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Abstract†

This paper presents a new method of obtaining quick and accurate values and deltas for discrete lookback options using Taylor series expansions. This method is applied to the case of ratchet guaranteed minimum death benefits attached to annuity contracts, and the method is extended to include annuities where a fixed fund is attached to the variable account. Finally, both the speed and the accuracy of the method are compared to Monte Carlo simulation and the exact analytic solution. The Taylor expansion method is shown to be faster and, in most cases, more accurate than the alternative methods.

Key words and phrases: *Taylor series, multivariate normal, lookback option, Monte Carlo simulation, risk, lognormal distribution, Black-Scholes formula, geometric Brownian motion*

1 Introduction

One of the biggest developments in the life insurance industry in the last decade or so has been the invention and growth of various equity options embedded in variable annuity contracts. These options range

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from simple (e.g., guaranteeing the return of principal invested should the annuitant die), to complex options such as a minimum guaranteed fund amount equal to some function of the past history of the fund should the annuitant choose to annuitize at guaranteed purchase rates (Milevsky and Posner, 2001).

The market for customers of variable annuities is competitive, and one of the ways producers attempt to distinguish themselves both from mutual fund providers and other variable annuity providers is by including a death benefit option with the contract. The cheapest and simplest case is one where the insurance company promises to pay out at least a return of the premium paid into the contract on the death of the contract owner, regardless of the actual performance of the underlying funds. The death benefit is commonly made more complicated, as well as more valuable and more expensive in several ways. Many companies offer not only a return of the premiums paid but an accumulation of the premiums at a minimal interest rate on death of the contract owner.

Even more generous provisions exist including reset and ratchet benefits. Death benefits on contracts with a reset provision will be reset to the fund value at various times during the life of the contract and can move up or down, but usually not below the return of premium. Death benefits on ratchet contracts ratchet to the value of the fund at various times during the life of the contract, but only if the resulting benefit is higher than the one in force before the ratchet. Otherwise, the death benefit remains where it was before the ratchet date. A good overview of the state of the market is contained in Milevsky and Posner (2001).

Many companies market and sell annuity products without recognizing the need to hedge the underlying risk and thereby expose themselves to unnecessary levels of equity risk. In addition, financial papers frequently present results for more traditional traded options that are not easily transferred to an insurance environment where the options are embedded in other contracts. Also, analytic results are not obtainable in all cases. When they are, they frequently require the calculation of the multivariate cumulative normal distribution function. This function is not directly computable. Although approximations do exist, they are slow in practice. Therefore, insurance companies usually resort to Monte Carlo simulations to value these options. It can be time-consuming to obtain even a passable value for all the options in all inforce contracts by this method. The situation becomes even worse when simulating the option value along many paths for cash flow testing or ALM purposes. The value of the option must be obtained by simulation at each time period along each testing path that could lead to billions of required simulations.

Much work has been done to attempt to value these options. An exact analytic solution was obtained for an individual discrete lookback put by Collin-Dufresne, Keirstad, and Ross (1997) by using a change of numeraire, which is theoretically valuable, but involves the cumulative multivariate normal distribution—a distribution that is difficult and time-consuming to evaluate in practice. In addition, the result has been obtained for a variable fund only, while this paper addresses the addition of a fixed fund to the account as well. Tiong (2000) uses the method of Esscher transforms pioneered by Gerber and Shiu (1994) to obtain analytic solutions for cliquet options in equity indexed annuities. Also, Milevsky and Posner (2001) have recently valued lookback guaranteed minimum death benefit (GMDB) options analytically in the specific case of an at-the-money continuous lookback option on a variable fund only, when mortality follows some simple analytic forms.

Why is there a need for a new paper on this subject? The approach of this paper addresses some of the major shortcomings in the practical implementation of the methods described above. The analytic solution is theoretically valuable but there are two major drawbacks involved with using it. First, it is time-consuming. There is no easy way to evaluate the function quickly and accurately because of the large number of multivariate normal functions that must be evaluated. The evaluation of the cumulative multivariate normal function has been addressed by many authors, including Gupta (1963), Wang and Kennedy (1990), Wang (1991), Terza and Welland (1991), Genz (1992), and Genz (2004). The method used in the actual comparisons is that of Somerville (1998a, 1998b). Second, the result has been obtained for a variable fund only, while this paper addresses the addition of a fixed fund to the account as well.

This paper describes a method to compute the Taylor coefficients of the value of a discrete lookback put option, a method that can easily be extended to the case of a ratchet GMDB where the value of the contract at death is the maximum value of the contract at any policy anniversary or the account value at death if larger. It can also be extended to fit the general case an insurance company faces where the variable account is attached to various other accounts that earn a fixed rate independent of equity performance.

While a Taylor series expansion is not as theoretically appealing as a closed form solution, it is just as valuable in practice, especially if it produces relatively quick and accurate results. The major problem, however, is determining the coefficients of the Taylor series expansion. The next section addresses this issue.

2 Taylor Expansion of Discrete Lookback Put

2.1 Black-Scholes Case

This paper primarily addresses the ratchet GMDB where the death benefit is the maximum value the contract attains on any policy anniversary or the contract value at death, whichever is greater. This is analogous to a series of discrete lookback puts where the notional amount of the put is equal to the original fund multiplied by the probability that the annuitant dies at that point without having previously lapsed his or her policy. This requires a model of surrender and mortality that also affects the GMDB value. We begin by valuing an arbitrary discrete lookback put and then show how it can be extended to cases more relevant to insurance company annuities.

Consider a put issued at time t_0 and coming due at some known and fixed time t_N. Its value at time t_N is the maximum of the underlying fund values at times $t_1 < t_2 < t_3 \cdots < t_{N-1} < t_N$ and the initial strike X_0. We want to determine V_0 (the ratchet GMDB value at time t_0) given F_0 (the total fund value at time t_0), X_0 (the strike at time t_0), and p_f and p_v (the initial percentages invested in fixed and variable accounts). In addition, the risk free rate (r), stock volatility (σ), asset charges (q), which are analogous to dividends in the analysis, and fixed growth rate (g) are assumed known. For $n = 1, 2, \ldots,$, we assume V_n is of the form:

$$V_n = F_n \sum_{j,k}^{k \le j} f_{jkn} \left(\ln \frac{S_n}{J_n} \right) \phi_n^j \left(\frac{S_n}{J_n} \right)^k \tag{1}$$

where J_n is the adjusted strike price given by

$$J_n = \left(\frac{X_n / F_n - p_f}{p_v} \right) S_n,$$

S_n is the stock index at time t_n, F_n is the account value at time t_n that behaves as

$$F_n = \left(p_{f_{n-1}} \left(1 + i_{n-1}\right) + p_{v_{n-1}} e^{y_n} \right) F_{n-1}$$

$$\phi_n = i_n \frac{p_{f_n}}{p_{v_n}} \tag{2}$$

where p_{f_n} and p_{v_n} are the percentages of the fund in the fixed and variable account at time t_n with $p_{f_n} + p_{v_n} = 1$, and

$$i_n = \exp\left(g\left(t_{n+1} - t_n\right)\right) - 1.$$

Note that J_n is equivalent to the strike on the variable fund only, taking out the effect of the fixed account. Assume the Taylor series coefficients of f_{jkn} are known, we show that if V_n is of the form given in equation (1) then so is V_{n-1}. As V_{N-1} is simply the Black-Scholes result and can be shown to be of the form given in equation (1), then V_0 can be shown to be of the same form by induction. The values of the Taylor coefficients are derived automatically during the induction step.

