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## Controlling the Solvency Interaction Among a Group of Insurance Companies

Alexandros Zimbidis\* and Steven Haberman†

### Abstract

Pooling of risks is an efficient risk management technique used by large employee benefit schemes of multinational companies to self-insure their retirement and other benefit obligations. This technique forms a basis for formulating a general control theoretic model for the interaction between insurance companies within a pooling network. The objective of these insurance companies is to avoid insolvency yet maintain stable premium and surplus processes. A general control system of equations that is used as a model for the interaction of  $m$  insurance companies within the network is first analyzed. An analytic solution is provided. Questions concerning the stability and optimal parameter design for the system are investigated. The special case of two identical companies is analyzed in detail.

*Key words and phrases: control theory, self-insurance, pooling, stability, optimal parameter design, feedback mechanism*

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## 1 Introduction

Since the 1980s, the rise in numbers, size, and shape of large multinational corporations have created a demand for special insurance products. As parent corporations exerted more control over their subsidiaries, the demand for insurance to cover contingencies in different countries grew. Insurance companies have responded to this demand by constructing multinational insurance networks. These networks are established through special reinsurance agreements between affiliated insurance companies (William M. Mercer, 1988).

One of the most important products sold through these networks is the pooling arrangement. Pooling is a special kind of self-insurance established to manage risks. For example, a multinational corporation with employee benefit schemes in two or more countries may use self-insurance to cover benefits as they are needed for all of their employees (Hart et al., 1996).

Two basic problems arise with developing models of pooling arrangements:

- Specifying a model to describe the premium rating process associated with sharing the claims experience of each insurance company in the pool; and
- Specifying a model for describing the interaction of the surpluses among the insurance companies participating in the pool.

The specification of these models is used as a starting point in the formulation of an optimal control theoretic model of the overall interactions among the group of insurance companies in the network.

Optimal control theory was developed in the late 1950s by scientists and engineers to investigate the properties of dynamic systems of difference or differential equations. As it is often difficult to obtain analytic solutions for many dynamic systems, control theory is concerned with the examination of the qualitative properties of these systems. One of the important qualitative properties is the stability of the system.

The stability of a system refers to the way the system reacts to different external input signals, the way it returns to its initial state or to a designated state, and whether or not it remains within an acceptable region of this state. In the insurance context, stability is directly related to the level of the surplus. A stable insurance system can react effectively by anticipating the appropriate premium (output variable) to any claim (input variable) pattern in order to maintain (in the long run) the

surplus (state variable) and consequently maintain the insolvency risk at an acceptable level.

Since 1980, actuaries have applied the results of control theory to actuarial problems. Balzer and Benjamin (1980), Martin-Löf (1983, 1994), Vandebroek and Dhaene (1990), Loades (1998), Runggaldier (1998), Schäl (1998), Chang (2000), and Zimbidis and Haberman (2001) have produced interesting actuarial papers using control theory methods and techniques to solve practical actuarial problems. Control theory may be used in other problems in which there exists an interaction between two or more insurance companies or between different lines of insurance businesses.

We have two main objectives for this paper: (i) to provide a comprehensive and convenient model for the interaction of the surplus among a group of insurance companies within the pooling network and the associated control actions that may be necessary for the management of the network; and (ii) to analyze the resulting system of equations that arise when we consider the control theory approach to solving this insurance problem.

The paper is organized as follows: Section 2 describes the assumptions and notation used throughout the paper. Section 3 introduces the general control model with  $m$  insurance companies in the network and the resulting system of equations and its solution. Certain properties of the solution, such as stability and optimality, are discussed in Section 4. Section 5 provides a detailed study of the model and its solution in the simpler case of two identical insurance companies in the network. A summary and conclusions are provided in Section 6. The appendix provides an algorithm for computing the determinant for a key matrix used in our analysis.

## 2 Assumptions and Notations

Suppose there are  $m$  insurance companies participating in a multinational insurance network that operates in  $m$  countries (one insurance company per country) covering the risks associated with the benefit payments from the multinational corporation's international employee benefits scheme. A typical employee benefits scheme may include some or all of the following benefits: term life insurance, accidental death and dismemberment insurance, permanent/temporary disability insurance, and medical benefits.

At the end of each year the accumulated surplus (whether it is positive or negative) is redistributed within the network of insurance com-

panies under a specific set of rules. The course of action mandated by these rules is enforced by the holding company or by a neutral central unit that coordinates the network in order to smooth the operational result and solvency requirement of each company.

The insurance companies all use the same experience rating procedure to calculate annual premiums. The experience rating procedure has the following characteristics:

- Experience rating is based on the most recently available claims experience;
- There is a time delay of  $f$  years, i.e., it takes  $f$  years for incurred claims to be fully reported, processed, and settled. Thus the available claim information at the beginning of the  $n^{\text{th}}$  year (or at the end of  $(n - 1)^{\text{th}}$  year) refers to the experience of the years  $n - f - 1$ ,  $n - f - 2$ ,  $n - f - 3$ , ..., 2, 1, 0, i.e., years prior to and inclusive of year  $n - f - 1$ ;
- Premiums are calculated annually at the beginning of each year according to a base premium and a profit sharing scheme;
- The base premium is calculated using the most recently available claims experience and taking into account the necessary expense margins;
- The profit-sharing scheme mandates an extra modification of the base premium through a refund (charge) to the policyholder a certain percentage of the benefit scheme's total accumulated surplus (deficit). This correction is aimed at driving the accumulated surplus to zero in the long run; and
- Each company passes to the other  $(m - 1)$  companies a pre-determined percentage of its accumulated surplus at the end of each year.

In general the predetermined percentages are not equally divided and are defined by a matrix  $\Lambda$ , called the *harmonization matrix*, that governs the surplus exchange. That is

$$\Lambda = [\lambda_{ij}] \in \mathbb{R}^{m \times m}$$

where  $\lambda_{ij} > 0$  is the predetermined percentage of surplus that the  $i^{\text{th}}$  company passes to  $j^{\text{th}}$  company. This obviously implies that

$$\sum_{j=1}^m \lambda_{ij} = 1 \quad \text{for } i = 1, 2, \dots, m. \quad (1)$$

The quantity  $\lambda_{ij}$ ,  $i = 1, 2, \dots, m$  determines the percentage of surplus retained by the  $i^{\text{th}}$  company. It is further assumed that each company has its own operational parameter values for expenses, feedback, accumulation, and inflation factors.

The following notations are used throughout the paper:

$m$  = Number of insurance companies participating in the multinational network.

$f$  = Length of time delay (measured in years).

$e_k$  = Expense factor for the  $k^{\text{th}}$  company, i.e.,  $(1 - e_k) \times$  Gross Premium is the margin for expenses. The expense factor vector is  $\mathbf{e} = (e_1, e_2, \dots, e_m)$ .

$R_k$  = Accumulation factor ( $R_k = 1 + j_k$ ), using an annual rate of investment return of  $j_k$  for the  $k^{\text{th}}$  company. The vector for the accumulation factor is  $\mathbf{R} = (R_1, R_2, \dots, R_m)$ .

$F_k$  = Inflation factor ( $F_k = 1 +$  inflation rate) of the  $k^{\text{th}}$  company. This factor indicates internal growth of the total annual claims, attributable to inflation or to business growth. The vector for inflation is  $\mathbf{F} = (F_1, F_2, \dots, F_m)$ .

$\lambda_{ij}$  = Interaction factor,  $i, j = 1, 2, \dots, m$ , is the proportion of surplus that the  $i^{\text{th}}$  company passes to the  $j^{\text{th}}$  company and constitutes the harmonization matrix  $\mathbf{\Lambda} = (\lambda_{ij})$

$\varepsilon_k$  = Profit sharing factor (feedback factor) for the  $k^{\text{th}}$  company, which includes both the local and international premium repayments and determines the percentage of accumulated surplus repaid to the policyholders. The vector of profit sharing factors is  $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$ .

$C_{k,n}$  = Actual total amount of annual incurred claims for the  $k^{\text{th}}$  company in the  $n^{\text{th}}$  year, for  $k = 1, \dots, m$ .

$\hat{C}_{k,n}$  = Estimated total expected annual incurred claims in year  $n$ , i.e., in  $(n - 1, n)$ . There is a delay of  $f$  years in updating information. The  $\hat{C}_{k,n}$  is a weighted average of the inflation-adjusted claims

over the two most recent years where data are available, i.e., for  $k = 1, \dots, m$

$$\hat{C}_{k,n} = \frac{1}{M_k} (F_k^{2+f} C_{k,n-f-2} + F_k^{1+f} C_{k,n-f-1}) \quad (2)$$

where

$$M_k = F_k^{1+f} (1 + F_k) \quad (3)$$

is a normalizing constant.

