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Dynamic Immunization and Transaction Costs With Different Term Structure Models

Eliseo Navarro* and Juan M. Nave[†]

Abstract[‡]

A bond portfolio selection model is developed in a dynamic framework using different term structures, but without transactions costs. We show that the optimal portfolios are consistent with Khang's dynamic immunization theorem, i.e., the optimal path consists of making portfolio duration equal to the investor's horizon planning period. The model is then extended to include transaction costs. The resulting optimal portfolios are no longer consistent with Khang's dynamic immunization theorem. In fact, the strategy for constructing the optimal portfolio consists of initially choosing a portfolio with a duration that is *smaller* than the horizon planning period.

Key words and phrases: *bond, portfolio, planning period, strategy, risk, interest rate, stochastic*

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1   Introduction

Suppose an investor in a fixed income market has certain obligations due at some specified future date, called the *investor's horizon planning period*. A key problem facing such an investor is the problem of immunizing[1] (protecting) his or her portfolio of bonds against interest rate risk.

Bierwag and Khang (1979) prove that the process of immunizing a bond portfolio can be described as a maxi-min strategy in a game against nature where the investor's target is to guarantee a minimum return over his or her planning period or, equivalently, to guarantee a minimum value at the end of his or her horizon planning period. Dantzig (1971) shows that this maxi-min solution can be determined by solving an equivalent linear program that depends on the assumption about the term structure of interest rates.

One of the main results concerning the development of portfolio immunization strategies against interest rate risk is due to Khang (1983) and is described by his dynamic global immunization theorem. Khang's strategy consists of a continuous portfolio rebalancing in order to keep portfolio duration equal to the length of the remaining planning period.

Specifically, consider an investor who has a horizon planning period of length H. Suppose the forward interest rates structure shifts up or down by a stochastic shift parameter at any time during the investor's planning period. If the investor follows Khang's strategy, then the investor's wealth at the end of his or her planning period will be no less than the amount anticipated on the basis of the forward interest rates structure observed initially (at time 0). Furthermore, the investor's wealth at time H will be greater than the amount anticipated initially if at least one interest shock takes place during the planning period.

The validity of Khang's strategy rests on two key assumptions: (i) If $g(t), t \geq 0$, denotes the forward interest rates structure, and the forward interest rates structure changes to $g^*(t)$, then

$$g^*(t) = g(t) + \delta$$

where δ is a stochastic shift parameter; and (ii) there are no transaction costs.

The first assumption avoids the problem of the risk of misestimating the term structure behavior, which Fong and Vasicek (1983) call

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[1]Immunization consists of making a portfolio's duration (properly defined) equal to the remaining horizon planning period.

the "immunization risk."[2] Assumption (ii) avoids the high costs that a strategy of continuous portfolio rebalancing may incur.

In this paper we investigate the applicability of Khang's strategy under both a static and a dynamic portfolio selection model. Each model is tested under three different assumptions about the term structure of interest rates behavior: a flat term structure, a diffusion process as in Vasicek (1977), and a diffusion process as in Cox, Ingersoll, and Ross (1985). The dynamic portfolio selection model under the flat term structure model behaves according to classical Fisher and Weil (1971) immunization theorem. It behaves according to Boyle's (1978) stochastic immunization in the two alternative stochastic cases. Finally the model is expanded to include transaction costs. We show, through an example, that if transaction costs are high enough, the optimal strategy may differ from that proposed by Khang.

2 The Term Structure Models

Three different term structure models are used in our analysis:

- The first and simplest model assumes a flat term structure and parallel term structure of interest rates shifts;

- The second model assumes a stochastic term structure with instantaneous spot interest rate following a diffusion process as in Vasicek (1977); and

- The third model assumes a stochastic term structure with instantaneous spot interest rate following a diffusion process as in Cox, Ingersoll, and Ross (1985).

2.1 Flat Term Structure Model

This model makes the following assumptions about process of the term structure of interest rates:

A1 The term structure is flat;

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[2]Bierwag (1987) calls it "stochastic process risk" and defines the stochastic process as "the way in which the term structure shifted from period to period," adding afterward that "it is conceivable that an investor could assume an incorrect stochastic process and, as a consequence, the perceived durations would be different from the actual ones. The investor ... losses from misestimation (or misguesstimation) of the correct process can be substantial" (Bierwag, 1987).

A2 Term structure of interest rates changes consist of parallel move-
ments of the entire term structure, i.e., short-term and long-term
interest rates changes are equal; and

A3 The pure expectations hypothesis[3] holds.

A4 There are m different levels of interest rates r_j $(j = 1, 2, \ldots m)$
with $r_1 < r_2 < \cdots < r_m$.

The implication of assumption (A3) is that under a flat term structure
model any interest rate change is considered to be unexpected.
    Let

$$
\begin{aligned}
r(t) &= \text{The spot rate of interest at time } t; \\
t_0 &= \text{The current time;} \\
r_c &= \text{The current spot rate of interest (at time } t_0); \\
P(r(t), t, s) &= \text{The price at time } t \text{ of a pure discount bond} \\
& \quad \text{maturing at time } s \ (t \le s).
\end{aligned}
$$

It follows that

$$
P(r(t), t, s) = e^{-(s-t)r(t)} \tag{1}
$$
$$
E[r(s) | r(t_0) = r_c] = r_c, \tag{2}
$$

for $s > t_0$, i.e., interest rates are expected to remain unchanged. Under
the flat term structure model, the relative basis risk[4] of a discount bond
is given by

$$
-\frac{1}{P}\frac{\partial P}{\partial r} = s - t.
$$

2.2   The Vasicek (1977) Model

Here we make the following assumptions about the term structure:

A5 Instantaneous spot interest rate $r(t)$ follows a diffusion process
so its behavior is described by the following stochastic differential
equation:

$$
dr = \beta(y - r)dt + \rho d\tilde{z} \tag{3}
$$

where β, y, and ρ are positive constants, and $d\tilde{z}$ is a Wiener pro-
cess with zero mean and variance dt; and

---

[3]For a thorough discussion of the different hypotheses about the term structure of
interest rates and their implications, see Cox, Ingersoll, and Ross (1981).

[4]*Basis risk* can be defined as the possibility that an institution's margin will rise or
fall as a consequence of market rate movements.

A6 There are no arbitrage opportunities.

Equation (3) yields the following expressions for $s > t$:

$$E[r(s)|r(t)] = \gamma + (r(t) - \gamma)e^{-\beta(s-t)} \qquad (4)$$

$$P(r(t), t, s) = \exp\left[\quad F(s-t)(G - r(t)) \right.$$

$$\left. -(s-t)G - \frac{\rho^2}{4\beta}F(s-t)^2 \right] \qquad (5)$$

where

$$F(x) = \frac{1}{\beta}[1 - \exp(-\beta x)]$$

$$G = \gamma - \frac{\rho^2}{2\beta^2}.$$

The relative basis risk of a discount bond is now given by

$$-\frac{1}{P}\frac{\partial P}{\partial r} = F(s-t).$$

2.3   The Cox-Ingersoll-Ross (1979) Model

In addition to assumption (A6), we assume the following:

A7 $r(t)$ satisfies the following stochastic differential equation:

$$dr = \kappa(\mu - r)dt + \sigma\sqrt{r}d\check{z} \qquad (6)$$

where κ, μ, and σ are positive constants.

Equation (6) yields the following expressions, for $s > t$:

$$E[r(s)|r(t)] = \mu + (r(t) - \mu)e^{-\kappa(s-t)} \qquad (7)$$

$$P(r(t), t, s) = A(s-t)\exp[-r(t)B(s-t)] \qquad (8)$$

where:

$$A(x) = \left[\frac{2\lambda\exp[(\kappa - \lambda)x/2]}{(\lambda + \kappa)[1 - \exp(-\lambda x)] + 2\lambda\exp(-\lambda x)} \right]^{2\kappa\mu/\sigma^2}$$

$$B(x) = \frac{2(1 - \exp(-\lambda x))}{(\lambda + \kappa)[1 - \exp(-\lambda x)] + 2\lambda\exp(-\lambda x)}$$

$$\lambda = \sqrt{\kappa^2 + 2\sigma^2}.$$

The relative basis risk is

$$-\frac{1}{P}\frac{\partial P}{\partial r} = B(s - t).$$

3   The Static Model

Consider an investor who wants to allocate an amount of I dollars in a market where n different default-free non-callable coupon-bearing bonds are available. The investor's objective is to construct a portfolio that guarantees a minimum return over his or her planning period or, equivalently, that guarantees minimum value at the end of the investor's horizon planning period.

In the static model, the market can be characterized by the following set of assumptions:

A8 Financial markets are competitive; Individual investors' decisions don't affect interest rates that are given exogenously;

A9 There is perfect divisibility of financial assets;

A10 There are no arbitrage opportunities;

A11 There are no transaction costs; and

A12 Short sales are not allowed.[5]

3.1   Notation

The notation introduced in this section will be used throughout this paper:

n = Number of default-free non-callable coupon bonds;

H = Horizon planning period, which spans the interval $(t_0, H]$; and

I = Investor's initial wealth at t_0.

We assume that the bonds are ordered according to their maturity so bond 1 is the bond with the shortest time to maturity and bond n is

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[5]This constraint is imposed in the model as a sufficient condition in order to guarantee that the net income generated by the portfolio is always nonnegative throughout the planning period, which is one of the hypotheses of Khang's theorem.

the bond with the longest term to maturity. For $i = 1, \ldots, n$, let

$$
\begin{aligned}
T_i &= \text{Time to maturity of bond } i \text{ with } T_i \leq T_{i+1} \\
&\quad \text{for } i = 1, \ldots, n-1; \\
\rho &= \text{Number of bonds maturing in } (t_0, H], \; \rho = 0, 1, \ldots, n; \\
n_i &= \text{Number of bond } i \text{ coupon payments made after } t_0; \\
\tau_s^{(i)} &= \text{Time of bond } i\text{'s } s\text{-th coupon payment after } t_0, \\
&= \text{for } s = 1, 2, \ldots, n_i; \\
\tau_{n_i}^{(i)} &= T_i \\
p_i &= \text{Current (at } t_0) \text{ price of one unit of bond } i; \\
x_i &= \text{Number of units of bond } i \text{ in the optimal portfolio; and} \\
C_i &= \text{Size of each coupon payment from bond } i.
\end{aligned}
$$

Clearly, in order to obtain a duration close to H, some of the T_is must exceed H. Also, bonds $\rho + 1, \ldots, n$ mature after H.

3.2   The Static Model's Linear Program

The investor's strategy consists of purchasing an allocation[6] vector $(x_1, x_2, \ldots, x_n)$ of bonds that satisfy the following budget constraint:

$$
\sum_{i=1}^{n} x_i p_i = I. \tag{9}
$$

If just after selecting a strategy at t_0, interest rates instantaneously change from r_c to r_j, then portfolio value at the end of the horizon planning period is V_j such that:

$$
V_j = \sum_{i=1}^{n} x_i v_{ij} \tag{10}
$$

where v_{ij} denotes the value at the end of the horizon planning period of an investment of p_i dollars in bond i, i.e.,

$$
v_{ij} = \frac{\sum_{s=1}^{n_i - 1} C_i e^{-r_j(\tau_s^{(i)} - t_0)} + (1 + C_i) e^{-r_j(T_i - t_0)}}{e^{-r_j H}} \tag{11}
$$

under the flat term structure model, and

---

[6] A portfolio allocation vector can be considered as a strategy of the investor.

$$v_{ij} = \frac{\sum_{s=1}^{n_i-1} C_i P(r_j, t_0, \tau_s^{(i)}) + (1 + C_i) P(r_j, t_0, T_i)}{P(r_j, t_0, H)}. \tag{12}$$

under the Vasicek and the Cox-Ingersoll-Ross models.

Note that v_{ij} is based on two assumptions: (i) the interest rates remain r_j until the end of the horizon planning period (in accordance with the pure expectations theory); and (ii) the coupon and principal payments made before the end of the horizon planning period are reinvested at rate r_j under the flat term structure model. Under stochastic models, coupon and principal payments are assumed to be reinvested at the forward rates corresponding to a term structure of interest rate derived from a instantaneous spot rate equal to r_j.

Let V denote the minimum final portfolio value the investor wishes to maximize, i.e., V is a lower bound for the final portfolio value. Thus, V is independent of interest rate changes and depends on the selected portfolio. The portfolio selection process can be modeled as the following linear program:

Static Model

$$\max_{x_i} V \tag{13}$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_i v_{ij} \geq V, \ j = 1, 2, \ldots, m$$

$$\sum_{i=1}^{n} x_i p_i = I$$

$$V \geq 0, \ x_i \geq 0, \ i = 1, 2, \ldots, n.$$

Cox, Ingersoll, and Ross (1979) point out that if we want stochastic duration to serve as a proxy for the basis risk of coupon bonds with the units of time, it is natural to define it as the maturity of a discount bond with the same risk. Therefore, portfolio duration at t_0 under Vasicek's term structure model is D_V given by:

$$D_V = F^{-1} \left(\frac{\sum_{i=1}^{n} x_i W_i^{(F)}}{\sum_{i=1}^{n} x_i \left[\sum_{s=1}^{n_i} C_i P(r, t_0, \tau_s^{(i)}) + P(r, t_0, T_i) \right]} \right) \tag{14}$$

where r is the interest rate at t_0,

$$W_i^{(F)} = \sum_{s=1}^{n_i} C_i P(r, t_0, \tau_s^{(i)}) F(\tau_s^{(i)} - t_0) P(r, t_0, T_i) + F(T_i - t_0)$$

and

$$F^{-1}(x) = -\frac{\ln(1 - \beta x)}{\beta}.$$

On the other hand, the portfolio duration under the Cox-Ingersoll-Ross term structure model is D_{CIR}:

$$D_{CIR} = B^{-1}\left(\frac{\sum_{i=1}^{n} x_i W_i^{(B)}}{\sum_{i=1}^{n} x_i \left[\sum_{s=1}^{n_i} C_i P(r, t_0, \tau_s^{(i)}) + P(r, t_0, T_i)\right]}\right) \qquad (15)$$

where:

$$W_i^{(B)} = \sum_{s=1}^{n_i} C_i P(r, t_0, \tau_s^{(i)}) B(\tau_s^{(i)} - t_0) P(r, t_0, T_i) + B(T_i - t_0)$$

and

$$B^{-1}(x) = \frac{1}{\lambda} \ln\left[\frac{2 - (\kappa - \lambda)x}{2 - (\kappa + \lambda)x}\right].$$

3.3 An Example

To illustrate our ideas, we apply them to a simple example. Assume an investor has $I = \$1,000,000$ and a horizon planning period of 18 months. There is a fixed income market with four default-free non-callable 10 percent coupon bonds, as described in Table 1.

Table 1
10% Coupon Bonds
With Coupons Paid Semi-Annually

	Maturity (In Years)	Macaulay Duration
Bond 1	0.5	0.5000
Bond 2	1.0	0.9762
Bond 3	1.5	1.4297
Bond 4	2.0	1.8616

In the case of the series flat term structure, we assume a nominal current interest rate level of 10 percent (compounded semiannually).

Interest rates may move up and down by 100 basis points to 9 percent or to 11 percent. In other words, nominal interest (compounded semiannually) may take only one of three values: $r_1 = 9$ percent; $r_2 = r_c = 10$ percent; $r_3 = 11$ percent.

The optimal solution of the linear program of equation (13) is shown in Table 2. This result is consistent with Fisher and Weil immunization theorem (which states that the optimal solution consists of a portfolio with a duration equal to the horizon planning period).

In the case of the series Vasicek (1977) and Cox-Ingersoll-Ross (1985) term structure model, we use the model parameters estimated in Nowman (1997) for both U.S. (from the Treasury bill market) and U.K. (sterling one month interbank rate). (See Table 3.)

Nature strategies consist of the different values that the current instantaneous spot rate can take which we assume can vary 100 basis points (up or down) from its current level (5.61 percent for the U.S. and 5.99 percent for the U.K.).[7] The optimal solutions are shown in Table 2.

It is important to see that, under stochastic term structure models, portfolio immunization consists of making portfolio duration (properly defined) equal to the remaining horizon planning period.

3.4 Immunization Risk

So far we have assumed a specific term structure behavior where the whole term structure is supposed to depend on a unique factor (short-term interest rate). The nature of the dynamics of interest rates, however, is more complex.[8] Immunization strategies may fail if the

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[7]We have assumed a one percent change in instantaneous spot interest rate and then recalculated the whole term structure of interest rates according to equations (5) and (8) to determine the new bond prices. Theoretically, the interest rate change considered in these models should be the largest change that is possible within a trading day (or any other suitably short time interval). We are aware that the probability of one percent change in interest rates within a day is, according to model parameters, negligible. Such a drastic change in interest rates is assumed, however, in order to have a similar level of interest rate risk in all three cases analyzed. Despite this, the interest rate risk assumed under the stochastic model is still lower due to the mean reversion effect. According to Boyle (1978), under flat term structure models, a small interest rate change may have a dramatic impact on the price of long-term bonds. The impact of instantaneous spot rate changes on long-term bonds under the stochastic models considered in this paper, however, is diminished by the expected mean reversion of short interest rates.

[8]There is some international evidence that at least 95 percent of term structure movements can be explained by three factors: parallel shifts, slope changes, and curvature changes. Depending on the country analyzed and the period covered by different studies, parallel shifts can explain between 72 and 97 percent of the variance on interest rate changes. For further detail see Steeley (1990), Strickland (1993), D'Ecclesia and Zenios

Table 2

Optimal Strategies in a Static Framework

Panel A: Nonstochastic Model				
x_1	x_2	x_3	x_4	Duration
2639.536	0	0	7360.465	1.5006

Panel B: Vasicek Model					
	x_1	x_2	x_3	x_4	Duration
U.S.	0	0	7886.288	1516.520	1.5010
U.K.	1044.207	0	4745.310	3676.038	1.4999

Panel C: Cox-Ingersoll-Ross Model					
	x_1	x_2	x_3	x_4	Duration
U.S.	2858.766	0	0	6843.708	1.5012
U.K.	0	3919.526	0	5545.140	1.5010

Notes: Portfolio durations are calculated as follows: Macaulay duration is used for Panel A; Equation (14) is used for Panel B; and Equation (15) is used for Panel C.

term structure of interest rates behaves differs significantly. This is known as *immunization risk*.[9]

To minimize the immunization risk from an unexpected behavior of the term structure, several proposals have been suggested. Most of them consist of selecting among the set of immunized portfolios those that generate payment streams as close as possible to the end of the horizon planning period. A trivial example would be a portfolio consisting entirely of zero coupon bonds maturing at the end of the horizon planning period.

There are several alternative measures of immunization risk. The usually accepted dispersion measure, however, is that proposed by Fong and Vasicek[10] known as M^2. By minimizing this quadratic dispersion measure, the effect on final portfolio value of a non-expected

(1994), Navarro and Nave (1995), and Sherris (1995) for the U.K., U.S., Italy, Spain, and Australia, respectively.

[9]For a review of the effects of nonparallel yield curve shifts on traditional immunization strategy, see Reitano (1992a and 1992b). Reitano (1991) generalizes Kang's result to any directional yield curve model and to a general multivariate nondirectional model.

[10]There are alternative dispersion measures, such as M-absolute, derived from different assumptions about term structure movements. Chalmers and Nawalka (1996) test the suitability of the M-absolute measure as a first order condition to protect an investment against interest rate risk instead of using it as a second order condition to minimize immunization risk. Other authors have criticized the M^2 measure, suggesting the convenience of including an asset with maturity at the end of the horizon planning

Table 3

Parameter Values of the

Vasicek and Cox-Ingersoll-Ross Models

(With $r(t)$ Determined in April 1997)

Panel A: Vasicek Model				
	β	γ	ρ^2	$r(t)$
U.S.	0.0506	0.0691700	0.0001	0.0561
U.K.	0.0311	0.1028939	0.0001	0.0599
Panel B: Cox-Ingersoll-Ross Model				
	β	γ	ρ^2	$r(t)$
U.S.	0.0373	0.0697051	0.0008	0.0561
U.K.	0.0279	0.1039427	0.0007	0.0599

Notes: The parameters β, γ, and ρ^2 were estimated by Nowman (1997) using a discrete time model that reduces some of the temporal aggregation bias. The data used are U.S. Treasury bill one month yields from June 1964 to December 1989 and the one month sterling interbank rate from March 1975 to March 1995.

term structure of interest rates movement is minimized. Fong and Vasicek analyze the effect of a shift consisting of a linear movement of the instantaneous forward rate around the end of the horizon planning period; in this case it is not possible to build an immunized portfolio, but there is a lower bound for the portfolio's final value that depends on M^2.

For $i = 1, \ldots, n$, the Fong-Vasicek dispersion measure for bond i, M_i^2, is defined as follows:

$$M_i^2 = \frac{\sum_{s=1}^{n_i} (\tau_s^{(i)} - H)^2 C_i P(r, t_0, \tau_s^{(i)}) + (T_i - H)^2 P(r, t_0, T_i)}{\sum_{s=1}^{n_i} C_i P(r(t), t, \tau_s^{(i)}) + P(r, t_0, T_i)}. \quad (16)$$

M_i^2 is introduced in the model by penalizing the objective function which becomes:

$$V - Q \sum_{i=1}^{n} M_i^2 x_i \quad (17)$$

---

period is the best strategy against immunization risk. (See Bierwag et al., 1993). In practical terms, the alternative measures lead to similar results.

where $Q > 0$ is a parametric constant that depends on the investor's immunization risk aversion. When M_i^2 is used in the model, the optimal portfolio path consists of immunized portfolios of minimum dispersion, independent of the term structure assumption.[11] (See Table 4.)

<div align="center">

Table 4

**Optimal Strategies of Minimum
Dispersion in a Static Framework**

</div>

Panel A: Nonstochastic Model				
x_1	x_2	x_3	x_4	Duration
0	0	8382.429	1617.571	1.4996
Panel B: Vasicek Model				
x_1	x_2	x_3	x_4	Duration
U.S. 0	0	7914.367	1488.932	1.4995
U.K. 0	0	7966.707	1494.853	1.4999
Panel C: Cox-Ingersoll-Ross Model				
x_1	x_2	x_3	x_4	Duration
U.S. 0	0	7916.721	1485.324	1.4999
U.K. 0	0	7956.605	1506.982	1.5005

Notes: Dispersion measure is calculated according to Fong and Vasicek M^2 equation (16). Portfolio durations are calculated as follows: Macaulay duration is used for Panel A; Equation (14) is used for Panel B; and Equation (15) is used for Panel C.

4   The Dynamic Model

The static portfolio selection model described in Section 3 provides a portfolio that is immunized against interest rate risk, but only at the beginning of the horizon planning period. The dynamic behavior of portfolio duration makes it impossible to keep that portfolio immunized during the entire planning period. Moreover, the immunization solution provided by the static model is valid only for the current interest rate, so the portfolio must be adjusted continuously as the rate

---

[11]Any other decreasing function of M^2 could be added to the objective function to penalize portfolio dispersion, as we are only trying to obtain the immunized portfolio of minimum dispersion.

of interest changes.[12] Our task now is to derive an optimal dynamic portfolio strategy that rebalances the portfolio in order to keep it free of interest rate risk.

4.1   The Rebalancing Points

Recall the notation from Section 3.1, i.e., bond i matures at T_i and pays coupons at times $\tau_s^{(i)}$ for $s = 1, 2, \ldots, n_i$. Consider all of the n bonds at t_0 and arrange the times of their coupon payments in ascending order so that t_s denotes the time of s-th coupon payment in $(t_0, H]$ so that

$$t_1 = \min\{\tau_1^{(1)}, \tau_1^{(2)}, \ldots, \tau_1^{(n)}\}.$$

Let t_k denote the time of the last coupon payment in $(t_0, H]$, i.e.,

$$k = \text{The integer such that } t_k \leq H \text{ and } t_{k+1} > H. \tag{18}$$

If one of the n bonds makes a coupon payment at H, then $t_k = H$; otherwise, $t_k < H$. Without loss of generality, we assume that at least one bond pays a coupon at H so that

$$t_k = H. \tag{19}$$

Equation (19) implies that $(t_0, H]$ is partitioned into k intervals. The s-th interval is $(t_{s-1}, t_s]$, for $s = 1, 2, \ldots, k$.