First, we derive the Taylor expansion for a Black-Scholes put at time t_{N-1} with payoff at time t_N. While this expansion could be obtained by repeated differentiation of the Black-Scholes formula, it will be derived in a more complicated manner similar to algorithmic differentiation [see Wengert (1964)] to illustrate some of the general concepts used in the inductive step. Let S_N be the stock level and F_N be the fund level at time t_N. Assume risk-neutral valuation, $\mu = r - q - \sigma^2/2$, and define

$$\begin{aligned} r_n &= \left(t_{n+1} - t_n\right) r, \\ \mu_n &= \left(t_{n+1} - t_n\right) \mu, \\ \sigma_n &= \sigma\sqrt{\left(t_{n+1} - t_n\right)}, \end{aligned}$$

and

$$y_n = \ln\left(S_n / S_{n-1}\right)$$

with i_n and ϕ_n defined as above. Then X_{N-1} is the strike on the fund at time t_{N-1}, and the put obeys the equation:

$$V_{N-1} = e^{-r_{N-1}} \mathbb{E}_{S_{N-1}}\left[\left(X_{N-1} - F_N\right)_+\right] \tag{3}$$

where $x_+ = \max(0, x)$, and $\mathbb{E}_{S_{N-1}}$ denotes expectation condition on (i.e., given) S_{N-1}. We will assume the stock index follows a geometric Brownian motion so that y_n is normally distributed with mean μ_{n-1} and variance σ^2_{n-1}, i.e., $y_n \sim \mathcal{N}(\mu_{n-1}, \sigma^2_{n-1})$. The integral for the value of this put option is given by:

$$V_{N-1} = F_{N-1}\left[\frac{p_{v_{N-1}}e^{-r_{N-1}}}{\sqrt{2\pi\sigma_{N-1}^2}}\right]\int_{-\infty}^{-\xi_{N-1}}\left(e^{-\xi_{N-1}} - e^{y_N}\right) \times \exp\left(\frac{-y_N^2}{2\sigma_{N-1}^2} + \frac{2\mu_{N-1}y_N}{2\sigma_{N-1}^2} + \frac{-\mu_{N-1}^2}{2\sigma_{N-1}^2}\right)\mathrm{d}y_N \quad (4)$$

where

$$\xi_n = -\ln\left(J_n/S_n - \phi_n\right). \quad (5)$$

Equation (4) can be explained by assuming the option is on the variable fund only (hence the p_v term in front of the integral), the stock market is normalized to 1 at time t_{N-1}, and the adjusted strike on the variable fund drops by ϕ_{N-1} due to an increase in the relative size of the fixed fund between t_{N-1} and t_N. Equation (4) can be divided into two integrals, one involving $e^{-\xi_{N-1}}$ and the other involving e^{y_N}. The terms independent of y_N can be pulled outside the integral. The remaining terms can be expanded into their individual Taylor expansions and multiplied term by term. This creates two integrals to evaluate:

$$F_{N-1}\left[\frac{p_{v_{N-1}}e^{-r_{N-1}}e^{-\mu_{N-1}^2/2\sigma_{N-1}^2}}{\sqrt{2\pi\sigma_{N-1}^2}}\right] \times\left[e^{-\xi_{N-1}}\int_{-\infty}^{-\xi_{N-1}}[1 - \frac{y_N^2}{2\sigma_{N-1}^2} + \frac{y_N^4}{2!\left(2\sigma_{N-1}^2\right)} - \cdots] \times[1 + \left(\frac{\mu_{N-1}}{\sigma_{N-1}^2}\right)y_N + \frac{1}{2!}\left(\frac{\mu_{N-1}}{\sigma_{N-1}^2}\right)^2 y_N^2 + \cdots]\mathrm{d}y_N\right] \quad (6)$$

and

$$F_{N-1}\left[\frac{p_{v_{N-1}}e^{-r_{N-1}}e^{-\mu_{N-1}^2/2\sigma_{N-1}^2}}{\sqrt{2\pi\sigma_{N-1}^2}}\right]\int_{-\infty}^{-\xi_{N-1}}[1 - \frac{y_N^2}{2\sigma_{N-1}^2} + \frac{y_N^4}{2!\left(2\sigma_{N-1}^2\right)} - \cdots] \times[1 + \left(\frac{\mu_{N-1}}{\sigma_{N-1}^2} + 1\right)y_N + \frac{1}{2!}\left(\frac{\mu_{N-1}}{\sigma_{N-1}^2} + 1\right)^2 y_N^2 + \cdots]\mathrm{d}y_N. \quad (7)$$

The integrals in expressions (6) and (7) can be split into two integrals, the first one from $-\infty$ to 0 and the second one from 0 to $-\xi_{N-1}$. A Taylor expansion in ξ_{N-1} can be obtained for the integral from 0 to $-\xi_{N-1}$ by multiplying the internal expansions in y_N term by term, integrating term by term and substituting $-\xi_{N-1}$ at the upper bound. Evaluation at the lower bound gives zero. The integrals from $-\infty$ to 0 can be done exactly from equation (4) without any expansions:

$$\sqrt{2\pi\sigma_{N-1}^2}\; e^{\mu_{N-1}^2/2\sigma_{N-1}^2}\, \mathcal{N}\left(-\frac{\mu_{N-1}}{\sigma_{N-1}}\right)$$

for expression (6) and

$$\sqrt{2\pi\sigma_{N-1}^2}\; e^{(\mu_{N-1}+\sigma_{N-1}^2)^2/2\sigma_{N-1}^2}\, \mathcal{N}\left(-\left(\frac{\mu_{N-1}}{\sigma_{N-1}}+\sigma_{N-1}\right)\right)$$

for expression (7).

Finally, the Taylor expansion for the integral in expression (6) can be multiplied by the expansion of $e^{-\xi_{N-1}}$ term by term, then added to the expansion for the integral in expression (7). Define a function $\alpha_{N-1}(\xi_{N-1})$ such that

$$\begin{aligned} V_{N-1} &= F_{N-1}p_{v_{N-1}}\alpha_{N-1}(\xi_{N-1}) \\ &= F_{N-1}p_{v_{N-1}}\alpha_{N-1}(-\ln[J_{N-1}/S_{N-1}-\phi_{N-1}]). \end{aligned} \tag{8}$$

To put this into the form of equation (1), we will assume the parameter $\phi_{N-1}/(J_{N-1}/S_{N-1})$ is sufficiently small so that all expansions are valid. (This parameter is small if the percentage of funds in the fixed fund is small.) In Section 3 we show how to expand the radius of convergence if this parameter is not small.

