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ARTICLES

Journal of Actuarial Practice

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A Study Note on the Actuarial Evaluation of Premium Liabilities

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Abstract‡

Several approaches have been used to estimate premium liabilities. The emphasis of these approaches has been on unearned premium and deferred policy acquisition expenses (DPAE), as such items represent the largest components of premium liabilities. The purpose of this paper is to provide a framework for the evaluation of premium liabilities and to augment the actuarial literature. We define and review the individual components of premium liabilities as well as the regulatory requirements and Canadian Institute of Actuaries recommendations and standards of practices related to premium liabilities. We also present an actuarial approach for estimating equity in the unearned premium, the premium deficiency, and DPAE. The approach here accords with Canadian Institute of Actuaries recommendations and standards of practice as well as statutory requirements as of December 31, 1997.

Key words and phrases: *unearned premium, deferred policy acquisition expenses, reinsurance, standards of practice, premium deficiency*

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1 Introduction

Since 1985, under Canada's Insurance Companies Act, the board of directors of each federally registered insurance company[1] has the duty to appoint an actuary, called an *appointed actuary*,[2] to perform the following duties:

- Value annually the policy liabilities of the company or other matters required by law;
- Monitor the financial position of the company;
- Report annually to the board of directors on the financial position and condition of the company; and
- Report to the board of directors on any transactions that may jeopardize the financial condition of the company.

Policy liabilities include both claim liabilities and premium liabilities.

There was no regulatory requirement in the United States for an actuarial opinion on premium liabilities until 1998. Several states now require an actuarial opinion on the adequacy of the unearned premium reserve for certain types of policies with terms exceeding 12 months.

Over the years several papers have been written and standard actuarial techniques have been developed to estimate claim liabilities and the various components of claim liabilities. Premium liabilities have, however, received little attention in the actuarial literature.

The Canadian Institute of Actuaries (CIA) standard of practice entitled *Recommendations for Property-Casualty Insurance Company Financial Reporting* provides a definition of premium liabilities as well as factors to consider in the evaluation of premium liabilities. Several approaches have been used to estimate premium liabilities, but none have been documented to date. The emphasis of these approaches has been on unearned premium and on deferred policy acquisition expenses (DPAE), as such items represent the largest components of premium liabilities. In particular, discussions between actuaries and some regulators have focused on the treatment of investment income in assessing equity in the unearned premium. Other components of premium liabilities (such as contingent commissions, retro-rated policies, and reinsurance adjustments) have received little attention.

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[1]The terms *insurance company* and *insurer*, as used throughout this paper, include stock insurance companies and mutuals.

[2]An actuary is defined under the act as a Fellow of the Canadian Institute of Actuaries. Note that for provincially registered companies (except in Quebec) the requirements are different, and the actuary is referred to as a *valuation actuary*.

The evaluation of premium liabilities encompasses more than assessing the adequacy of the excess of the pro-rata unearned premium over DPAE. It consists of examining all related assets and liabilities to ensure proper provision is made for the anticipated net costs incurred to discharge an insurer's obligations with respect to its insurance and reinsurance contracts, except its claim liabilities.

The purpose of this paper is to provide a framework for the evaluation of premium liabilities and to fill a gap in the actuarial literature. This paper defines and reviews the individual components of premium liabilities as well as the regulatory requirements and CIA standards of practice related to premium liabilities. It also presents an actuarial approach for estimating equity in the unearned premium, the premium deficiency, and DPAE.

The approach here accords with CIA recommendations and standards of practice as well as statutory requirements as of December 31, 1997.

2 Definition of Premium Liabilities

Premium liabilities generally have been defined as the cost of running off the unexpired portion of an insurer's policies and reinsurance contracts.

The following definition from the CIA standards of practice *Recommendations for Property-Casualty Insurance Company Financial Reporting* is broader, as it does not restrict premium liabilities to policies inforce. Therefore, liabilities can arise from policies already expired:

> Premium liabilities represent all the anticipated net costs to discharge the insurance company's obligations with respect to its insurance policies and reinsurance contracts except its claim liabilities.[3]

According to this definition, premium liabilities consist of all assets and liabilities resulting from an insurer's policies (direct, assumed, and ceded) other than those resulting from the collection of premiums currently due or payment of claims already incurred.

For most companies, premium liabilities, which are found on either side of the balance sheet (asset and liability), are composed of the following items:

[3]The Canadian Institute of Actuaries' *Recommendations for Property-Casualty Insurance Company Financial Reporting*, Part 4, Section 4.0.1.

- Unearned premiums (UP);
- Premium deficiency;
- Deferred policy acquisition expenses (DPAE);
- Provision for retro-rated policies;
- Earned but not recorded premiums (EBNR);
- Audit premiums;
- Premium development on reinsurance assumed;
- Ceded reinsurance retro-rated contracts (swing-rated contracts/sliding scale);
- Provision for contingent commissions; and
- Unearned reinsurance commissions.

In practice, these items can be grouped into these four larger categories:

- Future claims and adjustment expenses on inforce policies;
- Administrative costs of servicing inforce policies (maintenance costs);
- Anticipated premium adjustments; and
- Anticipated reinsurance expense (or commission) adjustments.

A simplified view of the balance sheet, highlighting the elements of premium liabilities, is shown in Table 1. Other elements of the balance sheet also are impacted by the various premium liability elements. For instance, a decrease in the unearned premium may increase the assets or the surplus of the company. The largest component of premium liabilities is the future claims and adjustment expense. For companies with large quota-share reinsurance, the unearned reinsurance commissions also may be a significant item on their balance sheet.

The provision for premium liabilities is not shown explicitly on the balance sheet of a Canadian insurer's annual statement (PC-1 or PC-2). Premium liabilities are the net total of the unearned premium, DPAE, and other related assets and liabilities on the balance sheet.

Finally, the equity in unearned premiums (EQUP) is defined as the expected profits on the unexpired policies. An example of a fictitious company is provided in Section 6, and the details of the EQUP are illustrated in Sections 7–10.

Table 1
Balance Sheet Items

Premium Liability Element	Asset	Liability	Surplus
Unearned premiums	Ceded unearned premium	Gross unearned premium	
Premium deficiency		Premium deficiency	
Deferred policy acquisition expenses (DPAE)	Deferred policy acquisition expenses		
Provision for retro-rated policies	*	*	
Earned but not recorded premiums (EBNR)		Gross unearned premium (negative amount)	
Audit premiums		Gross unearned premium (negative amount)	
Premium development on reinsurance assumed	Reinsurance receivables		
Ceded reinsurance retro-rated contracts	*	*	
Provision for contingent commissions		Provision for contingent commissions	
Unearned reinsurance commissions		Unearned reinsurance commissions	
	Cash		Additional policy reserve
	Investment		Contributed surplus
	Receivables	Payables	Earned surplus

Notes: An * denotes that, depending on the adjustment, this can be either an asset or a liability item.

3 Deferred Policy Acquisition Expenses (DPAE or DPAC)

The policy liabilities of an insurer, which include claim liabilities and premium liabilities, also can be thought of in terms of liabilities for past events and liabilities for future events. Liabilities for past events are provided by the unpaid claim provision (outstanding case provision, IBNR, and supplemental provision), the accounts payable (expenses), and premium or commission adjustments on policies that are expired. Liabilities for future events are the expected losses and maintenance expenses on the unexpired portion of the policies inforce at the end of the year. The unearned premium provides for these future liabilities. In the event that the unearned premium is less than the liabilities for future events, then a premium deficiency exists.

Premiums should be earned on a basis consistent with the occurrence of losses. For most lines, this translates into earning the premiums on a pro-rata basis. For some lines, however, earning premiums evenly throughout the year is not appropriate. For example, motorcycle premiums cannot be earned evenly over the year, as the bulk of the exposure is from April to October. Similarly, extended warranty premiums should be earned as losses are incurred, i.e., the risk increases with the elapsed time on the warranty. (For example, a three-year warranty will have more exposure to losses in the third year and may not have any exposure in year one, as manufacturers may provide coverage for that year). In those instances, the actuary should ensure that the unearned premiums for these lines reflect their exposure to risk, i.e., the potential incurral of losses.

An insurer's income is recognized on a pro-rata basis over the term of a policy, e.g., a 12-month policy written on July 1 is 50 percent earned at December 31. The expenses are also pro-rated over the term of the policy. Claims are accounted for as they occur. Some expenses are incurred over the term of the policy, e.g., endorsements, changes to coverage, mid-term cancellations, changes in reinsurance programs. All prepaid expenses (i.e., all the front-end expenses incurred by an insurer to write business and issue policies) are incurred at the time the policy is issued. These expenses, also referred to as acquisition expenses, include commissions, taxes, renewal costs, advertising, licenses and fees, associations and dues, etc.

The deferred policy acquisition expense (DPAE) provision is an asset that amortizes the prepaid expenses over the policy period, provided that such costs are recoverable from expected profits. This results in

Table 2
Two Scenarios on July 1, 1997

	Scenario 1	Scenario 2
Premium	$100	$100
Claims & LAE*	$60	$70
Prepaid expenses	$20	$20
Maintenance expenses	$10	$20
Profit/(loss)	$10	($10)

Notes: *LAE = Loss Adjustment Expenses.

a better match of income (premiums) and expenses. The DPAE provision cannot exceed the expected profits on the unexpired policies, i.e., it cannot exceed the equity in the unearned premiums. The deferred expenses are equal to the proportion of prepaid expenses, which relates to the unexpired portion of the policy (unearned).

Therefore, if a profit is expected, it is declared on a pro-rata basis in the income statement and the balance sheet. If a loss occurs, however, it is declared immediately. This is consistent with conservative accounting principles.

A simple example will illustrate this concept. Assume a policy that is written July 1, 1997 for a 12-month term, under the two scenarios given in Table 2.[4]

Table 3 shows how the various cash flows associated with this policy for Scenario 1 are accounted for in the income statement and in the balance sheet. The top part of Table 3 represents the policy's income statement, which shows that half of the premium, half of the losses (Claims & LAE), and half of the maintenance expenses are incurred by year-end, six months after the inception. The bottom part of Table 3 provides a view of the balance sheet item related to the policy after six months. Because the EQUP or expected profit of $15 (= $50 − $30 − $5) is higher than the portion of prepaid expenses that are deferrable $10 (= 50% × $20), the DPAE is equal to $10.

[4]Further assume that claims and adjustment expenses are incurred evenly over the term of the policy.

Table 3
Income Statement and Balance Sheet Items Under Scenario 1

Income Statement Cash Flows

	July 1–Dec. 31 1997	Jan. 1–June 30 1998	July 1–Dec. 31 1998
Premium (Revenue)	$50 (Earned)	$50 (Earned)	
Claims & LAE	$30 (Incurred)	$30 (Incurred)	
Expenses	$5 (Maintenance) $20 (Prepaid)	$5 (Maintenance) $0 (Prepaid)	

Issue
July 1, 1997

Expiry
June 30, 1998

Balance Sheet

	December 31, 1997	December 31, 1998
Premium	$50 (Unearned)	$0 (Unearned)
Claims & LAE	$30 (Expected Future Losses)	
Expenses	$5 (Future Maintenance) $15 (Equity in UP) $10 DPAE	$0 DPAE

Table 4
Insurer's Profit or Loss Under Scenario 1

	Dec. 31, 1997	Dec. 31, 1998
Earned premium	$50	$50
Less incurred claims & LAE	$30	$30
Less incurred expenses*	$25	$5
Plus change in DPAE**	$10	($10)
Profit/(loss)	$5	$5

Notes: * Includes maintenance and prepaid expenses; ** DPAE at year-end less DPAE at the beginning of the year.

The profit or loss for the insurer is shown in Table 4. The $10 profit is recognized pro-rata over the term of the policy. Without the provision for DPAE, there would be a loss of $5 recorded at December 31, 1997 and a profit of $15 recorded at December 31, 1998. The deferral of expenses results in a better match between revenue and expenses.

In Scenario 2, the expected profit on this policy is a loss of $10. Table 5 shows how the various cash flows associated with the policy are accounted for in the income statement and in the balance sheet. DPAE is decreased to the expected profit of $5 even though the deferrable expenses amount to $10 (= 50%×$20). Prepaid expenses can be deferred only to the extent they are recoverable from expected future profits.

The profit/(loss) by year under Scenario 2 is shown in Table 6. Note that a loss is declared in the first year under Scenario 2 compared to a profit under Scenario 1, using the accounting principle that a premium deficiency first should be recognized by writing-off any deferred acquisition costs. If insurance accounting were done on a policy year basis, no DPAE provision would exist. All premiums would be earned when the policy is inforce; thus all expected claims and all future expenses would have to be recognized in the liabilities and all commissions, taxes, and other issuing costs would be expensed immediately.

The DPAE provision is equal to the unearned acquisition costs. These can be approximated by:

$$\text{DPAE Provision} = \frac{\text{Paid Acquisition Costs}}{\text{Written Premium}} \times UP$$

where UP is the unearned premium.

Table 5
Income Statement and Balance Sheet Items Under Scenario 2

Income Statement Cash Flows

	July 1–Dec. 31 1997	Jan. 1–June 30 1998	July 1–Dec. 31 1998
Premium (Revenue)	$50 (Earned)	$50 (Earned)	
Claims & LAE	$35 (Incurred)	$35 (Incurred)	
Expenses	$10 (Maintenance) $20 (Prepaid)	$10 (Maintenance) $0 (Prepaid)	

Issue
July 1, 1997

Expiry
June 30, 1998

Balance Sheet

	December 31, 1997	December 31, 1998
Premium	$50 (Unearned)	$0 (Unearned)
Claims & LAE	$35 (Expected Future Losses)	
Expenses	$10 (Future Maintenance) $5 (Equity in UP) $5 DPAE	$0 DPAE

Table 6
Insurer's Profit or Loss Under Scenario 2

	Dec. 31, 1997	Dec. 31, 1998
Earned premium	$50	$50
Less incurred claims & LAE	$35	$35
Less incurred expenses*	$30	$10
Plus change in DPAE**	$5	($5)
Profit/(loss)	($10)	$0

Notes: * Includes maintenance and prepaid expenses; ** DPAE at year-end less DPAE at the beginning of the year.

The insurer calculates the DPAE amount usually equal to commissions and taxes. These are prepaid and easy to calculate. Some insurers also include additional prepaid expenses in their DPAE. These expenses, however, are more difficult to determine. The insurers may complete detailed reviews of the general expenses by categories and assign a portion of each category that may be deferrable to estimate these other costs. In practice, insurers approximate them.

The actuary's role is to determine if the DPAE as calculated by the insurer is recoverable from expected future profits, i.e., the actuary's role is to determine if the equity in the unearned premium (i.e., expected future profits) is sufficient to cover the calculated DPAE.

There is no regulatory limitation on the DPAE asset. But DPAE cannot exceed EQUP.

4 Other Components

The largest component of premium liabilities is future claims and adjustment expense. The importance of other components varies by insurer, depending on the book of business or the reinsurance programs.

These other components can be grouped into two major categories: those that relate to commission adjustments and those that relate to premium adjustments.

Although some practitioners may not consider some of these items (e.g., contingent commissions) as premium liabilities, they are liabilities related to the insurer's business. Thus, they should be included in the calculation. Moreover, the Office of the Superintendent of Financial

Institutions (OSFI) requires that the actuary comment on all actuarial liabilities, other than claims and premiums, which include all of the items below.

Contingent commissions: These commissions are what insurers pay their agents or brokers based on the results and volume of business of individual producers (agents/brokers), i.e., they are profit-sharing commissions. These agreements vary by company and are often established over one-year or three-year periods. If the agreement is over a three-year running period, then some commissions may be incurred as of the statement date, and they should be accrued. Contingent commissions are often not accrued in the balance sheet, but these liabilities can be significant.

Unearned commissions: Some insurers with large quota share treaties may have significant unearned commissions on the ceded premiums. These commissions may vary depending on the ultimate loss ratios of the business. The actuary should assess the calculated unearned commissions using his/her estimate of the loss ratios. The unearned commissions are booked as a liability and are earned pro-rata over the terms of the policies.

Provision for retro-rated policies: A liability provision for retro-rated policies is required when insurers issue policies for which the premium is adjusted yearly based on the actual experience on the policy. The final premium is not known until all losses are reported and settled. The provision to be accrued is equal to the difference (either positive or negative) between the estimated final premium and the paid premium at the date of the statement.

Other examples of premium development to be evaluated as part of the premium liabilities are:

EBNR premiums: In some instances the insurers will be at risk on insurance contracts but the transactions are processed only after the effective date of the policy. This may happen because of reporting or processing delays or because of the nature of the insurance product. These earned but not recorded premiums (EBNR) are also part of the premium liabilities. This item is usually small and mostly arises from reinsurance assumed business.

Audit premium and other: For audit premiums, the final premium is not known until the coverage expires. Sources of premium development on reinsurance assumed or ceded contracts include the following: (i) changes in subject matter premium[5] (usually unknown until the end of the contract period), (ii) swing-rated excess of loss treaties[6] with a rate adjustment based on the loss experience during the coverage period, and (iii) reinstatement premium for catastrophe treaty, i.e., additional premium to be paid when the limit of coverage provided by the layer has been exhausted.

5 CIA Recommendations and Regulatory Requirements

In previous sections we have introduced the concept of premium liability and discussed its components. We now turn our attention to regulatory requirements specific to premium liabilities. This section will focus on items where differences exist between regulatory requirements and CIA standards of practice.[7]

DPAE asset: Federally and provincially registered insurers (except provincial insurers in Alberta) may establish a DPAE asset up to the equity in unearned premium. Alberta regulators require insurers to record 80 percent of the unearned premiums in their balance sheet, which is equivalent to having an asset for DPAE equal to 20 percent of UP. The actuary is responsible for determining that 80 percent of UP is sufficient to cover future losses and maintenance expenses on the unexpired policies. If not, then an additional liability should be recorded for the difference.

Investment income: The CIA standards of practice requires actuaries to recognize the time value of money in evaluating the policy liabilities, except when regulators do not allow discounting. Under CIA requirements the expected losses should be discounted not only up to the average occurrence date of the losses arising from the unearned premiums, but to the average payment date of all future losses.

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[5] *Subject matter premiums* are the annual direct written premiums related to the business subject to the reinsurance arrangement for that contract year.

[6] A *swing-rated excess of loss treaty* is one where the reinsurance premium rate or commission rate is adjusted based on the actual experience on the treaty. For example, the commission rate increases if the loss ratio is lower than anticipated.

[7] We refer the reader to the *Consolidated Standards of Practice* and to the *Recommendations for Property-Casualty Insurance Company Financial Reporting* that are listed in the references below.

OSFI does not currently allow discounting of claims liabilities (except for some lines, e.g., accident benefits). For premium liabilities, OSFI allows limited recognition of discounting.

Under OSFI guidelines, investment income can be included in determining equity on the unearned premium only if the unearned premium reserve is sufficient to cover future undiscounted claims and expenses (i.e., if there is no premium deficiency). OSFI guidelines allow for investment income to be recognized only from the valuation date to the average earning date of the unearned premium (or average accident date of future claims). For one-year policies this results in approximately four months of investment income. (Some Canadian practitioners might not agree with these guidelines.)

The Inspecteur General des Institutions Financieres (IGIF) has different rules for Quebec provincially registered companies. IGIF's position on the issue is that actuaries should follow the CIA recommendations, thus effectively accepting discounting.

This issue will disappear only when all regulators allow discounted policy liabilities in the balance sheet.

For statutory purposes (and except for Quebec provincially registered insurers) the calculation of premium liabilities should recognize investment income on the unearned premium only for the period between the valuation date and the average earning date (or the average occurrence date of losses on the unexpired policies), i.e., three to four months.

Other liabilities versus premium liabilities: The actuarial opinion prescribed by OSFI shows other policy liabilities as a separate item.[8] This opinion is shown in Appendix H, Sheet 1. The actuarial opinion required from IGIF is shown in Appendix H, Sheet 2. At this time IGIF and OSFI have different views on what constitutes premium liabilities versus other liabilities.

The CIA definition, and the one we adopt in this paper, is the broad definition. Premium liabilities include all assets and liabilities related to future costs arising from all insurance or reinsurance contracts of an insurer. These contracts can either be inforce or expired.

At this time we understand that OSFI includes only liabilities related to the unexpired portion of the policies inforce. OSFI's po-

[8] See OSFI's *Instructions for Actuarial Reports on Property Casualty Business.*

sition is that the unearned premiums should not be charged with future costs or development on policies/contracts that are already expired. Instead, a separate item (other policy liabilities) should be shown for those premium liabilities that are not related to unearned premiums. IGIF, on the other hand, uses the broad definition. Although we agree that future liabilities related to expired policies should not be charged against the inforce policies when calculating the equity in unearned premiums, these liabilities (assets) should be part of total premium liabilities as they relate to the insurance (reinsurance) contracts of the insurer.

All lines combined versus by line equity: For regulatory purposes equity in unearned premiums may be calculated on an all lines combined basis. This means that deficiencies in some lines are offset by redundancies in other lines. This approach is appropriate on an ongoing concern basis when a company's mix of business does not change significantly from year to year. It is appropriate because it is unlikely that a company would stop writing its more profitable lines.

A more rigorous and conservative approach consists of evaluating the equity by line of business, split in a manner consistent with the way the insurer acquires business and measures profitability.

The current position of some regulators on the recognition of investment income in calculating the equity in UPR creates a mismatch between expected future costs and premiums, however, especially for long-tail lines. Thus, insurers with large portfolios of long-tail risks would be penalized using a by line approach. For the long-tail line, full recognition of investment income needs to be accepted before using a by-line calculation because investment income is an important pricing consideration for these products.

Subsequent Events: The major Quebec ice storm of January 1998 raised the issue of subsequent events and their treatment with regard to premium liabilities in the actuarial opinion.

CSOP Section 4.6 (second exposure draft, May 1997) offers the following guidelines.

> The actuary should correct any data defect or calculation error, which a subsequent event reveals.
>
> For work with respect to an entity, the actuary should take a subsequent event into account in the selection of

> methods and assumptions for a calculation, other than a pro forma calculation, if the subsequent event:
>
> - Provides information about the entity as it was at the calculation date, or
> - Retroactively makes the entity a different entity at the calculation date, or
> - Makes the entity a different entity after the calculation date and a purpose of the work is to report on the entity as it will be as a result of the event.
>
> The actuary should not so take the subsequent event into account if it makes the entity a different entity after the calculation date and a purpose of the work is to report on the entity as it was at the calculation date, but the actuary should report that event.

According to this guideline, each subsequent event must be analyzed separately. No general rule can be applied. The first step is to classify the event according to the three criteria listed above:

- Does it provide information about the entity as it was?
- Does it retroactively make the entity different?
- Does it make the entity different after the calculation date?

Reporting a claim incurred on or before the statement date provides information about the insurer as it was. On the other hand, reporting a claim incurred after the statement date, especially when it cannot be expected, makes an entity different after the fact.

In the case of the ice storm, although the actual premium liabilities are likely to be much larger than the premium liability anticipated at December 31, 1997 (due to the ice storm), the calculation should not reflect the impact of the ice storm. The actuarial guidance was that the appropriate course of action was to disclose the impact of the ice storm in the notes to financial statements, but make no changes to the premium liabilities calculation.

The considerations leading to this conclusion were that:

- The ice storm did not make the insurance company different retroactively, and
- The purpose of the actuarial report was to report on the insurance company as it was at December 31st.

A storm that would be predicted to occur or continue after the statement date should be considered in the premium liabilities on the basis that it provides information on the insurer as it was at December 31, 1997.

An example of a subsequent event that was considered in the evaluation of premium liabilities was the implementation of a new automobile compensation system—Bill 164 in Ontario on January 1, 1994. In this case, the key event was the announcement of Bill 164 effective date, which definitively occurred in 1993 and was known in advance at the time of calculating the premium liabilities. It was thus taken into account in the December 31, 1993 evaluation.

Each event is different, and no general rule can be applied to the treatment of such events. One criterion remains, however—the potential size of claims resulting from the event must exceed the materiality level.[9]

6 Data for the Example

Dubois Fire & Casualty Insurance Company (DF&C) is a federally registered insurance company writing business primarily in Ontario. It is wholly owned by Kosciuzsko Insurance Company (KIC), which is also federally registered. DF&C's book of business comprises automobile insurance [split among third party liability (TPL), accident benefits (AB), and physical damage (PD) coverages], personal property (PP), and general liability (GL) exposures. Its book is split 70 percent/30 percent between one-year and six-month policies, respectively. DF&C also underwrites aviation business but cedes it all to TupolevInsure (TvI), a specialty aviation writer for which DF&C acts as a fronting company.[10]

DF&C is reinsured under two different treaties:

- Proportional reinsurance for all lines with 75 percent retention.
- Excess-of-loss treaty for general liability covering losses in excess of $250,000 up to $1,000,000. The applicable reinsurance rate is 1.25 percent of the subject written premiums.

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[9]According to CIA recommendations, "A difference is material if it is significant to the user of the financial statements. The member should choose a standard of materiality which will reasonably satisfy each normal user of the financial statements."

[10]Dubois Fire & Casualty Insurance Company, Kosciuzsko Insurance Company, and TupolevInsure are fictitious financial entities. Any resemblance to real companies is purely coincidental.

DF&C and KIC have entered into an intercompany reinsurance arrangement whereby KIC assumes 40 percent of DF&C's exposures (net of all reinsurance) and cedes 25 percent of its exposures to DF&C (also net of all reinsurance). To simplify the calculation, we have assumed that internal adjustment expenses and maintenance expenses are ceded on the same basis.

DF&C has a contingent commission agreement with its independent brokers. Under this agreement, commissions are adjusted on a three-year rolling average basis.

Finally, DF&C participates in the facility association and in the risk sharing pool. The facility association (FA), risk sharing pool (RSP), and plan de répartition des risques (PRR) are residual market pools for automobile insurance in Canada.

Residual markets have been established primarily to ensure insurance availability to high-risk insureds who otherwise would be unable to find affordable insurance. Under the RSP and the PRR, insurers transfer risks written at the insurer's own rates to the pool and receive from the pool a share of all insurers' cessions based on their market share.

These are risks that the insurer deems unacceptable according to its own criteria. The business ceded to these pools is subject to a maximum percentage of direct written exposures or premiums. Under FA, risks are underwritten by the FA servicing carriers at FA rates, and losses and expenses are allocated to insurers licensed to write automobile insurance based on their market share.[11]

In the following sections we present an actuarial approach for determining equity in the unearned premium (EQUP). This calculation, in turn, determines the premium deficiency and DPAE. We believe the method and calculations covered represent approaches currently in use by actuaries in their actuarial evaluation.

Section 7 outlines a step by step approach to calculate EQUP for DF&C as of December 31, 1997. Considerations and assumptions involved in the calculations (expected loss ratios, future expenses, contingent commissions, etc.) are discussed in detail.

Later sections deal with discounting, gross premium liability calculations, and the treatment of assumed business in calculating EQUP.

[11] For further information, see Facility Association's *Plan of Operations* and the PRR's *Procedures Manual.*

7 Equity in the Unearned Premium

7.1 Overall Calculations

Exhibit 1 illustrates the process of calculating the equity in the net unearned premium. Similar calculations (shown in Exhibit 4) are done to obtain EQUP on a gross basis.[12] These calculations are in accordance with the CIA standards of practice.

The process starts with unearned premiums. To the extent possible, premiums should be adjusted for retro-rated policies, reinsurance assumed and ceded, or for any other future development on unexpired policies. These adjustments should be done on a line by line basis.

An expected loss ratio, including external (allocated) adjustment expenses (ALAE), by line of business is estimated based on historical experience and current considerations. This calculation and the related assumptions are covered in Section 7.2.

The unearned premium is converted to expected losses by multiplying the unearned premium by the overall estimated ultimate loss ratio. Internal (unallocated) adjustment expenses (IAE), maintenance expenses, and contingent commission adjustments, as well as all other cost adjustments (such as reinsurance costs) are added to the total estimated expected losses. In cases where ALAE is not included in the loss ratio, it should be added to the total as well.

EQUP is calculated as the difference between the unearned premiums and the expected losses and expenses (IAE, ALAE, maintenance expenses, contingent commissions, etc.). Investment income is factored in by discounting future claims and expenses. The maximum allowable DPAE asset is equal to the equity in unearned premium.

In cases where EQUP is negative (i.e., a premium deficiency exists), DPAE must be reduced by the amount of the deficiency. If DPAE is reduced to zero and EQUP remains negative (in other words, if the absolute value of negative EQUP exceeds the deferrable expenses), a premium deficiency must be booked as a liability for the remaining deficiency. Negative EQUP indicates that the unearned premium reserve will not be sufficient to cover future claims and expenses on the unexpired portion of the inforce policies.

Under current OSFI requirements, investment income can be included in the equity calculation only if there is no premium deficiency. We have included the statutory calculations in Exhibit 1.

[12]Exhibits are located after the references and immediately before the appendices.

7.2 Estimated Ultimate Loss Ratio

Exhibit 2 shows the estimation of ultimate loss ratios, including ALAE, for third party liability (TPL). Calculations for the other lines of business are shown in Appendix A, Tables A1–A5. The starting point is the company's historical experience. Because losses tend to be cyclical and the experience of a single year is too small to be reliable, our selection is based on the latest three calendar/accident years. The historical loss ratios are adjusted to the current and expected conditions for the period over which the unearned premium will be earned. These adjustments are discussed below.

For small, volatile, or new lines of business, industry experience can be used to select the loss ratios, with appropriate adjustments for differences between the insurer's operations and industry averages.

On-Level Factors: Premiums are adjusted to their current rate level using on-level factors. These factors are derived from the insurer's rate change history.

In April 1995 DF&C increased accident benefits (AB) rates 30 percent. Following the introduction of Ontario's Bill 59 (Automobile Insurance Rate Stability Act) in 1996 DF&C decreased its rates for both accident benefits (AB) and physical damage (PD) automobile coverages and increased its rates for TPL. The resulting on-level factors exceed 1.00 for TPL and are below 1.00 for AB (except in 1995) and PD coverages.

Catastrophe (CAT) Loading: Historical loss ratios need to be adjusted for catastrophic losses. These losses are rare but large and can significantly distort loss ratios. The losses are smoothed by removing the actual CAT losses from the historical data and adding an appropriate loading. The CAT loading is derived from the experience over a long time period to account for the infrequent nature of these losses. This loading, which varies by line of business, increases the historical loss ratio for each year.

As shown in Exhibit 2, DF&C experienced CAT losses of $435,000 during 1996. We removed this amount from the incurred losses before developing them to ultimate. For TPL, a judgmental loading of 0.3 percent was selected and added to ultimate losses. The CAT losses were not developed to ultimate. We assumed that, because of their unusual nature, case reserves are adequate.

Historical loss ratios also should be adjusted for the impact of large, noncatastrophic losses. A procedure similar to the one described above may be used whereby a judgmental threshold is set. Individual losses in excess of that threshold are considered large losses, and the amount in excess is removed from historical losses before computing the loss ratios. Selected thresholds ideally should reflect the time value of money and be detrended for older years. For example, assuming a \$200,000 threshold for general liability for 1997 and a 10 percent loss trend, the thresholds for 1996 and 1995 should be \$181,818 and \$165,289, respectively.

Loss Development Factor (LDF): These factors are used to develop reported losses to the ultimate. It is appropriate and often practical to select the reporting pattern implied by IBNR projections, as long as the pattern reflects future claims reporting development.

Trend Factors: Trend factors that reflect inflation in the cost of claims need to be taken into account when projecting ultimate loss ratios. Although business plans may be used to estimate trends, industry data or the company's historical data are probably a better starting point because these data are unbiased and cannot be distorted by pessimistic or optimistic assumptions used by management. Alternatively, trend factors used for ratemaking purposes also can be used.

The smoothed ultimate loss ratios are trended to the average accident date of losses arising from unearned premiums. For one-year policies, the average accident date (AAD) is six months after the policy inception date. The same logic can be applied to determine the accident date of losses that will arise from the unearned premium. Calculations, shown in Appendix F, result in average accident dates of May 1, 1998 and March 1, 1998 for one-year and six-month policies, respectively, assuming premiums are written evenly throughout the year.

Trends are assumed to impact losses uniformly over the year. Losses are trended from the experience period's AAD (July 1) to the AAD of losses arising from the unearned premium (May 1). The last leg of the trending period may not cover a full year (but most likely covers about ten months). Even if some lines could exhibit seasonal trends, it is unlikely that selected trends would be materially different if seasonality were considered.

Loss trends under Bill 59 are expected to differ from those under Bill 164. As a result, DF&C uses two trends for each coverage. Selected TPL trends for Bill 164 and Bill 59 are 5.0 percent and 0.0 percent, respectively. The accident-year 1995 trend factor of

1.068 was calculated by first bringing losses from the average accident date (July 1, 1995) to the effective date of Bill 59 (November 1, 1996) using the 5.0 percent trend. From there, losses were trended for an additional 17 months at 0.0 percent, to the average accident date of the unearned premium (May 1, for one-year policies).

Historical premiums also should be trended to the average writing date (AWD) of the unearned premium, which is September 1, 1997 for one-year policies (November 1, 1997 for six-month policies).[13] The premium trends account for rate group drifts (physical damage), change in insured value (personal property), and policy limit drifts (third party liability). We assume the impact of these factors is not material.

Benefit Changes: Bill 59, which became effective November 1, 1996, introduced significant changes in benefits for Ontario automobile drivers. Assuming that premiums were adjusted to reflect the full impact of Bill 59 on loss costs, the historical loss ratios do not need to be adjusted. In those instances where premium changes do not keep up with loss cost changes, however, historical loss ratios should be adjusted accordingly.

Other Adjustments: There are several other adjustments, including:

- Seasonality—Most of the unearned premium is earned from January to June, with a large portion being earned during the winter months. Seasonal variations in loss ratios impact our selections as the claims level varies by quarter. For example, there are usually more automobile collision claims during the winter months than during the summer months.

  Appendix B shows the distribution of expected loss ratios by month. Table B1 shows that, using the 24th method, the average loss ratio applicable to the unearned premium for automobile is 79.6 percent. The average loss ratio, assuming no seasonality or exposure growth, is 80.4 percent (simple average of the monthly ratios). This implies that a seasonality adjustment factor of 0.990 (79.6 percent/80.4 percent) is applied to the selected loss ratios to account for the difference in the loss ratio levels by month. This reflects the fact that, on average, unearned premiums will generate lower loss

[13] Appendix F shows how these dates were derived.

ratios than if premiums were earned evenly throughout the year.

- Policy Term—Another factor relates to the composition of the insurer's portfolio. The bulk of policies are still 12-month terms. There are companies, however, that primarily offer three-month and six-month policies. For example, niche companies targeting higher risk insureds typically offer three-month and six-month policies. This mix should be taken into account as it impacts trending periods, on-level factors, and seasonality adjustments, among others.
- Changes in Reinsurance Program—For reinsurance contracts made on an accident-year basis, consideration also should be given to changes in the insurer's reinsurance program. Most reinsurance contracts are effective at the beginning of the calendar year. Losses occurring during 1998, arising from a policy underwritten during 1997 (hence attributable to unearned premiums), will be subjected to the 1998 reinsurance program. Adjustment should be made to the historical loss ratios to reflect the prevailing reinsurance program conditions.

  For example, DF&C might decide to double its excess-of-loss (XOL) retention from $250,000 to $500,000, effective January 1, 1998. Assume a $350,000 loss occurs January 15 on a policy that was underwritten during 1997. Under the previous treaty, DF&C's liability was limited to $250,000; under the 1998 terms, DF&C is liable for the full amount. Therefore, the increased retention may or may not increase the loss ratio on the unearned premium depending on the terms of the contract. The selected loss ratio should be adjusted accordingly.

  In this example, the loss ratios on the unearned premium should be increased by the ratio of expected losses under the new XOL treaty to the expected losses under the current XOL treaty.
- Premium Development—As noted earlier, unearned premiums used in EQUP calculations should be fully developed before being multiplied by the ultimate expected loss ratios. Examples of premium development are audit premiums, where the final premium is unknown until the expiration of the coverage. Premium development also may exist on reinsurance assumed business due to a time lag between the time the pri-

mary insurer records the premiums and the time the assuming party reports them. Swing-rated excess-of-loss treaties, which provide for a rate adjustment based on the loss experience, are another example.

- Other—There are other factors that could require adjustments to historical loss ratios. This paper has focused on the factors that actuaries are most likely to encounter. No list, however extensive, can be expected to cover all situations. Actuarial judgment and skills should be used to determine the required adjustment if it is felt the impact is material.

7.3 Internal Adjustment Expenses

Internal adjustment expenses (IAE) will be incurred on future claims. They need to be taken into account when calculating future losses and expenses arising from the unearned premiums.

Future losses should be increased by the ratio of IAE to losses. Ratios of IAE to losses are usually stable. As a result, the IAE loading used in connection with claim liability calculations is a good proxy for the IAE loading on the unearned premium. As can be seen from Exhibit 1, line 10, the selected IAE percentage loading applied to the expected losses yields IAE of \$271,000.[14]

7.4 Maintenance Expenses

Maintenance expenses are necessary to maintain and service policies inforce. They must be estimated and accrued as part of the unearned premium. Servicing costs include expenses associated with endorsement, mid-term cancellations, and changes in reinsurance contracts.

These expenses should be expressed as a ratio of the premium, called the maintenance expense ratio (MER):

$$MER = \frac{\text{Maintenance Expenses on Inforce Policies}}{\text{Net Unearned Premiums}}.$$

This ratio is rarely used, given that an accurate estimate of maintenance expenses requires detailed expense studies that can be costly to produce. Instead, one can rely on the P&C-1 Expense Exhibit,[15] which is

[14] As will be seen later, the intercompany reinsurance agreement between DF&C and its parent KIC provides for IAE cession. The \$271,000 IAE provision on Exhibit 1 includes \$83,000 IAE assumed from KIC (based on the IAE ratio used by KIC's actuary).

[15] P&C-1, Page 80.20.

shown in Appendix G, and identify for each expense category (classification) the portion that belongs to policy maintenance. These expenses are divided by the earned premiums to obtain the maintenance expense ratio to be applied to the unearned premiums. As a result, the maintenance expense ratio shown above can be approximated by calculating for a given period:

$$MER \approx \frac{33\% \times \text{General Expenses}}{\text{Net Unearned Premiums}}.$$

This is based on the assumption that two-thirds of general expenses are front-end expenses and remaining expenses relate to maintenance and servicing policies. The considerations that should be taken into account when selecting this ratio include the insurer's distribution method (companies dealing with brokers may have fewer maintenance expenses than direct writers) and the degree of automation of the servicing insurer's operations.

The resulting maintenance provision is \$286,000, which is equal to the selected maintenance expense ratio of 2.5 percent multiplied by the \$11.45M net unearned premium provision (excludes FA unearned premiums).[16]

7.5 Contingent Commission

These commissions arise from agreements between insurers and their brokers or agents whereby the insurer may pay additional commissions based on the level and profitability of business produced. There are several kinds of contingent commission arrangements or contracts. In our example, the results are measured in terms of loss ratios and contracts are on a three-year rolling average basis.

Contingent commissions, available from the annual return,[17] are expressed as a percentage of premiums earned during the year. The resulting ratio is applied to unearned premiums. For DF&C, the 0.2 percent ratio yields a \$14,000 provision.

7.6 Net Reinsurance Costs

Net reinsurance costs are costs associated with reinsurance such as commissions paid to reinsurance brokers. These costs are reduced by

[16] As mentioned before, the intercompany reinsurance agreement between DF&C and KIC provides for the cession of maintenance expenses.

[17] From page 80.10 row 83.

the reinsurance commissions received from reinsurers. Such costs can be negative (and thus increase EQUP) for those insurers receiving large reinsurance commissions from their reinsurers. A loading approach is used whereby net reinsurance costs incurred during the year are divided by premiums earned during the year. The resulting ratio is applied to the unearned premium reserve.

If the risk transfer is at the expected loss level, no additional expense is included in the reinsurance premium. Therefore, EQUP calculations do not show any reinsurance cost item. If not, there might be a provision for the premium adjustment as a result of the experience level.

Finally, costs associated with the purchase of excess-of-loss protection also should be included. In the DF&C example, the premium is equal to 1.25 percent of the subject written premiums. This translates into a $4,000 provision, which reduces EQUP.

7.7 Adjustment for Retro-Rated Policies

Retro-rated policies allow for premium adjustment based on actual loss experience. The difference between the ultimate premium and the paid premium at the valuation date will dictate the magnitude of the premium adjustment. DF&C does not have retro-rated policies.

8 Discounting

CIA recommends that the premium liabilities provision be established on a present value basis using expected payment patterns. *Recommendations for Property-Casualty Insurance Company Financial Reporting* provides guidance related to the selection of a discount rate and provisions for adverse deviations (PFAD). CIA recognizes, however, that its position is different from some regulators and that its recommendations do not apply in instances where the regulators preclude present value liabilities.[18]

As noted earlier, the statutory premium deficiency must be calculated using undiscounted claims and expenses. The approach shown here is consistent with CIA recommendations. Exhibit 3 shows the

[18]"Pending better definition by the profession of an appropriate provision for adverse deviations, regulation in some jurisdictions requires the liabilities in government financial statements to be the sum, rather than the present value, of those payments. Where there is such a requirement, the recommendation in this section to establish a present value provision does not apply to the valuation of liabilities in government financial statement and (...) it likewise does not apply to the valuation of liabilities in published financial statements." (Section 5.04 of the CIA's *Recommendations* ... .)

calculations required to obtain discount factors applicable to future expected claims and expenses for auto-third party liability (TPL). Appendix D includes calculations for each line of business.

First, an expected payment pattern is selected for each line of business. It is appropriate, and often practical, to select the payment pattern implied by the IBNR projections, as long as it reflects future claims payment.

If future settlements are expected to behave differently than historical paid claims development, the selected patterns should reflect future paid claims development. This could arise from a change in legislation that affects both claims already reported and future claims. This was the case with the implementation of Bill 59 (discussed later). Another good example is found in medical malpractice, where the time allowed for filing a lawsuit after the discovery of an injury is prescribed by the statute of limitations. Extending the statute over a longer period also points to different payout patterns than those used in IBNR projections as, under the revised statute, one would expect claims to be paid over a longer time period.

The payouts are discounted to reflect the time value of money. CIA, without specifically defining an appropriate discount rate, provides guidance in selecting an investment rate of return. Among other things, the selected rate of return should depend on the projected rate of return on the insurer's assets, market rates, the method of reporting investment return and valuing assets, the expected investment expenses, and the expected losses arising from asset default.[19] Based on these considerations, a discount rate of 7 percent for the first five years and 5 percent for future years was selected for DF&C.

When claim liabilities are discounted, the inherent uncertainty again increases. In addition to the risk of underestimating or overestimating the overall amount of the claim liabilities, there are the additional risks that the timing of the future payment of those liabilities or the expected return on investments will differ materially from the assumptions underlying the calculation. Actual claim and external adjustment expense payments could occur more or less rapidly than projected due to random variations and the timing of large claim payments. Also, the yield on assets supporting the liabilities may be affected by capital gains or losses or by significant changes in economic conditions.

CIA standards require that a provision for adverse deviations (PFAD) be included to account explicitly for the uncertainty in the three following variables:

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[19]The *Recommendations for Property-Casualty Insurance Company Financial Reporting* provides an extensive list of considerations in Section 5.04.

Table 7
LOB Selected Margins

	Margins		
Line of Business	Claim Development	Reinsurance Recovery	Interest Rate
Auto-TPL	12.5%	5.0%	50 basis points
Auto-AB	10.0%	5.0%	50 basis points
Auto-PD	5.0%	5.0%	50 basis points
Personal property	5.0%	5.0%	50 basis points
General liability	12.5%	5.0%	50 basis points

- Claims development;
- Reinsurance recovery; and
- Interest rate.

Exhibit 3 illustrates how each PFAD is included in the calculation for auto-TPL. The claims development margin, judgmentally selected between 2.5 percent and 15 percent, increases the discounted loss ratio.[20] The reinsurance recovery margin, which varies between 0 percent and 15 percent, provides for the possibility that the insurer will not be able to recover reinsurance receivables. Hence, it is applied to the expected ceded claims (as a percentage of the net unearned premium), and the resulting margin is added to the discounted loss ratio (already loaded with the claims development). Finally, the interest rate margin (varying between 50 and 200 basis points) is treated as an additive factor that decreases the selected discount rate. Table 7 lists the selected margins by LOB.

The selected loss ratios are discounted to the average accident date (AAD) of the unearned premium by multiplying the discounted payment pattern [Column (7) in Exhibit 3] by the undiscounted loss ratios loaded for claims development and reinsurance recovery margins, as described above.

A further step is needed to discount the loss ratio from the average accident date to the evaluation date. The average accident date is four months after the evaluation date. These four months recognize the investment income generated on the unearned premium when the

[20]These selections are based on considerations mentioned in the CIA's *Memorandum on Provision for Adverse Deviations (P&C)* released January 1, 1994.

unearned premium is fully invested. Because part of the unearned premiums is held by brokers for up to 60 days after the policy inception, however, the investment income on premium receivables is credited to the brokers, not to the insurer. The larger the premium receivables as a proportion of the unearned reserve provision, the larger is the offset to the four month additional discount.

The methodology described in this section produces discounted loss ratios, which find their way back in Exhibit 1, where they are applied to the unearned premiums to yield discounted losses. For TPL, the selected undiscounted loss ratio of 72.5 percent, once discounted and loaded with PFAD, is 70.4 percent. As only 50 percent of the unearned premium is held by DF&C, an extra two months (instead of four) of investment income is credited to DF&C, resulting in a 69.6 percent discounted loss ratio. This loss ratio is used in Exhibit 1 to calculate the expected discounted losses arising from the unearned premium. As seen previously, regulators allow investment income in the EQUP calculation as long as the unearned premium reserve is sufficient to cover future undiscounted claims and expenses, i.e., that there is no premium deficiency.

Expenses are also discounted under similar circumstances. Maintenance expenses are incurred until the policy expires. Given that the average earning date of the unearned premium is May 1, 1999, the maintenance expenses provision is discounted four months.

Internal adjustment expenses are discounted using a factor equal to the ratio of the total discounted losses to the total undiscounted losses (excluding any pools such as the facility association where IAE is paid by the pool).

The discount factor applicable to the contingent commissions depends on the length of the period over which the underwriting results (which influence the commissions) are measured. DF&C's agreement with its broker provides for commissions to be determined on a three-year rolling average basis. The average accident date of that period is assumed to be the period's midpoint.[21] The discount rate, the interest rate margin, and the reinsurance recovery margin are the same as those used to discount losses. This is not true of the claims development margin, however. Although the contingent commissions ultimately depend on claims development, they are subject to less volatility than the un-

[21] This assumption does not differ significantly from the theoretically correct answer of 1.48 years

$$(1.065^{-1.48} = \frac{1.065^{-.5} + 1.065^{-1.5} + 1.065^{-2.5}}{3}.$$

derlying losses because the agreement provides for a minimum and a maximum commission. Hence, even though GL losses can be volatile, the impact of their variability on the contingent commissions' level is dampened by these limits. As a result, the claims development margin included in the contingent commissions discount factor is lower than those used in the claims discount. In the DF&C case, the claims development margin was judgmentally set at 5.0 percent, keeping in mind that the impact of the contingent commissions on the resulting EQUP is not significant.

The maximum allowable DPAE, after discounting and subject to the limitation of 30 percent of the total unearned premium, is calculated as the difference between the unearned premium reserve and the sum of the discounted losses and expenses.

9 Gross Calculations

The appointed actuary also must provide an opinion on the gross unearned premium provision, gross DPAE and deferred reinsurance commission, and the gross statutory premium deficiency. The same calculations described earlier must be performed on a gross basis.

The considerations and assumptions used to perform EQUP calculations on a gross basis are similar in most respects to those used for the net calculations described in the previous two sections. This section focuses on the differences and on the issues related to gross calculations.

9.1 Overall Calculations

Exhibit 4 illustrates the calculations needed to derive equity in the gross unearned premiums. It is similar in many respects to Exhibit 1, although there are a number of differences worth noting.

Additional Lines of Business: Insurance companies can act as fronting companies. (They write the business and cede it to the other party.) Companies with low acquisition expenses could follow that strategy when they expect the ceding commissions to outweigh the costs incurred to underwrite the business. Whatever the rationale, the fronting company, even though it has ceded the business to a third party, remains liable to the insureds should the third party go bankrupt or default on its obligations to indemnify the cedant under the agreement. As such, the gross claims provision needs to account for this liability and, therefore, the calculations underlying the equity in the gross unearned premium should include the additional exposures.

An extra line of business appears on Exhibit 4 to account for the fact that DF&C acts as a fronting company for TupolevInsure (TvI). The undiscounted expected loss ratios should be derived in a manner consistent with the approach described above, using, if possible, the historical loss experience.

