2005

An Application of Control Theory to the Individual Aggregate Cost Method

Alexandros A. Zimbidis
Athens University of Economics and Business, Greece, aaz@aueb.gr

Steven Haberman
Cass Business School, City University, s.haberman@city.ac.uk

Journal of Actuarial Practice Vol. 12, 2005

An Application of Control Theory to the Individual Aggregate Cost Method

Alexandros A. Zimbidis* and Steven Haberman†

Abstract

The paper investigates the individual aggregate cost method (also known as the individual spread-gain method), which is normally applicable in small pension funds or fully contributory schemes, using a control theoretical framework. We construct the difference equations describing the mechanisms of the respective funding method and then calculate the optimal control path of the contribution rate assuming (first) a stochastic and (second) a deterministic pattern for the future investment rates of return. For the first case, the optimal solution is achieved through a linear approximation and using stochastic optimization techniques. It is proved that the contribution rate is (optimally) controlled through the control of the valuation rate (which is determined incorporating a certain feedback mechanism of the past contribution rate). The optimal solution for the deterministic case is obtained using standard calculus and the method of Lagrange multipliers.

Key words and phrases: *individual aggregate funding, linear approximation, optimal stochastic control, Lagrange multipliers*

*Alexandros A. Zimbidis Ph.D., F.H.A.S., is a lecturer at Athens University of Economics and Business, Greece. He received his B.Sc. (first class) in mathematics from the University of Athens and his masters and Ph.D. in actuarial science from City University, London. He worked as a director in a large multinational insurance company. Dr. Zimbidis is a Fellow of the Hellenic Actuarial Society and is the chair of the President of the National Actuarial Authority in Greece. His research interests include applications of stochastic analysis and control theory to insurance and financial systems.

Dr. Zimbidis's address is: Athens University of Economics and Business, Department of Statistics, 76 Patission St., Athens 10434, GREECE. E-mail: *aaz@aueb.gr*

†Steven Haberman M.A., Ph.D., D.Sc., F.I.A., is professor and deputy dean at Cass Business School, City University. He has worked at Prudential Assurance and the Government Actuary's Department in London. He has published extensively in the fields of actuarial science and applied statistics. His current research interests include modeling pension plans (defined benefit and defined contribution); mortality forecasting, longevity risk and annuities; modeling guarantees in life insurance; non-life insurance premiums in competitive markets.

Dr. Haberman's address is: Cass Business School, City University, 106 Bunhill Row, London, EC1Y 8TZ, UNITED KINGDOM. E-mail: *s.haberman@city.ac.uk*

1 Introduction

Over the past fifteen years or so, control theory has been used to study different types of insurance systems. Researchers have found that this specific theory provides a powerful research framework to analyze the evolution of insurance and pension systems, as well as to determine optimal strategies for determining, for example, the level of insurance premiums, the level of pension fund contributions, or asset allocation. Several authors have used control theory for the investigation of the properties of different pension funding methods; e.g., Benjamin (1989), Zimbidis and Haberman (1993), Haberman and Sung (1994), Loades (1998), Owadally and Haberman (1999, 2004), Cairns (2000), and Taylor (2002).

This paper focuses on the application of control theory to the individual aggregate cost method, which is also known as the individual spread-gain method. The individual aggregate cost method is normally applicable to two broad categories of pension funds: (i) small pension funds where the number of members participating in the plan is so small or the membership is so heterogeneous that the average contribution rate obtained by the aggregate (or other) funding method is not reliable or sufficient; or (ii) pension funds (whether small or large) where the individual members contribute the major share of the total annual contributions; see McGill et al. (1989).

Consider an employee who was hired at age e and will retire at the normal retirement age r, i.e., the employee is expected to give $m = r - e$ years of service. If the employee is currently age x at time n, where $x = n + e$, then under the individual aggregate cost method, an individual's contribution rate at the beginning of year $n+1$ for $n = 0, 1, 2, \ldots, m-1$, is C_n, where:

$$C_n = \begin{cases} \frac{\mathrm{PVTB}_n - F_n}{{}^s\ddot{a}_{x:\overline{m-n}|}} \\ \frac{\mathrm{PVTB}_n - F_n}{\ddot{a}_{x:\overline{m-n}|}} \end{cases} \tag{1}$$

for $n = 0, 1, 2, \ldots, m-1$ while $C_n = 0$ for $n = m, m+1, \ldots$, F_n is the accumulated fund assets at time n, PVTB_n is the actuarial present value of total retirement benefits earned at time n, ${}^s\ddot{a}_{x:\overline{m-n}|}$ is the actuarial present value of a life annuity with payments increasing according to a salary scale, and $\ddot{a}_{x:\overline{m-n}|}$ is the actuarial present value of a life annuity with level payments. All actuarial present values are assumed to be discounted at the valuation rate of interest of i.

We assume each plan member has his/her own separate account that changes due to the employee's own contributions, investment returns on the fund, and expenses associated with managing the fund. For example, in Greece the fund value for each member, F_n, is calculated by crediting the employee's own contributions, debiting management and other expenses either as a flat amount for each person or as a standard percentage based upon the employee's own contributions, and finally crediting investment proceeds proportionally according to the prior fund value F_{n-1}. A full description of the individual aggregate cost method may be found in McGill et al. (1989) and Winklevoss (1993).

2 Description of The Model

We now present some of the assumptions used throughout this paper.

A.1: The pension plan is a defined benefit plan with normal retirement age r.

A.2: The plan uses the individual aggregate funding cost method for plan valuations.

A.3: We consider a plan member who was hired at age e and has a future working lifetime of m years, where $m = r - e$.

A.4: There are no pre-retirement mortality, disability, or other decrements.[1]

A.5: The normal retirement benefit is $B/12$ per month.

A.6: The unit of currency used is such that the product of the annuity factor and the annual retirement benefit is equal to one monetary unit, i.e.,

$$B\ddot{a}_r^{(12)} = 1.$$

A.7: As the normal retirement benefit is independent of salary, normal cost is calculated on the basis of a level-dollar amount.

A.8: The contribution rate for the plan year $[n, n+1)$ is C_n monetary units paid at time n and is equal to the plan's normal cost.

A.9: The total funding period of m years is divided in two subperiods: $[0, T)$ and $[T, m)$. In the first sub-period (up to

[1]This assumption may be justified because, in fully contributory plans (where this specific method is normally applicable), the ancillary non-retirement (death, disability, or other) benefits are normally equal to (or approximately equal to) the accrued liability at the date of decrement, resulting in no gain or loss to the plan.

time T) the investment return for the year $[n-1,n)$, j_n, is considered to be a stationary stochastic process where $\mathbb{E}(j_n) = j > 0$ and $\mathbb{V}\mathrm{ar}(j_n) = \sigma^2 > 0$, and the j_ns are mutually independent. In the second sub-period (after time T) the annual investment returns, j_n, are deterministic for $n = T, t+1, \ldots, m-1$.