$P_{k,n}$  = Gross annual premium paid at the end of the  $n^{\text{th}}$  year for the  $k^{\text{th}}$  company. The gross premium is determined as an expense-adjusted premium  $P_{k,n}^{(e)}$  less the surplus adjustment, where

$$P_{k,n}^{(e)} = \hat{C}_{k,n} + (1 - e_k) P_{k,n} = \frac{\hat{C}_{k,n}}{e_k}.$$

It follows that

$$\begin{aligned} P_{k,n} &= P_{k,n}^{(e)} - \varepsilon_k S_{k,n-f-1} \\ &= \frac{1}{e_k} \hat{C}_{k,n} - \varepsilon_k S_{k,n-f-1} \end{aligned} \quad (4)$$

for  $k = 1, 2, \dots, m$ . Equation (4) is also called the *decision function*.

$S_{k,n}$  = Accumulated surplus at the end of the  $n^{\text{th}}$  year for the  $k^{\text{th}}$  company where

$$S_{k,n} = R_k \sum_{i=1} \lambda_{ik} S_{i,n-1} + e_k P_{k,n} - C_{k,n} \quad (5)$$

for  $k = 1, 2, \dots, m$ .

The quantities  $m$ ,  $f$ ,  $e_k$ ,  $R_k$ ,  $\varepsilon_k$ , and  $\lambda_{ij}$  are assumed to be constant over time.

This set of assumptions is used as a basis to derive a model and a system of equations and to examine the analytical solution of this system, its stability, and the optimal parameter design with respect to

the interaction arising from the surplus exchange process. The formulation of the problem is similar to that of Balzer and Benjamin (1980), Balzer (1982), Benjamin (1984), and Zimbidis and Haberman (2001). These authors have investigated the stability and parameter design of a single company, consequently without the presence of any interaction phenomenon.

### 3 The Model and System of Equations

From the point of view of control theory, claims may be considered as an input variable, the surplus as a state variable, and premiums as an output variable. The whole system (i.e., the multinational company's employee benefit scheme) starts from an initial value for the first year's premium, then claims data provide the input background for the development of the surplus level—the surplus represents the state of the system. Finally, using both claims (directly) and surplus information through a feedback<sup>1</sup> mechanism, a decision function is built for premium development. The amount of feedback action is not obviously determined. The level of the state variable and how much is fed back to the system must be evaluated carefully in order to achieve and/or maintain the required stability.

For the  $k^{\text{th}}$  company, the  $n^{\text{th}}$  year's premium and surplus are determined using the following model:

$$P_{k,n} = \frac{F_k^{2+f}}{M_k e_k} C_{k,n-f-2} + \frac{F_k^{1+f}}{M_k e_k} C_{k,n-f-1} - \varepsilon_k S_{k,n-f-1} \quad (6)$$

and

$$S_{k,n} = R_k \lambda_{11} S_{1,n-1} + \dots + R_k \lambda_{m1} S_{m,n-1} + \frac{F_k^{2+f}}{M_k} C_{k,n-f-2} + \frac{F_k^{1+f}}{M_k} C_{k,n-f-1} - e_k \varepsilon_k S_{k,n-f-1} - C_{k,n}, \quad (7)$$

for  $k = 1, 2, \dots, m$ .

Each of the  $m$  insurance companies generates its own system of equations. These systems, however, cannot be solved independently

<sup>1</sup>In this context, a feedback mechanism can be used to measure the surplus level and calculate how much of this surplus (deficit) should be refunded (charged) to policyholders. In other words, through a feedback mechanism we decide how much of the state information must be fed back to the system.



because of the existence of the interaction factor  $\lambda_{ij}$ . Combining equations (6) and (7) leads to a system of  $2m$  simultaneous equations that describes the premium rating, the surplus process, and the interaction within the group of companies:

$$\left. \begin{aligned}
 S_{1,n} &= R_1 \lambda_{11} S_{1,n-1} + \dots + R_1 \lambda_{m1} S_{m,n-1} \\
 &\quad + \frac{F_1^{2+f}}{M_1} C_{1,n-f-2} + \frac{F_1^{1+f}}{M_1} C_{1,n-f-1} \\
 &\quad - e_1 \varepsilon_1 S_{1,n-f-1} - C_{1,n} \\
 &\vdots \\
 S_{m,n} &= R_m \lambda_{1m} S_{1,n-1} + \dots + R_m \lambda_{mm} S_{m,n-1} \\
 &\quad + \frac{F_m^{2+f}}{M_m} C_{m,n-f-2} + \frac{F_m^{1+f}}{M_m} C_{m,n-f-1} \\
 &\quad - e_m \varepsilon_m S_{m,n-f-1} - C_{m,n} \\
 P_{1,n} &= \frac{F_1^{2+f}}{M_1 e_1} C_{1,n-f-2} + \frac{F_1^{1+f}}{M_1 e_1} C_{1,n-f-1} - \varepsilon_1 S_{1,n-f-1} \\
 &\vdots \\
 P_{m,n} &= \frac{F_m^{2+f}}{M_m e_m} C_{m,n-f-2} + \frac{F_m^{1+f}}{M_m e_m} C_{m,n-f-1} - \varepsilon_m S_{m,n-f-1}.
 \end{aligned} \right\} \tag{8}$$

Let  $\mathbf{x}_n$  denote the state vector,  $\mathbf{y}_n$  denote the output vector, and  $\mathbf{u}_n$  denote the input vector for the system, i.e.,

$$\mathbf{x}_n = \begin{bmatrix} S_{1,n} \\ S_{1,n-1} \\ \vdots \\ S_{1,n-f} \\ \text{---} \\ S_{2,n} \\ S_{2,n-1} \\ \vdots \\ S_{2,n-f} \\ \text{---} \\ \vdots \\ \text{---} \\ S_{m,n} \\ S_{m,n-1} \\ \vdots \\ S_{m,n-f} \end{bmatrix}, \mathbf{y}_n = \begin{bmatrix} P_{1,n} \\ P_{2,n} \\ \vdots \\ P_{m,n} \end{bmatrix} \text{ and, } \mathbf{u}_n = \begin{bmatrix} C_{1,n} \\ C_{1,n-1} \\ \vdots \\ C_{1,n-f-2} \\ \text{---} \\ C_{2,n} \\ C_{2,n-1} \\ \vdots \\ C_{2,n-f-2} \\ \text{---} \\ \vdots \\ \text{---} \\ C_{m,n} \\ C_{m,n-1} \\ \vdots \\ C_{m,n-f-2} \end{bmatrix},$$

where  $\mathbf{x}_n \in \mathbb{R}^{m(1+f)}$ ,  $\mathbf{y}_n \in \mathbb{R}^m$ , and  $\mathbf{u}_n \in \mathbb{R}^{m(3+f)}$ .

It must be understood that the inputs,  $\mathbf{u}_n$ , are determined using the actual  $C_{k,n-j}$ s when they are available or  $\hat{C}_{k,n-j}$ s when the actual  $C_{k,n-j}$ s are not available. In other words the following substitution is used:

$$C_{k,n-j} \begin{cases} \text{is replaced by } \hat{C}_{k,n-j} & \text{for } j = 0, 1, \dots, f; \\ \text{remains unchanged} & \text{for } j = f + 1, f + 2, \dots \end{cases}$$

For ease of exposition, we introduce four matrices **A**, **B**, **C**, **D** whose elements are themselves matrices.

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1m} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2m} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{m1} & \mathbf{A}_{m2} & \cdots & \mathbf{A}_{mm} \end{bmatrix} \in \mathbb{R}^{m(1+f)} \times \mathbb{R}^{m(1+f)},$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \cdots & \mathbf{B}_{1m} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \cdots & \mathbf{B}_{2m} \\ \vdots & \vdots & & \vdots \\ \mathbf{B}_{m1} & \mathbf{B}_{m2} & \cdots & \mathbf{B}_{mm} \end{bmatrix} \in \mathbb{R}^{m(1+f)} \times \mathbb{R}^{m(3+f)},$$

$$\mathbf{C} = [\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_m] \in \mathbb{R}^m \times \mathbb{R}^{m(1+f)},$$

$$\mathbf{D} = [\mathbf{D}_1, \mathbf{D}_2, \dots, \mathbf{D}_m] \in \mathbb{R}^m \times \mathbb{R}^{m(3+f)}.$$

The elements of the super-matrices **A**, **B**, **C**, and **D** are defined below:

$$\mathbf{A}_{ii} = \begin{bmatrix} R_i \lambda_{ii} & 0 & 0 & \cdots & 0 & -e_i \varepsilon_i \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{(1+f)} \times \mathbb{R}^{(1+f)},$$

$$\mathbf{A}_{i,j} = \begin{bmatrix} R_i \lambda_{ji} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{(1+f)} \times \mathbb{R}^{(1+f)},$$