First, we expand the logarithm:

$$\begin{aligned} V_{N-1} &= F_{N-1}p_{v_{N-1}}\alpha_{N-1}\left(\ln\left(\frac{S_{n-1}}{J_{n-1}}\right)-\ln\left(1-\frac{\phi_{N-1}}{J_{N-1}/S_{N-1}}\right)\right) \\ &= F_{N-1}p_{v_{N-1}}\alpha_{N-1}\left(\ln\left(\frac{S_{n-1}}{J_{n-1}}\right)+\psi\right) \end{aligned}$$

where

$$\psi=\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\phi_{N-1}}{J_{N-1}/S_{N-1}}\right)^k.$$

As $\phi_{N-1}/(J_{N-1}/S_{N-1})$ is small so is ψ. A Taylor series expansion of α_{N-1} about $\ln(S_{n-1}/J_{n-1})$ yields:

$$V_{N-1} = F_{N-1} p_{v_{N-1}} \left[\alpha_{N-1}\left(\ln\left(\frac{S_{N-1}}{J_{N-1}}\right)\right) + \sum_{k=1}^{\infty} \alpha_{N-1}^{(k)}\left(\ln\left(\frac{S_{N-1}}{J_{N-1}}\right)\right)\frac{\psi^k}{k}\right],$$

where $\alpha_{N-1}^{(k)}$ denotes the k^{th} derivative of α_{N-1}. Rearranging this expression for V_{N-1} as a power series in $\phi_{N-1}/(J_{N-1}/S_{N-1})$ gives

$$\begin{aligned} V_{N-1} = F_{N-1} p_{v_{N-1}} \Bigg\{ & \alpha_{N-1}\left(\ln\left(\frac{S_{N-1}}{J_{N-1}}\right)\right) + \alpha_{N-1}^{(1)}\left(\ln\left(\frac{S_{N-1}}{J_{N-1}}\right)\right)\frac{\phi_{N-1}}{J_{N-1}/S_{N-1}} \\ & + \left[\frac{1}{2}\alpha_{N-1}^{(1)}\left(\ln\left(\frac{S_{N-1}}{J_{N-1}}\right)\right) + \frac{1}{2}\alpha_{N-1}^{(2)}\left(\ln\left(\frac{S_{N-1}}{J_{N-1}}\right)\right)\right]\left(\frac{\phi_{N-1}}{J_{N-1}/S_{N-1}}\right)^2 \\ & + \left[\frac{1}{3}\alpha_{N-1}^{(1)}\left(\ln\left(\frac{S_{N-1}}{J_{N-1}}\right)\right) + \frac{1}{2}\alpha_{N-1}^{(2)}\left(\ln\left(\frac{S_{N-1}}{J_{N-1}}\right)\right) + \frac{1}{6}\alpha_{N-1}^{(3)}\left(\ln\left(\frac{S_{N-1}}{J_{N-1}}\right)\right)\right] \\ & \times \left(\frac{\phi_{N-1}}{J_{N-1}/S_{N-1}}\right)^3 + \cdots \Bigg\} \end{aligned} \tag{9}$$

As the Taylor coefficients of α_{N-1} are known from the term by term integrations of expressions (6) and (7), so are those of the $\alpha_{N-1}^{(k)}$s. From the definition of ϕ_{N-1} the quantity $p_{v_{N-1}}$ can be expanded as:

$$p_{v_{N-1}} = \frac{1}{1+\frac{\phi_{N-1}}{i_{N-1}}} = 1 - \frac{\phi_{N-1}}{i_{N-1}} + \left(\frac{\phi_{N-1}}{i_{N-1}}\right)^2 - \left(\frac{\phi_{N-1}}{i_{N-1}}\right)^3 + \cdots \tag{10}$$

assuming $|\phi_{N-1}/i_{N-1}| < 1$. Substituting equation (10) into (9) then multiplying term by term, and summing gives:

$$V_{N-1} = F_{N-1}\sum_{j,k}^{k\le j} f_{jkN-1}\left(\ln\left(\frac{S_{N-1}}{J_{N-1}}\right)\right)\phi_{N-1}^{j}\left(\frac{S_{N-1}}{J_{N-1}}\right)^k \tag{11}$$

as hoped. Equation (10) dominates the convergence properties, and, therefore, equation (11) converges only for $p_{f_{N-1}}/p_{v_{N-1}} < 1$.

2.2 The Induction Step

For the induction step, it is important to know how variables at time t_n depend on those same variables at t_{n-1}. For instance:

$$F_n = \left(p_{f_{n-1}}(1+i_{n-1}) + p_{v_{n-1}}e^{y_n}\right)F_{n-1} \tag{12}$$

and

$$\phi_n = \frac{(1+i_{n-1})}{e^{y_n}}\left(\frac{i_n}{i_{n-1}}\right)\phi_{n-1}. \tag{13}$$

Writing the integral for the ratchet put at time t_{n-1} gives:

$$\begin{aligned}
V_{n-1} &= e^{-r_{n-1}}\mathbb{E}_{S_{n-1}}\left[V_n(y_n)\right]\\
&= e^{-r_{n-1}}\mathbb{E}_{S_{n-1}}\left[F_n\sum_{j,k}^{k\le j} f_{jkn}\left(\ln\left(\frac{S_n}{J_n}\right)\right)\phi_n^j\left(\frac{S_n}{J_n}\right)^k\right]\\
&= F_{n-1}e^{-r_{n-1}}\int_{-\infty}^{\infty}\left[p_{f_{n-1}}(1+i_{n-1}) + p_{v_{n-1}}e^{y_n}\right]\\
&\quad\times\sum_{j,k}^{k\le j} f_{jkn}\left(\ln\left(\frac{S_n}{J_n}\right)\right)\phi_n^j\left(\frac{S_n}{J_n}\right)^k\frac{(1+i_{n-1})^j}{e^{jy_n}}\left(\frac{i_n}{i_{n-1}}\right)^j\\
&\quad\times\frac{e^{-(y_n-\mu_{n-1})^2/2\sigma_{n-1}^2}}{\sqrt{2\pi\sigma_{n-1}^2}}\mathrm{d}y_n.
\end{aligned} \tag{14}$$

During the time interval (t_{n-1}, t_n), the strike X_{n-1} remains the same. The adjusted strike, however, falls from J_{n-1} to $J_{n-1} - \phi_{n-1}S_{n-1}$ because the fixed fund has risen the time interval (t_{n-1}, t_n) and takes up a larger percentage of the strike X_{n-1}. If $X_{n-1} > F_{n-1}$ then the strike does not rachet to a higher value and $J_n = J_{n-1} - \phi_{n-1}S_{n-1}$ if $S_n < J_{n-1} - \phi_{n-1}S_{n-1}$. If $F_{n-1} > X_{n-1}$, then the strike does ratchet, and $J_n = S_n$ when $S_n > J_{n-1} - \phi_{n-1}S_{n-1}$. So:

$$\begin{aligned}
V_{n-1} &= F_{n-1}\frac{e^{-r_{n-1}}}{\sqrt{2\pi\sigma_{n-1}^2}}\\
&\times\left[p_{f_{n-1}}(1+i_{n-1})\int_{-\infty}^{\ln(J_{n-1}/S_{n-1}-\phi_{n-1})}\sum_{j,k}^{k\le j} f_{jkn}\left(\ln\left(\frac{e^{y_n}}{J_{n-1}/S_{n-1}-\phi_{n-1}}\right)\right)\right.\\
&\times\frac{\phi_{n-1}^j(1+i_{n-1})^j}{(J_{n-1}/S_{n-1}-\phi_{n-1})^k e^{-ky_n}e^{jy_n}}\left(\frac{i_n}{i_{n-1}}\right)^j e^{-(y_n-\mu_{n-1})^2/2\sigma_{n-1}^2}\mathrm{d}y_n
\end{aligned}$$