A.10: The valuation rate for the plan year $[n, n+1)$, i_n, is determined at the end of the previous year (i.e., at time n) and is based on the information and experience available at n, the i_n is used to determine the contribution rate C_n for the plan year $[n, n+1)$.

Under the traditional approach to determining the contribution rate for the individual aggregate cost method (as formulated in equation (1)), we assume a constant valuation rate of interest i for each year n and an initial fund value F_0, which is normally zero. The contributions rates (C_n, $n = 0, 1, 2, \ldots, m-1$) determined by equation (1) vary because of fluctuations in the investment returns on the accumulated assets, F_n. In order to reduce the fluctuations in the C_n, $n = 0, 1, 2, \ldots, m-1$, we control the contribution rates by adapting the valuation rate of interest. This is justified because the valuation rate of a pension fund is highly correlated to the long-term interest rates (see Wilkie, 1995 or Ang and Sherris, 1997) so that the estimation of these rates will influence the determination of the valuation rate.

Our approach follows the standard practice, which is commonly called the life-style investment strategy.[2] Following assumption A.9, at the beginning of the first sub-period, we choose a high risk high expected return investment policy. Once in the second sub-period, the assets are switched to assets with a lower risk lower expected return investment policy in order to secure the benefits of the member at the date of retirement. The value of T is determined by the pension fund manager, and we suggest that T should be close to m in order to be able to obtain an investment product with guaranteed rates for the second period.

In the first sub-period, where the returns follow a stochastic process, we control the contribution rate by adopting a control feedback mechanism for the annual valuation rates i_n. In the second sub-period, where the returns follow a deterministic process, we control the contribution rate directly by calculating the optimal path that produces the promised retirement benefit.

[2]See Vigna and Haberman (2001) for a discussion in the context of defined contribution pension schemes.

Because there are no pre-retirement decrements (assumption A. 4) and the retirement benefit is independent of salary, we obtain the following system of difference equations:

$$C_n = \frac{v_{(i_n)}^{m-n} B \ddot{a}_r^{(12)} - F_n}{\ddot{a}_{\overline{m-n}|\, i_n}}, \tag{2}$$

which is the equation for the normal cost, and

$$F_n = (F_{n-1} + C_{n-1})(1 + j_n) \tag{3}$$

where $v_{(i_n)}^{m-n} = (1 + i_n)^{-(m-n)}$ is the discount factor. Equation (2) can be rewritten as $F_n = v_{(i_n)}^{m-n} - C_n \ddot{a}_{\overline{m-n}|\, i_n}$ and substituted into equation (3) to yield, after some elementary algebra,

$$C_n = (1 + j_n) \frac{a_{\overline{m-n}|\, i_{n-1}}}{\ddot{a}_{\overline{m-n}|\, i_n}} C_{n-1} + \frac{v_{(i_n)}^{m-n}}{\ddot{a}_{\overline{m-n}|\, i_n}} - (1 + j_n) \frac{v_{(i_{n-1})}^{m-n+1}}{\ddot{a}_{\overline{m-n}|\, i_n}}. \tag{4}$$

It is clear from equation (4) that the system is non-linear in the valuation rates i_n and i_{n-1} and linear in j_n and C_{n-1}. The state variable of the system is the contribution rate C_n while the input variable is the actual rate of investment return j_n. The valuation rate of interest i_n that appears in equation (4) is the source of non-linearity and operates as the control variable, which attempts to balance the system.

In a steady state, where $j_n = j$, a constant for $n = 0, 1, 2, \ldots$, and, if the valuation rate also is equal to j (i.e., $i_n = j$), then we obtain from equation (4)

$$C_{m-1} = C_{m-2} = \cdots = C_0. \tag{5}$$

The initial contribution (C_0) then is calculated using equation (2) and assuming $F_0 = 0$, i.e.,

$$C_0 = \frac{v_{(j)}^m}{\ddot{a}_{\overline{m}|\, j}} = \frac{1}{\ddot{s}_{\overline{m}|\, j}}. \tag{6}$$

Then the fund value at each time point n is equal to the respective accumulation of contributions, i.e.,

$$F_n = C_0 \ddot{s}_{\overline{n}|\, j} = \frac{\ddot{s}_{\overline{n}|\, j}}{\ddot{s}_{\overline{m}|\, j}} \tag{7}$$

for $n = 1, \ldots, m$. As the recursive equation (4) is non-linear, we propose to employ linear approximation techniques in order to solve the problem.

3 Mean and Variance of Contribution Rate

Before going further with the control theoretical analysis of the model and the linear approximation, we investigate the mean and variance of the contribution rates under the traditional approach where the valuation rate of interest is assumed to be constant for each year and equal to the expected rate of investment return, i.e.,

$$i_0 = i_n = j = \mathbb{E}(j_n) \tag{8}$$

for $n = 1, \ldots, m-1$. Substituting equation (8) in equation (4) we obtain

$$C_n = \frac{1+j_n}{1+j} C_{n-1} + \frac{j - j_n}{1+j} f_n(j) \tag{9}$$

where

$$f_n(j) = \frac{1}{\ddot{s}_{\overline{n|}j}}. \tag{10}$$

In order to facilitate our calculations, we introduce the filtration H_n, which represents all the available information generated by the entire funding process (the annual investment rates, the decisions for the contribution rates, etc.) up to and including time n. We also use two well known results from the theory of conditional probabilities:

$$\mathbb{E}(X) = \mathbb{E}[\mathbb{E}(X|Y)] \quad \text{and} \tag{11}$$

$$\mathbb{V}\text{ar}(X) = \mathbb{V}\text{ar}[\mathbb{E}(X|Y)] + \mathbb{E}[\mathbb{V}\text{ar}(X|Y)] \tag{12}$$

where X, Y are random variables.