$(i \neq j)$

$$\mathbf{C}_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -\varepsilon_i \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^m \times \mathbb{R}^{(1+f)},$$

$$\mathbf{B}_{ii} = \begin{bmatrix} -1 & 0 & \cdots & 0 & \frac{F_i^{1+f}}{M_i} & \frac{F_i^{2+f}}{M_i} \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{(1+f)} \times \mathbb{R}^{(3+f)},$$

with  $\mathbf{B}_{ij} = \mathbf{O}$  for  $i \neq j$ , and

$$\mathbf{D}_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \frac{F_i^{1+f}}{M_i e_i} & \frac{F_i^{2+f}}{M_i e_i} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^m \times \mathbb{R}^{(3+f)}.$$

The system (equation (8)) can now be written as:

$$\left. \begin{aligned} \mathbf{x}_n &= \mathbf{A}\mathbf{x}_{n-1} + \mathbf{B}\mathbf{u}_n \\ \mathbf{y}_n &= \mathbf{C}\mathbf{x}_{n-1} + \mathbf{D}\mathbf{u}_n \end{aligned} \right\}. \quad (9)$$

Following Cadzow (1973), the analytical solution to equation (9) is given by:

$$\left. \begin{aligned} \mathbf{x}_n &= \mathbf{A}^n \mathbf{x}_0 + \sum_{k=0}^{n-1} \mathbf{A}^k \mathbf{B} \mathbf{u}_{n-k-1}, \\ \mathbf{y}_n &= \mathbf{C} \mathbf{A}^{n-1} \mathbf{x}_0 + \mathbf{C} \sum_{k=0}^{n-2} \mathbf{A}^{n-2-k} \mathbf{B} \mathbf{u}_k + \mathbf{D} \mathbf{u}_n \end{aligned} \right\} \quad (10)$$

## 4 Properties of the Solution

Obtaining the analytical solution, equation (10), is not always necessary in order to understand the important properties of the system. We will now explore the stability of the system. This requires extensive use of eigenvalues and the characteristic function of a matrix.

The expression  $|\rho \mathbf{I} - \mathbf{A}|$ , which is the determinant of matrix  $(\rho \mathbf{I} - \mathbf{A})$ , can be expressed as a polynomial of  $\rho$ . This polynomial,  $\phi_m(\rho)$ , is called the characteristic polynomial of  $\mathbf{A}$  and is written as

$$\phi_m(\rho) = \sum_{r=0}^{m(f+1)} a_r \rho^r. \quad (11)$$

It follows that  $\rho_r$  is an eigenvalue of the matrix  $\mathbf{A}$  if and only if  $\phi_m(\rho_r) = |\rho_r \mathbf{I} - \mathbf{A}| = 0$ .

The necessary definitions, theorems, and results of linear algebra used in the remainder of the paper may be found in Healy (1995). The appendix contains an algorithm for calculating the characteristic function of  $\mathbf{A}$ .

### 4.1 Stability Analysis

For a dynamic system of the form described in equation (9), a point  $\mathbf{x}_*$  is called an equilibrium point if and only if  $\mathbf{x}_*$  satisfies the equation

$$\mathbf{A} \mathbf{x}_* = \mathbf{x}_*.$$

This equation clearly has at least one solution, i.e., the zero solution. Under certain conditions, however, the zero solution is the unique solution. Specifically, if  $\det(\mathbf{A}) \neq 0$ , then zero solution is the unique solution.

This statement is proved by considering the determinant of matrix  $\mathbf{A}$  and confirming that  $\det(\mathbf{A})$  differs from zero.

It is easy to show (see Section 5.2 for an outline of this) that

$$\det(\mathbf{A}) = (-1)^{m(1+f)} e_1 e_2 \cdots e_m \varepsilon_1 \varepsilon_2 \cdots \varepsilon_m,$$

which is different from zero for  $e_r, \varepsilon_r \neq 0$  for  $r = 1, 2, \dots, m$ . Consequently, in most practical situations  $\mathbf{0}$  is the only equilibrium point of the system.

According to Cadzow (1973, Chapter 3, page 106), a dynamic system of the form described in equation (9) is said to be stable at a state point  $\mathbf{x}$  (also called a stability point) if and only if the trajectory of the system that starts within a neighborhood of  $\mathbf{x}$  remains close to  $\mathbf{x}$  at all future times. The mathematical implication of this definition is that a state point  $\mathbf{x}$  is a stability point if and only if all the roots of the characteristic polynomial of the  $\mathbf{A}$  matrix have modulus less than unity.

It follows that the dynamic system of equation (9), is stable if the modulus of each eigenvalue of  $\mathbf{x}$  is less than unity, i.e.,

$$|\rho_r| < 1 \tag{12}$$

for  $r = 1, 2, \dots, m(1+f)$  where  $\rho_r$  is the  $r^{\text{th}}$  eigenvalue of  $\mathbf{A}$ . It follows that the system is unstable if  $|\rho_r| > 1$  for any  $k$ . Hence a sufficient condition for the system to be unstable is  $\prod_{r=1}^{m(1+f)} |\rho_r| > 1$ . But, as

$$\phi_m(\rho) = \prod_{r=1}^{m(1+f)} (\rho - \rho_r),$$

a sufficient condition for the system to be unstable is  $|\phi_m(0)| > 1$ .

It is easy to prove that the first and last coefficients of  $\phi_m(\rho)$  are  $a_{m(1+f)} = 1$  and  $a_0 = e_1 e_2 \dots e_m \varepsilon_1 \varepsilon_2 \dots \varepsilon_m$ . Applying the above criterion for instability requires

$$\prod_{r=1}^{m(1+f)} |\rho_r| = \prod_{r=1}^m |e_r \varepsilon_r| > 1. \tag{13}$$

In practice, expenses and profits will almost always be such that  $0 < e_r < 1$  and  $0 < \varepsilon_r < 1$  for  $r = 1, 2, \dots, m$ , so most practical systems will not satisfy equation (13)'s criterion for instability. This does not mean, however, that the system will automatically be stable.

## 4.2 Optimal Parameter Choices

The criterion for parameter optimality is defined in terms of the *fastest response time* of the system to different input signals.<sup>2</sup> A set of the parameter values is optimal if and only if the state vector moves to a desirable state (normally toward a stability point) faster than under any other choice of the parameter values, irrespective of the form, nature, or magnitude of the input vector. Below we describe a method that is useful in finding the approximate values of the optimal set of parameters.

Let  $\mathbf{s} = (\mathbf{e}, \boldsymbol{\varepsilon}, \mathbf{R}, \boldsymbol{\Lambda})$  denote a particular choice for the parameter values and  $\mathbb{S}$  denote the closed set of all possible choices for  $\mathbf{s}$ . Define  $\mathbf{A}(\mathbf{s})$  be the  $\mathbf{A}$  matrix derived from the choice of  $\mathbf{s}$ . If  $\rho_r(\mathbf{s})$  is the  $r^{\text{th}}$  eigenvalue of  $\mathbf{A}(\mathbf{s})$ , let  $\rho^{\max}(\mathbf{s})$  be the maximum absolute value of the eigenvalues of matrix  $\mathbf{A}(\mathbf{s})$ , i.e.,

$$\rho^{\max}(\mathbf{s}) = \max\{|\rho_1(\mathbf{s})|, |\rho_2(\mathbf{s})|, \dots, |\rho_{m(1+f)}(\mathbf{s})|\},$$

then the speed of the response of the system depends on the maximum absolute value of the eigenvalue,  $\rho^{\max}(\mathbf{s})$ . The smaller the value of  $\rho^{\max}(\mathbf{s})$ , the faster the response of the system.