For example, we have three coupon bonds bought at t_0. Bond 1 (initially a 10 year bond) matures in 7.1 years and makes its remaining coupon payments at times $(0.1, 0.6, 1.1, \ldots, 7.1)$; Bond 2 (initially a 20 year bond) matures in 15.75 years and makes its remaining coupon payments at times $(0.25, 0.75, 1.25, \ldots, 15.75)$; and finally Bond 3 (initially a 30 year bond) matures in 17 years and makes its remaining coupon payments at times $(0.5, 1.0, 1.5, \ldots, 17)$. Ordering all of the coupon payment times gives $t_0 = 0$, $t_1 = 0.1$, $t_2 = 0.25$, $t_3 = 0.5$, $t_4 = 0.6$, etc. If the investor's planning period is $H = 1.5$ years, then $k = 6$ and $t_k = H = 1.5$. If $H = 1.7$ years, however, then $k = 7$ because $t_6 < H$ and $t_7 = 1.75 > H$.

Let us now express the time of maturity of bond i, T_i, in terms of the t_s's for those $T_i \leq H$. For $i = 1, 2, \ldots, n$ define

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[12]Only portfolios consisting entirely of zero coupon bonds with maturity at the end of the horizon planning period can be kept immunized over the horizon planning period without any additional rearrangements.

$$s_i = \begin{cases} s & \text{if } \exists s \in \{1, 2, \ldots, k\} \text{ such that } t_s = T_i \leq H; \\ k & \text{if } T_i > H = t_k \end{cases} \tag{20}$$

Finally, we assume that portfolio rebalancing is only allowed at the beginning of each interval t_s, $s = 0, 1, \ldots, k - 1$.

4.2   The Constraints

Next we need to construct the constraints for the linear program. To this end, the following notation will be used:

$$
\begin{aligned}
x(s, i) \quad &= \quad \text{Number of units of bond } i \text{ in the portfolio immediately} \\
&\qquad \text{after the rebalancing at time } t_s; \\
b(s, i) \quad &= \quad \text{Number of units of bond } i \text{ bought at } t_s; \\
z(s, i) \quad &= \quad \text{Number of units of bond } i \text{ sold at } t_s; \\
y(s, i) \quad &= \quad \text{Number of units of bond } i \text{ maturing at } t_s; \\
\bar{r}_s \quad &= \quad E[r(t_s) | r(t_0)]; \quad \text{and} \\
p(s, i) \quad &= \quad \text{Price of one unit of bond } i \text{ at } t_s \text{ assuming that} \\
&\qquad r(u) = E[r(u) | r(t_0)] \text{ for } u \geq t_0.
\end{aligned}
$$

The definition of $p(s, i)$ assumes the actual interest rates are equal to the expected interest rates throughout the planning period.

The following set of constrains must be satisfied:

(i) $x(0, i) = b(0, i)$ for $i = 1, \ldots, n$;

(ii) $x(s, i) = x(s - 1, i) + b(s, i) - z(s, i) - y(s, i)$
       for $s = 1, \ldots, k$ and $i = 1, \ldots, n$;

(iii) $x(s, i) = b(s, i)$ for $s = s_i, s_i + 1, \ldots, k$ and $i = 1, \ldots, n$;

(iv) $z(s, i) = 0$ for $s = s_i, \ldots, k$ and $i = 1, \ldots, \rho$;

(v) $y(s, i) = 0$ for $s \neq s_i, \ldots, k$ and $i = 1, \ldots, \rho$;
       or for $s = 1, \ldots, k$ and $i = \rho + 1, \ldots, n$.

Constraint (iii) represents the number of units of bond s maturing at T_i or being sold at t_k. (Note that all bonds must be sold at the end of the horizon planning period.) Constraint (iv) indicates that bond i cannot be held or traded after it matures. Constraint (v) states that bond i matures only at a single point in time.

Constraint (i) indicates the number of bonds bought at the beginning of the horizon planning period. Constraint (ii) indicates the purchases

and sales at each subsequent t_s. Constraint (iii) indicates that bond i cannot be held or traded after it matures. Constrain (iv) indicates that those bonds with maturity before or at t_k cannot be sold after their maturity. Note that constraints (ii), (iii), and (iv) imply that $x(s_i - 1, i) = y(s_i, i)$ for $i = 1, \ldots, \rho$, because those bonds with maturity at or before t_k mature at s_i. Constraint (v) indicates that bonds 1 to ρ can only mature at s_i. Meanwhile, bonds $\rho + 1$ to n do not mature at any point during the planning period. Note that for bonds $\rho + 1$ to n constraints (ii) and (v) imply that $x(s_i - 1, i) = z(s_i, i)$, i.e., all bonds outstanding at t_k have to be sold at the end of the horizon planning period).

The initial budget constraint is now:

$$\sum_{i=1}^{n} x(0, i) p(0, i) = I \qquad (21)$$

where I is the amount of money available at the beginning of the horizon planning period. The budget constraint must be satisfied not only at t_0 but during the whole planning period, so we must add the following set of budget constraints for $s = 1, \ldots, k - 1$

(vi) $\sum_{i=1}^{n} [b(s, i) p(s, i) - z(s, i) p(s, i) - y(s, i) p(s, i)$
$- C_i x(s - 1, i)] = 0$

(vii) $\sum_{i=1}^{n} [z(k, i) p(k, i) + y(k, i) p(k, i) + C_i x(k - 1, i)] = V_k$

where V_k is the portfolio value at t_k, i.e., at the end of the horizon planning period. Note that $p(s_i, i)$ is the amount of face value of bond i maturing at t_{s_i}; The constraint (vi) shows that the amount of money invested in new purchases at each t_s, $\sum b(s, i) p(s, i)$, must come from coupon payments, $\sum C_i x(s - 1, i)$, sales, $\sum z(s, i) p(s, i)$, and principal repayment $\sum y(s, i) p(s, i)$. Constraint (vii) shows the expected value of the portfolio at H.

4.3   The Optimal Portfolio

As in the static model, the investor's aim at each t_s is to maximize the guaranteed portfolio value at the end of the horizon planning period assuming unexpected interest rate changes only occur immediately after portfolio rebalancing, i.e., just after each t_s.

If we let V_s be the minimum final portfolio value to guarantee at t_s, then the following set of constraints must be satisfied:

$$\sum_{i=1}^{n} x(s, i) v_{ij}(s) \geq V_s, \quad s = 0, \ldots, k - 1, \quad j = 1, \ldots, m,$$

where $v_{ij}(s)$ denotes the final value (at H) of an investment of $p(s,i)$ dollars in bond i at t_s. In addition, the definition of $v_{ij}(s)$ assumes that the instantaneous spot interest rate changes at t_s+ from $\tilde{r}_s$ to r_j and no additional unexpected interest rate change occurs until the end of the horizon planning period. Thus

$$v_{ij}(s) = \frac{\sum_{u:\tau_u^{(i)}>t_s} C_i P(r_j, t_s, \tau_u^{(i)}) + P(r_j, t_s, T_i)}{P(r_j, t_s, H)}. \tag{22}$$

As the investor's aim is to maximize the minimum portfolio final values at each t_s and to simultaneously minimize immunization risk at each t_s, the objective function can restated as:

$$\sum_{s=0}^{k} V_s - Q \sum_{s=0}^{k-1} \sum_{i=1}^{n} M^2(s,i) x(s,i) \tag{23}$$

where $M^2(s,i)$ is the Fong-Vasicek dispersion measure for bond i at t_s, defined as follows:

$$M^2(s,i) = \frac{\sum_{u:\tau_u^{(i)}>t_s} (\tau_u^{(i)} - H)^2 C_i P(\tilde{r}_s, t_s, \tau_u^{(i)}) + (T_i - H)^2 P(r_j, t_s, T_i)}{\sum_{u:\tau_u^{(i)}>t_s} C_i P(\tilde{r}_s, t_s, \tau_u^{(i)}) + P(r_j, t_s, T_i)} \tag{24}$$

for $s = 0,\ldots,k-1$; $i = s+1,\ldots,n$. The complete dynamic model is:

The Dynamic Model:
$$\max \sum_{s=0}^{k} V_s - Q \sum_{s=0}^{k-1} \sum_{i=1}^{n} M^2(s,i) x(s,i)$$
subject to
$$\sum_{i=1}^{n} x(0,i) p(0,i) = I$$

(i) $x(0,i) = b(0,i)$ for $i = 1,\ldots,n$;

(ii) $x(s,i) = x(s-1,i) + b(s,i) - z(s,i) - y(s,i)$
 for $s = 1,\ldots,k$ and $i = 1,\ldots,n$;

(iii) $x(s,i) = b(s,i)$ for $s = s_i, s_i+1,\ldots,k$ and $i = 1,\ldots,n$;

(iv) $z(s,i) = 0$ for $s = s_i,\ldots,k$ and $i = 1,\ldots,\rho$;

(v) $y(s,i) = 0$ for $s \neq s_i,\ldots,k$ and $i = 1,\ldots,\rho$;
 or for $s = 1,\ldots,k$ and $i = \rho+1,\ldots,n$;

(vi) $\sum_{i=1}^{n} [b(s,i) p(s,i) - z(s,i) p(s,i) - y(s,i) p(s,i)$
 $- C_i x(s-1,i)] = 0$ for $s = 1,\ldots,k-1$

(vii) $\sum_{i=1}^{n} [z(k,i) p(k,i) + y(k,i) p(k,i) + C_i x(k-1,i)] = V_k$

(viii) $x(s,i), b(s,i), z(s,i), y(s,i), V_s \geq 0$
 for $s = 1,\ldots,k$ and $i = 1,\ldots,n$.

4.4   The Example Continued

Here $H = 1.5$ years and $t_0 = 0$. From the data in Table 1 we see that $T_1 = 0.5$, $T_2 = 1.0$, $T_3 = 1.5$, and $T_4 = 2.0$. There are only three coupon payments in $(0, 1.5]$ at $t_1 = 0.5$, $t_2 = 1.0$, and $t_3 = 1.5$; it follows that $k = 3$. It is easily seen that, because of the way the bonds are labeled, $s_i = i$ for those bonds with maturity at or before t_3.

In the case of the series flat term structure, we again assume a current interest rate level of 10 percent and $r_1 = 9$ percent; $r_2 = r_c = 10$ percent; $r_3 = 11$ percent. The optimal solution paths are reported in Panel A of Table 5. We can see that this result is consistent with Khang's theorem: the optimal portfolio duration consists of making duration equal to the remaining horizon planning period at every t_s. The small difference between these two variables is due to the finite number of scenarios of interest rate changes considered.

In the case of the Vasicek (1977) and Cox-Ingersoll-Ross (1985) term structure models, we use the parameters estimated in Nowman (1997) for both U.S. (from the Treasury bill market) and U.K. (sterling one month interbank rate). (See Table 3.) The expected interest rate at the beginning of each interval is given by equation (4) for the Vasicek model and equation (7) for the Cox-Ingersoll-Ross model.

The results are displayed in Panel A of Tables 6 to 9. Again, Khang's theorem is still valid. These results also are consistent with those obtained by Gagnon and Johnson (1994) under a stochastic interest rate in a discrete time framework.[13]

5   Transaction Costs

The static and dynamic models described in Sections 3 and 4 do not include transaction costs and lead to solutions consistent with Khang's theorem. The next step is to introduce transaction costs into the model and analyze their effects on the optimal solution.

We assume all transaction costs are incurred only at portfolio rearrangement times and are a constant proportion, α ($0 \le \alpha < 1$), of the volume traded (in dollars) at each t_s. We also assume that principal and coupon repayments don't generate transaction costs.

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[13]In particular, Gagnon and Johnson (1994) assume the Black, Derman, and Toy (1990) arbitrage-free evolution model.

Table 5
Optimal Portfolio Path
Under the Flat Term Structure

Panel A: No Transactions Costs ($\alpha = 0.00\%$)

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	8382.429	1617.571	1.4996
0.5	0	9951.796	548.204	0	0.9999
1	11025.000	0	0	0	0.5000

Panel B: With Transactions Costs

$\alpha = 0.15\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	9448.628	536.395	1.4529
0.5	0	9947.132	536.395	0	0.9994
1	10470.523	536.395	0	0	0.5232

$\alpha = 0.30\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	9434.535	535.554	1.4529
0.5	0	9931.548	535.554	0	0.9994
1	10453.338	535.554	0	0	0.5232

$\alpha = 0.45\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	9420.485	534.717	1.4529
0.5	0	9916.016	534.717	0	0.9994
1	10436.212	534.717	0	0	0.5232

$\alpha = 0.60\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	9940.359	0	1.4297
0.5	0	10434.412	0	0	0.9762
1	10953.022	0	0	0	0.5000

Notes: (a) The α value represents the level of transaction costs as a percentage of the volume traded; $\alpha = 0$ means the absence of transaction costs. In this case the optimal strategy is consistent with Khang's theorem, i.e., at each rebalancing point the portfolio has to be restructured in order to keep its duration equal to the remaining horizon planning period; (b) Macaulay duration is used for this table.

Table 6
Optimal Portfolio Path
Under the Vasicek Model Using U.S. Data

Panel A: No Transactions Costs ($\alpha = 0.00\%$)

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	7914.367	1488.932	1.4995
0.5	0	9363.004	510.227	0	0.9999
1	10366.868	0	0	0	0.5000

Panel B: With Transactions Costs

$\alpha = 0.15\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	8893.123	513.351	1.4543
0.5	0	9344.520	513.351	0	1.0002
1	10348.526	0	0	0	0.5000

$\alpha = 0.30\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	8879.864	512.549	1.4543
0.5	0	9329.912	512.549	0	1.0002
1	9810.844	512.549	0	0	0.5238

$\alpha = 0.45\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	9387.450	0	1.4305
0.5	0	9836.590	0	0	0.9764
1	10316.518	0	0	0	0.5000

$\alpha = 0.60\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	9373.477	0	1.4305
0.5	0	9821.278	0	0	0.9764
1	10299.744	0	0	0	0.5000

Notes: (a) The α value represents the level of transaction costs as a percentage of the volume traded; $\alpha = 0$ means the absence of transaction costs. In this case the optimal strategy is consistent with Khang's theorem, i.e., at each rebalancing point the portfolio has to be restructured in order to keep its duration equal to the remaining horizon planning period; (b) Equation (14) is used for duration in this table.

Table 7
Optimal Portfolio Path
Under the Vasicek Model Using U.K. Data

Panel A: No Transactions Costs ($\alpha = 0.00\%$)

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	7966.707	1497.853	1.4999
0.5	0	9426.207	511.094	0	0.9999
1	10433.992	0	0	0	0.5000

Panel B: With Transactions Costs

$\alpha = 0.15\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	8960.509	505.285	1.4542
0.5	0	9416.569	505.285	0	0.9998
1	10415.415	0	0	0	0.5000

$\alpha = 0.30\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	8947.112	504.493	1.4542
0.5	0	9401.809	504.493	0	0.9998
1	9886.797	504.493	0	0	0.5233

$\alpha = 0.45\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	8933.754	503.703	1.4542
0.5	0	9387.090	503.703	0	0.9998
1	9870.596	503.703	0	0	0.5233

$\alpha = 0.60\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	9431.205	0	1.4308
0.5	0	9882.670	0	0	0.9764
1	10365.058	0	0	0	0.5000

Notes: (a) The α value represents the level of transaction costs as a percentage of the volume traded; $\alpha = 0$ means the absence of transaction costs. In this case the optimal strategy is consistent with Khang's theorem, i.e., at each rebalancing point the portfolio has to be restructured in order to keep its duration equal to the remaining horizon planning period; (b) Equation (14) is used for duration in this table.

Table 8
Optimal Portfolio Path
Under the Cox-Ingersoll-Ross Model Using U.S. Data

Panel A: No Transactions Costs ($\alpha = 0.00\%$)

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	7916.720	1485.324	1.4999
0.5	0	9364.264	507.666	0	1.0000
1	10365.467	0	0	0	0.5000

Panel B: With Transactions Costs

$\alpha = 0.15\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	8910.619	494.905	1.4539
0.5	0	9361.767	494.905	0	0.9995
1	10346.979	0	0	0	0.5000

$\alpha = 0.30\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	8897.360	494.133	1.4539
0.5	0	9347.161	494.133	0	0.9995
1	9827.833	494.133	0	0	0.5230

$\alpha = 0.45\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	8884.090	493.360	1.4539
0.5	0	9332.549	493.360	0	0.9995
1	9811.754	493.360	0	0	0.5230

$\alpha = 0.60\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	9372.248	0	1.4309
0.5	0	9819.790	0	0	0.9764
1	10297.982	0	0	0	0.5000

Notes: (a) The α value represents the level of transaction costs as a percentage of the volume traded; $\alpha = 0$ means the absence of transaction costs. In this case the optimal strategy is consistent with Khang's theorem, i.e., at each rebalancing point the portfolio has to be restructured in order to keep its duration equal to the remaining horizon planning period; (b) Equation (15) is used for duration in this table.

Table 9
Optimal Portfolio Path
Under the Cox-Ingersoll-Ross Model Using U.K. Data

Panel A: No Transactions Costs ($\alpha = 0.00\%$)

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	7956.605	1506.982	1.5005
0.5	0	9427.115	509.365	0	0.9999
1	10433.109	0	0	0	0.5000

Panel B: With Transactions Costs

$\alpha = 0.15\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	8959.023	505.935	1.4542
0.5	0	9414.835	505.935	0	0.9998
1	9901.047	505.935	0	0	0.5234

$\alpha = 0.30\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	8958.877	492.139	1.4537
0.5	0	9413.339	492.139	0	0.9992
1	9898.075	492.139	0	0	0.5228

$\alpha = 0.45\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	8945.533	491.371	1.4537
0.5	0	9398.639	491.371	0	0.9992
1	9881.896	491.371	0	0	0.5228

$\alpha = 0.60\%$

s	$x(s,1)$	$x(s,2)$	$x(s,3)$	$x(s,4)$	Duration
0	0	0	9430.492	0	1.4309
0.5	0	9882.616	0	0	0.9764
1	10364.794	0	0	0	0.5000

Notes: (a) The α value represents the level of transaction costs as a percentage of the volume traded; $\alpha = 0$ means the absence of transaction costs. In this case the optimal strategy is consistent with Khang's theorem, i.e., at each rebalancing point the portfolio has to be restructured in order to keep its duration equal to the remaining horizon planning period; (b) Equation (15) is used for duration in this table.

The budget constraints must be modified as follows:

$$\sum_{i=1}^{n}(1+\alpha)x(0,i)p(0,i) = I$$

(i) $x(0,i) = b(0,i)$ for $i = 1,\ldots,n$;

(ii) $x(s,i) = x(s-1,i) + b(s,i) - z(s,i) - y(s,i)$
for $s = 1,\ldots,k$ and $i = 1,\ldots,n$;

(iii) $x(s,i) = b(s,i)$ for $s = s_i, s_i + 1,\ldots,k$ and $i = 1,\ldots,n$;

(iv) $z(s,i) = 0$ for $s = s_i,\ldots,k$ and $i = 1,\ldots,\rho$;

(v) $y(s,i) = 0$ for $s \neq s_i,\ldots,k$ and $i = 1,\ldots,\rho$;
or for $s = 1,\ldots,k$ and $i = \rho + 1,\ldots,n$;

(vi) $\sum_{i=1}^{n}[(1+\alpha)b(s,i)p(s,i) - (1-\alpha)z(s,i)p(s,i)$
$- y(s,i)p(s,i) - C_i x(s-1,i)] = 0$ for $s = 1,\ldots,k-1$

(vii) $\sum_{i=1}^{n}[((1-\alpha)z(k,i) + y(k,i))p(k,i) + C_i x(k-1,i)] = V_k$

(viii) $x(s,i), b(s,i), z(s,i), y(s,i), V_s \geq 0$
for $s = 1,\ldots,k$ and $i = 1,\ldots,n$.

Transaction costs have the effect of increasing asset purchase prices by α while reducing sale prices by α. This new purchase (sale) price can be understood as the bid (ask) price of the bonds plus (minus) fees paid to intermediaries.

5.1   The Example Continued

The dynamic model with transactions costs is applied to the example, and the results are presented in Panel B of Tables 5 to 9 for different α values (0.15 percent, 0.3 percent, 0.45 percent, and 0.6 percent).

The optimal path depends on the level of the transaction costs, i.e., on the level of α. For $\alpha = 0$ we reach Khang' optimal solution: at each rebalancing point portfolio duration must be equal to the remaining horizon planning period. But for values of α greater than 0.05 percent the optimal path has an initial portfolio that is not immunized because its duration is less than the horizon planning period. The fact that values of α as low as 0.05 percent lead to a non-immunized portfolio implies that the immunization strategy cannot be optimal, in practical terms.

The difference between the initial portfolio duration and the horizon planning period increases as the level of transaction costs rises. In this simple example four different solutions are obtained; the initial portfolio durations range from 1.5 years (for $\alpha = 0$) to approximately 1.43 years (for $\alpha = 0.6$ percent). For sufficiently high transaction costs, the optimal solution is to invest the entire initial budget in a bond with

maturity at the end of the horizon planning period. This is because by investing in bonds with maturity at the end of the horizon planning period, we avoid the transaction costs generated by the reinvestment of those bonds with maturity before the end of the horizon planning period as well as the losses derived from the sales of those bonds still outstanding at the end of the horizon planning period. In the example this is the result we get when $\alpha = 0.6$ percent. To avoid transaction costs the optimal strategy consists of investing in bond 3, i.e., the bond with maturity at $t_3 = H = 1.5$ years

A possible explanation of why the optimal strategy consists of portfolios with an initial duration less than the horizon planning period[14] is that if no cash payment occurs, portfolio duration is equal to the remaining horizon planning period. But if a coupon payment occurs, portfolio duration is increased a finite amount and becomes greater than the horizon planning period.

If portfolio duration is long enough, this problem may be solved by reinvesting coupon payments in bonds with a short duration. If the horizon planning period is short, it will not be possible to keep duration equal to the horizon planning period unless we sell bonds with long durations and invest the proceeds in bonds with shorter duration.

The optimal solution of this model provides an initial portfolio with a duration less than the horizon planning period. As coupon payments are due, its duration increases approaching the horizon planning period without any additional rebalancing, thereby avoiding transaction costs. Also, this fact can be helped by an optimal reinvestment of coupon payments.

These findings are common to all cases analyzed, i.e., they are independent of the term structure of interest rate model assumed. These models provide a first hint to answer the question posed by Maloney and Logue (1989) with respect to the "mismatch duration that is tolerable, given that allowing a modest mismatch will certainly reduce trading costs."

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[14]At $t = 1$ all optimal portfolios have a duration greater or equal to the remaining horizon planning period. This is caused by the characteristics of the set of bonds considered in this counterexample. At $t = 1$ all the bonds have a duration greater than 0.5. If we had included bonds with a duration at $t = 1$ less than 0.5 (i.e., bonds with quarterly coupon payments) this result could not hold.

6 Summary

This paper develops a dynamic portfolio selection model for interest rate risk management under different term structure of interest rate regimes. This model's results are consistent with Khang's dynamic immunization strategy which consists of a continuous rebalancing to keep portfolio duration equal to the investor's horizon planning period.

The model is then extended in order to analyze the effects of transaction costs on the optimal strategy. Our results suggest that if transaction costs are considered, the strategy of making portfolio duration equal to the horizon planning period is not optimal. Moreover, the optimal path has an initial solution with a portfolio duration less than the horizon planning period. Furthermore, the bigger the level of transaction costs, the bigger the difference between the initial portfolio duration and the horizon planning period. This result holds under different term structure of interest rate models.

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A New Approach for Determining Claim Expense Reserves in Workers Compensation

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Abstract†

This paper describes a new approach for determining a reserve for claim expenses. While the discussion focuses on workers compensation claims, the methodology is equally applicable to other lines of business. The approach also can be applied to the calculation of the reserve for all claims (including IBNR claims) and the reserve for claims reported to date (excluding IBNR claims). In addition, a methodology for pricing claims-handling services is discussed. The implications of pricing claims-handling services on a handle-to-conclusion basis versus pricing claims-handling services on a limited time handling basis are examined.

Finally, the paper discusses a methodology for tracking the duration so that the rate of claim closing can be monitored. This, in turn, allows targets to be set. Departments that are interested in implementing new techniques for shortening the duration can use the monitoring techniques to determine if their new claim-closing techniques are successful.

Key words and phrases: *closed claims, open claims, duration*

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1   Introduction

The determination of a claim expense reserve is an important issue for workers compensation because of the length of time for which workers compensation claims remain open. The duration has been increasing over the last several years. As duration increases, so does the expense of handling the claim for the remainder of the claim's life.

Self-insurance and large deductible plans are now common means of financing risk. Few self-insureds handle their own claims, however. Risk managers are increasingly aware of the expense of handling claims. As insurance companies and third party administrators (TPAs) are under tremendous pressure to cut expenses, the need to know the total cost of handling claims becomes more important. Companies that are able to estimate their cost of handling claims will be more successful in reducing costs.