From equation (9) and conditioning on H_{n-1} give

$$
\begin{aligned}
\mathbb{E}(C_n) &= \mathbb{E}(\mathbb{E}(C_n|H_{n-1})) \\
&= \mathbb{E}\left[\mathbb{E}\left[\frac{1+j_n}{1+j}C_{n-1}+\frac{j-j_n}{1+j}f_n(j)|H_{n-1}\right]\right] \\
&= \mathbb{E}\left[\frac{1+\mathbb{E}(j_n)}{1+j}C_{n-1}+\frac{j-\mathbb{E}(j_n)}{1+j}f_n(j)\right] \\
&= \mathbb{E}(C_{n-1})
\end{aligned}
$$

for $n = 1,\ldots,m-1$. Consequently, we determine that the expected contribution rate is constant over time n, i.e.,

$$
\mathbb{E}(C_n) = C_0 \quad \text{for} \quad n = 1,\ldots,m-1. \tag{13}
$$

To obtain the variance we note that, using equation (9) and conditioning on H_{n-1} yields

$$
\mathbb{V}\mathrm{ar}\,(C_n) = \mathbb{V}\mathrm{ar}\,[\mathbb{E}\,(C_n|H_{n-1})] + \mathbb{E}\,[\mathbb{V}\mathrm{ar}\,(C_n|H_{n-1})]
$$

where $\mathbb{V}\mathrm{ar}\,[\mathbb{E}\,(C_n|H_{n-1})] = \mathbb{V}\mathrm{ar}\,(C_{n-1})$ and

$$
\begin{aligned}
&\mathbb{E}\,[\mathbb{V}\mathrm{ar}\,(C_n|H_{n-1})] \\
&\quad= \mathbb{E}\left[\mathbb{V}\mathrm{ar}\left(\frac{1+j_n}{1+j}C_{n-1}+\frac{j-j_n}{1+j}f_n(j)|H_{n-1}\right)\right] \\
&\quad= \mathbb{E}\left[\frac{C_{n-1}^2}{(1+j)^2}\mathbb{V}\mathrm{ar}(j_n)+\frac{f_n^2(j)-2f_n(j)C_{n-1}}{(1+j)^2}\mathbb{V}\mathrm{ar}(j_n)\right] \\
&\quad= \sigma^2\left[\frac{\mathbb{E}(C_{n-1}^2)}{(1+j)^2}+\frac{f_n^2(j)-2f_n(j)\mathbb{E}(C_{n-1})}{(1+j)^2}\right] \\
&\quad= \sigma^2\left[\frac{\mathbb{V}\mathrm{ar}(C_{n-1})+[\mathbb{E}(C_{n-1})]^2}{(1+j)^2}+\frac{f_n^2(j)-2f_n(j)\mathbb{E}(C_{n-1})}{(1+j)^2}\right] \\
&\quad= \left(\frac{\sigma}{1+j}\right)^2\mathbb{V}\mathrm{ar}\,(C_{n-1})+\left(\frac{\sigma}{1+j}\right)^2\left(C_0^2+f_n^2(j)-2f_n(j)C_0\right)
\end{aligned}
$$

for $n = 1,\ldots,m-1$. Hence

$$
\mathbb{V}\mathrm{ar}\,(C_n) = \left[1+\left(\frac{\sigma}{1+j}\right)^2\right]\mathbb{V}\mathrm{ar}\,(C_{n-1})+\frac{\sigma^2}{(1+j)^2}\left(C_0-f_n(j)\right)^2.
$$

For notational convenience, let $\psi_n = \mathbb{V}\mathrm{ar}\,(C_n)$ and

$$A_n = \frac{\sigma^2}{(1+j)^2}\left[C_0 - f_n(j)\right]^2 = \frac{\sigma^2}{(1+j)^2}\left[\frac{1}{\ddot{s}_{\overline{m}|j}} - \frac{1}{\ddot{s}_{\overline{m-n}|j}}\right]^2.$$

Thus we have the following difference equation for the variance:

$$\psi_n = \left[1 + \left(\frac{\sigma}{1+j}\right)^2\right]\psi_{n-1} + A_n \tag{14}$$

for $n = 1, 2, \ldots, m-1$, which has the solution

$$\psi_n = \left[1 + \left(\frac{\sigma}{1+j}\right)^2\right]^n \psi_0 + \sum_{k=0}^{n-1}\left[1 + \left(\frac{\sigma}{1+j}\right)^2\right]^k A_{n-k}. \tag{15}$$

As $1 + (\sigma/(1+j))^2 > 1$ and $A_{n-k} > 0$, it is clear from equation (15) that ψ_n increases as n increases up to m. We also observe that the rate of increase in ψ_n depends on the ratio $\sigma/(1+j)$.

In order to restrict the magnitude of the ψ_ns, we now consider a control theory approach based on a variable valuation rate of interest.

4 Optimal Control Strategy

4.1 The Objective Function

As described in assumption A.9, we split the total funding period into two sub-periods. In the first sub-period we apply a stochastic model for the investment rate of return, while in the second sub-period we apply a deterministic model. In the first sub-period, we use the valuation rate of interest as a control mechanism, while in the second sub-period we directly determine the contribution rates. In other words, we either control the valuation rate of interest or the contribution rate.

Our main objective in the control problem is the minimization of the contribution rate risk, which is defined as the total mean square deviation of contribution levels from their target values.[3] Following Vandebroek (1990) and Haberman and Sung (1994), we adopt a weighted quadratic objective function of minimizing total mean square deviations of contribution levels from their target values (because our basic

[3] According to Haberman and Sung (1994), the contribution rate risk is one of the two main risks with which a pension plan is confronted, while the other basic risk is the solvency risk.

aim is to reduce the fluctuations of the contribution rate) and also total mean square deviations of valuation rates from their target levels.[4]

Let C_{st} and i_{st} denote the contribution rate and valuation rate, respectively, for the steady state of the system (see equation (20)), and let β be a weighting factor, where $0 \le \beta \le 1$. The choice of the factor β reflects the preference of the pension scheme manager between the contribution rate and the valuation interest rate and which of them may exhibit more or less fluctuations. That is, if the manager chooses β to be close to zero that means the manager prefers a more stable contribution rate and pays almost no attention to possible large fluctuations in the valuation interest rate. For the first sub-period with the stochastic investment rates, the objective function under the quadratic performance criterion has the following form:

$$O_1 = \min_{C_k, i_k} \mathbb{E}\left[(1-\beta)\sum_{k=1}^{T-1}(C_k - C_{st})^2 + \beta\sum_{k=1}^{T-1}(i_k - i_{st})^2\right]. \tag{16}$$

For the second sub-period (from (T) up to (m)) the objective function includes no stochastic elements (hence no expectation operator) and has the following form:

$$O_2 = \min_{C_k} \sum_{k=T}^{m-1} (C_k - C_{st})^2 . \tag{17}$$

It is easy to argue that for $\beta = 1$, the new (controlled) model almost corresponds to the traditional approach of the individual aggregate cost method.