Suppose there is an  $\mathbf{s}^* \in \mathbb{S}$  such that

$$\rho^{\max}(\mathbf{s}^*) = \rho^* = \min\{\rho^{\max}(\mathbf{s}) : \mathbf{s} \in \mathbb{S}\}, \quad (14)$$

it follows that

$$\rho^* \geq \sqrt[m(1+f)]{\prod_{r=1}^m e_r \varepsilon_r}. \quad (15)$$

The minimization of the maximum root of  $\phi_m(\rho)$  is easily obtained in two special cases:

**Case 1:** If  $\phi_m(\rho)$  has a single real root with multiplicity  $m(1 + f)$ , i.e.,

$$\phi_m(\rho) = (\rho - \rho_0)^{m(1+f)}$$

<sup>2</sup>The response time refers to the time it takes for the system output (or state) variables to return to the initial state or move to a designated point.

where

$$\rho^* = \left( \prod_{r=1}^m e_r \varepsilon_r \right)^{\frac{1}{m(1+f)}}$$

Using a binomial expansion yields

$$\phi_m(\rho) = \sum_{r=0}^{m(1+f)} \binom{m(1+f)}{r} (-\rho^*)^{m(1+f)-r} \rho^r,$$

i.e., the coefficient of  $\rho^r$  is  $a_r$  where

$$a_r = \binom{m(1+f)}{r} (-\rho_0)^{m(1+f)-r}$$

for  $r = 0, 1, \dots, m(1+f)$ . This gives a system of  $m(1+f) + 1$  equations for  $a_r$  that contains  $m^2 + 3m$  (control) parameters, i.e.,  $e_1, \dots, e_m, \varepsilon_1, \dots, \varepsilon_m, R_1, \dots, R_m, \lambda_{11}, \dots, \lambda_{mm}$ . Some of these may be fully controlled (the  $\varepsilon$  vector and the  $\Lambda$  matrix) or partially controlled (the  $e$  and  $R$  vectors). Our aim should be the optimal selection of all the controlled parameters such that the system becomes solvable.

**Case 2:** If  $\phi_m(\rho)$  is such that all of its roots lie on the circumference of the circle in the complex plane centered at the origin and with radius  $\rho^*$ , where

$$\rho^* = \left( \prod_{r=1}^m e_r \varepsilon_r \right)^{\frac{1}{m(1+f)}}$$

In this case  $\phi_m(\rho)$  has the form

$$\phi_m(\rho) = \rho^{m(1+f)} + \prod_{r=1}^m e_r \varepsilon_r \quad (16)$$

and its roots are proportional to the complex roots of  $m(1+f)\sqrt{-1}$ , i.e.,

$$\rho_j = \rho^* \left( \cos \left( \frac{(2r-1)\pi}{m(1+f)} \right) + i \sin \left( \frac{(2r+1)\pi}{m(1+f)} \right) \right)$$

where  $j = 1, 2, \dots, m(1+f)$ . Notice that this case appears when

$$a_{m(1+f)-1} = a_{m(1+f)-2} = \dots = a_2 = a_1 = 0, \tag{17}$$

i.e., all the  $a_r$  coefficients are zero except the first and last ones.

For large values of  $m$  the system of equations is rather complicated and there is no obvious choice of a choice of  $\mathbf{s}^*$  that results in a root with multiplicity  $m(1+f)$ . In such situations, we are forced to follow a trial and error procedure to determine  $\mathbf{s}^*$ . In other words, if  $\hat{\mathbb{S}}$  is the set of all *practical parameter choices* then we can calculate  $\rho^{\max}(\mathbf{s})$  for each  $\mathbf{s} \in \hat{\mathbb{S}}$  then choose the  $\mathbf{s}^*$  that produces the minimum  $\rho^{\max}(\mathbf{s})$ .

## 5 The Special Case of Two Identical Companies

To further illustrate the ideas described in Section 4, let us consider a simple situation with two insurance companies ( $m = 2$ ) in the network and a one year delay factor ( $f = 1$ ). In order to facilitate the calculations, we assume that the companies are identical with respect to operational parameters, i.e.,  $e_1 = e_2 = e$ ,  $R_1 = R_2 = R$ ,  $F_1 = F_2 = F$ ,  $M = F^2 + F^3$ ,  $\epsilon_1 = \epsilon_2 = \epsilon$ , and  $\lambda_{12} = \lambda_{21} = \lambda$ . As we assumed each company passes the same percentage  $\lambda$  of its surplus fund to the other company, the harmonization matrix  $\Lambda$  is

$$\Lambda = \begin{bmatrix} 1 - \lambda & \lambda \\ \lambda & 1 - \lambda \end{bmatrix}.$$

These operational assumptions are reasonable because multinational networks tend to be composed of similar companies with respect to operational matters. The assumption of identical companies is necessary in order to obtain closed form analytical solutions and results.



## 5.1 The Solution

The matrix  $A$  is given by

$$A = \begin{pmatrix} R(1-\lambda) & -e\varepsilon & R\lambda & 0 \\ 1 & 0 & 0 & 0 \\ R\lambda & 0 & R(1-\lambda) & -e\varepsilon \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and its characteristic polynomial is

$$\phi_2(\rho) = \begin{vmatrix} \rho - R(1-\lambda) & e\varepsilon & -R\lambda & 0 \\ -1 & \rho & 0 & 0 \\ -R\lambda & 0 & \rho - R(1-\lambda) & e\varepsilon \\ 0 & 0 & -1 & \rho \end{vmatrix}.$$

Developing this determinant across the second row, yields

$$\phi_2(\rho) = (\rho^2 - R\rho + e\varepsilon)(\rho^2 - R(1-2\lambda)\rho + e\varepsilon).$$

The four roots of this quartic polynomial are

$$\rho_1, \rho_2 = \frac{R \pm \sqrt{R^2 - 4e\varepsilon}}{2} \quad (18)$$

and

$$\rho_3, \rho_4 = \frac{R(1-2\lambda) \pm \sqrt{R^2(1-2\lambda)^2 - 4e\varepsilon}}{2}. \quad (19)$$

We now examine the behavior of the system with respect to three types of inputs: spike signals, step signals, and sine signals assuming the zero initial condition  $\mathbf{x}_0 = \mathbf{0}$  and  $\mathbf{y}_0 = \mathbf{0}$  for any situation.

### 5.1.1 Spike Signals

Let us assume that a spike signal<sup>3</sup> appears as the input of the first subsystem while the second subsystem has a zero input, i.e.,

<sup>3</sup>In practice, a spike input signal may be interpreted as the appearance of an isolated unexpected claim into the system.

$$C_{1,n} = \begin{cases} 1, & n = 0 \\ 0, & n = 1, 2, \dots \end{cases}$$

and  $C_{2,n} = 0$  for  $n = 0, 1, \dots$ . The input vectors are

$$\mathbf{u}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and  $\mathbf{u}_n = \mathbf{0}$  for  $n = 4, 5, \dots$ . Substituting these values of  $\mathbf{u}_n$  for  $n = 0, 1, 2, \dots$  in the general solution given in equation (10) gives:

$$\begin{aligned} \mathbf{x}_n &= \mathbf{A}^n \mathbf{x}_0 + \mathbf{A}^{n-1} \mathbf{B} \mathbf{u}_0 + \mathbf{A}^{n-2} \mathbf{B} \mathbf{u}_1 + \mathbf{A}^{n-3} \mathbf{B} \mathbf{u}_2 + \mathbf{A}^{n-4} \mathbf{B} \mathbf{u}_3 \\ \mathbf{y}_n &= \mathbf{C} \mathbf{A}^{n-1} \mathbf{x}_0 + \mathbf{C} \mathbf{A}^{n-2} \mathbf{B} \mathbf{u}_0 + \mathbf{C} \mathbf{A}^{n-3} \mathbf{B} \mathbf{u}_1 \\ &\quad + \mathbf{C} \mathbf{A}^{n-4} \mathbf{B} \mathbf{u}_2 + \mathbf{C} \mathbf{A}^{n-5} \mathbf{B} \mathbf{u}_3 \end{aligned}$$

for  $n = 5, 6, \dots$ . Because  $\mathbf{x}_0 = \mathbf{0}$ , and  $\mathbf{B} \mathbf{u}_1 = \mathbf{0}$ , the solution takes the form

$$\begin{aligned} \mathbf{x}_n &= \mathbf{A}^{n-1} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mathbf{A}^{n-3} \begin{bmatrix} \frac{F^2}{M} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mathbf{A}^{n-4} \begin{bmatrix} \frac{F^3}{M} \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{y}_n &= \mathbf{C} \mathbf{A}^{n-2} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mathbf{C} \mathbf{A}^{n-4} \begin{bmatrix} \frac{F^2}{M} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mathbf{C} \mathbf{A}^{n-5} \begin{bmatrix} \frac{F^3}{M} \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \quad (20)$$

If the modulus of each of the eigenvalues of  $\mathbf{A}$  is less than unity,  $\mathbf{x}_n$  and  $\mathbf{y}_n$  in equation (20) will converge to zero as  $n$  increases to  $\infty$ .