There are several ways to estimate claim expense reserves, including the use of automated work measurement and paid-to-paid ratios. Automated work measurement[1] studies show that there are differing levels of work effort necessary for claims in the first 30 days than on claims that have been open for, say, five years. On the other hand, the paid-to-paid methodology assumes that claims incur expense only when initially opened and when closed. While this may not be an unreasonable assumption for claims from short-tailed lines, this assumption is not true for liability claims. Moreover, the paid-to-paid ratio is subject to distortion when a company is growing or shrinking or when a line of business is in transition.[2]

Throughout the rest of the paper, I will describe a methodology for setting a reserve for claim expenses. The method is straightforward and it opens the door to several related issues: specifically, a claim department's monitoring of closing claims and the pricing of claims service. Although this methodology is applicable to any line of business, the discussion and the examples that follow focus on workers compensation lost time claims.

My methodology shares many features with the methodology documented in Wendy Johnson's 1989 paper,[3] e.g., both use claim reporting and claim closure patterns to calculate the reserve. The differences in-

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[1]Automated work measurement, also known as time and motion studies, may be used to determine the key drivers in the cost of handling claims.

[2]This was the case for workers compensation throughout the early 1990s as many large customers moved to deductible policies or toward self-insurance.

[3]Johnson, W. "Determination of Outstanding Liabilities for Unallocated Loss Adjustment Expense"*Procedings of the Casualty Actuarial Society* 76 (1989): 111-125.

clude differing assumptions/applications of the expense loads for claim costs. Also, my paper broadens the reserving concepts to pricing and also to the monitoring of claim department efficiency.

2 Some Key Definitions

The following definitions are provided for the convenience of the reader:

Allocated loss adjustment expense (ALAE): Expenses associated with settling a claim that are allocable to a specific claim, e.g., attorneys' fees, investigative fees, independent medical examinations, many managed care expenses, and court and other legal fees;

Created claims: Claims reported to an insurance company or third party administrator. Also known as reported claims;

Duration: The amount of time that a claim remains open. Also known as the life of the claim;

Handle-to-conclusion: A term used by third party administrators to denote claims service that will continue for as long as the claim remains open. The fee charged for handle-to-conclusion, unless otherwise stated, also covers the handling of any reopened claims for as long as they remain (re-)opened;

Intake expense: The cost of setting up a newly created claim in the system;

Limited time handling: A term used by third party administrators to signify claims service for some specified time limit, after which time an additional fee will be charged for the continued handling of the claim;

Outstanding fee: The expense of handling a claim for as long as it remains open. This could be expressed in various ways, e.g., as a fee per month or a quarterly fee;

Reported claims: Claims of which the insurance company or third party administrator has been made aware. Also known as created claims;

Third party administrator (TPA): A company that is in the business of handling and servicing claims. Such a company may also provide services other than claims services such as loss control, risk management information systems, actuarial services, etc. A TPA may

be affiliated with an insurance carrier or operate as a stand-alone entity;

Unallocated loss adjustment expense (ULAE): Expenses associated with settling claims but not allocable to a specific claim, e.g., claim adjusters salaries, heat, light, rent, etc.

3   The New Reserve Methodology

The basic steps of the new reserve methodology are as follows:

Step 1 Construct the closed claim count and created claim[4] count triangles. Ideally, these triangles should have quarterly evaluations; also the created claim counts and the closed claim counts will be net of both canceled claims and claims closed with no loss or ALAE payment. For the sake of brevity, the example presented here is based on annual data; see Table 1.

Either accident year, report year, or policy year triangles may be used, but I prefer the report year version because the accompanying statistics are more useful. Report year triangles result in a ULAE reserve that makes no provision for IBNR claims. Later in the paper I will discuss some of these statistics, e.g., the number of months claims will remain open.

Step 2 Calculate loss development factors (LDF);

Step 3 Use the LDFs to project ultimate claims: Because the example uses report year claims, the ultimate number of claims is identical to the claims reported after twelve months. The number of report year claims could change after the end of the report year, however, due to re-openings, the re-assignment of initially medical only claims to lost time claims (and vice versa), and the removal of canceled or claims closed with no loss or ALAE payment claims. With accident year data, one can use either closed claims, created claims, or a combination of these to project the ultimate number of claims.

Step 4 Calculate the projected open claims: There are at least two methods for calculating the projected open claims. The first is to fill in the bottom of each of the created and closed triangles, i.e., use the LDFs from the first step to estimate the future created

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[4]Created claims are reported claims.

claims and use a similar procedure to estimate the future closed claims. Taking the difference of the projected created and the projected closed claims provides the projected open claims. In my experience, this can lead to some unreasonable results, e.g., more than 10 percent of claims remaining open after ten years for a line where this is not reasonable, which makes additional re-selection of the LDFs necessary.

My preferred method for projecting the open claims is to calculate another triangle which is the ratio of the (actual) open claims to the ultimate claims. By selecting the percentage of open claims at each evaluation and then applying this percentage to the ultimate number of claims for each year, one derives the projected number of open claims. This is illustrated in Tables 2 and 3.

Step 5 Estimate the average number of in-force claims during a year. One way of estimating the number of in-force claims during a year is to average the number of open claims at the beginning and end of a year as shown in Table 4.

Step 6 Calculate the reserve for each year by multiplying the number of open claims by the outstanding cost per claim: Multiplying the number of in-force claims in each year (Table 4) by the outstanding cost per claim per year (Table 5) gives the cost of handling claims in that particular year. This calculation produces the incremental cost per year as shown in Table 6. Summing all of these costs after a particular point in time, e.g., as of 12 months, results in the reserve for claim expenses as of 12 months (only for claims open through ten years); see Table 7.

4 An Example

This example assumes that the outstanding claim expense per year is $600 in 1995 dollars.[5] Future expenses are assumed to increase at 4 percent per year. The nominal value of the reserve can be calculated by using $600 consistently for as long as claims are expected to remain open.

One way of determining the outstanding cost per claim is an automated work measurement study within the claim department. Such a study would determine standards to complete various tasks rather

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[5]This is not a true standard that will apply to any company nor should it be construed to be my company's standard.

than dollar amounts because many costs are inflation-sensitive. For example, one may determine that a typical workers compensation claim requires fifteen hours to settle (which could be translated into a cost using the most current hourly rates) rather than saying its ultimate handling cost is $600.

The reserve calculated in Table 7 covers only the expense in the first ten years the claims are open because the triangles used in the example end at ten years. Because there are claims remaining open after ten years and there will likely be claims open for as many as 40 years (or more), the reserve must be adjusted to account for the claims open after ten years.

The assumption to be used in calculating this tail reserve is that any workers compensation claim still open after ten years is a tabular claim for which benefits will be paid for the claimant's or the survivor's lifetime. Ten years is used in this example only; it is not meant to be a standard. For example, if one has data through 15 or 20 years, one could make the same assumption at 15 or 20 years.

One can obtain historical information about the age of the claimant or survivor ten years after the claim is reported (for report year statistics) or ten years after the claim occurs (for accident year statistics). Additionally, an assumption must be made about the average age at death to determine how many years the claims will remain open. Refinements to this methodology are possible, e.g., one can apply mortality tables to each claim open after ten years.

We will assume that claims open for ten years will remain open, on average, for an additional 25 years. The tail reserve is the product of the number of claims open after ten years multiplied by 25 times the annual cost of handling the claim. The tail reserve calculated in this manner is sensitive to the number of years used in the calculation. The significant dollar amounts produced by this methodology beg the question "Will it really cost this much to handle tabular claims?"

While tabular claims incur expense, these claims are generally less expensive to handle than newer claims. The work typically involved in maintaining an open tabular claim is an annual or semi-annual review of the reserve and the mail delivery of a monthly or weekly check (typically an automated process). Discussions with my claim department indicate that tabular claims incur roughly one-third of the expense of a newer claim. This may differ from company to company; this also will differ for cases involving ongoing intensive medical treatment.

Table 1

Created Claims

Year	12	24	36	48	60	72	84	96	108	120	Ultimate	Tail Reserve
1986	101,909	101,909	101,909	101,909	101,909	101,909	101,909	101,909	101,909	101,909	101,909	17,653,849
1987	96,869	96,869	96,869	96,869	96,869	96,869	96,869	96,869	96,869		96,869	17,450,111
1988	102,346	102,346	102,346	102,346	102,346	102,346	102,346	102,346			102,346	19,150,355
1989	107,315	107,315	107,315	107,315	107,315	107,315	107,315				107,315	20,820,107
1990	111,029	111,029	111,029	111,029	111,029	111,029					111,029	22,425,567
1991	107,345	107,345	107,345	107,345	107,345						107,345	22,565,617
1992	113,367	113,367	113,367	113,367							113,367	24,743,281
1993	107,084	107,084	107,084								107,084	24,368,548
1994	107,687	107,687									107,687	25,500,196
1995	104,446										104,446	25,695,792

Table 2

Ratio of Open to Ultimate Claims

Year	12	24	36	48	60	72	84	96	108	120
1986	0.4581	0.2382	0.1364	0.0903	0.0585	0.0440	0.0346	0.0267	0.0242	0.0200
1987	0.4677	0.1991	0.1130	0.0655	0.0455	0.0355	0.0273	0.0229	0.0180	
1988	0.4567	0.2122	0.1184	0.0608	0.0576	0.0434	0.0318	0.0233		
1989	0.4647	0.2083	0.1284	0.0823	0.0589	0.0424	0.0323			
1990	0.4607	0.2284	0.1412	0.0971	0.0703	0.0509				
1991	0.4631	0.2294	0.1452	0.0990	0.0675					
1992	0.4669	0 2500	0.1594	0.0990						
1993	0.4616	0.2181	0.1359							
1994	0.4643	0.2476								
Average	0.4626	0.2257	0.1347	0.0877	0.0597	0.0432	0.0315	0.0243	0.0211	0.0200
Selected	0.4626	0.2257	0.1347	0.0877	0.0597	0.0432	0.0315	0.0243	0.0211	0.0200

Table 3

Actual and Estimated Open Claims

Year	12	24	36	48	60	72	84	96	108	120
1986	46,680	24,271	13,903	9,203	5,960	4,479	3,525	2,726	2,468	2,038
1987	45,306	19,287	10,949	6,341	4,410	3,438	2,646	2,219	1,742	1,937
1988	46,745	21,720	12,115	8,265	5,900	4,445	3,255	2,385	2,160	2,047
1989	49,865	22,355	13,775	8,835	6,320	4,550	3,470	2,608	2,264	2,146
1990	51,152	25,355	15,678	10,780	7,810	5,647	3,497	2,698	2,343	2,221
1991	49,710	24,630	15,585	10,630	7,245	4,637	3,381	2,608	2,265	2,147
1992	52,935	28,340	18,070	11,225	6,768	4,897	3,571	2,755	2,392	2,267
1993	49,428	23,352	14,554	9,391	6,393	4,626	3,373	2,602	2,259	2,142
1994	50,002	26,668	14,505	9,444	6,429	4,652	3,392	2,617	2,272	2,154
1995	49,155	23,573	14,069	9,160	6,235	4,512	3,290	2,538	2,204	2,089

Notes: (1) Numbers above the jagged line are actual data, while numbers below are estimates, and; (2) For example, for year 1995 at 24 months: $23,573 = 0.2257 \times 104,446$, where 0.2257 is the selected open ratio and 104,446 is the estimate of ultimate claims for 1995.

Table 4

Average Open Claims

Year	12	24	36	48	60	72	84	96	108	120
1986	23,340	35,476	19,087	11,553	7,582	5,220	4,002	3,126	2,597	2,253
1987	22,653	32,297	15,118	8,645	5,376	3,924	3,042	2,433	1,981	1,840
1988	23,373	34,233	16,918	10,190	7,083	5,173	3,850	2,820	2,273	2,104
1989	24,933	36,110	18,065	11,305	7,578	5,435	4,010	3,039	2,436	2,205
1990	25,576	38,254	20,517	13,229	9,295	6,729	4,572	3,098	2,521	2,282
1991	24,855	37,170	20,108	13,108	8,938	5,941	4,009	2,995	2,437	2,206
1992	26,468	40,638	23,205	14,648	8,997	5,833	4,234	3,163	2,574	2,330
1993	24,714	36,390	18,953	11,973	7,892	5,510	4,000	2,988	2,431	2,201
1994	25,001	38,335	20,587	11,975	7,937	5,541	4,022	3,005	2,445	2,213
1995	24,578	36,364	18,821	11,615	7,698	5,374	3,901	2,914	2,371	2,147

Notes: Average Open Claims = Average number of claims at the beginning and the end of the year.

Table 5

Cost Per Open Claim

Year	12	24	36	48	60	72	84	96	108	120
1986	422	438	456	474	493	513	533	555	577	600
1987	438	456	474	493	513	533	555	577	600	624
1988	456	474	493	513	533	555	577	600	624	649
1989	474	493	513	533	555	577	600	624	649	675
1990	493	513	533	555	577	600	624	649	675	702
1991	513	533	555	577	600	624	649	675	702	730
1992	533	555	577	600	624	649	675	702	730	759
1993	555	577	600	624	649	675	702	730	759	790
1994	577	600	624	649	675	702	730	759	790	821
1995	600	624	649	675	702	730	759	790	821	854

Notes: Cost per open claim is assumed to be $600 per year in 1995 dollars. Prior and subsequent expenses are derived assuming 4 percent inflation.

Table 6

Incremental Cost Per Year

Year	12	24	36	48	60	72	84	96	108	120
1986	9,839,025	15,552,960	8,702,731	5,478,302	3,738,864	2,676,990	2,134,658	1,733,820	1,498,269	1,351,800
1987	9,931,395	14,725,611	7,168,785	4,263,336	2,757,000	2,093,053	1,687,500	1,403,365	1,188,300	1,147,848
1988	10,656,707	16,232,665	8,342,971	5,226,273	3,777,790	2,869,360	2,221,154	1,692,000	1,418,040	1,365,087
1989	11,822,710	17,807,873	9,265,223	6,030,062	4,203,495	3,135,577	2,406,000	1,896,336	1,580,867	1,488,195
1990	12,612,965	19,619,551	10,943,456	7,338,572	5,362,500	4,037,100	2,852,928	2,010,154	1,701,132	1,601,770
1991	12,747,695	19,826,397	11,154,308	7,562,019	5,362,500	3,707,184	2,601,681	2,021,043	1,710,216	1,610,362
1992	14,117,707	22,542,992	13,387,500	8,788,500	5,613,816	3,785,059	2,857,605	2,220,158	1,878,634	1,768,536
1993	13,709,689	20,994,231	11,371,800	7,470,840	5,121,592	3,718,463	2,807,310	2,180,850	1,845,215	1,737,425
1994	14,423,654	23,001,000	12,845,976	7,770,972	5,356,490	3,888,961	2,936,027	2,280,991	1,930,077	1,817,186
1995	14,746,800	22,691,136	12,214,076	7,838,840	5,402,992	3,922,611	2,961,606	2,300,775	1,946,926	1,833,083

Notes: For example, for year 1995, expenses as of 12 months is equal to $14,746,800 = 24,578 \times 600$, where 24,578 is the number of average open claims for 1995 as of 12 months and 600 is the estimated cost per open claim (as shown on Table 5).

Table 7
Tail Reserve by Report Year
Workers Compensation Lost Time Claims

Report Year	(1)	(2)	(3)	(4)
1986	200	43.3117	2,038	$17,653,849
1987	208	43.3117	1,937	$17,450,111
1988	216	43.3117	2,047	$19,150,355
1989	224	43.3117	2,146	$20,820,107
1990	232	43.3117	2,221	$22,445,567
1991	244	43.3117	2,147	$22,565,627
1992	252	43.3117	2,267	$24,743,281
1993	264	43.3117	2,142	$24,368,548
1994	272	43.3117	2,154	$25,500,196
1995	284	43.3117	2,089	$25,695,792

Col. (1) = Estimated annual claim expense after 10 years and is one-third of the expense of handling newer claims. Col. (2) = Inflationary factor for 25 years, and is $\sum_1^{25} 1.04^k$. Col. (3) = Projected number of claims open after ten years and is taken from Table 2. Col. (4) = Cols. (1) × (2) × (3).

The tail reserve is estimated as the number of claims open after ten years multiplied by the outstanding expense per year multiplied by the number of years the claim is expected to remain open. In this example, we assume claims open after ten years will remain open, on average, for an additional 25 years. Note that the resulting tail reserve is sensitive to the number of years used. For example, for report year 1986:

$$\text{Tail Reserve} \quad = \quad 2,038 \times \$600 \times 43.3117$$
$$= \quad \$52,961,547.$$

As discussed in the paper, the tail or tabular claims incur roughly one-third the expense of a newer claim. Then the tail reserve for report year 1986 would be $17,653,849. Similarly, the tail reserve for other report years may be calculated.

The tail reserve for each report year is calculated as shown above. In Table 8 this tail reserve is shown for each report year after 120 months. The total reserve is calculated by summing the cost per quarter after a particular quarter. The reserve for older report years (or accident years) may be calculated using the procedure described above. See Tables 8, 9, and 10.

5   Duration

We have presented a methodology for calculating the total reserve, which is the sum of the expenses in handling claims in the first ten years and the tail reserve for the tabular claims. The example is based on report year data. If this methodology is used with accident or policy year data, the reserve will be for all claims, whether reported or not. For a company that does not wish to hold reserves for incurred but not reported (IBNR) claims or for claims that are not yet incurred, a variation of this methodology is necessary.

The concept of duration is introduced to illustrate the calculation of a reserve per claim. Duration is the average life of a claim or the length of time, on average, that a claim remains open. Duration has a different and distinct meaning in the financial community from that offered here. Because a claim incurs expense for as long as it remains open, duration is a key factor in calculating both the reserve and the cost of handling of a claim.

One way of computing the duration of a claim involves counting the number of days between the date of report and the date of closure using many years. This method of computing the duration may understate a company's duration if the claims system began in (for example) 1970 or if the company has not been writing workers compensation claims since the early 1900s. (It is not uncommon for workers compensation claims to remain open for 50 years or more). Even for a company writing business for many years, the duration may be misstated if the volume has changed significantly over time or if the nature of claims has changed.

Another way of estimating duration is to use triangles of claim count data. For each report year, one takes the weighted average over time of the incremental closed claims in each quarter as well as the weighted average over time of the incremental reported claims in each quarter. The difference of the closed weighted average and the created weighted average gives an estimate of the duration for each report year.

A company with only 20 years of workers compensation experience could compute the truncated duration of the first 20 years worth of claims and then make the assumption that claims still open after 20 years are tabular claims. One could estimate the length of time the tabular claims will remain open using annuity tables or use a method similar to that illustrated above for the tail reserve. The total duration could then be calculated using a simple weighted average.

Table 8

Reserve as of Year

Year	12	24	36	48	60	72	84	96	108	120
1986	42,868,395	27,315,435	18,612,704	13,134,402	9,395,538	6,718,548	4,583,889	2,850,069	1,351,800	0
1987	36,434,799	21,709,187	14,540,402	10,277,066	7,520,066	5,427,013	3,739,513	2,336,148	1,147,848	0
1988	43,145,341	26,912,675	18,569,704	13,343,431	9,565,641	6,696,281	4,475,127	2,783,127	1,365,087	0
1989	47,813,627	30,005,754	20,740,532	14,710,469	10,506,975	7,371,398	4,965,398	3,069,062	1,488,195	0
1990	55,467,164	35,847,613	24,904,156	17,565,584	12,203,084	8,165,984	5,313,056	3,302,902	1,601,770	0
1991	55,555,710	35,729,313	24,575,005	17,012,986	11,650,486	7,943,302	5,341,621	3,320,578	1,610,362	0
1992	62,842,799	40,299,807	26,912,307	18,123,807	12,509,991	8,724,932	5,867,328	3,647,170	1,768,536	0
1993	57,247,725	36,253,495	24,881,695	17,410,855	12,289,262	8,570,799	5,763,490	3,582,639	1,737,425	0
1994	61,827,679	38,826,679	25,980,703	18,209,731	12,853,241	8,964,281	6,028,254	3,747,263	1,817,186	0
1995	61,112,045	38,420,909	26,206,833	18,367,993	12,965,001	9,042,390	6,080,785	3,780,010	1,833,083	0

Notes: This reserve calculated in this example only covers claim expenses through the first ten years. For example, for year 1995, the reserve as of 36 months is $26,206,833 = 7,838,840 + 5,402,992 + 3,922,611 + \ldots$, which is the sum of the incremental cost per year for each year after 36 months.

Table 9

Incremental Cost Per Year

Year	12	24	36	48	60	72	84	96	108	120	Reserve
1986	9,839,025	15,552,960	8,702,731	5,478,302	3,738,864	2,676,990	2,134,658	1,733,820	1,498,269	1,351,800	17,653,849
1987	9,931,395	14,725,611	7,168,785	4,263,336	2,757,000	2,093,053	1,687,500	1,403,365	1,188,300	1,147,848	17,450,111
1988	10,656,707	16,232,665	8,342,971	5,226,273	3,777,790	2,869,360	2,221,154	1,692,000	1,418,040	1,365,087	19,150,355
1989	11,822,710	17,807,873	9,265,223	6,030,062	4,203,495	3,135,577	2,406,000	1,896,336	1,580,867	1,488,195	20,820,107
1990	12,612,965	19,619,551	10,943,456	7,338,572	5,362,500	4,037,100	2,852,928	2,010,154	1,701,132	1,601,770	22,445,567
1991	12,747,695	19,826,397	11,154,308	7,562,019	5,362,500	3,707,184	2,601,681	2,021,043	1,710,216	1,610,362	22,565,627
1992	14,117,707	22,542,992	13,387,500	8,788,500	5,613,816	3,785,059	2,857,605	2,220,158	1,878,634	1,768,536	24,743,281
1993	13,709,689	20,994,231	11,371,800	7,470,840	5,121,592	3,718,463	2,807,310	2,180,850	1,845,215	1,737,425	24,368,548
1994	14,423,654	23,001,000	12,845,976	7,770,972	5,356,490	3,888,961	2,936,027	2,280,991	1,930,077	1,817,186	25,500,196
1995	14,746,800	22,691,136	12,214,076	7,838,840	5,402,992	3,922,611	2,961,606	2,300,775	1,946,926	1,833,083	25,695,792

Notes: For example, for year 1995, expenses as of 12 months is $14,746,800 = 24,578x600$, where 24,578 is the number of average open claims for 1995 as of 12 months and 600 is the estimated cost per open claim (as shown on Table 5).

Table 10

Reserve as of Year

Year	12	24	36	48	60	72	84	96	108	120
1986	60,522,244	44,969,284	36,266,553	30,788,251	27,049,387	24,372,397	22,237,738	20,503,918	19,005,649	17,653,849
1987	53,884,910	39,159,298	31,990,513	27,727,177	24,970,177	22,877,124	21,189,624	19,786,259	18,597,959	17,450,111
1988	62,295,696	46,063,030	37,720,059	32,493,786	28,715,996	25,846,636	23,625,482	21,933,482	20,515,442	19,150,355
1989	68,633,734	50,825,861	41,560,639	35,530,576	31,327,082	28,191,505	25,785,505	23,889,169	22,308,302	20,820,107
1990	77,912,731	58,293,180	47,349,723	40,011,151	34,648,651	30,611,551	27,758,623	25,748,469	24,047,337	22,445,567
1991	78,121,337	58,294,940	47,140,632	39,578,613	34,216,113	30,508,929	27,907,248	25,886,205	24,175,989	22,565,627
1992	87,586,080	65,043,088	51,655,588	42,867,088	37,253,272	33,468,213	30,610,609	28,390,451	26,511,817	24,743,281
1993	81,616,273	60,622,043	49,250,243	41,779,403	36,657,810	?2,939,347	30,132,038	27,951,187	26,105,973	24,368,548
1994	87,327,875	64,326,875	51,480,899	43,709,927	38,353,437	34,464,477	31,528,450	29,247,459	27,317,382	25,500,196
1995	86,807,837	64,116,701	51,902,625	44,063,785	38,660,793	34,738,182	31,776,577	29,475,802	27,528,875	25,695,792

Notes: For example, for year 1995, the reserve as of 36 months is $51,902,625 = 7,838,840 + 5,402,992 + 3,922,611 + \ldots$, which is the sum of the incremental cost per year for each year after 36 months.

As an example, assume the duration of the report year 1977 closed claims as of December 31, 1996 is 12.6 months and that 99.5 percent of report year 1977 claims are closed. The remaining 0.5 percent of claims are open and are expected to remain open for an additional 21 years. The total duration would be 15 months.[6]

Duration differs by state because of the different laws in each state for workers compensation benefits. For example, the duration of the permanent total claims in the ten states in the 1994 National Council on Compensation Insurance (NCCI) Closed Claims Studies[7] ranged from 21.3 months (South Carolina) to 50.2 months (Wisconsin). Industry data from these 1994 NCCI studies show increasing durations for all of the ten states in the study. This study measures the duration in median number of days for permanent disability claims through closure year 1992. It seems likely that managed care will have some impact on decreasing the overall claim duration, but it is too soon to determine the validity of this hypothesis.