4.2 Optimal Control During the First Sub-Period

The difference equation (4) may be linearized in the neighborhood of a certain steady state, defined by:

$$(j_n, i_{n-1}, i_n, C_n) = (j_{st}, i_{st}, i_{st}., C_{st}) \tag{18}$$

In the steady state, the valuation rate of interest is equal to the actual investment rate of return, and normally we choose j_{st} to equal the mean of the j_n, i.e.,

[4] A quadratic objective function has the advantage of leading to mathematically tractable results but we acknowledge that it has the inherent disadvantage of treating deviations below and above the target in an equivalent manner. The use of semi-variance type measures would allow more flexibility in this direction but at the expense of tractability.

$$j_{st} = i_{st} = \mathbb{E}\left(j_n\right) = j. \tag{19}$$

The respective contribution rate is given by

$$C_{st} = \frac{1}{\ddot{s}_{\overline{m|}j}}. \tag{20}$$

By considering infinitesimally small (∇) changes[5] about the steady state for C_n, C_{n-1}, i_n, i_{n-1}, and j_n, i.e., $C_n = C_{st} + \nabla C_n$, $C_{n-1} = C_{st} + \nabla C_{n-1}$, $i_n = j + \nabla i_n$, $i_{n-1} = j + \nabla i_{n-1}$, and $j_n = j + \nabla j_n$, we obtain the following linear approximation (after using equation (19))

$$\nabla C_n = \nabla C_{n-1} + \xi_n \nabla j_n + \varphi_n \nabla i_{n-1} + \zeta_n \nabla i_n + \text{nonlinear terms} \tag{21}$$

where

$$\begin{aligned}
\xi_n &= \frac{1}{1+j}\left(\frac{1}{\ddot{s}_{\overline{m|}j}} - \frac{1}{\ddot{s}_{\overline{m-n|}j}}\right) \\
\varphi_n &= \frac{m-n+1}{1+j}\frac{1}{\ddot{s}_{\overline{m-n|}j}} + \frac{m-n}{j}\frac{1}{\ddot{s}_{\overline{m|}j}}\frac{1}{\ddot{s}_{\overline{m-n|}j}} - \frac{1}{j}\frac{1}{\ddot{s}_{\overline{m|}j}} \\
\zeta_n &= -\frac{m-n}{1+j}\frac{1}{\ddot{s}_{\overline{m-n|}j}} - \frac{m-n}{j}\frac{1}{\ddot{s}_{\overline{m|}j}}\frac{1}{\ddot{s}_{\overline{m-n|}j}} + \frac{1}{j(1+j)}\frac{1}{\ddot{s}_{\overline{m-n|}j}}
\end{aligned}$$

Note that $\varphi_n = -(\xi_n + \zeta_n)$; hence equation (21) may be rewritten as

$$\nabla C_n = \nabla C_{n-1} + \xi_n \nabla j_n - (\xi_n + \zeta_n)\nabla i_{n-1} + \zeta_n \nabla i_n. \tag{22}$$

At this point it is important to briefly describe the solution to a general linear dynamic difference equation of the form

$$\mathbf{x}_n = \mathbf{A}_n \mathbf{x}_{n-1} + \mathbf{B}_n \mathbf{u}_n + \mathbf{e}_n \tag{23}$$

for $n = 1, 2, \ldots, N$, where $\mathbf{x}_n \in \mathbb{R}^n$ is the state variable, $\mathbf{u}_n \in \mathbb{R}^k$ is the control variable, $\mathbf{A}_n \in \mathbb{R}^{r\times r}$ and $\mathbf{B}_n \in \mathbb{R}^{r\times k}$ are known non-random matrices, and $\mathbf{e}_n \in \mathbb{R}^r$ is a random vector with $\mathbb{E}(\mathbf{e}_n) = 0$ and finite covariance matrix. In addition, we assume $\mathbf{e}_n$ is independent of $\mathbf{x}_n$ and $\mathbf{u}_n$.

The problem is to search for the optimal control $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{N-1}$ that minimizes the following expectation:

[5]Here ∇ is the backward difference operator, i.e., for any function $f(x)$, $\nabla f(x) = f(x+1) - f(x)$ and $\nabla^{n+1} f(x) = \nabla^n f(x+1) - \nabla^n f(x)$ for $n = 1, 2, \ldots$.

$$\underset{e_1,e_2,\ldots,e_N}{\mathbb{E}}\left\{\sum_{n=1}^{N}\mathbf{x}_n^{\mathsf{T}}\mathbf{K}_n\mathbf{x}_n+\sum_{n=1}^{N-1}\mathbf{u}_n^{\mathsf{T}}\mathbf{R}_n\mathbf{u}_n\right\} \tag{24}$$

where the symbol $^{\mathsf{T}}$ denotes the transpose operator, $\mathbf{K}_n$ is a symmetric positive semi-definite matrix, and $\mathbf{R}_n$ is a symmetric positive definite matrix. Following Aoki (1989, pp. 131–148) and Bertsekas (1976, pp. 70–80), the optimal solution, given the initial condition $\mathbf{x}_0$, is described by the following equations:

$$\mathbf{u}_n = \mathbf{M}_n\mathbf{x}_{n-1} \tag{25}$$

where $\mathbf{H}_N = \mathbf{K}_N$, and, for $n = N-1, N-2, \ldots, 1, 0$, we have

$$\mathbf{M}_n = -\left(\mathbf{B}_n^{\mathsf{T}}\mathbf{H}_{n+1}\mathbf{B}_n+\mathbf{R}_n\right)^{-1}\left(\mathbf{B}_n^{\mathsf{T}}\mathbf{H}_{n+1}\mathbf{A}_n\right) \tag{26}$$

$$\mathbf{H}_n = A_n^{\mathsf{T}}\left[\mathbf{H}_{n+1}-\mathbf{H}_{n+1}\mathbf{B}_n\left(\mathbf{B}_n^{\mathsf{T}}\mathbf{H}_{n+1}\mathbf{B}_n+\mathbf{R}_n\right)^{-1}\mathbf{B}_n^{\mathsf{T}}\mathbf{H}_{n+1}\right]\mathbf{A}_n+\mathbf{K}_n. \tag{27}$$

In order to fit the last equation with the format of the linear system, which appears in equation (23), we write

$$\nabla C_n = \nabla C_{n-1} + \begin{pmatrix} \zeta_n & 0 \\ 0 & -(\xi_n+\zeta_n) \end{pmatrix}\begin{pmatrix} \nabla i_n \\ \nabla i_{n-1} \end{pmatrix} + \xi_n \nabla j_n, \tag{28}$$

in other words $\mathbf{x}_n = \nabla C_n$ and $\mathbf{A}_n = 1$, $\mathbf{e}_n = \xi_n\nabla j_n$ are scalars, and

$$\mathbf{B}_n = \begin{pmatrix} \zeta_n & 0 \\ 0 & -(\xi_n+\zeta_n) \end{pmatrix} \quad \text{and} \quad \mathbf{u}_n = \begin{pmatrix} \nabla i_n \\ \nabla i_{n-1} \end{pmatrix}.$$