### 5.1.2 Step Signal

We assume a step signal<sup>4</sup> for the first input variable while zero for the second one, i.e.,

$$C_{1,n} = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \quad \text{and } C_{2,n} = 0 \quad \text{for } n = 1, 2, \dots$$

then,

$$\mathbf{u}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_n = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{for } n \geq 3.$$

Thus  $\mathbf{Bu}_0 = \mathbf{Bu}_1$ , and  $\mathbf{Bu}_n = \mathbf{0}$ ,  $n \geq 3$ . The latter equality holds as  $M = F^2 + F^3$ . The solution can be written as

$$\left. \begin{aligned} \mathbf{x}_n &= [\mathbf{A}^{n-1} + \mathbf{A}^{n-2}] \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mathbf{A}^{n-3} \begin{bmatrix} \frac{F^2}{M} - 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\ \mathbf{y}_n &= \mathbf{C}[\mathbf{A}^{n-2} + \mathbf{A}^{n-3}] \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mathbf{CA}^{n-4} \begin{bmatrix} \frac{F^2}{M} - 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \right\} \quad (21)$$

for  $n = 5, 6, \dots$ . If the modulus of each of the eigenvalues of  $\mathbf{A}$  is less than unity,  $\mathbf{x}_n$  and  $\mathbf{y}_n$  in equation (21) will asymptotically converge to zero as  $n$  increases.

### 5.1.3 Sine Signal

Let us consider the case where the input variable can be expressed as a sine signal. The assumption of a sine input signal may be more

<sup>4</sup>A step signal may be interpreted as a claim of size one occurring annually.

realistic in some cases as it may represent the underlying underwriting cycle that occurs in many insurance markets (Berger 1988).

For  $n = 0, 1, \dots$ , let  $C_{1,n} = \sin(\omega_1 n + \phi_1)$  and  $C_{2,n} = \sin(\omega_2 n + \phi_2)$  with  $\omega_1 = \omega_2 = \pi$ ,  $\phi_1 = -\frac{\pi}{6}$  and  $\phi_2 = \frac{\pi}{2}$ . This leads to the following:

$n$	0	1	2	3	4
$C_{1,n}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\dots$
$C_{2,n}$	1	-1	1	-1	$\dots$

Consequently the input vectors,

$$\mathbf{u}_0 = \begin{bmatrix} -0.5 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 0.5 \\ -0.5 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -0.5 \\ 0.5 \\ -0.5 \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and}$$

$$\mathbf{u}_{2k+1} = \begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \text{ and } \mathbf{u}_{2k+2} = \begin{bmatrix} -0.5 \\ 0.5 \\ -0.5 \\ 0.5 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

for  $k = 1, 2, \dots$ . It follows that

$$\mathbf{B}\mathbf{u}_0 = \begin{bmatrix} 0.5 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{B}\mathbf{u}_1 = \begin{bmatrix} -0.5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{B}\mathbf{u}_2 = \begin{bmatrix} 0.5 \left(1 - \frac{F^2}{M}\right) \\ 0 \\ -\left(1 - \frac{F^2}{M}\right) \\ 0 \end{bmatrix},$$

$$\mathbf{Bu}_{2k+1} = \begin{bmatrix} -0.5 \left( 1 - \frac{F^2}{M} + \frac{F^3}{M} \right) \\ 0 \\ 1 - \frac{F^2}{M} + \frac{F^3}{M} \\ 0 \end{bmatrix}, \text{ and } \mathbf{Bu}_{2k+1} = \begin{bmatrix} 0.5 \left( 1 - \frac{F^2}{M} + \frac{F^3}{M} \right) \\ 0 \\ - \left[ 1 - \frac{F^2}{M} + \frac{F^3}{M} \right] \\ 0 \end{bmatrix},$$

for  $k = 1, 2, \dots$ . As we observe for the vectors calculated before  $\mathbf{Bu}_1 = -\mathbf{Bu}_0$  and  $\mathbf{Bu}_{k+1} = -\mathbf{Bu}_k$  for  $k = 2n + 1, n \geq 1$ . Now assuming again the zero initial condition, i.e.,  $\mathbf{x}_0 = \mathbf{0}$  and  $\mathbf{y}_0 = \mathbf{0}$ , we obtain the general solution for the state of the system

$$\mathbf{x}_n = \mathbf{A}^{n-1}\mathbf{Bu}_0 - \mathbf{A}^{n-2}\mathbf{Bu}_0 + \mathbf{A}^{n-3}\mathbf{Bu}_2 + \mathbf{A}^{n-4}\mathbf{Bu}_3 - \mathbf{A}^{n-5}\mathbf{Bu}_3 + \dots \\ + (-1)^{n-1}\mathbf{A}\mathbf{Bu}_3 + (-1)^n\mathbf{Bu}_3.$$

Rearranging the terms of this equation we obtain

$$\mathbf{x}_n = \mathbf{A}^{n-2}(\mathbf{A} - \mathbf{I})\mathbf{Bu}_0 + \mathbf{A}^{n-3}\mathbf{Bu}_2 + (\mathbf{A}^{n-4} - \mathbf{A}^{n-3} + \dots \\ + (-1)^n\mathbf{I})\mathbf{Bu}_3.$$

It follows that

$$\mathbf{x}_n = \mathbf{A}^{n-2}(\mathbf{A} - \mathbf{I})\mathbf{Bu}_0 + \mathbf{A}^{n-3}\mathbf{Bu}_2 \\ + (\mathbf{I} + \mathbf{A}^2 + \dots + \mathbf{A}^{n-3})(\mathbf{A} - \mathbf{I})\mathbf{Bu}_3 \quad \text{if } n \text{ is odd;} \\ \mathbf{x}_n = \mathbf{A}^{n-2}(\mathbf{A} - \mathbf{I})\mathbf{Bu}_0 + \mathbf{A}^{n-3}\mathbf{Bu}_2 \\ + [(\mathbf{A} - \mathbf{I})(\mathbf{A} + \mathbf{A}^3 + \dots + \mathbf{A}^{n-3}) + \mathbf{I}]\mathbf{Bu}_3; \quad \text{if } n \text{ is even.}$$

If the modulus of the eigenvalues of  $\mathbf{A}$  are less than unity, then  $\mathbf{x}_n$  does not converge as  $n$  increases. In fact it fluctuates between two limits

$$\mathbf{x}_n \rightarrow \begin{cases} \mathbf{Q}(\mathbf{A} - \mathbf{I})\mathbf{Bu}_3 & \text{if } n = 2k + 1 \text{ and } k \rightarrow \infty, \\ [(\mathbf{A} - \mathbf{I})\mathbf{A}\mathbf{Q} + \mathbf{I}]\mathbf{Bu}_3 & \text{if } n = 2k + 2 \text{ and } k \rightarrow \infty, \end{cases}$$

where  $\mathbf{Q} = \mathbf{I} + \mathbf{A}^2 + \mathbf{A}^4 + \dots$ . It can be easily proved via the appropriate definition of a norm that as  $n$  goes to infinity the sequence of solutions  $\mathbf{x}_n$  is bounded.

5.2 Stability and Optimality

5.3 The Zero Stability Point

First we will establish that the system has only one equilibrium point at the zero point,  $\mathbf{0}$ .

$$\mathbf{A} = \left[ \begin{array}{ccccc|cccc} R_1\lambda_{11} & 0 & \cdots & 0 & -e_1\varepsilon_1 & R_1\lambda_{21} & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & & & & & \\ \vdots & \vdots & & \vdots & \vdots & & & & 0 & \\ 0 & 0 & \cdots & 0 & 0 & & & & & \\ 0 & 0 & \cdots & 1 & 0 & & & & & \\ \hline R_2\lambda_{12} & 0 & \cdots & 0 & 0 & R_2\lambda_{22} & 0 & \cdots & 0 & -e_2\varepsilon_2 \\ & & & & & 1 & 0 & \cdots & 0 & 0 \\ & & & & & \vdots & \vdots & & \vdots & \vdots \\ & & & 0 & & 0 & 0 & \cdots & 0 & 0 \\ & & & & & 0 & 0 & \cdots & 1 & 0 \end{array} \right]$$

$$= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

where  $\mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_{21}, \mathbf{A}_{22} \in \mathbb{R}^{(1+f) \times (1+f)}$  are defined in an obvious manner. Developing  $\det(\mathbf{A})$  across the second row and again the minor across the second row continuously after  $f$ -steps we obtain

$$\det(\mathbf{A}) = (-1)^f \begin{vmatrix} -e_1\varepsilon_1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \mathbf{A}_{22} & \\ \vdots & & & \\ 0 & & & \end{vmatrix}$$

or

$$\det(\mathbf{A}) = (-1)^{f+1} e_1\varepsilon_1 \det(\mathbf{A}_{22}).$$

Similarly, we develop  $\det(\mathbf{A}_{22})$  and obtain:

$$\det(\mathbf{A}) = (-1)^{2(1+f)} e_1e_2\varepsilon_1\varepsilon_2 = e_1e_2\varepsilon_1\varepsilon_2.$$

### 5.3.1 Conditions on the Roots

To investigate the stability of the system, we focus on the pairs of roots  $(\rho_1, \rho_2)$  and  $(\rho_3, \rho_4)$  separately. There are two cases to consider for each pair of roots. As the analysis is similar for each pair, we provide a detailed treatment only for the pair  $(\rho_1, \rho_2)$ .

