{2}}\Big)\\
&+ \Big(|E_n^1||F_n^1|^{\frac{1}{2}}|G_n^0|^{\frac{1}{2}},\ |E_n^1||F_n^0|^{\frac{1}{2}}|G_n^1|^{\frac{1}{2}},\ |E_n^0||F_n^1|^{\frac{1}{2}}|G_n^1|^{\frac{1}{2}}\Big) + |E_n^1|F_n^1|^{\frac{1}{2}}|G_n^1|^{\frac{1}{2}}.
\end{aligned}$$

The second step is by inductive assumption. It is easy to check that $|E_1 + F_1 + G_1| \geq |E_1||F_1|^{\frac{1}{2}}|G_1|^{\frac{1}{2}}$. We hope that the last expression of (4) is not less than $|E_n||F_n|^{\frac{1}{2}}|G_n|^{\frac{1}{2}}$, which is equivalent to

Lemma. *Suppose* $0 \leq a,\ b,\ c \leq 1$. *Then*

$$\begin{aligned}
&ab^{\frac{1}{2}}c^{\frac{1}{2}} + \Big(a(1-b)^{\frac{1}{2}}c^{\frac{1}{2}},\ ab^{\frac{1}{2}}(1-c)^{\frac{1}{2}},\ (1-a)b^{\frac{1}{2}}c^{\frac{1}{2}}\Big) \qquad (5)\\
&+\Big(a(1-b)^{\frac{1}{2}}(1-c)^{\frac{1}{2}},\ (1-a)(1-b)^{\frac{1}{2}}c^{\frac{1}{2}},\ (1-a)b^{\frac{1}{2}}(1-c)^{\frac{1}{2}}\Big)\\
&+(1-a)(1-b)^{\frac{1}{2}}(1-c)^{\frac{1}{2}} \geq 1.
\end{aligned}$$

The lemma can be proved by Theorem 3.12 in a similarly way of the proof of (4) in §3. Then we get

$$|E_n + F_n + G_n| \geq |E_n||F_n|^{\frac{1}{2}}|G_n|^{\frac{1}{2}}$$

for each n. The proof of Theorem 3.11 is completed by a limit argument. ∎

Remark. 1. If α, β, γ are not all located in one of the three intervals mentioned in Theorem 3.11, we do not have the conclusion in general. However, if (α, β, γ) can be express as $c(\alpha', \beta', \gamma') + (1-c)(\alpha'', \beta'', \gamma'')$, we also have

$$m(E+F+G) \geq \nu_1(E)^{\alpha}\nu_1(F)^{\beta}\nu_1(G)^{\gamma}$$

where $0 < c < 1$ and $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ satisfy the assumption of Theorem 3.11.

2. Suppose For any Borel subsets E, F, G we have

$$m(E+F+G) \geq \nu_1(E)^{\alpha}\nu_1(F)^{\beta}\nu_1(G)^{\gamma}$$

then it must be that $\min\{\alpha, \beta, \gamma\} \geq 2 - \frac{\log 3}{\log 2} = 0.415037\cdots$. To see this let

$$E = F = \Big\{\sum_{n=1}^{\infty} \frac{\epsilon_n}{4^n} | \epsilon_n = 0, 1\Big\},$$
$$G = \Big\{\sum_{n=2}^{\infty} \frac{\epsilon_n}{4^n} | \epsilon_n = 0, 1\Big\}.$$

Then $\nu_1(E) = \nu_1(F) = 1$, $\nu_1(G) = \frac{1}{2}$ and $E+F+G = [0, \frac{3}{4}]$. Therefore, $m(E+F+G) = \frac{3}{4}$ and $\nu_1(E)^{\alpha}\nu_1(F)^{\beta}\nu_1(G)^{\gamma} = \left(\frac{1}{2}\right)^{\gamma}$. In order that $\frac{3}{4} \geq \left(\frac{1}{2}\right)^{\gamma}$, we must have $\gamma \geq 2 - \frac{\log 3}{\log 2}$.

CHAPTER 4. Measure Inequalities of Sumsets, II

§1. Introduction.

In this chapter we deal with measure inequalities of sumsets for coin tossing measures. There are four sections in this chapter. Our main result appears in section 2 and is proved in section 3. We get that if $(p_1,\ 1-p_1)+(p_2,\ 1-p_2) \leq \frac{3}{2}$, then for any Borel subsets E and F one has

$$m(E+F) \geq \mu_{p_1}(E)^2 \mu_{p_2}(F)^2. \tag{1}$$

If the condition $(p_1,\ 1-p_1)+(p_2,\ 1-p_2) \leq \frac{3}{2}$ fails and both p_1 and p_2 are greater than (or less than) $\frac{1}{2}$, then (1) is not true.

In section four we discuss coin tossing measures on Cantor set. We get that

$$m(E+F) \geq \nu_{p_1}(E)^2 \nu_{p_2}(F)^2$$

holds for all Borel subsets E, F if and only if $p_2 = 1-p_1$ and $\frac{1}{3} \leq p_1 \leq \frac{2}{3}$, where ν_p is defined in the coin tossing way on the set of numbers in whose base 3 expansion only 0 and 1 appear.

§2. Result.