We assume that the countrywide duration for a workers compensation lost time (WCLT) claim is 15 months, the cost per month of handling a claim is $50, and there is no inflation. Every reported claim needs a reserve of $750 (= 15 × $50) set aside. Therefore, the reserve at any point in time would be: Number of Created Claims × $750 − Reserve Released for Open Claims. This concept is probably easier to illustrate than to explain.

Assume that one claim is reported at the beginning of each quarter and that the number of open claims at the end of each quarter is as shown below. Also assume for simplicity that claims close at the end of the quarter.

In the example above, the reserve is increased $750 whenever a claim is reported and the reserve is drawn down $50 every month a claim is open. So each quarter the reserve is computed as the reserve at the beginning of the quarter plus the addition to the reserve (from newly-reported claims) minus the claim expenses incurred during the quarter.

In the example above, the assumption is made that claim expense is incurred if the claim is open at the end of the month. Because one claim was closed before the end of the first month of the quarter in the fourth quarter, no money is released from the reserve for this claim. In this way, the money set aside for claims that close early (before 15 months) is there for the claims that remain open late (after 15 months).

---

[6]Duration = 0.995 × 12.6 + 0.005 × (21 + 19.5) × 12 = 15 months.

[7]Hartwig, R.P., Kahley, W.J. and Retrepo, T.E. "Workers Compensation Loss Ratios and the Business Cycle." *NCCI Digest* 9, no. 2 (December 1994): 1-13.

Table 11
Quarterly Reserve Calculations

	(1)	(2)	(3)	(4)	(5)
Q1	1	1	$750	$150	$600
Q2	1	2	$750	$300	$1,050
Q3	1	3	$750	$450	$1,350
Q4	1	3	$750	$450	$1,650

Q1 = First Quarter, etc.; Col. (1) = Number of Reported Claims; Col. (2) = Number of Open Claims (at the end of each month of the quarter; Col. (3) = Addition to the Claim Reserve; Col. (4) = Subtraction from Claim Reserve; and Col. (5) = Reserve at the End of the Quarter.

6 Pricing Claims Service

The concept of duration is used to compute the reserve per claim, which can easily be modified to derive the price of handling a claim. For many customers today and for virtually all national accounts customers, claims service is an unbundled, separately negotiated piece of the risk-financing program.

The methodology described here is only for the basic claim expenses. The total cost of adjusting claims is the sum of basic unallocated and the sundry allocated types of loss adjustment expenses such as legal expenses, managed care expenses, nurse case managers, etc.

In the examples presented thus far, we have assumed that claims incur uniform expenses each month for the first ten years. Discussions with my claim department indicate that this is an overly simplistic assumption. Rather, a claim generally incurs the most expense during the first month in which it is open, during which time the file must be set up, various phone calls must be made, investigative work is necessary, etc. Therefore, the expense incurred by a claim may better be modeled by assuming an intake expense and then several months of outstanding expense for as long as the claim is open. One could also incorporate a closing expense for the cost necessary in closing a claim.

A further refinement in modeling the claim expense would be to differentiate outstanding expenses. Again, the idea is that the first few months a claim is open are more labor-intensive than are later months. Thus, there may be discriminatory standards for outstanding expenses.

The cost of handling a claim (excluding ALAE) would be:

$$\text{Cost} = \text{Intake Expense} + (OS_1 \times x) + (OS_2 \times (\text{Duration} - 1 - x)),$$

where x is the number of months early in the claim's life when the claim is more expensive, OS_1 is the higher cost of handling claims in the first few months and OS_2 is the lower cost of handling claims later. Note that we are assuming the cost of handling a claim in the first month is included in the intake expense, so we only must account for (Duration $- 1$) months of outstanding expenses.

In setting the reserve using the reserve per claim concept, a reserve equal to

$$\text{Reserve Per Claim} = (OS_1 \times x) + (OS_2 \times (\text{Duration} - 1 - x))$$

would be set aside for each claim in the month in which the claim is reported. If the claim closes in the first month, then the full reserve would be banked for claims remaining open longer than the average life of the claim. If the claim remains open at the end of the second (or third) month, then OS_1 dollars would be released from the reserve. If the claim remains open at the end of the fourth and succeeding months, then OS_2 dollars would be released from the reserve for each month the claim is open.

These additional claim standards will have to be determined based on some type of work measurement study. Although these standards conceivably will differ by state due to differences in wage levels, rent, etc., the most significant difference by state is due to duration. One could take these differing durations into account in pricing claims service to avoid adverse selection in problem states.

The formula presented above is for handle-to-conclusion pricing, i.e., the fee is sufficient to cover the expenses of handling the claim for as long as the claim is open. Today many third party administrators (TPAs) also price claims on a limited time handling basis. Under this option, an additional fee would be levied to service claims remaining open after (for example) two years. This additional fee typically is negotiated at the time of sale.

Today most large (self-)insureds separately negotiate the cost of claims service with an insurance company TPA or a stand-alone TPA. The stand-alone TPA will partner with an insurance company who is willing to unbundle its claims service. While an insurance company TPA would be willing to offer this limited time handling option, many insurance companies would not want the insured to take its claims elsewhere to be serviced because these claims are the insurance company's liability (or conceivably could be if serviced under a deductible policy).

Given a handle-to-conclusion fee, how could one quickly estimate the limited time handling fee? The statistics in Table 2 show that 22.6 percent of claims remain open after two years. We could then estimate the limited time handling fee for two years as $(1 - 0.226) \times \text{HTC}$, where HTC is the handle-to-conclusion fee. The claims remaining open after two years would begin to incur a monthly fee and would continue to do so as long as the claim stayed open. The flaw of this quick estimate is the 77.4 percent of claims closed in the first two years have lower average claim handling cost than do the 22.6 percent of claims still open after two years.

Claims still open at 24 months likely will remain open an additional 24 months. This is a key statistic because it allows you to price the claim handling expense for these claims. Many persons find it surprising when told the cost to handle a claim that has been open for 24 months is higher than the cost to handle a new claim. A new claim will be open, on average, for a shorter duration than an old claim, i.e., a claim remaining open after 24 months. If a customer chooses to pay a one-time fee to handle a claim remaining open after 24 months, the necessary fee assuming a monthly outstanding expense of $50 will be $1,200 = 24 \times $50 (per claim).

This one-time fee also could be calculated as the cost of handling takeover claims. A customer who has limited time handling option who chooses to take its claims to another TPA would be subject to a takeover claim fee.

7   Monitoring the Duration

There is some evidence that duration has increased during the 1990s. It also seems likely that managed care will play some part in decreasing duration. Because it is generally true that the longer a claim remains open, the higher will be the expense of handling that claim, it is a good idea for claim departments to monitor progress or slippage in duration.

A process for monitoring the duration would be to use outstanding claims by report quarter and to monitor the percentage open at three, six, nine, and 12 months. In the absence of change in claims handling, one would expect to see the same percentages throughout a column.

By using report quarter instead of accident quarter, there is no issue with claim development. Also, by using report quarter rather than report year, the analyst can more quickly discern changes in outstanding rates (because of the frequency with which these reports will be produced) or any seasonality that may exist.

While this type of triangulation may be used to monitor duration, it also may be used by claim departments or third party administrators in setting goals for the future. The goal could be to continue to close claims at the same rate or the goal could be to close claims more quickly. The longer claims stay open, the higher is the total cost of handling claims although this could be a trade-off as closing claims too quickly could lead to more reopened claims and/or higher settlement values.

A claim department or third party administrator who is interested in more sophisticated monitoring techniques could use the same types of report quarter comparisons at successive evaluations to monitor:

- Average incurred claim size;

- Average paid claim size:

- Average outstanding claim size:

- Ratio of paid ALAE to paid loss;

- Average ALAE per reported claim;

- Average recovery per claim;

- Recovery as a percentage of loss; and

- Ratio of closed claims to the number of claims handlers.

By monitoring the claim closing rate as well as the claim costs and other measures at like points in time, a claim department can monitor not just the closing of the claims but the full range of statistics bearing on a claim department's performance.

8   Closing Comments

By using the techniques described here, a claim department or third party administrator can price claim service based on the total cost of handling the claim. This will allow the company to set up and maintain an adequate reserve and to monitor the success in handling the claims.

As claim prices have become unbundled in insurance and service proposals, insurance companies and TPAs have become more aware of the expenses involved in handling claims. The concepts presented in this paper provide a framework for pricing and reserving for claims, as well as for monitoring the efficiency of the claim handling process.

Pricing Earthquake Exposure Using Modeling

Debra L. Werland* and Joseph W. Pitts[†]

Abstract[‡]

This paper demonstrates a practical methodology for determining a statewide rate level indication for the earthquake insurance and for determining more equitable territorial relativities within a state. The methodology is based on the output from a certain commercially available earthquake modeling software package. The methodology addresses some of the complex issues involved in pricing earthquake insurance exposure and potential regulatory acceptance. The paper also features a section dealing with the net cost of reinsurance in the proposed direct rates. A final consideration is the treatment of a model's output when it is believed the modeled results are less than fully credible.

Key words and phrases: *catastrophe modeling, reinsurance, target rate of return, zone relativities*

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References to and descriptions of some of the inner workings of the earthquake computer simulation model developed by Applied Insurance Research, Inc. of Boston, Massachusetts, are done with their express written permission.

1   Introduction

Pricing hurricane and earthquake risk has never been an easy task. No insurer's loss history is adequate to cover the expectation of all possible type and size of events. Any ratemaking formula based on actual loss experience for such rare events will fail to capture the scope of possible events that could affect an insurer's financial results. Catastrophe hazard modeling represents a way of developing the scope of possible catastrophic events. The financial impact of these events is based on characteristics of the underlying peril and their interaction with the insured properties.

Actuaries are relying more than ever on the use of modeling in pricing catastrophic risks such as hurricanes and earthquakes. As a result, catastrophe hazard modeling has become an important tool for ratemaking in lines of business subject to low frequency, high severity type losses. Natural hazard events such as hurricanes and earthquakes rarely occur, but their devastation can be overwhelming when they do. Few insurance companies have enough historical loss data to sufficiently price these events.

In this paper we will focus on the earthquake peril and its pricing. The approach adopted is to use an earthquake computer simulation model. In particular we use an earthquake model developed by Applied Insurance Research, Inc. (AIR) of Boston, a leading computer simulation/modeling firm. While it is not necessary for one to completely understand the intricacies of all functions and assumptions used in the simulation model, it is important nonetheless to present an overview of the AIR model. Briefly, the AIR earthquake model is composed of three separate component models: an earthquake occurrence model, a shake damage model, and a fire-following model. The overall model uses sophisticated mathematical techniques to estimate the probability distribution of losses resulting from earthquakes anywhere in the 48 contiguous states. The AIR earthquake model is described in more detail later in the appendix.

For ratemaking purposes, the output from the model includes loss costs applicable to a specific location, type of construction, and policy form. Our interest is in a single family dwelling as covered under a typical homeowners policy. The loss costs generated by the model are the basic building blocks in the development of an appropriate rate.

We will discuss target underwriting profit provisions, reinsurance costs, and other components of developing an adequate rate per $1,000 of dwelling coverage for a typical book of homeowners business. The credibility of the results will be addressed in the derivation of the in-

dicated rates, and the state will be partitioned into geographic zones based on the relative difference in loss costs determined from the modeled results.

2   Proposed Methodology

The goal of this paper is to present a methodology for developing a rate per $1,000 of earthquake coverage. We assume that the indicated rate is based on Coverage A (the dwelling limit of a typical homeowners single family dwelling). The modeled results include all coverages (dwelling, other structures, personal property, time element expenses), and the figures have been ratioed to Coverage A, in 1000s.

2.1   Statewide Indicated Rate

The statewide indicated rate is determined using the pure premium method. The losses are based on an insurer's own exposure distribution within the state. The first input into the methodology is the statewide modeled expected losses stated at a base deductible level. In this example the base deductible is 10 percent applicable to the dwelling limit. The expected annual losses represent the average annual amount of losses an insurer could expect from writing the earthquake line of business in state X if each insured had a 10 percent deductible.

The modeled results are generally available on an individual state basis as well as on a zip code or county basis within the state. The expected annual losses are trended (severity only) and adjusted for loss adjustment expense (LAE), then ratioed to the total trended value of insured dwellings to develop a projected pure premium which is used to determine the indicated rate as shown in Table 1. (A viable alternative would be to trend the insured values first and use these trended values as input to the catastrophe model, thus yielding an estimate of trended severity within the model results). In this example, the current rate is assumed to be $2.50 per $1,000 of dwelling coverage. The indicated rate is calculated by taking the projected pure premium and grossing it up to include reinsurance costs net of reinsurance recoveries, trended fixed expenses, and variable expenses. These calculations show that the indicated statewide rate is $3.77 per $1,000 of dwelling coverage.

Some of the rows of Table 1 are described in more detail as follows:

(1) This is the main output received from the modeling firm. It is an estimate of the expected annual losses at a base deductible for an

Table 1
Statewide Indicated Rate

(1)	Modeled Expected Annual Losses, 10% Deductible, 12/31/95	$19,500,000
(2)	Total Dwelling Coverage, 12/31/95	$10,965,281,000
(3)	Proposed Effective Date	7/1/96
(4)	LAE Factor	1.150
(5)	Loss Trend Factor Trended to 7/1/97	1.250
(6)	Exposure Trend Factor Trended to 7/1/97	1.190
(7)	State X Earthquake Share of Net Cost of Reinsurance	$7,592,703
(8)	Trended Fixed Expense Provision Per $1000 of Coverage	0.265
(9)	Pure Premium Per $1000 of Coverage	$2.99
(10)	Variable Permissible Loss and LAE Ratio	0.794
(11)	Indicated Rate: (9)/(10)	$3.77
(12)	Current Statewide Rate Per $1000 of Dwelling Coverage	$2.50
(13)	Indicated Percentage Change: (11)/(12) − 1	50.8%
(14)	Proposed Change	50.8%
(15)	Proposed Statewide Rate: (12) × [1 + (14)]	$3.77

insurer, given the current book of business within the state for the earthquake line of business;

(2) The total value of insured dwellings is provided to the modeling firm by the insurer and is used to determine the average expected annual losses per $1,000 of coverage in the pure premium method;

(3) The proposed effective date as selected by the insurer;

(4) The LAE factor is calculated based on a comparison of estimated ultimate loss adjustment expenses to estimated ultimate losses from the most recent earthquake events faced by the insurer;

(5) The modeled losses are trended using historical homeowners severity data. Earthquake loss trend data are not used because of their instability. Losses should not be trended for frequency, unless the insurer is confident there exists an increased period of seismicity in the future;

(6) The exposure trend is based on historical changes in the average amount of insurance for the earthquake line of business;

(7) The state X earthquake share of the expected net cost of reinsurance is calculated as described in Table 2;

(8) The trended fixed expense provision per $1,000 of coverage is calculated by trending fixed expenses to a point in time appropriate for the proposed effective date and dividing it by trended insured value, using an annualized fixed expense trend of 5 percent;

(9) The formula for Row (9) is:

$$\text{Pure Premium per } \$1,000 = (8) + \frac{[(1) \times (4) \times (5) + (7)] \times 1000}{(2) \times (6)},$$

which combines the modeled expected losses with the net cost ·of reinsurance for the state and line of business with the trended fixed expense provision to provide an estimate of the projected pure premium to be expected during the time the proposed rates are to be in effect; and

(10) The variable permissible loss and LAE ratio are calculated based on historical variable expenses and a consideration of the relative riskiness of the earthquake line of business compared to other lines being written and the overall required return on surplus. An 18.2 percent underwriting profit provision is used along with a 2.4 percent provision for variable expenses.

2.2 Net Cost of Reinsurance

An important component that we reflect in the rate indication is the net cost of reinsurance. An insurer should decide whether to include this component based on the costs and anticipated recoveries associated with its reinsurance program. The net cost of reinsurance should be included as a cost if the expected reinsurance recovery is less than the amount of premium paid to the reinsurer for reinsurance protection. This relationship generally will be the case due to the presence of transaction costs that include a margin for reinsurance risk load and profit.

The expected reinsurance recovery represents the average annual amount an insurer could expect to recover from the reinsurer(s) due to insured events and can be determined using catastrophe modeling. The expected reinsurance recovery needs to be calculated considering

the attachment points or quota share percentages associated with an insurer's reinsurance program. An insurer's reinsurance program often is structured to provide protection against many types of hazards; however, some reinsurance contracts are designed to provide protection against only one hazard.

To accurately measure the net cost of reinsurance for a particular hazard, the reinsurance premium from all programs that provide protection for the hazard should be included. If other catastrophic hazards such as hurricanes are a large proportion of an insurer's exposure to catastrophe loss, the reinsurance premium for multihazard contracts could be segregated for each hazard. The reinsurance premium for each hazard then could be included with each net cost of reinsurance calculation for every line of business. In the example, however, the net cost of reinsurance is allocated to the earthquake line of business and to the appropriate state.

The allocation to line of business in the example shown in Table 2 is based on model results comparing expected earthquake reinsurance recovery to the total expected reinsurance recovery. This ratio is applied to the net cost of reinsurance to obtain the earthquake-only net cost of reinsurance. The allocation to a state level uses earthquake written premium. This allocation may introduce a distortion if the state in question has a different level of premium adequacy than countrywide premium adequacy. In addition, a premium base allocation may not adequately represent the riskiness of expected earthquake losses by state.

The rows of Table 2 are described in more detail as follows:

(1) This is the total of all reinsurance premium paid for reinsurance contracts that provide protection for earthquake losses;

(2) This is a model output number. It is determined based on the attachment point or quota share arrangement an insurer has with its reinsurer(s);

(3) The net cost of reinsurance is the difference between the reinsurance premium paid for contracts providing earthquake protection and the expected total reinsurance recovery;

(4) Model results are used to determine what portion of the expected recovery is due to earthquake;

(5) The earthquake proportion of the total expected reinsurance recovery is expressed as a factor to be applied to the total net cost of reinsurance;

Table 2
Estimated Net Cost of Reinsurance

(1)	1995 Countrywide Reinsurance Premium for Contracts Covering the Earthquake Peril	$37,890,000
(2)	Expected Reinsurance Recovery	$17,481,970
(3)	Net Cost of Reinsurance: (1) − (2)	$20,408,030
(4)	Expected Earthquake Reinsurance Recovery	$9,154,600
(5)	Proportion of Earthquake Recovery to Total Recovery: (4)/(2)	52.4%
(6)	Earthquake Share of Net Cost of Reinsurance: (3) × (5)	$10,693,808
(7)	1995 State X Earthquake Written Premium	$27,271,677
(8)	1995 Countrywide Earthquake Written Premium	$38,551,154
(9)	State X Earthquake Share of Net Cost of Reinsurance: [(7)/(8)] × (6)	$7,592,703

(6) The earthquake share of the net cost of reinsurance is the proportion of the earthquake recovery to the total recovery multiplied by the total net cost of reinsurance;

(7) The latest year state X earthquake written premium is used to allocate the earthquake share of the net cost of reinsurance to a state level; and

(8) The latest year countrywide earthquake written premium is used to find what proportion is represented by state X. Each state's written premium is first adjusted to current rate levels, if applicable.

The concept of including the net cost of reinsurance in a rate indication is relatively new and likely will be challenged or subjected to additional scrutiny by regulatory agencies. It does represent a cost of doing business, however; therefore, we include its net costs. Reinsurance costs also may be considered in conjunction with the selected rate of return.

2.3 Target Rate of Return

To develop an underwriting profit provision, we choose a total rate of return methodology. We are not proposing one method over another; we have selected this particular method for the development

of a reasonable profit target for the earthquake line of business. The target rate of return on GAAP equity is developed using a discounted cash flow (dividend yield) method and the capital asset pricing model (CAPM). The selected rate of return, averaged from the results of these two methods, is 13.0 percent. From this selected rate of return we have subtracted 8.0 percent (which represents the post-tax investment rate of return from all investable funds). Table 3 converts this difference to a pre-tax basis, using a corporate tax rate of 35 percent. For an insurer's total book of business this percentage is divided by the company's premium-to-surplus ratio to convert the target underwriting profit provision to a percentage of premium. Although we do not endorse the divisibility of surplus or leverage ratios, we propose this method for calculating a reasonable earthquake underwriting profit provision.

We have selected a company whose underwriting results resemble the years 1985-1994 for all property and casualty insurers writing personal lines automobile, homeowners multiperil, and earthquake coverages. (It would be appropriate for more years to be used; however, the earthquake line of business was not segregated prior to 1985). The data are from *Best's Aggregate and Averages*. A company's own data also can be used for this purpose.

<div align="center">

Table 3
Target Underwriting Profit Provision

</div>

A. Target Rate of Return (% of GAAP Surplus)	
1. Dividend Yield Model	12.0%
2. Capital Asset Pricing Model	14.0%
3. Selected Target Rate of Return	13.0%
B. Target Underwriting Rate of Return (% of GAAP Surplus)	
1. Investment Rate of Return After Tax	8.0%
2. Target U/W Return After Tax $(A3) - (B1)$	5.0%
3. Target U/W Return Before Tax $(B2)/(1 - 0.35)$	7.7%
C. Target Underwriting Profit Provision (% of Direct Earned Premium)	
1. Net Written Premium/GAAP Surplus Ratio	1.30
2. Indicated U/W Profit Provision $(B3)/(C1)$	5.9%
3. Selected U/W Profit Provision	5.9%

Note: Insurers are chosen that resemble the mix of business written by the filing insurer. Company betas and projected dividend yields are from Value Line. Both the dividend yield method and CAPM are used in determining an appropriate rate of return. The selected target rate of return is a straight average of the two methods.

A company's underwriting profit provision should vary based on the riskiness of the line of business. A measure of risk we have chosen is the coefficient of variation (measured as standard deviation/mean) of a series of underwriting results for each line. Alternatively, combined ratios could be used, where a 100.0 combined ratio reflects a 0 percent underwriting result. Because the selected period includes the effects of Hurricane Andrew and the Northridge Earthquake, we adjust the losses so that Andrew reflects a 1-in-30 year event and Northridge a 1-in-50 year event. We did not adjust for Hurricane Hugo.

Table 4 shows the industry's yearly (1985-1994) underwriting gains and losses as a percent of net earned premium. Table 5 shows the coefficient of variation of each line, the weighted average of the coefficients of variation using the latest ten years of premium, and a risk index (the ratio of each line's coefficient of variation to the weighted coefficient of variation).

Table 4
Annual Underwriting Results as a Percentage of Premium

Year	Private Passenger Automobile	Homeowners Multiperil	Earthquake
1985	-11.0%	-11.7%	60.0%
1986	-8.3%	-3.5%	58.0%
1987	-6.0%	3.3%	44.2%
1988	-6.8%	0.0%	57.5%
1989	-8.9%	-13.9%	-42.1%
1990	-9.1%	-12.9%	43.8%
1991	-4.6%	-17.7%	55.3%
1992	-1.9%	-58.4%	61.4%
1993	-1.8%	13.5%	68.0%
1994	-1.3%	-18.4%	-222.2%

Assume the company's premium-to-surplus ratio corresponds to the industry's at 1.30, so that its inverse is 0.77. The risk indices are used to adjust each line's surplus ratio (surplus-to-premium) in the total rate of return methodology, resulting in target underwriting profit provisions that reflect the risk of each line of business. The resulting earthquake profit provision will be used in the derivation of the variable permissible loss and loss adjustment expense provision. Table 6 summarizes this information.

Table 5
Coefficient of Variation (CV) and Risk Index (RI)

Line of Business	PD	CV*	RI
Private Passenger Automobile	80.1%	0.550	0.92
Earthquake	0.5%	1.854	3.09
Homeowners Multiperil	19.4%	0.780	1.30
Total	100.0%	0.600	1.00

Notes: PD = Premium Distribution; *Absolute value.

Table 6
Target Underwriting Profit Provision

Line of Business	RI	S/P	TUPP
Private Passenger Automobile	0.92	0.71	5.4%
Earthquake	3.09	2.38	18.2%
Homeowners Multiperil	1.30	1.00	7.7%
Total	100.0%	0.77	5.9%

Notes: RI = Risk Index; S/P = Implied Surplus Ratio; TUPP = Target
Underwriting Profit Provision.