The objective function is

$$\min_{\{\nabla^2 i_1,\nabla^2 i_2,\ldots,\nabla^2 i_{T-1}\}} \mathbb{E}\left[(1-\beta)\sum_{k=1}^{T-1}\nabla C_k^2+\beta\sum_{k=1}^{T-1}\nabla^2 i_k^2\right], \tag{29}$$

which can be rewritten in matrix form as

$$\underset{e_1,e_2,\ldots,e_N}{\mathbb{E}}\left\{\sum_{n=1}^{N}\mathbf{x}_n^{\mathsf{T}}\mathbf{K}_n\mathbf{x}_n+\sum_{n=1}^{N}\mathbf{u}_n^{\mathsf{T}}\mathbf{R}_n\mathbf{u}_n\right\} \tag{30}$$

where $N = T-1$, $\mathbf{K}_n = (1-\beta)$, a scalar, and

$$\mathbf{R}_n = \beta \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The optimal control for equation (28), which minimizes the objective function (30), is then given by

$$\begin{pmatrix} \nabla i_n \\ \nabla i_{n-1} \end{pmatrix} = M_n \nabla C_{n-1} \tag{31}$$

where M_n is calculated according to equations (25) to (27).

If $\beta = 0$ then we obtain the special case where we pay no attention to the development of the valuation rate and we are fully interested to the development of the contribution rate. Equation (31) becomes

$$\nabla^2 i_n = \nabla i_n - \nabla i_{n-1} = -\frac{1}{\zeta_n} \nabla C_{n-1}. \tag{32}$$

Hence, the valuation rate of interest should be controlled using a feedback mechanism of the state variable (contribution rate). As the contribution increases, the proposed valuation rate of interest decreases.

Substituting the feedback mechanism of equation (32) into equation (22) yields

$$\begin{aligned} \nabla C_n &= \nabla C_{n-1} + \zeta_n \left[-\frac{1}{\zeta_n} \nabla C_{n-1} \right] + \xi_n (\nabla j_n - \nabla i_n) \\ &= \xi_n (\nabla j_n - \nabla i_n). \end{aligned} \tag{33}$$

4.3 Optimal Control During the Second Sub-Period

Having controlled the system for the first sub-period through an optimal path under a stochastic pattern of investment rates of return, we arrive at the time point T with a fund value of F_T. During the second sub-period the rates of return $j_{T+1}, j_{T+2}, \ldots, j_m$ are assumed to follow a deterministic process. Our problem now is to guide the fund value from F_T to $F_m = 1$ while minimizing the objective function:

$$\min_{\{C_T, C_{T+1}, \ldots, C_{m-1}\}} \sum_{k=T}^{m-1} (C_k - C_{st})^2. \tag{34}$$

Combining equation (3) and the requirement $F_m = 1$ yields the following constraint:

$$F_m = F_T \prod_{k=T+1}^{m} (1 + j_k) + \sum_{k=T}^{m-1} C_k \prod_{l=k+1}^{m} (1 + j_l) = 1. \tag{35}$$

Using Lagrange multipliers our problem is translated into the minimization of the Lagrangian function, $\Lambda (C_T, C_{T+1}, \ldots, C_{m-1}, \rho)$ with respect to $C_T, C_{T+1}, \ldots, C_{m-1}$ and ρ:

$$\begin{aligned}\Lambda(C_T, \ldots, C_{m-1}, \rho) &= \sum_{k=T}^{m-1} (C_k - C_{st})^2 \\ &+ \rho \left\{ F_T \prod_{k=T+1}^{m} (1 + j_k) + \sum_{k=T}^{m-1} C_k \prod_{l=k+1}^{m} (1 + j_l) - 1 \right\}.\end{aligned} \tag{36}$$

We find the minimum of Λ by equating the partial derivatives with respect to $C_T, C_{T+1}, \ldots, C_{m-1}$, ρ to zero. It is then straightforward (although tedious) to solve the resulting system of equations to give:

$$\rho = 2 \frac{\left[F_T \prod_{k=T+1}^{m} (1 + j_k) + C_{st} \sum_{k=T}^{m+1} \prod_{l=k+1}^{m} (1 + j_l) - 1 \right]}{\sum_{k=T}^{m-1} \prod_{l=k+1}^{m} (1 + j_l)^2} \tag{37}$$

$$C_k = C_{st} - \frac{\rho}{2} \prod_{l=k+1}^{m} (1 + j_l) \tag{38}$$

for $k = T, \ldots, m-1$. The case where $j_{T+1} = j_{T+2} = \ldots = j_m = j_*$, where j_* is the risk free rate (normally $j > j_*$) leads to

$$C_k = C_{st} - \frac{F_T (1 + j_*)^{m-T} + C_{st} \ddot{s}_{\overline{m-T}|j} - 1}{\ddot{s}_{\overline{m-n}|j_\phi}} (1 + j_*)^{m-k} \tag{39}$$

where $j_\phi = (1 + j_*)^2 - 1$.

5 The Mean and Variance of C_n with $\beta = 0$

In order to obtain a direct comparison between the traditional and control approach, we calculate the mean and variance of the contribution rate under the traditional approach, using the linearized difference equation (22). Under the traditional approach

$$\nabla i_0 = \nabla i_n = \mathbb{E}\left(\nabla j_n\right) \quad \text{for} \quad n = 0, 1, \ldots. \tag{40}$$

Hence, the traditional contribution rate, ∇C_n^{trad}, given in equation (22) becomes

$$\begin{aligned} \nabla C_n^{\text{trad}} &= \nabla C_{n-1}^{\text{trad}} + \xi_n \left(\nabla j_n - \nabla i_{n-1}\right) \\ &= \nabla C_0^{\text{trad}} + \xi_1 \left(\nabla j_1 - \nabla i_0\right) + \cdots + \xi_n \left(\nabla j_n - \nabla i_{n-1}\right). \end{aligned} \tag{41}$$

Taking the expectation of both sides of equation (41) we obtain

$$\mathbb{E}\left[\nabla C_n^{\text{trad}}\right] = \mathbb{E}\left[\nabla C_0^{\text{trad}} + \xi_1 \left(\nabla j_1 - \nabla i_0\right) + \cdots + \xi_n \left(\nabla j_n - \nabla i_{n-1}\right)\right] = 0$$

by using equation (40) and $\mathbb{E}(\nabla C_0^{\text{trad}}) = 0$. The last condition holds as the initial condition C_0^{trad} is constant so that ∇C_0^{trad} is equal to zero. Hence,

$$\mathbb{E}\left(C_n^{\text{trad}}\right) = \mathbb{E}\left(C_{st}^{\text{trad}} + \nabla C_n^{\text{trad}}\right) = \mathbb{E}\left(C_{st}^{\text{trad}}\right) = C_{st} = C_0.$$

The result is the same as in Section 3 where we used the full non-linear equation (9) for C_n.