For $0 < p < 1$ define

$$\mu_p = \mathop{*}_{n=1}^{\infty} (p\delta_0 + (1-p)\delta_{\frac{1}{2^n}}).$$

Then μ_p can be considered as the distribution of a random variable in the unit interval the digits of whose binary expansion are determined by tossing a biased coin (0 with probability p, 1 with probability $1-p$). It is known that μ_p is concentrated on the set

$$E_p = \left\{ \sum_{n=1}^{\infty} \frac{\epsilon_n}{2^n} \middle| \epsilon_n = 0 \text{ or } 1,\ \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \epsilon_k}{n} = 1 - p \right\}.$$

And we have

$$s(\mu_p) = -\frac{\log(p,\ 1-p)}{\log 2}$$

where $(p,\ 1-p) = \max\{p,\ 1-p\}$. [10] showed that for any $0 < p < 1$, there exist a positive integer n and $\alpha > 0$ such that for any Borel subsets $E_1,\ E_2, \cdots,\ E_n$ one has

$$m(E_1 + E_2 + \cdots + E_n) \geq \mu_p(E_1)^\alpha \mu_p(E_2)^\alpha \cdots \mu_p(E_n)^\alpha.$$

Suppose E and F are Borel subsets with $\mu_{p_1}(E) > 0$ and $\mu_{p_2}(F) > 0$, what can we say about the Lebesgue measure of $E + F$? Are there any real numbers $\alpha,\ \beta$ such that for any Borel subsets E, F one has $m(E+F) \geq \mu_{p_1}(E)^\alpha \mu_{p_2}(F)^\beta$? Our main result is

Theorem 4.1. *Assume that* $p_1,\ p_2 \in (0,\ 1)$ *satisfy*

$$(p_1,\ 1-p_1)+(p_2,\ 1-p_2) \le \frac{3}{2}. \tag{1}$$

Then for any Borel subsets E, F *of* $\mathbf{T}$ *we have*

$$m(E+F) \ge \mu_{p_1}(E)^2 \mu_{p_2}(F)^2. \tag{2}$$

If in addition,

$$(p_1,\ 1-p_1)\cdot(p_2,\ 1-p_2) \le \frac{1}{2} \tag{3}$$

then

$$m(E+F) \ge \mu_{p_1}(E)\mu_{p_2}(F). \tag{4}$$

The condition $(p_1,\ 1-p_1)+(p_2,\ 1-p_2) \le \frac{3}{2}$ is a necessary condition for Theorem 4.1 when both $p_1,\ p_2 > \frac{1}{2}$ or $< \frac{1}{2}$. In fact we have

Proposition 4.2. *Suppose that* $p_1,\ p_2 > \frac{1}{2}$ *or* $< \frac{1}{2}$ *with*

$$(p_1,\ 1-p_1)+(p_2,\ 1-p_2) > \frac{3}{2}.$$

Then there exist E, F *with* $\mu_{p_1}(E) = \mu_{p_2}(F) = 1$, *such that*

$$m(E+F) = 0.$$

Proof. First we suppose $p_1,\ p_2 > \frac{1}{2}$ and $p_1 + p_2 > \frac{3}{2}$. We show that

$$m(E_{p_1} + E_{p_2}) = 0.$$

Since $m = \mu_{\frac{1}{2}}$, we show that

$$(E_{p_1} + E_{p_2}) \bigcap E_{\frac{1}{2}} = \emptyset.$$

For $x = \sum_{n=1}^{\infty} \frac{\epsilon_n}{2^n} \in E_{p_1}$ and $y = \sum_{n=1}^{\infty} \frac{\eta_n}{2^n} \in E_{p_2}$, suppose $x + y = \sum_{n=1}^{\infty} \frac{\omega_n}{2^n} (\text{mod } 1)$, where e_n, η_n and ω_n are 0 or 1. Then

$$\sum_{k=1}^{n} \omega_k \leq \sum_{k=1}^{n} \epsilon_k + \sum_{k=1}^{n} \eta_k + 1.$$

Therefore

$$\limsup_{n \to \infty} \frac{\sum_{k=1}^{n} \omega_k}{n} \leq \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \epsilon_k}{n} + \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \eta_k}{n}$$
$$= (1 - p_1) + (1 - p_2) = 2 - (p_1 + p_2) < \frac{1}{2}.$$

which implies $(E_{p_1} + E_{p_2}) \bigcap E_{\frac{1}{2}} = \emptyset$.

When $p_1,\ p_2 < \frac{1}{2}$ with $(1 - p_1) + (1 - p_2) > \frac{3}{2}$, we have

$$E_{p_i} = 1 - E_{1-p_i}, \qquad i = 1,\ 2.$$

Then

$$m(E_{p_1} + E_{p_2}) = m(1 - (E_{1-p_1} + E_{1-p_2}))$$
$$= m(E_{1-p_1} + E_{1-p_2}) = 0. \qquad \blacksquare$$

When $p_1 > \frac{1}{2}$, $p_2 < \frac{1}{2}$ we do not know if (1) is necessary for (2), but conjecture that this is, in fact, the case.

§3. Proof.

The method we will use is similar to [10]. By the same reason of Chapter 3, we assume E and F are closed. Like in §3 last chapter, let

$$A_n = \Big\{\sum_{k=1}^{n} \frac{\epsilon_k}{2^k} | \text{there exist } x \in E, \text{ s.t. } x = \sum_{k=1}^{\infty} \frac{\epsilon_k}{2^k},\ \epsilon = 0 \text{ or } 1\Big\},$$

and

$$B_n = \Big\{\sum_{k=1}^{n} \frac{\epsilon_k}{2^k} | \text{there exist } x \in F, \text{ s.t. } x = \sum_{k=1}^{\infty} \frac{\epsilon_k}{2^k},\ \epsilon = 0 \text{ or } 1\Big\}.$$