In this example industry net underwriting results are used to deter-
mine an appropriate underwriting profit provision for the earthquake
line of business. A larger earthquake underwriting profit provision
would result if direct results were used. The variability of net under-
writing results is removed by the stabilization of reinsurance. Using our
methodology it is reasonable to conclude that part of the difference
between underwriting profit provisions calculated using net or direct
underwriting results would be due to reinsurance costs. An insurer
should expect a lower net cost of reinsurance if part of the reinsur-
ance cost is reflected in the earthquake underwriting profit provision
calculated using direct underwriting results. Efforts could be made to
quantify what portion of the net cost of reinsurance is contained in an
earthquake underwriting profit provision based on direct underwriting
results. One possible approach would be to compare the difference in
earthquake underwriting profit provisions calculated using net and di-
rect underwriting results to a net cost of reinsurance as calculated in
this example.

2.4   Zone Relativities

Model results also can be used to determine revised earthquake zone definitions and earthquake zone relativities. The data used to establish earthquake zone definitions are model results at a five digit zip code level. The sum of all the five digit zip code modeled losses and dwelling insured values should balance to the statewide totals used to determine the statewide indicated rate.

Table 7
State X Earthquake Model Results, Zip Code Level

Zip	DIV (in $000)	EAL at 10% Deductible	Loss Cost
1	$ 921,339	$ 2,303,348	$2.50
2	1,096,528	1,644,792	1.50
3	258,481	387,722	1.50
4	548,264	603,090	1.10
5	922,272	830,045	0.90
6	79,839	98,897	1.24
7	722,114	902,643	1.25
8	103,211	232,225	2.25
9	803,112	3,011,670	3.75
10	801,247	721,122	0.90
11	552,322	359,009	0.65
12	402,178	623,376	1.55
13	700,659	1,156,087	1.65
14	1,102,321	2,369,990	2.15
15	200,321	490,786	2.45
16	402,111	1,105,805	2.75
17	727,727	1,928,477	2.65
18	202,001	490,786	1.03
19	112,007	123,768	1.11
20	307,227	399,088	1.30
Total	$10,965,281	$ 19,500,000	$1.78

Notes: Zip = Five Digit Zip Code Area; DIV = Dwelling Insured Value; and EAL = Expected Annual Loss.

In the example we assume the state comprises 20 distinct five digit zip codes. Table 7 shows the data segregated by five digit zip code. We

use a SAS clustering program to determine the new earthquake zone definitions and zone relativities. The SAS procedure we used is described in the SAS user's manual (1989).

PROCFASTCLUS performs a joint cluster analysis on the basis of Euclidean distances computed from one or more quantitative variables. The observations are divided into clusters such that every observation belongs to one and only one cluster. The procedure is intended for use with large data sets, from approximately 100 to 100,000 observations. With small data sets the results may be highly sensitive to the order of the observations in the data set.

PROCFASTCLUS uses a method referred to as *nearest centroid sorting*. A set of points called *cluster seeds* is selected as a first guess of the means of the clusters. Each observation is assigned to the nearest seed to form temporary clusters. The seeds are replaced by the means of the temporary cluster, and the process is repeated until no further changes occur in the cluster.

Specifying the desired number of earthquake zones and using the SAS procedure yields the results in Table 8. The number of zones to be used in a real application will depend on the size of the insurer's earthquake book of business, geographic spread, and the level of seismic variation within the state. The proposed earthquake zones probably will not be contiguous because five digit zip codes from different parts of the state will fall into the same cluster in the SAS procedure. We only use 20 zip codes in our example; however, the SAS procedure has the capability to handle a much larger number of zip codes. The relativities shown in Table 8 are applied to the statewide indicated rate previously calculated to determine each zone's earthquake rate.

The resultant earthquake zone rates should display a wider variance, as it could be argued that risk margins should vary by geographic location for the earthquake peril. We view this as another area deserving further consideration and an important aspect of determining adequate earthquake rates.

3   Shortcomings Inherent in Modeling

3.1   Data Problems

Modeled results can be understated for many reasons, most of which can be attributed to company issues or to adjustments not made within the models. We first will discuss company shortcomings and then follow with model shortcomings. Where appropriate, we will make sug-

Table 8
State X Earthquake Zone Relativities

Zone	(1)	(2)	(3)	(4)	(5)
1	$552,322	$359,009	$0.65	0.37	$1.38
2	3,694,971	3,886,713	1.05	0.59	2.23
3	3,560,167	6,181,967	1.74	0.98	3.68
4	2,354,709	6,060,641	2.57	1.45	5.46
5	803,112	3,011,670	3.75	2.11	7.95
Statewide	$10,965,281	$19,500,000	$1.78	1.00	$3.77

Notes: Zone = Earthquake Zone; Column (1) = Dwelling Insured Value in ($000); Column (2) = Expected Annual Loss at 10% Deductible; Column (3) = Loss Cost = (2)/(1); Column (4) = Indicated Relativity to Statewide = (3)/1.78; and Column (5) = Indicated Earthquake Zone Rate = (4) × 3.77.

gestions on how to handle quantifiable and supportable adjustments to the modeled input or output. The following list is not meant to be exhaustive, but is typical of company issues. Company shortcomings include:

- Underinsurance (homes insured less than their value) or overinsurance (homes insured more than their value);

- Demand surge for labor and materials after a catastrophic event;

- The need for extra claims adjusters following catastrophic events;

- No data collecting or coding for retrofitting safety features; and

- Invalid or incomplete data.

The major company shortcoming may be the problem of underinsurance. Expected loss to a particular structure in a particular area is based on applying an average damage ratio (defined as the ratio of the repair cost of a building to its total replacement value) to the total insured value of the structure. It is assumed that the insured value of a building represents its true replacement cost. A company should estimate its underinsurance (or overinsurance) problem before providing data to a modeling firm. If, on average, it is determined that a book of business is underinsured by 10 percent, then all limits should be adjusted before the model is run.

The effects of demand surge can be significant and should be factored into all modeled results. (It is not clear whether this adjustment

should be made by the insurer or by the modeler.) The demand for labor and materials will vary depending on the location and magnitude of each earthquake. The additional cost probably varies between 0 percent and 30 percent, but the highest demand is associated with events that have the lowest expected probability; therefore, the effect on average annual aggregate losses should be minimal (albeit the effect could be substantial for large catastrophic events). We believe this adjustment to the modeled loss costs is important, yet is an uncertain aspect of the process. Studies should be conducted to determine the impact of demand surge factors, perhaps by studying the payout of events such as Loma Prieta and Northridge, if data are available. Either overall average demand surge factors should be applied to the resultant loss costs or variable demand surge factors should be determined and applied by location and event.

The need for independent claims adjusters is a real cost of settling claims following large catastrophic events. It is not clear which loss adjustment expense (LAE) factors should be applied to the modeled expected loss costs—there has not been enough loss experience to determine appropriate factors. We suggest using either the ratio of LAE to losses of past events (which may understate the true ratio) or the underlying policy average LAE factor, given earthquake coverages are normally endorsed to a homeowners or dwelling fire program.

Modeled results should account for retrofitting safety features of an insured structure. This is especially applicable to buildings made of unreinforced masonry. Average damage ratios should be adjusted for these features. It is not clear how the effects of retrofitting can be measured, but research should be conducted and insurers should encourage their installation. A strongly built and reinforced home should withstand the initial impact and aftershocks of an earthquake, as opposed to a home whose frame is not bolted to the foundation, for example. Most insurance companies do not request information on retrofitting mechanisms, nor do they store the data. We would encourage the Institute for Business and Home Safety (IBHS)[1] to study the effects of such safety features and simulate an earthquake under monitored laboratory conditions to determine the extent of damage on the structure and its contents. The Institute for Business and Home Safety is a nonprofit organization sponsored by the insurance industry. The mission of IBHS is "to reduce injuries, deaths, property damage, economic losses and human suffering caused by natural disasters."

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[1]Institute for Business and Home Safety, 73 Tremont Street, Suite 510, Boston MA 02108.

   Finally, there is always the possibility of invalid data, incomplete
data, or no data at all. Invalid data are most prominent if zip code,
county, or street address is not validated before being stored on the
insurer's database. Either the data should be cleaned before the in-
put files are created or the data should be eliminated from analysis.
Alternatively, invalid data could be proportionally distributed through-
out the state by county or zip code based on the distribution of the
insurer's valid data. Most companies do not have enough insureds lo-
cated in all areas of the state. Therefore, there will be many locations
with no modeled loss costs. In these situations, modeling firms have
access to an inventory of typical building structures by location, aver-
age dwelling limit, type of construction, average year of construction,
building height, etc. Modeled loss costs from this generic inventory can
supplement an insurer's results where few or no insureds reside.
   There will also be locations with insufficient data. Assume for a mo-
ment that an insurer's book of business is mapped to the geographic
zip code centroid of each zip code within the state. Although modeled
results are assumed to be 100 percent credible by location, the reader
could question whether one, ten, or even 100 exposures are enough
to deem the results credible. An insurer's database could be comple-
mented with the results of the generic inventory. The authors have
chosen to consider data 100 percent credible by zip code with more
than 100 exposures; otherwise, the generic inventory is given full cred-
ibility.

3.2   Inadequate Information

   These brief remarks are not intended to criticize any model or mod-
eler, but to highlight the importance of their impact on modeled results.
The following list is also not meant to be exhaustive, but does represent
typical shortcomings:

- Factor for unknown faults;

- Inclusion of debris removal expenses;

- Effects of aftershocks; and

- Parameter risk within the model.

   The 1994 Northridge Earthquake is a perfect example of an unknown
fault, a blind thrust fault that does not break the earth's surface. Not
even seismologists know the extent of undiscovered fault lines beneath

the earth's surface. How understated could the modeled results be? No one knows for sure, and we propose no solution to handle this uncertainty. Although the models account for possible earthquakes in all historical seismic source zones, it is questionable if distributions in the model account for all potential seismicity. With the passage of time and with advancing technology, perhaps these models may account for all possible faults some day. For now we must assume that a model's results may understate expected average annual losses and, hence, expected loss costs per $1,000 of coverage.

Debris removal expenses, although small, should be added to the model's expected loss costs. More prominent would be the effects of aftershocks that follow moderate to large earthquakes. Claims often are reopened months later due to weakened structures repeatedly damaged from aftershocks. Future modifications to catastrophe models should account for this possibility.

Because catastrophe modeling is based on incomplete distributions developed from historical information, parameter risk always will exist. This risk may lead to gross understatement (or overstatement) of potential insured losses and represents a potential shortcoming of modeling.

3.3   Additional Considerations

There will always exist areas that deserve further consideration. While we have presented a practical procedure for developing adequate earthquake rates, some areas deserve additional research and attention. We will divide these topics into four categories: (1) shortcomings of models, (2) credibility of data, (3) necessary target rate of return, and (4) net reinsurance costs.

We devote an entire section of this paper to model shortcomings and company data issues. We repeat them to emphasize their importance and the need for further study. The cooperation of the insurance industry, modeling firms, and the IBHS is necessary to quantify the impact of outstanding issues on expected loss costs. Perhaps special data calls or cooperative studies can be conducted and the results shared with all interested parties.

Computer modeling simulates thousands of possible events, and its results are generally considered credible. The earthquake peril is unique by location, especially in California, so a feasible complement of credibility to augment a local result does not exist. Perhaps a regional complement could be used, but its applicability is questionable, given local soil conditions and proximity to fault lines. We believe that

an industry inventory database represents the best alternative for a complement.

Insuring the earthquake peril is much riskier than insuring auto physical damage coverages. Due to the relationship between risk and return, a higher rate of return (and therefore a higher underwriting profit and contingency provision) should be allowed to cover a company's earthquake exposure. This provision also should vary by location. We have presented a simplified method for deriving a reasonable profit provision, but we encourage more research in this important area.

Should rates include the costs of reinsurance on an insurer's book of business? Their inclusion could be viewed as a pass-through to the consumer. Also, in the long run neither the insurer nor the reinsurer(s) should be worse off for engaging in a reinsurance program; otherwise, neither party would enter the contract. In the short run, however, reinsurance costs are a legitimate expense of doing business, and we believe that all parties should share in that expense, including policyholders. Policyholders benefit from financially strong companies.

4   Summary

Catastrophe hazard modeling has become an integral part of the ratemaking process. Casualty Actuarial Society ratemaking principles (1988) state that "other relevant data may supplement historical experience. These other data may be external to the company or to the insurance industry." We have entered the realm of that other relevant data. *Actuarial Standard of Practice (SOP) No. 9* (1991) states that "an actuary should take reasonable steps to ensure that an actuarial work product is presented fairly ... if it describes the data, material assumptions, methods, and material changes in these with sufficient clarity that another actuary practicing in the same field could make an appraisal of the reasonableness and the validity of the report." With the advent of modeling, however, the actuary must rely on the work of another person. *SOP No. 9* states that "reliance on another person means using that person's work without assuming responsibility therefore." These other persons now include experts in the fields of geology, seismology, and structural engineering, to name a few. Actuaries, however, can play a key role in contributing to the development of the models and, more importantly, the interpretation and communication of their valuable results.

Catastrophe hazard modeling has become a necessary tool for the pricing of large catastrophic events such as hurricanes and earthquakes.

Their frequency is so low and their severity so potentially high that not even all of the property and casualty companies in a state could have enough loss history upon which to base rates. Despite any shortcomings models may have, they hold the key to the future and the pricing of nature's perilous attacks.

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Appendix: The Applied Insurance Research Model

Overview

The model developed by Applied Insurance Research uses sophisticated mathematical techniques to estimate the probability distribution of losses resulting from earthquakes anywhere in the 48 contiguous states. The earthquake model is composed of three separate components: an earthquake occurrence model, a shake damage model, and a fire-following model. The earthquake occurrence component of the model uses a probabilistic simulation to generate a synthetic catalog of earthquake events that is consistent with the historical record. The shake damage estimation component uses analytical numerical techniques to calculate the distribution of losses for individual buildings given the characteristics of the event. The fire-following component uses simulation to estimate fire losses following an earthquake. Together these techniques allow the estimation of a wide range of information about potential earthquake losses in the United States.

The earthquake simulation model incorporates descriptions of a large number of variables that define both the originating event (the earthquake) and its effect on structures. Some of these variables are random and others are deterministic. We will describe the key aspects

of the model, the main variables affecting the outcomes, and the relationships between the primary variables in the rest of this appendix.

Earthquake Occurrence in the USA

For earthquakes there are three key types of variables that describe the physical phenomenon. In broad terms, these variables describe where earthquakes can occur, the size of the earthquake, and the likelihood of seeing an earthquake of a particular size. In other words, the variables describe where, how big, and how often earthquakes occur.

The issue of where earthquakes occur is handled by identifying *faults* or *seismic zones* where actual earthquakes have been observed. On the West Coast earthquakes tend to occur along well-defined geological features called faults, which are places where the surface of the earth has been ruptured by past earthquakes and which are observable at the ground surface or by subsurface sounding techniques.

Not all faults are active, i.e., not all faults are believed capable of rupturing in the present, although they have ruptured in the distant past. Where faults are observed and where the historical catalog (record) of earthquakes indicate that the faults are still capable of rupturing, the surface trace of the fault defines a possible location for future earthquakes.

Not all earthquakes occur on identifiable faults, however. Many earthquakes, especially those east of the Rocky Mountains, occur on faults that are not visible at the surface. Such faults are inferred from the occurrence of actual earthquakes in the historical record. For these areas, a source zone is created, which is an area with fuzzy boundaries within which future earthquakes are possible.

The AIR model contains approximately 250 seismic source zones covering the 48 contiguous states. Each source zone is defined by a line on the surface of the earth with probability distributions describing the variability of potential epicenters both along and perpendicular to that line. A potential earthquake is not limited to occur along a known fault line, but can occur anywhere in the vicinity of a fault or anywhere within a seismic source zone, depending on the degree of uncertainty associated with the historical record of earthquakes in that area. The central line of the source zone does define the dominant direction of faults in the area and characterizes the orientation of the rupture surface.

The size of an earthquake is usually measured by one of several magnitude scales. In the AIR model the surface wave magnitude Ms

scale[2] is used to characterize the earthquake magnitude. For every fault and source zone the frequency of earthquakes of different magnitudes must be described. Seismologists generally agree that, over a considerable magnitude range, the logarithm of the number of historic earthquakes that exceed a given *magnitude scales* varies linearly with magnitude. This indicates that the frequency-magnitude relationship is approximately exponential.

Additionally, prehistoric seismologic data have been interpreted by some researchers to indicate that the frequency-magnitude relationship for large earthquakes differs from exponential scaling, leading to the notion of characteristic earthquakes in certain geographic areas. The AIR model incorporates a truncated exponential distribution, or truncated Gutenberg-Richter relationship, to represent potential seismicity in each source zone. Where appropriate we incorporate a characteristic earthquake model.

The AIR earthquake model is calibrated to a catalog of historical earthquakes that covers the historical record from the mid-1600s to the present. Because the completeness of the catalog varies both in time and as a function of magnitude (larger earthquakes are more likely to be included in the historical record), the fitting of the frequency-magnitude distribution is adjusted to account for the variation in historical completeness.

Earthquake Attenuation

After earthquakes are simulated using the probability distributions of the different earthquake parameters, the shaking intensity of the earthquake at every location affected by the earthquake is calculated using a relationship called an *attenuation function*.[3] The local intensity is corrected to reflect local soil conditions, as some types of soil amplify the shaking intensity relative to other soil types. This section discusses the variable interrelationships required to calculate the local shaking intensity.

From the characteristics of the earthquake the local shaking intensity is calculated using an attenuation relationship. The attenuation relationship depends on the location of the source zone, as earthquake shaking attenuates more quickly in the western U.S. than in the east-

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[2]The Ms scale measures the strength of an earthquake as determined by observations of its local surface waves.

[3]This function measures the reduction in the shaking intensity as we move away from the epicenter of the earthquake.

ern part of the country. The same magnitude earthquake will affect a smaller area in California than in the northeast.

The attenuation calculation starts by spreading the energy released by the earthquake over the rupture surface and integrating over the entire rupture surface to calculate the total effect of the earthquake. In effect, energy is assumed to be released uniformly over the rupture, and each incremental piece of energy is attenuated separately to obtain the effect at some distant point. This results in contours of equal intensity that are elongated along the orientation of the rupture.

The calculation of local shaking intensity consists of two parts. First, a basic intensity is calculated that assumes uniform soil conditions at every location. This intensity (called a *Rossi-Forel intensity*) depends on the distance of the site from the earthquake rupture, the orientation of the rupture, and the earthquake magnitude and focal depth. The rupture length is calculated from the basic earthquake parameters. Second, the Rossi-Forel intensity is modified to reflect the soil conditions at the site. Soil conditions for the entire country are digitized on grids varying from 0.1 degree latitude/longitude squares to 0.5 minute latitude/longitude squares. The local soil condition can significantly affect shaking intensity. The final intensity is identified as a *modified Mercalli intensity* (MMI).

The MMI is a generally accepted unit of shaking intensity. It describes, in general terms, the type of damage that might be expected to buildings of usual design and other effects of earthquakes that would be expected at a particular location. The MMI is a good metric for estimating damages to structures.

Exposure Characterization

In order to calculate damages from an earthquake, the AIR model incorporates an extensive description both of the structural characteristics of an exposure and of the policy conditions describing the treatment of deductibles and other factors.

The seismic performance of a building depends primarily on the structural system resisting the lateral loads, but is also affected by other factors (including, in the AIR model, the age of the building and the height of the building). The age of the building is used to determine the likely code provisions under which the building was designed and constructed. Newer buildings, which may have been built to more exacting code provisions for seismic performance, are expected to perform better than older buildings.

The AIR model incorporates damageability relationships for many different classes of exposures, with up to three height categories in each class. There are 42 different damage relationships for each coverage type, plus several different age categories. The categories of structural types are based in part on the structural types defined in ATC-13,[4] although the actual damage relationships are modified and extended beyond those covered in that reference.

The exposures are characterized by policy limits for four different coverages:

- Coverage A refers to the dwelling limit;

- Coverage B refers to the appurtenant structures;

- Coverage C refers to personal property; and

- Coverage D refers to additional living expense.

Most commonly, Coverage B is combined with Coverage A for calculation purposes and is assumed to apply to the same structural type as Coverage A. The policy limit for each coverage may be defined by both a replacement value and a policy limit. The replacement value may rise in time without the policy limit being adjusted to reflect inflation. Damage is always calculated with respect to replacement value and then is capped at the policy limit if appropriate.

The location of the risk can be defined by a latitude and longitude point or by the five digit zip code in which the risk is located. The risk also can be associated with a line of business (homeowners, renters, commercial multiperil, etc.) in order to report losses separately in categories meaningful to the insurer.

Damage Estimation

Given the local shaking intensity in MMI units, damages to structures at a particular location can be calculated if sufficient information is available about the structure. Two types of damages are calculated by AIR: shake damage due to the lateral and vertical motions of the ground and fire damage due to earthquake-induced fires.

In order to calculate shake damage, the exposure information is combined with the level of shaking intensity at the building. Information on the structural characteristics of the properties at risk is used to

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[4]The Applied Technology Council is a 13 member advisory project engineering panel established in 1982 to develop earthquake damage/loss estimates for facilities in California.

select an appropriate damageability relationship (also sometimes called a *damage function* or a *fragility curve*) relating the probability of different levels of damage to the local shaking intensity (MMI). The damageability relationship is a complete probability distribution of damage, ranging from no damage to complete destruction (0 to 100 percent damage), with a probability corresponding to every level of damage. Thus the probability distribution is a continuous function of the local MMI level.

The earthquake damageability relationships have been derived and refined over a period of several years. They incorporate well-documented engineering studies by earthquake engineers and other experts both within and outside AIR. These damageability relationships also incorporate the results of post-earthquake field surveys performed by AIR engineers and others as well as detailed analyses of actual loss data provided to AIR by its client companies. These relationships are continually refined and validated.

Fire-Following Loss Estimation

Once the shake damages have been calculated for a particular earthquake, fire-following losses are estimated. This part of the model uses a separate simulation to estimate fire losses for each event.

First, the number of fires spawned by the earthquake is generated. The fire ignition rate is based on the local MMI intensity and the total population in the area. A number of fires is simulated for each affected zip code. The mean ignition rate increases as the MMI increases. The probability distribution of ignition rates is assumed to be uniform in some interval around the mean rate. Once the number of fires is simulated, each fire is randomly placed within a zip code and is assigned to affect either residential properties, commercial properties, and/or mobile homes.

The fire simulation then simulates the spread of the fires as well as the actions taken by local fire departments to control the fires. The fire spread rate is affected by a randomly selected wind speed appropriate for the location of the earthquake. Higher wind speeds increase the rate of spread of the fire.

Some of the factors included in the fire simulation are the time to report the fire, the time for one or more fire engines to reach the fire, and the availability of water to fight the fire. All of these factors are affected by the local MMI, as areas experiencing high shaking intensity are more likely to have obstructed roads and broken water mains. Also, the influence of fire breaks—wide roads or other natural impediments

to fire spread—is included in the simulation. Fire engines can move from fire to fire as fires are controlled.

Because the fire losses are determined by simulation, different levels of fire loss can be calculated for a given earthquake. Typically, the variability of fire losses is large, at least for the larger earthquakes, such that fire losses can vary by at least a factor of two if the same earthquake is simulated several times. This reflects the uncertainty in fire losses for larger earthquakes.

Actuarial Model Assumptions for Australian Inflation, Equity Returns, and Interest Rates

Michael Sherris*

Abstract[†]

Though actuaries have developed several types of stochastic investment models for inflation, stock market returns, and interest rates, there are two commonly used in practice: autoregressive time series models with normally distributed errors, and autoregressive conditional heteroscedasticity (ARCH) models. ARCH models are particularly suited when there is heteroscedasticity in inflation and interest rate series. In such cases nonnormal residuals are found in the empirical data. This paper examines whether Australian univariate inflation and interest rate data are consistent with autoregressive time series and ARCH model assumptions.

Key words and phrases: *stochastic investment models, heteroscedasticity, unit roots, ARCH, inflation, interest rates*

1   Introduction to ARCH Models

In recent years actuaries have developed and applied time series models of inflation, interest rates, and stock market returns to assist with pension and insurance financial management. Some of the earliest work in developing models for actuarial applications was performed by Wilkie (1986, refined in 1995). Carter (1991) develops an Australian version of the Wilkie model using traditional time series analysis of Australian time series data for inflation, equity markets, and interest rates. See Geoghegan *et al.*, (1992), Daykin and Hey (1989, 1990), and Boyle *et al.*, (1998, Chapter 9) for a discussion of these and other models and their actuarial applications.