The rate used to discount aviation expected claims theoretically should be derived by considering the projected return on TvI's assets and other factors described earlier. This is rarely practical, however, and the returns generated on DF&C's assets are used instead. This is generally a reasonable proxy. The same can be said of the interest rate margin, which should be selected based on TvI's portfolio, but instead is chosen by considering DF&C's portfolio. The claims development margin should reflect the uncertainty of the LOB; the reinsurance recovery margin does not apply.

Maintenance Expenses: Even though the insurer cedes part or all of a policy, it is still responsible for servicing and maintaining the inforce policy. This also holds true for aviation policies underwritten through a fronting agreement. Hence, in order to yield the same expense provision, the maintenance expense ratio will be a lower proportion of the gross unearned premium than it is of the net unearned premium.

Internal Adjustment Expenses: Typically, internal adjustment expenses are not subject to reinsurance and cost the same to the insurer on both gross and net bases. The IAE loading will be a higher proportion of the net unearned premium than it is of the gross unearned premium in order to yield the same IAE provision.

For those less frequent treaties that allow insurers to cede part of their internal adjustment expenses, the IAE ratio will be lower than in the circumstances above and will depend on how many IAE are ceded. Both gross and net loadings could be equal in cases where these expenses are ceded on a quota-share basis.

Discounting: The selected paid loss development factors are not usually the same for gross and net bases. DF&C has a $250,000 excess-of-loss treaty protecting its GL exposures. The gross payment pattern could be longer than the net pattern due to the fact that DF&C stops paying claims once they exceed $250,000. Also, there is no need for the reinsurance recovery PFAD when discounting gross policy liabilities.

9.2 The Discounting Paradigm

The previous subsection highlights the major differences between gross and net calculations. This subsection will briefly discuss a conceptual problem that arises from discounting gross policy liabilities.

As seen before, the discount rate used on a net basis reflects the insurer's projected rate of return, its method of reporting investment return and valuing assets, etc. When selecting a discount rate for gross calculations, the actuary effectively selects a discount rate for the ceded business, which is added to the net business to produce gross figures. Hence, the actuary is implicitly required to make assumptions about the reinsurer's investment portfolio, returns and valuation methods. Although this is conceptually problematic, it often will be reasonable to use the same discount rate on both gross and net bases even though the actuary has little or no knowledge of the reinsurer's investment returns.

In a similar fashion, although the interest rate margin should be based on the reinsurer's portfolio, it often will be reasonable to assume the same margin as the one used for net calculations. On the other hand, the claims development margin could differ between net and gross bases. Under the $250,000 GL excess-of-loss treaty mentioned previously, ceded losses are expected to be more volatile than net losses. In this case, claims development margins used in discounting gross policy liabilities should be at least as high as those used to discount net policy liabilities. If reinsurance were proportional, the claims development margins would be equal under both gross and net bases.

GL exposures are protected under a $250,000 XOL treaty. The gross claims development margin has been set at 15.0 percent, which is higher than the 12.5 percent margin used on a net basis. On the other hand, the proportional treaty under which DF&C cedes 25 percent of its premium (for all LOB) does not warrant selecting different claims development margins for gross discounting calculations.

10 Assumed Business

This section will focus on issues and considerations that arise from situations where the insurer participates in pools and associations or assumes business from other companies. More specifically:

- Facility association and other residual markets; and
- Intercompany reinsurance arrangements.

Under each of these situations, the insurer assumes business from a third party. Although different in nature, a number of analogies can be established between considerations related to ceded business and those that the actuary needs to take into account when factoring in the impact of assumed business on EQUP calculations.

10.1 Facility Association and other Residual Markets

Premiums and claims written by FA and other residual market pools are shared among insurers, also based on each insurer's total market share. Administrative expenses are reimbursed to the carriers, subject to certain limits. Part of the claims expenses also can be refunded.[22]

These pools typically provide participating insurers with a report that indicates the unpaid claims provision and the unearned premium reserve. The selected loss ratio and the discount factor used by the pool's actuary, in connection with his or her year-end valuation of the pool's liabilities, to calculate EQUP are provided to participating insurers. In addition, the pool's actuary provides those insurers with his/her estimates of the pool's premium deficiency. In his/her policy liability report, the insurer's actuary should disclose that he/she has relied on assumptions made by the pool's actuary.

The 92.6 percent loss ratio shown in Exhibit 1 is already discounted and was provided by the pool's actuary. An actuary also could perform a separate calculation instead of using the figure provided by the pool.

[22]For example, the PRR states that "insurers are also entitled to a full reimbursement of outside settlement expenses they have paid on transferred risks, except those expenses relating to claim adjusters; Insurers are however entitled to the reimbursement of fees paid to claim adjusters retained to make the original appraisal of a claim involving bodily injury covered under an Automobile Third Party Liability policy, or to make a supplemental appraisal in exceptional circumstances where an inadmissible or fraudulent claim is suspected, or to uphold the original appraisal of the claim against a formal contestation." Under RSP the allowance is calculated on the basis of the insurer's last approved private passenger automobile rate filing, subject to a maximum.

10.2 Intercompany Reinsurance Arrangements

Intercompany reinsurance arrangements are similar to ceding reinsurance, but to an affiliate or a parent company. They can take many forms. Our example will focus on DF&C's arrangement, which is analogous to proportional reinsurance. Considerations raised by including these arrangements in EQUP calculations are best understood by examining the DF&C example.

Under the agreement, DF&C assumes 25 percent of KIC's exposures (net of any other reinsurance). This increases DF&C's gross unearned premium reserve by $4,250,000. The selected undiscounted loss ratio of 72.5 percent and the 0.931 discount are identical to those used by KIC's actuary in his/her own EQUP calculations. The KIC actuary may use (but he/she is not required to) the same assumptions as used in DF&C calculations when including the exposures KIC is assuming from DF&C. The agreement also will specify if other items such as IAE and maintenance expenses are subject to cession by the parties. Computations of these items should follow the same process.

11 Closing Comments

As our paper illustrates, estimating policy liabilities encompasses much more than calculating the adequacy of the pro-rata unearned premiums in relation to deferred policy acquisition expenses. It consists of examining all assets and liabilities related to an insurer's insurance and reinsurance contracts and ensuring that these assets and liabilities make proper provisions to cover the obligations other than claim liabilities on the contracts. Our approach attempts to address all relevant causes. There may be circumstances particular to some insurers that may necessitate variations in the approach.

We hope this paper has achieved one of our goals, which is to generate more interest in this topic so that eventually more work will be done in developing or refining actuarial approaches to evaluating premium liabilities.

References

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Exhibit 1
Dubois Fire & Casualty Insurance Company
Equity in Net Unearned Premium Reserve as of December 31, 1997 ($000s)

A. Claims and External Adjustment Expense Data

Line of Business	Net Unearned Premium (a)	Estimated Ultimate Loss Ratio (b) 1995	1996	1997	Selected Undiscounted Loss Ratio (b)	Discount Factor	Discounted Loss Ratio (c)
Auto - Third Party Liability	1,500	0.656	0.681	0.740	0.725	0.960	0.696
Auto - Accident Benefits	2,100	0.958	0.944	0.870	0.900	0.858	0.772
Auto - Physical Damage	2,700	0.620	0.636	0.650	0.650	1.039	0.676
Auto - Total	6,300	0.741	0.749	0.745	0.751	0.949	0.713
Personal Property	600	0.667	0.641	0.594	0.600	1.060	0.636
Liability	300	0.886	0.860	0.978	0.950	0.985	0.936
(1)Total - Voluntary Business	7,200	0.741	0.745	0.742	0.747	0.958	0.716
(1a)Facility	350				0.926	1.000	0.926
(1b)Assumed from KIC	4,250				0.725	0.931	0.675

B. Actual Data Other Than Claims

	1995	1996	1997	Selected
(2)Earned Premiums - Voluntary Business (a)	19,487	22,543	24,546	
(3)Maintenance Exp. [1/3 of Gen. Exp.] (a)	521	540	580	
(4)Maintenance Expense Ratio [(3) / (2)]	2.7%	2.4%	2.4%	2.5%
(5)Selected Internal Adjustment Expense Ratio (d)				3.5%
(6)Contingent Commission Ratio (e)				0.2%

Exhibit 1 (cont.)
Dubois Fire & Casualty Insurance Company
Equity in Net Unearned Premium Reserve as of December 31, 1997 ($000s)

C. Equity in Unearned Premium Reserve

		Undiscounted	Discounted
(7)	Unearned Premiums – Voluntary Business [(1)]	7,200	7,200
(7a)	Unearned Premiums - Facility Association [(1a)]	350	350
(7b)	Unearned Premiums – Assumed from KIC [(1b)]	4,250	4,250
(8)	Expected Claims & ALAE – Voluntary Business [(7) × (1) disc.]	5,378	5,152
(8a)	Expected Claims & ALAE – Facility Association [(7a) × (1a)]	324	324
(8b)	Expected Claims & ALAE – Assumed from KIC [(7b) × (1b)]	3,081	2,869
(9)	Maintenance Expenses (f)	286	280
(10)	Internal Adjustment Expenses [(5) × (8)] + [2.7% × (8b)] (g)	271	258
(11)	Contingent Commissions [(6) × (7)]	14	14
(12)	Cost of Excess-of-Loss (h)	1,291	1,537
(13)	Equity in Unearned Premium Reserve (i)	4	4
(14)	Actual Deferred Policy Acquisition Expenses (a)	2,441	2,900
(15)	Statutory Premium Deficiency (j)	1,510	1,510

(a) From DF&C
(b) From Appendix A, Rows (16) and (17)
(c) From Appendix C, Row (17)
(d) From DF&C Policy Liabilities Report as of December 31, 1997
(e) From P&C-1, Page 80.10, Row 83
(f) (4) x [(7) + (7b)] × [Discounted: Appendix C, Row (16)]
(g) KIC's actuary uses a 2.7% IAE ratio
(h) Based on 1.25% of Subject Written Premiums
(i) [(7) + (7a) + (7b) - (8) - (8a) - (8b) - (9) - (10) - (11) - (12)]
(j) Max [(14) - (13) Undiscounted, 0]

Exhibit 2
Dubois Fire & Casualty Insurance Company
Selection of Net Loss Ratios
Auto – Third Party Liability ($000s)

	1995	1996	1997
(1) Earned Premiums (a)	3,413	3,823	4,013
(2) On-Level Factors (b)	1.321	1.342	1.078
(3) Drift Factors (c)	1.004	1.002	1.000
(4) Ultimate Premium [(1)×(2)×(3)]	4,529	5,140	4,328
(5) Incurred Losses (a)	2,482	3,300	2,454
(6) Incurred CAT Losses (a)	-	435	-
(7) Incurred Normal Losses [(5)-(6)]	2,482	2,865	2,454
(8) Loss Development Factor (d)	1.130	1.210	1.315
(9) Trend Factor (e)	1.068	1.017	1.000
(10) Other Adjustment Factors (f)	1.000	1.000	1.000
(11) Projected Ultimate Losses [(7)×(8)×(9)×(10)]	2,994	3,524	3,227
(12) Projected Loss Ratio [(11)/(4)]	66.1%	68.6%	74.6%
(13) CAT Loading (g)	0.3%	0.3%	0.3%
(14) Projected Smoothed Loss Ratio (12)×[1+(13)]	66.3%	68.8%	74.8%
(15) Seasonality Adjustment (h)	0.990	0.990	0.990
(16) Adjusted Loss Ratio [(14)×(15)]	65.6%	68.1%	74.0%
(17) Selected Loss Ratio (g)	72.5%		

Notes:
(a) From DF&C
(b) From DF&C's rate change history, using the parallelogram method
(c) Limit drift from Table E2, column (5)
(d) From DF&C's policy liabilities @12/31/97
(e) From Table E1, column (7)
(f) Estimated impact of Bill 59
(g) Judgmentally selected
(h) From Table B1, row (7)

Exhibit 3
Dubois Fire & Casualty Insurance Company
Discounting of Loss Ratios on Unearned Premium
December 31, 1997

Evaluation Point in Months (1)	Selected Age to Age Factors (a) (2)	Age to Ultimate Factors (a) (3)	Estimated Percentage Paid [1/(3)] (4)	Incremental Percentage Paid (5)	Discount Factor to Average Accident Date (b) (6)	Discounted Percentage Paid (5)×(6) (7)
Auto – Third Party Liability						
12	2.275	4.349	22.99%	22.99%	0.983	22.61%
24	1.180	1.912	52.31%	29.32%	0.935	27.40%
36	1.035	1.620	61.72%	9.42%	0.873	8.22%
48	1.027	1.565	63.88%	2.16%	0.816	1.76%
60	1.035	1.524	65.61%	1.72%	0.763	1.32%
72	1.035	1.473	67.90%	2.30%	0.717	1.65%
84	1.045	1.423	70.28%	2.38%	0.683	1.62%
96	1.050	1.362	73.44%	3.16%	0.651	2.06%
108	1.050	1.297	77.12%	3.67%	0.620	2.28%
120	1.042	1.235	80.97%	3.86%	0.590	2.28%
132	---	1.000	100.00%	19.03%	0.562	10.70%
Total				100.00%		81.89%
(8) Selected Undiscounted Loss Ratio (c)						72.5%
(9) Ratio of Expected Ceded Claims to Net UPR (d)						88.6%
(10) Reinsurance PFAD (e)						5.0%
(11) Reinsurance Recovery Margin [(9)×(10)]						4.4%
(12) Selected Claim Development Margin Factor (e)						12.5%
(13) Loss Ratio with Margin Discounted to Average Accident Date (f)						70.4%
(14) Average Earning Period for UPR (g)						4
(15) Percentage of Unearned Premium in Invested Assets (h)						50.0%
(16) Discount from Average Accident Date to Evaluation Date (g)						0.978
(17) Discounted Loss Ratio with Margins (13)×[1- (15)×{1-(16)}]						69.6%

Notes:
(a) Payment pattern from DF&C's paid triangles
(b) Yield rate from DF&C investment returns; three month payment lag in the first year
(c) From Exhibit I – Auto – Third Party Liability
(d) From Exhibits I and IV [(Gross UPR×Gross LR)-(Net UPR×Net LR)]/Net UPR
(e) Judgmentally selected based on CIA standards of practice on PFAD
(f) [Total for Column (7)]×[(8)×{1+(12)}+(11)]
(g) Assumptions: UPR is discounted four months, assuming 12 month policies
(h) From DF&C P&C-1: (Unearned Premium-Premium Receivables)/Unearned Premium

Exhibit 4
Dubois Fire & Casualty Insurance Company
Equity in Gross Unearned Premium Reserve as of December 31, 1997 ($000s)

A. Claims and External Adjustment Expense Data

Line of Business	Gross Unearned Premium (a)	Estimated Ultimate Loss Ratio (b) 1995	1996	1997	Selected Undiscounted Loss Ratio (b)	Discount Factor	Discounted Loss Ratio (c)
Auto - Third Party Liability	3,333	0.656	0.681	0.740	0.725	0.911	0.660
Auto - Accident Benefits	4,667	0.950	0.937	0.863	0.900	0.812	0.731
Auto - Physical Damage	6,000	0.620	0.636	0.650	0.650	0.982	0.638
Auto – Total	14,000	0.739	0.747	0.743	0.751	0.898	0.675
Personal Property	1,333	0.666	0.640	0.593	0.600	1.002	0.601
Liability	667	0.881	0.855	0.972	0.950	0.955	0.907
Aviation (g)	1,650	0.810	0.592	0.643	0.700	0.981	0.687
(1) Total - Voluntary Business	17,650	0.745	0.729	0.731	0.742	0.914	0.679
(1a) Facility	350				0.926	1.000	0.926
(1b) Assumed from KIC	4,250				0.725	0.931	0.675

Exhibit 4 (cont.)
Dubois Fire & Casualty Insurance Company
Equity in Gross Unearned Premium Reserve as of December 31, 1997 ($000s)

B. Equity in Unearned Premium Reserve

		Undiscounted	Discounted
(2)	Unearned Premiums - Voluntary Business [(1)]	17,650	17,650
(2a)	Unearned Premiums - Facility Association [(1a)]	350	350
(2b)	Unearned Premiums - Assumed from KIC [(1b)]	4,250	4,250
(3)	Expected Claims & ALAE - Voluntary Business [(7) × (1)]	13,105	11,984
(3a)	Expected Claims & ALAE - Facility Association [(7a) × (1a)]	324	324
(3b)	Expected Claims & ALAE - Assumed from KIC [(7b) × (1b)]	3,081	2,869
(4)	Maintenance Expenses (f)	286	280
(5)	Internal Adjustment Expenses [(5) × (8)] + [2.7% × (8b)] (g)	271	258
(6)	Contingent Commissions [(6) × (7)]	14	14
(7)	Equity in Unearned Premium Reserve (i)	5,168	6,522
(8)	Actual Deferred Policy Acquisition Expenses (a)	3,267	3,267
(9)	Statutory Premium Deficiency (j)		N/A

(a) From DF&C
(b) From Appendix A, Rows (16) and (17)
(c) From Appendix D, Row (14)
(d) From Exhibit I, Rows (9) through (11)
(e) [(2) + (2a) + (2b) - (3) - (3a) - (3b) - (4) - (5) - (6)]
(f) Max [(8) – (7), 0)
(g) Underwritten through DF&C's fronting agreement with TvI

Appendix A

Table A1
Dubois Fire & Casualty Insurance Company
Selection of Net Loss Ratios
Auto-Third Party Liability ($000s)

	1995	1996	1997
(1) Earned Premiums (a)	3,413	3,823	4,013
(2) On-Level Factors (b)	1.321	1.342	1.078
(3) Drift Factors (c)	1.004	1.002	1.000
(4) Ultimate Premium [(1)×(2)×(3)]	4,529	5,140	4,328
(5) Incurred Losses (a)	2,482	3,300	2,454
(6) Incurred CAT Losses (a)	-	435	-
(7) Incurred Normal Losses [(5)-(6)]	2,482	2,865	2,454
(8) Loss Development Factor (d)	1.130	1.210	1.315
(9) Trend Factor (e)	1.068	1.017	1.000
(10) Other Adjustment Factors (f)	1.000	1.000	1.000
(11) Projected Ultimate Losses [(7)×(8)×(9)×(10)]	2,994	3,524	3,227
(12) Projected Loss Ratio [(11)/(4)]	66.1%	68.6%	74.6%
(13) CAT Loading (g)	0.3%	0.3%	0.3%
(14) Projected Smoothed Loss Ratio (12)×[1+(13)]	66.3%	68.8%	74.8%
(15) Seasonality Adjustment (h)	0.990	0.990	0.990
(16) Adjusted Loss Ratio [(14)×(15)]	65.6%	68.1%	74.0%
(17) Selected Loss Ratio (g)	72.5%		

Notes:
(a) From DF&C
(b) From DF&C's rate change history, using the parallelogram method
(c) Limit drift from Table E2, column (5)
(d) From DF&C's policy liabilities @12/31/97
(e) From Table E1, column (7)
(f) Estimated impact of Bill 59
(g) Judgmentally selected
(h) From Table B1, row (7)

Table A2
Dubois Fire & Casualty Insurance Company
Selection of Net Loss Ratios
Auto – Accident Benefits
($000's)

	1995	1996	1997
(1) Earned Premiums (a)	4,631	6,245	7,499
(2) On-Level Factors (b)	1.026	0.857	0.954
(3) Drift Factors (c)	1.000	1.000	1.000
(4) Ultimate Premium [(1)×(2)×(3)]	4,751	5,350	7,153
(5) Incurred Losses (a)	3,001	3,432	3,888
(6) Incurred CAT Losses (a)	-	-	-
(7) Incurred Normal Losses [(5)-(6)]	3,001	3,432	3,888
(8) Loss Development Factor (d)	1.128	1.237	1.494
(9) Trend Factor (e)	1.358	1.202	1.083
(10) Other Adjustment Factors (f)	1.000	1.000	1.000
(11) Projected Ultimate Losses [(7)×(8)×(9)×(10)]	4,597	5,101	6,288
(12) Projected Loss Ratio [(11)/(4)]	96.8%	95.4%	87.9%
(13) CAT Loading (g)	0.0%	0.0%	0.0%
(14) Projected Smoothed Loss Ratio (12)×[1+(13)]	96.8%	95.4%	87.9%
(15) Seasonality Adjustment (h)	0.990	0.990	0.990
(16) Adjusted Loss Ratio [(14)×(15)]	95.8%	94.4%	87.0%
(17) Selected Loss Ratio (g)	90.0%		

Notes:
(a) From DF&C
(b) From DF&C's rate change history, using the parallelogram method
(c) Limit drift from Table E2, column (5)
(d) From DF&C's policy liabilities @12/31/97
(e) From Table E1, column (7)
(f) Estimated impact of Bill 59
(g) Judgmentally selected
(h) From Table B1, row (7)

Table A3
Dubois Fire & Casualty Insurance Company
Selection of Net Loss Ratios
Auto- Physical Damage ($000s)

	1995	1996	1997
(1) Earned Premiums (a)	7,501	8,211	8,464
(2) On-Level Factors (b)	0.950	0.951	0.986
(3) Drift Factors (c)	1.007	1.004	1.001
(4) Ultimate Premium [(1)×(2)×(3)]	7,172	7,835	8,347
(5) Incurred Losses (a)	4,411	5,226	5,914
(6) Incurred CAT Losses (a)	-	225	525
(7) Incurred Normal Losses [(5)-(6)]	4,411	5,001	5,389
(8) Loss Development Factor (d)	1.000	0.999	1.012
(9) Trend Factor (e)	1.013	1.003	1.000
(10) Other Adjustment Factors (f)	1.000	1.000	1.000
(11) Projected Ultimate Losses [(7)×(8)×(9)×(10)]	4,470	5,013	5,454
(12) Projected Loss Ratio [(11)/(4)]	62.3%	64.0%	65.3%
(13) CAT Loading (g)	0.5%	0.5%	0.5%
(14) Projected Smoothed Loss Ratio (12)×[1+(13)]	62.6%	64.3%	65.7%
(15) Seasonality Adjustment (h)	0.990	0.990	0.990
(16) Adjusted Loss Ratio [(14)×(15)]	62.0%	63.6%	65.0%
(17) Selected Loss Ratio (g)	65.0%		

Notes:
(a) From DF&C
(b) From DF&C's rate change history, using the parallelogram method
(c) Limit drift from Table E2, column (5)
(d) From DF&C's policy liabilities @12/31/97
(e) From Table E1, column (7)
(f) Estimated impact of Bill 59
(g) Judgmentally selected
(h) From Table B1, row (7)

Table A4
Dubois Fire & Casualty Insurance Company
Selection of Net Loss Ratios
Personal Property ($000s)

	1995	1996	1997
(1) Earned Premiums (a)	3,007	3,251	3,578
(2) On-Level Factors (b)	1.000	1.000	1.000
(3) Drift Factors (c)	1.000	1.000	1.000
(4) Ultimate Premium [(1)×(2)×(3)]	3,007	3,251	3,578
(5) Incurred Losses (a)	2,144	1,986	2,351
(6) Incurred CAT Losses (a)	263	-	411
(7) Incurred Normal Losses [(5)-(6)]	1,881	1,986	1,940
(8) Loss Development Factor (d)	0.992	0.991	1.050
(9) Trend Factor (e)	1.043	1.028	1.012
(10) Other Adjustment Factors (f)	1.000	1.000	1.000
(11) Projected Ultimate Losses [(7)×(8)×(9)×(10)]	1,946	2,023	2,062
(12) Projected Loss Ratio [(11)/(4)]	64.7%	62.2%	57.6%
(13) CAT Loading (g)	1.0%	1.0%	1.0%
(14) Projected Smoothed Loss Ratio (12)×[1+(13)]	65.4%	62.8%	58.2%
(15) Seasonality Adjustment (h)	1.020	1.020	1.020
(16) Adjusted Loss Ratio [(14)×(15)]	66.7%	64.1%	59.4%
(17) Selected Loss Ratio (g)	60.0%		

Notes:
(a) From DF&C
(b) From DF&C's rate change history, using the parallelogram method
(c) Limit drift from Table E2, column (5)
(d) From DF&C's policy liabilities @12/31/97
(e) From Table E1, column (7)
(f) Estimated impact of Bill 59
(g) Judgmentally selected
(h) From Table B1, row (7)

Table A5
Dubois Fire & Casualty Insurance Company
Selection of Net Loss Ratios
Liability ($000s)

	1995	1996	1997
(1) Earned Premiums (a)	935	1,013	992
(2) On-Level Factors (b)	1.000	1.000	1.000
(3) Drift Factors (c)	1.004	1.002	1.000
(4) Ultimate Premium [(1)×(2)×(3)]	939	1,015	992
(5) Incurred Losses (a)	642	652	592
(6) Incurred CAT Losses (a)	-	-	-
(7) Incurred Normal Losses [(5)-(6)]	642	652	592
(8) Loss Development Factor (d)	1.055	1.173	1.542
(9) Trend Factor (e)	1.227	1.142	1.062
(10) Other Adjustment Factors (f)	1.000	1.000	1.000
(11) Projected Ultimate Losses [(7)×(8)×(9)×(10)]	832	873	970
(12) Projected Loss Ratio [(11)/(4)]	88.6%	86.0%	97.8%
(13) CAT Loading (g)	0.0%	0.0%	0.0%
(14) Projected Smoothed Loss Ratio (12)×[1+(13)]	88.6%	86.0%	97.8%
(15) Seasonality Adjustment (h)	1.000	1.000	1.000
(16) Adjusted Loss Ratio [(14)×(15)]	88.6%	86.0%	97.8%
(17) Selected Loss Ratio (g)	95.0%		

Notes:

(a) From DF&C

(b) From DF&C's rate change history, using the parallelogram method

(c) Limit drift from Table E2, column (5)

(d) From DF&C's policy liabilities @12/31/97

(e) From Table E1, column (7)

(f) Estimated impact of Bill 59

(g) Judgmentally selected

(h) Judgmentally selected

Appendix B

Table B1
Dubois Fire & Casualty Insurance Company
Seasonality Adjustment Factor
Automobile - All Lines

Month (1)	Monthly Loss Ratios (a) (2)	Unearned Premium Weight (b) (3)	Earned Premium Weight (4)
January	88.0%	0.958	1.000
February	86.4%	0.875	1.000
March	81.5%	0.792	1.000
April	74.3%	0.708	1.000
May	68.1%	0.625	1.000
June	70.1%	0.542	1.000
July	76.7%	0.458	1.000
August	82.2%	0.375	1.000
September	77.4%	0.292	1.000
October	79.3%	0.208	1.000
November	88.8%	0.125	1.000
December	92.2%	0.042	1.000
(5) Average Loss Ratio on the Unearned Premium (c)			79.6%
(6) Average Loss Ratio on the Earned Premium (d)			80.4%
(7) Seasonality Adjustment [(5)/(6)]			0.990

Notes:
(a) From DF&C, based on latest three accident years experience
(b) Based on the 24th method
(c) Weighted average of columns (2) and (3)
(d) Weighted average of columns (2) and (4)

Table B2
Dubois Fire & Casualty Insurance Company
Seasonality Adjustment Factor
Property

Month (1)	Monthly Loss Ratios (a) (2)	Unearned Premium Weight (b) (3)	Earned Premium Weight (4)
January	69.1%	0.958	1.000
February	66.4%	0.875	1.000
March	62.9%	0.792	1.000
April	61.1%	0.708	1.000
May	59.4%	0.625	1.000
June	57.5%	0.542	1.000
July	54.3%	0.458	1.000
August	52.1%	0.375	1.000
September	55.9%	0.292	1.000
October	59.4%	0.208	1.000
November	60.6%	0.125	1.000
December	64.8%	0.042	1.000
(5) Average Loss Ratio on the Unearned Premium (c)			61.5%
(6) Average Loss Ratio on the Earned Premium (d)			60.3%
(7) Seasonality Adjustment [(5)/(6)]			1.020

Notes:
(a) From DF&C, based on latest three accident years experience
(b) Based on the 24th method
(c) Weighted average of columns (2) and (3)
(d) Weighted average of columns (2) and (4)

Appendix C

Notes for Appendix C:
(a) Payment pattern from DF&C's paid triangles
(b) Yield rate from DF&C investment returns; three month payment lag in the first year
(c) From Exhibit I
(d) From Exhibits I and IV [(Gross UPR×Gross LR)-(Net UPR×Net LR)]/Net UPR
(e) Judgmentally selected based on CIA standards of practice on PFAD
(f) [Total for Column (7)]×[(8)×{1+(12)}+(11)]
(g) Assumptions: UPR is discounted four months, assuming 12 month policies
(h) From DF&C P&C-1: (Unearned Premium-Premium Receivables)/Unearned Premium

Table C1
Dubois Fire & Casualty Insurance Company
Discounting of Net Premium Liabilities
Discounted Loss Ratios on the Unearned Premium
December 31, 1997

Evaluation Point in Months (1)	Selected Age to Age Factors (a) (2)	Age to Ultimate Factors (a) (3)	Estimated Percentage Paid [1/(3)] (4)	Incremental Percentage Paid (5)	Discount Factor to Average Accident Date (b) (6)	Discounted Percentage Paid (5)×(6) (7)
Auto - Third Party Liability						
12	2.275	4.349	22.99%	22.99%	0.983	22.61%
24	1.180	1.912	52.31%	29.32%	0.935	27.40%
36	1.035	1.620	61.72%	9.42%	0.873	8.22%
48	1.027	1.565	63.88%	2.16%	0.816	1.76%
60	1.035	1.524	65.61%	1.72%	0.763	1.32%
72	1.035	1.473	67.90%	2.30%	0.717	1.65%
84	1.045	1.423	70.28%	2.38%	0.683	1.62%
96	1.050	1.362	73.44%	3.16%	0.651	2.06%
108	1.050	1.297	77.12%	3.67%	0.620	2.28%
120	1.042	1.235	80.97%	3.86%	0.590	2.28%
132	---	1.000	100.00%	19.03%	0.562	10.70%
Total				100.00%		81.89%
(8) Selected Undiscounted Loss Ratio (c)						72.5%
(9) Ratio of Expected Ceded Claims to Net UPR (d)						88.6%
(10) Reinsurance PFAD (e)						5.0%
(11) Reinsurance Recovery Margin [(9)×(10)]						4.4%
(12) Selected Claim Development Margin Factor (e)						12.5%
(13) Loss Ratio with Margin Discounted to Average Accident Date (f)						70.4%
(14) Average Earning Period for UPR (g)						4
(15) Percentage of Unearned Premium in Invested Assets (h)						50.0%
(16) Discount from Average Accident Date to Evaluation Date (g)						0.978
(17) Discounted Loss Ratio with Margins (13)×[1- (15)×{1-(16)}]						69.6%

Table C2
Dubois Fire & Casualty Insurance Company
Discounting of Net Premium Liabilities
Discounted Loss Ratios on the Unearned Premium
December 31, 1997

Evaluation Point in Months (1)	Selected Age to Age Factors (a) (2)	Age to Ultimate Factors (a) (3)	Estimated Percentage Paid [1/(3)] (4)	Incremental Percentage Paid (5)	Discount Factor to Average Accident Date (b) (6)	Discounted Percentage Paid (5)×(6) (7)
Auto - Accident Benefits						
12	4.000	21.863	4.57%	4.57%	0.983	4.50%
24	1.850	5.466	18.30%	13.72%	0.935	12.82%
36	1.300	2.954	33.85%	15.55%	0.873	13.58%
48	1.180	2.273	44.00%	10.15%	0.816	8.29%
60	1.130	1.926	51.92%	7.92%	0.763	6.04%
72	1.090	1.704	58.67%	6.75%	0.717	4.84%
84	1.070	1.564	63.95%	5.28%	0.683	3.61%
96	1.060	1.461	68.43%	4.48%	0.651	2.91%
108	1.050	1.379	72.53%	4.11%	0.620	2.54%
120	1.045	1.313	76.16%	3.63%	0.590	2.14%
132	---	1.000	100.00%	23.84%	0.562	13.40%
Total				100.00%		74.69%

(8) Selected Undiscounted Loss Ratio (c)	90.0%
(9) Ratio of Expected Ceded Claims to Net UPR (d)	110.0%
(10) Reinsurance PFAD (e)	5.0%
(11) Reinsurance Recovery Margin [(9)×(10)]	5.5%
(12) Selected Claim Development Margin Factor (e)	10.0%
(13) Loss Ratio with Margin Discounted to Average Accident Date (f)	78.0%
(14) Average Earning Period for UPR (g)	4
(15) Percentage of Unearned Premium in Invested Assets (h)	50.0%
(16) Discount from Average Accident Date to Evaluation Date (g)	0.978
(17) Discounted Loss Ratio with Margins (13)×[1- (15)×{1-(16)}]	77.2%

Table C3
Dubois Fire & Casualty Insurance Company
Discounting of Net Premium Liabilities
Discounted Loss Ratios on the Unearned Premium
December 31, 1997

Evaluation Point in Months (1)	Selected Age to Age Factors (a) (2)	Age to Ultimate Factors (a) (3)	Estimated Percentage Paid [1/(3)] (4)	Incremental Percentage Paid (5)	Discount Factor to Average Accident Date (b) (6)	Discounted Percentage Paid (5)×(6) (7)
Auto - Physical Damage						
12	2.250	2.555	39.14%	39.14%	0.983	38.48%
24	1.130	1.136	88.05%	48.92%	0.935	45.72%
36	1.004	1.005	99.50%	11.45%	0.873	10.00%
48	1.001	1.001	99.90%	0.40%	0.816	0.32%
60	1.000	1.000	100.00%	0.10%	0.763	0.08%
72	1.000	1.000	100.00%	0.00%	0.717	0.00%
84	1.000	1.000	100.00%	0.00%	0.683	0.00%
96	1.000	1.000	100.00%	0.00%	0.651	0.00%
108	1.000	1.000	100.00%	0.00%	0.620	0.00%
120	1.000	1.000	100.00%	0.00%	0.590	0.00%
132	---	1.000	100.00%	0.00%	0.562	0.00%
Total				100.00%		94.60%

(8) Selected Undiscounted Loss Ratio (c)	65.0%
(9) Ratio of Expected Ceded Claims to Net UPR (d)	79.4%
(10) Reinsurance PFAD (e)	5.0%
(11) Reinsurance Recovery Margin [(9)×(10)]	4.0%
(12) Selected Claim Development Margin Factor (e)	5.0%
(13) Loss Ratio with Margin Discounted to Average Accident Date (f)	68.3%
(14) Average Earning Period for UPR (g)	4
(15) Percentage of Unearned Premium in Invested Assets (h)	50.0%
(16) Discount from Average Accident Date to Evaluation Date (g)	0.978
(17) Discounted Loss Ratio with Margins (13)×[1- (15)×{1-(16)}]	67.6%

Table C4
Dubois Fire & Casualty Insurance Company
Discounting of Net Premium Liabilities
Discounted Loss Ratios on the Unearned Premium
December 31, 1997

Evaluation Point in Months (1)	Selected Age to Age Factors (a) (2)	Age to Ultimate Factors (a) (3)	Estimated Percentage Paid [1/(3)] (4)	Incremental Percentage Paid (5)	Discount Factor to Average Accident Date (b) (6)	Discounted Percentage Paid (5)×(6) (7)
Personal Property						
12	1.375	1.420	70.45%	70.45%	0.983	69.27%
24	1.014	1.032	96.86%	26.42%	0.935	24.69%
36	1.008	1.018	98.22%	1.36%	0.873	1.18%
48	1.005	1.010	99.01%	0.79%	0.816	0.64%
60	1.002	1.005	99.50%	0.50%	0.763	0.38%
72	1.001	1.003	99.70%	0.20%	0.717	0.14%
84	1.002	1.002	99.80%	0.10%	0.683	0.07%
96	1.000	1.000	100.00%	0.20%	0.651	0.13%
108	1.000	1.000	100.00%	0.00%	0.620	0.00%
120	1.000	1.000	100.00%	0.00%	0.590	0.00%
132	---	1.000	100.00%	0.00%	0.562	0.00%
Total				100.00%		96.50%
(8) Selected Undiscounted Loss Ratio (c)						60.0%
(9) Ratio of Expected Ceded Claims to Net UPR (d)						73.3%
(10) Reinsurance PFAD (e)						5.0%
(11) Reinsurance Recovery Margin [(9)×(10)]						3.7%
(12) Selected Claim Development Margin Factor (e)						5.0%
(13) Loss Ratio with Margin Discounted to Average Accident Date (f)						64.3%
(14) Average Earning Period for UPR (g)						4
(15) Percentage of Unearned Premium in Invested Assets (h)						50.0%
(16) Discount from Average Accident Date to Evaluation Date (g)						0.978
(17) Discounted Loss Ratio with Margins (13)×[1- (15)×{1-(16)}]						63.6%

Table C5
Dubois Fire & Casualty Insurance Company
Discounting of Net Premium Liabilities
Discounted Loss Ratios on the Unearned Premium
December 31, 1997

Evaluation Point in Months (1)	Selected Age to Age Factors (a) (2)	Age to Ultimate Factors (a) (3)	Estimated Percentage Paid [1/(3)] (4)	Incremental Percentage Paid (5)	Discount Factor to Average Accident Date (b) (6)	Discounted Percentage Paid (5)×(6) (7)
Liability						
12	2.350	6.984	14.32%	14.32%	0.983	14.08%
24	1.500	2.972	33.65%	19.33%	0.935	18.07%
36	1.405	1.981	50.47%	16.82%	0.873	14.69%
48	1.150	1.410	70.91%	20.44%	0.816	16.69%
60	1.075	1.226	81.55%	10.64%	0.763	8.12%
72	1.050	1.141	87.67%	6.12%	0.717	4.39%
84	1.040	1.086	92.05%	4.38%	0.683	3.00%
96	1.025	1.045	95.73%	3.68%	0.651	2.40%
108	1.010	1.019	98.13%	2.39%	0.620	1.48%
120	1.009	1.009	99.11%	0.98%	0.590	0.58%
132	---	1.000	100.00%	0.89%	0.562	0.50%
Total				100.00%		83.98%

(8)	Selected Undiscounted Loss Ratio (c)	95.0%
(9)	Ratio of Expected Ceded Claims to Net UPR (d)	116.1%
(10)	Reinsurance PFAD (e)	5.0%
(11)	Reinsurance Recovery Margin [(9)×(10)]	5.8%
(12)	Selected Claim Development Margin Factor (e)	12.5%
(13)	Loss Ratio with Margin Discounted to Average Accident Date (f)	94.6%
(14)	Average Earning Period for UPR (g)	4
(15)	Percentage of Unearned Premium in Invested Assets (h)	50.0%
(16)	Discount from Average Accident Date to Evaluation Date (g)	0.978
(17)	Discounted Loss Ratio with Margins (13)×[1- (15)×{1-(16)}]	93.6%

Appendix D

Table D1
Dubois Fire & Casualty Insurance Company
Discounting of Gross Premium Liabilities
Discounted Loss Ratios on the Unearned Premium
December 31, 1997

Evaluation Point in Months (1)	Selected Age to Age Factors (a) (2)	Age to Ultimate Factors (a) (3)	Estimated Percentage Paid [1/(3)] (4)	Incremental Percentage Paid (5)	Discount Factor to Average Accident Date (b) (6)	Discounted Percentage Paid (5)×(6) (7)
Auto - Third Party Liability						
12	2.275	4.349	22.99%	22.99%	0.983	22.61%
24	1.180	1.912	52.31%	29.32%	0.935	27.40%
36	1.035	1.620	61.72%	9.42%	0.873	8.22%
48	1.027	1.565	63.88%	2.16%	0.816	1.76%
60	1.035	1.524	65.61%	1.72%	0.763	1.32%
72	1.035	1.473	67.90%	2.30%	0.717	1.65%
84	1.045	1.423	70.28%	2.38%	0.683	1.62%
96	1.050	1.362	73.44%	3.16%	0.651	2.06%
108	1.050	1.297	77.12%	3.67%	0.620	2.28%
120	1.042	1.235	80.97%	3.86%	0.590	2.28%
132	---	1.000	100.00%	19.03%	0.562	10.70%
Total				100.00%		81.89%
(8) Selected Undiscounted Loss Ratio (c)						72.5%
(9) Selected Claim Development Margin Factor (d)						12.5%
(10) Loss Ratio with Margin Discounted to Average Accident Date						66.8%
(11) Average Earning Period for UPR (f)						4
(12) Percentage of Unearned Premium in Invested Assets (g)						50.0%
(13) Discount from Average Accident Date to Evaluation Date (f)						0.978
(14) Discounted Loss Ratio with Margins (13)×[1- (15)×{1-(16)}]						66.0%

Notes:

(a) Payment pattern from paid triangles in appendices

(b) Yield rate from DF&C investment returns; three month payment lag in the first year

(c) From Exhibit IV

(d) Judgmentally selected based on CIA standards of practice on PFAD

(e) [Total for Column (7)]×[(8)×[1+(9)]

(f) Assumptions: UPR is discounted four months, assuming 12 month policies

(g) From DF&C P&C-1: (Unearned Premium - Premium Receivables)/Unearned Premium

Table D2
Dubois Fire & Casualty Insurance Company
Discounting of Gross Premium Liabilities
Discounted Loss Ratios on the Unearned Premium
December 31, 1997

Evaluation Point in Months (1)	Selected Age to Age Factors (a) (2)	Age to Ultimate Factors (a) (3)	Estimated Percentage Paid [1/(3)] (4)	Incremental Percentage Paid (5)	Discount Factor to Average Accident Date (b) (6)	Discounted Percentage Paid (5)×(6) (7)
Auto - Accident Benefits						
12	4.000	21.863	4.57%	4.57%	0.983	4.50%
24	1.850	5.466	18.30%	13.72%	0.935	12.82%
36	1.300	2.954	33.85%	15.55%	0.873	13.58%
48	1.180	2.273	44.00%	10.15%	0.816	8.29%
60	1.130	1.926	51.92%	7.92%	0.763	6.04%
72	1.090	1.704	58.67%	6.75%	0.717	4.84%
84	1.070	1.564	63.95%	5.28%	0.683	3.61%
96	1.060	1.461	68.43%	4.48%	0.651	2.91%
108	1.050	1.379	72.53%	4.11%	0.620	2.54%
120	1.045	1.313	76.16%	3.63%	0.590	2.14%
132	---	1.000	100.00%	23.84%	0.562	13.40%
Total				100.00%		74.69%
(8) Selected Undiscounted Loss Ratio (c)						90.0%
(9) Selected Claim Development Margin Factor (d)						10.0%
(10) Loss Ratio with Margin Discounted to Average Accident Date						73.9%
(11) Average Earning Period for UPR (f)						4
(12) Percentage of Unearned Premium in Invested Assets (g)						50.0%
(13) Discount from Average Accident Date to Evaluation Date (f)						0.978
(14) Discounted Loss Ratio with Margins (13)×[1- (15)×{1-(16)}]						73.1%

Notes:

(a) Payment pattern from paid triangles in appendices

(b) Yield rate from DF&C investment returns; three month payment lag in the first year

(c) From Exhibit IV

(d) Judgmentally selected based on CIA standards of practice on PFAD

(e) [Total for Column (7)]×[(8)×[1+(9)]

(f) Assumptions: UPR is discounted four months, assuming 12 month policies

(g) From DF&C P&C-1: (Unearned Premium - Premium Receivables)/Unearned Premium

Table D3
Dubois Fire & Casualty Insurance Company
Discounting of Gross Premium Liabilities
Discounted Loss Ratios on the Unearned Premium
December 31, 1997

Evaluation Point in Months (1)	Selected Age to Age Factors (a) (2)	Age to Ultimate Factors (a) (3)	Estimated Percentage Paid [1/(3)] (4)	Incremental Percentage Paid (5)	Discount Factor to Average Accident Date (b) (6)	Discounted Percentage Paid (5)×(6) (7)
Auto – Physical Damage						
12	2.250	2.555	39.14%	39.14%	0.983	38.48%
24	1.130	1.136	88.05%	48.92%	0.935	45.72%
36	1.004	1.005	99.50%	11.45%	0.873	10.00%
48	1.001	1.001	99.90%	0.40%	0.816	0.32%
60	1.000	1.000	100.00%	0.10%	0.763	0.08%
72	1.000	1.000	100.00%	0.00%	0.717	0.00%
84	1.000	1.000	100.00%	0.00%	0.683	0.00%
96	1.000	1.000	100.00%	0.00%	0.651	0.00%
108	1.000	1.000	100.00%	0.00%	0.620	0.00%
120	1.000	1.000	100.00%	0.00%	0.590	0.00%
132	---	1.000	100.00%	0.00%	0.562	0.00%
Total				100.00%		94.60%
(8) Selected Undiscounted Loss Ratio (c)						65.0%
(9) Selected Claim Development Margin Factor (d)						5.0%
(10) Loss Ratio with Margin Discounted to Average Accident Date						64.6%
(11) Average Earning Period for UPR (f)						4
(12) Percentage of Unearned Premium in Invested Assets (g)						50.0%
(13) Discount from Average Accident Date to Evaluation Date (f)						0.978
(14) Discounted Loss Ratio with Margins (13)×[1- (15)×{1-(16)}]						63.8%

Notes:

(a) Payment pattern from paid triangles in appendices

(b) Yield rate from DF&C investment returns; three month payment lag in the first year

(c) From Exhibit IV

(d) Judgmentally selected based on CIA standards of practice on PFAD

(e) [Total for Column (7)]×[(8)×[1+(9)]

(f) Assumptions: UPR is discounted four months, assuming 12 month policies

(g) From DF&C P&C-1: (Unearned Premium – Premium Receivables)/Unearned Premium

Table D4
Dubois Fire & Casualty Insurance Company
Discounting of Gross Premium Liabilities
Discounted Loss Ratios on the Unearned Premium
December 31, 1997

Evaluation Point in Months (1)	Selected Age to Age Factors (a) (2)	Age to Ultimate Factors (a) (3)	Estimated Percentage Paid [1/(3)] (4)	Incremental Percentage Paid (5)	Discount Factor to Average Accident Date (b) (6)	Discounted Percentage Paid (5)×(6) (7)
Personal Property						
12	1.375	1.420	70.45%	70.45%	0.983	69.27%
24	1.014	1.032	96.86%	26.42%	0.935	24.69%
36	1.008	1.018	98.22%	1.36%	0.873	1.18%
48	1.005	1.010	99.01%	0.79%	0.816	0.64%
60	1.002	1.005	99.50%	0.50%	0.763	0.38%
72	1.001	1.003	99.70%	0.20%	0.717	0.14%
84	1.002	1.002	99.80%	0.10%	0.683	0.07%
96	1.000	1.000	100.00%	0.20%	0.651	0.13%
108	1.000	1.000	100.00%	0.00%	0.620	0.00%
120	1.000	1.000	100.00%	0.00%	0.590	0.00%
132	---	1.000	100.00%	0.00%	0.562	0.00%
Total				100.00%		96.50%
(8) Selected Undiscounted Loss Ratio (c)						60.0%
(9) Selected Claim Development Margin Factor (d)						5.0%
(10) Loss Ratio with Margin Discounted to Average Accident Date						60.8%
(11) Average Earning Period for UPR (f)						4
(12) Percentage of Unearned Premium in Invested Assets (g)						50.0%
(13) Discount from Average Accident Date to Evaluation Date (f)						0.978
(14) Discounted Loss Ratio with Margins (13)×[1- (15)×{1-(16)}]						60.1%

Notes:

(a) Payment pattern from paid triangles in appendices

(b) Yield rate from DF&C investment returns; three month payment lag in the first year

(c) From Exhibit IV

(d) Judgmentally selected based on CIA standards of practice on PFAD

(e) [Total for Column (7)]×[(8)×[1+(9)]

(f) Assumptions: UPR is discounted four months, assuming 12 month policies

(g) From DF&C P&C-1: (Unearned Premium - Premium Receivables)/Unearned Premium

Table D5
Dubois Fire & Casualty Insurance Company
Discounting of Gross Premium Liabilities
Discounted Loss Ratios on the Unearned Premium
December 31, 1997

Evaluation Point in Months (1)	Selected Age to Age Factors (a) (2)	Age to Ultimate Factors (a) (3)	Estimated Percentage Paid [1/(3)] (4)	Incremental Percentage Paid (5)	Discount Factor to Average Accident Date (b) (6)	Discounted Percentage Paid (5)×(6) (7)
Liability						
12	2.350	6.984	14.32%	14.32%	0.983	14.08%
24	1.500	2.972	33.65%	19.33%	0.935	18.07%
36	1.405	1.981	50.47%	16.82%	0.873	14.69%
48	1.150	1.410	70.91%	20.44%	0.816	16.69%
60	1.075	1.226	81.55%	10.64%	0.763	8.12%
72	1.050	1.141	87.67%	6.12%	0.717	4.39%
84	1.040	1.086	92.05%	4.38%	0.683	3.00%
96	1.025	1.045	95.73%	3.68%	0.651	2.40%
108	1.010	1.019	98.13%	2.39%	0.620	1.48%
120	1.009	1.009	99.11%	0.98%	0.590	0.58%
132	---	1.000	100.00%	0.89%	0.562	0.50%
Total				100.00%		83.98%
(8) Selected Undiscounted Loss Ratio (c)						95.0%
(9) Selected Claim Development Margin Factor (d)						15.0%
(10) Loss Ratio with Margin Discounted to Average Accident Date						91.8%
(11) Average Earning Period for UPR (f)						4
(12) Percentage of Unearned Premium in Invested Assets (g)						50.0%
(13) Discount from Average Accident Date to Evaluation Date (f)						0.978
(14) Discounted Loss Ratio with Margins (13)×[1- (15)×{1-(16)}]						90.7%