**Case 1:** The roots  $\rho_1, \rho_2$  are real, i.e.,  $R^2 - 4e\varepsilon > 0$ . Here we want  $|\rho_1| < 1$  and  $|\rho_2| < 1$ . This implies that

$$\left| \frac{R \pm \sqrt{R^2 - 4e\varepsilon}}{2} \right| \leq 1,$$

i.e.,  $R < 1 + e\varepsilon$ . Hence the pair of roots are real with absolute value less than one if and only if

$$4e\varepsilon < R^2 < (1 + e\varepsilon)^2. \quad (22)$$

**Case 2:** When  $\rho_1, \rho_2$  are complex roots, i.e.,  $R^2 - 4e\varepsilon < 0$ , the complex conjugate roots are

$$\rho_1, \rho_2 = \frac{R}{2} \pm i \frac{\sqrt{4e\varepsilon - R^2}}{2}$$

and consequently

$$|\rho_1|, |\rho_2| \leq 1 \Leftrightarrow \left[ \left( \frac{R}{2} \right)^2 + \left( \frac{\sqrt{4e\varepsilon - R^2}}{2} \right)^2 \right]^{\frac{1}{2}} \leq 1 \Leftrightarrow$$

which implies that

$$\frac{R^2}{4} + \frac{4e\varepsilon - R^2}{4} \leq 1 \Leftrightarrow e\varepsilon \leq 1$$

and

$$\frac{R^2}{4} < e\varepsilon < 1. \quad (23)$$

For the second pair of roots  $\rho_3$  and  $\rho_4$ , we follow the same procedure and replace  $R$  with  $R(1 - 2\lambda)$  to give

$$4e\varepsilon < (R(1 - 2\lambda))^2 < (1 + e\varepsilon)^2 \quad (24)$$

for real roots, and

$$\frac{R^2(1 - 2\lambda)^2}{4} < e\varepsilon < 1 \quad (25)$$

for complex roots.

### 5.3.2 Fastest Response Solution

Next we turn our attention to the determination of the optimal parameter values according to the fastest response criteria. Suppose we require a solution to the system such that the solution has no oscillations.

The speed at which the state variables respond to the different input signals depends on the maximum modulus of the eigenvalues of  $A$ : the smaller the maximum modulus of the eigenvalues, the faster the response. We note that the minimum value of maximum modulus of the eigenvalues is obtained when the quadratic polynomials have double roots, i.e., when

$$R^2 - 4e\varepsilon = 0 \text{ and } R^2(1 - 2\lambda)^2 - 4e\varepsilon = 0.$$

This may occur if and only if  $\lambda = 0$  or  $\lambda = 1$ , i.e., when there is either no interaction or full interaction between the two insurance companies, and in either case we have a root of multiplicity four:

$$\rho_1 = \rho_2 = \rho_3 = \rho_4 = \frac{R}{2}. \quad (26)$$

In practical situations we cannot choose  $\lambda = 0$  or  $\lambda = 1$ , yet we still have to minimize the maximum modulus of the roots. This means that we must choose which of the equations  $R^2 - 4e\varepsilon = 0$  and  $R^2(1 - 2\lambda)^2 - 4e\varepsilon = 0$  is more important and minimize the maximum modulus of the roots according to the chosen equation.

If we choose the equation  $R^2 - 4e\varepsilon = 0$  then  $\rho_1 = \rho_2 = R/2$  while  $\rho_3, \rho_4$  are complex numbers such that  $|\rho_3|, |\rho_4| < R/2$ . The root with



the maximum absolute value is the real double root at  $R/2$ . As there are two complex roots, there will be oscillations in the solution.

The other option of choosing  $R^2(1 - 2\lambda)^2 - 4e\varepsilon = 0$  produces four real roots, a double root

$$\rho_1, \rho_2 = \frac{R(1 - 2\lambda)}{2},$$

and two different roots

$$\rho_3, \rho_4 = \frac{R \pm \sqrt{R^2 - 4e\varepsilon}}{2}.$$

The root with the maximum absolute value is

$$\rho_3 = \frac{R + \sqrt{R^2 - 4e\varepsilon}}{2}.$$

As there are no complex roots, there will be no oscillations in the solution.

Thus we can conclude that:

1. The fastest response with oscillations occurs if we choose  $R^2 - 4e\varepsilon = 0$ . In this case the maximum modulus is  $R/2$ ; and
2. The fastest response with no oscillations occurs if we choose  $(1 - 2\lambda)^2 R^2 - 4e\varepsilon = 0$ . In this case the maximum modulus is  $(R + \sqrt{R^2 - 4e\varepsilon})/2 > R/2$ .

Note that the overall fastest response with or without oscillations occurs when  $R^2 - 4e\varepsilon = 0$ , i.e., it yields oscillations.

A compromise is thus needed to reduce the oscillations to an acceptable level, but without unduly reducing the speed. The approach suggested is to choose the fastest overall response (i.e.,  $R^2 - 4e\varepsilon = 0$ ) and then choose  $\lambda$  to reduce the amplitude of the oscillations caused by the two complex conjugate roots

$$\begin{aligned} \rho_3, \rho_4 &= \frac{R}{2} \left[ (1 - 2\lambda) \pm i\sqrt{4(\lambda - \lambda^2)} \right]. \\ &= \frac{R}{2} (\cos \theta \pm i \sin \theta) \end{aligned} \tag{27}$$

where

$$\tan \theta = \frac{\sqrt{\lambda - \lambda^2}}{0.5 - \lambda}. \quad (28)$$

The choice of  $\lambda$  affects both the frequency and amplitude of the oscillations.

At this point we digress in order to discuss the importance of  $\theta$  in connection with the general solution of the system. If  $z_1, z_2$  are complex eigenvalues of matrix  $A$  of a dynamic system, then the general solution  $y_n$  contains a linear combination of the powers of  $z_1$  and  $z_2$ , i.e.,

$$y_n = \mu_1 z_1^n + \mu_2 z_2^n$$

where

$$\begin{aligned} \mu_1 &= a (\cos \beta + i \sin \beta) \\ z_1 &= z (\cos \theta + i \sin \theta) \\ \mu_2 &= a (\cos \beta - i \sin \beta) \\ z_2 &= z (\cos \theta - i \sin \theta), \end{aligned}$$

and

$$y_n = 2az^n \cos(n\theta + \beta).$$

## 5.4 Numerical Example

This example illustrates the methodology for the special case of two identical companies described in Section 5.1. The system of difference equations is:

$$\begin{aligned} S_{1,n} &= R(1 - \lambda)S_{1,n-1} + R\lambda S_{2,n-1} \\ &\quad + \frac{F^3}{M}C_{1,n-3} + \frac{F^2}{M}C_{1,n-2} - e\epsilon S_{1,n-2} - C_{1,n} \\ S_{2,n} &= R\lambda S_{1,n-1} + R(1 - \lambda)S_{2,n-1} \\ &\quad + \frac{F^3}{M}C_{2,n-3} + \frac{F^2}{M}C_{2,n-2} - e\epsilon S_{2,n-2} - C_{2,n} \\ P_{1,n} &= \frac{F^3}{Me}C_{1,n-3} + \frac{F^2}{Me}C_{1,n-2} - \epsilon S_{1,n-2} \\ P_{2,n} &= \frac{F^3}{Me}C_{2,n-3} + \frac{F^2}{Me}C_{2,n-2} - \epsilon S_{2,n-2} \end{aligned}$$

The parameters used are  $R_1 = R_2 = R = 1.04$ ,  $e_1 = e_2 = e = 0.8$ ,  $F_1 = F_2 = F = 1$ , and feedback factor  $\varepsilon_1 = \varepsilon_2 = \varepsilon = 0.34$ , in order to obtain the fastest response as indicated in Sections 5.1. The interaction parameter  $\lambda$  is allowed to vary and the time horizon is  $n = 20$  years.

A spike input of 1 is used to model the claim variable of the first company. The mean and variance of the surplus variables of the two companies are calculated over the twenty years for several values of  $\lambda$ . The question of interest is: How does the system (i.e., the surplus variables) react to the occurrence of an unexpected claim in the first company?

Tables 1 and 2 show the development of  $S_{1,n}$  and  $S_{2,n}$ . Observe that both surplus variables return to the stability point  $\mathbf{0}$ . Finally, we also observe that the summation of the variances in Tables 1 and 2 ( $\text{Var}[S_{1,n}] + \text{Var}[S_{2,n}]$ ) is minimized for  $\lambda = 0.8$ . The last result means that for  $\lambda = 0.8$  the system has the optimal behavior with regard to solvency requirements.