The standard assumption in actuarial models is that the model errors are independent and identically distributed (i.i.d.) normal random variables. Inflation rates and interest rates are then modeled using *autoregressive time series*. A discrete time stochastic process $\{Y_t, t = 0, 1, \ldots, n, \ldots\}$, where Y_t is a real valued random variable at time t, is called an *autoregressive process* of order p, AR(p), if it can be represented as

$$Y_t = \mu + \sum_{k=1}^{p} \phi_1 (Y_{t-k} - \mu) + \epsilon_t \tag{1}$$

where $\mu = E[Y_t]$, p is a positive integer, and $\phi_1, \ldots, \phi_p$ are constants with $\phi_p \neq 0$. In addition, the ϵ_ts form a sequence of uncorrelated normal random variables with mean 0 and variance σ^2. The time series in equation (1) is stationary in the sense that it has a constant unconditional mean and variance. In practice the series used in actuarial applications, such as the inflation or interest rate, are assumed to be autoregressive and have constant unconditional means.

If the level of a series in equation (1) is not stationary, but the difference of the series (i.e., ΔY_t) is stationary, then the series is said to contain a unit root (or said to be integrated or order 1, or to be difference stationary). The existence of unit roots determines the nature of the trends in the series. If a series contains a unit root, then the trend in the series is stochastic and shocks to the series will be permanent. If the series does not contain a unit root, then the series is trend stationary. The trend in the series will be deterministic, and shocks to the series will be transitory.

When the i.i.d. error assumption is not practical, other models must be considered. One such model is the autoregressive conditional heteroscedasticity (ARCH) model. The ARCH model, introduced by Engle

(1982), allows for time-varying conditional variance by modeling the variance of the errors of a series, ν_t, as a function of past model errors, ϵ_t, using the equation:

$$\nu_t = \alpha_0 + \sum_{j=1}^{q} \alpha_j \epsilon_{t-j}^2 \qquad (2)$$

where q is the order of the ARCH process, or simply an ARCH(q) process. The errors of the series are obtained after fitting a mean equation to allow for mean reversion.

The GARCH model, introduced by Bollerslev (1986), allows the variance of the errors to depend on previous values of the variance as well as past errors using the equation:

$$\nu_t = \alpha_0 + \sum_{j=1}^{q} \alpha_j \epsilon_{t-j}^2 + \sum_{j=1}^{q} \phi_j \nu_{t-j}$$

which is referred to as a GARCH(p, q) process. Many other volatility models have been proposed: the exponential GARCH model (Nelson, 1991) and the nonlinear asymmetric GARCH model (Engle and Ng, 1993).

The models used for scenario generation as described in the actuarial literature typically use ARCH models. For example, Mulvey (1996) describes the Towers Perrin model where inflation is modeled as an autoregressive process with ARCH errors. Sherris, Tedesco, and Zehnwirth (1996), Harris (1994, 1995), and others support the need to model heteroscedasticity in Australian inflation and interest rates.

This paper will consider using ARCH models for Australian time series data. Specifically, the models assume ARCH and normal distribution of errors using Australian inflation, stock market, and interest rate time series data. The paper does not examine assumptions of independence of errors or model selection, and models will need to satisfy wider criteria than are examined in this paper. Carter (1991) and Harris (1994, 1995) have considered some of these issues for Australian data.

2 Australian Time Series Data

The data used for the empirical analysis in this paper are taken from the *Reserve Bank of Australia Bulletin* database. The study uses quarterly data. This is the highest frequency for which the inflation series

is available in Australia. The Australian Consumer Price Index is determined quarterly—a frequency suitable for many actuarial applications.

Different series are available over different time periods. The longest time period for which data are available on a quarterly basis for all of the financial and economic series is from September 1969.[1] The series considered are:

- The Consumer Price Index–All Groups (CPI);

- The All Ordinaries Share Price Index (SPI);

- Share dividend yields;

- The 90 day bank bill yields;

- The two year Treasury bond yields;

- The five year Treasury bond yields; and

- The ten year Treasury bond yields.

An index of dividends is constructed from the dividend yield and the Share Price Index series. Logarithms and differences of the logarithms are used in the analysis of the CPI, SPI, and dividends. The difference in the logarithms of the level of a series is the continuously compounded equivalent growth rate of the series.

Figures 1 through 8 provide time series plots of the series. An examination of the plots for the CPI, SPI and the Dividend Index series shows exponential growth. The plot of the logarithms of these series suggests that the series could be fluctuations around a linear trend in the logarithms. Such a series is referred to as *trend stationary*. The plot of the differences of the logarithms of these series appears to indicate a nonconstant variance or heterogeneity. Table 1 provides summary statistics for all of the series.

The interest rate series all show a changing level as interest rates rose during the 1970s and 1980s. Models of interest rates that incorporate mean-reversion, i.e., models that assume that the level of interest rates has constant unconditional mean and variance, are often used. This is not intuitive from our examination of the time series plots of the interest rates. The differences in the levels of the interest rates seem to fluctuate around a constant value, but the series appear to be heteroscedastic.

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[1]Individual series are available for differing time periods. For example, Phillips (1994) fits Bayes models to Australian macroeconomic time series. The data used are similar to those used here but cover different time periods.
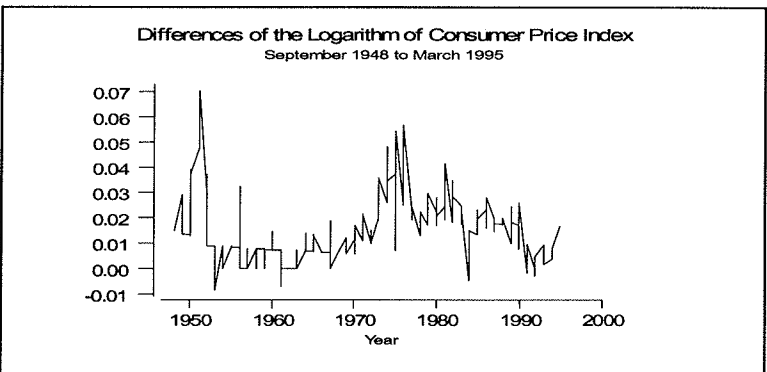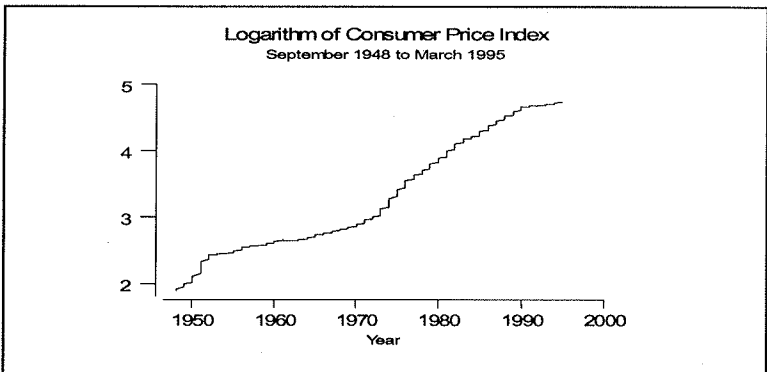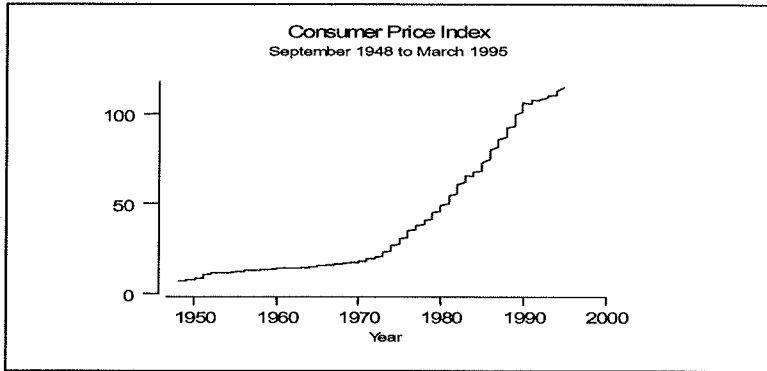
Figure 1
Consumer Price Index

Figure 2
All Ordinaries Share Price Index

All Ordinaries Share Price Index
September 1939 to March 1995

Logarithm of Share Price Index
March 1939 to March 1995
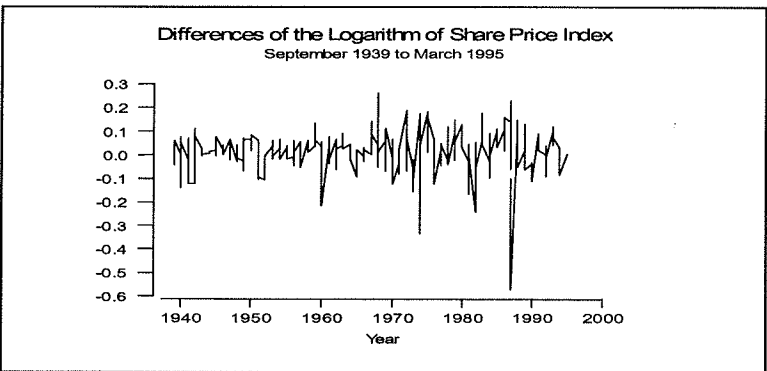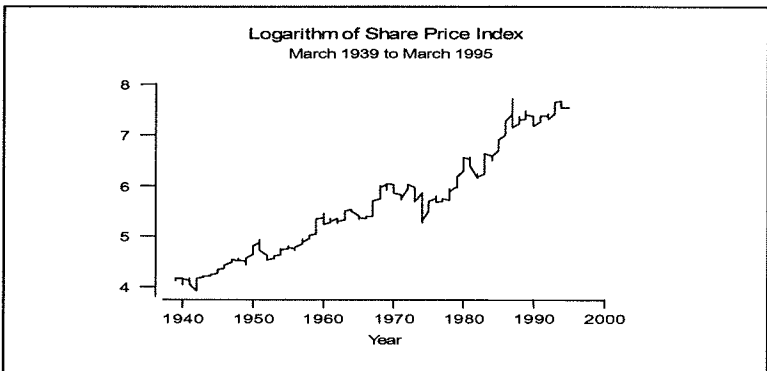

Differences of the Logarithm of Share Price Index
September 1939 to March 1995
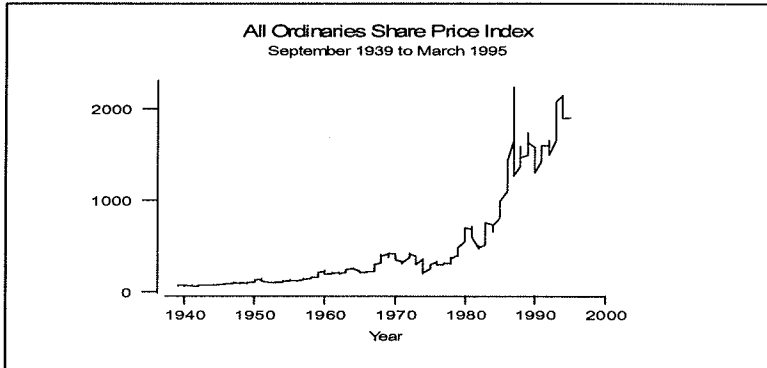
Figure 3
Share Price Dividend Index

Share Price Dividend Index
September 1967 to December 1994

Logarithm of Share Price Dividend Index
September 1967 to December 1994

Differences of the Logarithm of Share Price Dividend Index
September 1967 to December 1994
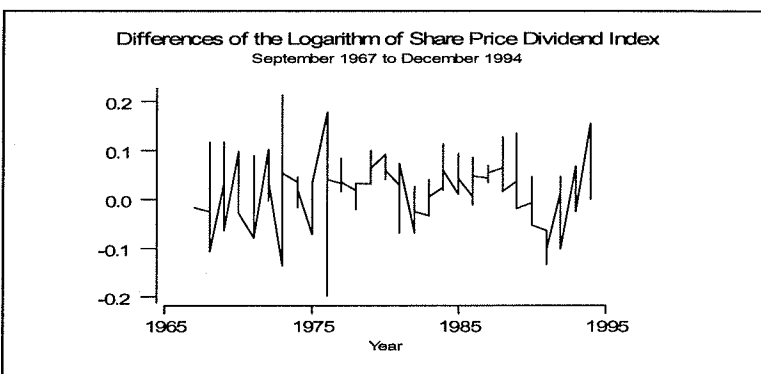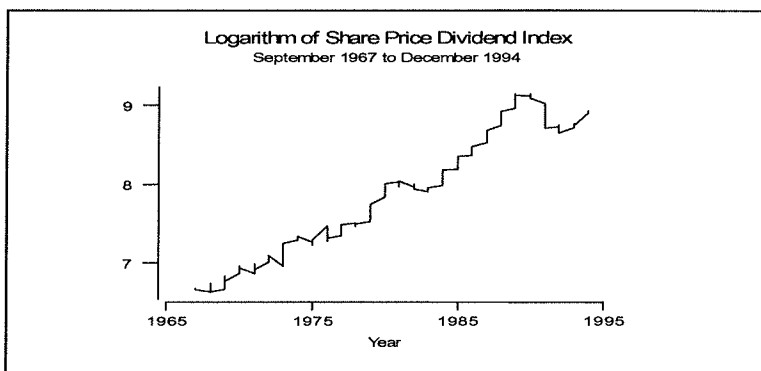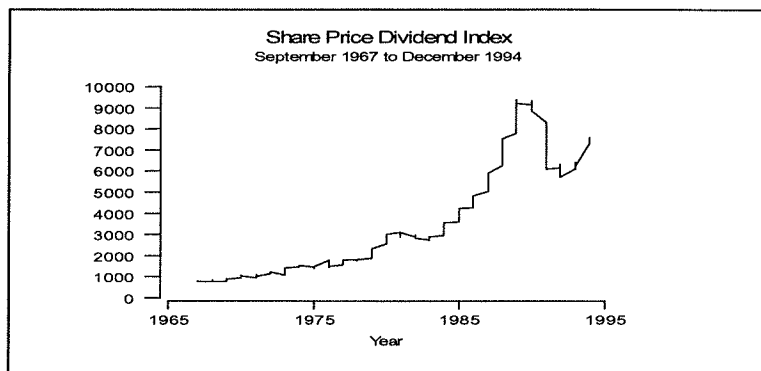
Figure 4
Dividend Yields

Dividend Yields
September 1967 to December 1994

Differences of the Dividend Yields
September 1967 to December 1994
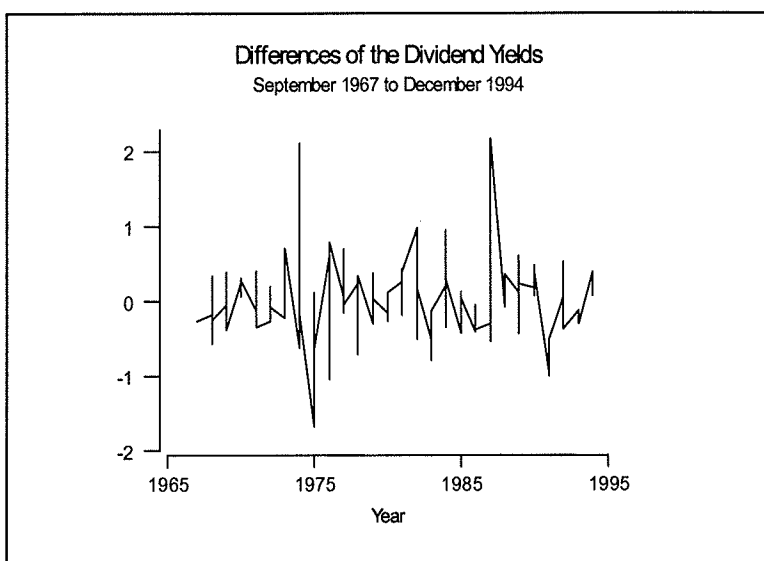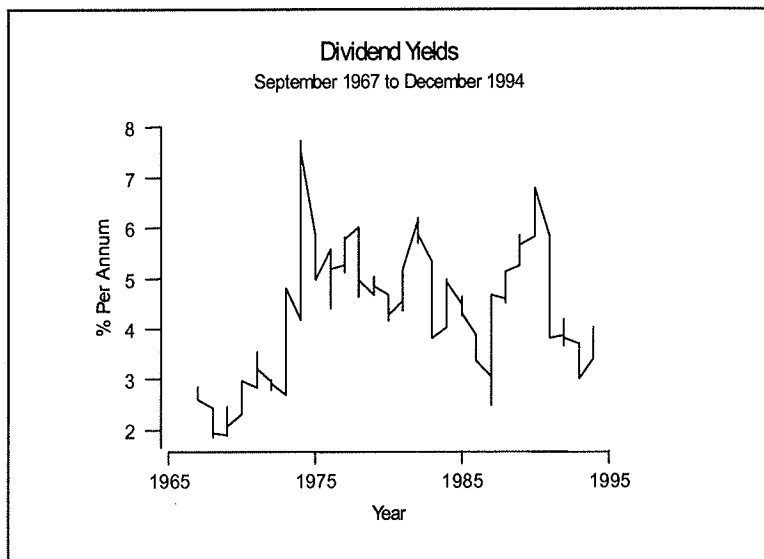
Figure 5
90 Day Bank Bill Yields

90 Day Bank Bill Yields
September 1967 to December 1994

Differences of 90 Day Bank Bill Yields
September 1967 to December 1994
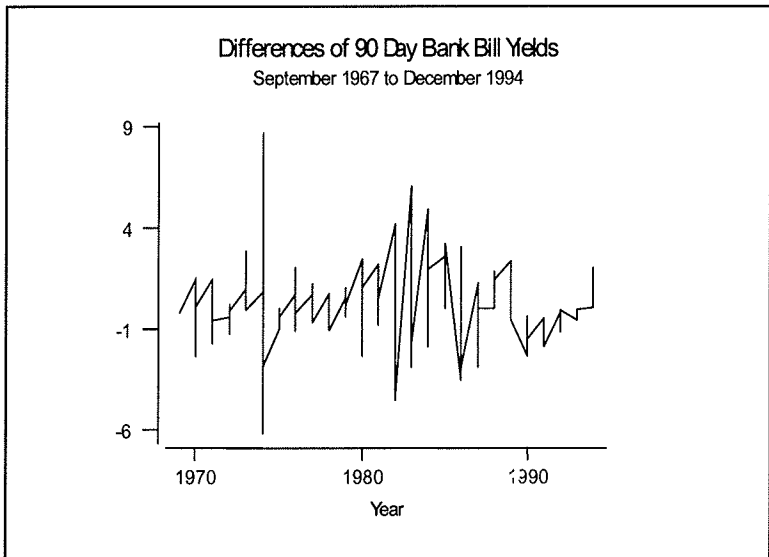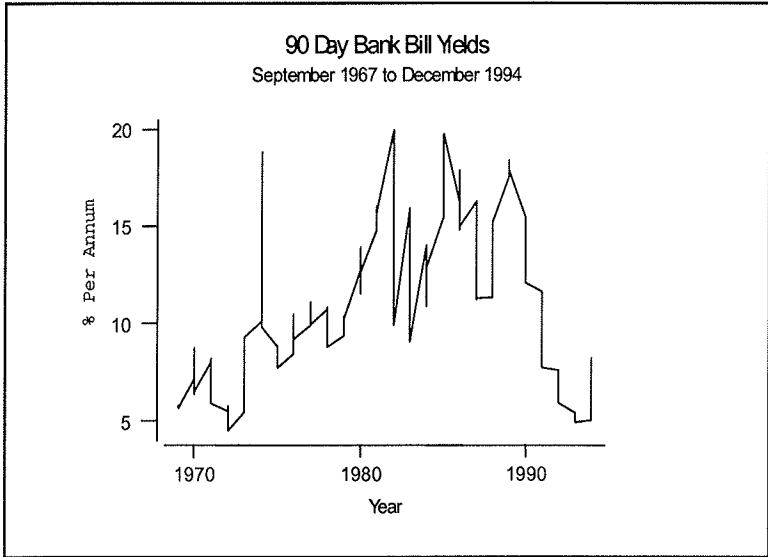
Figure 6
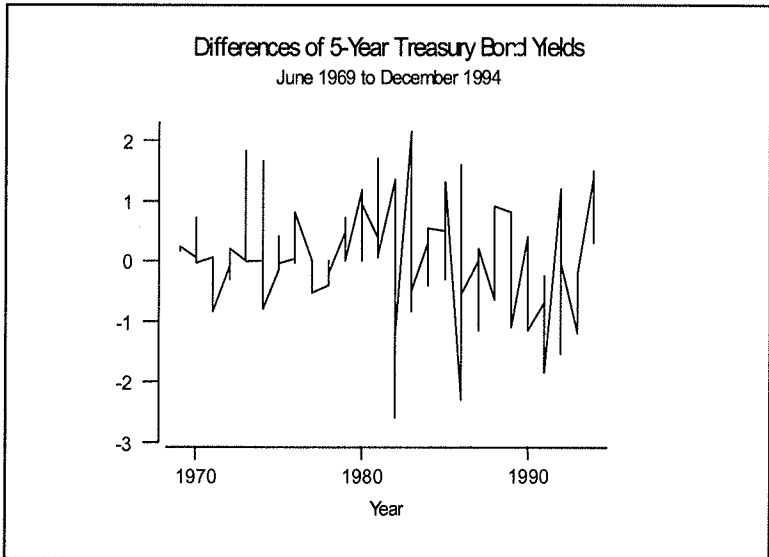## Two Year Treasury Bond Yields

2-Year Treasury Bond Yields
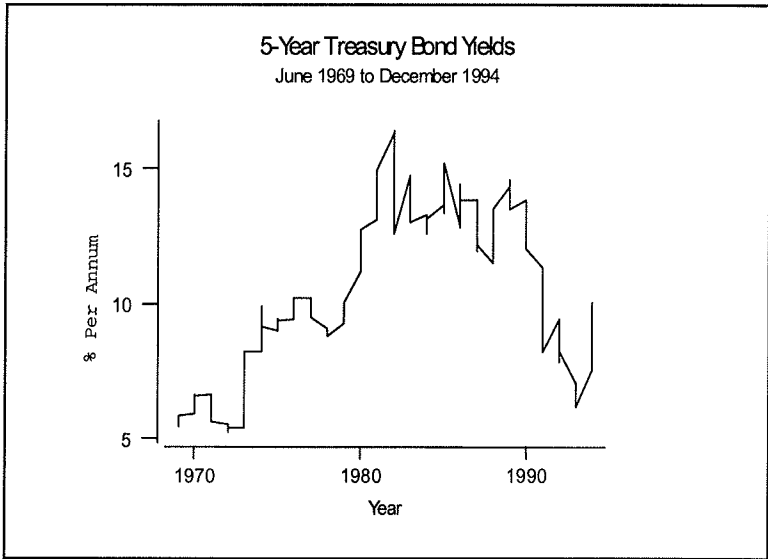September 1964 to December 1994

Differences of 2-Year Treasury Bond Yields
September 1964 to December 1994
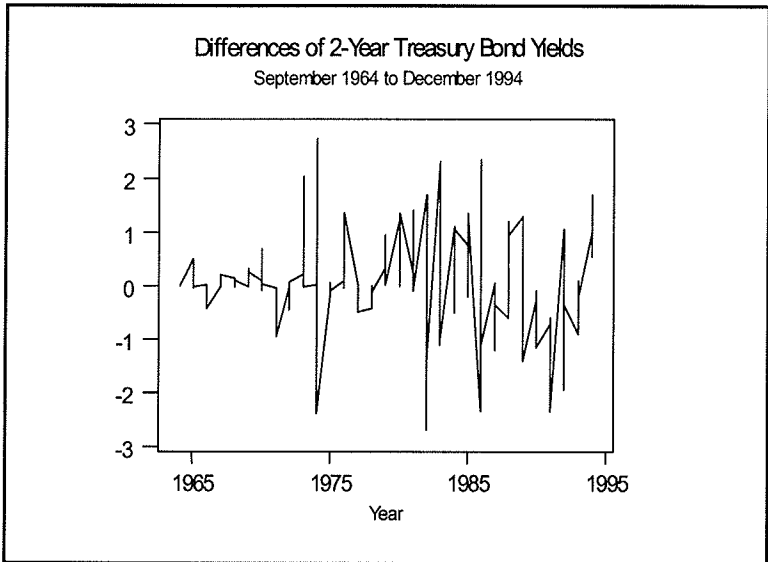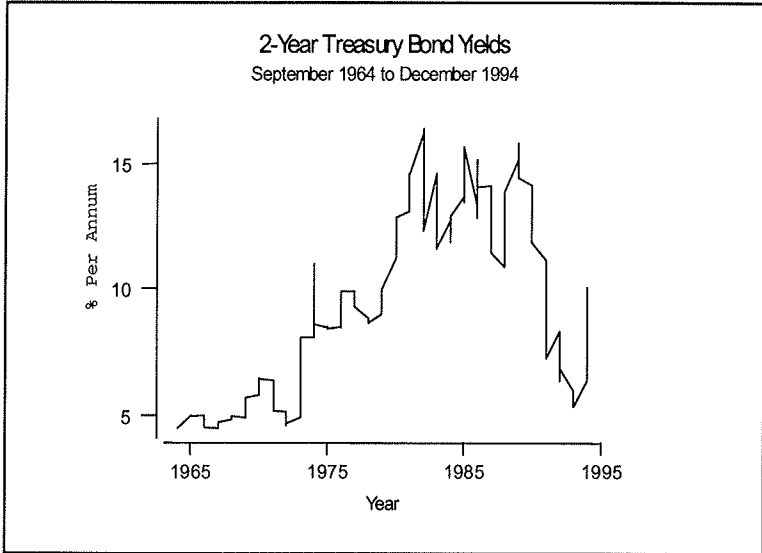
Figure 7
Five Year Treasury Bond Yields