Notes:

(a) Payment pattern from paid triangles in appendices

(b) Yield rate from DF&C investment returns; three month payment lag in the first year

(c) From Exhibit IV

(d) Judgmentally selected based on CIA standards of practice on PFAD

(e) [Total for Column (7)]×[(8)×[1+(9)]

(f) Assumptions: UPR is discounted four months, assuming 12 month policies

(g) From DF&C P&C-1: (Unearned Premium – Premium Receivables)/Unearned Premium

Table D6
Dubois Fire & Casualty Insurance Company
Discounting of Gross Premium Liabilities
Discounted Loss Ratios on the Unearned Premium
December 31, 1997

Evaluation Point in Months (1)	Selected Age to Age Factors (a) (2)	Age to Ultimate Factors (a) (3)	Estimated Percentage Paid [1/(3)] (4)	Incremental Percentage Paid (5)	Discount Factor to Average Accident Date (b) (6)	Discounted Percentage Paid (5)×(6) (7)
Aviation						
12	2.371	5.176	19.32%	19.32%	0.983	19.00%
24	1.450	2.183	45.81%	26.49%	0.935	24.76%
36	1.160	1.505	66.43%	20.62%	0.873	18.01%
48	1.097	1.297	77.09%	10.66%	0.816	8.70%
60	1.060	1.182	84.60%	7.51%	0.763	5.73%
72	1.031	1.115	89.68%	5.08%	0.717	3.64%
84	1.019	1.081	92.50%	2.82%	0.683	1.93%
96	1.023	1.061	94.24%	1.74%	0.651	1.13%
108	1.018	1.038	96.37%	2.13%	0.620	1.32%
120	1.019	1.019	98.10%	1.73%	0.590	1.02%
132	---	1.000	100.00%	1.90%	0.562	1.07%
Total				100.00%		86.31%
(8) Selected Undiscounted Loss Ratio (c)						70.0%
(9) Selected Claim Development Margin Factor (d)						15.0%
(10) Loss Ratio with Margin Discounted to Average Accident Date						69.5%
(11) Average Earning Period for UPR (f)						4
(12) Percentage of Unearned Premium in Invested Assets (g)						50.0%
(13) Discount from Average Accident Date to Evaluation Date (f)						0.978
(14) Discounted Loss Ratio with Margins (13)×[1- (15)×{1-(16)}]						68.7%

Notes:

(a) Payment pattern from paid triangles in appendices

(b) Yield rate from DF&C investment returns; three month payment lag in the first year

(c) From Exhibit IV

(d) Judgmentally selected based on CIA standards of practice on PFAD

(e) [Total for Column (7)]×[(8)×[1+(9)]

(f) Assumptions: UPR is discounted four months, assuming 12 month policies

(g) From DF&C P&C-1: (Unearned Premium – Premium Receivables)/Unearned Premium

Appendix E

Table E1
Dubois Fire & Casualty Insurance Company
Calculation of Loss Trend Factors
December 31, 1997

		Selected Annual Trend		Time Spent Under (b)		
	Average	Bill	Bill	Bill	Bill	
Accident	Accident	164	59	164	59	Trend
Year	Date	(a)	(b)	1-Nov-96	1-May-98	Factor (c)
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Auto-Third Party Liability						
1995	1-Jul-95	5.0%	0.0%	1.339	1.495	1.068
1996	1-Jul-96	5.0%	0.0%	0.337	1.495	1.017
1997	1-Jul-97	5.0%	0.0%	0.000	0.832	1.000
Auto-Accident Benefits						
1995	1-Jul-95	13.0%	10.0%	1.339	1.495	1.358
1996	1-Jul-96	13.0%	10.0%	0.337	1.495	1.202
1997	1-Jul-97	13.0%	10.0%	0.000	0.832	1.083
Auto – Physical Damage						
1995	1-Jul-95	1.0%	0.0%	1.339	1.495	1.013
1996	1-Jul-96	1.0%	0.0%	0.337	1.495	1.003
1997	1-Jul-97	1.0%	0.0%	0.000	0.832	1.000
Personal Property						
1995	1-Jul-95	1.5%	1.5%	1.339	1.495	1.043
1996	1-Jul-96	1.5%	1.5%	0.337	1.495	1.028
1997	1-Jul-97	1.5%	1.5%	0.000	0.832	1.012
Liability						
1995	1-Jul-95	7.5%	7.5%	1.339	1.495	1.227
1996	1-Jul-96	7.5%	7.5%	0.337	1.495	1.142
1997	1-Jul-97	7.5%	7.5%	0.000	0.832	1.062

Notes:
(a) Bill 164 and Bill 59 impact only automobile coverages
(b) Time span starts at average accident date
(c) [1+(3)]^(5)×[1+(4)]^(6)

Table E2
Dubois Fire & Casualty Insurance Company
Calculation of Drift Factors
December 31, 1997

Accident Year (1)	Average Written Date (2)	Selected Drift Factor (3)	Time Span from Average Written Date to 1-Sep-96 (4)	Drift Factor (5)
Auto-Third Party Liability				
1995	1-Jul-95	0.2%	2.171	1.004
1996	1-Jul-96	0.2%	1.169	1.002
1997	1-Jul-97	0.2%	0.170	1.000
Auto-Accident Benefits				
1995	1-Jul-95	na		na
1996	1-Jul-96	na		na
1997	1-Jul-97	na		na
Auto - Physical Damage				
1995	1-Jul-95	0.3%	2.171	1.007
1996	1-Jul-96	0.3%	1.169	1.004
1997	1-Jul-97	0.3%	0.170	1.001
Personal Property				
1995	1-Jul-95	0.0%	2.171	1.000
1996	1-Jul-96	0.0%	1.169	1.000
1997	1-Jul-97	0.0%	0.170	1.000
Liability				
1995	1-Jul-95	0.2%	2.171	1.004
1996	1-Jul-96	0.2%	1.169	1.002
1997	1-Jul-97	0.2%	0.170	1.000

Notes:
(a) [1+(3)]^(4)

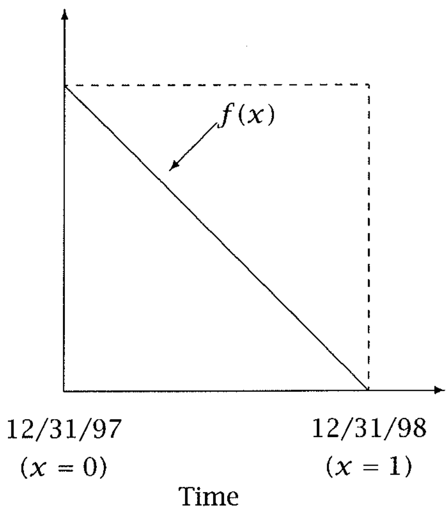
Appendix F

Average Accident Date (AAD) of the Unearned Premium

Figure F1 displays the earning pattern of the 12/31/97 unearned premium reserve through 1998 assuming that the unearned premium density function, $f(x)$, is given by

$$f(x) = \begin{cases} 1 - x & \text{if } 0 \le x < 1 \\ 0 & \text{if } x \ge 1. \end{cases}$$

Figure F1
Earnings Pattern for 1998



The average earning date of the unearned premium can be found by calculating the area of the lower triangle. The following integral calculates the lower triangle's average, which is equal to the average earning date of the 12/31/97 unearned premium reserve:

$$\int_0^1 x f(x) dx = \frac{1}{6}.$$

As this is the mean of the triangle whose area is equal to half a year, the average earning period for the unearned premium is

$$\frac{1}{6} \times 2 \times 12 \text{ months} = 4 \text{ months},$$

and the AAD is thus May 1, 1998. Generally it is assumed that the average accident date of losses is equal to the average earning date of the premium. We can conclude that the average accident date of losses that will arise from the unearned premium is May 1, 1998. The calculation is similar for a six-month policy. It easily can be shown that the resulting AAD is March 1, 1998. This calculation assumes that there is no unusual growth/decline in premium volume.

Average Writing Date (AWD) of the Unearned Premium

Figure F2
Written Premium Pattern for 1998

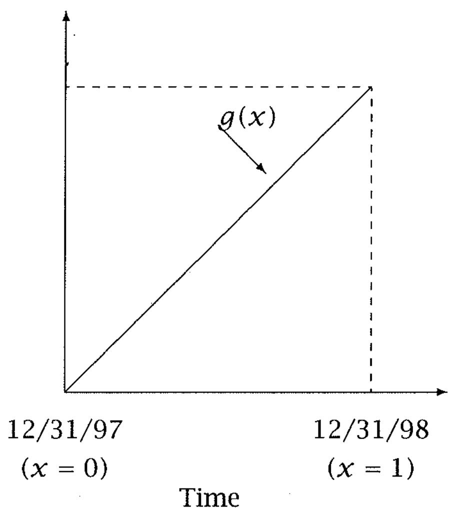


Figure F2 displays the writing pattern, $g(x)$, of the 12/31/97 unearned premium reserve through 1998, where

$$g(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 0 & \text{if } x \ge 1. \end{cases}$$

The average writing date of the unearned premium can be found by calculating the area of the lower triangle. The lower triangle's average, which is equal to the average writing date of the 12/31/97 unearned premium reserve, is given by:

$$\int_0^1 x g(x) dx = \frac{1}{3}.$$

As this is the mean of the triangle whose area is one half of the year, the average earning period for the unearned premium is

$$\frac{1}{3} \times 2 \times 12 \text{ months} = 8 \text{ months},$$

and the AWD is thus September 1, 1997.

The calculation is similar for a six-month policy. It easily can be shown that the resulting AWD is November 1, 1998.

Appendix G

.... Insurer

.... Year

Expenses - Total
($000)

Expense Classification		Applicable to Insurance Operations: Acquisition Expenses: Deferred at End of Year (01)	Applicable to Insurance Operations: Acquisition Expenses: Attributable to the Year* (02)	Applicable to Insurance Operations: General Expenses (04)	Applicable to Insurance Operations: Net Internal Adjustment Expenses (06)	Investment Expenses (08)
Salaries	02					
Employee benefits	04					
Employee acquisition	06					
Occupancy	08					
Advertising	10					
Agency (excl. commiss.)	12					
Auto & travel	14					
Allowance-doubtful accnts	16					
Bureaus & associations	18					
Directors remuneration	20					
Donations	22					
EDP & statistical	24					
Furniture & equipment	26					
Home office overhead	28					
Inspections, investigations	30					
Insurance	32					
Management fees	34					
Postage & courier	36					
Printing & stationery	38					
Professional fees	40					
Telephone, other comm.	42					
Miscellaneous	44					
	46					
Regulatory assessments	48					
Total	89					

*Total on line 89 to be reported on page 20.30. line 14

Appendix H—Sheet 1
Expression of Opinion

I have valued the policy liability of XYZ Insurance Company for its balance sheet at December 31, 19xx and their change in the statement of income for the year then ended in accordance with accepted actuarial practice, including selection of appropriate assumptions and methods. I am satisfied that the data utilized are reliable and sufficient for the valuation of these liabilities. I have verified the consistency of the valuation data with the company's financial records.

(Qualifications should be included here.)

The results of my valuation with items from the Annual Return are the following:

	Carried in Annual Return ($000)	Actuary's Estimate ($000)
Policy liabilities in connection with unpaid claims		
Direct unpaid claims and adjustment expenses	$	$
Direct unpaid claims and adjustment expenses	$	$
Assumed unpaid claims and adjustment expenses	$	$
Gross unpaid claims and adjustment expenses	$	$
Ceded unpaid claims and adjustment expenses	$	$
Net unpaid claims and adjustment expenses	$	$
Policy liabilities in connection with unearned premiums		
Gross policy liabilities in connection with unearned premiums		$
Net policy liabilities in connection with unearned premiums		$
Gross unearned premiums	$	
Net unearned premiums	$	
Deferred policy acquisition expenses	$	
Maximum policy acquisition expenses deferrable		$
Premium deficiency	$	$
Other policy liabilities - Net	$	$

In my opinion, the amount of policy liabilities makes appropriate provision for all policyholders' obligations and the annual return fairly presents the results of the valuation.

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Signature of Actuary Date Opinion was Rendered

Fellow, Canadian Institute of Actuaries

Printed name of Actuary

Appendix H—Sheet 2
Expression of Opinion

I have valued the policy liability of XYZ Insurance Company for its balance sheet at December 31, 19xx and their change in the statement of income for the year then ended in accordance with accepted actuarial practice, including selection of appropriate assumptions and methods. I am satisfied that the data utilized are reliable and sufficient for the valuation of these liabilities. I have verified the consistency of the valuation data with the company's financial records.

The results of my valuation with items from the Annual Return are the following:

	Carried in Annual Return ($000)	Actuary's Estimate ($000)
Policy liabilities in connection with unpaid claims		
Direct unpaid claims and adjustment expenses	$	$
Assumed unpaid claims and adjustment expenses	$	$
Gross unpaid claims and adjustment expenses	$	$
Unpaid claims recoverable from other insurers under the loss transfer provisions	$	$
Ceded unpaid claims and adjustment expenses	$	$
Net unpaid claims and adjustment expenses	$	$
Policy liabilities in connection with unearned premiums		
Gross policy liabilities in connection with unearned premiums		$
Net policy liabilities in connection with unearned premiums		$
Gross unearned premiums	$	
Net unearned premiums	$	
Deferred policy acquisition expenses	$	
Unearned commission	$	
Maximum policy acquisition expenses deferrable		$
Premium deficiency	$	$

In my opinion, the amount of policy liabilities makes appropriate provision for all policyholders' obligations and the annual return fairly presents the results of the valuation.

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Joe Doe, Montreal, Quebec
Fellow, Canadian Institute of Actuaries

Recognizing Actuarial Assumptions

Victoria Stachowski* and Alice Underwood†

Abstract‡

As assumptions underlie every aspect of actuarial calculations, actuaries must be aware of the assumptions they are using and understand their importance and the possible effects of changing assumptions on the results of their calculations.

This paper explores the nature of assumptions in: (i) mathematical models, (ii) data selection, (iii) actuarial methods, and (iv) the business environment. We examine reasons for making assumptions such as convenience, tradition, indications in the data, or lack of data. In addition, we discuss (i) how actuaries can judge whether these reasons are sufficient; (ii) methods that can help actuaries quantify the impact of their assumptions, such as what-if scenarios, simulation, and stress testing; and (iii) the circumstances for which testing is most important.

Key words and phrases: *hypotheses, postulates, methods, standards, models, and analysis*

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1 The Importance of Assumptions

Understanding actuarial assumptions is a requirement of the professional standards of practice. According to one actuarial compliance guideline, "If there is a change in the actuarial assumptions or methods from those previously employed, the change should be mentioned in the actuarial statement of opinion" (Actuarial Standards Board 4).

Actuaries are often told to check their assumptions but they may not know what checking assumptions entails. This article suggests ways to check assumptions and also explains where to look for assumptions, as some assumptions are less obvious than others. We explore common assumptions in several areas: mathematical models, data selection, actuarial methods, and the business environment. Methods to quantify the impact of assumptions, such as what-if scenarios, simulation, and stress testing are discussed, as are the circumstances in which such testing is most important. We examine reasons for making certain assumptions—such as convenience, historical practice, indications in the data, or simply lack of data—and discuss how actuaries can judge whether these reasons are sufficient.

Understanding assumptions can assist actuaries in choosing the most appropriate methods for pricing, reserving, and other tasks; deciding the next steps to take in an analysis; determining the level of confidence for estimates; and in creating financial products that protect against the chance that certain assumptions are incorrect.

2 Reasons for Making Assumptions

When actuaries recognize their assumptions, they must also recognize their reasons for making them. Though there are many reasons for making actuarial assumptions, we will group our examples into two general categories: (i) assumptions are dictated by external or internal factors affecting the analysis and (ii) assumptions are dictated by convenience or circumstance.

2.1 Assumptions Dictated by the Analysis

The data may suggest or necessitate certain assumptions. For example, perhaps you are fitting a claim size severity distribution to experience data. Having trended the historical claims, you use one or several statistical methods to fit different distributions: perhaps a lognormal, a two-parameter Pareto, and a loggamma. If you determine the lognormal is the best choice, then assuming a lognormal distribution is logical, at least for claim sizes within or not far outside the range of available data.

Gaps in the data may force certain assumptions. Suppose you need to make a calculation using the standard deviation of annual loss ratios by line of business, but you have only the plan loss ratios. Some assumption will therefore be necessary. One possibility would be the use of related data, such as standard deviations from industry data—but then you are assuming that the variability of your own book is similar to that of the industry book.

Anecdotal evidence. In some lines of business, common knowledge plays a large role. You may encounter assertions such as "everyone knows motor liability business has Pareto severity with alpha parameter 2.5," or "claims in that line take five years to pay out." In the absence of strong evidence, either supporting or contradictory, it may be reasonable to heed the general wisdom. Although you should not let folklore override empirical evidence, informed actuarial judgment is one of the cornerstones of the profession.

Anecdotal evidence, however, often contains implicit assumptions that have not been tested recently, if ever. Investigating these assumptions may lead to new insights.

Significance. If the value of a particular parameter will have only a minor effect on the final result, estimating the parameter's value precisely may not be worth the effort. In such cases, you may consider the tradeoff between time and accuracy with respect to making a reasonable or standard assumption for the value. The accurate assessment of the significance of the value in question distinguishes this type of assumption from one made solely for convenience's sake.

2.2 Assumptions of Convenience

The problem can't be solved otherwise. Some calculations are intractable without simplifying assumptions. For example, assuming independence of property losses is often not correct. Unfortunately, there is rarely good information about the correlation between the frequency and/or severity of individual losses. Even if the correlation were known, including it in your calculations could be complicated. For these reasons we often assume independence. If the correlation is weak, this assumption may be harmless. But if losses are materially correlated, the extent of the dependence should be considered somehow—possibly through a Monte Carlo simulation or judgmental loading.

Simplicity. Ockham's Razor states that, all things being equal, a simple hypothesis is preferable to a complicated one. This is also called *the principle of parsimony.* A simpler assumption is easier to work with and easier to explain to others. For example, we may select a distribution with few parameters rather than a distribution with many, if the goodness-of-fit is similar. In this case we gain the advantage of simplicity and also avoid possible over-fitting.

It's what you can do. Suppose you only learned one method for doing a particular analysis. To take a far-fetched example, maybe the only kind of average you can calculate is a simple average. Perhaps you have heard that there are exotic types of averages such as weighted average or average ex high/low, but you don't know how they work. Still, if you recognize the assumptions underlying the method you do know—for a simple average, all the data points are assumed to be equally valid—and feel confident that these assumptions are satisfied, there may be no problem. If you suspect that your assumptions are violated, however, seek assistance and advice from more experienced colleagues, research papers, or outside experts.

It's what the relevant authority will accept. This is the flip side of the situation above. Here are two examples: (i) your boss, or another executive in your company, may be comfortable only with one particular method—such as a simple average; or (ii) regulatory bodies may require a certain calculation method, thereby dragging along its fundamental assumptions. When regulation explicitly requires certain assumptions, such as a specific interest rate for discounting, we have "prescribed assumptions" (Actuarial Standards Board 1996). In such cases you must make the calculation using the assumptions required by those directing the work product.

If you believe a prescribed assumption is unrealistic or unwarranted, however, you may want to recalculate using the assumption you consider best. Then you can compare the results and, if appropriate and material, explain any differences to the relevant person or authority.

It's what the client will accept. Insurance companies are in the business of making money, and one part of this process is selling products to clients. Sometimes the client is in a position to dictate which assumptions the actuary will use—and sometimes the actuary has insufficient information to refute the client's demands.

The historical treatment of credibility illustrates this situation. Credibility methods are used in property-casualty business to balance a client's observed experience against the a priori expectation. Simply put, credibility is the percentage weight given to the client's observed experience. Hewitt (1989) explains that while the theoretical Bayesian credibility formula never assigns 100 percent weight or full credibility to the data, historically "buyers with better-than-average experience wanted full recognition in their rates." These buyers were by definition the better risks and usually were the larger customers as well. So, "an arbitrary assignment was made—the point at which exposures were sufficient to admit of 'full' credibility—and, of course, on the basis of convenience." Such methods allowing full credibility remain part of the standard actuarial repertoire.

Resources. The investigation and refinement of assumptions take time and money. These resources are finite, however, particularly for those trying to analyze an issue that appears at the last minute. Sometimes an answer that is too late is as bad as no answer at all. This is why it is important for actuaries to determine which of their assumptions will have the most impact on their model and to prioritize their efforts accordingly.

3 Assumptions in the Mathematical Methods

Because assumptions are fundamental to mathematics and statistics, whenever actuaries use mathematics, they rely on fundamental mathematical assumptions. In addition, actuaries must make assumptions about which mathematical methods are appropriate in given situations.

The latter issue is of more practical concern. While it would be impossible to discuss all such instances, we will give some examples to illustrate what we mean.

3.1 Choice of Techniques

Suppose you want to make a best estimate of the coming year's loss ratio. Premium and loss data for several years have been on-leveled and trended to produce loss ratios as if at today's premium rates and trended loss levels.

Table 1
As-If Loss Ratios

Underwriting Year	On-Level Premium	Trended Loss	As-If Loss Ratio
1	1,000,000	620,000	62%
2	1,250,000	850,000	68%
3	1,300,000	650,000	50%
4	1,400,000	798,000	57%
5	1,500,000	1,575,000	105%
6	1,600,000	1,216,000	76%
7	1,800,000	1,170,000	65%
8	1,750,000	717,500	41%
9	2,000,000	1,420,000	71%
10	2,000,000	1,600,000	80%

You want to take some sort of average as your best estimate for next year. There are several types of averages used by actuaries; what follows is a look at the implicit assumptions made by a few of these averaging methods. The data used are displayed in Table 1.

Straight average. This method assumes that: (i) all the historical years are equally predictive or equally credible, (ii) the fundamental loss situation has not changed over time, and (iii) none of the historical data points is an outlier (a fluke) that should contribute less to the final prediction. This average of Column (4) of Table 1 is

$$\frac{62 + 68 + 50 + 57 + 105 + 76 + 65 + 41 + 71 + 80}{10 \times 100} = 67.5\%.$$

Average excluding high and low values. This method accepts the first two assumptions of the straight average. It assumes, however, that the high and low values are outliers and thus not predictive. This average of Column (4) of Table 1 is

$$\frac{62 + 68 + 50 + 57 + 76 + 65 + 71 + 80}{8 \times 100} = 66.1\%.$$

Premium-weighted average. This method assumes years of larger premium volume are more predictive than years with smaller premium volume. If there are large year-to-year differences in on-level premium, this method can be more appropriate than others. This average is

$$\frac{\sum(\text{premium}_t \times \text{loss ratio}_t)}{\sum \text{premium}_t} = \frac{\sum \text{losses}_t}{\sum \text{premium}_t} = \frac{\sum \text{Column (3)}}{\sum \text{Column (2)}}.$$

Summing down Columns (2) and (3) of Table 1 yields

$$\frac{620 + 850 + 650 + \cdots + 717.5 + 1420 + 1600}{1000 + 1250 + 1300 + \cdots + 1750 + 2000 + 2000} = 68\%.$$

General weighted average with greater weight on more recent years. This method assumes that recent years are more predictive. This assumption may be warranted if there is a trend in the data. A common method is using only the last three years, with more weight on the most recent years. For example, take weights for the last three years of 20 percent, 30 percent, and 50 percent. Our result is then

$$(0.2)41\% + (0.3)71\% + (0.5)80\% = 69.5\%.$$

One problem with this approach, especially for longer tailed lines of business, is that the most recent years are not as reliable because the development of losses to ultimate is based on immature data.

These examples demonstrate that a procedure as common as taking an average may be full of assumptions.

3.2 Choice of Distributions

A common and important actuarial task is the selection of probability distributions for the frequency and severity of individual losses or for the aggregate amount of annual losses.

The most commonly used distribution to describe frequency of claims is the Poisson process distribution. If $N(t)$ is the number of claims in $(0, t]$, then

$$\Pr[N(t) = n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, 2, \ldots$$

where $\lambda > 0$ is a parameter. The mean and variance of this distribution are both equal to λ.

The Poisson process is characterized by three assumptions:

1. In an infinitesimally small time interval $(t, t + dt)$, the probability of having one claim is approximately λdt;

2. In an infinitesimally small time interval $(t, t + dt)$, the probability of two or more claims is essentially zero; and

3. The numbers of claims in non-overlapping time intervals are independent.

The Poisson process is popular in part because of its simplicity and mathematical tractability. Unfortunately, in many real world situations the variance is not likely to be equal to the mean. And, as indicated by Hogg and Klugman (1984, Chapter 2, p. 25),

> ... while Poisson postulates (1) and (2) are reasonably accurate in [many] kinds of situations, assumption (3) concerning the independence of the number of [claims] in non-overlapping intervals is often questionable. For example, a car may be so badly damaged at a given time that it has no chance of being damaged again in the near future because it is being repaired.

The Poisson process may nonetheless be appropriate for an insurer with a large portfolio of auto policies and auto claims. If a situation arises where assumption (3) is clearly false, the Poisson process may be modified as a mixed Poisson process or a blocked Poisson process—the above quote from Hogg and Klugman refers to a blocked Poisson process; see Ramsay (1991)—or other frequency distributions must be investigated.

Loss frequency is only one aspect of the actuarial analysis. It may also be necessary to select a form for the severity distribution. There may be several possibilities including the gamma, Pareto, or lognormal. The assumptions behind each of these models are important and should be considered. For example, the Pareto and lognormal are heavily skewed and have thick right tails; this results in relatively high probabilities of catastrophic claims. See, for example, Panjer and Willmot (1992, Chapter 4) for other distributions that may be appropriate for loss frequency and loss severity and the various Poisson processes.

Clustering of claim sizes is another potential problem. The mathematical models are generally smooth, but actual claims tend to cluster at round numbers. You may need to smooth the data or group the data in order to avoid distortion. Beware of assuming that your real-world data are driven only by the purity of a mathematical process.

3.3 Model Risk and Parameter Risk

It is common knowledge that models, regardless of their complexities, are only idealizations of the real world phenomena they purport to describe. Following Daykin, Pentikäinen, and Pesonen (1994, p. 18) we define the following:

Model risk arises from the fact that a model is only an approximation to reality. This results in unavoidable errors because some meaningful variable has been omitted from the model. As some degree of model risk is always present, this risk must be recognized.

Parameter risk arises because we must rely on statistical estimates from observed data to determine the parameters of the model.

Process risk arises because the actual data (losses, investment returns, etc.) are inherently random, even if the model and parameters were exactly correct.

Parameter risk and model risk are important components of the risk taken by an insurance company. Unlike process risk, they are to some extent under the actuary's control through the assumptions about model and parameters and deserve special consideration.

4 Assumptions in the Data

Once your mathematical model is created, it must be tested with real data in order to determine whether it is useful in a practical setting. Use of these data may involve as many assumptions as those in your mathematical model.

The concern with respect to data quality and appropriateness is important. An Actuarial Standards Board bulletin on this issue states that although it is not the actuary's task to audit the data, he or she

> may be aware that the data are incomplete, inaccurate, or not appropriate as desired. In such cases, the actuary should consider whether the use of such imperfect data may produce material biases in the results of the study, or whether the data are so inadequate that the data cannot be used to satisfy the purpose of the study (Actuarial Standards Board 1993b, p. 4).

The following are some of the assumptions most relevant to data selection and use.

4.1 Assuming the Data Are Clean

It is common to assume that the data are clean, i.e., that the person(s) who entered the data corrected any typographical errors. If the data set is not too large, you can inspect it visually for obvious errors.

Sometimes the data that are initially free of errors are sullied at a later stage, as might happen when printed information is converted to machine-readable form via scanning. Scanned data may contain errors resulting from misinterpreting numbers and letters. Data sent by fax can present similar difficulties. Sometimes data stored on magnetic media such as hard drives and floppy disks may become corrupted if they are used often, improperly stored, or are left unattended for a long time.

A comparison to the original data, including checking totals, is advisable to make sure that such errors are found and corrected.

4.2 Assuming the Data Are Appropriate

Even if you are confident that the data are clean, they still may not be appropriate for your particular situation. Here are some examples to consider.

Using data from a statistical collection agency. The use of data from a statistical collection agency such as Insurance Services Office (ISO) assumes that the companies submitting data to the statistical collection agency reflect the same sort of business that your own company writes. Is this reasonable? For example, if your company engages in target marketing, the ISO classification relativities might not be appropriate. If you have excellent loss control measures, the ISO relativities might be correct, while the overall loss levels are too high.

Using data from another state or country. Regional differences may affect the insurance data. Consider the amount of litigation in California versus the amount in North Carolina. On an international scale, the definition of workers compensation in the United States means something quite different from the "accidents du travail" found in Belgium.

Data behind standard tables. There may be data behind standard relationships, formulas, or tables you plan to use. If these data are not applicable to your situation, consider whether and how you should use the results derived from them.

For years, actuaries in the United States used the Salzmann Tables in pricing property excess-of-loss reinsurance. These tables were compiled by Ruth Salzmann and based on 1960 accident year data for homeowner fire claims. As Ludwig (1991) has pointed out, these tables have been used to price many exposures and perils not similar to those studied by Salzmann.

4.3 Different Assumed Meanings Behind the Data

It is important that you understand the meaning of the data. Differences in meanings can arise in several ways.

Definitions. Did the person who created the data use the definition you have in mind? For example, there are several different meanings for IBNR. Many Europeans use the term IBNR to refer to pure IBNR or incurred but not yet reported (IBNYR). In the United States IBNR is used more commonly to refer to broad IBNR or all future development to come: this would include the IBNYR and also pipeline claims, incurred but not enough reported (IBNER), and so on. There may also be definitional differences with respect to claim counts, exposure units, earned premium, incurred claims, and so on.

Interpretation. Even when definitions are the same, interpretations may differ. For example, reserving actuaries may be interested in expected loss ratios (ELR) by line of business. Suppose in a company's computer system these ELRs have been entered by the pricing actuaries. A problem will ensue if pricing actuaries think that they are supposed to enter a conservative ELR estimate whereas the reserving actuaries interpret the entries as the most likely value. Ideally the two different actuarial groups would work with the same assumptions (e.g., ELR estimates are always best point estimate, neither conservative nor aggressive). If this is impossible or impractical, the two groups should at least be aware of their differences.

Multiple Meanings. Particularly when working with data from different sources, you may have to combine items that you would prefer not to combine. Pinto and Gogol (1987), in their paper on excess development factors, discuss the data from the Reinsurance Association of America (RAA). The RAA information goes back more than 20 years for some lines of business and is calculated by pulling together statistics supplied by member companies. But the member companies have written different types of policies over the years. In order to combine and compare the information, assumptions are needed with regard to the treatment of ALAE, different policy limits, different attachment points, different reporting patterns, and so on.

One way to avoid data-meaning difficulties is by asking questions, to make sure that there are no ambiguities. Another good idea is to check through the numbers and formulas. A calculation can be worth a thousand words.

5 Assumptions in the Actuarial Methods

Most actuarial methods involve assumptions, whether explicit or implicit. In this section we explore a few examples of assumptions in common reserving and ratemaking techniques.

5.1 Reserving Methods

Calculating loss reserves, i.e., the amount of money that must be set aside in the present to make claim payments in the future, is an important actuarial task. The actuary must consider the reported value of claims, any development likely to occur on these claims, and the projected additional amount for claims which have not yet been reported.

Here we will consider three different commonly used methods for creating the reserve for broad IBNR (including development on known claims, pipeline claims, etc., in addition to IBNYR). These are the loss ratio method, the link ratio method, and the Bornhuetter-Ferguson method. Assorted variations on these methods exist, but we will focus on basic versions of each.

The loss ratio method: This method is described by the following equation:

$$IBNR_{\text{LossRatio}} = \text{Selected ELR\%} \times \text{Premium} - \text{Reported Incurred}$$

where ELR is the expected loss ratio. Notice that this method makes a simple but powerful assumption: the selected ELR is the correct ultimate loss ratio, and thus ultimate losses can be calculated as the premium multiplied by ELR. The IBNR is just the difference between the losses originally expected and those losses that have already been reported.

This assumption has consequences. For example, the IBNR is inversely related to the reported incurred. The larger the reported incurred loss amount, the smaller the calculated IBNR—the IBNR may even be negative.

The link ratio method: This method (based on reported incurred) can be expressed as follows:

$$IBNR_{\text{LinkRatio}} = \text{Reported Incurred}_t \times (LDF_t - 1)$$

where LDF_t is the factor needed to develop losses at time t to their ultimate value.

The key assumption behind this method is that ultimate losses are directly related to the reported incurred at time t through a multiplicative loss development factor. For a larger reported incurred loss amount, the calculated IBNR will also be larger. The fundamental assumption leads in the opposite direction from the assumption for the loss ratio method.

The Bornhuetter-Ferguson method: This method is a combination of the two previous methods:

$$IBNR_{\text{BF}} = \text{Selected ELR\%} \times \text{Premium} \times \left(1 - \frac{1}{LDF_t}\right).$$

The assumption behind this method is a combination of the previous two. The ultimate loss is assumed equal to the reported incurred at time t plus an IBNR that is independent of the reported incurred. To produce the IBNR, the ELR is applied to that percentage of the premium that is not yet reported according to the development pattern.

If development patterns are steady from year to year, and if you pick the same (and the correct) percentage of premium as your loss ratios for all three methods, then they will give the same results. Tables 2 and 3 provide a simplified example, with only a few accident years.

Table 2
A Perfect World (in $000s)

				Methods		
	Earned	Reported	Age-Ult	Loss	Link	
AY	Premium	Losses	LDF	Ratio	Ratio	B-F
1	50,000	39,375	1.0000	0	0	0
2	50,000	31,500	1.2500	7,875	7,875	7,875
3	50,000	18,000	2.1875	21,375	21,375	21,375
Total	150,000	88,875		29,250	29,250	29,250
Selected ELR		78.75%				

Notes: AY = Accident Year; Age-Ult = Age to Ultimate; and B-F = Bornhuetter-Ferguson

Why did we call this a perfect world? All three methods produce the same estimated IBNR, but that is no coincidence.

The oldest year, Year 1, is fully developed: its age-to-ultimate factor is 1.000 so its losses are already at ultimate. Its loss ratio is 78.75 percent. Year 2 is not completely developed; if we apply the age-to-ultimate factor of 1.250 to the reported incurred (stated in thousands) of 31,500 as we would in the link ratio method, we obtain an estimated ultimate loss of 39,375. The ultimate loss ratio is again 78.75 percent. Similarly, developing Year 3 losses to ultimate yields $2.1875 \times 18,000 = 39,375$ for an ultimate loss ratio of 78.75 percent once more.

There is simply no place for the three methods to differ. In a world as perfect as this, where losses develop consistently over time and we have a known and unchanging ultimate loss ratio for all years, the assumptions of all three methods are equivalent and the calculations must produce identical estimates for IBNR.

But most real world situations will not produce such clean results. Most likely you will obtain the selected ELR and the loss development factors from industry data, or from a larger body of historical data at your own company, or from a credibility weighting of the two. It is improbable that they will fit together with your three years of data as beautifully as in the last example. Something like this is far more realistic:

Table 3
A More Realistic World (in $000s)

				Methods		
AY	Earned Premium	Reported Losses	Age-Ult LDF	Loss Ratio	Link Ratio	B-F
1	50,000	39,375	1.0000	-1,875	0	0
2	50,000	31,500	1.3000	6,000	9,450	8,654
3	50,000	18,000	2.3400	19,500	24,120	21,474
Total	150,000	88,875		23,625	33,570	30,128
Selected ELR		75.00%				

Notes: AY = Accident Year; Age-Ult = Age to Ultimate; and B-F = Bornhuetter-Ferguson

As you can see, small changes in the loss development factors or loss ratios can make a big difference. Using the data and selections in this table, there is a difference of about $10 million between the IBNRs calculated by the link ratio method and the loss ratio method—a difference of about 42 percent in the reserves.

What can you do in this situation?

Consider the fundamental assumptions of each method. The loss ratio method assumes the selected ELR is correct. Do you have strong confidence in this? If so, this method is a reasonable choice. But even if you don't, you may still select the loss ratio method. You might choose it because you have little confidence in the values of the loss development factors or have reason to believe that losses could be reported significantly faster or slower than the selected development pattern indicates.

The link ratio method has no ELR assumption, but instead assumes a direct relationship between reported losses and the ultimate loss value. This method is more appropriate if you think something has changed in the environment to cause losses to be higher or lower than the original expectation—as opposed to losses simply being reported faster or slower than expected. If you have higher confidence in your development pattern and link ratios than in your ELR, this method is preferred.

The Bornhuetter-Ferguson method uses a link ratio assumption to determine what portion of the premium represents unreported losses at a certain point in time; it then applies an ELR assumption to this portion of the premium to produce the IBNR. This method assumes that IBNR is independent of the losses reported to date. The assumption is reasonable if you think that nothing has changed in the environment or the reporting of losses—but instead you were just lucky or unlucky in the low or high level of losses reported to date.

Use more than one method. You may opt to take a weighted average of results produced by the different methods. Or you may use the link ratio method for the older periods, where the losses are most completely developed; the loss ratio method on the younger periods, where the link ratios are most uncertain and unstable; and the Bornhuetter-Ferguson method on the intermediate time periods.

Test methods under different assumptions. If you are unsure about that 75 percent expected loss ratio, try the methods with 70 percent and 80 percent and get a feel for the sensitivity. The same goes for the loss development factors.

If you can, ask for more information, such as possible changes in premium rates and reporting patterns over time, to refine your estimates or at least develop a range of estimates.

Consider changes in the environment. Perhaps the assumed 75 percent loss ratio was appropriate in past years, but is now deteriorating. On the other hand, the claims department may have changed its policies about setting reserves for known claims—calling into question the assumed development pattern. Fisher and Lester (1975) explain how these three methods perform under two different situations: loss reserve strengthening and a deteriorating loss ratio.

5.2 Tail Factors

The selection of the tail factor—the development factor that takes losses from the oldest reported age to their ultimate value—is critical in reserving. According to one American Academy of Actuaries survey, the selection of the loss development tail factor was second out of four major causes of reserve deficiency (American Academy of Actuaries 1995). Unfortunately, tail factor selection is hampered by the fact that there are scarce data on older accident years and that the older accident years may bear little resemblance to more recent accident years. Thus, assumptions are necessary, and informed actuarial judgment is critical.

In both of the examples in Subsection 5.1, the tail factor (age-3-to-ultimate) is 1.000. But we did not discuss how these tail factors, or the age-to-age development factors in general, were chosen. If you're not simply copying development factors out of an ISO circular, you probably will analyze a triangle similar to that in Table 4.

Table 4
Data for Short-Tail Line

	Paid Losses at Month				
UY	12	24	36	48	60
1	23,000	25,000	27,000	27,000	27,000
2	29,000	35,000	36,000	36,000	
3	40,000	47,000	58,000		
4	34,000	38,000			
5	29,000				

	Age-to-Age Ratios				
UY	12-24	24-36	36-48	48-60	60-Ult.
1	1.087	1.080	1.000	1.000	
2	1.207	1.029	1.000		
3	1.175	1.234			
4	1.118				
5					

	Age-to-Age Ratios				
	12-24	24-36	36-48	48-60	60-Ult.
Simple Avg	1.147	1.114	1.000	1.000	
Avg Ex Hi Lo	1.146	1.080	N/A	N/A	
Weighted Avg	1.151	1.131	1.000	1.000	
Selected	1.150	1.120	1.000	1.000	1.000

Notes: UY = Underwriting Year

In Table 4, the paid losses are our given data. We have constructed the triangle of age-age ratios: for example, the 12 to 24 ratio for year 1 is 25,000/23,000 = 1.087 and so on. At the bottom of Table 4 we show some averages of the age-to-age ratios thus obtained. Considering the assumptions behind different averaging methods, we have made the final selections shown in the last row (Selected) of Table 4.

We have selected a factor of 1.000 for 60–Ultimate, even though we have no information about how the losses develop beyond age 60. In this case the assumption may be reasonable, because the previous two age-to-age factors were already 1.000, suggesting that loss development has stopped.

A factor of 1.000 was reasonable above. What might we do with a (more realistic) triangle such as the one in Table 5?

What link ratio for 120-Ultimate—in other words what tail factor—do these data suggest?

The point here is that we should not take a tail factor of 1.000 for development 120-Ultimate simply because the data end at age 120. Though your software may suggest 1.000 as a default tail factor, you must not make the software's default assumption your own assumption without giving the matter some thought. With Table 5, as the development has not ended by age 120 and the last few link ratios do not show a strongly decreasing pattern, an assumption of 1.000 for the tail factor may be too optimistic.

On what could you reasonably base your tail factor assumptions? Here are a few possibilities: industry data; data from related business in your company; informed actuarial judgment; or fitting a curve to the selected age-age factors and extrapolating.

Our point here is not to tell you how a tail factor should be selected. Rather, we want to call your attention to the fact that assumptions will be necessary and deserve careful consideration.

Table 5
Data for Long-Tail Line

UY	Paid Losses at Month									
	12	24	36	48	60	72	84	96	108	120
1	12,354	18,386	22,553	26,588	32,204	33,329	34,891	35,084	35,682	36,788
2	16,381	21,863	30,020	31,119	31,576	35,088	35,455	37,693	38,934	
3	15,478	26,142	36,586	38,387	43,819	46,166	49,140	50,535		
4	17,457	17,854	23,476	24,580	26,561	26,683	27,841			
5	16,453	25,084	29,315	35,299	35,695	38,072				
6	17,864	33,809	39,996	46,772	57,971					
7	18,934	24,829	33,962	38,400						
8	18,462	20,076	28,765							
9	19,035	35,080								
10	19,512									
	Age-to-Age Ratio									
Simple Avg	1.466	1.308	1.117	1.116	1.054	1.041	1.032	1.025	1.031	
Avg Ex Hi Lo	1.468	1.311	1.115	1.112	1.052	1.045	1.028	N/A	N/A	
Weighted Avg	1.464	1.301	1.117	1.124	1.056	1.043	1.032	1.025	1.031	
Selected	1.466	1.308	1.117	1.117	1.054	1.043	1.031	1.025	1.031	?

Note: We have omitted the triangle of age to age factors.

5.3 Catastrophe Pricing and the Excess Wind Procedure

Before the advent of computer modeling, actuaries used other methods to price catastrophe-exposed business. These methods typically involved isolating the catastrophe-related portions of historical losses and then spreading these losses over a long time period to produce an average catastrophe load for the rates. One of the best known such methods, the excess wind procedure, is still in use today. But, as Musulin (1997) has pointed out, the excess wind procedure rests on at least four assumptions that may not be appropriate and which, therefore, call its accuracy into question.

Before we explore these assumptions, here is a brief explanation of the excess wind procedure. In this method, the actuary collects 20 to 30 years of statewide loss data by accident year and separates these data into wind and nonwind components. A yearly ratio of wind to nonwind losses is computed. For those years having an excessively high wind-to-nonwind ratio, the actuary removes the excess wind losses from the yearly totals and spreads the excess losses over the time period to produce an average yearly wind loading. This procedure smoothes the rates and prevents large swings in the rate indication. [See Musulin (1997) or Homan (1990).]

As Musulin (1997) points out,

> ... this method makes several assumptions about the 20–30 year period used in the 'excess' calculation including:
>
> - Catastrophic activity was 'normal;'
> - Population demographics were stable;
> - Insured losses by peril were stable;
> - Changes in coverage or construction practices did not affect the ratio of wind to nonwind losses.

Musulin shows each of these assumptions to be questionable. An examination of weather history over a 100-year time frame shows that the hurricane activity in the period 1960–1987 was unusually low. Population demographics have certainly not remained stable in the past few decades; a far higher percentage of people today live in coastal areas, especially in Florida, than 30 years ago. And there have been changes in standard insurance coverages and construction practices that render the last two assumptions dubious as well.

Consider these points carefully and make appropriate adjustments before using the excess wind procedure in your own ratemaking calculations. But also be aware that similar assumptions underlie other traditional catastrophe rating procedures.

In recent years the trend has been toward using computer models to assess likely catastrophe losses. Actuaries who use such models still must be cognizant of crucial assumptions. Catastrophe models, due to their computational complexity and their sophisticated meteorological and/or seismological underpinnings, are notoriously prone to inducing a black box mentality.

A black box mentality occurs, in large part, due to a user's failure to recognize and understand a method's assumptions (coupled with ignorance or incomprehension of the calculations based on those assumptions). Actuaries need not be experts in seismology and meteorology to use computerized catastrophe models, and they need not be able to follow all the details of the programming to obtain reasonable results. Users of such a model, however, *should* have a good grasp of the fundamental assumptions and methods stuffed into the box in order to provide the proper input and then correctly interpret the output.

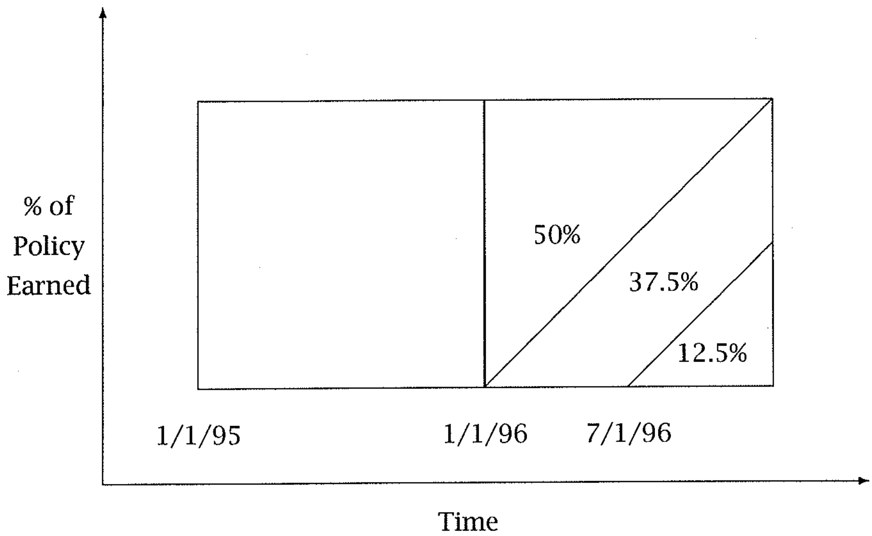
5.4 Parallelogram Method for On-Leveling of Premium

The parallelogram method is often used to convert premium from policies written and earned in the past to what the premium would be if those policies were written today (or some other selected date). This step is necessary for ratemaking. Losses and premium both need to be made current in order to begin the ratemaking process.

The parallelogram approach graphically demonstrates how policies with a term of one year that are written after January 1st will be earning during the next calendar year. Using the relative areas indicated by the parallelogram method simplifies the calculation of on-level factors. For example, suppose your company takes rate changes on January 1, 1996 and again on July 1, 1996. We can use the parallelogram method to see what percentage of the calendar year 1996 earned premium is at the different rate levels. In Figure 1, time runs along the horizontal axis while the vertical axis represents the percentage of a policy that has been earned. Each policy can be thought of as a diagonal line running from lower left to upper right.

The parallelogram method is elegant, but it makes an important assumption: that policies are written uniformly throughout the year. Depending on the line of business, this may not be the case. Commercial policies tend to clump around January 1st and the start of the other quarters. In the following example, we compare calculations to create on-level factors for the 1996 accident year earned premium.

Figure 1
Earned Premium with Uniform Writings Assumption



Using Figure 1 and assuming uniform inception dates produces Table 6. You can see that 50 percent of the premium earned in calendar year 1996 comes from policies that started in calendar year 1995. This means that the on-level factor of 1.155 needs to be applied to 50 percent of the earned premium from calendar year 1996. Another 37.5 percent of the earned premium comes from those policies that started between 01/01/96 and 07/01/96. We use the on-level factor 1.050 to adjust this amount of the calendar year 1996 premium. Finally, the remaining portion, 12.5 percent of the premium earned in calendar year 1996, needs no adjustment because it is already on-level.

The accuracy of this method depends on the assumption of uniform premium writings throughout the year. If this assumption fails, the parallelogram's sub-areas do not represent the right proportion of earned premium. For example, if 40 percent of the premium was written on January 1 (a fairly reasonable assumption), and the rest was written evenly throughout the year, we would use the method similar to that of Table 7 to calculate a calendar year on-level factor.[1]

[1]Here's clarification on the math: 40 percent of the premium starts on January 1. This means that 60 percent of the premium remains and we assume it is written in a uniform manner. The amounts of premium for each category use the same proportions as shown in Table 6, but are simply adjusted for the reduced amount. According to

Table 6
On-Leveling Premium to 12/31/96: Uniform Assumption

Date of Rate Change	Amount of Rate Change	Amount of Premium	Interval On-Level Factor*
01/01/95	0.0%	50.0%	1.155
01/01/96	10.0%	37.5%	1.050
07/01/96	5.0%	12.5%	1.000
Weighted Average for 1996			1.096

*The on-level factor is calculated by adding 1.00 to each of the future rate changes and taking the product; e.g., 1.100 × 1.050 = 1.155.