$$+p_{f_{n-1}}(1+i_{n-1}) \int_{\ln(J_{n-1}/S_{n-1}-\phi_{n-1})}^{\infty} \sum_{j,k}^{k\leq j} f_{jkn}(0) \frac{\phi_{n-1}^{j}(1+i_{n-1})^{j}}{e^{jy_n}} \left(\frac{i_n}{i_{n-1}}\right)^{j} \times e^{-(y_n-\mu_{n-1})^2/2\sigma_{n-1}^2} \mathrm{d}y_n$$

$$+ p_{v_{n-1}} \int_{-\infty}^{\ln(J_{n-1}/S_{n-1}-\phi_{n-1})} e^{y_n} \sum_{j,k}^{k\leq j} f_{jkn}\left(\ln\left(\frac{e^{y_n}}{J_{n-1}/S_{n-1}-\phi_{n-1}}\right)\right) \times \frac{\phi_{n-1}^{j}(1+i_{n-1})^{j}}{(J_{n-1}/S_{n-1}-\phi_{n-1})^{k} e^{-ky_n} e^{jy_n}} \left(\frac{i_n}{i_{n-1}}\right)^{j} e^{-(y_n-\mu_{n-1})^2/2\sigma_{n-1}^2} \mathrm{d}y_n$$

$$+ p_{v_{n-1}} \int_{\ln(J_{n-1}/S_{n-1}-\phi_{n-1})}^{\infty} e^{y_n} \sum_{j,k}^{k\leq j} f_{jkn}(0) \frac{\phi_{n-1}^{j}(1+i_{n-1})^{j}}{e^{jy_n}} \left(\frac{i_n}{i_{n-1}}\right)^{j} \left. \times e^{-(y_n-\mu_{n-1})^2/2\sigma_{n-1}^2} \mathrm{d}y_n \right] . \quad (15)$$

Let us introduce a new dummy variable δ such that $\delta = 1$ if an integral in equation (15) is a variable integral (i.e., the third and fourth integrals), and $\delta = 0$ if an integral in equation (15) is a fixed integral (i.e., the first and second integrals). For a given arbitrary j and k, each of the integrals in equation (15) is proportional to:

$$e^{-\mu_{n-1}^2/2\sigma_{n-1}^2} \phi_{n-1}^{j} (1+i_{n-1})^{j} \left(\frac{i_n}{i_{n-1}}\right)^{j} \times \int_{-\infty}^{-\xi_{n-1}} e^{\left(\frac{\mu_{n-1}}{\sigma_{n-1}^2}+\delta+k-j\right)y_n} e^{-y_n^2/2\sigma_{n-1}^2} f_{jkn}(y_n+\xi_{n-1}) e^{k\xi_{n-1}} \mathrm{d}y_n \quad (16)$$

where ξ is defined in equation (5). Changing variables to $v = y_n + \xi_{n-1}$ in the integral in expression (16) and ignoring external constants gives:

$$e^{[j-(\delta+\mu_{n-1}/\sigma_{n-1}^2)]\xi_{n-1}} e^{-\xi_{n-1}^2/2\sigma_{n-1}^2} \times \int_{-\infty}^{0} \exp\left[\frac{\xi_{n-1}v}{\sigma_{n-1}^2} + \left(\frac{\mu_{n-1}}{\sigma_{n-1}^2}+\delta+k-j\right)v\right] f_{jkn}(v) e^{-v^2/2\sigma_{n-1}^2} \mathrm{d}v. \quad (17)$$

Using the expansions:

$$\exp\left[\frac{\xi_{n-1}v}{\sigma_{n-1}^2}\right] = \sum_{k=0}^{\infty} a_k \left(\xi_{n-1}v\right)^k$$

and

$$\exp\left[\left(\frac{\mu_{n-1}}{\sigma_{n-1}^2} + \delta + k - j\right)v\right] f_{jkn}(v) = \sum_{k=0}^{\infty} b_k v^k$$

and multiplying these expansions term by term yields the following integral:

$$\int_{-\infty}^{0} \Big[(a_0b_0) + (a_1b_0\xi_{n-1} + a_0b_1)v + \left(a_2b_0\xi_{n-1}^2 + a_1b_1\xi_{n-1} + a_0b_2\right)v^2 + \cdots\Big] \times e^{-v^2/2\sigma_{n-1}^2} dv. \quad (18)$$

The definite integral for each independent power of v can be found by integration by parts and the normal integral. This turns this integral into a power series in ξ_{n-1}:

$$\begin{aligned} & a_0\left[b_0\left(\sqrt{\frac{\pi}{2}}\right)\sigma_{n-1} + b_1\left(-\sigma_{n-1}^2\right) + b_2\left(\sqrt{\frac{\pi}{2}}\right)\sigma_{n-1}^3 + \cdots\right] \\ & + a_1\left[b_0\left(-\sigma_{n-1}^2\right) + b_1\left(\sqrt{\frac{\pi}{2}}\right)\sigma_{n-1}^3 + b_2\left(-2\sigma_{n-1}^4\right) + \cdots\right]\xi_{n-1} \\ & + a_2\left[b_0\left(\sqrt{\frac{\pi}{2}}\right)\sigma_{n-1}^3 + b_1\left(-2\sigma_{n-1}^4\right) + b_2\left(3\sqrt{\frac{\pi}{2}}\right)\sigma_{n-1}^5 + \cdots\right]\xi_{n-1}^2 \\ & + \cdots \quad (19) \end{aligned}$$

Unfortunately, the interior coefficients are sequences in b_n. Several methods are available for accelerating the convergence of these terms, but it is still necessary to compute a substantial number of them. Expression (19) can then be multiplied term by term with the Taylor expansions of the external terms in expression (17). This gives:

$$V_{n-1} = F_{n-1}\left[p_{f_{n-1}}\left(1 + i_{n-1}\right)\sum_{j,k}^{k\le j} \phi_{n-1}^{j} g_{j,k,n-1,\delta=0}\left(\xi_{n-1}\right)\right.$$

$$+ p_{v_{n-1}} \sum_{j,k}^{k\le j} \phi_{n-1}^{j} g_{j,k,n-1,\delta=1}\left(\xi_{n-1}\right)\Bigg]. \quad (20)$$

Some of the external coefficients have been pulled into the function g in equation (20).