Equation (41) also can be used to obtain the variance of the contribution rate under the traditional approach as follows:

$$\begin{aligned} \mathbb{V}\text{ar}\left[\nabla C_n^{\text{trad}}\right] &= \mathbb{V}\text{ar}\left[\nabla C_0^{\text{trad}} + \xi_1 \left(\nabla j_1 - \nabla i_0\right) + \ldots + \xi_n \left(\nabla j_n - \nabla i_{n-1}\right)\right] \\ &= \mathbb{V}\text{ar}\left[\nabla C_0^{\text{trad}}\right] + \xi_1^2 \sigma^2 + \xi_2^2 \sigma^2 + \ldots + \xi_n^2 \sigma^2 \\ &= \sigma^2 \sum_{k=1}^{n} \xi_k^2 \end{aligned} \tag{42}$$

because $\mathbb{V}\text{ar}\left(\nabla C_0^{\text{trad}}\right) = 0$.

Let C_n^{ctrl} denote the contribution rate under the control approach. We use equation (33) for ∇C_n^{ctrl}, i.e.,

$$\nabla C_n^{\text{ctrl}} = \xi_n \left(\nabla j_n - \nabla i_{n-1}\right) = \xi_n (j_n - i_{n-1}). \tag{43}$$

Proceeding as before,

$$\mathbb{E}(\nabla C_n^{\text{ctrl}}) = \mathbb{E}\left(\mathbb{E}\left(\xi_n (j_n - i_{n-1}) \,|H_{n-1}\right)\right) = \xi_n \left(j - \mathbb{E}\left(i_{n-1}\right)\right) \tag{44}$$

while the variance is given by

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left(\nabla C_n^{\mathrm{ctrl}}\right) &= \mathbb{V}\mathrm{ar}\left(\mathbb{E}\left(\nabla C_n^{\mathrm{ctrl}}|H_{n-1}\right)\right) + \mathbb{E}\left(\mathbb{V}\mathrm{ar}\left(\nabla C_n^{\mathrm{ctrl}}|H_{n-1}\right)\right)\\ &= \mathbb{V}\mathrm{ar}\left(\xi_n(j - i_{n-1})\right) + \mathbb{E}\left(\xi_n^2\sigma^2\right)\\ &= \xi_n^2 \cdot \mathbb{V}\mathrm{ar}\left(i_{n-1}\right) + \sigma^2\xi_n^2. \end{aligned} \tag{45}$$

From equations (44) and (45) we observe that the mean and the variance of the contribution rate depend upon the mean and the variance of the valuation rate of interest.

Recall the control law for the valuation rate of interest, i.e., equation (32). Substituting the expression for $\nabla C_{n-1}^{\mathrm{ctrl}}$ from equation (43) in equation (32) we obtain

$$i_n = i_{n-1} + \frac{\xi_{n-1}}{\zeta_n} i_{n-2} - \frac{\xi_{n-1}}{\zeta_n} j_{n-1}, \tag{46}$$

which is a difference equation of time-varying format with initial conditions $i_0 = i_1 = j$. We now directly can obtain a recursive relationship for the means by taking the expectations of both sides of equation (46), i.e.,

$$\mathbb{E}(i_n) = \mathbb{E}(i_{n-1}) + \frac{\xi_{n-1}}{\zeta_n}\mathbb{E}(i_{n-2}) - \frac{\xi_{n-1}}{\zeta_n}\mathbb{E}(j_{n-1}).$$

Using the initial conditions $i_0 = i_1 = j$ and $\mathbb{E}(j_{n-1}) = j$, we obtain (by induction) that

$$\mathbb{E}(i_n) = j \quad \text{for} \quad n = 0, 1, 2, \ldots, m-1. \tag{47}$$

Hence, combining equation (47) and equation (44) we obtain

$$\mathbb{E}(\nabla C_n^{\mathrm{ctrl}}) = \mathbb{E}(\nabla C_n^{\mathrm{trad}}) = 0 \tag{48}$$

for any $n = 0, 1, \ldots, m-1$, which is the same result as for the traditional approach.

Considering the difference equation (46) and the initial conditions $i_0 = i_1 = j$, we obtain (by induction) the following relationship for the variance of the valuation rate:

$$\mathbb{V}\mathrm{ar}(i_n) = \sigma^2\varphi_n \tag{49}$$

where

$$\varphi_n = \sum_{k=2}^{n} \left(\frac{\xi_{k-1}}{\zeta_k}\right)^2 + \sum_{k_1,k_2} \left(\frac{\xi_{k_1-1}}{\zeta_{k_1}}\right)^2 \left(\frac{\xi_{k_2-1}}{\zeta_{k_2}}\right)^2$$
$$+ \sum_{k_1,k_2,k_3} \left(\frac{\xi_{k_1-1}}{\zeta_{k_1}}\right)^2 \left(\frac{\xi_{k_2-1}}{\zeta_{k_2}}\right)^2 \left(\frac{\xi_{k_3-1}}{\zeta_{k_3}}\right)^2 + \dots \tag{50}$$

for $k_r = 2, 3, \dots, n$ and $1 + k_r < k_{r+1}$ for $r = 1, 2, \dots$. Substituting equation (49) into equation (45) we obtain

$$\mathbb{V}\text{ar}\left(\nabla C_n^{\text{ctrl}}\right) = (1 + \varphi_n)\xi_n^2\sigma^2. \tag{51}$$

Compare the variances under the traditional and the control approaches in the first sub-period; we would expect to see

$$\mathbb{V}\text{ar}\left(\nabla C_n^{\text{trad}}\right) > \mathbb{V}\text{ar}\left(\nabla C_n^{\text{ctrl}}\right) \tag{52}$$

because $\xi_n < \zeta_{k+1}$ for any $n < m$ and $k < n$.

6 Numerical Example

Consider an employee age 25 who will retire at age 65, i.e., $m = 40$. We assume $T = 36$, i.e., it might possible for the fund manager to find in the market a 4-year guaranteed interest rate deposit account; $i = j = 4\%$, which we assume reflects the level of long-term rates in the market; j_n is log-normally distributed[6] with parameters $\mu = -3.2492$ and $\sigma = 0.2462$ for $n = 1, 2, \dots$, i.e., $\mathbb{E}(j_n) = 4\%$ and $\mathbb{V}\text{ar}(j_n) = 0.0001$.