Table 7
On-Leveling Premium to December 31, 1996
40 Percent on January 1, Uniform Thereafter

Date of Rate Change	Amount of Rate Change	Amount of Premium	Interval On-Level Factor
01/01/95	0.0%	30.0%	1.155
01/01/96	10.0%	62.5%	1.050
07/01/96	5.0%	7.5%	1.000
Weighted Average for 1996			1.078

This may not seem like a large difference. But let's go a little further with this calculation and track the difference in the indicated rate change caused by the two assumptions.

The difference between a rate indication of 10.5 percent and 8.6 percent may not seem large, but it is a large difference to the marketing department, underwriters, agents, state regulators, and your customers. This amount of difference in the future premium can have a significant impact on the bottom line. Finally, the uniform writings assumption is not appropriate in this situation, so why use it?

With the power and availability of today's computers, it may not be necessary to make these approximations when bringing premium on-level. The extension of exposures method, which re-rates each past policy at today's rates, is more accurate. Systems limitations or signifi-

each date of rate change, we have the following: 1/1/95: 60% × 50% = 30%; 1/1/96: 60% × 37.5% + 40% = 62.5%; 7/1/96: 60% × 12.5% = 7.5%.

Table 8
Comparison of Indicated Rate Changes Under Different Writings Assumptions

	Uniform Writings	40% on January 1
(1) Losses trended to future accident period:*	500,000	500,000
(2) Earned premium for 1996 accident year:*	700,000	700,000
(3) Calculated on-level rate factor (Table 7):	1.096	1.078
(4) Converted premium ((2) × (3)):	767,200	754,600
(5) Calculated loss ratio ((1) ÷ (4)):	65.2%	66.3%
(6) Target loss ratio:*	60.0%	60.0%
(7) Calculated rate indication: ((5) ÷ (6))	1.086	1.105
(8) Selected rate increase:	8.6%	10.5%

Notes: *These items were selected arbitrarily; we did not need them in the parallelogram calculations in Table 7.

cant changes in the class plan, however, may make it difficult to update premium on this detailed basis. In these cases, understanding the impact of the uniform writings assumption may save you from avoidable errors.

6 Assumptions in Software Tools

Increasing numbers of actuaries are studying complex problems using computer programs that are either off-the-shelf or written specifically for the task. Examples include the modeling of catastrophes, asset liability management, and dynamic financial analysis. Using such programs can save time and effort, enabling a more in-depth analysis with greater speed. As pointed out earlier in Section 5.3, blindly using software tools as black boxes can lead to misuse and error.

The needed level of understanding depends on the use of the tool. For example, generally it is not necessary to question how spreadsheet functions have been implemented. But if you are performing extensive Monte Carlo simulations, however, you should understand the workings of the random number generator. In particular you must be certain that the generator has a sufficiently large period and is capable of producing a sufficiently random stream. It may be prudent to perform statistical tests on the generator to judge how well its output mimics the behavior of a truly random sequence.[2]

It's important to be aware of general software issues such as the default values, the functional approximations used, and the number of significant digits used in calculations. Also, to the extent that the tool incorporates actuarial or statistical techniques, the user should be aware of assumptions inherent in these—for example the probability distibutions selected, the type of IBNR calculation used, assumptions about parameter uncertainty, and the theory behind and practical effect of any statistical tests.

7 Assumptions about the Business Environment

Actuaries must take into account the business and economic environment when performing calculations; this entails making assumptions. These assumptions can be broken down into three categories: (i) departures from the past, (ii) the future economic environment, and (iii) company-specific assumptions.

7.1 Departures from the Past

Occasionally a new situation arises and the old actuarial assumptions are no longer valid and you require new assumptions. In such cases the new assumptions become very significant, as there are few, if any, data to support them. For example:

Brand new coverages. When a new coverage is introduced, actuaries generally have few, if any, relevant historical data. In order to create a price for this new coverage, underwriters and actuaries must estimate frequency and severity of future claims. Selecting appropriate frequencies, severities, and exposure bases, however, means relying heavily on assumptions.

Law changes and unique settlements. Pollution liabilities have proven difficult to estimate due to Superfund, Superfund reform, and the prospect of further Superfund reform. Additionally, there are data and data interpretation problems.

One suggestion when dealing with new situations is to use the same methods being used by others in the industry. For example, you might

[2]For more on random number generators see, for example, Kalos and Whitlock (1986, Appendix) and Bratley, Fox, and Schrage (1983, Chapter 6).

use the multiple-of-current-payments approach described by Bouska and McIntyre (1994) in evaluating pollution liabilities. A standard approach may prevent unpleasant situations with regulatory bodies and rating agencies. But if the question is critical for the insurance company, the actuary may want to consult with outside experts about the validity of applying standard assumptions and methods to the particular situation.

7.2 Assumptions about Future Economic Environments

The economic environment, particularly its future outlook, affects a number of actuarial areas including valuations, the discount of loss payment patterns, and estimation of future inflation rates for pricing. Given the complexity of our economy, some assumptions are necessary.

Interest rate curve or fixed interest rate. Although life and pension actuaries have always been aware of the importance of interest rates, property and casualty actuaries have historically paid less heed to the interest rate assumption.[3] U.S. statutory accounting principles do not allow discounting of many property-casualty loss reserves.

With the advent of asset-liability management and dynamic financial analysis, property-casualty actuaries are increasingly involved in creating interest rate assumptions. Issues to consider include flat versus upward sloping yield curves, the use of a risk-free versus a risk-adjusted rate, inclusion or exclusion of inflation, and methods for measuring duration. For a discussion of these issues see, for example, Panjer et al., (1998) and, from a European perspective, Daykin, Pentikäinen, and Pesonen (1994, Part 2).

With life, pensions, annuities, or any other long-tail business, actuaries also make assumptions about reinvestment rates of return. Reinvestment risk may be the greatest risk to profitability. Assets that are intended to support future liabilities often provide

[3]This may seem surprising to life actuaries. But the neglect of interest rate assumptions in the older property-casualty literature should be put in context. Developing as it did from fire and marine insurance, property-casualty practice was originally much more concerned with the shorter-tail lines of business. Even for the longer-tail property-casualty lines, the uncertainty in the amount to be paid has generally been much greater than the potential amount of interest discount. Because the interest discount was perceived to have a relatively small effect, it did not receive much attention in the early literature. The emphasis on appropriate interest rate selection has changed as time to payment has lengthened, as inflation has become more important, and as techniques have become more refined.

interim cash flows, e.g., bond coupons. Assuming that future income flows can be reinvested at today's prevailing interest rates could lead to poor decisions. Yield curves may shift or change shape. Actuaries need to take into account relevant economic forecasts and the expected pattern of future cash flows. Margins for model risk, parameter risk, and process risk should be considered; see, for example, McClenahan (1990) and Geske (1999) for more on this.

Trends for premiums and losses. In the 1980s and the early 1990s, U.S. insurer losses in their workers compensation line of business appeared to be out of control. Annual medical costs were climbing, and actuaries could clearly see an upward trend.

Then things changed. Whether you credit the U.S. federal government, the health care system, or the various state legislatures around the country, workers compensation costs suddenly decreased. This is an example in which the assumption that tomorrow would be like yesterday was incorrect.

As this example illustrates, inflation rates for losses (and premiums) do change (sometimes suddenly) over time. Nevertheless, a constant trend is assumed in many actuarial models to project future cost levels. This can produce distortions in loss development, loss ratio estimates, and payment patterns.

Other economic situations. There are many more situations in which actuaries have to make economic assumptions: future currency exchange rates for international business; change in property values for residual value insurance; and the effect of the broader economy on sensitive lines of business such as credit and surety.

7.3 Assumptions about Your Business

It is important for actuaries to keep abreast of the changes in their business and to translate these changes into appropriate actuarial assumptions.

Here are some areas that may be worth inspection.

The underwriting mix. This refers to the types of products sold, where the products are sold, the demographics of those purchasing, and/or the limits of the policies. Most actuarial methods assume that the underwriting mix is constant. If the mix changes it may affect how the losses and premiums develop.

For example, an actuary may warn that rates are too low for a certain territory. The underwriters, in response to the warning, reduce their writings in this territory. Unless this action is communicated to the actuary, inappropriate assumptions may be used in the next analysis.

Other possibilities abound. For example, if offered policy limits are increased, the actuary must allow for increased development of higher losses. A decision to stop offering higher limits would have the opposite effect. Turnover in the underwriting staff may shift the amounts of business written in certain territories or lines of business. Demographic shifts over time, such as increasing percentages of the population moving to urban or coastal areas, also can have a major impact.

The cost of reinsurance. Many actuarial methods are based on an analysis of direct data, so they assume implicitly that the cost of reinsurance is zero. The transaction costs associated with reinsurance, however, can be significant. If no allowance for such costs is made in the analysis, the overall return realized on the net book of business may not achieve the goals outlined in the rate filing.

The company's business plan. One possible source of information is your company's business plan. You need to take care in using this plan; some business plans are compiled as a matter of form and are not adequate reflections of the intentions of management. You also need to be cognizant of the many assumptions used in making the plan and whether these assumptions are reasonable.

Assumptions about your business may find their way into your work, possibly through communication—or lack of it—with other departments in your company. To be aware of these and other business effects, you need regular contact with underwriting and other departments. You may need to improve your communication with other actuaries.

8 You are Aware of Assumptions—What Next?

Once you are aware of your assumptions you must (i) document them, (ii) check to see that they are correct and appropriate, (iii) check for consistency among them, and (iv) quantify their impact on your work.

Documentation: The objective here is to make sure that others who rely on your work are aware of your assumptions. You should state and explain all assumptions, emphasizing the most significance ones. A good rule of thumb to follow is to make sure that another actuary practicing in your field can follow and understand your work. This is the level of documentation expected by the Actuarial Standards Board of the American Academy of Actuaries (1991). Even if no one else examines your work, going through this process may help you carefully choose your assumptions.

Correctness: Each assumption should be checked to see if it is correct and appropriate. Some assumptions, though technically incorrect, may be made in order to simplify calculations.

Consistency: There are two types of consistency to look for: consistency within an analysis and consistency across analyses. Consistent assumptions within an analysis simply means that, for the particular analysis, the assumptions make sense. Consistent assumptions across different analyses, however, mean that similar assumptions must be used in different analyses. For example, suppose the loss ratios and development patterns used to set your reserves are markedly different from the ones used to calculate rate indications. Then you might forecast a bad profit result for the year even if your whole rate change is approved. Similarly, if you are using one set of interest rate assumptions for estimating your liabilities, you may want to use the same set for valuing your assets. Consistent assumptions across analyses may not be required but should be considered.

Another important area for consistency is in financial planning. Anderson (1998) discusses this aspect of actuarial practice. He argues that if planning assumptions "such as catastrophe loads, loss trends and the effects of variability are not explicitly linked to the assumptions used for ratemaking on the product and state level, a built-in bias may be created for either rate adequacy or rate redundancy." Thus the planned results are not achieved.

Choosing planning assumptions consistently with pricing and reserving assumptions may help management to better understand the company's results and to see why operating differences from plan arise.

Quantification: In order to understand the importance of your assumptions you must be able to quantify their impact. The greater the impact of your assumption, the more important it is that you test the assumption. But what does it mean to test an assumption?

How can you go about quantifying its importance? Here are some possibilities.

- **What-if scenarios (stress testing).** One way to judge the importance of assumptions is through stress testing. You can test and quantify the impact of assuming different loss ratios, different lapse rates, and different tail factors. How much does the tail factor have to change in order to create a 5 percent change in the estimate of the reserves? Is such a change in the tail factor likely or not?
- **Different methods.** Using various approaches to solving a problem, such as link ratio versus Bornhuetter-Ferguson versus loss ratio reserving methods, allows you to determine whether different methods are giving approximately the same answer. If they do, you can work with greater confidence. If they do not, check to determine which assumptions underlying the methods might be leading to the discrepancy.
- **Different data sets.** This is similar to the idea behind using different methods. Using various parts of your data—such as incurred versus paid losses, incremental versus cumulative, primary versus excess—is another way to test your results and your assumptions. You also may want to investigate data resampling techniques such as the jackknife and the bootstrap which are based on repeated sampling of parts of the data; see Efron (1982).
- **Simulations.** Stochastic simulations are often the only way to proceed when working with relationships that are complicated and involve many variables. Simulation can help you get a handle on the results. Additionally, simulation allows you to perform stress tests using thousands of different what-if scenarios, whether the underlying relationships are complex or simple.
- **Effect of changing assumptions over time.** Sometimes assumptions change from one actuarial report to the next. Khury (1980) describes the idea of actuarial gain or loss, i.e., the change in a final amount (such as a reserve) that is due to a change in an assumption rather than due to changes in the data. It is important to communicate the size and nature of the actuarial gain or loss to management, so that they have a clear understanding of the current business situation.

- **Range of estimates.** It is standard actuarial practice to establish a range of values for estimates, not just a single number. The range may reflect (i) statistical confidence interval, (ii) the impact of different assumptions, or (iii) uncertainty about your assumptions.

9 Benefits from Understanding Assumptions

There are many benefits to better understanding your assumptions. We have divided them into three categories: professional, practical, and business. The boundaries between these categories, however, are fluid.

Professional Benefits: Understanding and testing your assumptions is a requirement of actuarial standards of practice.

Practical Benefits: Knowing the assumptions behind the various methods can help determine which one is most appropriate or which method will lead to greater certainty and/or reliability regarding the final outcome. Understanding your assumptions helps you determine the level of confidence you can have in your estimates. If you have too little confidence, an understanding of the assumptions can help you determine the next steps you ought to take (if any) in an investigation.

Business Benefits: These are many; only a few are cited here.

- **Early warning system.** Thoroughly understanding your assumptions can show the areas in your company that need the greatest attention. Key assumptions can be monitored as experience emerges, possibly allowing for corrections to your analysis before the experience is mature enough for a complete review. This helps to prevent surprises when the situation does not develop as originally planned.

  Awareness of these assumptions may also allow for strategic action with respect to the business. Of course, there may be some things over which the company has little control, such as inflation or mortality rates. But other items may be open to company influence. For example, a scenario test may demonstrate that a lower lapse rate among term life policyholders is required in order to reach a profit goal. Efforts can then be directed toward achieving this lower lapse rate.

- **Creating a safety margin.** In other situations, you may be able to safeguard future results. For example, if you are creating a specialty insurance product, you may be able to make sure that the product design protects you in case your more significant assumptions are incorrect.

  Suppose that you are designing a reinsurance cover, but your client has only limited data. You are forced to rely on industry data and assume that your client will have average results. If you are uncomfortable with this assumption, maybe you can build in a special sub-limit for the lines of business with the scantiest data or the lines where you fear your client may have worse-than-average experience. Or perhaps you can charge a higher up-front premium and offer a generous profit share in the event that ceded losses are small.

- **Planning.** As discussed earlier, understanding your assumptions can help you ensure that your financial plan is consistent with your pricing and reserving operations (Anderson 1998). This makes your company's operations more transparent and can help management to more easily identify reasons for deviations from plan.

Actuarial science is inexact at best. By understanding your assumptions you can avoid unnecessary errors, increase your level of certainty in your results, and improve the decisions made by your company.

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Commissioners Annuity Reserve Valuation Method (CARVM)

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Abstract

This paper describes the commissioners annuity reserve valuation method (CARVM) and highlights the fundamental contrast with insurance valuation. Numerical examples illustrate methods of applying CARVM to particular annuity designs. The application of NAIC Actuarial Guideline 13 on bailouts is given particular attention.

Key words and phrases: *cash surrender values, single premium deferred annuity, antiselection, election*

1 Introduction

Annuity business has shown substantial growth over the past two decades. According to the *Life Insurance Fact Book 1998* the total U.S. industry reserves for annuities were more than twice as high as reserves for life insurance policies (ACLI 1999, p. 114).

Annuity products generally are designed so that one optional benefit is a series of payments till death, a life annuity. Despite their name, however, most annuities bought by individuals are purchased as tax-favored cash accumulation vehicles. Most annuity contracts terminate by surrender rather than annuitization and eventual death.

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The commissioners annuity reserve valuation method (CARVM) was first defined by the National Association of Insurance Commissioners (NAIC) in the 1976 amendments to the Standard Valuation Law (American Academy of Actuaries, 1997). The CARVM reserve was defined to be:

> ... the greatest of the respective excesses of the present values, at the date of valuation, of the future guaranteed benefits, including guaranteed nonforfeiture benefits, provided for by such contracts over the present value at the date of valuation, of any future valuation considerations derived from gross premiums ....

The valuation considerations are defined to be " ... the portions of the respective gross considerations applied under the terms of such contracts to determine nonforfeiture values ... ." These valuation considerations are here given the symbol P^{NFV}. This definition of the CARVM reserve will be represented here by the following formula, valid in some simple cases:

$$_tV_x^{CARVM} = \max_{n \geq t}\{B_n v^{n-t} - P^{NFV} \ddot{a}_{\overline{n-t}|}\} \tag{1}$$

where x is the issue age, t is the number of policy years that have elapsed at valuation, B_r is the surrender benefit at the end of policy year r, $r = 1, 2, \ldots$, and n is the policy year-end being tested to determine whether it gives the maximum. The maximum is taken over all possible future policy year-ends, n, including the possibility $n = t$ of immediate surrender.

All the policy year-ends at which the annuity holder may elect to receive a benefit are included. For most annuities the most important mode of termination is surrender at a date n chosen by the policyholder, hence B_n. Detailed consideration of non-elective benefits, for example those available on death, are considered in Sharp (1999).

CARVM differs greatly from the usual prospective definition of a life insurance reserve.

$$\begin{aligned} {}_tV_x^{INS} = & \sum_{m=t+1}^{\infty} DB_m \times {}_{m-t-1}p_x \times q_{x+m-t-1} v^{m-t} \\ & - \sum_{m=t+1}^{\infty} P^{NET} \times {}_{m-t-1}p_x \times v^{m-t-1}. \end{aligned} \tag{2}$$

For insurance the dominant benefit may be the death benefit DB_m.

A related matter is that the insurance reserve in equation (2) uses probabilities (e.g., ${}_{m-t-1}p_x$ and $q_{x+m-t-1}$) while the CARVM reserve equation (1) has no probabilities. CARVM replaces the probabilities with a maximum. The CARVM philosophy is to assume that with 100 percent certainty the policyholder will antiselect at the worst time for the insurance company. This is reasonable, though conservative, in the case of an annuity and its surrender benefit. In the life insurance case it would be unreasonable to assume that the policyholder is so eager to antiselect at the optimum time that he or she will die to do so. Thus the CARVM calculations in this paper do not use probabilities.

Jaffe (1983) gave an early analysis of CARVM. Much of that paper and its discussion are still relevant today, though the impact of the relatively recent Actuarial Guidelines 33 and 34 (see e.g., National Association of Insurance Commissioners, 1998) should be kept in mind.

2 CARVM Reserve in a Simple Case

The application of CARVM is most readily illustrated through numerical examples. First, consider a single premium deferred annuity (SPDA), which is an annuity more straightforward than would usually be met in practice. In particular there are no explicit loads. The valuation assumptions are listed in Table 1.

The guaranteed credited rates are those specified in the contract at issue, possibly with stronger guarantees made subsequent to issue. The guaranteed annuitization basis is less generous than the valuation basis. Thus, the lump sum equivalent to the guaranteed annuity will be, on the valuation basis, of lesser value than the actual lump sum benefit option. Therefore, we can ignore the annuitization guarantee in calculating the reserve. The converse case will be considered later.

We calculate the reserve required under CARVM if the single premium were \$10,000 and accumulation during 1996 and 1997 were at the guaranteed 8 percent p.a. Thus we calculate the cash surrender values (CSV) that would apply on surrender at the end of each policy year. The present value at the valuation date, two years after issue, is found by discounting the CSVs at the 6 percent p.a. valuation rate. For years when the 8 percent p.a. accumulation is operative, an additional year of accumulation increases the present value because 6 percent is less than 8 percent. Once the accumulation credited rate falls to 5 percent p.a., a year's delay in surrendering reduces the present value because 5 percent is less than 6 percent.

Table 1
Valuation Assumptions for
A Single Premium Deferred Annuity (SPDA) Without Loads

Single premium:	\$10,000
Front-end load:	0 percent of single premium
Surrender charge (back-end load):	0 percent
Guaranteed credited rates:	8 percent p.a. for 5 years; 5 percent thereafter
Actual credited rates:	8 percent p.a. in 1996; 8 percent p.a. in 1997
Guaranteed annuitization basis:	4 percent p.a., IA 83 (Table a)*
Issue date:	December 31, 1995
Maturity date:	December 31, 2019
Valuation date:	December 31, 1997
Valuation rate:	6 percent p.a.
Valuation mortality	
Before maturity:	Zero mortality
After maturity:	IA83 (Table a)

Notes: p.a. = per annum; *Published by the Society of Actuaries, Schaumburg, Ill., USA.

The minimum reserve is the largest value in the right column of Table 2. This column corresponds to a surrender on December 31, 2000 and is

$$12,337 = 10,000 \times 1.08^2 \times 1.08^3/(1.06^3).$$

This surrender contains no probabilities; the maximum occurs at the point at which the guaranteed credited rate falls below the valuation rate.

The above calculation method has sometimes been used if nonzero pre-maturity mortality is assumed, but the death benefit equals the cash surrender value. Then, effectively, the above calculation ignores death as a separate decrement and includes it as a surrender. This is conservative because the annuitant won't choose the most expensive time to die, so the 100 percent certainty philosophy shouldn't apply to the death benefit. Under Actuarial Guidelines 33 and 34 (NAIC, 1998) this conservative approximation is not the approved method.

Table 2
CARVM Cash Surrender Value (CSV) and Present Value (PV) Calculations For the Single Premium Deferred Annuity (SPDA) Without Loads

Dec. 31	Cash Surrender Value (CSV)	PV at Dec. 31, 1997
1995	$10,000 = 10,000$	-
1996	$10,800 = 10,000 \times 1.08$	-
1997	$11,664 = 10,000 \times 1.08^2$	$11,664 = 10,000 \times 1.08^2$
1998	$12,597 = 10,000 \times 1.08^2 \times 1.08$	$11,884 = 10,000 \times 1.08^2 \times 1.08/(1.06)$
1999	$13,605 = 10,000 \times 1.08^2 \times 1.08^2$	$12,108 = 10,000 \times 1.08^2 \times 1.08^2/(1.06^2)$
2000	$14,693 = 10,000 \times 1.08^2 \times 1.08^3$	$12,337 = 10,000 \times 1.08^2 \times 1.08^3/(1.06^3)$
2001	$15,428 = 10,000 \times 1.08^2 \times 1.08^3 \times 1.05$	$12,220 = 10,000 \times 1.08^2 \times 1.08^3 \times 1.05/(1.06^4)$
2002	$16,199 = 10,000 \times 1.08^2 \times 1.08^3 \times 1.05^2$	$12,105 = 10,000 \times 1.08^2 \times 1.08^3 \times 1.05^2/(1.06^5)$

3 CARVM Reserve in the Presence of Loads

Let us now continue the example with a few complications added:

1. There is a front-end load charged to the policyholder;
2. There is a surrender charge (back-end load) in the early policy years;
3. Accumulation before the valuation date was at a higher credited rate than that originally guaranteed.

The complete set of valuation assumptions is given in Table 3.

Table 3
Valuation Assumptions for
A Single Premium Deferred Annuity (SPDA) With Loads

Single premium:	$10,000
Front-end load:	4 percent of single premium
Surrender charge (back end load):	8 percent for the first six years from issue and zero on and after January 1, 2002
Guaranteed credited rates:	8 percent p.a. for five years; 5 percent p.a. thereafter
Actual credited rates:	9 percent p.a. in 1996 and in 1997
Guaranteed annuitization basis:	4 percent p.a., IA83 (Table a)
Issue date:	December 31, 1995
Maturity date:	December 31, 2019
Valuation date:	December 31, 1997
Valuation rate:	6 percent p.a.
Valuation mortality:	
Before maturity:	Zero mortality
After maturity:	IA83 (Table a)

The reserve required under CARVM is calculated using a single premium of $10,000 and accumulation during 1996 and 1997 of 9 percent p.a., which is higher than the 8 percent p.a. that was guaranteed at issue. The minimum reserve is the largest value in the right column of Table 4.

Table 4
CARVM Cash Surrender Value (CSV) and Present Value (PV) Calculations For the Single Premium Deferred Annuity (SPDA) With Loads

Dec. 31	Cash Surrender Value (CSV)	PV at Dec. 31, 1997
1995	$8,832 = 0.92 \times 0.96 \times 10,000$	-
1996	$9,627 = 0.92 \times 0.96 \times 10,000 \times 1.09$	-
1997	$10,493 = 0.92 \times 0.96 \times 10,000 \times 1.09^2$	$10,493 = 0.92 \times 0.96 \times 10,000 \times 1.09^2$
1998	$11,333 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08$	$10,691 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08/(1.06)$
1999	$12,239 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^2$	$10,893 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^2/(1.06^2)$
2000	$13,219 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3$	$11,099 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3/(1.06^3)$
2001	$13,879 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05$	$10,994 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05/(1.06^4)$
2002	$15,841 = 1.00 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05^2$	$11,837 = 1.00 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05^2/(1.06^5)$

This is a competition between December 31, 2000 at

$$11,099 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3/(1.06^3)$$

and December 31, 2002 at

$$11,837 = 1.00 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05^2/(1.06^5).$$

The latter wins because the effect of removing the 0.92 back-end load factor outweighs the additional two years of discounting at 6 percent p.a. a sum accumulating at only 5 percent p.a. ($1.05^2/1.06^2$). As usual, the valuation is of only the contract guarantees applying at the valuation date. Thus, the 5 percent accumulation is used even though after December 31, 1997 the insurance company may choose to credit more than 5 percent in order to discourage surrenders.

4 New York Continuous CARVM Reserve

Under standard CARVM the maximum in equation (1) is taken over all possible future policy year-ends. Equation (1) is often referred to as *curtate CARVM.* Under New York's version of CARVM (New York Insurance Law, Section 4217(6)(D)) the maximum in equation (1) is taken over each possible future day of surrender; continuous CARVM. This can make a significant difference if there is a back-end load that reduces at the end of a policy year. The valuation assumptions are given in Table 5.

The minimum reserve is the largest value in the right column of Table 6 visualized as being produced for each possible surrender day. This is the continuous CARVM method prescribed by New York. This maximum value is just after the surrender charge is removed, January 1, 2002, in view of the 6 percent discounting beating the 5 percent accumulation:

$$\$11,950 = 1.00 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05/(1.06^4).$$

Some insurance commissioners in other states including Illinois and Virginia require the use of New York continuous CARVM in at least some situations. NAIC Actuarial Guideline 33 is neutral on this topic.

Table 5
New York CARVM Reserve Valuation Assumptions for A Single Premium Deferred Annuity (SPDA) With Loads

Single premium:	\$10,000
Front-end load:	4 percent of single premium
Surrender charge (back end load):	8 percent for the first six years from issue; zero from Jan. 1, 2002
Guaranteed credited rates:	8 percent p.a. for five years; 5 percent p.a. thereafter
Actual credited rates:	9 percent p.a. in 1996 and in 1997
Guaranteed annuitization basis:	4 percent p.a., IA83 (Table a)
Issue date:	December 31, 1995
Maturity date:	January 1, 2020
Valuation date:	December 31, 1997
Valuation rate:	6 percent p.a
Valuation mortality:	
Before maturity:	Zero mortality
After maturity:	IA83 (Table a)

5 CARVM Reserve Under Annuitization Option

Under CARVM it is necessary to consider all options that can be exercised (elected) by the annuity owner. Frequently an annuity contract includes a current settlement provision that at annuitization the company's then in-force annuitization rates will be used if more favorable than those guaranteed in the contract.

We calculate the reserve required under (non-New York) CARVM assuming the single premium was \$10,000 and the accumulation during 1996 and 1997 (i.e., January 1, 1996 to December 31, 1997) was at 9 percent p.a. We are given that at m, the age at maturity,

$$a_m(6\%)/a_m(7\%) = 1.085$$

and that annuitization is allowed only at the maturity date.

Table 6
New York CARVM Cash Surrender Value (CSV) and Present Value (PV) Calculations For the Single Premium Deferred Annuity (SPDA) With Loads

Dec. 31	Cash Surrender Value (CSV)	PV at Dec. 31, 1997
1995	$8,832 = 0.92 \times 0.96 \times 10,000$	-
1996	$9,627 = 0.92 \times 0.96 \times 10,000 \times 1.09$	-
1997	$10,493 = 0.92 \times 0.96 \times 10,000 \times 1.09^2$	$10,493 = 0.92 \times 0.96 \times 10,000 \times 1.09^2$
1998	$11,333 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08$	$10,691 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08/(1.06)$
1999	$12,239 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^2$	$10,893 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^2/(1.06^2)$
2000	$13,219 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3$	$11,099 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3/(1.06^3)$
2001	$13,879 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05$	$10,994 = 0.92 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05/(1.06^4)$
1/1/2002	$15,086 = 1.00 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05$	$11,950 = 1.00 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05/(1.06^4)$
12/31/2002	$15,841 = 1.00 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05^2$	$11,837 = 1.00 \times 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05^2/(1.06^5)$

Table 7
Valuation Assumptions for an Annuitization Option
Single Premium Deferred Annuity (SPDA) With Loads

Single premium:	\$10,000
Front-end load:	4 percent of single premium
Surrender charge (back end load):	0 percent
Guaranteed credited rates:	8 percent p.a. for five years; 5 percent p.a. thereafter
Annuitization age:	Allowed only maturity date
Guaranteed annuitization basis:	7 percent p.a., IA83 (Table a)
Issue date:	December 31, 1995
Maturity date:	December 31, 2002
Valuation date:	December 31, 1997
Valuation rate:	6 percent p.a.
Valuation mortality:	
Before maturity:	Zero mortality
After maturity:	IA83 (Table a)

In reality, individual deferred annuity contracts often allow considerable flexibility in the timing of annuitization. The valuation assumptions are listed in Table 7.

The minimum reserve is the largest value in the right column of Table 8. This corresponds to the December 31, 2002 annuitization on a basis more generous (7 percent) than the valuation basis (6 percent p.a.). The amount of annual annuity purchased with an amount M is $M/a_n(7\%)$. When valued at 6 percent p.a. this gives a reserve $a_n(6\%)/a_n(7\%)$, discounted back to the valuation date:

$$12,843 = 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05^2 \times 1.085/(1.06^5).$$

If such a current settlement provision is present, then consideration must be given to Actuarial Guideline 33. Under AG 33 the provision would trigger the application of the reserve floor of 93 percent of the contract fund value at valuation.

Table 8
Annuitization Option Cash Surrender Value (CSV) and Present Value (PV) Calculations For the Single Premium Deferred Annuity (SPDA) With Loads

Dec. 31	Cash Surrender Value (CSV)	PV at Dec. 31, 1997
1995	$9,600 = 0.96 \times 10,000$	-
1996	$10,464 = 0.96 \times 10,000 \times 1.09$	-
1997	$11,405 = 0.96 \times 10,000 \times 1.09^2$	$11,405 = 0.96 \times 10,000 \times 1.09^2$
1998	$12,318 = 0.96 \times 10,000 \times 1.09^2 \times 1.08$	$11,620 = 0.96 \times 10,000 \times 1.09^2 \times 1.08/(1.06)$
1999	$13,304 = 0.96 \times 10,000 \times 1.09^2 \times 1.08^2$	$11,840 = 0.96 \times 10,000 \times 1.09^2 \times 1.08^2/(1.06^2)$
2000	$14,368 = 0.96 \times 10,000 \times 1.09^2 \times 1.08^3$	$12,063 = 0.96 \times 10,000 \times 1.09^2 \times 1.08^3/(1.06^3)$
2001	$15,086 = 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05$	$11,950 = 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05/(1.06^4)$
2002	$15,841 = 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05^2$	$11,837 = 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05^2/(1.06^5)$
2002	Annuitize	$12,843 = 0.96 \times 10,000 \times 1.09^2 \times 1.08^3 \times 1.05^2 \times a_n(6\%)/(a_n(7\%) \times 1.06^5)$

Table 9
Bailout Significant: Valuation Assumptions for A Single Premium Deferred Annuity (SPDA) With Loads

Single premium:	$100,000
Guaranteed interest rate	
Years 1 to 5:	8 percent
Years 6 to 10:	6 percent
Years 11 and above:	3 percent
Front-end load:	4 percent
Surrender charge	
Years 1 to 4:	5 percent
Years 5 to 10:	2 percent
Years 11 and above:	0 percent
Bail-out rate:	7 percent
Valuation rates:	
This SPDA:	$6\frac{1}{2}$ percent
Whole life policy:	$5\frac{1}{2}$ percent

6 Actuarial Guideline 13 on Bailouts

6.1 Bailout Significant

It is common for an annuity contract to include a bailout feature. This is a response to potential policyholder fear that he or she will be trapped by the surrender charge into holding an annuity with a below-market-credited rate. The bailout generally allows for a surrender gross of surrender charge (back-end load) if the credited rate falls below a bailout rate specified in the contract.

There was some confusion about whether CARVM required the reserve in such circumstances to be gross of the back-end load. NAIC Actuarial Guideline 13 (NAIC, 1998) clarifies this by requiring that the reserve be gross of the load only if the bailout is significant. The term 'significant' is not used in the guideline, but it refers to a situation where it is thought that there is chance that the bailout clause will come into play. Under the guideline the bailout is, generally, significant if the bailout rate is higher than the long-life rate. The long-life rate is the standard valuation law valuation interest rate used for policies with guarantee durations of more than 20 years.

Thus, in reserving for an annuity with bailout, the first step is to determine whether the bailout is significant. In the example below, the bailout is significant and the present value is taken of the surrender value gross of surrender charge. The valuation assumptions are listed in Table 9 and the reserve at issue under CARVM is shown in Table 10.

The bailout rate is 7 percent, greater than the $5\frac{1}{2}$ percent long valuation rate. The bailout is significant, so the surrender charge is contingent and not available to reduce the reserve. In years 1 through 5 the guaranteed credited rate of 8 percent is higher than the bailout rate of 7 percent, so bailout cannot occur. From year 6 we have 6 percent less than 7 percent and the bailout is significant. The maximum of PV (CSV) for years 1 to 5 and PV (Fund) for years above 5 is $102,470. (See Table 10.)

6.2 Bailout Not Significant

Let us now consider an annuity with a lower bailout rate, 4 percent p.a. Table 11 shows the valuation assumptions. We calculate the reserve at issue under CARVM in light of Actuarial Guideline 13 on Bailouts. The results are shown in Table 12.

The bailout is not significant because the bailout rate, 4 percent p.a., is less than the whole life valuation rate (long life rate) 5.5 percent p.a. Thus we can ignore the bailout and calculate the reserve assuming the surrender charge to be available to reduce the reserve. The reserve thus calculated is $98,205. (See Table 12.)

Table 10

Bailout Significant: Fund, Cash Surrender Value (CSV), and Present Value (PV) Calculations For the Single Premium Deferred Annuity (SPDA) With Loads

Duration	Fund	CSV	PV (Fund)	PV (CSV)
0	$100,000 \times 0.96 = 96,000$	91,200	96,000	91,200
1	$100,000 \times 0.96 \times 1.08 = 103,680$	98,496	97,352	92,485
2	$100,000 \times 0.96 \times 1.08^2 = 111,974$	106,376	98,723	93,787
3	$100,000 \times 0.96 \times 1.08^3 = 120,932$	114,886	100,113	95,108
4	$100,000 \times 0.96 \times 1.08^4 = 130,607$	124,077	101,524	96,448
5	$100,000 \times 0.96 \times 1.08^5 = 141,055$	138,234	102,953	100,894
6	$100,000 \times 0.96 \times 1.08^5 \times 1.06 = 149{,}519$	146,529	102,470	100,421
7	$100,000 \times 0.96 \times 1.08^5 \times 1.06^2 = 158,490$	155,320	101,989	99,949

Table 11
Bailout Not Significant: Valuation Assumptions for A Single Premium Deferred Annuity (SPDA) With Loads

Single premium:	\$100,000
Guaranteed interest rate	
Years 1 to 3:	8 percent
Years 4 to 10:	6 percent
Years 11 and above:	3 percent
Front-end load:	2 percent
Surrender charge	
Years 1 to 4:	5 percent
Years 5 to 10:	3 percent
Years 11 and above:	0 percent
Bail-out rate:	4 percent
Valuation rates:	
This SPDA:	$6\frac{1}{2}$ percent
Whole life policy:	$5\frac{1}{2}$ percent

7 Conclusion

CARVM methodology differs from that used for traditional insurance policies and for annuities in payment. The differences are needed because the time of receipt of benefits can be elected by the policyholder without the necessity of dying.

Annuity designs in practice often include the option to take a partial withdrawal. There is usually also a death benefit. Actuarial Guidelines 33 and 34 clarify techniques to be used with such a mix of benefits. These guidelines are the focus of Sharp (1999), which immediately follows this paper in this issue.

Table 12
Bailout Not Significant
Fund, CSV, and PV Calculations
For the Single Premium Deferred Annuity With Loads

PY	GR	SC	FV	CSV	PV(Fund)	PV(SCV)
0		0.05	98,000	93,100	98,000	93,100
1	0.08	0.05	105,840	100,548	99,380	94,411
2	0.08	0.05	114,307	108,592	100,780	95,741
3	0.08	0.05	123,452	117,279	102,199	97,089
4	0.06	0.05	130,859	124,316	101,720	96,634
5	0.06	0.03	138,710	134,549	101,242	98,205
6	0.06	0.03	147,033	142,622	100,767	97,744
7	0.06	0.03	155,855	151,179	100,294	97,285
8	0.06	0.03	165,206	160,250	99,823	96,828
9	0.06	0.03	175,119	169,865	99,354	96,374
10	0.06	0.03	185,626	180,057	98,888	95,921
11	0.03	0.00	191,195	191,195	95,638	95,638
12	0.03	0.00	196,930	196,930	92,495	92,495
13	0.03	0.00	202,838	202,838	89,455	89,455
14	0.03	0.00	208,924	208,924	86,515	86,515
15	0.03	0.00	215,191	215,191	83,672	83,672
16	0.03	0.00	221,647	221,647	80,922	80,922
17	0.03	0.00	228,296	228,296	78,263	78,263
18	0.03	0.00	235,145	235,145	75,691	75,691
19	0.03	0.00	242,200	242,200	73,203	73,203
20	0.03	0.00	249,466	249,466	70,798	70,798
21	0.03	0.00	256,950	256,950	68,471	68,471
Maximum						98,205

Notes: PY = Policy Year; GR = Guaranteed Credited Rate; SC = Surrender Charge; FV = Fund Value at End of Policy Year; CSV = CSV at End of Poilcy Year; PV(Fund) = PV(Fund) at Issue Gross of Surrender Charge; and PV(SCV) = PV(SCV) at Issue Ignoring Bailout.

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CARVM and NAIC Actuarial Guidelines 33 & 34

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Abstract

Annuity valuation under the NAIC Standard Valuation Law is determined according to methods different from those methods used for life insurance. The CARVM assumption of efficient policyholder selection is clarified under NAIC Actuarial Guidelines 33 and 34 to allow for non-elective (e.g., death) benefits. In particular, Actuarial Guideline 34 is oriented toward variable annuities and prescribes methods to be used in the presence of a minimum guaranteed death benefit. In this paper these methods are examined and illustrated with examples.

Key words and phrases: *annuity, elective benefit, valuation, reserves*

1 Introduction

In the previous article in this volume, Sharp (1999) explained the calculations involved in determining annuity reserves under the commissioners annuity reserve valuation method (CARVM). These reserves are calculated by a method different from that used for insurance reserves (American Academy of Actuaries, 1997). CARVM assumes that for elective benefits such as surrender, the policyholder will select with 100 percent efficiency the best time to make the election, if the comparisons are made using the company's valuation rate of interest. More

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concisely, this is the worst time for the insurance company. In some simple cases the CARVM reserve is calculated using formulas containing no probabilities.

In this paper we consider the treatment of annuities with a (non-elective) benefit on death under National Association of Insurance Commissioners (NAIC) Actuarial Guideline (AG) 33 (NAIC, 1998). In two examples we consider the case of a fixed (nonvariable) annuity. The treatment is extended in later examples to the valuation under NAIC Actuarial Guideline 34 of variable annuities with a minimum guaranteed death benefit (MGDB).

2 Actuarial Guideline 33

After its 1976 introduction there was some disagreement about how CARVM should be applied to the situation where there were potentially elective and non-elective (e.g., death) benefits. After the issue of Actuarial Guideline 33 (formerly GGG) (see, e.g., Lalonde, 1995) there continued to be some confusion on this issue. The method of using CARVM in complicated situations, however, now has been largely resolved.

At its September 1995 meeting, the NAIC Life and Health Actuarial Task Force interpreted Actuarial Guideline 33 to require consideration of integrated benefits in the CARVM stream(s). Here *integrated* refers to the consideration of the present value of benefit streams under which certain proportions of policyholders are dying and the remaining policyholders are selecting the optimum time of surrender. Under the revised version of Actuarial Guideline 33 effective December 31, 1998, benefits are classified as either elective or non-elective. Each possible set of elections then is considered. This may result in a large tree of possible sets of elections.

For example, there may be a policy provision for annual surrender of up to 10 percent of the annuity value without imposition of a back-end load (surrender charge). One possible branch of the tree would correspond to a 10 percent surrender at the end of policy year one, a 5 percent surrender at the end of policy year two, a 10 percent surrender at the end of policy year three, etc. Typically one can use linearity to cut the number of branches to be tested. In other words, the reserve candidate is likely to be a linear function of the surrender proportion. Hence, the reserve candidate is a monotonic (increasing or decreasing) function of the surrender proportion. In this case the maximum corresponds to either the lowest or the highest possible surrender proportion. In this example, the CARVM maximum would likely correspond to either a 0 percent or 10 percent surrender at the end of year two, not a 5 percent surrender. Superimposed on this structure would be the probabilities of death, a non-elective benefit for which the use of expected values is appropriate. A valuable intuitive analysis of complicated situations like this is given by Backus (1998).

3 Example 1: Simple CARVM with Zero Deaths

The following notation is used throughout all examples:

$$
\begin{aligned}
SC_t &= \text{Surrender charge;}\\
AV_t &= \text{Account value at the end of year } t;\\
&= AV_{t-1} \times 1.06\\
AVAVG_t &= \text{Average account value at the end of year } t\\
&= (AV_{t-1} + AV_t)/2;\\
CSV_t &= \text{Cash surrender value at the end of year } t;\\
&= AV_t \times (1 - SC_t);\\
DB_t &= \text{Death benefit at the end of year } t;\\
NAR_t &= \text{Net amount at risk at the end of year } t;\\
NARAVG_t &= \text{Average net amount at risk at the end of year } t\\
&= (NARAV_{t-1} + NARAV_t)/2;\\
P_t &= \text{Probability of survival from}\\
&\quad\ \text{Dec. 31, 1999 to Jan. 1, } t;\\
q_t &= \text{Assumed annual mortality in year } t;\\
PV_2(NARAVG_t) &= \text{Present value at Dec. 31, 1999 of } NARAV_t\\
&\quad\ \text{paid on midyear death in year } t;\\
&= NARAV_t \times P_t \times q_t \times 1.07^{-(t-1999.5)}\\
PV_2(CSV_t) &= \text{Present value at Dec. 31, 1999}\\
&\quad\ \text{of future Dec. 31 CSVs in year } t;\\
&= CSV_t \times P_t \times (1.07)^{-(t-1999)}\\
PV_2(AVAVG_t) &= AVAVG_t \times P_t \times q_t \times 1.07^{-(t-1999.5)}
\end{aligned}
$$

We first present an example using simple CARVM assuming zero deaths. The assumptions are given in Table 1.

Table 1
Valuation Assumptions for
A Single Premium Deferred Annuity (Fixed)

Issue date:	January 1,1998
Single premium:	$60,000
Accumulation	
Guaranteed:	6 percent per annum
Actual for 1998 and 1999:	6 percent per annum
Death benefit:	$100,000
Front end load:	0 percent
Back end load	
Policy year 1:	8 percent
Policy year 2:	4 percent
Policy year 3:	0 percent
Policy year 4:	0 percent
Valuation date:	December 31, 1999
Valuation mortality rate	
Policy year 1:	0.000
Policy year 2:	0.000
Policy year 3:	0.000
Policy year 4:	0.000
Valuation interest rate:	7 percent per annum

Specifically, assume a January 1, 1998 issue of a $60,000 single premium deferred annuity credited with a guaranteed 6 percent per annum. In other words, the account value (fund value) visible to the policyholder is credited at a rate of at least 6 percent per annum.

Assume that the contract specifies that the policy matures after four years. To motivate a later discussion of a variable minimum guaranteed death benefit we discuss a policy with (perhaps unrealistically) a $100,000 minimum death benefit.

A valuation is to be performed on December 31, 1999, and we need to consider possible surrender on December 31, 1999, December 31, 2000, or December 31, 2001. The immediate December 31, 1999 cash surrender value (CSV) forms a floor for the CARVM reserve. The two dates December 31, 2000 and December 31, 2001 represent two candidates for the status of maximum present value at December 31, 1999.

The CARVM reserve is the greater of these two but with a floor of the immediate CSV. The reserve calculations are shown in Table 2.

Surrender on December 31, 1999

We have accumulation at 6 percent per annum so $AV_t = AV_{t-1} \times 1.06$, where AV_t is the account value at the end of year t. By December 31, 1999 the account value (Rows (3) and (4) of Table 2) has grown to $60,000 \times 1.06^2 = \$67,416$. An immediate surrender would be for $67,416 \times (1-0.04) = \$64,719$, which is a floor to the CARVM reserve.

Surrender on December 31, 2000

A surrender at December 31, 2000 is projected to give a CSV of $60,000 \times 1.06^3 = \$71,461$ and hence a reserve candidate at December 31, 1999 of $71,461/1.07 = \$66,786$.

Surrender on December 31, 2001

A surrender at December 31, 2001 is projected to give a CSV of $60,000 \times 1.06^4 = \$75,749$ and hence a reserve candidate at December 31, 1999 of $75,749/1.07^2 = \$66,162$.

Table 2
Reserve Using Assumption of Zero Mortality

	Policy year from Jan. 1, (t)			
	1998	1999	2000	2001
SC_t:	8%	4%	0%	0%
AV_{t-1} at Jan. 1:	60,000	63,600	67,416	71,461
AV_t at Dec. 31:	63,600	67,416	71,461	75,749
$AVAVG_t$:	61,800	65,508	69,438	73,605
CSV_t:	58,512	64,719	71,461	75,749
$PV_2(CSV_t)$:		64,719	66,786	66,162

CARVM Maximum of the Candidates:

The largest of these candidates is $66,786, which is the CARVM reserve at December 31, 1999 if we are assuming zero mortality. Here we are using strict noncontinuous CARVM, examining only surrenders on the last day of each contract year.

We assume that no decision has been made to use continuous CARVM, that is, to use the maximum over all possible days of surrender. New York requires the use of continuous CARVM. Many actuaries (including the author) believe that continuous CARVM gives more appropriate reserves. In this case a surrender on January 1, 2000, the day after valuation, would give a CSV of $67,416 because the surrender charge is then zero. In reality, it would be preferable to use the $67,416 as floor to the standard CARVM reserve which otherwise is $66,786.

4 Example 2: Assuming Non-Zero Deaths

Let us extend our example to highlight the elective/non-elective distinction. Now the previous contract is revalued assuming nonzero deaths, as indicated below. The fixed $100,000 death benefit is now integrated into the reserve calculation. The assumptions are given in Table 3.

The reserve will consist mainly of the present value of the elected cash surrender value (CSV), but we add also the value of deaths by those who otherwise would make the optimal selection. Surrender on December 31, 2000 or 2001 gives two candidates for the status of maximum present value at December 31, 1999. With consideration of the floor of the immediate December 31, 1999 CSV we have three candidates for the CARVM reserve: surrender on December 31, 1999, surrender on December 31, 2000, or surrender on December 31, 2001.

Surrender on December 31, 1999

Here we consider an immediate surrender on the valuation day. As in Table 2, we have a CSV of $64,719. Under New York (continuous) CARVM (New York Insurance Law, Section 4217(6)(D)) we consider also a surrender a day later, on January 1, 2000. Under noncontinuous CARVM, however, we do not consider a January 1, 2000 surrender even though that would give a higher value because the 4 percent load has then become zero. The December 31, 1999 candidate to be the CARVM maximum is $64,719.