The second and fourth integrals in equation (15) can be treated similarly:

$$(1+i_{n-1})^{j}\left(\frac{i_n}{i_{n-1}}\right)^{j}\phi_{n-1}^{j}e^{-\mu_{n-1}^2/2\sigma_{n-1}^2}f_{jkn}(0) \times \int_{-\xi_{n-1}}^{\infty} e^{\left(\delta-j+\frac{\mu_{n-1}}{\sigma_{n-1}^2}\right)y_n}e^{-y_n^2/2\sigma_{n-1}^2}\mathrm{d}y_n. \quad (21)$$

The integral in expression (21) can be separated into an integral from 0 to ∞ and an integral from $-\xi_{n-1}$ to 0. The exponential functions inside the integral from $-\xi_{n-1}$ to 0 can be expanded, multiplied term by term, and finally integrated term by term analogous to the procedure introduced above. The constant of integration is obtained by calculating the integral from 0 to ∞ exactly:

$$\sqrt{2\pi\sigma_{n-1}^2}\,e^{[\mu_{n-1}+\sigma_{n-1}^2(\delta-j)]/2\sigma_{n-1}^2}\mathcal{N}\left(\frac{\mu_{n-1}}{\sigma_{n-1}}+\delta-j\right),$$

finally giving:

$$\begin{aligned}V_{n-1} = F_{n-1}\Bigg\{& p_{f_{n-1}}(1+i_{n-1}) \\ &\times \sum_{j,k}^{k\le j}\phi_{n-1}^{j}\left[g_{j,k,n-1,0}\left(\ln\left(\frac{1}{J_{n-1}/S_{n-1}-\phi_{n-1}}\right)\right)\right. \\ &\left. + h_{j,k,n-1,0}\left(\ln\left(\frac{1}{J_{n-1}/S_{n-1}-\phi_{n-1}}\right)\right)\right] \\ &+ p_{v_{n-1}}\sum_{j,k}^{k\le j}\phi_{n-1}^{j}\left[g_{j,k,n-1,1}\left(\ln\left(\frac{1}{J_{n-1}/S_{n-1}-\phi_{n-1}}\right)\right)\right. \\ &\left. + h_{j,k,n-1,1}\left(\ln\left(\frac{1}{J_{n-1}/S_{n-1}-\phi_{n-1}}\right)\right)\right]\Bigg\}. \quad (22)\end{aligned}$$

We now define the function $G = g + h$, which can be expanded similarly to the method in equations (8) to (9) yielding:

$$\begin{aligned}
V_{n-1} = F_{n-1} \Bigg\{ p_{f_{n-1}} (1 + i_{n-1}) \sum_{j,k}^{k \le j} \phi_{n-1}^{j} \Bigg[& G_{j,k,n-1,0} \left(\ln \left(\frac{S_{n-1}}{J_{n-1}} \right) \right) \\
& + G^{(1)}_{j,k,n-1,0} \left(\ln \left(\frac{S_{n-1}}{J_{n-1}} \right) \right) \left(\frac{\phi_{n-1}}{J_{n-1}/S_{n-1}} \right) \\
& + \frac{1}{2} \left(G^{(1)}_{j,k,n-1,0} + G^{(2)}_{j,k,n-1,0} \right) \left(\frac{\phi_{n-1}}{J_{n-1}/S_{n-1}} \right)^2 \\
+ \left(\frac{1}{3} G^{(1)}_{j,k,n-1,0} + \frac{1}{2} G^{(2)}_{j,k,n-1,0} + \frac{1}{6} G^{(3)}_{j,k,n-1,0} \right) & \left(\frac{\phi_{n-1}}{J_{n-1}/S_{n-1}} \right)^3 + \cdots \Bigg] \\
+ p_{v_{n-1}} \sum_{j,k}^{k \le j} \phi_{n-1}^{j} \Bigg[& G_{j,k,n-1,1} \left(\ln \left(\frac{S_{n-1}}{J_{n-1}} \right) \right) \\
& + G^{(1)}_{j,k,n-1,1} \left(\ln \left(\frac{S_{n-1}}{J_{n-1}} \right) \right) \left(\frac{\phi_{n-1}}{J_{n-1}/S_{n-1}} \right) \\
& + \frac{1}{2} \left(G^{(1)}_{j,k,n-1,1} + G^{(2)}_{j,k,n-1,1} \right) \left(\frac{\phi_{n-1}}{J_{n-1}/S_{n-1}} \right)^2 \\
+ \left(\frac{1}{3} G^{(1)}_{j,k,n-1,1} + \frac{1}{2} G^{(2)}_{j,k,n-1,1} + \frac{1}{6} G^{(3)}_{j,k,n-1,1} \right) & \left(\frac{\phi_{n-1}}{J_{n-1}/S_{n-1}} \right)^3 + \cdots \Bigg] \Bigg\}
\end{aligned} \tag{23}$$

where the superscript (k) denotes k^{th} order differentiation. We can then group like terms as:

$$\begin{aligned}
V_{n-1} = F_{n-1} \Bigg\{ & p_{f_{n-1}} (1 + i_{n-1}) \\
& \left[\sum_{j,k}^{k \le j} Q_{j,k,n-1,0} \left(\ln \left(\frac{S_{n-1}}{J_{n-1}} \right) \right) \left(\frac{(\phi_{n-1})^j}{(J_{n-1}/S_{n-1})^k} \right) \right] \\
& + p_{v_{n-1}} \left[\sum_{j,k}^{k \le j} Q_{j,k,n-1,1} \left(\ln \left(\frac{S_{n-1}}{J_{n-1}} \right) \right) \left(\frac{(\phi_{n-1})^j}{(J_{n-1}/S_{n-1})^k} \right) \right] \Bigg\}
\end{aligned} \tag{24}$$

where

$$Q_{j,0,n-1,\delta} = \sum_{k=0}^{j} G_{j,k,n-1,\delta}$$
$$Q_{j,1,n-1,\delta} = Q^{(1)}_{j-1,0,n-1,\delta}$$
$$Q_{j,2,n-1,\delta} = \frac{1}{2}\left(Q^{(1)}_{j-2,0,n-1,\delta} + Q^{(2)}_{j-2,0,n-1,\delta}\right)$$
$$Q_{j,3,n-1,\delta} = \frac{1}{3}Q^{(1)}_{j-3,0,n-1,\delta} + \frac{1}{2}Q^{(2)}_{j-3,0,n-1,\delta} + \frac{1}{6}Q^{(3)}_{j-3,0,n-1,\delta},$$

etc. Finally, we substitute equation (10) into (24), which leads to:

$$\begin{aligned} V_{n-1} = F_{n-1} \Bigg[& \left(\frac{\phi_{n-1}}{i_{n-1}} - \frac{\phi^2_{n-1}}{i^2_{n-1}} + \frac{\phi^3_{n-1}}{i^3_{n-1}} - \cdots\right)(1+i_{n-1}) \\ & \times \left(\sum_{j,k}^{k\le j} Q_{j,k,n-1,0}\left(\ln\left(\frac{S_{n-1}}{J_{n-1}}\right)\right)\frac{(\phi_{n-1})^j}{(J_{n-1}/S_{n-1})^k}\right) \\ & + \left(1 - \frac{\phi_{n-1}}{i_{n-1}} + \frac{\phi^2_{n-1}}{i^2_{n-1}} - \frac{\phi^3_{n-1}}{i^3_{n-1}} + \cdots\right) \\ & \times \left(\sum_{j,k}^{k\le j} Q_{j,k,n-1,1}\left(\ln\left(\frac{S_{n-1}}{J_{n-1}}\right)\right)\frac{(\phi_{n-1})^j}{(J_{n-1}/S_{n-1})^k}\right)\Bigg]. \end{aligned} \quad (25)$$