Let $E_n = A_n + [0, \frac{1}{2^n}]$, $F_n = B_n + [0, \frac{1}{2^n}]$. Define measures $\mu_{p_1}^{(n)}$, $\mu_{p_2}^{(n)}$ on $S_n = \Big\{\sum_{i=1}^{n} \frac{\epsilon_i}{2^i} | \epsilon_i = 0,\ 1\Big\}$ by

$$\mu_{p_i}^{(n)} = \mathop{*}_{k=1}^{n} (p_i\delta_0 + (1-p_i)\delta_{\frac{1}{2^k}}), \qquad i = 1,\ 2.$$

Then

$$\mu_{p_1}(E) = \lim_{n \to \infty} \mu_{p_1}^{(n)}(A_n), \text{ and } \mu_{p_2}(F) = \lim_{n\to\infty} \mu_{p_2}^{(n)}(B_n).$$

If we can show that

$$m(E_n + F_n) \geq \left[\mu_{p_1}^{(n)}(A_n)\mu_{p_2}^{(n)}(B_n)\right]^\alpha \tag{1}$$

then we obtain

$$m(E+F) = \lim_{n\to\infty} m(E_n + F_n) \geq \mu_{p_1}(E)^\alpha \mu_{p_2}(F)^\alpha.$$

Denote $(E+F)_n = (A_n + B_n) + [0, \frac{1}{2^n}]$. Then $E_n + F_n \supset (E+F)_n$.

Lemma 1. *If* $(p_1,\ 1-p_1)+(p_2,\ 1-p_2)\le \frac{3}{2}$, *then*

$$m((E+F)_n)\ge \left[\mu_{p_1}^{(n)}(A_n)\mu_{p_2}^{(n)}(B_n)\right]^2. \tag{2}$$

If $(p_1,\ 1-p_1)\cdot(p_2,\ 1-p_2)\le \frac{1}{2}$, *then*

$$m((E+F)_n)\ge \mu_{p_1}^{(n)}(A_n)\mu_{p_2}^{(n)}(B_n). \tag{3}$$

Proof. First we check (2) and (3) hold for $n=1$. We have $m((E+F)_1)=\frac{1}{2}$, if $|A_1|=|B_1|=1$; or $m((E+F)_1)=1$ otherwise. We need only check the first case. Now

$$\mu_{p_1}^{(1)}(A_1)\mu_{p_2}^{(1)}(B_1)\le (p_1,\ 1-p_1)(p_2,\ 1-p_2).$$

Hence, (3) holds when $n=1$, if $(p_1,\ 1-p_1)(p_2,\ 1-p_2)\le \frac{1}{2}$. If $(p_1,\ 1-p_1)+(p_2,\ 1-p_2)\le \frac{3}{2}$, then $(p_1,\ 1-p_1)(p_2,\ 1-p_2)\le \left(\frac{3}{4}\right)^2=\frac{9}{16}$. Therefore $\left[\mu_{p_1}^{(1)}(A_1)\mu_{p_2}^{(1))}(B_1)\right]^2\le\frac{1}{2}$. Assume (2) and (3) hold for n. We show they also hold for $n+1$. Denote

$$A_{n+1}^i=\left\{\sum_{k=1}^{n}\frac{\epsilon_k}{2^k}\Big|\sum_{k=1}^{n+1}\frac{\epsilon_k}{2^k}\in A_{n+1}^i,\text{ with }\epsilon_{n+1}=i\right\}$$

and

$$B_{n+1}^i=\left\{\sum_{k=1}^{n}\frac{\epsilon_k}{2^k}\Big|\sum_{k=1}^{n+1}\frac{\epsilon_k}{2^k}\in B_{n+1}^i,\text{ with }\epsilon_{n+1}=i\right\}$$

for $i=0,\ 1$. Let

$$(E+F)_{n+1}^{ij}=(A_{n+1}^i+B_{n+1}^j)+[0,\frac{1}{2^{n+1}}]+\frac{i+j}{2^{n+1}},\qquad i,\ j=0,\ 1.$$

Then

$$(E+F)_{n+1}=\left((E+F)_{n+1}^{00}\cup(E+F)_{n+1}^{11}\right)\bigcup\left((E+F)_{n+1}^{01}\cup(E+F)_{n+1}^{10}\right).$$

Obviously, $(E+F)_{n+1}^{00}\cup(E+F)_{n+1}^{11}$ and $E+F)_{n+1}^{01}\cup(E+F)_{n+1}^{10}$ are disjoint. Hence

$$m((E+F)_{n+1})\ge$$

$$\ge\left(m((E+F)_{n+1}^{00}),\ m((E+F)_{n+1}^{11})\right)+\left(m((E+F)_{n+1}^{01}),\ m((E+F)_{n+1}^{10})\right).$$

By inductive assumption,

$$2m((E+F)^{ij}_{n+1}) \geq \left[\mu^{(n)}_{p_1}(A^i_{n+1})\mu^{(n)}_{p_2}(B^j_{n+1})\right]^\alpha, \qquad \alpha = 1 \text{ or } 2.$$

On the other hand, there exist $0 \leq x,\ y \leq 1$ such that

$$\mu^{(n)}_{p_1}(A^0_{n+1}) = \frac{1}{p_1}\mu^{(n+1)}_{p_1}(A^0_{n+1}) = \frac{x}{p_1}\mu^{(n+1)}_{p_1}(A_{n+1})$$

and

$$\mu^{(n)}_{p_2}(B^0_{n+1}) = \frac{1}{p_2}\mu^{(n+1)}_{p_2}(B^0_{n+1}) = \frac{y}{p_2}\mu^{(n+1)}_{p_2}(B_{n+1}).$$