## 6 Summary and Conclusions

We construct an input/output control model of multinational pooling arrangements. A key aspect of these types of arrangements is the interaction of their premium, claims, and surplus processes. The objectives of this interaction are to:

- Smooth, as far as possible, the fluctuations of the surplus fund of each company participating in the pool; and to
- Spread each company's premium income and claims experiences to the block of the other companies.

The specific modeling also may be used generally for subsidiary insurance companies whose parent company wants to smooth the solvency requirement of each individual company. It can also be used for capital allocation between different lines of business.

We have derived several important results:

- Given  $e_1 \neq 0$ ,  $e_2 \neq 0$ , ...,  $e_m \neq 0$  and  $\varepsilon_1 \neq 0$ ,  $\varepsilon_2 \neq 0$ , ...,  $\varepsilon_m \neq 0$ , which is normally the case in practice, the general model (for any value of  $f$  and  $m$ ) has one equilibrium point, the zero point, and consequently one potential point of stability. If

**Table 1**  
 $S_{1,n}$  for Various Values of  $\lambda$

$n$	0	0.1	0.2	0.3	0.4
0	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000
1	-1.0400	-0.9360	-0.8320	-0.7280	-0.6240
2	-0.3112	-0.1165	0.0349	0.1431	0.2080
3	0.4576	0.6238	0.6820	0.6593	0.5825
4	0.5600	0.5981	0.5113	0.3783	0.2553
5	0.4587	0.3872	0.2511	0.1538	0.1286
6	0.3256	0.2081	0.1139	0.1048	0.1485
7	0.2146	0.1023	0.0709	0.1036	0.1316
8	0.1351	0.0512	0.0581	0.0817	0.0765
9	0.0825	0.0290	0.0452	0.0481	0.0365
10	0.0493	0.0188	0.0297	0.0237	0.0212
11	0.0289	0.0130	0.0165	0.0122	0.0150
12	0.0168	0.0087	0.0083	0.0077	0.0094
13	0.0096	0.0055	0.0042	0.0051	0.0049
14	0.0055	0.0032	0.0024	0.0031	0.0025
15	0.0031	0.0018	0.0015	0.0016	0.0015
16	0.0017	0.0009	0.0009	0.0008	0.0009
17	0.0010	0.0005	0.0005	0.0004	0.0005
18	0.0005	0.0002	0.0003	0.0003	0.0003
19	0.0003	0.0001	0.0001	0.0002	0.0001
20	0.0002	0.0001	0.0001	0.0001	0.0001
$\mathbb{E}[S_{1,n}]$	0.0000	0.0000	0.0000	0.0000	0.0000
$\text{Var}[S_{1,n}]$	0.1546	0.1422	0.1254	0.1092	0.0951

**Table 1 (continued)**  
 $S_{1,n}$  for Various Values of  $\lambda$

$n$	0.5	0.6	0.7	0.8	0.9	1.0
0	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000
1	-0.5200	-0.4160	-0.3120	-0.2080	-0.1040	0.0000
2	0.2296	0.2080	0.1431	0.0349	-0.1165	-0.3112
3	0.4788	0.3750	0.2983	0.2755	0.3338	0.5000
4	0.1759	0.1513	0.1703	0.1993	0.1821	0.0400
5	0.1617	0.2166	0.2562	0.2671	0.2824	0.4056
6	0.1910	0.2003	0.1813	0.1611	0.1451	0.0444
7	0.1256	0.1030	0.0924	0.0982	0.1077	0.1828
8	0.0600	0.0583	0.0688	0.0737	0.0721	0.0211
9	0.0363	0.0444	0.0448	0.0399	0.0374	0.0692
10	0.0267	0.0265	0.0228	0.0238	0.0266	0.0082
11	0.0158	0.0132	0.0143	0.0153	0.0139	0.0241
12	0.0078	0.0081	0.0090	0.0081	0.0084	0.0029
13	0.0044	0.0052	0.0046	0.0048	0.0050	0.0080
14	0.0029	0.0027	0.0027	0.0028	0.0026	0.0010
15	0.0016	0.0014	0.0016	0.0015	0.0016	0.0025
16	0.0008	0.0009	0.0009	0.0009	0.0008	0.0003
17	0.0005	0.0005	0.0005	0.0005	0.0005	0.0008
18	0.0003	0.0003	0.0003	0.0003	0.0003	0.0001
19	0.0002	0.0001	0.0001	0.0002	0.0001	0.0002
20	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000
$\mathbb{E} [S_{1,n}]$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\text{Var} [S_{1,n}]$	0.0834	0.0742	0.0675	0.0637	0.0644	0.0777

**Table 2**  
 $S_{2,n}$  for Various Values of  $\lambda$

$n$	0	0.1	0.2	0.3	0.4
1	0.0000	-0.1040	-0.2080	-0.3120	-0.4160
2	0.0000	-0.1947	-0.3461	-0.4543	-0.5192
3	0.0000	-0.1662	-0.2245	-0.2017	-0.1250
4	0.0000	-0.0381	0.0487	0.1817	0.3047
5	0.0000	0.0715	0.2076	0.3048	0.3301
6	0.0000	0.1175	0.2118	0.2208	0.1771
7	0.0000	0.1123	0.1437	0.1110	0.0830
8	0.0000	0.0840	0.0771	0.0534	0.0587
9	0.0000	0.0536	0.0373	0.0344	0.0460
10	0.0000	0.0304	0.0196	0.0256	0.0280
11	0.0000	0.0160	0.0124	0.0167	0.0139
12	0.0000	0.0081	0.0084	0.0091	0.0073
13	0.0000	0.0041	0.0054	0.0045	0.0047
14	0.0000	0.0023	0.0031	0.0024	0.0030
15	0.0000	0.0013	0.0016	0.0015	0.0016
16	0.0000	0.0008	0.0008	0.0009	0.0008
17	0.0000	0.0005	0.0004	0.0005	0.0005
18	0.0000	0.0003	0.0003	0.0003	0.0003
19	0.0000	0.0002	0.0002	0.0001	0.0002
20	0.0000	0.0001	0.0001	0.0001	0.0001
$E[S_{2,n}]$	0.0000	0.0000	0.0000	0.0000	0.0000
$\text{Var}[S_{2,n}]$	0.0000	0.0060	0.0166	0.0268	0.0352

**Table 2 (continued)**  
 $S_{2,n}$  for Various Values of  $\lambda$

$n$	0.5	0.6	0.7	0.8	0.9	1.0
1	-0.5200	-0.6240	-0.7280	-0.8320	-0.9360	-1.0400
2	-0.5408	-0.5192	-0.4543	-0.3461	-0.1947	0.0000
3	-0.0212	0.0825	0.1593	0.1820	0.1238	-0.0424
4	0.3842	0.4087	0.3897	0.3607	0.3779	0.5200
5	0.2969	0.2421	0.2025	0.1916	0.1763	0.0531
6	0.1346	0.1253	0.1443	0.1645	0.1805	0.2812
7	0.0890	0.1116	0.1223	0.1164	0.1069	0.0318
8	0.0752	0.0768	0.0664	0.0614	0.0631	0.1141
9	0.0462	0.0382	0.0377	0.0426	0.0451	0.0133
10	0.0226	0.0228	0.0264	0.0254	0.0227	0.0411
11	0.0131	0.0157	0.0147	0.0136	0.0151	0.0049
12	0.0089	0.0086	0.0078	0.0087	0.0084	0.0139
13	0.0052	0.0044	0.0050	0.0048	0.0046	0.0017
14	0.0026	0.0027	0.0028	0.0026	0.0029	0.0045
15	0.0014	0.0016	0.0015	0.0016	0.0015	0.0005
16	0.0009	0.0008	0.0009	0.0009	0.0009	0.0014
17	0.0005	0.0005	0.0005	0.0005	0.0005	0.0002
18	0.0003	0.0003	0.0003	0.0003	0.0003	0.0004
19	0.0001	0.0002	0.0001	0.0001	0.0002	0.0001
20	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
$E[S_{2,n}]$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$Var[S_{2,n}]$	0.0417	0.0464	0.0499	0.0530	0.0577	0.0726

$$\prod_{r=1}^m e_r \varepsilon_r > 1$$

then the system is unstable regardless of the other parameter values, i.e., the surplus and premium levels fluctuate, with the surplus diverging to infinity.