This yields as a final result through interchange of summation indices:

$$V_{n-1} = F_{n-1}\sum_{j,k}^{k\le j} f_{jkn-1}\left(\ln\left(\frac{S_{n-1}}{J_{n-1}}\right)\right)\phi^j_{n-1}\left(\frac{S_{n-1}}{J_{n-1}}\right)^k \quad (26)$$

where

$$\begin{aligned} f_{j,k,n-1} = & \sum_{l=k}^{j}(-1)^{(l-j)}\frac{Q_{l,k,n-1,1}}{(i_{n-1})^{(j-l)}} \\ & + \sum_{l=k;k<j}^{j-1}(-1)^{(l-j+1)}\frac{Q_{l,k,n-1,0}}{(i_{n-1})^{(j-l)}}(1+i_{n-1}). \end{aligned}$$

Once the function can be shown to be of the form:

$$V_n = F_n\sum_{j,k}^{k\le j} f_{jkn}\left(\ln\left(\frac{S_{n-1}}{J_{n-1}}\right)\right)\phi^j_{n-1}\left(\frac{S_{n-1}}{J_{n-1}}\right)^k$$

$$= F_n \sum_{j,k}^{k \le j} f_{jkn}(\xi_n)\, \phi_n^j e^{-k\xi_n}, \tag{27}$$

the final term exponential term can be expanded, multiplied term by term, and then summed over k to give:

$$V_n(\xi_n, \phi_n) = F_n \sum_j f_{jn}(\xi_n)\, \phi_n^j \tag{28}$$

where the functions f_{jn} result from the Taylor series expansions and from the functions f_{jkn} and, therefore, are known.

The biggest practical issue in the determination of the coefficients is the summation of the series in equation (19). We found that keeping 128 terms and using several iterations of Euler's method followed by one iteration of Levin's method from Fessler, Ford, and Smith (1983) worked best. Accuracy of the coefficients, and, therefore, of the final result, might be improved further by the use of infinite precision arithmetic.

3 Comparison to Monte Carlo Results

The first issue that needs to be addressed when comparing the results of the Taylor expansion to the results of Monte Carlo simulation is the issue of convergence of the series. As each step involves only functions that are analytic everywhere, it seems reasonable that the final function for the lookback value should also be analytic everywhere. In practice, however, the convergence range might be limited. In nearly all cases, the agreement is poor outside of the range $0.5 < S/X < 2.0$, and, in many cases, the effective range of convergence is even smaller than this.

3.1 Increasing the Range of Convergence to $+\infty$

The convergence radius can be extended to infinity on both ends by considering the limits. The limit $\xi \to +\infty$ corresponds to an extremely low strike. In this case, the strike is nearly certain to ratchet to the fund value at time t_1. This implies a value at time t_0 of:

$$V_0(\infty, \phi) = e^{-rt_1} \int_{-\infty}^{\infty} F_1 \sum_j \phi_1^j f_{j1}(0) \frac{e^{-(y_1-\mu_0)^2/2\sigma_0^2}}{\sqrt{2\pi\sigma_0^2}} \mathrm{d}y_1. \tag{29}$$

As F_1 and ϕ_1 are defined in equations (12) and (13), this gives:

$$V_0(\infty,\phi) = F_0\left[e^{-rt_1}\sum_j f_{j1}(0)\,\phi_0^j \right.$$
$$\left.\times\int_{-\infty}^{\infty}\left(p_f(1+i_0)+p_v e^{y_1}\right)\frac{(1+i_0)^j}{e^{jy_1}}\left(\frac{i_1}{i_0}\right)^j \frac{e^{-(y_1-\mu_0)^2/2\sigma_0^2}}{\sqrt{2\pi\sigma_0^2}}\mathrm{d}y_1\right]. \quad (30)$$

The integral can be evaluated directly. Remembering to expand p_f and p_v as in equation (11) and multiplying term by term gives:

$$V_0(\infty,\phi) = F_0 H(\phi) = F_0\left(H_0 + H_1\phi + H_2\phi^2 + H_3\phi^3 + \cdots\right) \quad (31)$$

where

$$\begin{aligned}
H_0 &= e^{-qt_1} f_{01}(0)\\
H_1 &= \left[e^{(g-r)t_1} - e^{-qt_1}\right]\frac{f_{01}(0)}{i_0} + e^{-qt_1} f_{11}(0)(1+i_0)\left(\frac{i_1}{i_0}\right)e^{-\mu_0}e^{-\sigma_0^2/2}\\
H_2 &= \left[e^{-qt_1} - e^{(g-r)t_1}\right]\frac{f_{01}(0)}{i_0^2} + e^{-qt_1} f_{21}(0)(1+i_0)^2\left(\frac{i_1}{i_0}\right)^2 e^{-2\mu_0}\\
&\quad + \left[e^{(g-r+\sigma^2)t_1} - e^{-qt_1}\right]\frac{f_{11}(0)}{i_0}(1+i_0)\left(\frac{i_1}{i_0}\right)e^{-\mu_0}e^{-\sigma_0^2/2}
\end{aligned}$$

and

$$\begin{aligned}
H_3 &= \left[e^{(g-r)t_1} - e^{-qt_1}\right]\frac{f_{01}(0)}{i_0^3}\\
&\quad + \left[e^{-qt_1} - e^{(g-r+\sigma^2)t_1}\right]\frac{f_{11}(0)}{i_0^2}(1+i_0)\left(\frac{i_1}{i_0}\right)e^{-\mu_0}e^{-\sigma_0^2/2}\\
&\quad + \left[e^{(g-r+2\sigma^2)t_1} - e^{-qt_1}\right]\frac{f_{21}(0)}{i_0}(1+i_0)^2\left(\frac{i_1}{i_0}\right)^2 e^{-2\mu_0}\\
&\quad + e^{-qt_1} f_{31}(0)(1+i_0)^3\left(\frac{i_1}{i_0}\right)^3 e^{-3\mu_0}e^{-\sigma_0^2/2}.
\end{aligned}$$

To extend the convergence of the function to the entire range $0 \le \xi < \infty$, we can create a multipoint Pade approximant [see Baker and Graves-Morris (1996, pages 335-361)] for the functions $f_{j0}(\xi) - H_j$. A multipoint Pade approximant is a rational function of two polynomials whose values and derivatives agree to given, possibly different, orders at two different points. The points chosen at this stage are $\xi = 0$ and $\xi = +\infty$.

The Taylor expansions are known at $\xi = 0$, and the limiting behavior must decay to 0 as $\xi \to +\infty$. It seems likely that the asymptotic expansion near $+\infty$ should be 0 as the probabilities depend on the cumulative normal distribution that has this asymptotic expansion. The approximant that has these features is simply the reciprocal of the Taylor expansion of the reciprocal, and this is the function we will use for comparison purposes in the range $0 \le \xi < \infty$.

It is true that in the limit $\xi \to \infty$, $\phi/(J/S)$ is no longer small, as was assumed in the derivation of equation (9) and equation (23). This difficulty is solved in the same manner, by creating a multipoint Pade approximation in ϕ at $\phi = 0$ and $\phi = \infty$ from the coefficients in equation (28).