We perform 3,000 simulations for each of three different values of beta ($\beta = 0.0$, 0.5 , 1.0) and then calculate $\mathbb{E}(C_n^{\text{trad}})$, $\mathbb{E}(C_n^{\text{ctrl}})$, $\mathbb{E}(F_n^{\text{trad}})$, and $\mathbb{E}(F_n^{\text{ctrl}})$, and the standard deviations $\sigma(C_n^{\text{trad}})$, $\sigma(C_n^{\text{ctrl}})$, $\sigma(F_n^{\text{trad}})$, and $\sigma(F_n^{\text{ctrl}})$ for the contribution rate and the fund levels under the traditional and the control approach, respectively. Results are provided in Tables 1 and 2.

[6]The assumption of log-normality for investment returns is a simple though realistic approximation to observations of actual investment rates; see, for example, Baxter and Rennie (1996).

Table 1
Standard Deviations of Contribution Rates Under Control (C_n^{ctrl}) and Traditional (C_n^{ctrl}) Approaches and Various Values of β

n	$\beta = 0.00$		$\beta = 0.50$		$\beta = 1.00$		$\beta = 0.00$	$\beta = 0.50$	$\beta = 1.00$
	C_n^{ctrl}	C_n^{trad}	C_n^{ctrl}	C_n^{trad}	C_n^{ctrl}	C_n^{trad}	$C_n^{\text{ctrl}}/C_n^{\text{trad}}$	$C_n^{\text{ctrl}}/C_n^{\text{trad}}$	$C_n^{\text{ctrl}}/C_n^{\text{trad}}$
1	0.00	0.01	0.00	0.01	0.00	0.01	0%	0%	0%
2	0.01	0.01	0.01	0.01	0.01	0.01	91%	90%	90%
3	0.02	0.02	0.02	0.02	0.02	0.02	82%	91%	96%
4	0.02	0.03	0.03	0.03	0.03	0.03	75%	87%	97%
5	0.03	0.04	0.04	0.04	0.04	0.04	70%	84%	99%
6	0.04	0.06	0.04	0.06	0.05	0.06	67%	81%	99%
7	0.05	0.07	0.06	0.07	0.07	0.07	64%	78%	98%
8	0.05	0.09	0.07	0.09	0.09	0.09	61%	75%	98%
9	0.06	0.11	0.08	0.11	0.11	0.11	56%	72%	98%
10	0.07	0.13	0.09	0.13	0.13	0.13	54%	70%	97%
11	0.08	0.16	0.10	0.16	0.15	0.15	53%	67%	97%
12	0.09	0.18	0.11	0.18	0.17	0.18	51%	63%	96%
13	0.10	0.22	0.13	0.21	0.20	0.21	47%	62%	96%
14	0.12	0.25	0.14	0.24	0.23	0.24	46%	60%	95%
15	0.13	0.27	0.16	0.27	0.26	0.28	48%	60%	95%
20	0.23	0.49	0.26	0.49	0.46	0.50	46%	53%	91%
25	0.38	0.84	0.45	0.86	0.76	0.87	45%	52%	87%
30	0.71	1.50	0.83	1.56	1.23	1.52	48%	53%	81%
35	1.83	3.14	2.03	3.24	2.11	3.07	58%	63%	69%

Table 2

Standard Deviations (and Expectations for $\beta = 0.50$ only) of Fund Levels Under Control (F_n^{ctrl}) and Traditional (F_n^{ctrl}) Approaches and Various Values of β

	Standard Deviations										
	$\beta = 0.00$		$\beta = 0.50$		$\beta = 1.00$		Ratios $F_n^{\text{ctrl}}/F_n^{\text{trad}}$			$\beta = 0.50$	
n	F_n^{ctrl}	F_n^{trad}	F_n^{ctrl}	F_n^{trad}	F_n^{ctrl}	F_n^{trad}	$\beta = 0.00$	$\beta = 0.50$	$\beta = 1.00$	$\mathbb{E}(F_n^{\text{ctrl}})$	$\mathbb{E}(F_n^{\text{trad}})$
1	0.1	0.1	0.1	0.1	0.1	0.1	100%	100%	100%	0.01	0.01
2	0.2	0.2	0.2	0.2	0.2	0.2	101%	101%	101%	0.02	0.02
3	0.4	0.4	0.4	0.4	0.4	0.4	101%	101%	101%	0.03	0.03
4	0.6	0.6	0.6	0.6	0.6	0.6	101%	101%	101%	0.04	0.04
5	0.8	0.8	0.8	0.8	0.8	0.8	102%	101%	100%	0.06	0.06
6	1.1	1.1	1.1	1.1	1.1	1.1	103%	101%	100%	0.07	0.07
7	1.4	1.4	1.4	1.3	1.4	1.3	104%	101%	100%	0.08	0.08
8	1.7	1.6	1.7	1.6	1.7	1.7	105%	102%	100%	0.10	0.10
9	2.1	2.0	2.0	2.0	2.0	2.0	106%	103%	100%	0.11	0.11
10	2.6	2.4	2.4	2.3	2.3	2.3	107%	103%	100%	0.13	0.13
11	3.0	2.8	2.9	2.8	2.7	2.7	108%	104%	100%	0.14	0.14
12	3.4	3.1	3.2	3.0	3.1	3.1	110%	105%	100%	0.16	0.16
13	4.1	3.7	3.7	3.5	3.6	3.6	111%	106%	100%	0.18	0.18
14	4.7	4.2	4.2	3.9	4.1	4.1	113%	108%	101%	0.19	0.19
15	5.1	4.5	4.7	4.4	4.6	4.5	116%	109%	101%	0.21	0.21
20	8.7	6.9	8.2	7.0	7.2	7.1	125%	117%	102%	0.32	0.32
25	13.5	9.7	12.7	10.0	10.4	10.0	139%	127%	104%	0.44	0.44
30	19.8	12.6	18.7	13.2	14.0	12.8	156%	142%	109%	0.60	0.59
35	26.7	14.5	25.0	15.0	17.3	14.2	184%	167%	122%	0.78	0.78

As we observe (and also anticipated by expressions (13) and (49)) the mean of the contribution rates under the two approaches (controlled or traditional) remain almost constant and close to the initial rate C_{st} (i.e., $\mathbb{E}(C_n^{\text{trad}}) = \mathbb{E}(C_n^{\text{ctrl}}) = 0.01$ when $\beta = 0.50$). As regards the expectations of the contribution rates, we present the simulation results only for the value of $\beta = 0.5$ because the results for the other values of β are almost identical to that of $\beta = 0.5$.