Surrender on December 31, 2000

A proportion $(1 - 0.019)$ of the policyholders survives to the end of calendar year 2000 and under this candidate they then surrender for a

Table 3
Valuation Assumptions for
A Single Premium Deferred Annuity (Fixed)

Issue date:	January 1,1998
Single premium:	$60,000
Death benefit:	$100,000
Valuation interest rate:	7 percent per annum
Accumulation	
Guaranteed:	6 percent per annum
Actual for 1998 and 1999:	6 percent per annum
Front end load:	0 percent
Back end load	
Policy year 1:	8 percent
Policy year 2:	4 percent
Policy year 3:	0 percent
Policy year 4:	0 percent
Valuation date:	December 31, 1999
Valuation mortality rate	
Policy year 1:	0.015
Policy year 2:	0.017
Policy year 3:	0.019
Policy year 4:	0.022
Deaths occur in the middle of the year	

CSV of $\$60,000 \times 1.06^3 = \$71,461$. The present value at December 31, 1999 is

$$71,461 \times (1 - 0.019) \times 1.07^{-1} = \$65,517.$$

Those who die (deaths are assumed to occur on June 30, 2000) receive a death benefit of $100,000; thus the present value of the death benefit is $100,000 \times 0.019 \times 1.07^{-0.5} = \$1,837$. This can also be calculated (Table 4) as the rounded sum of the present value $(1,275 + 561 = \$1,837)$ of two components:

- The average account value at death in 2000,

$$(67,416 + 67,416 \times 1.06)/2 = \$69,438,$$

with present value

$$69,438 \times 0.019 \times (1.07)^{-0.5} = \$1,275.$$

- The average excess of the death benefit over the average account value,

$$(100,000\text{–}69,438) \times 0.019 \times 1.07^{-0.5} = \$561.$$

The Table 4 approach is comparable to that used later in valuing a minimum guaranteed death benefit.

Hence the total value of this candidate is 65,517 + 1,837 = \$67,354.

Surrender on December 31, 2001

A proportion $(1 - 0.019) \times (1 - 0.022)$ of the policyholders survives calendar years 2000 and 2001 and under this candidate they then surrender for a CSV of $\$60,000 \times 1.06^4 = \$75,749$. The present value at December 31, 1999 is

$$75,749 \times (1 - 0.019) \times (1 - 0.022) \times (1.07)^{-2} = \$63,477.$$

Those who die during 2001 (on June 30, 2001) receive \$100,000; thus the present value of the year 2001 death benefit is

$$100,000 \times (1 - 0.019) \times 0.022 \times 1.07^{-1.5} = \$1,949.$$

Some fellow cohorts of those surrendering on December 31, 2001 die also in 2000, so we add that present value also (above), $1,837 + 1,949 = \$3,787$, rounded to agree with the $1,076 + 2,711 = \$3,787$ of Table 4. Hence the total value of this candidate is $63,477 + 3,787 = \$67,264$.

Table 4
Integrated Reserve Including Fixed $100,000 Death Benefit

Policy year commencing Jan. 1 (t):	1998	1999	2000	2001	2002
CSV_t at Dec. 31:	58,512	64,719	71,461	75,749	
DB_{t-1} at Jan. 1:	100,000	100,000	100,000	100,000	
DB_t (death benefit) at Dec. 31:	100,000	100,000	100,000	100,000	
NAR_{t-1} (benefit top-up) at Jan. 1:	44,800	38,944	32,584	28,539	
NAR_t (benefit top-up) at Dec. 31:	41,488	35,281	28,539	24,251	
$NARAVG_t$ (benefit top-up):	43,144	37,112	30,562	26,395	
q_t:	0.015	0.017	0.019	0.022	
P_t:			1.000	0.981	0.959
$PV_2(NARAVG_t)$*:			561	515	
$PV_2(AVAVG_t)$:			1,275	1,435	
$CUMPV_2(NARAVG_t) = \sum_{s=1999}^{t}(NARAVG_s)$:			561	1,076	
$CUMPV_2(AVAVG_t) = \sum_{s=1999}^{t}(AVAVG_s)$:			1,275	2,711	
$PV_2(CSV_t)$**:		64,719	65,517	63,477	
Total-Integrated reserve (V_2^{INT}):		64,719	67,354	67,264	

Notes: V_2^{INT} is the maximum of this row of candidates $V_{2,t}^{CAND}$, and is the sum of the previous three rows of this table.

$*PV_2(NARAVG_t) = NARAVG_t \times P_t \times q_t \times 1.07^{-(t-1999.5)}$.

$**PV_2(CSV_t)$ at Dec. 31, 1999 of Dec. 31 CSV $= CSV_t \times P_t \times 1.07^{-(t-1999.5)}$.

CARVM Maximum of the Candidates

Hence we have for our valuation at December 31, 1999 three possible elections for surrender:

- December 31, 1999: \$64,719
- December 31, 2000: \$67,354
- December 31, 2001: \$67,264.

For this fixed annuity under CARVM, in marked contrast with life insurance valuation, we use the greatest of these three candidates, \$67,354, as our CARVM reserve at December 31, 1999.

5 Example 3: Minimum Guaranteed Death Benefits

Under certain annuity designs, often called *variable annuities*, the account value, and hence the CSV, varies with the investment performance of the underlying assets. Commonly the contract specifies that on death the benefit will be the greater of the account value and a minimum guaranteed death benefit.

Consider a single premium variable annuity with valuation assumptions given in Table 5. We will look at various possible contract provisions defining the death benefit. The benefit on surrender on August 14, 2000 will be the account value net of a 4 percent back-end load:

$$10,000 \times (1 + 0.12) \times (1 - 0.13) \times (1 - 0.04) = \$9,354.$$

If the contract provisions provide for the surrender charge to be waived on death, then the benefit on death on August 14, 2000 is:

$$10,000 \times (1 + 0.12) \times (1 - 0.13) = \$9,744.$$

If on death the surrender charge is waived and there is a minimum benefit of the return of premium (one possible design of minimum guaranteed death benefit), the benefit on death on August 14, 2000 is \$9,744 with a floor of \$10,000; hence the death benefit is \$10,000.

If on death the surrender charge is waived and there is an annual reset of the minimum guaranteed death benefit on the policy anniversary, the benefit on death on August 14, 2000 is \$11,200. It was reset

Table 5
Valuation Assumptions for
A Single Premium Deferred Annuity (Variable)

Issue date:	August 15, 1998
Single premium:	$10,000
Surrender charge	
During policy year 1:	6 percent
During policy year 2:	4 percent
During policy year 3:	2 percent
Thereafter:	0 percent
Actual credited rate	
August 15, 1998 to Aug 14, 1999:	12 percent
August 15, 1999 to Aug 14, 2000:	-13 percent
August 15, 2000 to Aug 14, 2001:	-8 percent
August 15, 2001 to Aug 15, 2002:	2 percent

August 15, 1999 to the then fund value $10,000 \times (1 + 0.12) = \$11,200$ and at August 15, 2000 will be set to $11,200 \times (1 - 0.13) = \$9,744$.

If on death the surrender charge is waived and there is an annual ratchet of the minimum guaranteed death benefit on the policy anniversary, then the benefit on death on August 14, 2001 is $11,200. On August 15, 1999 the minimum guaranteed death benefit was ratcheted up to the then fund value $10,000 \times (1 + 0.12) = \$11,200$ and was left unchanged at August 15, 2000—the ratchet means that the minimum guaranteed death benefit cannot be reduced.

The design of the minimum guaranteed death benefit can vary widely; the above set of illustrations is only a small subset of the possible designs. Actuarial Guideline 34 is intended to apply to all such designs.

6 NAIC Actuarial Guideline 34

AG 34 (NAIC, 1998) requires that minimum guaranteed death benefits be projected by assuming an immediate drop in the values of the assets supporting the variable annuity contract, followed by a subsequent recovery in asset values at a net assumed return until the maturity of the contract. The amounts of the drops and subsequent increase are specified and depend on the types of assets. This immediate drop methodology was adopted for AG 34 after discussion of the risk of a long-term bear market in stocks (American Academy of Actuaries, 1996).

Not all observers would agree, however, that it is appropriate to assume a recovery at a rate higher than the rate of return that would apply if there had been no drop.

The basic reserve for the annuity is to be calculated by methods consistent with CARVM provisions in the standard valuation law and AG 33. This reserve is held in a separate account. For the projection of account values, most companies use the valuation rate of interest less asset charges or, more commonly, mortality and expense charges. The base policy reserve generally equals the CSV obtainable at the date of valuation.

Under AG 34 any additional reserve held for the minimum guaranteed death benefit is held in a general account. We consider an example to illustrate the workings of AG 34.

7 Example 4: No Minimum Guaranteed Death Benefit

Consider the example of a variable SPDA with no minimum guaranteed death benefit. The valuation assumptions are described in Table 6. This example is based partly on that given in American Academy of Actuaries (1996). We are performing a valuation at December 31, 1999, two years after issue. The actual credited rate is known: 9 percent in 1998 and -3 percent in 1999, after reduction by the 1.75 percent asset charge. The results of the reserve calculations are given in Table 7.

The account value at December 31, 1999 is

$$60,000 \times (1 + 0.09) \times (1 - 0.03) = \$63,438.$$

The CSV at December 31, 1999 is $63,438 \times (1 - 0.05) = \$60,266$ where the 0.05 is for the 5 percent surrender charge (back-end load). For projections of years after 1999 we use the assumed investment return of 5.25 percent (= 7 − 1.75 percent). The projected CSV at December 31, 2001 is

$$63,438 \times (1 + 0.0525)^2 \times (1 - 0.02) = \$68,868$$

because the surrender charge has dropped to 2 percent. (See Table 7.)

Table 6
Valuation Assumptions for
A Single Premium Deferred Annuity (Variable)
No Minimum Guaranteed Death Benefit

Minimum guaranteed death benefit rollup rate:	6 percent
Single premium:	$60,000
Issue date:	Jan. 1, 98
Asset charge:	1.75 percent
Investment return:	
Policy year 1 (net of 1.75 percent):	9.00 percent
Policy year 2 (net of 1.75 percent):	-3.00 percent
Future assumed (net of 1.75 percent):	5.25 percent
Immediate drop:	-23.00 percent
Subsequent:	15.00 percent
Valuation rate:	7.00 percent

Assume that the reserve at December 31, 1999 is the greatest of the present values at December 31, 1999 of all possible future surrender values without reduction for probability of death. In effect, we make a valuation assumption of zero deaths. This is equivalent to an assumption that on death the CSV is paid if we ignore the small correction for the fact that a death may occur in a non-optimal year. Assume we are using noncontinuous CARVM, so we are considering only surrenders on the last day of each policy year.

The valuation rate of 7 percent exceeds the assumed accumulation rate of 5.25 percent. Therefore most likely to be the greatest is the immediate CSV at December 31, 1999 of $60,266 or the present value

$$\$68,868 \times (1 + 0.07)^{-2} = \$60,152$$

after the surrender charge drops from 5 percent to 2 percent. This is taken at December 31, 2001, although a surrender at January 1, 2001 would also have a charge of only 2 percent and so would give a higher reserve. Thus $60,266 is the greater of the two values and is confirmed by Table 7 to be the greatest of all values.

In the absence of a minimum guaranteed death benefit, this may have been considered an appropriate CARVM reserve before AG 33 and AG 34. The rough treatment of the death benefit would now not conform with AG 33 and AG 34.

Table 7
Reserve Calculation Ignoring Minimum Guaranteed Death Benefit

	Policy year from Jan. 1 (t)								
	1998	1999	2000	2001	2002	2003	2004	2005	2006
(SC_t):	5%	5%	5%	2%	1%	0%	0%	0%	0%
AV_{t-1} at Jan. 1:	60,000	65,400	63,438	66,768	70,274	73,963	77,846	81,933	86,235
AV_t at Dec. 31:	65,400	63,438	66,768	70,274	73,963	77,846	81,933	86,235	90,762
$AVAVG_t$:	62,700	64,419	65,103	68,521	72,119	75,905	79,890	84,084	88,498
CSV_{t-1} at Jan. 1:	57,000	62,130	60,266	65,433	69,571	73,963	77,846	81,933	86,235
CSV_t at Dec. 31:	62,130	60,266	63,430	68,868	73,224	77,846	81,933	86,235	90,762
$*PV_2(CSV_t)$:		60,266	59,280	60,152	59,772	59,389	58,417	57,462	56,522

Notes: $*PV_2(CSV_t)$ at Dec. 31, 1999 of Dec. 31 = $CSV_t \times 1.07^{-(t-1999)}$.

It is common for the valuation of variable annuities to be performed using a credited rate equal to the valuation rate minus charges. In the absence of a major drop in back-end load in some subsequent year, the CARVM maximum often corresponds to surrender on the valuation date. Hence the CSV often forms the reserve.

8 Example 5: Guaranteed Minimum Death Benefit

We again consider the valuation at December 31, 1999 of the single premium deferred annuity of Example 4 above. Now the contract specifies that on death the benefit equals the greater of:

- Asset value, and
- Minimum guaranteed death benefit of the single premium of $60,000 accumulated at 6 percent p.a.

The results of the reserve calculations are shown in Table 8. The minimum guaranteed death benefit at, for example, December 31, 2001 is given by:

$$60,000 \times (1 + 0.06)^4 = \$75,749.$$

As part of the calculation of the integrated reserve we follow Actuarial Guideline 34. We calculate base asset values on the assumption that at January 1, 2000 for our particular fund type there is an immediate drop of 23 percent in the asset value, followed by a recovery at 15 percent per annum. These particular values are not among those given in AG 34, but are adequate for illustrating the process. The base asset value at December 31, 2001 is:

$$60,000 \times (1 + 0.09) \times (1 - 0.03) \times (1 - 0.23) \times (1 + 0.15)^2 = \$64,601.$$

The asset value is assumed to be subject to a maximum of (capped by) the asset value calculated assuming no immediate drop. The base uncapped asset value at December 31, 2003 thus calculated is:

$$60,000 \times (1 + 0.09) \times (1 - 0.03) \times (1 - 0.23) \times (1 + 0.15)^4 = \$85,434.$$

Table 8
Calculation of Minimum Guaranteed Death Benefit Amounts

		Policy year from Jan. 1 (t)							
1.		1998	1999	2000	2001	2002	2003	2004	2005
2.	(SC_t):	5%	5%	5%	2%	1%	0%	0%	0%
3.	AV_{t-1} at Jan. 1:	60,000	65,400	63,438	66,768	70,274	73,963	77,846	81,933
4.	AV_t at Dec. 31:	65,400	63,438	66,768	70,274	73,963	77,846	81,933	86,235
5.	$AVAVG_t$:	62,700	64,419	65,103	68,521	72,119	75,905	79,890	84,084
6.	CSV_{t-1} at Jan. 1:	57,000	62,130	60,266	65,433	69,571	73,963	77,846	81,933
7.	CSV_t at Dec. 31:	62,130	60,266	63,430	68,868	73,224	77,846	81,933	86,235
8.	$PV_2(CSV_t)$		60,266	59,280	60,152	59,772	59,389	58,417	57,462
9.	Base AV at Jan. 1:[1]	60,000	65,400	48,847	56,174	64,601	74,291	85,434	98,249
10.	Base AV at Dec. 31:[2]	65,400	63,438	56,174	64,601	74,291	85,434	98,249	112,987

Notes: $^*PV_2(CSV_t)$ at Dec. 31, 1999 of Dec. 31 is equal to $CSV_t \times 1.07^{-(t-1999)}$; [1]Base AV at Jan. 1 if Jan. 1, 2000 drop, no cap; [2]Base AV at Dec. 31 if Jan. 1, 2000 drop, no cap.

Table 8 (Continued)
Calculation of Minimum Guaranteed Death Benefit Amounts

		Policy year from Jan. 1 (t)							
		1998	1999	2000	2001	2002	2003	2004	2005
11.	Base AV at Jan. 1:[3]	60,000	65,400	48,847	56,174	64,601	73,963	77,846	81,933
12.	Base AV at Dec. 31:[4]	65,400	63,438	56,174	64,601	73,963	77,846	81,933	86,235
13.	Base AV average:[5]	62,700	64,419	52,511	60,387	69,282	75,905	79,890	84,084
14.	$MGDB_{t-1}$ at Jan. 1:	60,000	63,600	67,416	71,461	75,749	80,294	85,111	90,218
15.	$MGDB_t$ at Dec. 31:	63,600	67,416	71,461	75,749	80,294	85,111	90,218	95,631
16.	DB_{t-1} at Jan. 1:	60,000	65,400	67,416	71,461	75,749	80,294	85,111	90,218
17.	DB_t at Dec. 31:	65,400	67,416	71,461	75,749	80,294	85,111	90,218	95,631
18.	NAR_{t-1} at Jan. 1:	-	-	18,569	15,287	11,148	6,330	7,265	8,285
19.	NAR_t at Dec. 31:	-	3,978	15,287	11,148	6,330	7,265	8,285	9,396
20.	$NARAVG_t$:	-	1,989	16,928	13,217	8,739	6,798	7,775	8,840

Notes: [3]Base AV at Jan. 1 if Jan. 1, 2000 drop, cap of non-drop AV (line 3); [4]Base AV at Dec. 31 if Jan. 1, 2000 drop, cap of non-drop AV (line 4); [5]Base AV average if Jan. 1, 2000 drop, cap of non-drop AV (average of lines 11 and 12); $MGDB_{t-1}$ at Jan. 1 = $60,000 \times 1.06^{(t-1998)}$; $MGDB_t$ at Dec. 31 = $60,000 \times 1.06^{(t-1997)}$; DB_{t-1} at Jan. 1 = max(line 11, line 14); DB_t at Dec. 31 = max(line 12, line 15); NAR_{t-1} at Jan. 1 = (line 16 - line 11); NAR_t Dec. 31 = (line 17 - line 12).

The cap on asset value at December 31, 2003 is given by:

$$60,000 \times (1 + 0.09) \times (1 - 0.03) \times (1 + 0.0525)^4 = \$77,846$$

so the cap is binding; the capped asset value at December 31, 2003 is \$77,846. The minimum guaranteed death benefit at December 31, 2003 is:

$$60,000 \times (1 + 0.06)^6 = \$85,111.$$

The capped asset value is \$77,846. Hence the net amount at risk (NAR benefit top-up) because of the minimum guaranteed death benefit at December 31, 2003 is $85,111 - 77,846 = \$7,265$.

Following the usual CARVM philosophy, we test a set of candidates to determine which is greatest. This candidate will be the legal minimum reserve at December 31, 1999. The candidates correspond to potential surrender at December 31, 1999, 2000, 2001, etc.

For example, we consider the possibility of the reserve at December 31, 1999 being given by a CARVM maximum occurring at December 31, 2001. Therefor we find the present value at December 31, 1999 (or January 1, 2000) of the NAR payouts on deaths in 2000 and 2001. We assume mid-year deaths and rates of mortality as given in Table 9.

For 2000, $q_t = 0.019$ and average $NAR = 16,928$ from Table 8:

$$PV = 0.019 \times 16,928/1.07^{0.5} = 311.$$

For 2001, the probability of dying is $(1 - 0.019) \times 0.022$ and average $NAR = 13,217$ from Table 8.

$$PV = (1 - 0.019) \times 0.022 \times 13,217/1.07^{1.5} = 258.$$

The total for 2000 and 2001 is $311 + 258 = \$569$. All present values (PV) are at the valuation date, Dec. 31, 1999.

As a further part of examining the December 31, 2001 candidate policy termination date, we find the present value at December 31, 1999 (or January 1, 2000) of the unreduced asset value payouts in 2000 and 2001. In other words, we consider the present value of a death benefit of the unreduced (no-drop) asset value.

The use of unreduced asset payouts on death is consistent with the use of the unreduced asset payouts in calculating the present value of surrenders.

At first sight this is inconsistent with using the reduced asset value in the value of the minimum guaranteed death benefit guarantee. Unreduced amounts are used for benefits, however, where the benefit is proportional to the assets accumulated at the actual credited rate. If the investment return is -50 percent, then these benefits are halved. But this doesn't mean that we need half the reserve. We needed to be holding the full amount of assets in the separate account because these also were halved in value.

The same logic does not apply to the NAR and the value of the minimum guaranteed death benefit. The minimum guaranteed death benefit is specified in dollars independent of asset performance. This is like the death benefit under a traditional whole life policy. Correspondingly, the minimum guaranteed death benefit reserve is held in the general account.

For 2000, $q_t = 0.019$ and the average base account value (AV) is 65,103 from Table 8:

$$PV = 0.019 \times 65,103/1.07^{0.5} = 1,196.$$

For 2001, the probability of dying is $(1-0.019) \times 0.022$ and the average base account value is \$68,521 from Table 8.

$$PV = (1 - 0.019) \times 0.022 \times 68,521/1.07^{1.5} = 1,336.$$

The total for 2000 and 2001 is $1,196 + 1,336 = \$2,532$. All the present values (PV) are at valuation date, Dec. 31, 1999.

The major portion of this candidate to be the reserve is the present value at December 31, 1999 of surrenders by all survivors at December 31, 2001.

We are assuming that everyone who survives to December 31, 2001 surrenders then. Hence the PV at December 31, 1999 is, using the Table 7 no-drop CSV of \$68,868:

$$PV = (1 - 0.019) \times (1 - 0.022) \times 68,868/1.07^{2} = \$57,711.$$

Table 9
The Integrated Reserve

	Policy year from Jan. 1 (t)						
	1998	1999	2000	2001	2002	2003	2004
<u>Account Value</u>							
1. $AVAVG_t$ (line 5 of Table 8):	62,700	64,419	65,103	68,521	72,119	75,905	79,890
2. CSV_t(line 7 of Table 8) :	62,130	60,266	63,430	68,868	73,224	77,846	81,933
<u>Integrated Reserve Calculation Including MGDB</u>							
3. $NARAVG_t$(line 20 of Table 8):	-	1,989	16,928	13,217	8,739	6,798	7,775
4. (q_t):	0.015	0.017	0.019	0.022	0.024	0.027	0.030
5. P_t:			1.000	0.981	0.959	0.936	0.911
6. $PV(NARAVG)_t$:			311	258	170	136	157
7. $PV(AVAVG)_t$:			1,196	1,336	1,402	1,514	1,610
8. Cumulative total of line 6:			311	569	739	874	1,031
9. Cumulative total of line 7:			1,196	2,532	3,934	5,449	7,059
10. $PV(S\&SCSV)_t$:		60,266	58,154	57,711	55,970	54,109	51,628
11. Total of lines 8, 9 and 10:		60,266	59,661	60,812	60,643	60,432	59,718
The integrated reserve is maximum of the above row.							

Notes: P_t is the probability of survival from Jan. 1, 2000 to Jan. 1, t; $PV(NARAVG)_t$ at Jan. 1, 2000 of NARs paid on death and is equal to $NARAVG_t \times q_t \times P_t \times 1.07^{-(t-1999.5)}$; $PV(AVAVG)_t$ of average unreduced account values paid at death (mid year discounting) and is equal to $AVAVG_t \times q_t \times P_t \times 1.07^{-(t-1999.5)}$; $PV(S\&SCSV)_t = CSV_t \times P_{t-1} \times 1.07^{-(t-1999)}$, where $PV(S\&SCSV)_t$ is the present value of the cash value of those who survive and surrender at year end.

We add the PV at December 31, 1999 of the sum to December 31, 2001 of deaths and of 2001 surrenders, from above:

$$PV = 569 + 2,532 + 57,711 = \$60,812.$$

We consider all candidate policy termination dates in deciding the minimum reserve to be held at December 31, 1999. But we notice that this \$60,812 at December 31, 2001 is the highest number in the integrated-reserve-is-the-maximum line (line 11); it so happens that we calculated for the correct year. In reality all years (all candidates) are calculated and the maximum taken.

We then take the maximum also of the separate account reserve, line 10. This is \$60,266, and it applies to a December 31, 1999 surrender. This is less than \$60,812, which applies to the December 31, 2001 candidate; therefore, the reserve held is \$60,812. Despite the difference in dates, \$546 of the \$60,812 is held as a general account minimum guaranteed death benefit reserve.

Note the CARVM philosophy that we assume the worst case about the elective decrement that is controlled by the policyholder, surrender. Deaths are calculated according to a mortality table. Policyholders won't elect to die to get the best return from their annuity. We have to add for each possible surrender year the value of deaths that would occur previous to that surrender date.

9 Conclusion

Actuarial Guideline 34 clarifies CARVM. Its provisions are consistent with the idea that surrender benefits will, with 100 percent efficiency, be timed by the policyholder to maximize his or her return. Death is a non-elective benefit, and the calculations resemble the traditional actuarial discounting of a product of a death probability and a benefit amount.

The logic of this view may be clearer if we consider an insurance company to be valuing a cohort of 100,000 policyholders. Perhaps the optimum strategy for the policyholders is to elect to surrender after three years. Only 98,801 of them, however, will be alive to do so, for example. The other 1,199 will have died and received the death benefit. Thus we value this cohort at issue assuming a 0.98801 probability of receipt of CSV after three years. The valuation must also take into account the benefits paid on death with probability 0.01199.

The traditional view is that an insurance company spreads risks over many individuals. It is possible to spread the mortality risk, but the CARVM view is that antiselective surrender will be performed efficiently and simultaneously by all living policyholders.

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Multilife Premium Calculation with Dependent Future Lifetimes

Michel Denuit* and Anne Cornet†

Abstract

Actuaries traditionally have calculated multilife (joint life) premiums by assuming the independence of the future lifetimes of insured persons. Recent studies, however, demonstrate dependence of the future lifetimes of couples (such as husbands and wives). This dependence materially affects the values of multilife annuities and insurances. Using the Fréchet-Hoeffding bounds and Norberg's Markov model, we determine the effect of this dependence in lifetimes on the actuarial present values of a widow's pension benefit.

Key words and phrases: *joint lives, reversionary annuity, widow's pension, Fréchet-Hoeffding bounds, Markov model*

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1 Introduction

For computational convenience, standard actuarial theory of multiple (joint) life insurance traditionally assumes the independence of the future lifetimes of the insured lives. This assumption, however, is unrealistic. An example of possible dependence between insureds' future lifetimes occurs when a policy is issued to a married couple. A husband and wife are more or less exposed to the same risks because they spend so much time together. Moreover, several clinical studies indicate that the "broken heart syndrome" may cause an increase in the mortality rate after the death of a spouse; see, for example, Parkes, Benjamin, and Fitzgerald (1969), and Jagger and Sutton (1991).

Investigations carried out by the Belgian National Institute of Statistics (NIS) established that marital status significantly affects an individual's mortality. Similar conclusions have been drawn from actuarial studies; see, for example, Maeder (1995, Section 2.3).

To illustrate this dependence, we have prepared Figures 1 and 2. These figures are based on the data collected by the Belgian NIS during 1991. The observed probabilities q_x (i.e., the probability that a life age x will die before age $x+1$) are plotted as a function of the age x (for $x = 25$ to 90), separately for Belgian men and women, split according to their marital status. These figures clearly show that the mortality depends on marital status, especially for men. The mortality experienced by the widows seems worse than the mortality experienced by the entire Belgian population.

Of course, one could convincingly argue that Belgian society's attitudes toward marriage and divorce have drastically changed during the last two or three decades. Marriage is no longer the obligatory prerequisite when a couple decides to start a life together. Consequently, many individuals counted as single by the Belgian NIS are in fact cohabiting with their partner and should be considered as "married" from a sociological point of view. Thus, marital status will not appear as the most relevant explanatory variable.

Still, the Belgian government's fiscal legislations often provide tax incentives only to insurance policies issued to officially married couples. Therefore, the data collected by the governmental statistical services are relevant as far as contracts such as the state's widow's pension are concerned.

Figure 1
Belgian Observed Mortality Rates (q_xs) for Males in 1991

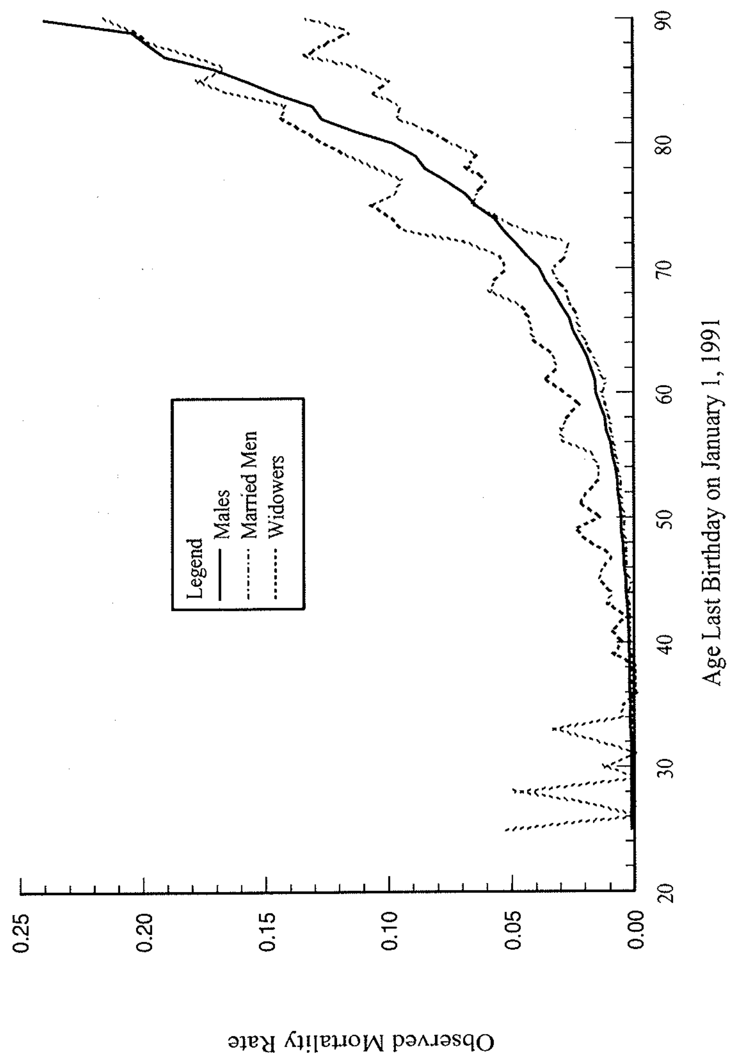
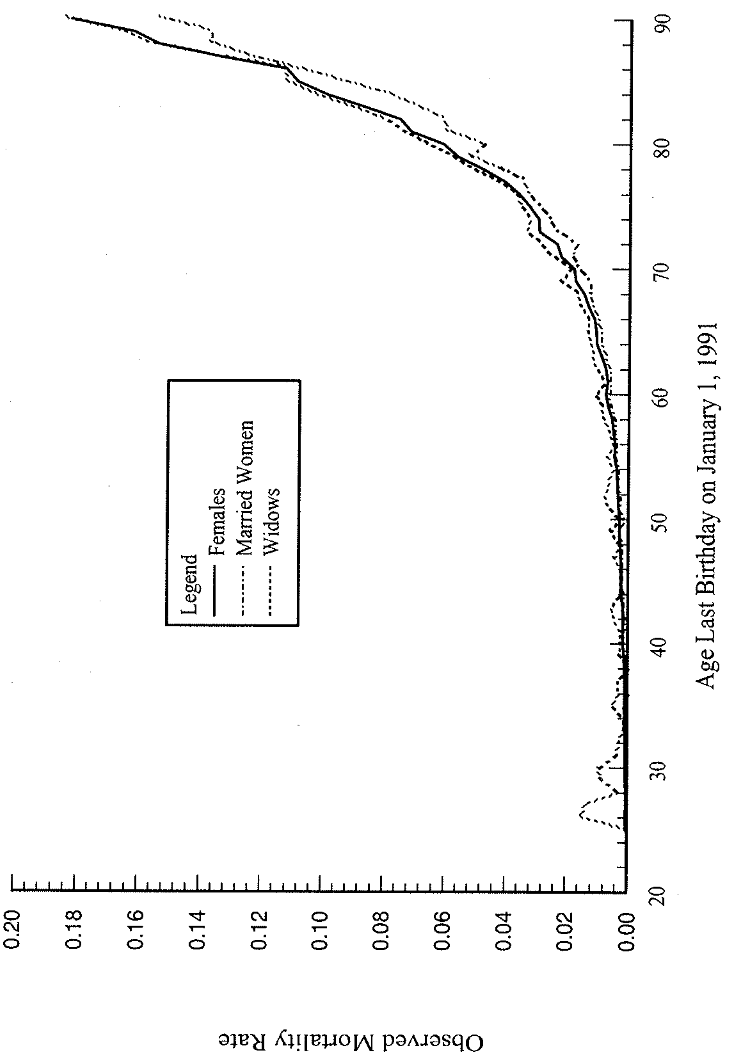


Figure 2
Belgian Observed Mortality Rates (q_xs) for Females in 1991



Recently, several articles have been devoted to the study of the impact of a possible dependence among insured risks in setting premium rates. Several authors have based their analysis on multivariate stochastic orderings; see, for example, Dhaene and Goovaerts (1996 and 1997); Dhaene, Vanneste, and Wolthuis (1997); Denuit and Lefèvre (1997); Denuit, Lefèvre, and Mesfioui (1999a and 1999b); Müller (1997); and Bäuerle and Müller (1998). Others have used copula models to take this dependence into account; see, for example, Carrière and Chan (1986); Carrière (1994); Frees, Carrière, and Valdez (1996); Frees and Valdez (1998); and Denuit and Teghem (1998).

In this paper, we quantify the effect of a possible dependence of future lifetime random variables on the amount of premium relating to the widow's pension. For this purpose, we use the Fréchet-Hoeffding bounds and a Markov model introduced by Norberg (1989) and Wolthuis (1994). We focus our attention on widow's pensions because Denuit and Teghem (1998) show that the dependent mortality may have a significant impact on the actuarial present value of these contracts. A similar study may be carried on the actuarial present value of other contracts involving married couples.

Our paper is organized as follows. In Section 2, we introduce the notation and the basic tenets of the model. In Section 3 we show that the Fréchet-Hoeffding bounds provide poor margins for widow's pensions.[1] Also, the assumption of positive quadrant dependence developed by Norberg (1989) is briefly explored. Then, in Section 4, we present the Markov model of the dependence between husband and wife mortality. The parameters of the Markov model are estimated using the Belgian NIS data. The actuarial present values of various annuities are calculated and displayed in Tables A1–A3 in the Appendix.

2 The Basic Notations and Definitions

The following notations are used throughout the paper: for x and y positive integers,

[1] Fréchet-Hoeffding bounds provide accurate margins for most other multiple life actuarial present values; see Denuit and Teghem (1998).

$$x = \text{Age of the husband at the start of the contract;}$$
$$y = \text{Age of the wife at the start of the contract;}$$
$$(x, y) = \text{The husband } (x) \text{ and wife } (y) \text{ couple;}$$
$$T_x = \text{Husband's future lifetime random variable;}$$
$$T_y = \text{Wife's future lifetime random variable;}$$
$$\omega_x = \text{Husband's maximum future lifetime, i.e., } 0 < T_x < \omega_x;$$
$$\omega_y = \text{Wife's maximum future lifetime, i.e., } 0 < T_y < \omega_y;$$
$$\omega_{xy} = \min(\omega_x, \omega_y);$$
$$\mathbb{R}^+ = (0, \infty);$$
$${}_tp_x = \Pr[T_x > t] = 1 - {}_tq_x, \quad \text{for } t \in \mathbb{R}^+;$$
$${}_tp_y = \Pr[T_y > t] = 1 - {}_tq_y, \quad \text{for } t \in \mathbb{R}^+;$$
$${}_tp_{xy} = \Pr[\min(T_x, T_y) > t] = \Pr[T_x > t, T_y > t], \quad \text{for } t \in \mathbb{R}^+;$$
$$i = \text{The constant annual effective interest rate; and}$$
$$v = \text{Discount factor } = (1 + i)^{-1}.$$

For our calculations we assume $i = 4.75$ percent, which is the maximal guaranteed rate according to the terms of Belgian legislation. In practice, since January 1999, however, most Belgian insurance companies have now adopted a rate around 3.25 percent based on long-term European public loans.

The widow's pension is a reversionary annuity with annual payments starting at the end of the year of the husband's death and terminating upon the death of his wife; if the wife dies first, no payments are made. Such annuities are used as post-retirement benefits in some pension plans and are also widely used in the European social security systems. The corresponding net single life premium for a couple (x, y) (i.e., an x-year old husband and his y-year old wife) is denoted as $a_{x|y}$, is given by

$$a_{x|y} = a_y - a_{xy}$$

where

$$a_y = \sum_{k=1}^{\omega_y} v^k {}_kp_y$$

and

$$a_{xy} = \sum_{k=1}^{\omega_{xy}} v^k {}_k p_{xy}.$$

Calculating the exact values of $a_{x|y}$ requires the knowledge of the joint distribution of the lifetime random vector (T_x, T_y). In practice, the actuary is only able to approximate $a_{x|y}$ with the help of various probabilistic models. The easiest approach certainly consists in considering T_x and T_y as independent, i.e.,

$${}_t p_{xy} = {}_t p_x \, {}_t p_y, \quad \text{for } t > 0.$$

In what follows, the superscript "ind" indicates that joint life annuities are calculated under the independence assumption. Thus

$$a^{\text{ind}}_{x|y} = \sum_{k=1}^{\omega_y} v^k {}_k p_y - \sum_{k=1}^{\omega_{xy}} v^k {}_k p_x {}_k p_y.$$

In this paper, we use life tables based on the Makeham formula and the mortality experienced in Belgium during 1991. The Makeham formula, for $t \in \mathbb{R}^+$, is

$$\begin{aligned} {}_t p_x &= s_1^t g_1^{c_1^x (c_1^t - 1)}, \quad c_1 > 1, \quad s_1, g_1 \in [0,1], && (1) \\ \mu_{x+t} &= A_1 + B_1 c_1^{x+t} \quad \text{and} \\ {}_t p_y &= s_2^t g_2^{c_2^y (c_2^t - 1)}, \quad c_2 > 1, \quad s_2, g_2 \in [0,1] && (2) \\ \mu_{y+t} &= A_2 + B_2 c_2^{y+t} \end{aligned}$$

where

$$A_i = -\ln(s_i) \quad \text{and} \quad B_i = -\ln(c_i) \ln(g_i), \quad i = 1, 2.$$

The parameters involved in equations (1) and (2) have been estimated using the Belgian NIS data collected in 1991.[2] The method used is the one proposed by Frère (1968) and the parameter estimates are given in Table 1.

It should be noted that the data collected by the Belgian NIS relate to the mortality experienced by the Belgian population during 1991. Such

[2]The details of these data can be found in a publication of the Belgian NIS (1992).

Table 1
Parameter Estimates of the Makeham Formulas

Parameter	Men ($i = 1$)	Women ($i = 2$)
s_i	0.999 408 439 685	0.999 767 237 352
g_i	0.999 598 683 466	0.999 831 430 984
c_i	1.102 904 035 923	1.106 730 646 873

data are suitable for pricing the widow's pensions included in social security systems. The data, however, could be unsuitable for policies issued by private insurance companies, so such companies must use their own data.[3]

3 Bounds on $a_{x|y}$

3.1 Fréchet-Hoeffding Bounds

This approach centers on quantifying the maximal impact of a possible dependence on actuarial values by using bounds for bivariate distributions. More precisely, it is well-known, since Hoeffding (1940) and Fréchet (1951), that

$$\max\{0, {}_tp_x + {}_tp_y - 1\} \leq {}_tp_{xy} \leq \min\{{}_tp_x, {}_tp_y\}, \quad \forall \quad t \in \boldsymbol{R}^+. \tag{3}$$

The leftmost and rightmost expressions of equation (3) are usually referred to as the Fréchet-Hoeffding lower and upper bounds, respectively. These bounds have been first applied by Carrière and Chan (1986) to different annuities and then placed by Denuit and Lefèvre (1997) and Dhaene, Vanneste, and Wolthuis (1997) in the context of bivariate stochastic orderings.

By inserting equation (3) in the net single premium $a_{x|y}$, we get

$$a_{x|y}^{\min} \leq a_{x|y} \leq a_{x|y}^{\max}, \tag{4}$$

where

[3]In fact, the Belgian authorities (i.e., the "Office de Contrôle des Assurances") cannot provide the researchers with specific data about widow's pensions sold by private companies because the statistics about such contracts are mixed with those of other life insurance operations.

$$a_{x|y}^{\min} = \sum_{k=1}^{\omega_y} v^k {}_kp_y - \sum_{k=1}^{\omega_{xy}} v^k \min\{{}_kp_x, {}_kp_y\} \tag{5}$$

and

$$a_{x|y}^{\max} = \sum_{k=1}^{\omega_y} v^k {}_kp_y - \sum_{k=1}^{\omega_{xy}} v^k \max\{0, {}_kp_x + {}_kp_y - 1\}. \tag{6}$$

The values of these annuities under three scenarios are listed in Tables A1–A3 in the appendix. The scenarios are: (i) for Table A1, $x = y = 25, 26, \ldots, 90$, i.e., the husband and his wife both have the same age; (ii) for Table A2, $x = y + 5 = 25, 26, \ldots, 90$, i.e., the husband is five years older than his wife; and (iii) for Table A3, $x = y - 5 = 25, 26, \ldots, 90$, i.e., the husband is five years younger than his wife.

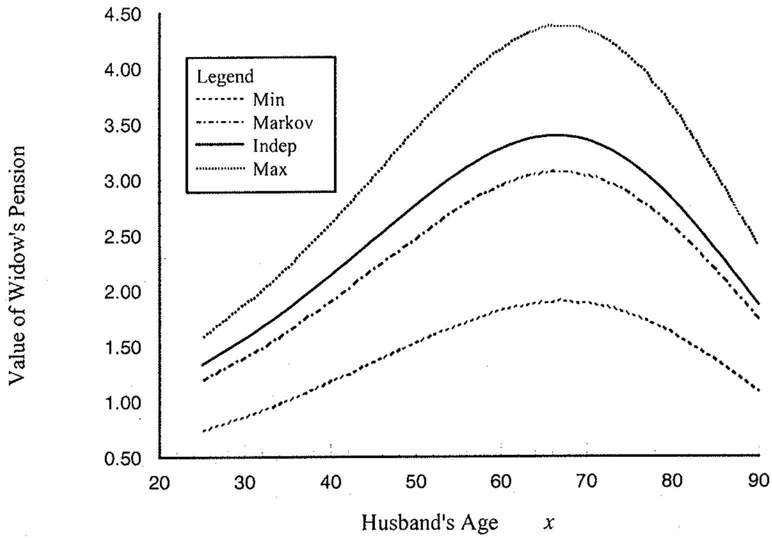
In order to determine the accuracy of the bounds provided in equation (4), we have prepared Figures 3, 4, and 5. Tables A1, A2, and A3 contain the numerical values used to plot Figures 3, 4, and 5.

The margins provided by $a_{x|y}^{\min}$ and $a_{x|y}^{\max}$ for the unknown $a_{x|y}$ are rather wide. For $x = y$ (Table A1), if the insurer decides to charge $a_{x|y}^{\mathrm{ind}}$ instead of the true premium $a_{x|y}$, the error the insurer makes consists in an overestimate up to 45 percent or in an undervaluation up to 30 percent (because $a_{x|y}^{\min}$ is about 55 percent to 59 percent of $a_{x|y}^{\mathrm{ind}}$, while $a_{x|y}^{\max}$ represents 120 percent to 130 percent of $a_{x|y}^{\mathrm{ind}}$). In such a case, the independence assumption thus may lead to a significantly erroneous amount of premium. Similar conclusions can be drawn for the two other scenarios: when the husband is older than his wife (Table A2), $a_{x|y}^{\min}$ is about 74 percent to 82 percent of $a_{x|y}^{\mathrm{ind}}$ and $a_{x|y}^{\max}$ represents 114 percent to 118 percent of $a_{x|y}^{\mathrm{ind}}$; when the husband is younger than his wife (Table A3), $a_{x|y}^{\min}$ is about 22 percent to 1 percent of $a_{x|y}^{\mathrm{ind}}$, while $a_{x|y}^{\max}$ represents 126 percent to 144 percent of $a_{x|y}^{\mathrm{ind}}$.

3.2 Positive Quadrant Dependence

A number of ideas of positive dependence between the two random future lifetimes T_x and T_y have been introduced in the literature in an effort to mathematically describe the property that large (small) values of T_x go together with large (small) values of T_y; see, for example, Joe (1997) or Scarsini and Shaked (1996). Most of these ideas are based on

Figure 3
Actuarial Present Values of Widow's Pension for
$x = y = 25, 26, \ldots, 90$



some comparisons of the joint distribution of the pair (T_x, T_y) with its distribution under the theoretical assumption that T_x and T_y are independent. The notion of positive quadrant dependence was introduced by Lehmann (1966) and is defined as follows:

Definition 1. *The random vector* (T_x, T_y) *is said to be positively quadrant dependent (PQD) if, and only if,*

$$\Pr[T_x \leq t_1, T_y \leq t_2] \geq \Pr[T_x \leq t_1] \times \Pr[T_y \leq t_2] \quad \forall t_1, t_2 \in \mathbb{R}^+, \tag{7}$$

or, equivalently, if, and only if,

$$\Pr[T_x > t_1, T_y > t_2] \geq \Pr[T_x > t_1] \times \Pr[T_y > t_2] \quad \forall t_1, t_2 \in \mathbb{R}^+. \tag{8}$$

Hence, by equations (7) and (8), saying that T_x and T_y are *PQD* means that the probability that T_x and T_y both realize small (resp. large) values is larger than the corresponding probability in the case

Figure 4
Actuarial Present Values of Widow's Pension for
$x = y + 5 = 25, 26, \ldots, 90$

of independent remaining lifetimes. From the introduction, the *PQD* assumption for the remaining lifetimes of married couples appears as natural.

When T_x and T_y are *PQD*, we get ${}_kp_{xy}^{\mathrm{PQD}} \geq {}_kp_{x\,k}p_y$ for any k yielding in turn

$$a_{x|y}^{\min} \leq a_{x|y}^{\mathrm{PQD}} \leq a_{x|y}^{\mathrm{ind}}. \tag{9}$$

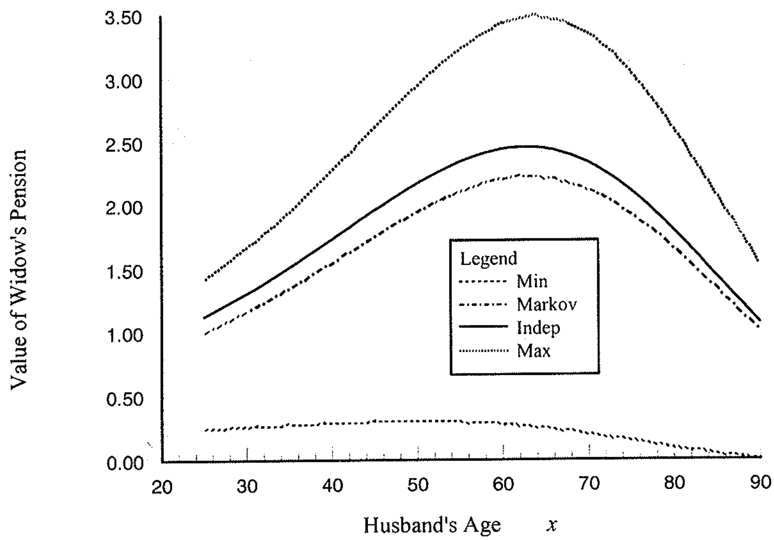
The independence assumption therefore appears to be conservative compared to the *PQD* assumption.

4 Markov Process Model

4.1 Description of the Model

Since the seminal lecture given by Amsler (1968) at the 18th International Congress of Actuaries and the paper by Hoem (1969), the Markov

Figure 5
Actuarial Present Values of Widow's Pension for
$x = y - 5 = 25, 26, \ldots, 90$



process model[4] has become an appreciated tool for the calculation of life contingencies functions. Markov processes have been extensively discussed in the actuarial literature; see, for example, the papers by Amsler (1988), Davis and Vellekoop (1995), Haberman (1983, 1984, 1988, and 1995), Hoem (1972,1977,1988), Hoem and Aalen (1978), Jones (1994, 1995, 1996, 1997a, and 1997b), Moller (1990 and 1992), Norberg (1988 and 1989), Panjer (1988), Pitacco (1995), Ramlau-Hansen (1988a, 1988b, 1991), Ramsay (1989), Tolley and Manton (1991), Waters (1984), Wilkie (1988), and Wolthuis and Van Hoeck (1986), as well as the references therein.

In order to price insurance contracts issued to married couples, Norberg (1989) and Wolthuis (1994) propose a Markov process model with

[4]Let $X = \{X_t,\ \ t \in \boldsymbol{R}^+\}$ be a stochastic process. In actuarial applications, X_t is the state of the insurance/annuity contract at time t (measured from the start of the policy). If $\mathcal{F}_t$ denotes the history of the process X up to time t, the Markov model assumes, roughly speaking, that the future of X is independent of all information contained in $\mathcal{F}_t$, except the state X_t at time t.

forces of mortality depending on marital status. They define the various states for the married couple (x, y) as follows:

State 0 = Both husband (x) and wife (y) are alive;
State 1 = Husband (x) is dead and wife (y) is alive;
State 2 = Husband (x) is alive and wife (y) is dead;
State 3 = Both husband (x) and wife (y) are dead.