3.2 Increasing the Range of Convergence to $-\infty$

Now, the limit $\xi \to -\infty$ corresponds to an extremely high strike. In this case, the GMDB is unlikely to ratchet, and the value of the option approaches the value of a simple Black-Scholes put. The Taylor expansion of this put is available by the same methods used in Section 2.1 with $t_1 = t_N$, the maturity of the option. Subtracting this expansion from the full Taylor expansion, an expansion for the value of the ratchet alone can be obtained. This value should drop to 0 as $\xi \to -\infty$ for the same reasons as the $\xi \to +\infty$ limit. The solution in the range $-\infty < \xi < 0$ will, therefore, be the sum of the exact Black-Scholes calculation and the multipoint Pade approximant for the excess contribution of the ratchet. Figure 1 shows a comparison of these values to Monte Carlo simulations using 32,000 antithetic scenarios. We typically use a risk-free rate of 5%, risk fees of 115 bp consistent with the market survey in Milevsky and Posner (2001), a stock market volatility of 20%, a fixed fund return of 5%, and time between ratchets (excluding the first and last) of 12 months. We use terms up to ξ^{64} and ϕ^2 and fixed fund percentages from 0% to 90% fixed in 10% increments. A range of strikes near 1 is presented to highlight the agreement, which is good, though, over the entire range $-\infty < \xi < 0$.

The final issue is how to connect the two approximations at $\xi = 0$. The two functions are discontinuous if $\phi \neq 0$. Strikes greater than 1 tend to produce greater agreement near $\xi = 0$, so the approximation for strikes less than one are scaled to this value. Figure 2 shows the comparison at low strike prices after the results have been rescaled.

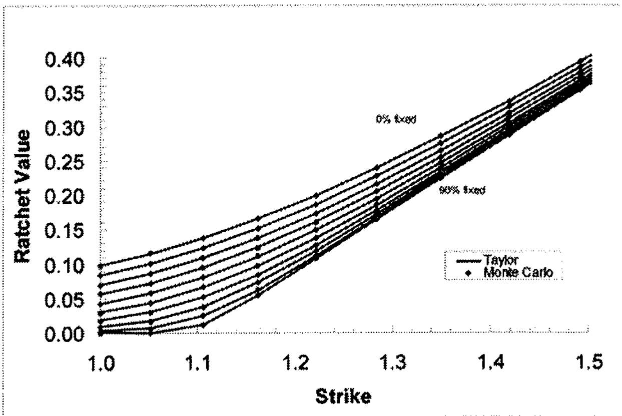

Figure 1: Comparison of Simulation and Taylor Expansion of a 24 Month Lookback Option with the First Ratchet in 12 Months at High Strike Prices

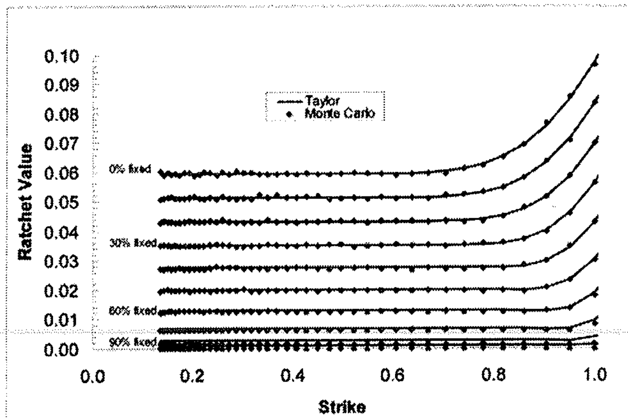

Figure 2: Comparison of Simulation and Taylor Expansion of a 24 Month Lookback Option with the First Ratchet in 12 Months at Low Strike Prices

Accuracy in the calculation of the value of the option is not the only consideration in evaluating the approximation. It is frequently necessary to calculate the value of various derivatives of the function in order to hedge the option. Figures 3 and 4 show the comparison of delta with the Monte Carlo values. For low strikes, delta is positive because the option is likely to ratchet at its next opportunity. The higher the stock market, the higher the value to which the strike will ratchet. On the other hand, for high strikes, delta is negative, because the ratchet is less relevant and the lookback delta approaches the delta of a simple put.

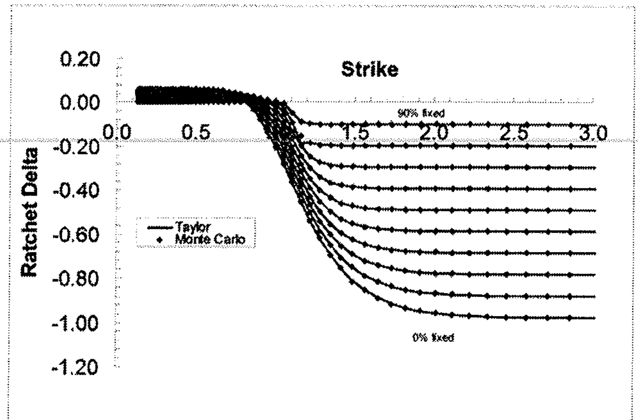

Figure 3: Comparison of Simulation and Taylor Expansion of the Value of Delta for a 24 Month Lookback Option with the First Ratchet in 12 Months

Finally, we need to show the results of a full calculation of the ratchet value for all ranges of strikes using a series of puts with maturities from zero to 96 months weighted by their expected exercise probability in the double decrement model. We use a mortality rate equal to that in the 2000 GAM table and lapse percentages that rise from 0% in year one to a maximum of 22.5% in year nine and settling in at a long-term rate of 19.5%. Instead of weighting the values of the individual lookback puts, a weighted average of the Taylor expansions is used as the Taylor expansion of the full GMDB value. This results in a time savings of a factor of 100 or so and is a feature of the Taylor expansion method that cannot be duplicated by the simulation or analytic methods. The value of the GMDB is then calculated from this new expansion in a manner

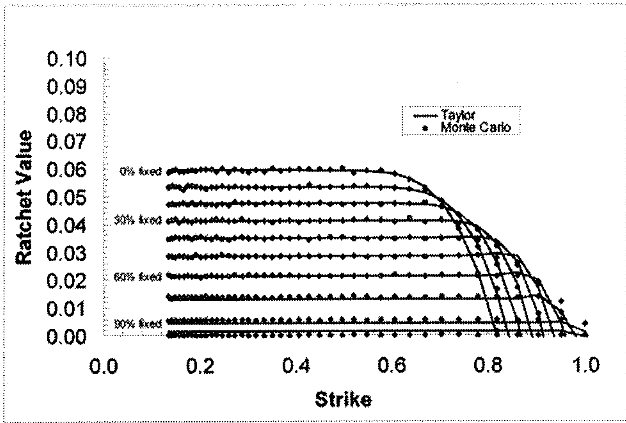

Figure 4: Comparison of Simulation and Taylor Expansion of the Value of Delta for a 24 Month Lookback Option with the First Ratchet in 12 Months at Low Strike Prices

similar to that of the individual puts. Figure 5 shows comparisons of the approximation to the simulated values in a range of strike values near 1. Similar agreement is obtained over the entire range. Figure 6 shows comparisons of delta to the simulated values. The values of delta show minor disagreements in the range of strikes between 1.5 and 2 on the order of perhaps a few basis points. We would consider this disagreement to be minor, as it would be dwarfed by any errors in assumptions of risk-free rate, volatility, lapse rates, and mortality used in the approximation.