5-Year Treasury Bond Yields
June 1969 to December 1994

Differences of 5-Year Treasury Bond Yields
June 1969 to December 1994

Figure 8
Ten Year Treasury Bond Yields

10-Year Treasury Bond Yields
March 1958 to December 1994

Differences of 10-Year Treasury Bond Yields
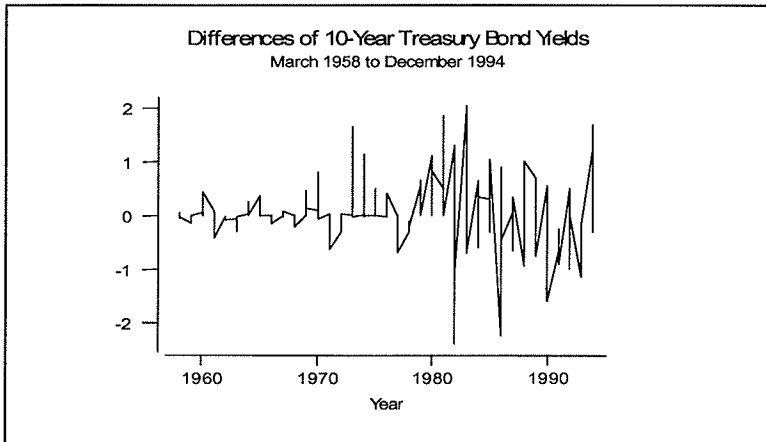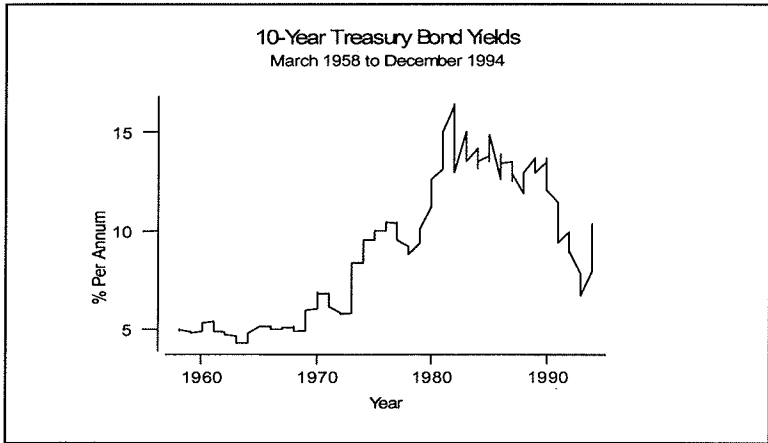March 1958 to December 1994

Table 1
Summary Statistics of All Series
Quarterly Data from September 1969 to December 1994

	Mean	STDEV	Max	Min	Median	Mode	SKEW	KURT
CPI	60.074	32.462	112.80	17.000	55.300	107.60	0.2375	-1.3631
C	3.9220	0.62386	4.7256	2.8332	4.0128	4.6784	-0.3408	-1.2288
SPI	865.01	595.05	2238.7	194.30	603.40	2238.7	0.6797	-1.0008
S	6.5177	0.71137	7.7137	5.2694	6.4026	7.7137	0.1667	-1.4523
DVY	4.4506	1.1496	7.7300	2.0700	4.5000	5.8500	0.2237	-0.1128
DVS	3741.5	2584.0	9398.3	861.74	2877.4	9398.3	0.7365	-0.7603
BB90	10.909	4.1029	19.950	4.4500	10.350	15.450	0.3310	-0.8313
TB2	10.185	3.2623	16.400	4.6000	9.9400	15.150	0.0137	-1.1443
TB5	10.465	2.9845	16.400	5.2000	10.030	13.850	-0.0775	-1.0843
TB10	10.648	2.8299	16.400	5.7500	10.180	9.5000	-0.0997	-1.0091

Notes: Quarterly data for all series were available from September 1969 to December 1994. The data are *CPI* = Consumer Price Index; *C* = ln(*CPI*); *SPI* = Share Price Index; *S* = ln(*SPI*); *DVY* = Share dividend yields; *DVS* = Share dividends series; *BB90* = 90 day bank bills yields; *TB2* = Two year treasury bond yields; *TB5* = Five year treasury bond yields; *TB10* = Ten year treasury bond yields. In addition, STDEV = Standard Deviation; SKEW = Coefficient of skewness; and KURT = Coefficient of excess kurtosis.

The following notation is used throughout the rest of the paper:

$$
\begin{aligned}
t &= \text{Number of quarters since January 1, 1969, } t = 1, 2, \ldots; \\
\epsilon_t &= \text{The error term at } t, \text{ for } t = 1, 2, \ldots; \\
CPI_t &= \text{Consumer Price Index for quarter } t; \\
C_t &= \ln(CPI_t); \\
\Delta f_t &= f_t - f_{t-1} \quad \text{for any function } f; \\
SPI_t &= \text{Share Price Index for quarter } t; \\
S_t &= \ln(SPI_t); \\
DVY_t &= \text{Dividend yield for quarter } t; \\
Y_t &= \ln(DVY_t); \\
DVI_t &= \text{Dividend index for the Australian data for quarter } t; \\
I_t &= \ln(DVI_t); \\
F_t &= \text{Force of interest for quarter } t.
\end{aligned}
$$

3   Analysis of the Australian Data

3.1   Inflation

Sherris, Tedesco, and Zehnwirth (1996) provide empirical evidence that the C_t series contains a unit root for Australian data. Although unit root tests can erroneously reject the hypothesis of a unit root in the presence of structural breaks[2] (Silvapulle, 1996) and are affected by additive outliers[3] (Shin, Sarkar, and Lee, 1996), this is not taken into account. Structural changes can lead to erroneous rejection of the hypothesis of a unit root.

An AR(1) model is fitted[4] (with a log-likelihood value of 331.778) to the CPI series to give

$$\Delta C_t = 0.0187 + 0.802(\Delta C_{t-1} - 0.0187) + 0.0090\epsilon_t. \tag{3}$$

This AR(1) model is examined first because it is used in actuarial applications with the assumption that the errors are normally distributed and with constant variance. Diagnostics for these model assumptions

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[2]A structural break occurs in the series where there is a discontinuity in the mean or the trend.

[3]An additive outlier is a single observation which is not consistent with the other observations in the series usually indicated by a highly significant t-ratio.

[4]All equations were fitted with the SHAZAM (1993) econometrics package.

are given in Table 2. The ARCH test of Engle (1982), is based on a regression of ϵ_t^2 on ϵ_{t-1}^2 and is a test for nonlinear dependence in the residuals.

The ARCH test regresses the squared residuals from the AR(1) model on a constant and the lagged squared residuals. The number of observations times the R^2 of this regression ($N \times R^2$) has an asymptotic χ^2 distribution with 1 degree of freedom.

The Jarque-Bera test is based on the statistic

$$N \times \left[\frac{y_1^2}{6} + \frac{y_2^2}{24} \right]$$

where y_1 is defined as the skewness and y_2 is defined as the excess kurtosis. This statistic has a χ^2 distribution with 2 degrees of freedom for large N. Skewness and excess kurtosis are defined as:

$$y_1 = \frac{m_3}{m_2^{3/2}} \quad \text{and} \quad y_2 = \frac{m_4}{m_2^2} - 3$$

where m_k is the k-th sample central moment, i.e.,

$$m_k = \frac{1}{N} \sum_{t=1}^{N} (\epsilon_t - \bar{\epsilon}).$$

Table 2
Quarterly Inflation Rate Autoregressive Model
AR(1) Model for C_t

Log-Likelihood Function Value		331.778
ARCH Test	2.535	(χ^2, 1 df, - 5% critical value 3.841)
Skewness	0.7781	(std. dev. is 0.240)
Excess Kurtosis	3.1785	(std. dev. is 0.476)
Jarque-Bera Test	46.8732	(χ^2, 2 df, 5% critical value 5.991)

The residuals for equation (3) are leptokurtic.[5] The statistical evidence for ARCH in this data series over this time period is not strong, although Sherris, Tedesco, and Zehnwirth (1996) find that a GARCH(1, 1) model fits ΔC_t well for the period September 1948 to March 1995.

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[5] A leptokurtic distribution is more peaked than the normal distribution and thus has fatter tails.

The inflation model described in Mulvey (1996) uses an ARCH model for volatility. An ARCH(1) model is fitted to the Australian quarterly CPI data to obtain

$$\Delta C_t = 0.0187 + 0.675(\Delta C_{t-1} - 0.0187) + \sigma_t \epsilon_t \qquad (4)$$
$$\sigma_t^2 = 0.00007 + 0.31\epsilon_{t-1}^2 \qquad (5)$$

with a log-likelihood value of 329.792. Diagnostics for ARCH and normal distribution of errors for this model are reported in Table 3.

Table 3
Quarterly Inflation Rate Autoregressive Model
AR(1) Model–ARCH(1) Model for C_t

Log-Likelihood Function Value		329.792
ARCH Test	0.294	$(\chi^2, 1 \text{ df})$
Skewness	0.6550	(std. dev. is 0.240)
Excess Kurtosis	3.7087	(std. dev. is 0.476)
Jarque-Bera Test	57.6464	$(\chi^2, 2 \text{ df})$

Although the model appears to capture ARCH in the volatility of the rate of inflation, the errors are still significantly nonnormal. The log-likelihood decreases. These results suggest that if an autoregressive model for the rate of inflation is used, the normality assumption for the errors will not be appropriate. An ARCH model with the assumption that errors are normally distributed is also not supported as an appropriate model for Australian inflation data. Because such an ARCH model is often used by actuaries in practice for inflation, some caution about the results from such a model is warranted.

3.2 Stock Market Series

The Wilkie (1986) approach to modeling stock returns uses a dividend yield and a dividend index. The model described in Mulvey (1996) divides stock returns into dividends and price appreciation. We consider models for price appreciation, dividend yields, and a dividend index for the Australian data. Sherris, Tedesco, and Zehnwirth (1996) present the results from unit root tests for the data considered here which indicate that the logarithm of the Australian Share Price Index, the logarithm of the dividends series, and dividend yields are difference

stationary. An important issue in equity market data is the allowance for share market crashes. In this paper we consider them as additive outliers.

Growth in an equity index and dividends are the two components of the return from equities that require modeling for actuarial applications. In this section models for the Australian equity market index and for dividends on the index are considered.

3.3   Share Price Index

Because we are interested in using volatility models for stock market returns we consider the following model for the Share Price Index:

$$\Delta S_t = \mu_S + \epsilon_t \sqrt{\nu_t} \qquad (6)$$

$$\nu_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 \qquad (7)$$

where $\mu_S = E[\Delta S_t]$.

Table 4 reports the results from fitting this model with ARCH(1) volatility. Note the α_1 parameter for ARCH volatility is significant at the 5 percent significance level. Based on the tests on the residuals given in Table 4, however, the residuals do not appear to be from a normal distribution. We have not tested these residuals for independence. Thus, although scenarios generated from a model using ARCH errors appear to be supported by the historical data, we should not use such a model in practice with the normal distribution of errors.

Because the quarter December 1987 appears in the residuals as an outlier corresponding to a stock market crash, it is of interest to determine the impact that this observation has on the results. This particular quarter is modeled as an additive outlier using a dummy variable denoted by $D(4, 87)$, i.e.,

$$D_t(4, 87) = \begin{cases} 1 & t \text{ denotes the quarter is December 1987;} \\ 0 & \text{otherwise.} \end{cases}$$

The AR(1) model is modified as:

$$\Delta S_t = \mu_S + \beta D_t(4, 87) + \epsilon_t. \qquad (8)$$

Table 5 reports the results of fitting equation (8) assuming constant variance.

The ARCH test indicates that an ARCH model should be considered for the volatility even after adjusting for the market crash outlier. The model used is

$$\Delta S_t = \mu_S + \beta D_t(4,87) + \epsilon_t \sqrt{\nu_t} \tag{9}$$

with equation (7) representing the ARCH(1) component. Table 6 reports the results of fitting equation (9). The ARCH parameter is not significant, and the results do not support ARCH errors in SPI returns after adjusting for the market crash using an additive outlier.

Table 4
ΔS_t with ARCH Errors

Log-Likelihood Function Value	83.992	
Mean Equation	Constant	
Coefficient	0.01634	
t-ratio	1.720	
Variance Equation ARCH	α_0	α_1
Coefficient	0.00795	0.40661
t-ratio	4.921	1.985
Diagnostics of Errors		
ARCH Test	0.105	(χ^2, 1 df)
Skewness	-0.6818	(std. dev. is 0.240)
Excess Kurtosis	1.2789	(std. dev. is 0.476)
Jarque-Bera Test	13.2316	(χ^2, 2 df)

3.4 Dividend Yields

Preliminary analysis using the unit root tests indicate that the logarithms of the dividend yields are difference stationary, so we consider the model:

$$\Delta Y_t = \mu_Y + \beta D_t(4,87) + \epsilon_t \sqrt{\nu_t} \tag{10}$$

with the ARCH(1) component as in equation (7). This model is fitted, and the ARCH test gives a significant result. An ARCH model is fitted for ν_t, and the results for the variance equation are reported in Table 7. The model appears satisfactory from the point of view of ARCH errors.

Autoregressive models for dividend yields are used in scenario generation for actuarial modeling. With this in mind, the following AR(1) model is used:

$$\Delta Y_t = \mu_Y + \psi \Delta Y_{t-1} + \beta D_t(4,87) + \epsilon_t \sqrt{\nu_t} \tag{11}$$

Table 5
ΔS_t with Constant Mean and Variance
and December 1987 Dummy Variable for Market Crash

Log-Likelihood Function Value	92.238	
Mean Equation	μ_S	β
Coefficient	0.02195	-0.59475
t-ratio	2.231	-6.035
Diagnostics of Errors		
ARCH Test	4.164	$(\chi^2, 1 \text{ df})$
Skewness	-0.7679	(std. dev. is 0.240)
Excess Kurtosis	1.4587	(std. dev. is 0.476)
Jarque-Bera Test	17.0620	$(\chi^2, 2 \text{ df})$

with the ARCH(1) component as in equation (7). Note that ψ is a constant.

We fit an AR(1) model to the dividend yield and check for outliers and ARCH. As would be expected given the share market index results, an outlier in the December 1987 quarter is detected corresponding to the share market crash. A dummy intervention variable is included for this observation and the residuals are tested for ARCH. The test is significant, so we fit an autoregressive model with ARCH errors as in equation (11). The residuals from this model do not reject the normal distribution assumption.

As noted earlier, in the actuarial literature models for scenario generation are based on autoregressive models for dividend yields and a normal distribution of errors. Such a model would have been considered satisfactory if no test for unit roots had been performed. Unit root tests, however suggest that the series is difference stationary and the difference stationary model would be preferred in this case.

3.5   Share Dividends

Sherris, Tedesco, and Zehnwirth (1996) construct a dividend index (DVI_t) for the Australian data. This index is defined as:

$$DVI_t = SPI_t \times DVY_t. \tag{12}$$

Modeling the rate of growth of dividends, $I_t = \ln(DVI_t)$, is difficult because dividends contain seasonal patterns. The difference series, ΔI_t,

Table 6

ΔS_t with Constant Mean, ARCH Errors,

and December 1987 Dummy Variable for Market Crash

Log-Likelihood Function Value	93.7266	
Mean Equation	μ_S	β
Coefficient	0.02195	-0.59475
t-ratio	2.395	-4.641
Variance Equation ARCH	α_0	α_1
Coefficient	0.00752	0.20405
t-ratio	5.262	1.346
Diagnostics of Errors		
ARCH Test	0.456	(χ^2, 1 df)
Skewness	-0.4209	(std. dev. is 0.240)
Excess Kurtosis	0.2998	(std. dev. is 0.476)
Jarque-Bera Test	3.10087	(χ^2, 2 df)

is first modeled as an AR(1) time series. The residuals from this model indicate ARCH and an outlier in the series in the June quarter of 1976. The cause of this outlier is not known. A dummy variable, $D_t(2,76)$, is defined as:

$$D_t(2,76) = \begin{cases} 1 & t \text{ denotes the quarter is June 1976;} \\ 0 & \text{otherwise.} \end{cases}$$

After including a dummy variable for the outlier, the model becomes:

$$\Delta I_t = \mu_I + \psi \Delta I_{t-1} + \beta D_t(2,76) + \epsilon_t \sqrt{\nu_t} \qquad (13)$$

with the ARCH(1) component as in equation (7). In this model of equation (13) the ARCH effect diminishes in significance. These results for the equity series are displayed in Table 8 support the point made in Chan and Wang (1996) that ARCH effects in share investment returns series are magnified by observations such as the crash that may be outliers.

3.6 Interest Rates

The interest rate series is transformed into a force of interest, F_t, using the transformations:

$$F_t = \begin{cases} \ln(1 + 90i_t/36500) & \text{for 90 day bank bill yields;} \\ \ln(1 + i_t/200) & \text{for 2, 5, and 10 year bond yields} \end{cases} \quad (14)$$

where i_t is the per annum percentage yield to maturity for the 90 day bank bill, two, five, and ten year bond for quarter t.

Sherris, Tedesco, and Zehnwirth (1996) present statistical support for these Australian bond yields containing a unit root and hence being difference stationary. In contrast, the assumption often used for scenario generation of future bill and bond yields in actuarial investment models is an autoregressive model. The standard unit root tests do not provide support for an autoregressive model for the Australian data series examined in this paper. These tests may have low power against close-to-stationary models.

For the interest rate series we consider models for the transformed interest rate series of the form

$$\Delta F_t = \mu_F + \epsilon_t \sqrt{\nu_t} \quad (15)$$

As before, models with constant volatility are considered initially.

For 90 day bank bills there is an outlier for the June 1994 quarter. This corresponds to a quarter when there was a significant tightening of monetary policy with the government raising short-term official interest rates dramatically. The series is adjusted for the effect of this outlier as follows:

$$F_t = \mu_I + \psi \Delta F_{t-1} + \beta D_t(2, 94) + \epsilon_t \sqrt{\nu_t} \quad (16)$$

where

$$D_t(2, 94) = \begin{cases} 1 & t \text{ denotes the quarter is June 1994;} \\ 0 & \text{otherwise.} \end{cases}$$

The adjusted series shows evidence of ARCH, so an ARCH model is fitted. Although this captures the ARCH effect, the normal distribution assumption for the residuals still is rejected.

Table 9 reports the fitted model and diagnostics for ARCH and normality for all of the bond series. For the two year bond yields there are no outliers and no evidence of ARCH, and the residuals appear to satisfy the normal distribution assumption. For the five year bond yields there are no outliers and no significant evidence of ARCH, but the residuals are negatively skewed and fat-tailed and reject the normal distribution assumption. In the case of the ten year bond yields there are no outliers

Table 7
ΔD_t (After Adjustment for Crash Dummy Variable)

Log-Likelihood Function Value	85.0154	
Variance Equation ARCH	α_0	α_1
Coefficient	0.00865	0.22236
t-ratio	5.006	1.448
Diagnostics of Errors		
ARCH Test	0.000	(χ^2, 1 df)
Skewness	0.170	(std. dev. is 0.240)
Excess Kurtosis	-0.0194	(std. dev. is 0.476)
Jarque-Bera Test	0.4984	(χ^2, 2 df)

and no evidence of ARCH. The residuals reject the normal distribution even more strongly than for the five year bond yields.

Autoregressive models are commonly used for interest rates in actuarial modeling. An AR(1) model of the form:

$$F_t = a_0 + a_1(F_{t-1} - a_0) + \epsilon_t \tag{17}$$

is fitted to the transformed yields for the Australian series. For the two year bond yields the parameter estimates (standard errors in parentheses) are $a_0 = 0.0534$ (0.0084) and $a_1 = 0.943$ (0.0301) with log-likelihood 399.3. This autoregressive model is used as the null hypothesis in a likelihood ratio test against the alternative of $a_1 = 1.0$ (a unit root), but the standard critical values reject the null hypothesis.

The AR(1) residuals reject the normal distribution assumption but show no significant statistical evidence of ARCH. This result holds for all of the autoregressive models fitted to the bond yield series. If an autoregressive model is used, then these results indicate that these interest rate models are not adequate and that adding ARCH volatility does not produce a better model.

4   Conclusions

The main aim of this paper has been to examine standard assumptions used in actuarial models for economic scenario generation. Quarterly Australian data for inflation, stock market, and interest rate series are examined to see if simple autoregressive models and ARCH models

Table 8
ΔI_t is AR(1) with ARCH errors and June, 1976 Dummy Variable

Log-Likelihood Function Value		142.19	
Mean Equation	μ_I	ψ	β
Coefficient	0.02419	-0.01256	-0.2206
t-ratio	4.175	-1.416	-2.584
Variance Equation ARCH		α_0	α_1
Coefficient		0.00299	0.17822
t-ratio		5.548	1.315
Diagnostics of Errors			
ARCH Test		0.001	$(\chi^2, 1 \text{ df})$
Skewness		-0.1549	(std. dev. is 0.240)
Excess Kurtosis		0.7948	(std. dev. is 0.476)
Jarque-Bera Test		2.4374	$(\chi^2, 2 \text{ df})$

of volatility with the assumption of a normal distribution of errors are reasonable. All of the analysis has been based on univariate series.

The results do not suggest that volatility in the series can be successfully modeled using an ARCH process. After allowing for additive outliers, some series do not show evidence of ARCH (for example, the rate of change of (transformed) bond yields). Equity returns show evidence of ARCH, even after adjusting for the effect of outliers such as the market crash. Outliers also increase the ARCH effect in the equity series.

The distribution assumed for errors in models used in practice must be considered carefully because the normal distribution assumption is not appropriate for errors based on the time series data for most of the models considered here. Alternative models and error distributions for economic scenario generation for actuarial applications require further investigation. It is not necessarily sufficient to use simple autoregressive models and a normal distribution for the errors. Even adding ARCH volatility in the hope that the normal distribution for errors will be adequate for modeling is not satisfactory.

This paper further demonstrates the need to model volatility in these series but indicates that the ARCH and normal distribution assumptions often used in practice and the actuarial literature are not supported by Australian historical data.

Comments on Some Parametric Models for Mortality Tables

Kam C. Yuen*

Abstract[†]

Parametric models for the entire age pattern of mortality have been suggested by Heligman and Pollard (1980) and Carriere (1992). The former is designed to fit the classical mortality pattern while the latter is supported by a statistical theory. Insights into their papers motivate us to consider a variation of the Heligman-Pollard model. We also apply these models to the 1993 Hong Kong Assured Lives Mortality Tables as well as the 1991 Hong Kong Female Life Table. This paper is not intended to construct a better parametric model for mortality tables; the main purpose is simply to provide insights into the potential of these models.

Key words and phrases: *Inverse-Weibull, Inverse-Gompertz, Gompertz, Weibull, mortality*

1   Introduction

The study of parametric models for mortality tables, sometimes referred to as the *law of mortality*, has been of interest to actuaries for many years. A good model can give us a better understanding of the underlying mechanism governing the mortality pattern. Recent development in this topic can be found in Forfar *et al.*, (1988), Renshaw (1991), Tenenbein and Vanderhoof (1980), and Wetterstrand (1981).

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A parametric model for mortality tables has many advantages: (i) it involves parameters having demographic and statistical interpretations; (ii) it is applicable to non-integral ages; (iii) it allows comparison among mortality tables by comparing only a few parameters, and; (iv) it provides a ready extrapolation beyond the range of the available data.