Similarly,

$$\mu^{(n)}_{p_1}(A^1_{n+1}) = \frac{1-x}{1-p_1}\mu^{(n+1)}_{p_1}(A_{n+1})$$

and

$$\mu^{(n)}_{p_2}(B^1_{n+1}) = \frac{1-y}{1-p_2}\mu^{(n+1)}_{p_2}(B_{n+1}).$$

Therefore,

$$\begin{aligned} &m((E+F)_{n+1}) \geq \\ &\geq \frac{1}{2}\left[\left(\left(\frac{x}{p_1}\frac{y}{p_2}\right)^\alpha,\ \left(\frac{1-x}{1-p_1}\frac{1-y}{1-p_2}\right)^\alpha\right) + \left(\left(\frac{x}{p_1}\frac{1-y}{1-p_2}\right)^\alpha,\ \left(\frac{1-x}{1-p_1}\frac{y}{p_2}\right)^\alpha\right)\right]\cdot \\ &\quad \cdot \left(\mu^{(n+1)}_{p_1}(A_{n+1})\mu^{(n+1)}_{p_2}(B_{n+1})\right)^\alpha. \end{aligned}$$

At this stage, we need

Lemma 2. *Suppose* $0 \leq x,\ y \leq 1$. *If* $(p_1,\ 1-p_1) + (p_2,\ 1-p_2) \leq \frac{3}{2}$, *then*

$$\left(\left(\frac{x}{p_1}\frac{y}{p_2}\right)^2,\ \left(\frac{1-x}{1-p_1}\frac{1-y}{1-p_2}\right)^2\right) + \left(\left(\frac{x}{p_1}\frac{1-y}{1-p_2}\right)^2,\ \left(\frac{1-x}{1-p_1}\frac{y}{p_2}\right)^2\right) \geq 2. \tag{4}$$

If $(p_1,\ 1-p_1)\cdot(p_2,\ 1-p_2) \leq \frac{1}{2}$, *then*

$$\left(\frac{x}{p_1}\frac{y}{p_2},\ \frac{1-x}{1-p_1}\frac{1-y}{1-p_2}\right) + \left(\frac{x}{p_1}\frac{1-y}{1-p_2},\ \frac{1-x}{1-p_1}\frac{y}{p_2}\right) \geq 2. \tag{5}$$

Proof. Without loss of generality, we may suppose that $p_1 \le x \le 1$ and $p_2 \le y \le 1$. So the left hand sides of (4) and (5) become

$$(\frac{x}{p_1}\frac{y}{p_2})^\alpha + \left((\frac{x}{p_1}\frac{1-y}{1-p_2})^\alpha, (\frac{1-x}{1-p_1}\frac{y}{p_2})^\alpha\right) \tag{6}$$

where $\alpha = 2$ or 1.

If $\frac{x}{p_1}\frac{1-y}{1-p_2} \ge \frac{1-x}{1-p_1}\frac{y}{p_2}$, then we need to prove

$$(\frac{x}{p_1}\frac{y}{p_2})^\alpha + (\frac{x}{p_1}\frac{1-y}{1-p_2})^\alpha \ge 2. \tag{7}$$

Since the left hand side of (7) is increasing with respect to x, we need to consider the case

$$\frac{x}{p_1}\frac{1-y}{1-p_2} = \frac{1-x}{1-p_1}\frac{y}{p_2}. \tag{8}$$

If $\frac{x}{p_1}\frac{1-y}{1-p_2} \le \frac{1-x}{1-p_1}\frac{y}{p_2}$, we can get the same conclusion by doing the same discussion on y. From (8),

$$x = \frac{p_1(1-p_2)y}{p_1(1-p_2)y + (1-p_1)p_2(1-y)} = \frac{p_1(1-p_2)y}{p_2(1-y) + p_1(y-p_2)}. \tag{9}$$

Substitute (9) into (7),

$$\frac{y^\alpha}{p_2^\alpha}\frac{(1-p_2)^\alpha y^\alpha + p_2^\alpha(1-y)^\alpha}{\left[p_2(1-y) + p_1(y-p_2)\right]^\alpha} \ge 2. \tag{10}$$

Clearly, the left hand side of (10) is decreasing with respect to p_1.

First we prove (5). Remember that when $\alpha = 1$ we require $(p_1, 1-p_1)\cdot(p_2, 1-p_2) \le \frac{1}{2}$. Then

$$p_1 \le \frac{1}{2}(p_2, 1-p_2)^{-1}.$$

When $p_2 \ge \frac{1}{2}$,

$$\begin{aligned}\text{LHS of (10)} \ge& \frac{y}{p_2}\frac{(1-p_2)y + p_2(1-y))}{p_2(1-y) + (y-p_2)/2p_2} \\ =& 2\cdot\frac{(1-2p_2)y^2 + p_2 y}{2p_2^2(1-y) + (y-p_2)}.\end{aligned}$$

Now we show that $\frac{(1-2p_2)y^2+p_2y}{2p_2^2(1-y)+(y-p_2)} \geq 1$, or equivalently

$$[(1-2p_2)y^2+p_2y]-[2p_2^2(1-y)+(y-p_2)] \geq 0. \tag{11}$$

Let

$$\begin{aligned} f_1(y) =& [(1-2p_2)y^2+p_2y]-[2p_2^2(1-y)+(y-p_2)] \\ =& (1-2p_2)y^2+(p_2+2p_2^2-1)y-2p_2^2+p_2. \end{aligned}$$

Then $f_1(p_2) = f_1(1) = 0$ and $f_1''(y) = 2(1-2p_2) \leq 0$(since $p_2 \geq \frac{1}{2}$). Therefore $f_1(y) \geq 0$ for $y \in [p_2,\ 1]$.