3.3 Time Comparisons

There are at least two advantages of using the Taylor expansion instead of Monte Carlo simulation. The first is that the expansion is more accurate in most cases because simulation contains random errors. This is by no means true in all cases, particularly those at high fixed fund percentages, as can be seen in the figures. The major advantage, however, is the increase in speed obtained by using a function that can be quickly evaluated, rather than performing a large number of simulations. To demonstrate this, we compared the results of 69,120 calculations of both the approximation and 32,000 antithetic Monte Carlo scenarios. We used ages ranging from 55 to 66, durations rang-

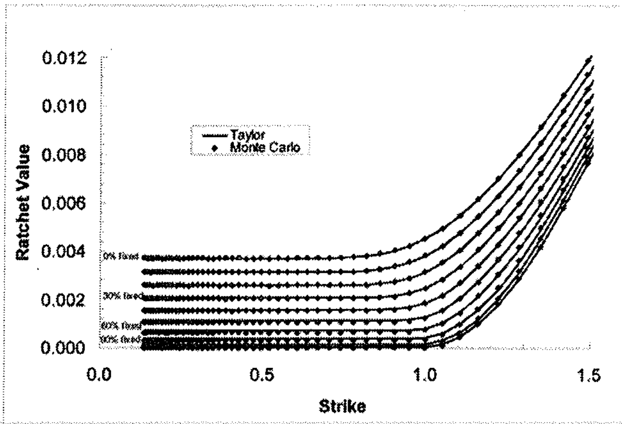

Figure 5: Comparison of Simulation and Taylor Expansion of a GMDB Ratchet Option on a 55 Year Old One Year After Issue

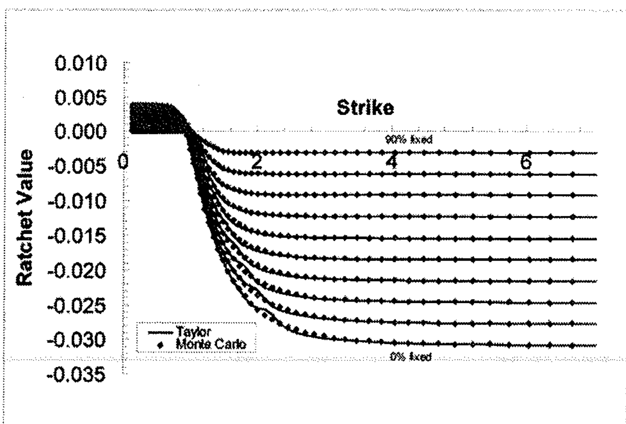

Figure 6: Comparison of Simulation and Taylor Expansion of the Value of Delta for a GMDB Ratchet Option on a 55 Year Old One Year After Issue

ing from one to 96 months, fixed percentages ranging from 0% to 50% in steps of 10%, and ten different strike values. The computations were performed on a Dell Computer with 2.8GHz Pentium processor. It took 40.203 seconds for the 69,120 Taylor series expansions to be computed for an average of 0.582 milliseconds per point. It took 199,606 seconds (2 days, 9 hours) to complete the 69,120 Monte Carlo simulations for an average of 2.888 seconds per point. This shows the approximation will improve runtimes by a factor of about 5,000 if 32,000 scenarios are used. Equivalently, the approximation runs about as fast as six to seven Monte Carlo runs.

4 Comparisons to the Analytic Solution

Now that we have shown the superiority of the Taylor expansion method over the most commonly used method of simulation, we wish to show its superiority over the analytic solution in Collin-Dufresne, Keirstad, and Ross (1997), hereafter referred to as CKR, in both speed and accuracy. First, the formulas will be reproduced here to prevent unnecessary cross-referencing. The price of a single discrete lookback put is given as:

$$P = V_{DM}^{n}\left(S_0, X_0, t_N, \sigma, r, q, t_1, \cdots, t_N\right) - S_0 e^{-q t_N} + X_0 e^{-r t_N} \tag{32}$$

where

$$\begin{aligned} V_{DM}^{n}\left(S_0, X_0, t_N, \sigma, r, q, t_1, \cdots, t_N\right) &= \sum_{j=1}^{N} H_j I_{n-j} S_0 e^{(r-q)t_j - r t_N} \\ &\quad - X_0\left[1 - \mathcal{N}_N\left(-d^Q\left(X_0, t_1\right), \cdots, -d^Q\left(X_0, t_N\right); \{C_{ik}^{(1)}\}\right)\right], \end{aligned}$$

$$H_j = \mathcal{N}_j\left(\frac{(\mu+\sigma^2)}{\sigma}\sqrt{t_j - t_1}, \cdots, \frac{(\mu+\sigma^2)}{\sigma}\sqrt{t_j - t_{j-1}}, d^R\left(X_0, t_j\right); \{C_{ik}^{(3)}\}\right)$$

$$I_{N-j} = \mathcal{N}_{N-j}\left(-\frac{\mu}{\sigma}\sqrt{t_{j+1} - t_j}, \cdots, -\frac{\mu}{\sigma}\sqrt{t_{j+N} - t_j}; \{C_{ik}^{(2)}\}\right)$$

$$d^Q\left(X_0, t_j\right) = \frac{\log\left(S_0/X_0\right) + \mu t_j}{\sigma\sqrt{t_j}}$$

$$d^R\left(X_0, t_j\right) = d^Q\left(X_0, t_j\right) + \sigma\sqrt{t_j}$$

and the covariance matrices are:

$$C_{ik}^{(1)} = \sqrt{\frac{t_{i\wedge k}}{t_{i\vee k}}}$$

$$C_{ik}^{(2)} = \sqrt{\frac{t_{j+(i\wedge k)} - t_j}{t_{j+(i\vee k)} - t_j}}$$

$$C_{ik}^{(3)} = \sqrt{\frac{t_j - t_{i\vee k}}{t_j - t_{i\wedge k}}}, \quad i,k \neq j$$

$$C_{ij}^{(3)} = C_{ji}^{(3)} = \sqrt{\frac{t_j - t_i}{t_j}}, \quad i \neq j.$$

The evaluation of one lookback put value requires the evaluation of $2N+1$ cumulative multivariate normal functions with an average of $N/2$ variables. They are evaluated in the comparison by the method of Somerville (1998a, 1998b). This by itself can be time-consuming. Using 10,000 directions, a point with $N = 2$ requires on average 92 milliseconds to compute and a point with $N = 8$ requires 1.52 seconds on the 2.8GHz Dell. This should be compared with about 0.5 milliseconds per point for the Taylor expansions. Figure 7 shows comparisons of the values obtained from the Taylor expansion with the CKR results for a 67 month lookback put 5 months after issue. No comparisons were done for any fixed fund percentages other than 0%, as the CKR function doesn't apply to this case. The Taylor expansion agrees well with the exact solution, and the disagreements are primarily due to random errors in the evaluation of the multivariate normal functions. These errors could be reduced if more directions were used in the CKR evaluation, but this would increase the time for a method that is already much slower than the Taylor expansion.

Next, we use the CKR formula to evaluate a complete ratchet GMDB. In the case of the Taylor expansion, the coefficients could be weighted and summed prior to evaluation, which results in a substantial time savings. There is no comparable procedure for the CKR formula. The value of the lookback put must be found every month and then multiplied by the probabilities of death and summed. A comparison of results for a 55 year old one month after issue is found in Figure 8. The agreement in values is better than for the individual puts because the random errors in each put value have a tendency to cancel out. The evaluation time, however, has grown enormously. The average time to compute one point is 54.9 seconds, compared with 0.582 milliseconds for the

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