- For the special case of two identical companies with time delay of one year  $f = 1$  the exact condition for stability (assuming typical values for  $R$  and  $\lambda$ , i.e.,  $0 < R < 2$  and  $0 < \lambda < 1$ , and considering equations (22) through (25)) is  $R - 1 < e\varepsilon < 1$ .
- For the case of the two identical companies ( $m = 2$ ), we show that the ultimate surplus level converges to zero under each of the spike and step input signals. This is a highly desirable result because it means that the system reacts properly and returns to its initial state. For the sine signal we show that the ultimate surplus fund fluctuates between two levels.
- For the special case ( $m = 2$ ), full investigation has been done with respect to the fastest response and oscillatory form of the solution. It has been shown that the fastest response is obtained when  $\varepsilon^* = \frac{R^2}{4e}$ .
- For the special case of the two identical companies and considering the optimal choice for the feedback factor  $\varepsilon^*$ , we also have shown that amplitude and frequency of the oscillations depend on the interaction factor  $\lambda$ .

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## Appendix: The Characteristic Polynomial $\phi_m(\rho)$

First we calculate the form of the  $\phi_2(\rho)$  then we may generalize our result for any value of  $m$ .

$$\phi_2(\rho) = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} = 0$$

where

$$\mathbf{H}_{11} = \begin{bmatrix} \rho - R_1\lambda_{11} & 0 & \cdots & 0 & e_1\varepsilon_1 \\ -1 & \rho & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & \rho & 0 \\ 0 & 0 & \cdots & -1 & \rho \end{bmatrix}$$

$$\mathbf{H}_{12} = \begin{bmatrix} -R_1\lambda_{21} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\mathbf{H}_{21} = \begin{bmatrix} -R_2\lambda_{12} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\mathbf{H}_{22} = \begin{bmatrix} \rho - R_2\lambda_{22} & 0 & \cdots & 0 & e_2\varepsilon_2 \\ -1 & \rho & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & \rho & 0 \\ 0 & 0 & \cdots & -1 & \rho \end{bmatrix}.$$

The analytical development of  $\phi_2(\rho)$  is difficult, but we can find the final form if we follow a simple rule and a recursive procedure. The simple rule is to develop the major determinant  $\phi_2(\rho)$  or the minor ones (which are produced by deleting rows and columns) across the (first) row or column that has the greatest number of zero elements. The recursive procedure is described with the following steps.

**Step 1:** Develop the  $\phi_2(\rho)$  across the second row that has only two non-zero elements the  $-1$  and  $\rho$ ,

$$\phi_2(\rho) = (-1)(-1)\Xi_1^{(2)} + \rho\Psi_1^{(2)}$$

where  $\Xi_1^{(2)}$  is the minor determinant of  $\phi_2(\rho)$ , produced by deleting the first column and the second row of  $\phi_2(\rho)$  and  $\Psi_1^{(2)}$  is the minor determinant of  $\phi_2(\rho)$ , produced by deleting the second column and the second row of  $\phi_2(\rho)$ . [The (2) superscript of  $\Xi_1$  and  $\Psi_1$  refers to the case  $m = 2$ .]

**Step 2:** Develop the minor determinant  $\Xi_1^{(2)}$  across the first column (having one non-zero element the  $-1$ ).

$$\Xi_1^{(2)} = (-1)(-1)\Xi_2^{(2)}.$$

**Step 3:** Continue the development of  $\Xi_i^{(2)}$

$$\Xi_i^{(2)} = (-1)(-1)\Xi_{i+1}^{(2)}$$

for  $i = 2, 3, \dots, f-1$  with  $\Xi_i$  being the minor determinant of  $\Xi_{i-1}$ , produced by deleting the first column and second row of  $\Xi_i$  and

$$\Xi_f^{(2)} = \begin{vmatrix} e_1\varepsilon_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \phi_1(\rho) & \\ 0 & & & \end{vmatrix}$$

or

$$\Xi_f^{(2)} = e_1\varepsilon_1\phi_1(\rho).$$

**Step 4:** Combine equations in Steps 1, 2, and 3 to obtain

$$\phi_2(\rho) = e_1\varepsilon_1\phi_1(\rho) + \rho\Psi_1^{(2)}.$$

**Step 5:** Develop the minor determinant  $\Psi_1^{(2)}$  across the second row (having only one non-zero element,  $\rho$ ).

$$\Psi_1^{(2)} = \rho \Psi_2^{(2)}.$$

**Step 6:** Continue the development of the determinants across the second row (similarly with  $\Psi_1^{(2)}$ )

$$\Psi_i^{(2)} = \rho \Psi_{i+1}^{(2)}$$

for  $i = 2, 3, \dots, f-1$ , where  $\Psi_i^{(2)}$  is the minor determinant of  $\Psi_{i-1}^{(2)}$ , produced by deleting the second column and the second row and

$$\Psi_f^{(2)} = \begin{vmatrix} \rho - R_1 \lambda_{11} & -R_1 \lambda_{21} & 0 & \cdots & 0 \\ -R_2 \lambda_{12} & & & & \\ 0 & & \phi_1(\rho) & & \\ \vdots & & & & \\ 0 & & & & \end{vmatrix}.$$

**Step 7:** Combining from Steps 5 and 6, we obtain

$$\phi_2(\rho) = e_1 \varepsilon_1 \phi_1(\rho) + \rho^f \Psi_f^{(2)}.$$

**Step 8:** Develop  $\Psi_f^{(2)}$  across the third row that has two non-zero elements  $-1$  and  $\rho$ .

$$\Psi_f^{(2)} = (-1)^{(2)} \begin{vmatrix} \rho - R_1\lambda_{11} & -R_1\lambda_{21} & 0 & \cdots & 0 \\ -R_2\lambda_{12} & & & & \\ 0 & & \Xi_1^{(1)} & & \\ \vdots & & & & \\ 0 & & & & \end{vmatrix} \\ + \rho \begin{vmatrix} \rho - R_1\lambda_{11} & -R_1\lambda_{21} & 0 & \cdots & 0 \\ -R_2\lambda_{12} & & & & \\ 0 & & \Psi_1^{(1)} & & \\ \vdots & & & & \\ 0 & & & & \end{vmatrix}$$

(The (1) superscripts of  $\Xi_1$  and  $\Psi_1$  refers to  $\phi_1(\rho)$ .)  $\Xi_1^{(1)}$  and  $\Psi_1^{(1)}$  are produced similarly to  $\Xi_1^{(2)}$  and  $\Psi_1^{(2)}$  from  $\phi_2(\rho)$ . So we follow similar steps and finally,

$$\Psi_f^{(2)} = \begin{vmatrix} \rho - R_1\lambda_{11} & 0 & 0 \\ -R_2\lambda_{12} & 0 & e_2\varepsilon_2 \\ 0 & -1 & \rho \end{vmatrix} \\ + \rho^{f-1} \begin{vmatrix} \rho - R_1\lambda_{11} & -R_1\lambda_{21} & 0 \\ -R_2\lambda_{12} & \rho - R_2\lambda_{22} & e_2\varepsilon_2 \\ 0 & 0 & \rho \end{vmatrix}$$

which implies

$$\Psi_f^{(2)} = e_2\varepsilon_2 (\rho - R_1\lambda_{11}) \\ + \rho^f [(\rho - R_1\lambda_{11})(\rho - R_2\lambda_{22}) - R_1R_2\lambda_{12}\lambda_{21}].$$

**Step 9:** Develop  $\phi_1(\rho)$  similarly with  $\phi_2(\rho)$  and obtain

$$\phi_1(\rho) = e_2\varepsilon_2 + \rho^f (\rho - R_2\lambda_{22}).$$

**Step 10:** Combining the equations from Steps 7, 8, and 9 so that we finally obtain the following equation

$$\begin{aligned} \phi_2(\rho) = & e_1 e_2 \varepsilon_1 \varepsilon_2 + e_1 \varepsilon_1 \rho^f (\rho - R_2 \lambda_{22}) + \rho^f e_2 \varepsilon_2 (\rho - R_1 \lambda_{11}) \\ & + \rho^{2f} (\rho - R_1 \lambda_{11}) (\rho - R_2 \lambda_{22}) - \rho^{2f} R_1 R_2 \lambda_{12} \lambda_{21}. \end{aligned} \quad (29)$$

We observe that  $\phi_2(\rho)$  is a polynomial with  $a_{2(1+f)} = 1$  (coefficient of  $\rho^{2(1+f)}$ ) and  $a_0 = e_1 e_2 \varepsilon_1 \varepsilon_2$  (constant term). It is straightforward to generalize the above procedure and obtain from the equation that appears in Step 4:

$$\phi_m(\rho) = e_1 \varepsilon_1 \phi_{m-1}(\rho) + \rho \Psi_1^m.$$

Finally,  $\phi_m(\rho)$  is a polynomial with  $a_{m(1+f)} = 1$  (coefficient of  $\rho_{m(1+f)}$ ) and  $a_0 = e_1 e_2 \dots e_m \varepsilon_1 \varepsilon_2 \dots \varepsilon_m$  (constant term).