The future development of the marital status for the couple (ignoring the possibility of divorce) may be regarded as a Markov process depicted in Figure 6.

For $0 \le t_1 \le t_2$, let $p_{ij}(t_1, t_2)$ denote the transition probabilities of the Markov process of Figure 6, i.e., for $i, j = 0, 1, 2, 3$

$$p_{ij}(t_1, t_2) = \Pr[(x, y) \text{ in state } j \text{ at } t_2 | (x, y) \text{ in state } i \text{ at } t_1].$$

Obviously, for any $0 \le t_1 \le t_2$, $0 \le p_{ij}(t_1, t_2) \le 1$ for all i and j, $p_{ij}(t_1, t_1) = 1$ if $i = j$ and 0 otherwise, and $\sum_j p_{ij}(t_1, t_2) = 1$ for all i. For $i, j = 0, 1, 2, 3$, let $\mu_{ij}(t)$ denote the force of transition from state i to j at time t. The forces of transition are related to transition probabilities through

$$p_{ij}(t, t + \Delta t) = \begin{cases} \mu_{ij}(t)\Delta t + o(\Delta t) & i \ne j \\ 1 - \sum_{\substack{r=0 \\ r \ne i}}^{3} \mu_{ir}(t)\Delta t + o(\Delta t) & i = j \end{cases} \tag{10}$$

where $o(\cdot)$ is a function such that $\lim_{h \to 0} o(h)/h = 0$. It can easily be shown that, for $0 \le t_1 \le t_2$, equation (10) yields the following expressions for the transition probabilities:

$$p_{00}(t_1, t_2) = \exp\left[-\int_{t_1}^{t_2} (\mu_{01}(\tau) + \mu_{02}(\tau))d\tau\right], \tag{11}$$

$$p_{11}(t_1, t_2) = \exp\left[-\int_{t_1}^{t_2} \mu_{13}(\tau)d\tau\right], \tag{12}$$

$$p_{22}(t_1, t_2) = \exp\left[-\int_{t_1}^{t_2} \mu_{23}(\tau)d\tau\right], \tag{13}$$

$$p_{33}(t_1, t_2) = 1 \tag{14}$$

and

$$p_{0j}(t_1, t_2) = \int_{t_1}^{t_2} p_{00}(t_1, \tau)\mu_{0j}(\tau)p_{jj}(\tau, t_2)d\tau. \tag{15}$$

Now, the joint survival function of (T_x, T_y) is given by

$$\Pr[T_x > t_1, T_y > t_2] = \begin{cases} p_{00}(0, t_2) + p_{00}(0, t_1)p_{01}(t_1, t_2) \text{ if } 0 \le t_1 \le t_2, \\ p_{00}(0, t_1) + p_{00}(0, t_2)p_{02}(t_2, t_1) \text{ if } 0 \le t_2 < t_1. \end{cases} \tag{16}$$

The marginal survival functions of T_x and T_y are respectively given by

$$\Pr[T_x > t_1] = \Pr[T_x > t_1, T_y > 0] = p_{00}(0, t_1) + p_{02}(0, t_1) \tag{17}$$

$$\Pr[T_y > t_2] = \Pr[T_x > 0, T_y > t_2] = p_{00}(0, t_2) + p_{01}(0, t_2), \tag{18}$$

for t_1, $t_2 \ge 0$.

Norberg (1989) showed that, under certain circumstances, equation (16) can yield independent T_x and T_y or PQD T_x and T_y. Specifically,

$$\mu_{01}(t) \equiv \mu_{23}(t) \text{ and } \mu_{02}(t) \equiv \mu_{13}(t) \Leftrightarrow T_x, T_y \text{ independent}, \tag{19}$$

while

$$\mu_{01}(t) \le \mu_{23}(t) \text{ and } \mu_{02}(t) \le \mu_{13}(t) \Rightarrow T_x \text{ and } T_y \text{ are } PQD. \tag{20}$$

Given our earlier comments in the introduction, it seems natural to assume that mortality dependence is PQD. In addition, we choose a mortality structure that is consistent with the PQD structure given in equation (20). Specifically, for $t \in \boldsymbol{R}^+$, we set

$$\mu_{01}(t) = (1 - \alpha_{01})\mu_{x+t} \tag{21}$$

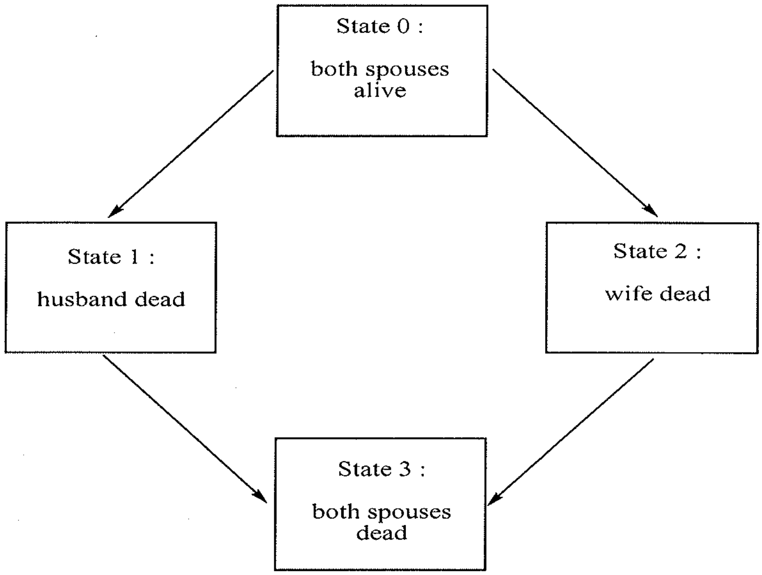
$$\mu_{23}(t) = (1 + \alpha_{23})\mu_{x+t} \tag{22}$$

$$\mu_{02}(t) = (1 - \alpha_{02})\mu_{y+t} \tag{23}$$

$$\mu_{13}(t) = (1 + \alpha_{13})\mu_{y+t} \tag{24}$$

where μ_{x+t} and μ_{y+t} are the male and female forces of mortality respectively in the entire Belgian population, the α_{ij}'s are nonnegative and the α_{0j}'s are less than 1. Therefore, we are assuming that the mortality intensities are lower than that in the entire Belgian population as long as both spouses are alive and become higher when a spouse dies. Setting $\alpha_{ij} \equiv \alpha$, we find the model proposed by Wolthuis (1994, page 62).

Figure 6
Markov Model with Forces of Mortality Depending on Marital Status



4.2 Estimation of the Parameters

Estimators for the four parameters α_{01}, α_{02}, α_{13}, and α_{23} are needed. To this end, we use data collected by the Belgian NIS and we follow the method of least squares proposed in Wolthuis (1994, Chapter 6). The estimator, $\hat{\alpha}_{ij}$, of α_{ij} minimizes the sum of the squared differences between the increments $\Delta\Omega_{ij}(t) = \Omega_{ij}(t+1) - \Omega_{ij}(t)$ of the transition functions and their estimations $\Delta\hat{\Omega}_{ij}(t)$, where

$$\Omega_{ij}(t) = \int_0^t \mu_{ij}(\tau)d\tau, \quad t \geq 0;$$

thus $\hat{\alpha}_{ij}$ minimizes

$$\sum_k \left(\Delta\hat{\Omega}_{ij}(k) - \int_{t=0}^1 \mu_{ij}(k+t)dt\right)^2. \tag{25}$$

We will now expand on the estimation of $\Delta\Omega_{ij}$. Let $L_i(t)$ be the number of couples in state i at age $t-$ (just prior to any transition from state i at time t), and let $L_{ij}(t)$ be the number of transitions from state i to state j over $[0, t]$. The Nelson-Aalen nonparametric estimator of $\Omega_{ij}(t)$ is

$$\hat{\Omega}_{ij}(t) = \int_0^t \frac{I[L_i(\tau) > 0]}{L_i(\tau)} dL_{ij}(\tau),$$

where $I[A]$ is the indicator function of the event A, and with the convention that the integrand is defined to be zero when $L_i(\tau) = 0$; for more details on the Nelson-Aalen estimator, see Nelson (1969) and Aalen (1978), or, for example, Jones (1997b) and the references therein.

The data are derived from the Belgian population during 1991. The data are split by age, sex, and marital status on January 1, 1991 and on January 1, 1992, as well as the number of deaths, the number of marriages and divorces in 1991 by age, sex, year of birth, and marital status. As the number of transitions is only available for a year, we use the uniform distribution assumption, i.e., we assume that for any integer k and $0 \leq t < 1$,

$$L_{ij}(k+t) = L_{ij}(k) + t\left\{L_{ij}(k+1) - L_{ij}(k)\right\}$$

and

$$L_i(k+t) = L_i(k) + t\left\{L_i(k+1) - L_i(k)\right\}.$$

These approximations yield

$$\begin{aligned}
\Delta\hat{\Omega}_{ij}(k) &= \int_0^1 \frac{I[L_i(k+\tau) > 0]}{L_i(k) + \tau\left\{L_i(k+1) - L_i(k)\right\}} \left\{L_{ij}(k+1) - L_{ij}(k)\right\} d\tau \\
&= \frac{L_{ij}(k+1) - L_{ij}(k)}{L_i(k+1) - L_i(k)} \left\{\ln L_i(k+1) - \ln L_i(k)\right\} \\
&\equiv \frac{L_{i:j}(k)}{L_i(k+1) - L_i(k)} \left\{\ln L_i(k+1) - \ln L_i(k)\right\}, \qquad (26)
\end{aligned}$$

where $L_i(k)$ represents the number of couples in state i at age k and $L_{i:j}(k) = L_{ij}(k+1) - L_{ij}(k)$ is the number of transitions from state i to state j observed for k-year old individuals. Equation (26) is in accordance with Wolthuis (1994, page 108, equation (33)).

We will now explain precisely how $\Delta\hat{\Omega}_{01}(k)$ and $\Delta\hat{\Omega}_{13}(k)$ are estimated.[5] Let us start with $\Delta\hat{\Omega}_{01}(k)$ and examine the different elements constituting equation (26):

1. The numerator $L_{0:1}(k)$ is the number of k-year old married men dying during 1991 (this number is directly available from the NIS).
2. The denominator $L_0(k+1) - L_0(k)$ is equal to

   – Number of k-year old married men dying during 1991
   – Number of k-year old widowers whose wife died during 1991
   + Number of k-year old men getting married during 1991
   – Number of k-year old married men getting divorced during 1991

   The number of couples with a k-year old man whose wife died during 1991 cannot be obtained from the NIS. Therefore, we estimate it as follows:

   Number of $(k+1)$-year old widowers at January 1, 1992
   – Number of k-year old widowers at January 1, 1991
   + Number of k-year old widowers dying during 1991
   + Number of k-year old widowers getting married during 1991.

3. Finally, concerning the difference of the logarithms in equation (26), $L_0(k)$ is the number of k-year old married men at January 1, 1991, and $L_0(k+1)$ is easily deduced from above.

Let us now examine $\Delta\hat{\Omega}_{13}(k)$:

1. The numerator $L_{1:3}(k)$ is the number of k-year old widows dying during 1991;
2. The denominator $L_1(k+1) - L_1(k)$ is equal to

   – Number of k-year old widows dying during 1991
   + Number of k-year old widows whose husband died during 1991
   – Number of k-year old widows getting married during 1991.

[5] $\Delta\hat{\Omega}_{02}(k)$ and $\Delta\hat{\Omega}_{23}(k)$ are estimated in a similar manner by switching the roles of the two spouses.

The number of couples with a k-year old woman whose husband died during 1991 is not available from the NIS. Therefore, we estimate it as follows:

Number of $(k+1)$-year old widows at January 1, 1992
− Number of k-year old widows at January 1, 1991
+ Number of k-year old widows dying during 1991
+ Number of k-year old widows getting married during 1991.

3. Finally, concerning the difference between the logarithms in equation (26), $L_1(k)$ is the number of k-year old widows at January 1, 1991, and $L_1(k+1)$ is easily obtained from above.

From equations (21) through (24), the estimators $\hat{\alpha}_{ij}$ of the parameters α_{ij} are:

$$\hat{\alpha}_{01} = 1 - \frac{\sum_k \left(A_1 + B_1 c_1^k \left(\frac{c_1 - 1}{\ln c_1}\right)\right) \Delta\hat{\Omega}_{01}(k)}{\sum_k \left(A_1 + B_1 c_1^k \left(\frac{c_1 - 1}{\ln c_1}\right)\right)^2}, \tag{27}$$

$$\hat{\alpha}_{02} = 1 - \frac{\sum_k \left(A_2 + B_2 c_2^k \left(\frac{c_2 - 1}{\ln c_2}\right)\right) \Delta\hat{\Omega}_{02}(k)}{\sum_k \left(A_2 + B_2 c_2^k \left(\frac{c_2 - 1}{\ln c_2}\right)\right)^2}, \tag{28}$$

$$\hat{\alpha}_{13} = \frac{\sum_k \left(A_2 + B_2 c_2^k \left(\frac{c_2 - 1}{\ln c_2}\right)\right) \Delta\hat{\Omega}_{13}(k)}{\sum_k \left(A_2 + B_2 c_2^k \left(\frac{c_2 - 1}{\ln c_2}\right)\right)^2} - 1, \tag{29}$$

and

$$\hat{\alpha}_{23} = \frac{\sum_k \left(A_1 + B_1 c_1^k \left(\frac{c_1 - 1}{\ln c_1}\right)\right) \Delta\hat{\Omega}_{23}(k)}{\sum_k \left(A_1 + B_1 c_1^k \left(\frac{c_1 - 1}{\ln c_1}\right)\right)^2} - 1. \tag{30}$$

Using the NIS data on individuals aged from 30 to 80 years, we get the actual estimates:

$$\hat{\alpha}_{01} = 0.092926, \qquad \hat{\alpha}_{02} = 0.133982$$
$$\hat{\alpha}_{13} = 0.041349, \qquad \text{and } \hat{\alpha}_{23} = 0.241033.$$

In other words, there is (on the basis of the NIS data collected during 1991) a reduction in mortality of about 9 percent for married men, of

13 percent for married women, and an increase in mortality of about 4 percent for the widows and of 24 percent for the widowers, when compared to the mortality experienced by the entire Belgian population.

4.3 Premium Calculation in the Markov Model

In order to price the widow's pension, we only need the probabilities $p_{00}(t, t + \Delta t)$, $p_{01}(t, t + \Delta t)$, and $p_{11}(t, t + \Delta t)$ for integers t and Δt. The p_{00}'s and p_{11}'s can be calculated recursively because they satisfy the recurrence relations:

$$p_{00}(0, k + 1) = p_{00}(0, k)p_{00}(k, k + 1), \tag{31}$$

$$p_{11}(0, k + 1) = p_{11}(0, k)p_{11}(k, k + 1), \tag{32}$$

starting with $p_{00}(0, 0) = p_{11}(0, 0) = 1$. We further assume that the transition intensities $\mu_{ij}(\cdot)$ are constant for each year of age, i.e.,

$$\mu_{ij}(k + \tau) = \mu_{ij}(k) \quad \text{for} \quad 0 \leq \tau < 1.$$

Thus, for each integer ages $x + k$ and $y + k$, we have

$$\mu_{x+k+\tau} = \mu_{x+k} \text{ and } \mu_{y+k+\tau} = \mu_{y+k} \text{ for } 0 \leq \tau < 1.$$

The one-year probabilities $p_{00}(k, k + 1)$ and $p_{11}(k, k + 1)$ are then respectively given by

$$p_{00}(k, k + 1) = \exp\{-\mu_{01}(k) - \mu_{02}(k)\},$$
$$p_{11}(k, k + 1) = \exp\{-\mu_{13}(k)\},$$

while the one-year transition probabilities $p_{01}(k, k+1)$ can be expressed as

$$p_{01}(k, k + 1) = \left(\frac{\mu_{01}(k)}{\mu_{13}(k) - \mu_{01}(k) - \mu_{02}(k)}\right) \times \left(\exp\{-\mu_{01}(k) - \mu_{02}(k)\} - \exp\{-\mu_{13}(k)\}\right).$$

Reformulated in the Markov model, the net single premium $a_{x|y}^{\text{mark}}$ relating to the widow's pension is given as:

$$a_{x|y}^{\text{mark}} = \sum_{k=0}^{\omega_{xy}} p_{00}(0,k)p_{01}(k,k+1) \sum_{j=0}^{\omega_y - k} p_{11}(k+1,k+1+j)v^{k+1+j}. \tag{33}$$

Returning to Figures 3-5, the lines labeled "Markov" depict the net single premiums $a_{x|y}^{\text{mark}}$. Notice that the $a_{x|y}^{\text{mark}}$s are indeed lower than $a_{x|y}^{\text{ind}}$ for the calculation based on the assumption of dependent remaining lifetimes. This can be explained as follows: recall that the $\hat{\alpha}_{ij}$'s are such that the implication in equation (20) is true so that the future lifetime random variables T_x and T_y are PQD. With PQD remaining lifetimes, the policy stays longer in state 0 (thus there is a longer time until possible annuity payments) and shorter in state 1 (less annuity payments).

The Markov model provides net single premiums $a_{x|y}^{\text{mark}}$ of about 90 percent of those computed on the independence assumption (i.e., $a_{x|y}^{\text{ind}}$); see Tables A1-A3 in the appendix for more details.

5 Summary and Conclusions

The present study aims to examine the effect on the premiums relating to the widow's pension when there is a departure from the usual assumption of independence of the lifetimes of a husband and wife. Using data from a large insurance company, Frees, Carrière, and Valdez (1996) show that the lifetimes of paired lives (e.g., husband and wife) are highly correlated. In our study, we adopt a different approach. After determining the maximal impact of a possible dependence with the help of the Fréchet-Hoeffding bounds, the premiums for the widow's pension is computed in a Markov model.

The numerical illustrations are based on the data collected by the Belgian NIS during 1991. The estimation results show an economically significant positive dependence between joint lives: in Norberg's model, the amounts of premium are reduced approximately 10 percent compared to the standard model that assumes independence. Whereas Denuit and Teghem (1998) showed that the effect of a possible dependence is rather moderate for classical multiple life contracts (at most 5 percent in the cases considered by the authors), the consequences on the amount of premium of the widow's pension could thus be more important in practice.

In conclusion, the Markov model allows the actuary to determine a more accurate value for $a_{x|y}$. It offers the actuary a yardstick to decide whether or not to grant a discount to the assured persons, as well as to select the amount of this discount, or to evaluate the level of the mortality benefits in profit testing. Finally, the value $a_{x|y}^{\text{mark}}$ is also of primary importance when the level of the safety loading is to be selected. Indeed, the manual premium $a_{x|y}^{\text{ind}}$ itself contains an implicit safety loading of about 10 percent. This has to be taken into account in order to avoid excessive safety margins.

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Appendix

The percentages in the brackets represent the relative importance of the premiums $a_{x|y}^{\text{min}}$, $a_{x|y}^{\text{mark}}$, and $a_{x|y}^{\text{max}}$ when $a_{x|y}^{\text{ind}} = 100$ percent.

Table A1
Reversionary Annuity Values for $x = y$

Age x	$a_{x\|y}^{\text{min}}$	$a_{x\|y}^{\text{mark}}$	$a_{x\|y}^{\text{ind}}$	$a_{x\|y}^{\text{max}}$
25	0.74071041 [55.5%]	1.19041068 [89.2%]	1.33396690	1.59044134 [119.2%]
30	0.86789658 [55.3%]	1.40024796 [89.2%]	1.56930532	1.88684152 [120.2%]
31	0.89562955 [55.3%]	1.44601003 [89.2%]	1.62056381	1.95180730 [120.4%]
32	0.92411391 [55.2%]	1.49301078 [89.2%]	1.67318352	2.01866170 [120.6%]
33	0.95333675 [55.2%]	1.54122697 [89.2%]	1.72713524	2.08738643 [120.9%]
34	0.98328112 [55.2%]	1.59062830 [89.2%]	1.78238166	2.15795402 [121.1%]
35	1.01392562 [55.1%]	1.64117686 [89.2%]	1.83887671	2.23032697 [121.3%]
36	1.04524409 [55.1%]	1.69282650 [89.3%]	1.89656489	2.30445678 [121.5%]
37	1.07720524 [55.1%]	1.74552223 [89.3%]	1.95538056	2.38028311 [121.7%]
38	1.10977224 [55.1%]	1.79919958 [89.3%]	2.01524725	2.45773281 [122.0%]
39	1.14290242 [55.1%]	1.85378398 [89.3%]	2.07607700	2.53671898 [122.2%]
40	1.17654683 [55.0%]	1.90919017 [89.3%]	2.13776966	2.61714005 [122.4%]
41	1.21064996 [55.0%]	1.96532159 [89.3%]	2.20021226	2.69872784 [122.7%]
42	1.24514933 [55.0%]	2.02206984 [89.3%]	2.26327842	2.78128761 [122.9%]
43	1.27997517 [55.0%]	2.07931415 [89.4%]	2.32682777	2.86481441 [123.1%]
44	1.31505016 [55.0%]	2.13692089 [89.4%]	2.39070548	2.94912917 [123.4%]
45	1.35028914 [55.0%]	2.19474322 [89.4%]	2.45474183	3.03403290 [123.6%]
46	1.38559887 [55.0%]	2.25262072 [89.4%]	2.51875194	3.11930614 [123.8%]
47	1.42087788 [55.0%]	2.31037921 [89.5%]	2.58253550	3.20470868 [124.1%]
48	1.45601633 [55.0%]	2.36783059 [89.5%]	2.64587679	3.28997948 [124.3%]
49	1.49089601 [55.0%]	2.42477294 [89.5%]	2.70854472	3.37483703 [124.6%]

Table A1 (Continued)
Reversionary Annuity Values for $x = y$

Age x	$a_{x\|y}^{\text{min}}$	$a_{x\|y}^{\text{mark}}$	$a_{x\|y}^{\text{ind}}$	$a_{x\|y}^{\text{max}}$
50	1.52539031 [55.1%]	2.48099061 [89.6%]	2.77029314	3.45898007 [124.9%]
51	1.55936441 [55.1%]	2.53625464 [89.6%]	2.83086126	3.54208894 [125.1%]
52	1.59267547 [55.1%]	2.59032325 [89.6%]	2.88997438	3.62382754 [125.4%]
53	1.62517300 [55.1%]	2.64294257 [89.7%]	2.94734481	3.70384613 [125.7%]
54	1.65669933 [55.2%]	2.69384765 [89.7%]	3.00267299	3.78178516 [125.9%]
55	1.68709020 [55.2%]	2.74276362 [89.8%]	3.05564902	3.85728030 [126.2%]
56	1.71617552 [55.3%]	2.78940718 [89.8%]	3.10595434	3.92996895 [126.5%]
57	1.74378029 [55.3%]	2.83348826 [89.9%]	3.15326379	3.99949860 [126.8%]
58	1.76972562 [55.4%]	2.87471208 [89.9%]	3.19724790	4.06394381 [127.1%]
59	1.79383004 [55.4%]	2.91278136 [90.0%]	3.23757556	4.12265380 [127.3%]
60	1.81591080 [55.5%]	2.94739886 [90.0%]	3.27391695	4.17658850 [127.6%]
61	1.83578549 [55.5%]	2.97827015 [90.1%]	3.30594670	4.22543567 [127.8%]
62	1.85327372 [55.6%]	3.00510667 [90.2%]	3.33334738	4.26895407 [128.1%]
63	1.86819904 [55.7%]	3.02762896 [90.2%]	3.35581321	4.30700000 [128.3%]
64	1.88039086 [55.7%]	3.04557006 [90.3%]	3.37305392	4.33956041 [128.7%]
65	1.88968663 [55.8%]	3.05867915 [90.4%]	3.38479875	4.36069712 [128.8%]
66	1.89593403 [55.9%]	3.06672518 [90.4%]	3.39080062	4.37356997 [129.0%]
67	1.89899323 [56.0%]	3.06950058 [90.5%]	3.39084019	4.38008920 [129.2%]
68	1.89873921 [56.1%]	3.06682497 [90.6%]	3.38472994	4.38087477 [129.4%]
69	1.89506404 [56.2%]	3.05854874 [90.7%]	3.37231811	4.37164242 [129.6%]

Table A1 (Continued)
Reversionary Annuity Values for $x = y$

Age x	$a_{x\mid y}^{\text{min}}$	$a_{x\mid y}^{\text{mark}}$	$a_{x\mid y}^{\text{ind}}$	$a_{x\mid y}^{\text{max}}$
70	1.88787910 [56.3%]	3.04455644 [90.8%]	3.35349232	4.34866396 [129.7%]
71	1.87711724 [56.4%]	3.02476993 [90.9%]	3.32818300	4.32084527 [129.8%]
72	1.86273471 [56.5%]	2.99915118 [91.0%]	3.29636626	4.28911419 [130.1%]
73	1.84471296 [56.6%]	2.96770463 [91.1%]	3.25806633	4.23591022 [130.0%]
74	1.82306014 [56.7%]	2.93047901 [91.2%]	3.21335735	4.18125005 [130.1%]
75	1.79781223 [56.9%]	2.88756862 [91.3%]	3.16236452	4.11958528 [130.3%]
76	1.76903392 [57.0%]	2.83911386 [91.4%]	3.10526443	4.04102401 [130.1%]
77	1.73681898 [57.1%]	2.78530112 [91.6%]	3.04228459	3.96910578 [130.5%]
78	1.70129022 [57.2%]	2.72636185 [91.7%]	2.97370211	3.86928550 [130.1%]
79	1.66259893 [57.3%]	2.66257088 [91.8%]	2.89984152	3.78138097 [130.4%]
80	1.62092392 [57.5%]	2.59424385 [92.0%]	2.82107167	3.66766506 [130.0%]
81	1.57646991 [57.6%]	2.52173400 [92.1%]	2.73780180	3.57100391 [130.4%]
82	1.52946554 [57.7%]	2.44542810 [92.3%]	2.65047684	3.44092317 [129.8%]
83	1.48016085 [57.8%]	2.36574177 [92.4%]	2.55957193	3.32784237 [130.0%]
84	1.42882432 [58.0%]	2.28311415 [92.6%]	2.46558644	3.19889202 [129.7%]
85	1.37573951 [58.1%]	2.19800211 [92.8%]	2.36903733	3.06639023 [129.4%]
86	1.32120146 [58.2%]	2.11087410 [93.0%]	2.27045230	2.95083750 [130.0%]
87	1.26551270 [58.3%]	2.02220376 [93.2%]	2.17036270	2.80530186 [129.3%]
88	1.20897920 [58.4%]	1.93246339 [93.4%]	2.06929638	2.66675798 [128.9%]
89	1.15190626 [58.5%]	1.84211760 [93.6%]	1.96777070	2.54460930 [129.3%]
90	1.09459437 [58.7%]	1.75161702 [93.9%]	1.86628586	2.41474525 [129.4%]

Table A2

Reversionary Annuity Values for $x = y + 5$

Age x	$a_{x\mid y}^{\min}$	$a_{x\mid y}^{\text{mark}}$	$a_{x\mid y}^{\text{ind}}$	$a_{x\mid y}^{\max}$
25	1.14227539 [74.0%]	1.37612564 [89.2%]	1.54342883	1.75504028 [113.7%]
30	1.36338692 [74.4%]	1.63282105 [89.1%]	1.83169288	2.09389586 [114.3%]
31	1.41199510 [74.5%]	1.68921887 [89.1%]	1.89496287	2.16851190 [114.4%]
32	1.46207932 [74.6%]	1.74731263 [89.1%]	1.96010956	2.24541854 [114.6%]
33	1.51363601 [74.7%]	1.80709457 [89.1%]	2.02712102	2.32464174 [114.7%]
34	1.56665579 [74.7%]	1.86854990 [89.1%]	2.09597709	2.40617056 [114.8%]
35	1.62112295 [74.8%]	1.93165598 [89.2%]	2.16664860	2.48998411 [114.9%]
36	1.67701481 [74.9%]	1.99638170 [89.2%]	2.23909659	2.57605046 [115.0%]
37	1.73430108 [75.0%]	2.06268666 [89.2%]	2.31327144	2.66432562 [115.2%]
38	1.79294324 [75.0%]	2.13052045 [89.2%]	2.38911199	2.75475241 [115.3%]
39	1.85289386 [75.1%]	2.19982184 [89.2%]	2.46654470	2.84725935 [115.4%]
40	1.91409588 [75.2%]	2.27051796 [89.2%]	2.54548271	2.94175948 [115.6%]
41	1.97648199 [75.3%]	2.34252356 [89.2%]	2.62582502	3.03814919 [115.7%]
42	2.03997385 [75.3%]	2.41574015 [89.2%]	2.70745553	3.13630702 [115.8%]
43	2.10448149 [75.4%]	2.49005528 [89.2%]	2.79024227	3.23609255 [116.0%]
44	2.16990259 [75.5%]	2.56534173 [89.3%]	2.87403649	3.33734523 [116.1%]
45	2.23612182 [75.6%]	2.64145687 [89.3%]	2.95867201	3.43988334 [116.3%]
46	2.30301028 [75.7%]	2.71824194 [89.3%]	3.04396439	3.54350304 [116.4%]
47	2.37042493 [75.7%]	2.79552151 [89.3%]	3.12971042	3.64797753 [116.6%]
48	2.43820804 [75.8%]	2.87310294 [89.3%]	3.21568757	3.75305642 [116.7%]
49	2.50618685 [75.9%]	2.95077605 [89.4%]	3.30165363	3.85846524 [116.9%]

Table A2 (Continued)

Reversionary Annuity Values for $x = y + 5$

Age x	$a_{x\|y}^{\text{min}}$	$a_{x\|y}^{\text{mark}}$	$a_{x\|y}^{\text{ind}}$	$a_{x\|y}^{\text{max}}$
50	2.57417319 [76.0%]	3.02831282 [89.4%]	3.38734644	3.96390541 [117.0%]
51	2.64196329 [76.1%]	3.10546727 [89.4%]	3.47248393	4.06905439 [117.2%]
52	2.70933771 [76.2%]	3.18197555 [89.5%]	3.55676416	4.17356642 [117.3%]
53	2.77606141 [76.3%]	3.25755618 [89.5%]	3.63986578	4.27707372 [117.5%]
54	2.84188397 [76.4%]	3.33191052 [89.5%]	3.72144866	4.37918848 [117.7%]
55	2.90654007 [76.5%]	3.40472351 [89.6%]	3.80115472	4.47950555 [117.8%]
56	2.96975011 [76.6%]	3.47566461 [89.6%]	3.87860924	4.57760625 [118.0%]
57	3.03122108 [76.7%]	3.54438907 [89.7%]	3.95342232	4.67306332 [118.2%]
58	3.09064772 [76.8%]	3.61053952 [89.7%]	4.02519080	4.76428185 [118.4%]
59	3.14771391 [76.9%]	3.67374783 [89.7%]	4.09350045	4.85072990 [118.5%]
60	3.20209433 [77.0%]	3.73363732 [89.8%]	4.15792863	4.93277147 [118.6%]
61	3.25345642 [77.1%]	3.78982532 [89.8%]	4.21804722	5.00993798 [118.8%]
62	3.30146269 [77.3%]	3.84192603 [89.9%]	4.27342600	5.08178959 [118.9%]
63	3.34577323 [77.4%]	3.88955379 [90.0%]	4.32363634	5.14793205 [119.1%]
64	3.38604861 [77.5%]	3.93232653 [90.0%]	4.36825530	5.20803748 [119.2%]
65	3.42195298 [77.7%]	3.96986969 [90.1%]	4.40686995	5.26187031 [119.4%]
66	3.45315751 [77.8%]	4.00182026 [90.1%]	4.43908204	5.30430385 [119.5%]
67	3.47934401 [77.9%]	4.02783112 [90.2%]	4.46451293	5.33796393 [119.6%]
68	3.50020876 [78.1%]	4.04757551 [90.3%]	4.48280856	5.36391886 [119.7%]
69	3.51546656 [78.2%]	4.06075167 [90.4%]	4.49364464	5.38238291 [119.8%]

Table A2 (Continued)
Reversionary Annuity Values for $x = y + 5$

Age x	$a_{x\mid y}^{\min}$	$a_{x\mid y}^{\text{mark}}$	$a_{x\mid y}^{\text{ind}}$	$a_{x\mid y}^{\max}$
70	3.52485478 [78.4%]	4.06708749 [90.4%]	4.49673180	5.39321698 [119.9%]
71	3.52813753 [78.5%]	4.06634504 [90.5%]	4.49182066	5.38636635 [119.9%]
72	3.52510976 [78.7%]	4.05832515 [90.6%]	4.47870672	5.37182863 [119.9%]
73	3.51560127 [78.9%]	4.04287159 [90.7%]	4.45723497	5.35091431 [120.1%]
74	3.49948050 [79.0%]	4.01987498 [90.8%]	4.42730408	5.31567424 [120.1%]
75	3.47665805 [79.2%]	3.98927636 [90.9%]	4.38887011	5.26688895 [120.0%]
76	3.44708985 [79.4%]	3.95107007 [91.0%]	4.34194954	5.21476466 [120.1%]
77	3.41077974 [79.6%]	3.90530612 [91.1%]	4.28662160	5.14443318 [120.0%]
78	3.36778163 [79.7%]	3.85209178 [91.2%]	4.22302978	5.06568899 [120.0%]
79	3.31820094 [79.9%]	3.79159240 [91.3%]	4.15138237	4.98179523 [120.0%]
80	3.26219525 [80.1%]	3.72403128 [91.5%]	4.07195204	4.87806485 [119.8%]
81	3.19997431 [80.3%]	3.64968871 [91.6%]	3.98507441	4.77864454 [119.9%]
82	3.13179904 [80.5%]	3.56889995 [91.7%]	3.89114550	4.65372832 [119.6%]
83	3.05797978 [80.7%]	3.48205235 [91.9%]	3.79061818	4.53676549 [119.7%]
84	2.97887359 [80.9%]	3.38958143 [92.0%]	3.68399750	4.39770993 [119.4%]
85	2.89488068 [81.0%]	3.29196608 [92.2%]	3.57183510	4.26113994 [119.3%]
86	2.80644005 [81.2%]	3.18972292 [92.3%]	3.45472273	4.12004326 [119.3%]
87	2.71402427 [81.4%]	3.08339992 [92.5%]	3.33328493	3.96177884 [118.9%]
88	2.61813363 [81.6%]	2.97356937 [92.7%]	3.20817104	3.81763990 [119.0%]
89	2.51928964 [81.8%]	2.86082033 [92.9%]	3.08004677	3.65708358 [118.7%]
90	2.41802801 [82.0%]	2.74575079 [93.1%]	2.94958532	3.48893402 [118.3%]

Table A3
Reversionary Annuity Values for $x = y - 5$

Age x	$a_{x\|y}^{\text{min}}$	$a_{x\|y}^{\text{mark}}$	$a_{x\|y}^{\text{ind}}$	$a_{x\|y}^{\text{max}}$
25	0.24520562 [21.8%]	1.00350369 [89.4%]	1.12307645	1.42060031 [126.5%]
30	0.26068097 [20.0%]	1.16692988 [89.4%]	1.30598852	1.67332321 [128.1%]
31	0.26383968 [19.6%]	1.20220237 [89.4%]	1.34540096	1.72832446 [128.5%]
32	0.26699798 [19.3%]	1.23828123 [89.4%]	1.38568799	1.78476501 [128.8%]
33	0.27014470 [18.9%]	1.27513086 [89.4%]	1.42680684	1.84260888 [129.1%]
34	0.27326761 [18.6%]	1.31270910 [89.4%]	1.46870723	1.90181061 [129.5%]
35	0.27635344 [18.3%]	1.65096669 [89.4%]	1.51133080	1.96231452 [129.8%]
36	0.27938782 [18.0%]	1.38984690 [89.4%]	1.55461066	2.02405391 [130.2%]
37	0.28235528 [17.7%]	1.42928502 [89.4%]	1.59847086	2.08695022 [130.6%]
38	0.28523925 [17.4%]	1.46920798 [89.4%]	1.64282594	2.15091229 [130.9%]
39	0.28802203 [17.1%]	1.50953390 [89.4%]	1.68758051	2.21583561 [131.3%]
40	0.29068483 [16.8%]	1.55017175 [89.5%]	1.73262883	2.28160160 [131.7%]
41	0.29320780 [16.5%]	1.59102099 [89.5%]	1.77785446	2.34807704 [132.1%]
42	0.29557002 [16.2%]	1.63197133 [89.5%]	1.82313002	2.41511356 [132.5%]
43	0.29774962 [15.9%]	1.67290245 [89.5%]	1.86831693	2.48254730 [132.9%]
44	0.29972381 [15.7%]	1.71368392 [89.6%]	1.91326532	2.55019878 [133.3%]
45	0.30146900 [15.4%]	1.75417511 [89.6%]	1.95781403	2.61787301 [133.7%]
46	0.30296088 [15.1%]	1.79422526 [89.6%]	2.00179070	2.68535994 [134.1%]
47	0.30417458 [14.9%]	1.83367363 [89.7%]	2.04501201	2.75243533 [134.6%]
48	0.30508483 [14.6%]	1.87234980 [89.7%]	2.08728409	2.81886212 [135.0%]
49	0.30566611 [14.4%]	1.91007412 [89.7%]	2.12840309	2.88335664 [135.5%]

Table A3 (Continued)

Reversionary Annuity Values for $x = y - 5$

Age x	$a_{x\|y}^{\text{min}}$	$a_{x\|y}^{\text{mark}}$	$a_{x\|y}^{\text{ind}}$	$a_{x\|y}^{\text{max}}$
50	0.30589290 [14.1%]	1.94665829 [89.8%]	2.16815591	2.94623322 [135.9%]
51	0.30573987 [13.9%]	1.98190617 [89.8%]	2.20632113	3.00740135 [136.3%]
52	0.30518218 [13.6%]	2.01561467 [89.9%]	2.24267018	3.06655030 [136.7%]
53	0.30419571 [13.4%]	2.04757494 [89.9%]	2.27696871	3.12336737 [137.2%]
54	0.30275741 [13.1%]	2.07757371 [90.0%]	2.30897814	3.17754430 [137.6%]
55	0.30084559 [12.9%]	2.10539482 [90.0%]	2.33845749	3.22878557 [138.1%]
56	0.29844028 [12.6%]	2.13082096 [90.1%]	2.36516544	3.27681879 [138.5%]
57	0.29552359 [12.4%]	2.15363567 [90.2%]	2.38886256	3.32140790 [139.0%]
58	0.29208006 [12.1%]	2.17362545 [90.2%]	2.40931380	3.36236966 [139.6%]
59	0.28809702 [11.9%]	2.19058205 [90.3%]	2.42629110	3.39645331 [140.0%]
60	0.28356503 [11.6%]	2.20430502 [90.4%]	2.43957627	3.42406709 [140.4%]
61	0.27847815 [11.4%]	2.21460426 [90.4%]	2.44896387	3.44680526 [140.7%]
62	0.27283434 [11.1%]	2.22130276 [90.5%]	2.45426429	3.46472262 [141.2%]
63	0.26663576 [10.9%]	2.22423936 [90.6%]	2.45530685	3.47804805 [141.7%]
64	0.25988910 [10.6%]	2.22327152 [90.7%]	2.45194281	3.48565830 [142.2%]
65	0.25260575 [10.3%]	2.21827808 [90.8%]	2.44404842	3.48020222 [142.4%]
66	0.24480209 [10.1%]	2.20916186 [90.9%]	2.43152781	3.46991894 [142.7%]
67	0.23649952 [9.8%]	2.19585221 [91.0%]	2.41431559	3.45588404 [143.1%]
68	0.22772461 [9.5%]	2.17830721 [91.1%]	2.39237927	3.43458482 [143.6%]
69	0.21850902 [9.2%]	1.15651564 [91.2%]	2.36572129	3.39893437 [143.7%]

Table A3 (Continued)
Reversionary Annuity Values for $x = y - 5$

Age x	$a_{x\mid y}^{\min}$	$a_{x\mid y}^{\text{mark}}$	$a_{x\mid y}^{\text{ind}}$	$a_{x\mid y}^{\max}$
70	0.20888943 [8.9%]	2.13049863 [91.3%]	2.33438061	3.36169111 [144.0%]
71	0.19890733 [8.7%]	2.10031078 [91.4%]	2.29843382	3.31944710 [144.4%]
72	0.18860880 [8.4%]	2.06604094 [91.5%]	2.25799572	3.26044836 [144.4%]
73	0.17804400 [8.0%]	2.02781233 [91.6%]	2.21321925	3.20540362 [144.8%]
74	0.16726675 [7.7%]	1.98578218 [91.8%]	2.16429483	3.13430321 [144.8%]
75	0.15633397 [7.4%]	1.94014073 [91.9%]	2.11144894	3.06170507 [145.0%]
76	0.14530503 [7.1%]	1.89110964 [92.0%]	2.05494216	2.98264084 [145.1%]
77	0.13424103 [6.7%]	1.83893977 [92.2%]	1.99506644	2.89631854 [145.2%]
78	0.12320409 [6.4%]	1.78390839 [92.3%]	1.93214179	2.80662089 [145.3%]
79	0.11225684 [6.0%]	1.72631590 [92.5%]	1.86651234	2.71368536 [145.4%]
80	0.10146033 [5.6%]	1.66648190 [92.7%]	1.79854195	2.61038276 [145.1%]
81	0.09087444 [5.3%]	1.60474105 [92.8%]	1.72860941	2.52314362 [146.0%]
82	0.08055668 [4.9%]	1.54143845 [93.0%]	1.65710323	2.40307339 [145.0%]
83	0.07056449 [4.5%]	1.47692488 [93.2%]	1.58441632	2.30134703 [145.2%]
84	0.06094652 [4.0%]	1.41155187 [93.4%]	1.51094059	2.20245182 [145.8%]
85	0.05174927 [3.6%]	1.34566685 [93.6%]	1.43706158	2.07964701 [144.7%]
86	0.04302653 [3.2%]	1.27960832 [93.9%]	1.36315330	1.97423032 [144.8%]
87	0.03481044 [2.7%]	1.21370137 [94.1%]	1.28957338	1.88633356 [146.3%]
88	0.02715750 [2.2%]	1.14825347 [94.4%]	1.21665865	1.76404200 [145.0%]
89	0.02011459 [1.8%]	1.08355072 [94.7%]	1.14472131	1.64874600 [144.0%]
90	0.01378185 [1.3%]	1.01985466 [95.0%]	1.07404564	1.54921654 [144.2%]

A Fuzzy Approach to Grouping by Policyholder Age in General Insurance

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Abstract‡

In general insurance, policyholder age is often treated as a factor with the number of levels requiring that the individual ages of the policyholders be grouped. Although the groups are usually defined by the existing underwriting structure, it should be investigated as part of any premium rating exercise that uses a model to assess past claims experience. It is possible that an incorrect grouping by policyholder age could bias the results of the risk premium estimation. On the other hand, it may not be computationally feasible to use separate ages in the premium model, making some form of grouping necessary. In this paper, we specify a data-based procedure for grouping by age using fuzzy set theory. An example is given that illustrates how the method can be used in practice.

Key words and phrases: *fuzzy set theory, general insurance, premium rating*

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1 Introduction

Policyholder age is one of the important underwriting factors used in many general (property/casualty) insurance policies, such as motor insurance. Data are usually available subdivided according to age last birthday, but policyholders are often rated according to the set of age groups in which they fall. For example, Brockman and Wright (1991) use eight age groups:

$$17-18, 19-21, 22-24, 25-29, 30-34, 35-44, 45-54, 55+.$$

It is usual practice for the boundaries and sizes of the age groups to be fixed a priori, usually based on the underwriter's or actuary's judgment. Sometimes the decision is based on no more than the previous rating age groups already in use by the company in their rating system. These assumptions represent part of the specification of the model used to investigate the claims experience and to set the relative premium levels.

An investigation into the appropriateness of the age groups should be part of any investigation of a general insurance portfolio. It is possible that an inappropriate choice for the age groupings could adversely affect the results from the premium rating model, leading to incorrect estimation of risk based on historical data.

There are several approaches that can be used to determine the appropriate policyholder age grouping. For example, parametric or non-parametric smoothing methods, as employed in the graduation of life tables, could be used. See Renshaw (1991) and Verrall (1996) for descriptions of these models applied to life tables within the framework of generalized linear models. Haberman and Renshaw (1996) provide a useful overview of the applications of generalized linear models in actuarial science.

We will consider rating based on the grouping of ages, as this is often the approach taken in practice. There are several reasons for retaining the age grouping procedure: (i) it can lead to simple rating structures, (ii) it is the familiar method, and (iii) it alleviates problems with sparse data. The latter problem can be overcome if a simple parametric model is used, but this may not be appropriate.

This paper shows how fuzzy set methods can be used to give an indication of suitable age groupings based on past data.[1] The advantage of this method is that the results are data-dependent, rather than being

[1] In spite of the importance of age groupings, we are not aware of other attempts to use statistical or other methods as aids in determining suitable age groups.

entirely subjective. It is intrinsic to the fuzzy approach that there is still an element of subjectivity, but we believe that the results of the fuzzy methods can lead to greater confidence in the groupings. It may be that a decision is made to retain the historical age groupings, but the method in this paper allows the suitability of these groupings to be assessed.

Our aim is to group individual ages into clusters that will form the rating groups for policyholder age. It is necessary to restrict such groupings to adjacent ages, and this requirement will limit the amount of clustering that is possible. Fuzzy clustering is suitable for this purpose and can be implemented using Bezdek's (1981) fuzzy c-means clustering algorithm. Derrig and Ostaszewski (1994) first applied this algorithm in the actuarial literature, but other methods of fuzzy clustering exist. The algorithm is described in the main part of this paper and is then applied to a set of general insurance data.

2 Fuzzy Set Theory

Fuzzy set theory was first developed by Zadeh (1965). It was introduced to the actuarial literature by DeWit (1982) and Lemaire (1990). Ostaszewski (1993) has detailed the possible applications of fuzzy methods in actuarial science. Recently several other papers have appeared (e.g., Cummins and Derrig, 1993, 1997; Derrig and Ostaszewski, 1994, 1995; and Young, 1996). In some cases fuzzy set theory is not used for inference, but to reach a decision on the basis of fuzzy information. An example of this type of application is Horgby et al., (1997) where a fuzzy expert system is defined in order to reach an underwriting decision in the case of persons with diabetes.[2] These papers have covered a wide range of contexts, including health underwriting, pricing of general insurance business, asset allocation, and marketing. Yakoubov and Haberman (1998) give a more recent comprehensive review of fuzzy techniques with actuarial applications.

We now give a brief introduction to the ideas from fuzzy set theory that will be needed in the following sections. We do not provide a full exposition of the theory of fuzzy sets. For an introduction, the reader should see Zimmerman (1991).

In the application considered in this paper, the aim is to decide to which group each individual age should be allocated. Thus, for example, should a 30 year old be allocated to the group containing 29 year

[2] In this paper, we use fuzzy set theory to make inferences from the data, where the resulting inference is fuzzy.

olds or to the group containing 31 year olds? Given the random nature of insurance data, the evidence from the data favoring one group over another is unlikely to be conclusive. In other words, there will be uncertainty, and this uncertainty can be quantified using the concept of fuzziness.

Thus, if the individual age under consideration is x and the age group is denoted by A, we can quantify the degree of membership of x in A by $\mu_A(x)$. This requires the generalization of the definition of a set (a crisp set) to a fuzzy set.

Definition 1. *Given a collection of objects and the universe of discourse, U, a fuzzy set A is defined by*

$$\mu_A(x) : U \to M$$

where $x \in A$, $\mu_A(x)$ is the membership function of A and M is an ordered set.

M is usually defined as the unit interval $[0, 1]$. This definition will be used throughout this paper. Thus, for a crisp set A, the membership function is defined as

$$\mu_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

The two extreme values, 0 and 1, represent the lowest and highest degrees of membership, respectively. The degree of membership can be interpreted as the truth value of the statement "x is a member of A". In the application to grouping by policyholder age, the membership function will indicate to which group(s) each of the individual ages should be considered as belonging (noting that this may not be a conclusive decision). In order to interpret the outcome of the algorithm that gives estimates of the membership function for each age, it is useful to discard possibilities where the membership function is low. This can be achieved using the notion of an α-cut of a fuzzy set, A_α, where

$$A_\alpha = \{x \in U : \mu_A(x) \geq \alpha\}.$$

This set contains the elements for which there is (at least) an 100α percent belief that they are in A.

When grouping by policyholder age, the problem is to choose the number of groups into which the policyholder age will be partitioned and then to assess to which group each age belongs. For ease of notation, denote the individual ages by $\{i : i = 1, 2, \ldots, n\}$ and the groups by $\{k : k = 1, 2, \ldots, c\}$, where we would expect c to be much less than n. The assessment of appropriate age groupings requires estimates of the elements of the $n \times c$ matrix $\mathbf{M}$, defined by

$$\mathbf{M} = \{\mu_{ik} : i = 1, 2, \ldots, n; k = 1, 2, \ldots, c\}$$

where μ_{ik} is the degree of membership of the ith individual age in the kth group.