If $p_2 < \frac{1}{2}$, then $p_1 \leq \frac{1}{2}(1-p_2)^{-1}$. Hence

$$\begin{aligned} \text{LHS of (10)} \geq& \frac{y}{p_2}\frac{(1-p_2)y+p_2(1-y))}{p_2(1-y)+(y-p_2)/2(1-p_2)} \\ =& 2\cdot\frac{(1-p_2)[(1-2p_2)y^2+p_2y]}{p_2[2p_2(1-p_2)(1-y)+(y-p_2)]}. \end{aligned}$$

Let

$$\begin{aligned} f_2(y) =& (1-p_2)[(1-2p_2)y^2+p_2y]-p_2[2p_2(1-p_2)(1-y)+(y-p_2)] \\ =& (1-3p_2+2p_2^2)y^2+(p_2^2-2p_2^3)y+(2p_2^3-p_2^2). \end{aligned}$$

We have $f_2(p_2) = 0$ and $f_2'(y) = 2(1-p_2)(1-2p_2)y+p_2^2(1-2p_2) \geq 0$ (since $p_2 \leq \frac{1}{2}$). Hence $f_2(y) \geq 0$ for $y \geq p_2$. (5) has been proved.

Next we prove (4). Substituting $\alpha = 2$ into (10),

$$\frac{y^2}{p_2^2}\frac{(1-p_2)^2y^2+p_2^2(1-y)^2}{[p_2(1-y)+p_1(y-p_2)]^2} \geq 2. \tag{12}$$

Notice that $(p_1,\ 1-p_1)+(p_2,\ 1-p_2) \leq \frac{3}{2}$. Then

$$p_1 \leq \frac{3}{2}-(p_2,\ 1-p_2) = \min\{\frac{3}{2}-p_2,\ \frac{1}{2}+p_2\}.$$

If $p_2 \geq \frac{1}{2}$ then

$$\text{LHS of (12)} \geq \frac{y^2}{p_2^2} \frac{(1-p_2)^2 y^2 + p_2^2(1-y)^2}{\left[p_2(1-y) + (\frac{3}{2} - p_2)(y-p_2)\right]^2}$$

Let

$$g_1(y) = y^2\left[(1-p_2)^2 y^2 + p_2^2(1-y)^2\right] - 2p_2^2\left[p_2(1-y) + (\frac{3}{2} - p_2)(y-p_2)\right]^2.$$

Checking that $g_1(p_2) = g_1'(p_2) = 0$, $g_1''(p_2) > 0$, $g_1^{(3)}(p_2) > 0$ and $g_1^{(4)}(y) > 0$, we know that $g_1(y) \geq 0$ when $y \geq p_2$.

When $p_2 < \frac{1}{2}$,

$$\text{LHS of (12)} \geq \frac{y^2}{p_2^2} \frac{(1-p_2)^2 y^2 + p_2^2(1-y)^2}{\left[p_2(1-y) + (\frac{1}{2} + p_2)(y-p_2)\right]^2}$$

Let

$$g_2(y) = y^2\left[(1-p_2)^2 y^2 + p_2^2(1-y)^2\right] - 2p_2^2\left[p_2(1-y) + (\frac{1}{2} + p_2)(y-p_2)\right]^2.$$

For g_2 we have $g_2(p_2) = 0$, $g_2'(p_2) > 0$ and

$$g_2^{(3)}(y) = 24y(1-p_2)^2 - 12p_2^2(1-y) + 12p_2^2 y > 0$$

when $y \geq p_2$. Hence $g_2(y) \geq g_2(p_2) = 0$ for $y \geq p_2$. ∎

§4. Coin Tossing Measures on Cantor Set.

In this section we consider measures based on the Cantor set but defined by coin tossing. Define

$$\nu_p = \mathop{*}_{n=1}^{\infty}(p\delta_0 + (1-p)\delta_{\frac{1}{3^n}}).$$

Then ν_p is carried on the set

$$E_p = \left\{\sum_{n=1}^{\infty} \frac{\epsilon_n}{3^n} | \epsilon_n = 0 \text{ or } 1, \ \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \epsilon_k}{n} = 1 - p\right\}$$

which is a subset of the Cantor set $\left\{\sum_{n=1}^{\infty} \frac{\epsilon_i}{3^n} | \epsilon_n = 0, \ 1\right\}$. Let us see under what condition the Lebesgue measure of the sumset of two Borel subset E and F with positive ν_{p_1} and ν_{p_2} measure respectively is positive. In Chapter 3 we proved when $p_1 = p_2 = \frac{1}{2}$ we have

$$m(E+F) \geq \nu_{\frac{1}{2}}(E)^{\alpha} \nu_{\frac{1}{2}}(F)^{\beta}$$

where α, β satisfy $\alpha + \beta \geq \frac{\log 3}{\log 2}$ and $3(\alpha^{-1} + \beta^{-1}) \leq 8$. In this section we will prove

Theorem 4.3. *Let ν_p be defined in the above. For any Borel subsets E, F one has*

$$m(E+F) \geq \nu_{p_1}(E)^2 \nu_{p_2}(F)^2$$

if and only if $p_2 = 1 - p_1$ and $\frac{1}{3} \leq p_1 \leq \frac{2}{3}$.