The purpose of this paper is to study two parametric models for modeling the pattern of mortality: one proposed by Heligman and Pollard (1980) and the other by Carriere (1992). A brief description of both models is given in Section 2. Insights into their papers and a variant on the Heligman-Pollard model are presented in Section 3. The Heligman-Pollard model is designed to fit the classical pattern of mortality; in some cases a modified version may perform better. In Section 4, we fit the models to the 1993 Hong Kong Assured Lives Mortality Tables presented by the Actuarial Society of Hong Kong (1993) and to the 1991 Hong Kong Female Life Table published by the Census and Statistics Department of Hong Kong (1992). To conclude this paper, we remark on various aspects of these two models.

Finally, the objective of this paper is not to build a better parametric model for mortality tables. Instead, we are interested in exploring modifications to these models that may be better. It is hard to say which model is the best. It all depends on the pattern of mortality, the theory behind the model, and the interpretation of the parameters. Therefore, it is wise to plot the mortality pattern and to consider various aspects of the visible mortality patterns before making a choice.

2   Heligman-Pollard and Carriere Models

We will present only a brief discussion of these models. For more details about the two models, we refer the reader to the original papers.

2.1   The Heligman-Pollard Model

Heligman and Pollard (1980) propose a mathematical expression for the graduation of the pattern of mortality that fits Australian mortality fairly well at all ages. Their law of mortality has the form

$$q_x/(1 - q_x) = A^{(x+B)^C} + D \exp\{-E(\ln x - \ln F)^2\} + GH^x \qquad (1)$$

where q_x is the probability that a person age x will die within a year.

Equation (1) contains three terms, each representing a distinct component of mortality. The first term reflects the fall in mortality during

childhood. The second term reflects the hump that generally exists between ages 10 and 40. This hump is a consequence of the elevated accident mortality for males and the increased accident mortality plus maternal mortality for females. The third term reflects the exponential pattern of mortality at adult ages.

Figure 1
Hong Kong 1991 Male Table: Plot of $\ln(q_x)$ vs. x
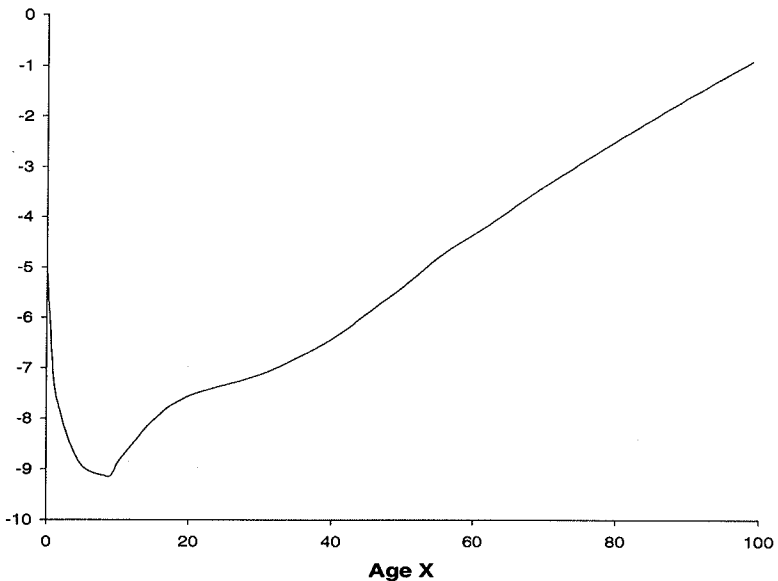
Many mortality tables exhibit the classical pattern suggested by equation (1). Such a pattern is illustrated by plotting $\ln(q_x)$ versus x, using the 1991 Hong Kong Male Life Table; see Figure 1. In Figure 1, we can immediately identify a fall at the early ages, a hump at around 22, and a linear component at the adult ages. The demographic interpretation of the eight parameters in Equation (1) is as follows: A measures the level of mortality in childhood; B is an age displacement to account for infant mortality; C measures the rate of mortality decline in childhood; D, E, and F represent the severity, spread, and location in the accident term, respectively; G represents the base level of mortality at the senior ages; while H reflects the rate of increase of mortality at the adult ages.

2.2   The Carriere Model

Carriere (1992) establishes another parametric model for life tables that also describes the entire age pattern of mortality. This model can be written as a mixture of n survival functions $(S_k(x),\ k = 1, \ldots, n)$, i.e.,

$$S(x) = \sum_{k=1}^{n} \omega_k S_k(x)$$

where the ω's are mixing probabilities with $\sum_{k=1}^{n} \omega_k = 1$. Then, q_x can be evaluated by the relationship

$$q_x = 1 - S(x + 1)/S(x).$$

For modeling the classical pattern of mortality, Carriere used $n = 3$, i.e.,

$$S(x) = \omega_1 S_1(x) + \omega_2 S_2(x) + \omega_3 S_3(x) \tag{2}$$

where the parameter ω_1 may be interpreted as the probability that a new life will die during childhood. Similar interpretations apply to ω_2 and ω_3.

Carriere argued that extreme-value survival functions are reasonable models for $S_k(x)$'s. Table 1 summarizes the distributions suggested by Carriere (1992). Note that one can choose either Inverse-Weibull or Inverse-Gompertz to depict the mortality for teenage years, i.e., the accident hump. From Table 1, we note that equation (2) represents an eight-parameter model just like equation (1).

The forms of the distributions in Table 1 look rather different from the standard ones, for example, $\mu_x = GH^x$, for the Gompertz distribution. Carriere claims that this reparametrization provides an insightful statistical interpretation in the sense that $m > 0$ is a measure of location and that $\sigma > 0$ is a measure of dispersion about m. For the Weibull distribution and the Inverse-Weibull distribution, however, m and σ are statistically informative only when σ is small relative to m.

3   Insights and Variations

It is well-known that the third term of equation (1) is the force of mortality of the Gompertz distribution. Helgiman and Pollard also mentioned in their paper that the second term of equation (1) is similar to the lognormal distribution.

Table 1
Component Distributions for Equation (2)

	Childhood ($S_1(x)$) Weibull	Teenage ($S_2(x)$) Inverse-Weibull	Teenage ($S_2(x)$) Inverse-Gompertz	Adult ($S_3(x)$) Gompertz
$S(x)$	$\exp\{-(\frac{x}{m})^{\frac{m}{\sigma}}\}$	$1-\exp\{-(\frac{x}{m})^{-\frac{m}{\sigma}}\}$	$\dfrac{1-\exp\{-e^{-\frac{(x-m)}{\sigma}}\}}{1-\exp\{-e^{\frac{m}{\sigma}}\}}$	$\exp\{e^{-\frac{m}{\sigma}}-e^{\frac{(x-m)}{\sigma}}\}$
$f(x)$	$\dfrac{\frac{1}{\sigma}(\frac{x}{m})^{\frac{m}{\sigma}-1}}{\exp\{(\frac{x}{m})^{\frac{m}{\sigma}}\}}$	$\dfrac{\frac{1}{\sigma}(\frac{x}{m})^{-\frac{m}{\sigma}-1}}{\exp\{(\frac{x}{m})^{-\frac{m}{\sigma}}\}}$	$\dfrac{\frac{1}{\sigma}\exp\{-\frac{x-m}{\sigma}-e^{-\frac{(x-m)}{\sigma}}\}}{1-\exp\{-e^{\frac{m}{\sigma}}\}}$	$\dfrac{\exp\{\frac{x-m}{\sigma}+e^{-\frac{m}{\sigma}}\}}{\sigma\exp\{e^{\frac{(x-m)}{\sigma}}\}}$
μ_x	$\frac{1}{\sigma}(\frac{x}{m})^{\frac{m}{\sigma}-1}$	$\dfrac{\frac{1}{\sigma}(\frac{x}{m})^{-\frac{m}{\sigma}-1}}{\exp\{(\frac{x}{m})^{-\frac{m}{\sigma}}\}-1}$	$\dfrac{\frac{1}{\sigma}\exp\{-\frac{x-m}{\sigma}\}}{\exp\{e^{-\frac{(x-m)}{\sigma}}\}-1}$	$\frac{1}{\sigma}\exp\{\frac{x-m}{\sigma}\}$

Note: Equation (4) uses a different parametrization for the Inverse-Gompertz density.

It appears that Helgiman and Pollard did not recognize the first term, which is equivalent to the three parameter Weibull survival function:

$$S(x) = A^{(x+B)^C}$$
$$= \exp\left(-(a(x+b))^c\right) \tag{3}$$

for $x > -b$, $a > 0$, and $c > 0$. Therefore, the parameters A, B and C can be rewritten as $\exp(-a^c)$, b, and c, respectively. If we let $b = 0$, then we have the two parameter (standard) Weibull distribution. Thus, the Heligman-Pollard model of equation (1) is related to three distinct lifetime distributions.

From the viewpoint of curve-fitting, the three parameter Weibull is better than the two parameter Weibull because the location parameter b can shift the two parameter Weibull curve back and forth. For example, if equation (1) does not have the parameter B, then q_0 will be fixed at $1/2$ no matter what values A and C may have (the value of G is usually small).

The implicit idea behind the Heligman-Pollard model is that there are three distinct components of human mortality. With this in mind, we may wish to find other functions to replace those in equation (1) provided that they can do the job better. In our opinion, the first and third terms of equation (1) fit extremely well. It is hard to find other functions to supersede them. We may use the Inverse-Gompertz or Inverse-Weibull, however, to handle the second component. For example, the model

$$\frac{q_x}{1-q_x} = A^{(x+B)^C} + D \times \frac{E\ln(\frac{1}{F})\,F^x\,\exp(-EF^x)}{1-\exp(-E)} + GH^x \tag{4}$$

fits the 1991 Hong Kong Female Life Table better than equation (1); see Tables 4 and 5. For computational and notational convenience, the Inverse-Gompertz density in the second term of equation (4) is reparametrized using E and F instead of the m and σ shown in Table 1. The parameters D, $-\ln(E)/\ln(F)$, $-1/\ln(F)$ can be interpreted as the severity, location, and spread in the accident component. This reparametrization also is used in the numerical examples given in the next section.

Under the Balducci assumption, the function $q_x/(1-q_x)$ is the same as the force of mortality at x. Hence, Carriere construes equation (1) as a total force of decrement that is equal to the sum of three forces of decrement from different causes. This interesting statistical idea still holds for equation (4). Provided that each of the three terms is

nonnegative for the range of x and that $\int_0^\infty q_x/(1 - q_x)dx = \infty$, other variations of the Heligman-Pollard model share the same interpretation.

Using the parametrization of the Weibull distribution in Table 1, it is possible to gain further insight by focusing on the mean, $m\Gamma(1 + \sigma/m)$, and the median, $m(\ln 2)^{\sigma/m}$. Each of these quantities is close to m when σ is small relative to m. With this restriction, the location parameter m and the dispersion parameter σ are statistically informative. The values of σ and m quoted in various applications of equation (2), however, do not conform with the assumption that $\sigma < m$. The same comments apply to the Inverse-Weilbull distribution.

In the theory of lifetime distribution (Lawless 1982), the parameter c in equation (3) or m/σ in equation (2) is known as the *shape* parameter because the shape of the Weibull density depends on the value of c. Furthermore, the parameter a in equation (3) or $1/m$ in equation (2) is called the *scale* parameter for the Weihull distribution because the effect of different values of a in equation (3) on the graph of the density is just to change the scale on the horizontal x-axis, and not the basic shape of the graph. The parameter b in equation (3) may be described as a *location* or *shift* parameter. In our opinion, the widely-used parametrization of equation (3) is more meaningful and natural.

4   Application to Hong Kong Mortality Tables

To illustrate the applications of the Heligman and Pollard (1980) and Carrier (1992) models, we now apply equations (1), (2), and (4) to several Hong Kong mortality tables. These equations are applied to the *smoothed mortality tables* and not to the raw mortality rates. The male and female tables are fitted separately.

4.1   1993 Hong Kong Assured Lives Mortality Tables

We apply equations (1), (2), and (4) to the 1993 Hong Kong Assured Lives Mortality Tables. Following the examples given in Heligman and Pollard (1980) and Carrier (1992), we estimate the parameters by minimizing the loss function L

$$L = \sum_{x=0}^{99} \left(1 - \frac{\hat{q}_x}{q_x}\right)^2 \tag{5}$$

where $\hat{q}_x$ is the estimate of q_x. This commonly-used loss function is based on the sum of squared relative errors. All parameter estimates

and the loss given in Tables 2 and 3 are calculated using SAS, a statistical software package.[1] For equation (2), we use the Inverse-Gompertz for teenage years because it fits both male and female tables better than the Inverse-Weibull. For the 1993 Hong Kong Assured Lives Mortality male table, the parameter values for the Weibull distribution are $m_1 = 719.42446$ and $\sigma_1 = 5956.388042$ in the Carriere model of equation (2). Hence, it is inappropriate to interpret m_1 and σ_1 as location and dispersion parameters, respectively, in this case.

The term GH^x can be expressed as H^{x-x_0} where x_0 is the age at which $q_x/(1 - q_x) = 1$ simply because the first and second terms of equations (1) and (4) are extremely small at that age. Admittedly, x_0 is close to the end of the life table.

<div align="center">

Table 2

Parameter Estimates Using Equations (1), (2), and (4)

for the 1993 Hong Kong Assured Lives Mortality Male Table
</div>

Heligman-Pollard Model:		
Equation (1) with Loss = 0.138381		
$A = 0.000474$	$B = 0.000475$	$C = 0.063875$
$D = 0.000221$	$E = 5.76254$	$F = 17.616056$
$G = 0.0000177511$	$H = 1.104655$	
Carriere Model:		
Equation (2) with Loss = 0.372211		
$\omega_1 = 0.023591$	$m_1 = 719.42446$	$\sigma_1 = 5956.388042$
$\omega_2 = 0.004303$	$m_2 = 17.825229$	$\sigma_2 = 6.361696$
$\omega_3 = 0.972106$	$m_3 = 87.200722$	$\sigma_3 = 10.121226$
Modified Heligman-Pollard Model:		
Equation (4) with Loss = 0.145928		
$A = 0.000474$	$B = 0.000484$	$C = 0.063961$
$D = 0.003088$	$E = 31.092184$	$F = 0.820805$
$G = 0.0000176876$	$H = 1.104667$	

In equation (4), x_0 is 109.9267 for males and 114.2788 for females. Based on the same model, comparison can be made between the two mortality tables. For instance, in equation (4), the value of G is higher and the values of x_0 is lower for males than for females, indicating higher male mortality. Actually, the same phenomenon can be seen in the parameter estimates derived from other models.

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[1]For more information on SAS see, for example, *SAS/STAT User's Guide*, Version 6.

Table 3
Parameter Estimates Using Equations (1), (2), and (4)
### for the 1993 Hong Kong Assured Lives Mortality Female Table

Heligman-Pollard Model:

Equation (1) with Loss = 0.387583

$A = 0.000474$	$B = 0.000378$	$C = 0.0622$
$D = 0.000244$	$E = 3.110173$	$F = 22.727553$
$G = 0.0000095144$	$H = 1.106253$	

Carriere Model:

Equation (2) with Loss = 0.580548

$\omega_1 = 0.019969$	$m_1 = 105.354377$	$\sigma_1 = 847.670466$
$\omega_2 = 0.008744$	$m_2 = 23.792527$	$\sigma_2 = 11.304304$
$\omega_3 = 0.971287$	$m_3 = 92.370704$	$\sigma_3 = 9.903073$

Modified Heligman-Pollard Model:

Equation (4) with Loss = 0.418212

$A = 0.000473$	$B = 0.000392$	$C = 0.062592$
$D = 0.006058$	$E = 13.989076$	$F = 0.892388$
$G = 0.0000091677$	$H = 1.106835$	

The pattern of mortality for the 1993 Hong Kong Assured Lives Mortality male and female tables can be described by the function $\ln(q_x)$ and is displayed in Figures 2 and 3, which plot $\ln(q_x)$ and $\ln(\hat{q}_x)$ for $x = 0,\ldots,99$ for male and female lives respectively. The male and female tables exhibit similar mortality pattern at childhood and adult ages except that the dip at around age 10 is lower for females.

Tables 2 and 3 show that the estimates of F in equation (1) are 17.6 for males and 22.7 for females. It does appear in Figures 2 and 3 that the hump has its peak at about these ages. The wider spread of the hump for female is reflected by the smaller value of E. The level of mortality in this region for both tables are more or less the same because there is only a slight difference in the values of D.

The estimated mortality rates fit the actual pattern reasonably well for each model. By comparing the loss, the Heligman-Pollard model of equation (1) is slightly better than equation (4). On the other hand, equation (2) does not fit as close as equation (1) and equation (4), especially for the 1993 Hong Kong Assured Lives Mortality male table.

Figure 2
1993 Hong Kong Assured Lives Mortality Male Table
**Plot of $\ln(q_x)$ and $\ln(\hat{q}_x)$ Using Equation (1)**
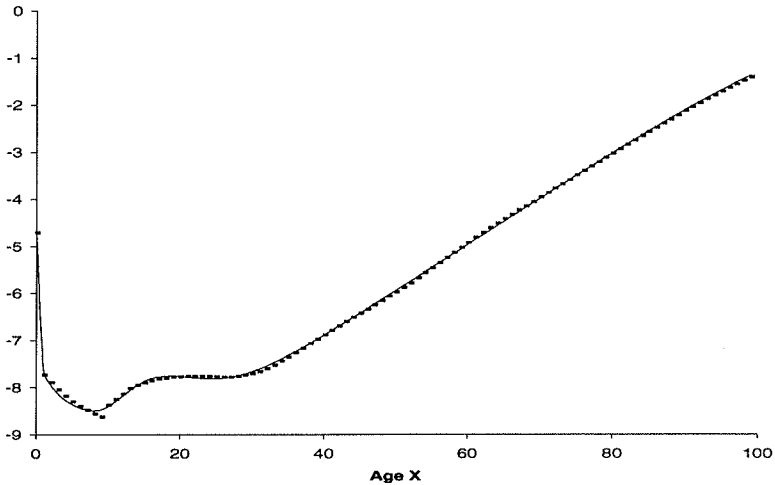
4.2   1991 Hong Kong Female Life Table

In addition, we fit equation (1) and equation (4) to the 1991 Hong Kong Female Life Table. Again, both models fit the pattern of mortality very well. Equation (4) has a smaller loss this time. This fact shows that equation (1) does not always give the best fit among the three models. It depends on the underlying pattern of mortality. The results are given in Tables 4 and 5.

The plots of residuals for Figures 2 and 3 are shown in Figures 4 and 5, respectively. Some of the residuals in the childhood ages are large. Particularly, equation (2) produces relatively large residuals in this region partly because the two parameter Weibull is used instead of the more flexible three parameter Weibull. The plots also indicate the presence of systematic bias at certain ages. This weakness is the price that we have to pay for using a parametric model to fit the entire mortality table that already contains smoothed values.

Figure 3
1993 Hong Kong Assured Lives Mortality Female Table
Plot of $\ln(q_x)$ and $\ln(\hat{q}_x)$ Using Equation (1)
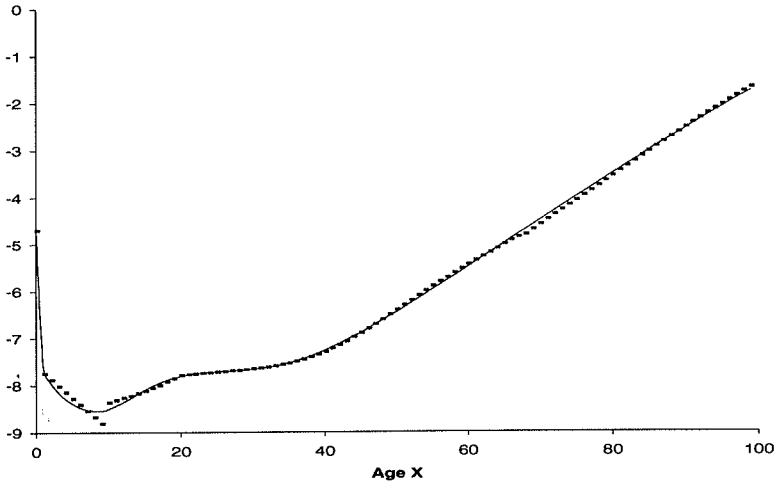
5 Some Closing Remarks

We should be aware that, in Section 4, the equations are fitted to the *smoothed mortality tables,* and not to the raw mortality rates, thereby constituting a rather unusual type of two stage smoothing process. If we fit the equations to the raw data, the loss may become larger than those shown in Tables 2 to 5. Under certain circumstances, the NLIN procedure of SAS is successful in minimizing equation (5) only if it has good starting values. In particular, the Carriere model of equation (2) converges slowly due to its complicated nature, compared to the other two models.

The fit during the childhood years could be improved for equation (2). This point was mentioned in Carriere's original paper as well. To obtain a better fit, we may employ the three parameter Weibull distribution with the idea of truncated survival distribution, instead of using the standard (two parameter) Weibull distribution. This change should provide at least a better interpretation of the parameters in the first term of the model.

Figure 4
1993 Hong Kong Assured Lives Mortality Male Table
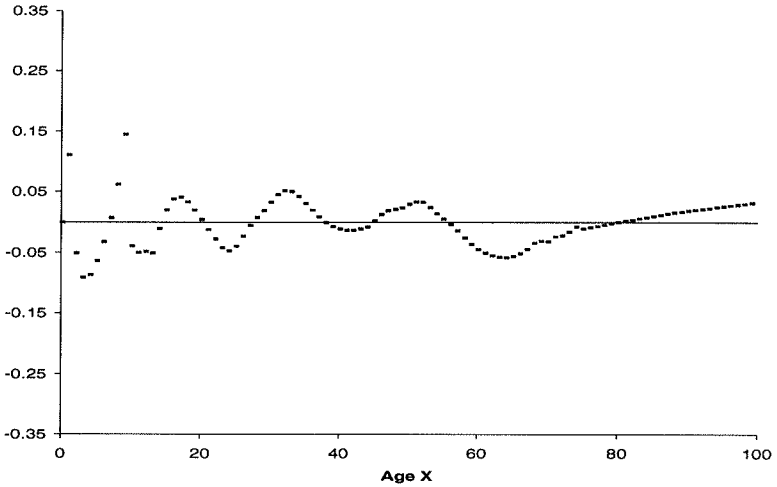**Residuals Resulting From Using Equation (1)**

Figure 5
1993 Hong Kong Assured Lives Mortality Female Table
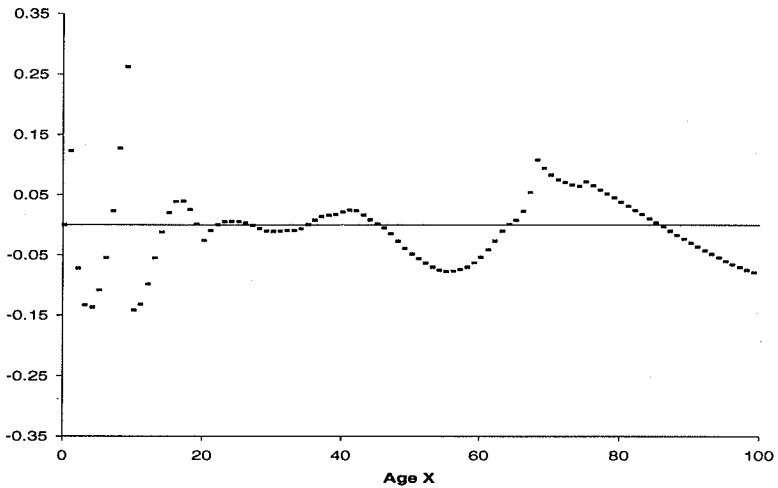**Residuals Resulting From Using Equation (1)**

Table 4
Parameter Estimates Using Equation (1)
for the 1991 Hong Kong Female Mortality Table

Heligman-Pollard Model:	
Equation (1) with Loss = 0.14638	
A = 0.000753	E = 0.888422
B = 0.158423	F = 23.71991
C = 0.167371	G = 0.0000076034
D = 0.000226	H = 1.11813

Table 5
Parameter Estimates Using Equation (4)
for the 1991 Hong Kong Female Mortality Table

Modified Heligman-Pollard Model:	
Equation (1) with Loss = 0.122038	
A = 0.001015	E = 4.946691
B = 0.317964	F = 0.933656
C = 0.229451	G = 0.000007981
D = 0.009157	H = 1.117485

Carriere (1994) proposes a select and ultimate parametric model which is based on equation (2). We feel that equation (1) also may be used to construct his select and ultimate parametric model because equation (1) has a much simpler form. This reconstruction seems feasible.

References

Carriere, J.F. "A Select and Ultimate Parametric Model." *Transactions of the Society of Actuaries* 46 (1994): 33–50.

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Forfar, D.O., McCutcheon, J.J. and Wilkie, A.D. "On Graduation by Mathematical Formula." *Journal of the Institute of Actuaries* 115 (1988): 1–149.