Definition 2. $\mathbf{M}$ *is a* fuzzy c-partition *(Bezdek, 1981) if its elements satisfy:*

1. $\mu_{ik} \in [0, 1]$ *for* $i = 1, 2, \ldots, n$ *and* $k = 1, 2, \ldots, c$;
2. $\sum_{k=1}^{c} \mu_{ik} = 1$, *for* $i = 1, 2, \ldots, n$; *and*
3. $0 < \sum_{i=1}^{n} \mu_{ik} < n$ *for* $k = 1, 2, \ldots, c$.

Elements can belong to two or more clusters to some extent, determined by the membership functions. The boundaries of the groups are not determined a priori. We also would expect adjacent ages to lie in the same or adjacent groups. This is not specified as part of the model, but will be considered after the matrix $\mathbf{M}$ has been estimated and an appropriate α-cut has been applied. If the results do not indicate that a relatively smooth transition between the groups is possible, then age grouping may not be appropriate (or it may not be possible to determine the groups from the data). The next section shows how the values of the membership functions can be estimated from the data.

3 The Fuzzy c-Means Algorithm

In order to calculate values of the membership functions for each individual age, an objective function is required to which an optimization criterion can be applied. In general, we may have a number of different features of the data that should be used to determine the age groups (for example, different categories of claims, claims frequency, and claims severity, etc.). To be as general as possible, we define the data set by the $n \times p$ matrix $\mathbf{X}$

$$\mathbf{X} = \{\mathbf{x}_i : i = 1, 2, \ldots, n\}$$

where each $\mathbf{x}_i$ observation consists of p features $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})$.

If we consider two ages, r and s ($r, s = 1, 2, \ldots, n$), the decision on grouping will be based on a measure of their dissimilarity, which is taken to be their distance apart

$$d(\mathbf{x}_r, \mathbf{x}_s) = \|\mathbf{x}_r - \mathbf{x}_s\|$$

for a suitably defined norm on $\mathbf{X}$. The degrees of membership of each individual age in each age group are defined by minimizing $z_m(\mathbf{M}, \mathbf{V})$ over $\mathbf{M}$ and $\mathbf{V}$, where

$$z_m(\mathbf{M}, \mathbf{V}) = \sum_{i}^{n} \sum_{k}^{c} (\mu_{ik})^m \|\mathbf{x}_i - \mathbf{v}_k\|^2 \tag{1}$$

and $\mathbf{V} = \{\mathbf{v}_k : k = 1, 2, \ldots, c\}$ denotes the $c \times p$ matrix, with $\mathbf{v}_k$ as the center of the kth cluster to be estimated in the optimization procedure. (See the steps outlined below.) The exponential weight m ($m > 1$) reduces the influence of noise in the membership values in relation to the clustering criterion. The larger m is, the more weight is assigned to elements with a higher degree of membership and the less weight is assigned to those with a lower degree of membership. As $m \to \infty$, the membership function tends toward the constant value of $1/C$, indicating that each element is assigned to each cluster with the same degree of membership. It is preferable to have $\mathbf{M}$ more uniform, and, usually, m is taken to be 2. To motivate the form of $z_m(\mathbf{M}, \mathbf{V})$, note that each term increases as $\|\mathbf{x}_i - \mathbf{v}_k\|$ increases and as μ_{ik} increases. Thus, minimizing will assign low membership values when $\|\mathbf{x}_i - \mathbf{v}_k\|^2$ is large and vice versa.

It can be shown (Bezdek, 1981) that a local minimum of equation (1) is obtained when the set of equations (2) and (3) are satisfied simultaneously:

$$\mathbf{v}_k = \frac{\sum_i^n (\mu_{ik})^m \mathbf{x}_i}{\sum_i^n (\mu_{ik})^m} \tag{2}$$

$$\mu_{ik} = \frac{\left(\|\mathbf{x}_i - \mathbf{v}_k\|^2\right)^{-1/(m-1)}}{\sum_i^n \left(\|\mathbf{x}_i - \mathbf{v}_k\|^2\right)^{-1/(m-1)}} \tag{3}$$

for $i = 1, 2 \ldots, n$; and $k = 1, 2 \ldots, c$. The fuzzy c-means algorithm solves these equations iteratively to converge to a (perhaps local) optimum value of equation (1). The number of groups, c, is not estimated from the data, but must be specified before the algorithm is applied. We have found, however, that a small range of values can be tried to choose the most suitable number of groups (see Section 4). Also, it is necessary to specify the norm on **X**, by choosing a suitable symmetric, positive definite $(p \times p)$ matrix **G**. This matrix indicates the relative importance of each element and the correlations between them. Examples of **G** that could be used are the identity matrix, a diagonal matrix (with appropriate terms), and the covariance matrix of $\mathbf{x}_i$. The norm is then defined by

$$\|\mathbf{x}\|_{\mathbf{G}} = \mathbf{x}'\mathbf{G}\mathbf{x}.$$

The algorithm may now be summarized as follows:

Step 1: Specify:

- The number of clusters, c $(c = 2, 3, \ldots, n)$,
- The parameter m $(m = 2, 3, 4, \ldots)$,
- The matrix **G**, and
- A small positive number, ϵ, to measure the convergence.

Step 2: Initialize:

- The membership function values, **M** as $\mathbf{M}^{(0)}$, and
- The counter $j = 0$.

Step 3: For $k = 1, 2, 3, \ldots, c$, calculate the centers of the fuzzy clusters, $\{\mathbf{v}_k^{(j)}\}$, using $\mathbf{M}^{(j)}$, and equation (2).

Step 4: Calculate $\mathbf{M}^{(j+1)}$, using $\mathbf{v}_k^{(j)}$ and equation (3) if $\{\mathbf{x}_i \neq \{\mathbf{v}_k^{(j)}\}$. Otherwise, set

$$\mu_{jk} = \begin{cases} 1 & \text{for } j = i; \\ 0 & \text{for } j \neq i. \end{cases}$$

Step 5: Calculate

$$\Delta = \|\mathbf{M}^{(j+1)} - \mathbf{M}^{(j)}\|_{\mathbf{G}}.$$

If $\Delta > \epsilon$, then set $j = j + 1$ and return to Step 2.

This algorithm will converge to a local optimum. The results must be checked using different initial partitions to ensure that the result is consistent. There are no computational problems, even when there are large numbers of elements.

The final problem is to consider the validity of the clustering resulting from the fuzzy c-means algorithm. Part of this problem is choosing an appropriate value for c, and Bezdek (1981) discusses the criteria for this selection. The next section provides some ways in which this choice may be approached. Also, the results of clustering must be interpreted in relation to the aim of selecting groups for the policyholder age factor.

4 An Example

For this example we use a set of data based on more than 50,000 motor policies. For each policy, we have, for each of two types of claims (material damage (MD), and bodily injury (BI)), the number of claims, the total cost of the claims, and the earned driver years (exposure). The data for the youngest ages (< 25) and oldest ages (> 82) are grouped in order to achieve exposures greater than 30. For this data set, the exposures were low at low ages, making this initial grouping advisable; a more comprehensive data set or one with a different distribution over exposure would remove this need.

First, two quantities of interest are derived from the data set, the frequency and severity for MD and BI claims. A summary of these values is given in Table 1. It is necessary at this stage to remove distortion effects due to the uneven mix of business by policyholder age, i.e., to calculate standardized frequencies and severities after accounting for other factors such as car group, gender, etc. Standard techniques such as generalized linear models (Brockman and Wright, 1991) or an approach similar to that of Taylor (1989) can be used. This involves the use of well-documented estimation methods, keeping the ages separate. In this example, however, it can be assumed that this has already been done and that policyholder age is the only significant factor. This allows us to concentrate on fuzzy clustering techniques.

Considering the data in Table 1, the claim frequency seems to give a reasonable indication of the variability of the risk by policyholder age, but that claim severity does not show any clear pattern and contains a high degree of variability.

Table 1
Frequency and Severity for MD and BI Claims

	Frequency		Severity		AdjFreq			
Age	MD	BI	MD	BI	MD	BI	CruPrem	Exposure
< 25	0.30691	0.04384	515.90	5381.68	121.10	248.14	369.25	43.54
25	0.27846	0.05967	439.60	2242.76	109.88	337.70	447.58	31.99
26	0.13580	0.01598	302.74	9742.90	53.59	9.42	144.01	79.66
27	0.18732	0.01767	364.29	4286.98	73.91	10.01	173.93	36.11
28	0.20380	0.01002	395.48	5254.42	8.42	56.73	137.15	571.41
29	0.18907	0.01126	362.36	4598.03	74.61	63.75	138.35	79.95
30	0.18175	0.01486	434.33	6041.05	71.72	84.10	155.82	1113.44
31	0.14277	0.01142	406.46	7614.30	56.34	64.64	12.98	1671.45
32	0.15469	0.00729	331.00	5928.38	61.04	41.26	102.30	2007.57
33	0.12644	0.00651	30.71	5423.97	49.89	36.85	86.74	1857.12
34	0.12914	0.00861	416.71	6037.58	5.96	48.73	99.69	1921.73
35	0.14105	0.00641	369.25	6043.16	55.66	36.29	91.94	1885.85
36	0.12895	0.00794	414.31	4742.69	5.88	44.96	95.85	2082.56
37	0.14444	0.00698	365.89	5473.63	57.00	39.53	96.52	2004.59
38	0.12641	0.00967	423.00	5445.77	49.88	54.74	104.62	1907.93
39	0.12772	0.00872	458.40	4757.70	5.40	49.37	99.77	1823.65
40	0.12218	0.00732	356.89	3381.56	48.21	41.40	89.61	1739.68
41	0.11796	0.01222	422.91	3424.66	46.55	69.14	115.69	1666.95
42	0.11471	0.00828	401.68	6085.11	45.26	46.85	92.11	1614.38
43	0.11017	0.00646	416.18	4804.83	43.47	36.55	8.02	1675.11

Notes: AdjFreq = Adjusted Frequency; and CruPrem = Crude Premium.

Table 1 (Continued)
Frequency and Severity for MD and BI Claims

	Frequency		Severity		AdjFreq			
Age	MD	BI	MD	BI	MD	BI	CruPrem	Exposure
44	0.11005	0.00758	385.92	5724.75	43.42	42.88	86.30	1596.01
45	0.11247	0.00287	345.49	4551.51	44.38	16.26	6.64	155.33
46	0.11479	0.00776	36.96	3789.87	45.29	43.89	89.19	164.99
47	0.11970	0.00874	385.04	7947.82	47.23	49.45	96.68	1456.71
48	0.11975	0.01023	411.26	3776.65	47.25	57.88	105.13	1493.30
49	0.12098	0.01052	343.41	4671.31	47.74	59.54	107.27	1209.86
50	0.11047	0.00440	337.79	8008.71	43.59	24.90	68.49	1301.91
51	0.14009	0.01302	386.38	415.16	55.28	73.69	128.96	1221.93
52	0.12067	0.00546	396.67	7867.67	47.61	3.90	78.52	1165.49
53	0.12340	0.00507	391.90	2685.65	48.69	28.70	77.40	1129.33
54	0.12615	0.00788	455.80	6436.11	49.78	44.62	94.40	887.81
55	0.10273	0.00654	535.29	6999.40	4.54	37.03	77.57	972.51
56	0.09071	0.00542	366.74	5385.20	35.79	3.65	66.44	94.04
57	0.07957	0.00918	387.84	3859.75	31.40	51.96	83.36	415.89
58	0.11671	0.00824	382.74	2726.71	46.05	46.63	92.68	463.46
59	0.10629	0.00518	496.70	11538.43	41.94	29.34	71.28	49.95
60	0.07552	0.00252	37.90	1599.23	29.80	14.25	44.05	505.56
61	0.08699	0.00829	619.72	9716.70	34.33	46.89	81.22	46.85
62	0.08507	0.00597	503.03	3677.61	33.57	33.79	67.36	426.37
63	0.06762	0.00432	353.15	3397.81	26.68	24.43	51.11	442.31

Notes: AdjFreq = Adjusted Frequency; and CruPrem = Crude Premium.

Table 1 (Continued)
Frequency and Severity for MD and BI Claims

	Frequency		Severity		AdjFreq			
Age	MD	BI	MD	BI	MD	BI	CruPrem	Exposure
64	0.06307	0.00293	696.58	27154.86	24.89	16.60	41.49	433.88
65	0.07488	0.00326	346.98	2545.45	29.55	18.42	47.97	39.95
66	0.08823	0.01307	514.26	6541.60	34.81	73.97	108.79	389.50
67	0.08404	0.00615	343.60	4642.38	33.16	34.80	67.96	31.47
68	0.08037	0.00423	304.66	1415.02	31.71	23.94	55.65	30.88
69	0.08252	0.00236	35.23	23768.18	32.56	13.34	45.91	269.91
70	0.06287	0.00286	31.02	7089.09	24.81	16.17	4.98	222.70
71	0.07759	0.00517	749.87	9937.03	3.62	29.27	59.89	246.06
72	0.07509	0.00901	401.62	3782.42	29.63	51.00	8.63	211.86
73	0.08597	0.00000	351.00	0.00	33.92	0.00	33.92	17.25
74	0.04838	0.00000	43.71	0.00	19.09	0.00	19.09	157.85
75	0.04505	0.00000	379.38	0.00	17.77	0.00	17.77	155.39
76	0.09957	0.01494	378.72	3991.06	39.29	84.53	123.82	127.82
77	0.06820	0.00000	662.06	0.00	26.91	0.00	26.91	74.65
78	0.11220	0.00863	276.11	636.36	44.27	48.85	93.12	73.73
79	0.04088	0.00000	172.09	0.00	16.13	0.00	16.13	46.70
80	0.05759	0.01920	857.47	6532.27	22.73	108.65	131.38	33.15
81	0.06099	0.02033	1782.07	636.36	24.07	115.06	139.13	31.30
82	0.04193	0.00000	193.53	0.00	16.55	0.00	16.55	3.35
83+	0.00854	0.00000	396.89	0.00	3.37	0.00	3.37	74.50

Notes: AdjFreq = Adjusted Frequency; and CruPrem = Crude Premium.

Table 2
Centers of the Six Clusters

Clusters	1	2	3	4	5	6
MDADJ	114.48	48.39	52.16	48.53	36.80	21.94
BIADJ	292.07	90.79	64.61	43.44	29.85	4.47
Crude premium	406.55	139.18	116.77	91.97	66.65	26.41

Notes: MDADJ = MD Adjusted Frequency; and BIADJ = BI Adjusted Frequency.

BI claims are, on average, more than 13 times as costly as MD claims. For this reason, the fuzzy c-means algorithm will be applied to the claim frequencies and to adjusted claim frequencies, defined as claim frequencies multiplied by the average claim severity for each type of claim. Thus,

$$\text{MD adjusted frequency} = \text{MD frequency} \times \text{MD severity}$$

and

$$\text{BI adjusted frequency} = \text{BI frequency} \times \text{BI severity}.$$

The claim frequencies and the adjusted claim frequencies for each type of claim are shown in Table 1. Also shown are the crude premiums, calculated as follows:

$$\text{crude premium} = \text{MD adjusted frequency} + \text{BI adjusted frequency}.$$

Although both unadjusted and adjusted frequencies were considered, the adjusted frequencies were expected to be superior. This adjustment could have been incorporated into the norm matrix, **G**.

The algorithm was applied, with the number of clusters, c, set at five, six, seven, and ten. A more rigorous method would have been to use optimality measures (Bezdek, 1981) to determine a suitable value for c, but we found the ad hoc approach to be sufficient. After studying the results, $c = 6$ was chosen as being the most suitable value. The centers of the six clusters used in Tables 3 and 4 are given in Table 2. The conclusions for the age groupings are summarized below.

Table 3 contains the membership values for the unadjusted frequencies, and Table 4 shows the same results after a 20 percent cut. These tables clarify the results and make interpretation easier.

Table 3
Membership Values for Unadjusted Frequencies

	Cluster					
Age	1	2	3	4	5	6
< 25	0.96053	0.01684	0.00800	0.00644	0.00466	0.00353
25	0.94411	0.02563	0.01106	0.00868	0.00606	0.00446
26	0.00155	0.01386	0.89201	0.07550	0.01229	0.00479
27	0.00179	0.98356	0.00766	0.00404	0.00187	0.00108
28	0.01999	0.89561	0.04013	0.02410	0.01251	0.00767
29	0.00042	0.99622	0.00174	0.00093	0.00043	0.00025
30	0.00433	0.94781	0.02615	0.01290	0.00564	0.00316
31	0.00264	0.02910	0.86827	0.07746	0.01586	0.00666
32	0.01149	0.19305	0.58377	0.14857	0.04292	0.02020
33	0.00157	0.01156	0.64341	0.31516	0.02124	0.00707
34	0.00087	0.00677	0.86603	0.11273	0.01009	0.00352
35	0.00188	0.01964	0.89489	0.06595	0.01250	0.00514
36	0.00095	0.00740	0.84832	0.12819	0.01124	0.00390
37	0.00365	0.04222	0.82768	0.09668	0.02086	0.00891
38	0.00147	0.01085	0.67236	0.28908	0.01968	0.00655
39	0.00121	0.00919	0.77329	0.19591	0.01521	0.00518
40	0.00136	0.00923	0.27280	0.68619	0.02329	0.00713
41	0.00092	0.00580	0.10639	0.86171	0.01964	0.00554
42	0.00000	0.00000	0.00000	1.00000	0.00000	0.00000
43	0.00050	0.00277	0.02945	0.94424	0.01886	0.00417
44	0.00049	0.00273	0.02903	0.94493	0.01869	0.00412
45	0.00068	0.00393	0.04530	0.92287	0.02192	0.00529
46	0.00000	0.00000	0.00000	1.00000	0.00000	0.00000
47	0.00081	0.00525	0.11804	0.85538	0.01592	0.00462
48	0.00095	0.00616	0.13883	0.83025	0.01844	0.00537
49	0.00126	0.00834	0.21512	0.74585	0.02265	0.00679
50	0.00066	0.00368	0.03891	0.92675	0.02450	0.00550
51	0.00164	0.01668	0.90733	0.05887	0.01099	0.00449
52	0.00111	0.00736	0.17229	0.79201	0.02100	0.00623
53	0.00167	0.01163	0.36768	0.58355	0.02698	0.00848
54	0.00152	0.01116	0.63637	0.32324	0.02082	0.00689

Table 3 (Continued)
Membership Values for Unadjusted Frequencies

	Cluster					
Age	1	2	3	4	5	6
55	0.00245	0.01234	0.09233	0.68962	0.17499	0.02827
56	0.00211	0.00921	0.04697	0.16092	0.73696	0.04383
57	0.00026	0.00102	0.00411	0.01022	0.97420	0.01017
58	0.00018	0.00111	0.01870	0.97459	0.00427	0.00115
59	0.00157	0.00828	0.07154	0.82148	0.08155	0.01558
60	0.00050	0.00191	0.00711	0.01633	0.94664	0.02750
61	0.00120	0.00506	0.02367	0.07126	0.86868	0.03013
62	0.00062	0.00254	0.01138	0.03245	0.93550	0.01752
63	0.00196	0.00695	0.02314	0.04739	0.71255	0.20800
64	0.00245	0.00840	0.02634	0.05106	0.48074	0.43101
65	0.00053	0.00200	0.00740	0.01684	0.94278	0.03044
66	0.00226	0.00953	0.04551	0.13744	0.75520	0.05006
67	0.00041	0.00169	0.00740	0.02053	0.95747	0.01250
68	0.00000	0.00000	0.00000	0.00000	1.00000	0.00000
69	0.00039	0.00158	0.00664	0.01754	0.96065	0.01320
70	0.00245	0.00839	0.02626	0.05077	0.47003	0.44210
71	0.00009	0.00034	0.00133	0.00319	0.99093	0.00413
72	0.00061	0.00229	0.00854	0.01950	0.93669	0.03237
73	0.00136	0.00565	0.02529	0.07151	0.85892	0.03726
74	0.00023	0.00070	0.00188	0.00320	0.01397	0.98003
75	0.00008	0.00025	0.00066	0.00110	0.00434	0.99357
76	0.00368	0.01760	0.11435	0.52436	0.29403	0.04599
77	0.00213	0.00760	0.02529	0.05179	0.69834	0.21485
78	0.00016	0.00092	0.01117	0.98135	0.00517	0.00123
79	0.00037	0.00111	0.00279	0.00453	0.01612	0.97507
80	0.00410	0.01324	0.03849	0.06887	0.34477	0.53053
81	0.00443	0.01464	0.04400	0.08067	0.43032	0.42594
82	0.00025	0.00076	0.00193	0.00315	0.01150	0.98241
83+	0.01064	0.02697	0.05574	0.07898	0.17474	0.65292

Table 4
Membership Values for Unadjusted Frequencies After a 20 Percent Cut

	Cluster					
Age	1	2	3	4	5	6
< 25	1.00	0.00	0.00	0.00	0.00	0.00
25	1.00	0.00	0.00	0.00	0.00	0.00
26	0.00	0.00	1.00	0.00	0.00	0.00
27	0.00	1.00	0.00	0.00	0.00	0.00
28	0.00	1.00	0.00	0.00	0.00	0.00
29	0.00	1.00	0.00	0.00	0.00	0.00
30	0.00	1.00	0.00	0.00	0.00	0.00
31	0.00	0.00	1.00	0.00	0.00	0.00
32	0.00	0.00	1.00	0.00	0.00	0.00
33	0.00	0.00	0.67	0.33	0.00	0.00
34	0.00	0.00	1.00	0.00	0.00	0.00
35	0.00	0.00	1.00	0.00	0.00	0.00
36	0.00	0.00	1.00	0.00	0.00	0.00
37	0.00	0.00	1.00	0.00	0.00	0.00
38	0.00	0.00	0.70	0.30	0.00	0.00
39	0.00	0.00	1.00	0.00	0.00	0.00
40	0.00	0.00	0.28	0.72	0.00	0.00
41	0.00	0.00	0.00	1.00	0.00	0.00
42	0.00	0.00	0.00	1.00	0.00	0.00
43	0.00	0.00	0.00	1.00	0.00	0.00
44	0.00	0.00	0.00	1.00	0.00	0.00
45	0.00	0.00	0.00	1.00	0.00	0.00
46	0.00	0.00	0.00	1.00	0.00	0.00
47	0.00	0.00	0.00	1.00	0.00	0.00
48	0.00	0.00	0.00	1.00	0.00	0.00
49	0.00	0.00	0.22	0.78	0.00	0.00
50	0.00	0.00	0.00	1.00	0.00	0.00
51	0.00	0.00	1.00	0.00	0.00	0.00
52	0.00	0.00	0.00	1.00	0.00	0.00
53	0.00	0.00	0.39	0.61	0.00	0.00
54	0.00	0.00	0.66	0.34	0.00	0.00

Table 4 (Continued)
Membership Values for Unadjusted Frequencies
After a 20 Percent Cut

	Cluster					
Age	1	2	3	4	5	6
55	0.00	0.00	0.00	1.00	0.00	0.00
56	0.00	0.00	0.00	0.00	1.00	0.00
57	0.00	0.00	0.00	0.00	1.00	0.00
58	0.00	0.00	0.00	1.00	0.00	0.00
59	0.00	0.00	0.00	1.00	0.00	0.00
60	0.00	0.00	0.00	0.00	1.00	0.00
61	0.00	0.00	0.00	0.00	1.00	0.00
62	0.00	0.00	0.00	0.00	1.00	0.00
63	0.00	0.00	0.00	0.00	0.77	0.23
64	0.00	0.00	0.00	0.00	0.53	0.47
65	0.00	0.00	0.00	0.00	1.00	0.00
66	0.00	0.00	0.00	0.00	1.00	0.00
67	0.00	0.00	0.00	0.00	1.00	0.00
68	0.00	0.00	0.00	0.00	1.00	0.00
69	0.00	0.00	0.00	0.00	1.00	0.00
70	0.00	0.00	0.00	0.00	0.52	0.48
71	0.00	0.00	0.00	0.00	1.00	0.00
72	0.00	0.00	0.00	0.00	1.00	0.00
73	0.00	0.00	0.00	0.00	1.00	0.00
74	0.00	0.00	0.00	0.00	0.00	1.00
75	0.00	0.00	0.00	0.00	0.00	1.00
76	0.00	0.00	0.00	0.64	0.36	0.00
77	0.00	0.00	0.00	0.00	0.76	0.24
78	0.00	0.00	0.00	1.00	0.00	0.00
79	0.00	0.00	0.00	0.00	0.00	1.00
80	0.00	0.00	0.00	0.00	0.39	0.61
81	0.00	0.00	0.00	0.00	0.50	0.50
82	0.00	0.00	0.00	0.00	0.00	1.00
83+	0.00	0.00	0.00	0.00	0.00	1.00

The centers of the clusters above have been calculated using the data for each age that may be in each cluster (i.e., for which the membership value is > 0). Underwriting procedures require that the fuzzy results are converted into crisp age groups. This requires us to decide to which group each individual age should be assigned, using the results above. Deciding on the borders and sizes of each group is not straightforward. We would expect and require that risk should progress smoothly with age, although some variability will always be present. Thus, the age groups should contain only adjacent ages. Considering the results for the adjusted frequencies in Table 5, which are considered to be more reliable, we define the age groupings as shown in Table 6.

Group 5 is questionable, as it indicates a higher risk compared to group 4 (as indicated by the risk cluster) and makes an unexpected progression in the risk rating. It is possible that this effect is real, but otherwise groups 4 and 5 could be amalgamated.

In order to assess whether adjacent groups should be amalgamated, and in order to assess the implications for the premium, we can calculate the following risk measure R_i for each age group i:

$$R_i = \frac{1}{\|i\|} \sum_{\text{ages in } i} \sum_{k}^{c} \mu_{jk} \|\mathbf{v}_k\|.$$

The values of this measure for the groups above are displayed in Table 7. These values can be used to measure the relative risk of each group. For example, it can be seen that the highest risk group (group 1) has a risk measure which is nearly seven times that of the lowest risk group (group 7). These values are presented graphically in Figure 1, which also shows the crude risk premiums for comparison purposes. For greater clarity, Figure 2 reproduces the results shown in Figure 1, omitting cluster 1. Although an analysis of the residuals is not appropriate, these figures can be used to identify any strange results that can be investigated further.

Table 5
Membership Values for Adjusted Frequencies After a 20 Percent Cut

	Cluster					
Age	1	2	3	4	5	6
< 25	1.00	0.00	0.00	0.00	0.00	0.00
25	1.00	0.00	0.00	0.00	0.00	0.00
26	0.00	1.00	0.00	0.00	0.00	0.00
27	0.00	0.69	0.31	0.00	0.00	0.00
28	0.00	0.00	0.56	0.44	0.00	0.00
29	0.00	0.00	0.66	0.34	0.00	0.00
30	0.00	0.51	0.49	0.00	0.00	0.00
31	0.00	0.00	1.00	0.00	0.00	0.00
32	0.00	0.00	0.00	1.00	0.00	0.00
33	0.00	0.00	0.00	0.76	0.24	0.00
34	0.00	0.00	0.00	1.00	0.00	0.00
35	0.00	0.00	0.00	0.76	0.24	0.00
36	0.00	0.00	0.00	1.00	0.00	0.00
37	0.00	0.00	0.00	1.00	0.00	0.00
38	0.00	0.00	0.38	0.62	0.00	0.00
39	0.00	0.00	0.00	1.00	0.00	0.00
40	0.00	0.00	0.00	1.00	0.00	0.00
41	0.00	0.00	1.00	0.00	0.00	0.00
42	0.00	0.00	0.00	1.00	0.00	0.00
43	0.00	0.00	0.00	0.44	0.56	0.00
44	0.00	0.00	0.00	1.00	0.00	0.00
45	0.00	0.00	0.00	0.00	0.71	0.29
46	0.00	0.00	0.00	1.00	0.00	0.00
47	0.00	0.00	0.00	1.00	0.00	0.00
48	0.00	0.00	0.61	0.39	0.00	0.00
49	0.00	0.00	0.73	0.27	0.00	0.00
50	0.00	0.00	0.00	0.00	1.00	0.00
51	0.00	0.00	1.00	0.00	0.00	0.00
52	0.00	0.00	0.00	0.39	0.61	0.00
53	0.00	0.00	0.00	0.37	0.63	0.00
54	0.00	0.00	0.00	1.00	0.00	0.00

Table 5 (Continued)
Membership Values for Adjusted Frequencies After a 20 Percent Cut

	Cluster					
Age	1	2	3	4	5	6
55	0.00	0.00	0.00	0.28	0.72	0.00
56	0.00	0.00	0.00	0.00	1.00	0.00
57	0.00	0.00	0.23	0.44	0.33	0.00
58	0.00	0.00	0.00	1.00	0.00	0.00
59	0.00	0.00	0.00	0.00	1.00	0.00
60	0.00	0.00	0.00	0.00	0.32	0.68
61	0.00	0.00	0.00	0.54	0.46	0.00
62	0.00	0.00	0.00	0.00	1.00	0.00
63	0.00	0.00	0.00	0.00	0.73	0.27
64	0.00	0.00	0.00	0.00	0.30	0.70
65	0.00	0.00	0.00	0.00	0.54	0.46
66	0.00	0.38	0.62	0.00	0.00	0.00
67	0.00	0.00	0.00	0.00	1.00	0.00
68	0.00	0.00	0.00	0.00	1.00	0.00
69	0.00	0.00	0.00	0.00	0.37	0.63
70	0.00	0.00	0.00	0.00	0.28	0.72
71	0.00	0.00	0.00	0.00	1.00	0.00
72	0.00	0.00	0.00	0.52	0.48	0.00
73	0.00	0.00	0.00	0.00	0.00	1.00
74	0.00	0.00	0.00	0.00	0.00	1.00
75	0.00	0.00	0.00	0.00	0.00	1.00
76	0.00	0.73	0.27	0.00	0.00	0.00
77	0.00	0.00	0.00	0.00	0.00	1.00
78	0.00	0.00	0.00	1.00	0.00	0.00
79	0.00	0.00	0.00	0.00	0.00	1.00
80	0.00	1.00	0.00	0.00	0.00	0.00
81	0.00	1.00	0.00	0.00	0.00	0.00
82	0.00	0.00	0.00	0.00	0.00	1.00
83+	0.00	0.00	0.00	0.00	0.00	1.00

Table 6
Age Groupings

Group	1	2	3	
Risk Cluster	1	2	3	
Ages	(<25, 25)	(26, 27)	(28, 31)	
Group	4	5	6	7
Risk Cluster	4	3	5	6
Ages	(32, 47)	(48, 51)	(52, 68)	(69, > 69)

Table 7
Values of Measure for Groups

Group, i	1	2	3	
R_i	406.29	135.65	114.79	
Group, i	4	5	6	7
R_i	90.15	100.15	71.78	60.92

5 Conclusion

This paper has shown how the fuzzy c-means algorithm can be used to investigate age groupings in general insurance. The fuzzy approach is well-suited to this problem, but there are other methods that can be used. The obvious candidates are parametric and nonparametric smoothing within the framework of generalized linear models, while treating the policyholder age as a continuous variable instead of as a factor. Thus, it would be possible to use, for example, a polynomial function of age to model the effect of policyholder age on the risk. It also would be possible to apply nonparametric smoothing methods, such as cubic smoothing splines, if a parametric model were not suitable. The problem is similar to that of graduating life tables.

Closer in spirit to our approach would be the use of (crisp) clustering methods such as the minimum variance method (van Eeghen et al., 1983) or the method proposed by Loimaranta et al. (1980). The first of these two methods uses an algorithm that aims to break the data into a number of clusters, so that within-cluster variance is small and between-cluster variance is large. The second method derives the posterior probability that each data point belongs to each cluster, using a Bayesian approach. We believe that the flexibility of the fuzzy approach makes it most suitable for grouping policyholder age.

Figure 1
Comparison of the Crude Risk Premium
And the Risk Premium Based on the Risk Groups

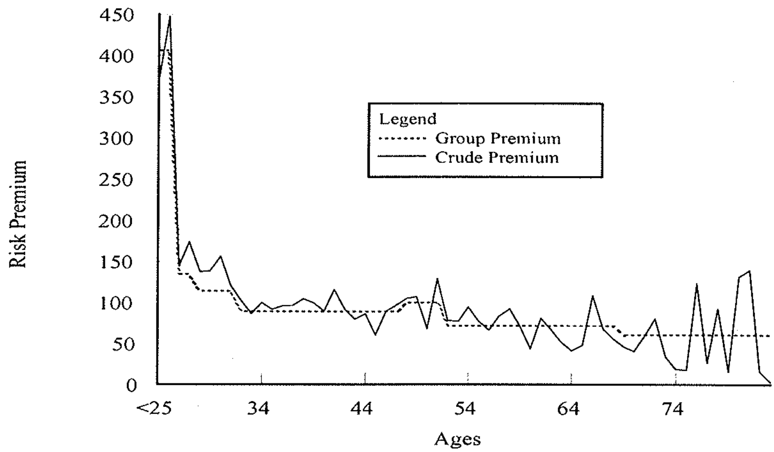
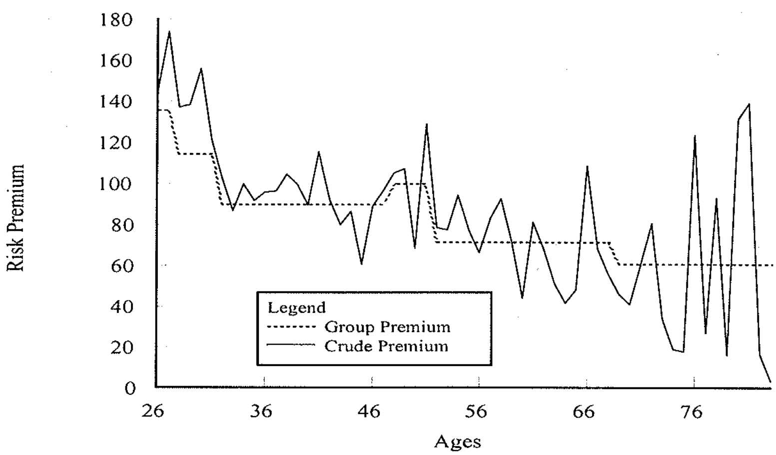


Figure 2
Comparison of the Crude Risk Premium
And the Risk Premium Based on the Risk Groups
(Risk Groups 2 to 6)



The problem of grouping by policyholder age has not been considered previously in the actuarial literature, and we believe that it forms an important part of the underwriting process. This paper has shown how the fuzzy c-means algorithm can be used to assess the groups used in practice, and we recommend that an investigation of this type should be part of any risk rating exercise for a general insurance portfolio.

This paper has emphasized the application of the fuzzy c-means algorithm to grouping by policyholder age, but algorithm also could be applied to other explanatory variables and in other types of insurance. For example, the classification of vehicles into vehicle rating groups, the grouping of car engine sizes, and the classification of excess mortality risk in life insurance according to blood pressure are all problems to which the approach described in this paper could be used.

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Determination of Optimal Premiums as a Constrained Optimization Problem

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Abstract†

A simple stochastic model of an insurer's underwriting and related investment operations is used to determine the optimal amounts of written premiums for one period for the insurer's book of business. The written premium for each class is determined by the solution of a constrained optimization problem. The insurer's objective function is the expected profit on a book of business over the period. The insurer has a safety constraint where a certain portion of capital and surplus can be depleted with a small probability. This paper provides an explicit solution for optimum expected profit and corresponding written premiums by classes. Due to the closed form nature of the solution for expected profit, insights are given as to how the optimum expected profit depends upon the model's parameters.

Key words and phrases: *written premiums, profits, risk, capital, surplus, objective function*

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1 Introduction

In a simple deterministic world with only one class of business, an underwriter would be concerned only with the profitability of this class of business. If the class is perceived as profitable, then the underwriter will commit as much available capacity as possible to writing that class of business. Considering even a single class of business in a real world (i.e., where risk exists), however, one has to balance profitability with risk.

Insurers typically have several classes of business, so insurance management considers business expediency as well as diversification. As these classes may be interdependent, management is interested in a balanced book of business consisting of different classes of business. Thus the management's concern with the overall profitability of a book of business must be balanced against the assumption of an acceptable level of risk. This approach to maximizing the profitability and controlling the amount of assumed risk is consistent with modern investment portfolio theory.

Markowitz (1952) introduced an important portfolio selection criterion in the field of finance. The Markowitz approach seeks to maximize the return on an investment portfolio by requiring the portfolio's standard deviation to be within an acceptable range. In this paper, we use a different portfolio criterion for selecting the optimal amounts of written premiums for an insurer book of business.

We define an insurer as any risk-bearing entity. An insurer may be a primary insurance company, a reinsurer, or a group of insurance companies. The insurer writes various classes of business. What constitutes a class of business depends upon the context of a given situation. For a primary insurer, the class of business may be fire, allied lines, private passenger auto liability, and other annual statement lines. For a reinsurer, the class of business may be property pro-rata, property excess, property cat covers, casualty pro-rata, casualty excess, and other suitable classes. For a group of insurers, the classes of business may be different profit centers.

The time horizon for which the optimal premiums are to be determined is short (for example, one year or less). There are certain reasons for this short time frame. First, for a fixed model it is unlikely that the functional relationship among variables would stay the same over a long time period. Second, due to changes in environment (economic, legal, regulatory, and technological), it would not be possible for a single model to represent the behavior of an insurer's operation over an extended time period.

Brubaker (1979) considered the same constrained optimization problem. His solution for optimal premium levels is based on a numeric iterative procedure that could be (but not easily) extended to situations where there are many classes of business. This paper improves Brubaker's approach by providing an explicit closed form solution to the constrained optimization problem. This solution provides insights into relationships between expected profits and input model variables. Other examples of constrained optimization problems have appeared in Ang and Lai (1987) and Meyers (1991).

For each class of business considered, the losses and expenses have a depleting effect upon the insurer's capital and surplus. Written premium and investment income attributable to a class of business increase the level of capital and surplus. The insurer's capital and surplus serve as safety margins against unexpected adverse underwriting and investment results. In addition, capital and surplus are needed to support future growth. It is assumed that management does not want the total capital and surplus to be depleted by more than a specified amount within a given time period. Losses, expenses, and investment income are considered to be random variables. Written premiums, portion of capital and surplus at risk, and the probability of depletion are viewed as deterministic decision variables to be controlled by management. The stochastic model introduced below considers the underwriting and investment income supporting the insurance operation of an insurer in a simple fashion.

2 The Model, Objective Function, and Constraint

Consider an insurance company with m classes of business. The total random profit, Π, over the single period of interest is

$$\begin{aligned}\Pi &= \sum_{i=1}^{m} (w_i + I_i - L_i - E_i) \\ &= \sum_{i=1}^{m} w_i R_i\end{aligned}$$

where, for the ith class, L_i, I_i, and E_i denote the losses, investment income, and the expenses, respectively, and w_i is the written premium and R_i is the profit per unit of written premium.

Let us define the following vectors:

$$\begin{aligned}\mathbf{r} &= (r_1, r_2, \ldots, r_m)^T \quad \text{where } r_i = E[R_i]\\ \mathbf{R} &= (R_1, R_2, \ldots, R_m)^T\end{aligned}$$

and

$$\mathbf{w} = (w_1, w_2, \ldots, w_m)^T.$$

The total profit for this period can be written in vector form as

$$\Pi = \sum_{i=1}^{m} w_i R_i = \mathbf{w}^T \mathbf{R}.$$

The total expected profit, π, is

$$\pi = E[\mathbf{w}^T \mathbf{R}] = \mathbf{w}^T \mathbf{r} \tag{1}$$

and the variance of the total profit is

$$\text{Var}[\mathbf{w}^T \mathbf{R}] = \mathbf{w}^T \mathbf{V} \mathbf{w} \tag{2}$$

where $\mathbf{V}$ is the variance-covariance matrix for $\mathbf{R}$. Note that $\mathbf{V}$ is an $m \times m$ symmetric matrix:

$$\mathbf{V} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mm} \end{pmatrix}$$

where

$$\begin{aligned}\sigma_{ij} &= \text{Cov}[R_i, R_j] = E[(R_i - r_i)(R_j - r_j)]\\ \sigma_{ii} &= \text{Var}[R_i].\end{aligned}$$

We seek to maximize the expected profit, $\mathbf{w}^T \mathbf{r}$, subject to restrictions on the use of capital and surplus as specified below. Let C denote the insurer's total capital and surplus at the beginning of the period under review.

We assume that during this period, management is not willing to have more than a specific portion, k, of capital and surplus be depleted with a small probability, α. In other words, the probability that the total loss and the total expense arising from all classes of business may exceed the total written premium, the total investment income attributable to its underwriting operation during the period, and a portion of capital and surplus, kC, is at most α. We can write this statement as

$$\Pr[\sum_{i=1}^{m}(L_i + E_i) > \sum_{i=1}^{m}(w_i + I_i) + kC] = \Pr[kC + \mathbf{w}^T\mathbf{R} \leq 0] \leq \alpha. \qquad (3)$$

Assuming that $\mathbf{R}$ has a multivariate normal distribution, equation (3) reduces to

$$\Pr[\mathbf{w}^T\mathbf{R} \leq -kC] = \Pr\left[Z \leq \frac{-\mathbf{w}^T\mathbf{r} - kC}{\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}}}\right] \leq \alpha. \qquad (4)$$

where

$$Z = \frac{\mathbf{w}^T\mathbf{R} - \mathbf{w}^T\mathbf{r}}{\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}}}$$

has a standard normal distribution. If we replace the inequality $(\leq \alpha)$ in equation (4) by equality then we have

$$\Pr\left[Z \leq -\frac{\mathbf{w}^T\mathbf{r} + kC}{\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}}}\right] = \alpha$$

which, by the symmetry of the normal distribution, implies that

$$\Pr\left[Z > \frac{\mathbf{w}^T\mathbf{r} + kC}{\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}}}\right] = \alpha. \qquad (5)$$

It then follows that

$$z_\alpha = \frac{\mathbf{w}^T\mathbf{r} + kC}{\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}}}$$

or

$$z_\alpha\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}} = \mathbf{w}^T\mathbf{r} + kC \qquad (6)$$

where z_α is the $100(1 - \alpha)$ percentile of standard normal random variable, i.e.,

$$\Pr[Z > z_\alpha] = \alpha.$$

Equation (6) defines the safety constraint condition, due to limitations on the use of capital and surplus in our optimization problem. Now we proceed with the solution of the optimization problem.

3 The Optimization Problem

Our optimization problem is the determination of the vector of written premiums so that our objective function, total expected profit, is maximized subject to constraint as specified by equation (6). That is,

$$\text{Maximize} \mathbf{w}^T \mathbf{r}$$

subject to the constraint

$$z_\alpha \sqrt{\mathbf{w}^T \mathbf{V} \mathbf{w}} = \mathbf{w}^T \mathbf{r} + kC.$$

Before proceeding with the derivation of the optimal solution for $\mathbf{w}$, the vector of written premiums, the following steps will facilitate the derivation of our results.

First, we note some special properties of the variance-covariance matrix $\mathbf{V}$. Second, we introduce vectors $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{w}}$ based on linear transformations of the original vectors $\mathbf{r}$ and $\mathbf{w}$. These transformed vectors facilitate solution of the optimization problem and make interpretation of our results easier. Third, we reformulate the original constrained optimization problem in terms of the transformed vectors $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{w}}$. The optimization problem is solved in terms of the transformed vectors. Finally, the solution of the constrained optimization problem is restated in terms of original vectors $\mathbf{r}$ and $\mathbf{w}$.

First, it is well-known that any variance-covariance matrix such as $\mathbf{V}$ is a nonnegative definite matrix; see, for example, Schott (1997, Chapter 1, page 23). Using Cholesky decomposition,[1] of $\mathbf{V}$ there exists an upper-triangular matrix $\mathbf{B}$ such that

[1] The Cholesky decomposition of a matrix is a well known algorithm for expressing a square matrix as a product of a lower-triangular matrix and an upper-triangular matrix; see, for example, Burden and Faires (1997, Chapter 6.6, page 410).

$$\mathbf{V} = \mathbf{B}^T\mathbf{B}. \tag{7}$$

If we assume further that $\mathbf{V}$ is nonsingular, i.e., $\mathbf{V}$ is a positive definite matrix, then $\mathbf{B}$ is also nonsingular.

Second, we introduce $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{w}}$ as follows:

$$\tilde{\mathbf{w}} = \mathbf{B}\mathbf{w} \tag{8}$$

$$\tilde{\mathbf{r}} = (\mathbf{B}^T)^{-1}\mathbf{r}. \tag{9}$$

The objective function in terms of transformed vectors is

$$\mathbf{w}^T\mathbf{r} = (\mathbf{B}^{-1}\tilde{\mathbf{w}})^T(\mathbf{B}^T\tilde{\mathbf{r}}) = \tilde{\mathbf{w}}^T\tilde{\mathbf{r}}.$$

Also, note that

$$\begin{aligned}\mathbf{w}^T\mathbf{V}\mathbf{w} &= (\mathbf{B}^{-1}\tilde{\mathbf{w}})^T\mathbf{B}^T\mathbf{B}(\mathbf{B}^{-1}\tilde{\mathbf{w}})\\ &= \tilde{\mathbf{w}}^T\tilde{\mathbf{w}}.\end{aligned}$$

Equation (6) can now be rewritten as

$$z_\alpha\sqrt{\tilde{\mathbf{w}}^T\tilde{\mathbf{w}}} = \tilde{\mathbf{w}}^T\tilde{\mathbf{r}} + kC. \tag{10}$$

To summarize, the optimization problem in terms of transformed vectors is to maximize $\tilde{\mathbf{w}}^T\tilde{\mathbf{r}}$ subject to the constraint of equation (10).

4 The Solution

The traditional approach to solving this optimization problem is by first defining a Lagrangian function $\mathcal{L}$:

$$\mathcal{L} = \tilde{\mathbf{w}}^T\tilde{\mathbf{r}} - \lambda[z_\alpha\sqrt{\tilde{\mathbf{w}}^T\tilde{\mathbf{w}}} - \tilde{\mathbf{w}}^T\tilde{\mathbf{r}} - kC]. \tag{11}$$

Next we determine the partial derivatives of H with respect to $\tilde{\mathbf{w}}$ and λ.

$$\frac{\partial\mathcal{L}}{\partial\tilde{w}_i} = \tilde{r}_i - \lambda[\frac{z_\alpha\tilde{w}_i}{\sqrt{\tilde{\mathbf{w}}^T\tilde{\mathbf{w}}}} - \tilde{r}_i] \tag{12}$$

and

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(z_\alpha \sqrt{\tilde{\mathbf{w}}^T \tilde{\mathbf{w}}} - \tilde{\mathbf{w}}^T \tilde{\mathbf{r}} - kC). \quad (13)$$

Setting these partial derivatives to zero yields

$$\tilde{r}_i - \lambda[\frac{z_\alpha \tilde{w}_i}{\sqrt{\tilde{\mathbf{w}}^T \tilde{\mathbf{w}}}} - \tilde{r}_i] = 0 \quad (14)$$

and

$$z_\alpha \sqrt{\tilde{\mathbf{w}}^T \tilde{\mathbf{w}}} - \tilde{\mathbf{w}}^T \tilde{\mathbf{r}} - kC = 0,$$

which is simply the constraint equation (10). Equation (14) implies

$$\tilde{w}_i = a\tilde{r}_i \quad (15)$$

where

$$a = \frac{(1+\lambda)\sqrt{\tilde{\mathbf{w}}^T \tilde{\mathbf{w}}}}{\lambda z_\alpha}. \quad (16)$$

Note that the a is a scalar and does not depend upon a particular i, though it does depend on the unknown premiums. Equation (15) also can be written in vector form as

$$\tilde{\mathbf{w}} = a\tilde{\mathbf{r}}. \quad (17)$$

Substituting equation (17) into the constraint equation (10) gives

$$a z_\alpha \sqrt{\tilde{\mathbf{r}}^T \tilde{\mathbf{r}}} - a\tilde{\mathbf{r}}^T \tilde{\mathbf{r}} - kC = 0. \quad (18)$$

Solving equation (18) for a, we have

$$a = \frac{kC}{z_\alpha \sqrt{\tilde{\mathbf{r}}^T \tilde{\mathbf{r}}} - \tilde{\mathbf{r}}^T \tilde{\mathbf{r}}} \quad (19)$$

which yields the vector of the optimum premium as

$$\tilde{\mathbf{w}}^* = a\tilde{\mathbf{r}} = \left[\frac{kC}{z_\alpha \sqrt{\tilde{\mathbf{r}}^T \tilde{\mathbf{r}}} - \tilde{\mathbf{r}}^T \tilde{\mathbf{r}}}\right] \tilde{\mathbf{r}}. \quad (20)$$

The maximum expected profit, $\pi^{\max}$, is

$$\pi^{\max} = \frac{kC}{\frac{z_\alpha}{\sqrt{\tilde{\mathbf{r}}^T\tilde{\mathbf{r}}}} - 1}. \tag{21}$$

We now rewrite our results in terms of the original vectors $\mathbf{r}$ and $\mathbf{w}$ using the equations (7), (8), and (9):

$$a = \frac{kC}{z_\alpha\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}} - \mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}} \tag{22}$$

and the vector of the optimum premium is

$$\mathbf{w}^* = \left[\frac{kC}{z_\alpha\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}} - \mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}}\right]\mathbf{V}^{-1}\mathbf{r} \tag{23}$$

and

$$\begin{aligned}\pi^{\max} &= \left[\frac{kC}{z_\alpha\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}} - \mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}}\right]\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r} \\ &= \frac{kC}{\frac{z_\alpha}{\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}}} - 1}.\end{aligned} \tag{24}$$

Note for the maximum expected profit to be positive, we must ensure that

$$\left(\frac{z_\alpha}{\sqrt{\tilde{\mathbf{r}}^T\tilde{\mathbf{r}}}} - 1\right) = \left(\frac{z_\alpha}{\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}}} - 1\right) > 0. \tag{25}$$

This imposes restrictions on the selection of α and of the portfolio of risks assumed in terms of the vector of expected profits $\mathbf{r}$ and the matrix of variance-covariance $\mathbf{V}$.