Proof. First we show the necessity of the condition of the theorem. We know that for almost all numbers $x \in [0, \ 1]$(with respect to Lebesgue measure m) 0, 1 and 2 appear with equal frequency in the base 3 expansion of x. For given $x \in E_{p_1}$

and $y \in E_{p_2}$, suppose 0, 1 and 2 appear with the same frequency in the base 3 expansion of $x+y$. If there is a 0 in some decimal place of $x+y$, then we can see that it is 0 in the same place of both x and y. Hence we get $p_1,\ p_2 \geq \frac{1}{3}$. Similarly we get $1-p_1,\ 1-p_2 \geq \frac{1}{3}$ by considering the frequency of 2 in $x+y$. Therefore $\frac{1}{3} \leq p_1,\ p_2 \leq \frac{2}{3}$. If there is a 1 in some decimal place of $x+y$, then it is a 0 in the same place of x and a 1 of y or the other way around. Hence we get

$$(p_1 - \frac{1}{3}) + (p_2 - \frac{1}{3}) \geq \frac{1}{3}$$

and

$$[(1-p_1) - \frac{1}{3}] + [(1-p_2) - \frac{1}{3}] \geq \frac{1}{3}$$

which deduce that $p_1 + p_2 = 1$.

Next we show that the condition is sufficient. Let $p_1 = p$. Then $p_2 = 1-p$. We will use the same method as in §2. We assume E and F are subsets of E_p and E_{1-p} respectively. Let

$$A_n = \left\{ \sum_{k=1}^{n} \frac{\epsilon_k}{3^k} | \text{there exist } x \in E, \text{ s.t. } x = \sum_{k=1}^{\infty} \frac{\epsilon_k}{3^k},\ \epsilon = 0 \text{ or } 1 \right\},$$

and

$$B_n = \left\{ \sum_{k=1}^{n} \frac{\epsilon_k}{3^k} | \text{there exist } x \in F, \text{ s.t. } x = \sum_{k=1}^{\infty} \frac{\epsilon_k}{3^k},\ \epsilon = 0 \text{ or } 1 \right\}.$$

Let $E_n = A_n + [0, \frac{1}{3^n}]$, $F_n = B_n + [0, \frac{1}{3^n}]$ and $(E+F)_n = (A_n + B_n) + [0, \frac{1}{3^n}]$. Define the measure $\mu_p^{(n)}$ on $S_n = \left\{ \sum_{i=1}^{n} \frac{\epsilon_i}{3^i} | \epsilon_i = 0,\ 1 \right\}$ by

$$\mu_p^{(n)} = \mathop{*}_{k=1}^{n} (p\delta_0 + (1-p)\delta_{\frac{1}{3^k}})$$

and $\mu_{1-p}^{(n)}$ similarly. We check that

$$m((E+F)_1) \geq [\mu_p^{(1)}(A_1)\mu_{1-p}^{(1)}(B_1)]^2. \tag{1}$$

If $|A_1| = |B_1| = 1$, then $m((E+F)_1) = \frac{1}{3}$ and

$$[\mu_p^{(1)}(A_1)\mu_{1-p}^{(1)}(B_1)]^2 \le (p^2,\ (1-p)^2)^2 \le \left(\frac{2}{3}\right)^4 < \frac{1}{3}.$$

If $|A_1| = 2,\ |B_1| = 1$ or $|A_1| = 1,\ |B_1| = 2$, then $m((E+F)_1) = \frac{2}{3}$ and

$$[\mu_p^{(1)}(A_1)\mu_{1-p}^{(1)}(B_1)]^2 \le (p,\ 1-p)^2 \le \left(\frac{2}{3}\right)^2 = \frac{4}{9} < \frac{2}{3}.$$

When $|A_1| = |B_1| = 1$, both sides of (1) equal 1.

By a similar discussion as §2, we know that Theorem 4.3 can be proved by the following lemma.

Lemma. *Assume* $\frac{1}{3} \le p \le \frac{2}{3}$. *Then for* $x,\ y \in [0,\ 1]$ *we have*

$$\frac{1}{p^2(1-p)^2}\left(x^2y^2 + (1-x)^2(1-y)^2\right) + \left(\frac{x^2(1-y)^2}{p^4},\ \frac{(1-x)^2y^2}{(1-p)^4}\right) \ge 3. \qquad (2)$$

Proof. Let

$$f(x,y) = x^2y^2 + (1-x)^2(1-y)^2 + \frac{(1-p)^2}{p^2}x^2(1-y)^2.$$

It is sufficient to show that $f(x,y) \ge 3p^2(1-p)^2$ for (x,y) satisfying $y-(1-p) \le x-p$ (shaded area of Fig. 1 or 2), since when $y-(1-p) > x-p$ we may interchange x, y and p, $1-p$.

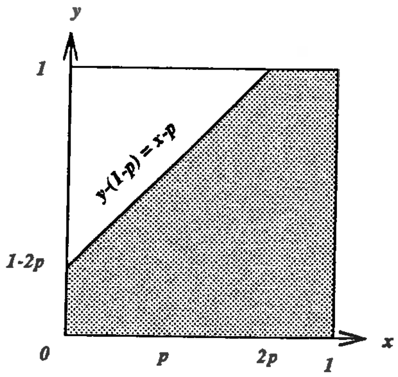

Fig 1. $p < 1/2$.

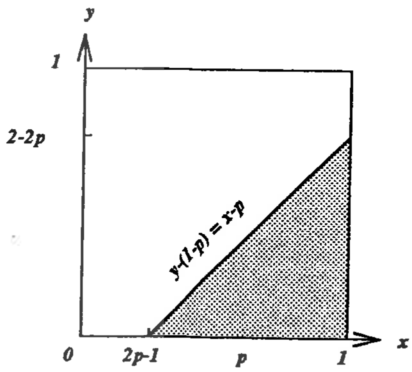

Fig 2. $p > 1/2$.