Next we must prove that $\pi^{\max}$ is indeed the maximum premium. To this end, we must prove that the Hessian matrix, $\mathbf{H}$, of second order partial derivatives is negative semidefinite. Specifically let

$$g_i = \frac{\partial}{\partial \tilde{w}_i}[z_\alpha\sqrt{\tilde{\mathbf{w}}^T\tilde{\mathbf{w}}} - \tilde{\mathbf{w}}^T\tilde{\mathbf{r}} - kC] \tag{26}$$

$$h_{ij} = \frac{\partial^2 \mathcal{L}}{\partial \tilde{w}_i \partial \tilde{w}_j} \quad \text{for } i, j = 1, 2, \ldots, m, \text{ and} \tag{27}$$

$$\mathbf{H} = \{h_{ij}\} \tag{28}$$

where these derivatives are evaluated at $\tilde{\mathbf{w}}^*$ (given in equation (20)). Following Varian (1992, Chapter 27), to prove that $\pi^{\max}$ is a maximum we must prove that

$$\mathbf{y}^T\mathbf{H}\mathbf{y} \le 0 \tag{29}$$

for any vector $\mathbf{y}^T = (y_1, \dots, y_m)$ satisfying

$$\mathbf{g}^T\mathbf{y} = 0 \tag{30}$$

where $\mathbf{g}^T = (g_1, \dots, g_m)$.

When evaluated at $\tilde{\mathbf{w}}^*$, it is easily seen that

$$g_i = \frac{z_\alpha \tilde{w}_i^*}{\theta^{1/2}} - \tilde{r}_i \tag{31}$$

$$h_{ij} = \begin{cases} \lambda \dfrac{z_\alpha \tilde{w}_i^* \tilde{w}_j^*}{\theta^{3/2}} & \text{if } i \ne j, \text{ and} \\ -\lambda \dfrac{z_\alpha}{\theta^{3/2}} (\theta + (\tilde{w}_i^*)^2) & \text{if } i = j \end{cases} \tag{32}$$

where $\theta = \tilde{\mathbf{w}}^{*T}\tilde{\mathbf{w}}^*$. In matrix terms,

$$\mathbf{g} = \left(\frac{a z_\alpha}{\theta^{1/2}} - 1 \right) \tilde{\mathbf{r}} \tag{33}$$

after using equation (20) and

$$\begin{aligned} \mathbf{H} &= -\lambda \frac{z_\alpha}{\theta^{3/2}} (\theta \mathbf{I} + \tilde{\mathbf{w}}^* \tilde{\mathbf{w}}^{*T}) \\ &= -\lambda \frac{z_\alpha}{\theta^{3/2}} (\theta \mathbf{I} + a^2 \tilde{\mathbf{r}} \tilde{\mathbf{r}}^T) \end{aligned} \tag{34}$$

where $\mathbf{I}$ is the identity matrix. Substituting equation (33) into equation (30) implies that the constraint reduces to

$$\tilde{\mathbf{r}}^T\mathbf{y} = \mathbf{y}^T\tilde{\mathbf{r}} = 0. \tag{35}$$

Hence for $\mathbf{y}$ satisfying equation (35),

$$\begin{aligned} \mathbf{y}^T\mathbf{H}\mathbf{y} &= -\lambda \frac{z_\alpha}{\theta^{3/2}} (\theta \mathbf{y}^T \mathbf{I} \mathbf{y} + a^2 \mathbf{y}^T \tilde{\mathbf{r}} \tilde{\mathbf{r}}^T \mathbf{y}) \\ &= -\lambda \frac{z_\alpha}{\theta^{1/2}} \mathbf{y}^T \mathbf{y}. \end{aligned} \tag{36}$$

Thus **H** is negative semidefinite if and only if $\lambda \geq 0$ (assuming, of course, that $\alpha > 0.5$, i.e., $z_\alpha > 0$).

To prove that $\lambda \geq 0$ when evaluated at $\mathbf{w}^*$, we substitute equation (20) into equation (16) to give

$$a = \frac{(1+\lambda)\sqrt{a^2\tilde{\mathbf{r}}^T\tilde{\mathbf{r}}}}{\lambda z_\alpha}$$

which reduces to

$$1 = \frac{(1+\lambda)\sqrt{\tilde{\mathbf{r}}^T\tilde{\mathbf{r}}}}{\lambda z_\alpha}$$

which implies

$$\lambda = \frac{1}{\dfrac{z_\alpha}{\sqrt{\tilde{\mathbf{r}}^T\tilde{\mathbf{r}}}} - 1}. \tag{37}$$

But under the requirement that equation (25) holds, $\lambda \geq 0$, and the proof is completed.[2]

Let us summarize our results with regard to the optimum premium vector and maximum expected profit as a proposition.

Proposition 1. *Based on the following three premises:*

1. *The vector of profits per unit of written premiums,* **R**, *has a multivariate normal distribution with mean vector* **r** *and variance-covariance matrix* **V**;
2. *The insurer's constraint for the time period under consideration is the probability that the insurer's total losses exceed kC dollars of its capital and surplus is at most α; and*
3. **r**, **V**, *and α are such that*

$$\left(\frac{z_\alpha}{\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}}} - 1\right) > 0.$$

The vector of optimum written premium ($\mathbf{w}^*$) *is given by*

$$\mathbf{w}^* = \left[\frac{kC}{z_\alpha\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}} - \mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}}\right]\mathbf{V}^{-1}\mathbf{r}$$

[2]The author thanks the editor for proving that $\pi^{\max}$ is maximum.

and the maximum expected profit, $\pi^{\max}$, for the book of business is

$$\pi^{\max} = \frac{kC}{\frac{z_\alpha}{\sqrt{\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r}}} - 1}.$$

Some implications of Proposition 1 are:

- As the portion of capital and surplus exposed to the risk of depletion, k, increases, then expected profit increases; and
- As α increases, then the expected profit increases.

Finally, expressions for the vector of written premiums and expected profit can be simplified in the case of uncorrelated classes of business because the variance-covariance matrix becomes a diagonal matrix with diagonal elements corresponding to the variances of the respective classes.

We note that

$$\begin{aligned}\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r} &= \sum_{i=1}^{m} \frac{r_i^2}{\sigma_{ii}} \\ &= \sum_{i=1}^{m} \frac{1}{\mathrm{CV}_i^2}\end{aligned}$$

where CV_i is the coefficient of variation for the ith class.

Thus, in the case of uncorrelated classes of business, the class with the smallest coefficient of variation contributes most to total expected profit. [See equation (24).] Furthermore,

$$\mathbf{V}^{-1}\mathbf{r} = \begin{pmatrix} \frac{r_1}{\sigma_{11}} \\ \vdots \\ \frac{r_m}{\sigma_{mm}} \end{pmatrix}.$$

It can be noted from equation (23) that the optimum amount of written premium, w_i, is inversely proportional to the variances σ_{ii}, in the case of uncorrelated classes of business.

5 Numerical Examples

5.1 Example 1

Let us consider the example stated as Case 7 by Brubaker (1979). Brubaker provides an iterative solution for optimal premiums. The setting is as follows:

- $k = 0.5$, $C = 300$ million, $\alpha = 0.001$, $z_\alpha = 3.1$;
- Three classes of business ($m = 3$);
- Expected profit per unit of premium: $r_i = 0.05$, for $i = 1, 2, 3$;
- Variance of each class: $\sigma_{ii} = (0.075)^2$, for $i = 1, 2, 3$;
- Correlation coefficients among classes: $\rho_{12} = -0.5$ and $\rho_{13} = \rho_{23} = 0$.

Using our notation,

$$\mathbf{r} = \begin{pmatrix} 0.05 \\ 0.05 \\ 0.05 \end{pmatrix}$$

and

$$\mathbf{V} = (0.075)^2 \begin{pmatrix} 1.0 & -0.5 & 0.0 \\ -0.5 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{pmatrix}.$$

As

$$\mathbf{V}^{-1} = \begin{pmatrix} 237.037 & 118.519 & 0.0 \\ 118.519 & 237.037 & 0.0 \\ 0.0 & 0.0 & 177.778 \end{pmatrix}$$

and $\mathbf{r}^T \mathbf{V}^{-1} \mathbf{r} = 2.222$, the optimum vector of premium is

$$\mathbf{w}^* = \begin{pmatrix} 1,111.6 \\ 1,111.6 \\ 555.8 \end{pmatrix}$$

and the maximum expected profit is $\pi^{\max} = 138.9$.

Note the upper-triangular matrix B is

$$\mathbf{B} = \begin{pmatrix} 0.07500 & -0.03750 & 0 \\ 0 & 0.06495 & 0 \\ 0 & 0 & 0.07500 \end{pmatrix}$$

and

$$\tilde{\mathbf{r}} = \mathbf{B}^{-1}\mathbf{r} = \begin{pmatrix} 0.667 \\ 1.155 \\ 0.667 \end{pmatrix}.$$

As the expected profit per unit of premium and the standard deviation for each class are the same, it is tempting to infer that the three classes contribute equally to total profitability. Considering the correlation structure among the three classes, $\mathbf{V}$, and replacing $\mathbf{r}$ by $\tilde{\mathbf{r}}$ that summarizes the information about $(\mathbf{r}, \mathbf{V})$, we note that the components of the vector $\tilde{\mathbf{r}}$ are not all equal. Thus, the three classes impact the insurer's total profitability in unequal ways.

Matrix operations and the Cholesky decompositions can be done using readily available software such as SAS, S-Plus, Maple, or Mathematica.[3]

5.2 Example 2

Let us consider an example of an insurer with four classes of business ($m = 4$) and with $C = 300$ million. The insurer faces the following expected profit per unit of premium vector and variance-covariance matrix:

$$\mathbf{r} = \begin{pmatrix} 0.05 \\ 0.06 \\ 0.07 \\ 0.08 \end{pmatrix}$$

[3]SAS is a registered trademark of: SAS Institute Inc., Cary, NC 27512-8000, USA; S-Plus is a registered trademark of: MathSoft, 101 Main Street, Cambridge, MA 02142, USA; MAPLE is a registered trademark of: Waterloo Maple Software, 450 Phillip Street, Waterloo ON N2L 5J2, CANADA; and Mathematica is a registered trademark of: Wolfram Research, Inc., 100 Trade Center Drive, Champaign IL 61820-7237, USA.

and

$$\mathbf{V} = (0.075)^2 \begin{pmatrix} 1.0 & -0.4 & -0.5 & -0.6 \\ -0.4 & 2.0 & -0.5 & 0.3 \\ -0.5 & -0.5 & 3.0 & 0.1 \\ -0.6 & 0.3 & 0.1, & 4.0 \end{pmatrix}$$

We will calculate the optimum premium and profits for various levels of α and k.

The following are easily derived:

$$\mathbf{V}^{-1} = \begin{pmatrix} 243.359 & 56.237 & 48.897 & 31.064 \\ 56.234 & 106.979 & 27.212 & -0.268 \\ 48.8971 & 27.212 & 71.827 & 3.498 \\ 31.064 & -0.268 & 3.498 & 49.037 \end{pmatrix}$$

$$\mathbf{B}^T = \begin{pmatrix} 0.075 & 0 & 0 & 0 \\ -0.03 & 0.1017 & 0 & 0 \\ -0.0375 & -0.0387 & 0.1182 & 0 \\ -0.045 & 0.0033 & -0.0084 & 0.1428 \end{pmatrix}$$

and

$$\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r} = 2.853.$$

Tables 1 through 5 display the results of our calculations.

Table 1
Optimum Premiums and Profits: Case k = 1.00

		Optimum Premiums $\mathbf{w}^*$				
α	z_α	w_1^*	w_2^*	w_3^*	w_4^*	Profit
0.001	3.090	2719.45	1409.03	1189.88	723.27	361.67
0.005	2.576	4295.32	2225.54	1879.38	1142.39	571.25
0.010	2.326	5981.06	3098.97	2616.97	1590.73	795.44
0.025	1.960	14058.64	7284.22	6151.26	3739.05	1869.70

Table 2
Optimum Premiums and Profits: Case $k = 0.75$

α	z_α	Optimum Premiums $\mathbf{w}^*$				Profit
		w_1^*	w_2^*	w_3^*	w_4^*	
0.001	3.090	2039.59	1056.78	892.41	542.45	271.25
0.005	2.576	3221.49	1669.15	1409.54	856.79	428.43
0.010	2.326	4485.80	2324.23	1962.73	1193.05	596.58
0.025	1.960	10543.98	5463.16	4613.44	2804.29	1402.27

Table 3
Optimum Premiums and Profits: Case $k = 0.50$

α	z_α	Optimum Premiums $\mathbf{w}^*$				Profit
		w_1^*	w_2^*	w_3^*	w_4^*	
0.001	3.090	1359.73	704.52	594.94	361.63	180.84
0.005	2.576	2147.66	1112.77	939.69	571.19	285.62
0.010	2.326	2990.53	1549.49	1308.49	795.37	397.72
0.025	1.960	7029.32	3642.11	3075.63	1869.53	934.85

In our model, the maximum expected profit and the vector of optimum premiums are functions of k, C, α (or z_α), $\mathbf{r}$, and $\mathbf{V}$. In Tables 1 to 5, C, $\mathbf{r}$, and $\mathbf{V}$ are held constant while k and α are varied. We shall now explain the effect of changes in k and α on the maximum expected profit.

First, keeping α constant, as k increases we are exposing a larger portion of our capital and surplus to a fixed level of risk, α, in order to support our insurance operation. Thus we see higher profit levels as shown in Tables 1 to 5. This is consistent with the idea that greater risk bearing should be compensated by higher profit (return) levels.

Similarly, keeping k constant, as α increases we are exposing a fixed portion of our capital and surplus (k) to a larger level of risk, α, in order to support our insurance operation. Thus we see higher profit levels as shown in Tables 1 to 5. Again, this is consistent with the idea that greater risk bearing should be compensated by higher profit (return) levels. To illustrate this in the extreme case: as $\alpha \to 0$ (i.e., the level of risk decrease to zero), $z_\alpha \to \infty$ and both the optimum premium vector (equation (23)) and the maximum expected profit (equation (24))

Table 4
Optimum Premiums and Profits: Case $k = 0.25$

		Optimum Premiums $\mathbf{w}^*$				
α	z_α	w_1^*	w_2^*	w_3^*	w_4^*	Profit
0.001	3.090	679.86	352.26	297.47	180.82	90.42
0.005	2.576	1073.83	556.38	469.85	285.60	142.81
0.010	2.326	1495.27	774.74	654.24	397.68	198.86
0.025	1.960	3514.66	1821.05	1537.81	934.76	467.42

Table 5
Optimum Premiums and Profits: Case $k = 0.10$

		Optimum Premiums $\mathbf{w}^*$				
α	z_α	w_1^*	w_2^*	w_3^*	w_4^*	Profit
0.001	3.090	271.95	140.90	118.99	72.33	36.17
0.005	2.576	429.53	222.55	187.94	114.24	57.12
0.010	2.326	598.11	309.90	261.70	159.07	79.54
0.025	1.960	1405.86	728.42	615.13,	373.91	186.97

decrease to zero. In other words, the best way to avoid all risk is to write no business altogether.

Note that these observations are easily derived mathematically because, in equation (24), $\mathbf{r}^T\mathbf{V}^{-1}\mathbf{r} = 2.853$, so $\pi^{\max}$ increases as k or α increases. Recall that as α increases, z_α decreases.

6 Conclusion

We have introduced a simple model to represent underwriting and related investment income for a class of business of an insurer. An important decision for the management of insurance companies is the determination of written premiums for respective classes of business. A rational solution to this problem requires balancing profitability against risk to the insurer. This paper explains risk-taking in terms of the extent to which management is willing to lose a portion of its capital and surplus during a short time horizon.

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Credibility Calculations Using Analysis of Variance Computer Routines

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Abstract

In this paper we present a method of calculating Bühlmann-Straub credibility factors using standard statistical techniques developed for the analysis of variance. Emphasis is placed on using readily available statistical packages such as SAS and SPSS. Additionally many other computational tools such as EXCEL can be programmed to make such calculations. An example and some sample SAS programs are provided.

Key words and phrases: *Bühlmann-Straub credibility factors, empirical Bayes, borrowing strength, random ANOVA model*

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1 Introduction

Casualty actuaries long have recognized the use of the methods of credibility theory as important in assisting them when setting premiums for (i) renewing business, (ii) blocks of new business, and (iii) determining experience-based refunds. The value of these methods also is gaining recognition among health actuaries.[1] Implementation of these credibility methods, however, is varied. Although formal methods of calculating credibility rates are well established, their implementation varies mathematically from ad hoc computations to simple approximations to detailed estimation of the model parameters. One of the reasons for this is the differences in computational complexity. Despite the fact that company experience is maintained in well-documented databases, use of computer programs on these databases to form credibility estimates is far from seamless and may be too complex to warrant the effort.

We present a method of calculating credibility factors under the Bühlmann-Straub (1970) model using readily available statistical software.[2] The Bühlmann-Straub model is one of a variety of credibility models and is based on a least squares argument. Though the least squares basis for credibility is adequate justification for the procedure, it has been shown that the Bühlmann-Straub method of calculating credibility is identical to the empirical Bayes method when the distribution of losses is a member of the linear exponential family, the loss is quadratic, and when the Bayesian prior used is the conjugate prior for this distribution (Ericson, 1970). Although software programs do not explicitly identify the credibility factors in the software documentation and are not part of the traditional statistical reports generated by these packages, Bühlmann-Straub credibility factors can be calculated from such packages with minimal effort. This paper illustrates these procedures.

A credibility premium uses data from two sources: the estimate of the pure premium based only on the data from a specific group of interest at a specific time and an estimate of the pure premium based on the other data sources and/or prior information. This second estimate may be the overall average of observed rates taken from samples of other groups of policies or the historical average of the group of policies of interest.

[1] There is an extensive literature on credibility in general (see, e.g., Longley-Cook, 1962; Norberg, 1979; Hossack et al., 1983; Herzog, 1996; Goulet, 1998).

[2] For other papers on the Bühlmann-Straub model see, for example, Morris and Slyke, (1978), and Venter (1985, 1990), and Klugman (1987).

The credibility premium classically takes the form

$$C = ZR + (1 - Z)H, \qquad 0 \leq Z \leq 1, \tag{1}$$

where C is the credibility premium; R is the estimate of pure premium using the data from the group of interest; H is a global premium (i.e., an exogenous estimate or assumed value of the average of observations); and Z is the credibility factor and denotes the weight assigned to R. If $Z = 1$ then the data are said to be fully credible, and no compromise estimate is needed.

Although the simple form given in equation (1) is found in most of the literature, there are many different approaches to calculate the credibility factor.[3] Bühlmann (1967) arrives at a credibility premium by finding the linear estimator that minimizes the expected squared error. The resulting credibility premium follows the form of the model shown in equation (1), with the credibility factor, Z, given as

$$Z = \frac{n \times \text{VHM}}{n \times \text{VHM} + \text{EPV}}. \tag{2}$$

where EPV is the expected value of the process variance and refers to the value of the variance of the pure premium within each group, averaged across all groups; and VHM is the variance of the hypothetical means, which is the mean square distance between the mean of the pure premium in each group and the mean over all groups. Bühlmann (1967) proposes this estimate of credibility for cases when the n_i are equal. The extension to the case where the n_i are not equal is presented by Bühlmann-Straub (1970).

2 The Analysis of Variance (ANOVA) Approach

The connection between credibility methods and analysis of variance (ANOVA)[4] has been alluded to in several papers. For example, both Venter (1990) and Morris and Van Slyke (1978) describe a model similar

[3] Morris and Van Slyke (1978) determine Z using a Bayesian framework to obtain a form of equation (1). Bühlmann (1970) suggests an alternative method that is also related to the empirical Bayes approach. Herzog (1996), Philbrick (1981), and Venter (1990) also describe this method.

[4] Analysis of variance is a standard statistical technique in the design and analysis of experiments. For more on analysis of variance, see, for example, Scheffé (1959) and Neter, Wasserman, and Craig (1990, Part 3.)

to the random one-way analysis of variance model. Dannenburg (1995) uses a one-way random effects model in a cross-classification credibility model that determines the credibility score using estimated variance components. Dannenburg et al. (1996) use the general variance components models of which this is a special case. (See also Goulet, 1998.)

Analysis of variance can be put into the context of the insurance model as follows: Consider an insurance company with I groups of policies. Suppose further that there are n_i individuals from group i who have a claim within a single period (a month, quarter, or year, say). For $i = 1, 2, \ldots, I$, the claim amount, Y_{iu}, associated with individual u in group i, is modeled as

$$Y_{iu} = \mu + \alpha_i + e_{iu}, \quad u = 1, ..., n_i, \tag{3}$$

where μ represents the mean over all groups and α_i represents the amount that the mean of the ith group varies from this overall mean, α_is are mutually independent random variables mean zero and variance σ_1^2, and the e_{iu}s are mutually independent random variables mean zero and variance σ_0^2. We also assume that α_i and e_{iu} are mutually independent.

If an assumption of normality of the distribution of α_i and e_{iu} were added to equation (3), this would be the standard formulation of the random one-way ANOVA model. This assumption is unnecessary to form the Bühlmann-Straub credibility premium.

Equation (3) implies that the unconditional expected value of Y_{iu} is μ. Conditional on α_i, however, the expected value of Y_{iu} is $\mu + \alpha_i$. It is the past experience that provides the basis for improving our estimate of the expected value of Y_{iu}, for each group by providing information regarding α.

In the ANOVA model of equation (3), the credibility factor is easy to estimate if we use the method of moments estimate of the variance components as suggested by Venter (1990). The method of moments estimate of σ_1^2 is referred to in the European literature as $\hat{a}$. Other than simplicity and unbiasedness, this method of estimation has no known optimality properties. Other estimates of σ_1^2 exist with optimality properties, however (see Goulet, 1998; and DeVylder and Goovaerts, 1992). We will use the simple method of moments estimator.

The following notation is used:

$$N = \sum_{i=1}^{t} n_i; \tag{4}$$

$$\begin{aligned}\bar{Y}_{i.} &= \text{Average of all observations in group } i;\\ &= \frac{\sum_{u=1}^{n_i} Y_{iu}}{n_i};\end{aligned} \tag{5}$$

$$\begin{aligned}\bar{Y}_{..} &= \text{Average of all observations, across all groups;}\\ &= \frac{1}{N}\sum_{i=1}^{t}\sum_{u=1}^{n_i} Y_{iu};\end{aligned} \tag{6}$$

$$s_i^2 = \frac{1}{n_i - 1}\sum_{u=1}^{n_i}(Y_{iu} - \bar{Y}_{i.})^2 \tag{7}$$

$$\text{MSE} = \frac{1}{N-t}\sum_{i=1}^{t}(n_i - 1)s_i^2, \tag{8}$$

$$\text{MS}(\alpha) = \frac{1}{t-1}\sum_{i=1}^{t} n_i(\bar{Y}_{i.} - \bar{Y}_{..})^2. \tag{9}$$

The last two expressions are referred to as the mean square for error (MSE) and the mean square for groups (MS(α)), respectively. The expected values of these mean squares are:[5]

$$E[\text{MSE}] = \sigma_0^2$$

and

$$E[\text{MS}(\alpha)] = \sigma_0^2 + n_0\sigma_1^2,$$

where

$$n_0 = \frac{N^2}{t-1}\left(1 - \sum_{i=1}^{t}\frac{n_i^2}{N^2}\right). \tag{10}$$

In Bühlmann' notation, σ_0^2 is the expected value of the process variance and σ_1^2 is the variance of the hypothetical means. Thus, Bühlmann's k is given as

[5]For a derivation of $E[\text{MSE}]$ and $E[\text{MS}(\alpha)]$ see Scheffé (1959, Chapter 3) or Neter, Wasserman, and Craig (1990, Chapters 14, pages 543–546).

$$k = \frac{n_0 \times \text{MSE}}{\text{MS}(\alpha) - \text{MSE}}.$$

From these expectations we can calculate the following method of moments estimators for the variance components:

$$\hat{\sigma}_0^2 = \text{MSE},$$

and

$$\hat{\sigma}_1^2 = \frac{\text{MS}(\alpha) - \text{MSE}}{n_0}. \tag{11}$$

Thus, for the simple one-way model in equation (3), the Bühlmann-Straub credibility factor, Z, given in equation (2) becomes

$$\begin{aligned} Z_i &= \frac{n_i}{n_i + k}, \\ &= \frac{n_i}{n_i + \hat{\sigma}_0^2/\hat{\sigma}_1^2} \\ &= \frac{n_i\hat{\sigma}_1^2}{n_i\hat{\sigma}_1^2 + \hat{\sigma}_0^2}, \end{aligned} \tag{12}$$

which can be rewritten as

$$Z_i = \frac{\text{MS}(\alpha) - \text{MSE}}{\text{MS}(\alpha) + (\frac{n_0}{n_i} - 1) \times \text{MSE}}. \tag{13}$$

Most analysis of variance routines calculate MSE and MS(α). Only the number of observations in the ith group, n_i , and the value of n_0 need to be determined.

The credibility factor is different for each group depending on the value of n_i. As n_i increases, Z_i goes to unity and the group becomes fully credible. On the other hand, as σ_1^2 increases, indicating a high degree of variability from group to group, Z_i approaches unity and the group becomes fully credible. When σ_1^2 is small relative to σ_0^2 and/or n_i is small relative to n_0, Z_i drops below unity and the group experience is less credible. In this case the compromise estimate borrows more strength from the experience of other groups.

Equation (13) provides a simple calculation of the credibility factor using output from ANOVA routines. Many times, however, the data have been summarized so that for each group i only the observed pure premium, say $\bar{Y}_{i.}$, the number insured, n_i, and the standard deviation, s_i, are known. In this case the formulas can be used by first observing that

$$\bar{Y}_{..} = \sum_{i=1}^{t} \frac{n_i}{N} \bar{Y}_{i.} \,. \tag{14}$$

Thus, $\mathrm{MS}(\alpha)$ is calculated as given in equation (9) using $\bar{Y}_{..}$ as given in equation (14). Rearranging the terms in equation (9) yields a formula that is often easier to use. Explicitly,

$$\mathrm{MS}(\alpha) = \frac{1}{t-1}\left(\sum_{i=1}^{t} n_i \bar{Y}_{i.}^2 - N\bar{Y}_{..}^2\right) . \tag{15}$$

Second, MSE is calculated as in equation (8).

The credibility factors Z_i can be calculated using equation (13) where the MSE is given by equation (8) and $\mathrm{MS}(\alpha)$ is calculated using equation (15) with $\bar{Y}_{..}$ as defined in equation (14).

3 Calculation of Z via Computer Programs

3.1 Individual Data Case

To illustrate the formulas and computer programs we consider the hypothetical data given in Table 1. The data sets are small and would not be seriously considered as reliable insurance experience. With such small data sets, however, the details of calculations are more apparent. The data in Table 1 represent four hypothetical groups with claims for each group. We wish to determine the credibility factors for each group assuming that the four groups represent the entire experience of interest for the insurer.

Table 2 gives the EXCEL[6] output for a one-way analysis of variance of the data in Table 1. To obtain this analysis we perform the following steps:

[6]EXCEL is a registered trademark of: Microsoft Corporation, One Microsoft Way, Redmond WA 98052-6399, USA.

Table 1
Hypothetical Individual Cost Data
For Four Groups of Insureds for a Single Year

Groups			
1	2	3	4
1550	1879	1440	1014
1325	2028	1601	1231
1417	2150	1790	1487
1824	2245	1852	1491
2138	2516	1998	
	2918	2081	
		2171	

Step 1: Click the *Data Analysis* menu selection under *Tools*;

Step 2: We then click *One-Way*;

Step 3: As each column represents a different group, we indicate the *Grouped by Columns* option and then proceed.

The output consists of one table (Table 2) with two panels, Panel A and Panel B. The first column in Panel A lists the group name. The second gives the value of n_i for group i, where i indicates the column of the group data. The fourth column gives $\bar{Y}_{i.}$ for group i as given by equation (5). The fourth column of Panel B lists the MS(α) in the first row and the MSE in the second row.

Using the second column of Table 2, Panel A we calculate n_0 using equation (10). For this equation $t-1 = 4-1 = 3$. The other components of the equation are given as:

$$N = 22$$

$$\sum n_i^2 = 126, \quad \text{and}$$

$$n_0 = (22^2 - 126)/(22 * 3) = 5.4242.$$

Table 2
Output from Excel Program of the
One-Way ANOVA Analysis of the Data in Table 1

Panel A: ANOVA Single Factor (Summary)

Groups	Count (n_i)	Sum	Average ($\bar{Y}_{i.}$)	Variance
Group 1	5	8254	1650.800	109582.70
Group 2	6	13736	2289.333	140929.50
Group 3	7	12933	1847.571	68661.60
Group 4	4	5223	1305.750	52624.92

Panel B: ANOVA

Source of Variation	SS	df	MSE	F-Value	P-Value	F-Crit
Between Groups	2527409	3	842469.6	8.853487	0.000805	3.159911
Within Groups	1712823	18	95156.81			
Total	4240231	21				

Notes: SS = Sum of Squares; *MSE(α) = Between Groups MSE; F-value = Test statistic to test whether mean costs are the same across groups under the ANOVA assumptions; P-value = Probability of a value greater than or equal to the F-value assuming the means are the same; F-Crit = The value which, if it is exceeded by the F-value, there is statistical evidence that the mean costs differ from between groups.

Using these values we calculate the Z_i for each group using equation (13). Explicitly, for group 1 we have

$$Z_1 = \frac{842469.6 - 95156.81}{842469.6 + (\frac{5.4242}{5} - 1) \times 95156.81}$$
$$= 0.878631$$

Thus, the credibility score for group 1 is about 87.9 percent. Relative to the complete set of data available, the data on group 1 are relatively credible—there is little difference between the compromise estimate of the group pure premium and the estimate using the observed average of the group.

3.2 Grouped Data Case

Suppose that only the summary data consisting of n_i, $\bar{Y}_{i.}$, and s_i^2 for each group are available (columns (2), (4), and (5) of Table 2, Panel A). In this case we can use equations (15) and (8) to calculate the components of equation (13). Explicitly we make the following calculations. First from equation (14) we have

$$\bar{Y}_{..} = (5 \times 1650.8 + 6 \times 2289.333 + 7 \times 1847.571 + 4 \times 1305.75)/22$$
$$= \frac{40146}{22}$$
$$= 1824.818182.$$

Using these in equation (15) we obtain

$$MS(\alpha) = \frac{75786559.41 - 73259150.73}{3}$$
$$= \frac{2527408.68}{3}$$
$$= 842469.56$$

This is close to the value given in Table 2, Panel B (row (1), column (4)). The difference is due to roundoff error.

Calculation of MSE follows similarly using equation (8). Explicitly, we get

$$\text{MSE} = \frac{1712822.66}{18}$$
$$= 95156.81$$

These results can be used to calculate the credibility scores as before.

Computer code for the same calculations using SAS are given in the appendix; no code is provided for SPSS.[7]

4 Discussion

We have illustrated how the Bühlmann-Straub credibility factors can be calculated using one-way ANOVA statistical routines common in many computer programs. In order to form such scores the mean squares reported in the ANOVA tables must be used as given in equation (13). Under certain situations estimated MS(α) can be negative. In this case the value of $Z_i = 0$ is used. This reduces the bias of the compromise estimate as shown by Morris (1983).

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[7]SAS is a registered trademark of: SAS Institute Inc., Cary, NC 27512-8000, USA; and SPSS is a registered trademark of: SPSS Inc., 444 North Michigan Avenue, Chicago IL 60611, USA.

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Appendix

The codes for making credibility calculations using SAS for the data in Table 1 are given below. First we use the individual data. We have used the *cards* option. In practice one would read a data file. Below we give the code for grouped data. In both cases the amount of work to get the SAS code seems long relative to the simple problem considered. For longer, more practical problems, however, the benefits of SAS routines are more apparent.

```
DATA costs;
   INFILE cards;
   INPUT cost group;
   CARDS;
1550 1
1325 1
1417 1
1824 1
2138 1
1879 2
2028 2
2150 2
2245 2
2516 2
2918 2
1440 3
1601 3
1790 3
1852 3
1998 3
2081 3
2171 3
```

```
1014 4
1231 4
1487 4
1491 4
;
RUN;

/*** Getting number of individuals per group ***/
PROC SQL;
   CREATE TABLE counts AS
   SELECT DISTINCT group,count(group) AS number
   FROM costs
   GROUP BY group;

/*** Calculating n_not ***/
PROC SQL;
   SELECT (sum(number)-(sum(number**2)/sum(number)))
          /(count(number)-1)
   INTO :n_not
   FROM counts;

/*** Calculating MSE, MSA ***/
PROC ANOVA DATA=costs OUTSTAT=results NOPRINT;
   CLASS group;
   MODEL cost=group;
RUN;

DATA _null_;
   SET results;
   mean_sqr=ss/df;
   SELECT (_source_);
      WHEN ("ERROR") CALL SYMPUT("MSE",mean_sqr);
      WHEN ("GROUP") CALL SYMPUT("MSA",mean_sqr);
   END;
RUN;

/*** Calculating credibilities ***/
DATA creds;
   SET counts;
   cred=(&MSA-&MSE)/(&MSA+(&n_not/number-1)*&MSE);
   KEEP group cred;
RUN;
```

```
PROC PRINT NOOBS DATA=creds;
   TITLE 'Credibility Factors for Individual Data';
RUN;

/******************
 USING GROUPED DATA
******************/

DATA grouped;
   INFILE cards;
   INPUT group number avg_cost var_cost;
   CARDS;
1 58 1666 49597893
2 115 5051 216276545
3 81 4670 193990984
4 108 8966 757144094
;
RUN;

/*** Calculating n_not and the overall mean ***/
PROC SQL;
   SELECT (sum(number)-(sum(number**2)/sum(number)))
          /(count(number)-1),
          sum(avg_cost*number)/sum(number)
   INTO :n_not,:y_bar2
   FROM grouped;

/*** Calculating MSE, MSA ***/
PROC SQL;
   SELECT 1/(count(group)-1)*(sum(number*avg_cost**2)
          -sum(number)*&y_bar2**2),
          1/(sum(number)-count(group))*sum((number-1)
          *var_cost)
   INTO :msa,:mse
   FROM grouped;

/*** Calculating credibilities ***/
DATA creds;
   SET grouped;
   cred=(&msa-&mse)/(&msa+(&n_not/number-1)*&mse);
   KEEP group cred;
```

time period. Recent examples of collective risk modeling in insurance include Butler, Gardner, and Gardner (1998); Butler and Worall (1991); and Cummins and Tennyson (1996).

The stochastic structure is two-pronged: both the size of the individual claims and the number of claims are considered random variables. Specifically, let S denote the aggregate claims random variable, i.e.,

$$S = \sum_{i=1}^{N} X_i \tag{1}$$

where N is the number of claims and X_i is the size of the ith individual claim. The X_is are assumed to be mutually independent and identically distributed (i.i.d.) and are mutually independent of N. In the literature equation (1) is referred to as a compound random variable; see, for example, Bowers et al. (1997, Chapter 12).

Theoretically, the distribution of S can be obtained from equation (1) as follows:

$$\Pr[S \leq s] = \sum_{n=0}^{\infty} p_n F^{*n}(s) \tag{2}$$

where $p_n = \Pr[N = n]$ and $F^{*n}(s) = \Pr[X_1 + \ldots + X_n \leq s]$, i.e., $F^{*n}(s)$ is the nth convolution of the X_is, with $F(x) = F^{*1}(x)$ being the cumulative distribution function of X_1.

The difficulty in using equation (2), however, often lies in calculating $F^{*n}(s)$. Thus, approximations are frequently used. There are several approximations used by actuaries, including discretizing the claim size distribution (Panjer 1981); using the Wilson-Hilferty approximation or Haldane Type A approximation (Pentikäinen, 1987); and, of course, the normal approximation. See Panjer and Willmot (1992, Chapter 6) and Bowers et al. (1997, Chapters 2 and 12) for a discussion of the actuarial approaches. Other methods such as the Edgeworth expansion (Feller, 1971) or the conjugate density method (Esscher, 1932) have been applied.

The methods mentioned above provide good approximations near the center of the distribution but can be slow or inaccurate for calculating tail probabilities of the form $\Pr[S > s]$ (for large values of s). For a discussion of the tail behavior of aggregate distributions; see Panjer and Willmot (1992, Chapter 10). A fairly quick and accurate method of calculating tail probabilities is the so-called saddlepoint approximation.

Since their introduction by Daniels (1954) saddlepoint approximations have been utilized to approximate tail probabilities in a variety of situations; see, for example, Goutis and Casella (1999), Huzurbazar (1999), Butler and Sutton (1998), Tsuchiya and Konishi (1997), and Wood, Booth, and Butler (1993). Field and Ronchetti (1990) document the accuracy of these procedures for small sample sizes (even of sample size one). In this paper a saddlepoint approximation is developed for $\Pr[S > s]$ and is applied to specific examples.

2 The Saddlepoint Approximation

The key assumption in the saddlepoint approximation is the assumption of the existence of the moment-generating functions corresponding to X_i and N, which are denoted by $M_X(\theta)$ and $M_N(\theta)$, respectively, where θ is a real valued parameter.[1] The moment-generating function of S, $M_S(\theta)$, is then given by

$$\begin{aligned} M_S(\theta) &= E[e^{\theta S}] \\ &= E[E[e^{\theta S}|N]] \\ &= M_N(log(M_X(\theta))). \end{aligned} \tag{3}$$

Equation (3) can be used to derive the well-known results on the moments of compound sums of i.i.d. random variables:

$$\mu_S = E[S] = E[N]E[X_1] \tag{4}$$

$$\sigma_S^2 = \text{Var}[S] = \text{Var}[N](E[X_1])^2 + E[N]\text{Var}[X_1]. \tag{5}$$

The saddlepoint approximation for the tail probability $\Pr[S > s]$ is adapted from Field and Ronchetti (1990) for sample size one. First let T denote the standardized random variable

$$T = \frac{S - \mu_S}{\sigma_S}$$

[1] The moment-generating function of a random variable Z is defined as

$$M_Z(\theta) = E[e^{\theta Z}], \quad \theta > 0.$$

where μ_S and σ_S and the mean and standard deviation of S respectively (which can be obtained from equations (4) and (5)). The moment-generating function for T is easily seen to be:

$$M_T(\theta) = e^{-\mu_S/\sigma_S} M_S(\theta/\sigma_S). \tag{6}$$

For a fixed value of s, let $t = (s - \mu_S)/\sigma_S$ and let β be the solution to the equation

$$M_T'(\beta) = t\, M_T(\beta) \tag{7}$$

where the $'$ denotes differentiation with respect to θ. Note that β is a function of t. Further, let

$$c = \frac{e^{\beta t}}{M_T(\beta)} \tag{8}$$

and

$$\sigma^2 = \frac{M_T''(\beta)}{M_T(\beta)} - t^2. \tag{9}$$

The saddlepoint approximation for P(S > s) is:

$$\Pr(S > s) \approx 1 - \Phi(\sqrt{2\ln(c)}) + \frac{1}{c\sqrt{2\pi}}\left[\frac{1}{\beta\sigma} - \frac{1}{\sqrt{2\ln(c)}}\right] \tag{10}$$

where $\Phi(\cdot)$ is the standard normal distribution function, and c and σ are defined in equations (8) and (9).

In practice, once s is chosen and t is computed, equation (7) is solved numerically using a technique such as Newton's method or the secant method; see, for example, Burden and Faires (1997, Chapter 2).

3 Examples

The saddlepoint approximations of tail probabilities are now applied to several specific collective risk models. These saddlepoint approximations are compared to the Haldane Type A and the normal approximations, and the exact probabilities. The exact calculations are found by simulation using 10,000 repetitions, which gives accuracy to four decimal places.

If X has mean μ_X, standard deviation σ_X, and coefficient of skewness γ_X, then the Haldane Type A approximation is as follows:

$$Pr[X \le x_0] \approx \Phi\left[\left((1+r\tilde{x}_0)^h - \mu(h,r)\right)/\sigma(h,r)\right] \tag{11}$$

where

$$\tilde{x}_0 = \frac{(x_0-\mu_X)}{\sigma_X}$$

$$r = \frac{\sigma_X}{\mu_X} \tag{12}$$

$$h = 1 - \frac{\gamma_X}{3r} \tag{13}$$

$$\mu(h,r) = 1 - \frac{1}{2}h(1-h)[1 - \frac{1}{4}(2-h)(1-3h)r^2]r^2 \tag{14}$$

$$\sigma(h,r) = hr\sqrt{1 - \frac{1}{2}(1-h)(1-3h)r^2}. \tag{15}$$

The Haldane approximation is chosen because Pentikäinen's (1987) results show it to be, under certain circumstances, an accurate approximation. Recall that the normal approximation is

$$Pr[X \le x_0] \approx \Phi\left[\tilde{x}_0\right]. \tag{16}$$

The relative errors shown in the tables are calculated as:

$$\text{Relative Error} = \left|\frac{\text{Approximation} - \text{Exact}}{\text{Exact}}\right|.$$

3.1 Light and Medium Tailed Claim Size Distributions

Example 1: X_1 is normal random variables with mean $\mu_X = 100$ and standard deviation $\sigma_X = 10$ while N is Poisson with mean $\lambda = 10$. From equation (3)

$$M_S(\theta) = \exp\left[\lambda\left(\exp\left(\mu_X\theta + \frac{1}{2}\sigma_X^2\theta^2\right) - 1\right)\right]. \tag{17}$$

Table 1
Approximating Tail Probabilities for
The Compound Normal-Poisson Model

t	β	Exact	Relative Error Normal	Relative Error HALD A	Relative Error SADP
0.5	0.4637	0.2964	0.0411	0.0039	0.0034
1.0	0.8672	0.1575	0.0074	0.0077	0.0070
1.5	1.2243	0.0750	0.1089	0.0062	0.0087
2.0	1.5445	0.0303	0.2498	0.0125	0.0082
2.5	1.8347	0.0112	0.4469	0.0019	0.0089
3.0	2.1001	0.0036	0.6351	0.0091	0.0084

In this setting the central limit theorem is known to hold for large λ.

Example 2: X_1 is a gamma random variable with a mean of $\mu_X = 100$ and standard deviation $\sigma_X = 10$. N is a negative binomial random variable with mean of $\alpha = 10$ and and standard deviation $\gamma = 20$. Here

$$M_S(\theta) = \left[\frac{1-q}{1-q(1-\beta)^{-\delta}}\right]^r \tag{18}$$

where $q = 0.5$, $\mu_X = \beta\delta$, $\sigma_X = \beta\sqrt{\delta}$, $\alpha = rq/(1-q)$ and $\gamma = rq/(1-q)^2$.

Table 2
Approximating Tail Probabilities for
The Compound Gamma-Negative Binomial

t	β	Exact	Relative Error Normal	Relative Error HALD A	Relative Error SADP
0.5	0.4284	0.2684	0.1494	0.0961	0.0417
1.0	0.7502	0.1548	0.0252	0.0284	0.0032
1.5	1.001	0.0796	0.1608	0.0515	0.0050
2.0	1.203	0.0375	0.3920	0.6907	0.0027
2.5	1.369	0.0166	0.6265	0.3012	0.0084
3.0	1.508	0.0070	0.8086	0.4571	0.0100

Example 3: X_1 is an inverse Gaussian random variable with mean $\mu_X = 100$ and standard deviation $\sigma_X = 10$. N is Poisson with mean $\lambda = 10$. The moment-generating function for the inverse Gaussian distribution is

$$M_X(\theta) = \exp\left[\left(\frac{\mu_X}{\sigma_X}\right)^2\left(1 - \left(1 - \frac{2\sigma_X^2\theta}{\mu_X}\right)\right)\right],$$

see Johnson and Kotz (1970, Chapter 15). Hence

$$M_S(\theta) = \exp\left[\lambda\left(\left(\left(\frac{\mu_X}{\sigma_X}\right)^2\left(1 - \left(1 - \frac{2\sigma_X^2\theta}{\mu_X}\right)\right)\right) - 1\right)\right]. \tag{19}$$

Table 3
Approximating Tail Probabilities for
The Compound Inverse Gaussian–Poisson Model

			Relative Error		
t	β	Exact	Normal	HALD A	SADP
0.5	0.4537	0.2998	0.0290	0.0153	0.0147
1.0	0.8671	0.1629	0.0258	0.0258	0.0264
1.5	1.2242	0.0775	0.1381	0.0387	0.0413
2.0	1.5444	0.03 16	0.2785	0.0285	0.0348
2.5	1.8345	0.0119	0.4790	0.0588	0.0672
3.0	2.0998	0.0038	0.6474	0.0526	0.0526

These examples show that the saddlepoint approximation is superior to the central limit theorem, but seems to be on par with the Haldane approximation in calculating tail probabilities. Next we consider a more difficult setting involving heavy tailed distributions.

4 Heavy Tailed Claim Size Distributions

The saddlepoint approximation requires the existence of the moment-generating function of the claim variable. For heavy tailed distributions, such as the Pareto (the moment-generating function does not exist) and lognormal (the moment-generating function is not in convenient a closed form), an approximation is required. For these problem cases a censoring limit is imposed on the claim distribution.

For cases where the moment-generating function does not exist, the distribution of the claim variable is approximated utilizing an upper tail censoring limit. For small ϵ the censoring limit, L, satisfies $\Pr[X_1 > L] = \epsilon$. Let us define the censored claim random variable as

$$Y_i = \begin{cases} X_i & \text{if } X_i \leq L \\ L & \text{if } X_i > L. \end{cases}$$

The distribution function for the Y_is is now

$$F_Y(x) = \begin{cases} F(x) & \text{if } x < L \\ 1 & \text{if } x \geq L. \end{cases}$$

The corresponding moment-generating function is

$$M_Y(\theta) = \int_{x=0}^{L} e^{\theta x} dF(x) + \epsilon e^{\theta L}. \tag{20}$$

The saddlepoint approximation is applied using the censoring moment-generating function in equation (20). This technique is now demonstrated on two examples of heavy tailed claim distributions. In both cases the number of claims is assumed to be Poisson with mean 5.

Example 4: Claims are assumed to follow a lognormal distributed with probability density function (pdf) of X_1 is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp[-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2] \quad -\infty < x < \infty. \tag{21}$$

where $\mu = 0$ and $\sigma = 1$. We assume that $\epsilon = 0.001$, which produces a censoring limit of $L = 59.7697$.

Example 5: Here we assume the claim size follows a Pareto distribution with distribution function given by

$$F(x) = 1 - \frac{1}{(1+x)^3}.$$

Table 4
Approximating Tail Probabilities for
The Compound Lognormal–Poisson Model

			Relative Error		
t	β	Exact	Normal	HALD A	SADP
0.5	0.7251	0.1628	0.8950	0.5565	0.0498
1.0	0.9501	0.0630	1.5190	1.3016	0.0825
1.5	1.0512	0.0241	1.7718	2.3361	0.0622
2.0	1.2001	0.0108	1.1111	1.0463	0.1574
2.5	1.4211	0.0047	0.3191	5.5319	0.3404

Again, $\epsilon = 0.001$, and this produces a censoring limit of $L = 9.0$.

As in the previous section, normalized tail probabilities and the saddlepoint approximations are compared to the exact values as obtained by simulation. These computations are listed in Tables 4 and 5.

Table 5
Approximating Tail Probabilities for
The Compound Pareto–Poisson Model

			Relative Error		
t	β	Exact	Normal	HALD A	SADP
0.5	0.6959	0.1664	0.8540	0.6280	0.0313
1.0	0.9880	0.0688	1.3067	1.2456	0.1933
1.5	1.1623	0.0327	1.0428	1.5199	0.1804
2.0	1.2842	0.0165	0.3818	1.5091	0.0727
2.5	1.3772	0.0094	0.3404	1.1064	0.1596

For the heavy tailed distributions, the saddlepoint approximation is superior to the central limit theorem and the Haldane approximation in calculating tail probabilities.

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