We have

$$\frac{\partial f}{\partial x} = 2xy^2 - 2(1-x)(1-y)^2 + 2\frac{(1-p)^2}{p^2}x(1-y)^2,$$
$$\frac{\partial f}{\partial y} = 2x^2y - 2(1-x)^2(1-y) - 2\frac{(1-p)^2}{p^2}x^2(1-y).$$

We first check that $f(x,y) \geq 3p^2(1-p)^2$ holds on the boundary of the shaded areas. On the line $x - p = y - (1-p)$ we let

$$\begin{aligned} g(x) &= f(x, x+1-2p) \\ &= x^2(x+1-2p)^2 + (1-x)^2(2p-x)^2 + \frac{(1-p)^2}{p^2}x^2(2p-x)^2. \end{aligned}$$

Checking that $g(p) = 3p^2(1-p)^2$, $g'(p) = 0$, $g''(p) > 0$, $g^{(3)}(p) = 0$ and $g^{(4)}(x) > 0$ we can get that $g(x) \geq g(p) = 3p^2(1-p)^2$. We also have

$$\begin{aligned} f(x,0) = (1-x)^2 + \frac{(1-p)^2}{p^2}x^2 &\geq f(\frac{p^2}{p^2+(1-p)^2}, 0) \\ ,= \frac{(1-p)^2}{p^2+(1-p)^2} &\geq (1-p)^2 > 3p^2(1-p)^2 \end{aligned}$$

since $\frac{1}{p^2+(1-p)^2} \geq \frac{9}{5} > 3p^2$ when $\frac{1}{3} \leq p \leq \frac{2}{3}$, and

$$f(1,y) = y^2 + \frac{(1-p)^2}{p^2}(1-y)^2 > 3p^2(1-p)^2$$

similarly.

When $p < \frac{1}{2}$, we need to check $f(0,y) \geq 3p^2(1-p)^2$ for $0 \leq y \leq 1-2p$, and $f(x,1) \geq 3p^2(1-p)^2$ for $2p \leq x \leq 1$. In fact

$$f(0,y) = (1-y)^2 \geq 4p^2 > 3p^2(1-p)^2 \qquad \text{when } 0 \leq y \leq 1-2p;$$

and

$$f(x,1) = x^2 \geq 4p^2 > 3p^2(1-p)^2 \qquad \text{for } 2p \leq x \leq 1.$$

Now we consider $f(x,y)$ for (x,y) inside of the shaded areas. Let

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2xy^2 - 2(1-x)(1-y)^2 + 2\dfrac{(1-p)^2}{p^2}x(1-y)^2 = 0 \\ \dfrac{\partial f}{\partial y} = 2x^2y - 2(1-x)^2(1-y) - 2\dfrac{(1-p)^2}{p^2}x^2(1-y) = 0. \end{cases} \tag{3}$$

Then

$$\begin{aligned} \frac{(1-p)^2}{p^2}x^2(1-x)^2 &= x(1-x)(1-y)^2 - x^2y^2 \\ &= x^2y(1-y) - (1-x)^2(1-y)^2. \end{aligned}$$

Therefore solutions of (3) (if any) will satisfy

$$x^2y = (1-x)(1-y)^2. \tag{4}$$

The only non-negative solution of (4) is $y = 1 - x$. Let

$$g(x) = f(x, 1-x) = 2x^2(1-x)^2 + \frac{(1-p)^2}{p^2}x^4.$$

Remember we require $x - p \geq y - (1-p)$ from which (recalling that $y = 1 - x$) we deduce that $x \geq p$. Hence it is suffices to show $g(x) \geq 3p^2(1-p)^2$ for $p \leq x \leq 1$. However, we have $g'(p) \geq 0$, $g''(p) \geq 0$, $g^{(3)}(p) > 0$ and $g^{(4)}(x) > 0$. Therefore, $g(x) \geq g(p) = 3p^2(1-p)^2$ when $p \leq x \leq 1$. ∎

REFERENCES

[1]. C. Bandt: *Self-similar Sets 3. Constructions with Sofic Systems,* Mh. Math. 108 (1989), 89-102.

[2]. M.F. Barnsley & S. Demko: *Iterated Function Systems and the Global Construction of Fractals,* Proc. R. Soc. Lond. A 399 (1985), 243-275.

[3]. M.F. Barnsley: *Fractals Everywhere,* Boston; Academic Press. 1988.

[4]. T. Bedford: *The Box Dimension of Self-affine Graphs and Repellers,* Nonlinearity. 2 (1989) 53-57.

[5]. P. Billingsley: *Ergodic Theory and Information,* John Wiley &Sons, Inc. 1965.

[6]. G. Brown: *Inequalities for Measures of Sum Sets,* Proc.CMA ANU 15 (1987), 15-20.

[7]. G. Brown: *Some Inequalities That Arise in Measure Theory,* J. Austral Math.Soc. 45 (1988), 83-94.

[8]. G. Brown & W. Moran:*Raikov Systems and Radicals in Convolution Measure Algebras,* J. London Math. Soc. 28(2) (1983), 531-542.

[9]. G. Brown & L.A. Shepp: *A Convolution Inequality,* Contributions to Probability and Statistics, Essays in Honor of Ingram Olkin, Springer N.Y. (1989), 51-57.

[10]. G. Brown & J.H. Williamson: *Coin-tossing and Sum Sets,* J. Austral Math. Soc. 43 (1987), 211-219.

[11]. G. Brown, W. Moran & R. Tijdeman: *Riesz Products are Basic Measures,* J. London Math. Soc. 30 (2) (1984), 105-109.

[12]. G. Brown & Q. Yin: *Some Metric Properties of Sum Sets,* Proc. BYU Conference on Markov Spectrum and Diophantine Approximation.

[13]. K.A. Bush: *Continuous Functions without Derivatives,* Amer. Math. Monthly 59 (1952), 222-225.

[14]. T.W. Cusick & M.E. Flahive: *The Markoff and Lagrange Spectra,* Math. Surveys & Monograph 30, American Math. Soc. 1989.

[15]. D. B. Ellis & M. G. Branton: *Non-self-similar Attractors of Hyperbolic Iterated Function Systems,* Lecture Notes in Math. 1342, Springer-Verlag 1988, 158-171.

[16]. J.H. Elton: *An Ergodic Theorem for Iterated Maps,* Ergod. Th. & Dynam. Sys. 7 (1987), 481-488.

[17]. K.J. Falconer: *The Geometry of Fractal Sets,* Cambridge Univ. Press 1985.

[18]. K.J. Falconer: *The Hausdroff Dimension of Self-affine Fractals.* Math. Proc. Camb. Phil. Soc. 103 (1988), 339-350.

[19]. K.J. Falconer: *Fractal Geometry: Mathematical Foundations and Applications,* John Wiley & Sons 1990.

[20]. K.J. Falconer: *The Dimension of Self-affine Fractals, II.* Math. Proc. Camb. Phil. Soc. 111 (1992), 169-179.

[21]. U. Feiste: *A Generalization of Mixed Invariant Sets,* Mathematika 30 (1988), 198-206.

[22]. F.R. Gantmacher: *The Theory of Matrices*, Vol. 2, Chelsea 1959.

[23]. S. Gibert & R. Massopust: *The Exact Hausdorff Dimension for a Class of Fractal Functions,* J. Math. Anal. & Appl. 168 (1992), 171-183.

[24]. W.J. Gilbert: *Complex Bases and Fractal Similarity,* Ann. Sc. Math. Quebec, 11 no.1 (1987), 65-77.

[25]. M. Hall, Jr: *On the Sum and Product of Continued Fractions,* Ann. Math. vol 48 (1947), 966–993.

[26]. J. E. Hutchinson: *Fractals and Self Similarity,* Indiana Univ. Math. J. 30 (1981), 713-747.

[27]. J. Keesling: *Hausdorff Dimension,* Topology Proc. 11 (1986), 349-383.

[28]. A.W. Kemp: *Certain Inequalities Involving Fractional Powers,* J. Austral Math. Soc. Ser. A, Vol. 53 (1992), 131–136.

[29]. J.H.B. Kemperman: *On Small Sumsets in an Abelian Group,* Acta Math. 103 (1960), 63–68.

[30]. A.M. MacBeath: *On the Measure of Product Set in a Topological Group,* J. London Math. Soc. 35 (1960), 403–407.

[31]. B.B. Mandelbrot: *Fractals: Form, Chance, and Dimension,* W. H. Freeman and Company, San Francisco 1977.

[32]. H.B. Mann: *Addition Theorems: the Addition Theorems of Group Theory and Number Theory,* Wiley-Interscience Tracts in Math. 18, New York 1965.

[33]. R.D. Mauldin & S.C. Williams: *Hausdorff Dimension in Graph Directed Constructions,* Trans. Amer. Math. Soc. 309 No.2 (1988), 811-829.

[34]. S.E. Newhouse: *Nondensity of Axiom A(a) on* $\mathbf{S}^2$, Global analysis, Symp. Pure Math., vol 14; Rhode Island: Am. Math. Soc. Providence, 1970

[35]. S.E. Newhouse: *Diffeomorphisms with Infinitely Many Sinks,* Topology 13 (1974), 9-18

[36]. S.E. Newhouse: *The Abundance of Wild Hyperbolic Sets and Non-smooth Stable Sets for Diffeomorphisms,* Publ. I.H.E.S. 50 (1979), 101–151.

[37]. D.M. Oberlin: *The Size of Sums of Sets,* Studia Math. T. LXXXIII (1986), 139–146.

[38]. D.M. Oberlin: *The Size of Sums of Sets, II,* Israel J. of Mathematics, 55 (1986), 305-316.

[39]. J. Palis: *Homoclinic Orbits, Hyperbolic Dynamics and Dimension of Cantor Sets,* Contemp. Math. vol 58, III (1987), 203–216.

[40]. J. Palis & F. Takens: *Cycles and Measure of Bifurcation Sets for Two-dimensional Diffeomorphisms,* Invent. Math. 82, 397–422 (1985)

[41]. C.A. Rogers: *Hausdorff Measures,* Cambridge Univ. Press 1970

[42]. I.Z. Ruzsa: *Diameter of Sets and Measure of Sumsets,* Mh. Math. 112 (1991), 323-328.

[43]. D.W. Spear: *Measures and Self Similarity,* Adv. Math. 91 (1992), 143-157

[44]. C.E. Sutherland: Private conversation.

[45]. P. Walters: *An Introduction to Ergodic Theory,* Springer-Verlag, New York Inc., 1982.

[46]. B. Weiss: *Subshift of Finite Type and Sofic Systems* Mh. Math. 77 (1973), 462-474.

[47]. R.F. Williams: *How Big is the Intersection of Two Thick Cantor Sets?* Contemp.

Math. vol 117 (1991), 163–175.

[48]. D.R. Woodall: *A Theorem on Cubes,* Mathematika 24 (1977), 60-62

[49]. Q. Yin: *On Hausdorff Dimension for Attractors of Iterated Function Systems,* J. Austral. Math. Soc., Ser. A, (to appear).

[50]. Q. Yin: *On Attractors of Hyperbolic Iterated Function Systems,* (submitted)