The standard deviation of the contribution rate under the controlled approach exhibits a slightly increasing pattern (as anticipated by expression (51)) but always remains (as anticipated by expression (52)) below the standard deviation of the contribution rate under the traditional approach which exhibits a steeper increasing pattern (as anticipated by expressions (15) and (42)). The proportional difference between the controlled and traditional approach decreases as the β parameter increases toward unity. Actually, under the extreme value of $\beta = 1$ the controlled approach is almost the same as the traditional approach. (See the first columns of Table 1.) With respect to the standard deviations of the contribution rates, we present the results for all the three simulated values of the beta factor ($\beta = 0.0$, 0.5, and 1.0).

Additionally, we also may observe the expectation of the fund level under the two approaches (controlled or traditional) that remains almost the same for any value of n. (See the last two columns of Table 2.) With respect to the mean fund level, we again present the simulation results only for the value of $\beta = 0.5$ because the results for the other values of β are almost identical.

The standard deviation of the fund level under the traditional approach exhibits a slightly increasing pattern but always remains below the standard deviation of the fund level under the controlled approach which exhibits a steeper increasing pattern (the opposite pattern of the contribution rate). The proportional difference between the traditional and controlled approach decreases as the β parameter increases toward unity. Actually, under the extreme value of $\beta = 1$ the controlled approach is almost the same as the traditional approach. (See the first columns of Table 2) As regards the standard deviations of the fund level, we present the results for all the three simulated values of the beta factor ($\beta = 0.0$, 0.5, and 1.0).

It is clear from the results above that both the traditional and the controlled approach succeed in achieving (in expected value terms) the target value of the fund in a very similar way. The controlled approach also succeeds in reducing the variance of the contribution rate but this advantage is balanced with the disadvantage of a higher variance for the fund value. It is also interesting to identify the important role of the weighting factor β which may act as a regulator. The extreme values of the weighing factor β produce two extreme versions of the model.

When $\beta = 0$, we obtain the absolute controlled version of the model with the minimum value for the standard deviation of the contribution rate and the maximum value for the standard deviation of the fund value. When $\beta = 1$, we obtain almost the traditional approach with the maximum value for the standard deviation of the contribution rate and the minimum value for the standard deviation of the fund value. Hence, the choice of β may balance the levels of standard deviations between the fund value and the contribution rate.

7 Summary and Areas for Further Research

The central concept of our paper is the consideration of the valuation rate of interest as a free control variable (as proposed by Benjamin, 1989). This concept may be deemed an attractive one if one wants to determine the actual position of a pension fund. Unfortunately, however, it may pose practical problems with legislative or other regulatory restrictions. The optimal path for the contribution rate (according to our objective function) is then determined by controlling the pattern of the valuation rate of interest through a feedback mechanism. Actually, our model process permits the actuary to adjust the initial valuation rate (which may be based on projections of long-term rates) to reflect the recent investment experience.

The model is solved using a standard linear approximation procedure for the basic equation of the system. The important result is provided by equation (31) where the valuation rate is optimally controlled through a feedback mechanism of the state variable (which is the contribution rate). Under this optimal control law, we observe that the expected contribution rate remains the same (as for the traditional approach) for the whole funding period, while the variance of the contribution rate exhibits a slightly increasing pattern. This increase in variance is less than the increase in variance under the traditional approach. Unfortunately, this advantage of the controlled approach is counterbalanced with the higher fluctuations of its fund levels over time.

It is also interesting to identify the regulatory role of the weighting factor β which under the extreme values $\beta = 0$ and $\beta = 1$ produces the absolute controlled and traditional version of the model. Hence, the specific approach illustrates that the traditional form of the individual aggregate cost method may be seen in a wider context as a special case (for $\beta = 0$) of a controlled model. In this new controlled model, we can design in advance the desired level of the variances for the contribution rate and the fund value by selecting the appropriate value for the weighting factor of the objective function named β.

The model may be extended further by relaxing some assumptions. For example, we can make the pension benefit dependent on final salary and assume fluctuations in the interest rates available to purchase the retirement annuity at the time of retirement.

It is clear from the model investigated above that control theory may be applicable to the individual aggregate cost method by providing a system with an improved performance relative to the traditional version of this specific method.

References

Ang, A. and Sherris, M. "Interest Rate Risk Management: Developments in Interest Rate Term Structure Modeling for Risk Management and Valuation of Interest-Rate-Dependent Cash Flows." *North American Actuarial Journal* 1 (1997): 1-21.

Aoki, M. *Optimization of Stochastic Systems,* 2nd Edition. New York, NY: Academic Press, 1989.

Benjamin, S. "Driving the Pension Fund." *Journal of the Institute of Actuaries* 116 (1989): 717-735.

Baxter, M. and Rennie, A. *Financial Calculus.* Cambridge, United Kingdom: Cambridge University Press, 1996.

Bertsekas, D.P. *Dynamic Programming and Stochastic Control.* Cambridge, United Kingdom: Academic Press, 1976.

Cairns, A. "Some Notes on the Dynamic and Optimal Control of Stochastic Pension Fund Models in Continuous Time." *ASTIN Bulletin* 30 (2000): 19-55.

Haberman, S. and Sung, J.H. "Dynamic Approaches to Pension Funding." *Insurance: Mathematics and Economics* 15 (1994): 151-162.

Loades, D.H. "Elementary Engineering Control and Pension Funding." *Transactions of the* 26th *International Congress of Actuaries* 5 (1998): 239-266.

McGill, D.M., Brown, K.N., Haley, J.J. and Schieber, S.J. *Fundamentals of Private Pension Funds.* Philadelphia, PA: Pension Research Council, University of Pennsylvania, 1989.

Owadally, M.I. and Haberman, S. "Pension Fund Dynamics and Gains/Losses Due to Random Rates of Return." *North American Actuarial Journal* 3, no. 3 (1999): 105-117.

Owadally, M.I. and Haberman, S. (2004). "Efficient Gain and Loss Amortization and Optimal Funding of Pension Plans." *North American Actuarial Journal* 8, no. 1 (2004): 21-36.

Taylor, G.C. "Stochastic Control of Funding Systems." *Insurance: Mathematics and Economics* 30 (2002): 323-350.

Vandebroek M. "Pension Funding and Optimal Control." *Mitteilungen der Vereinigung Schweizerischer Versichrungsmathematiker* (1990): 313-325.

Vigna, E. and Haberman, S. "Optimal Investment Strategy For Defined Contribution Pension Schemes." *Insurance: Mathematics and Economics* 28 (2001): 233-262.

Wilkie, A.D. "More on a Stochastic Asset Model for Actuarial Use." *British Actuarial Journal* 1 (1995): 777-945..

Winklevoss, H.E. *Pension Mathematics with Numerical Illustrations.* Philadelphia, PA: Pension Research Council, University of Pennsylvania, 1993.

Zimbidis, A. and Haberman, S. "Delay, Feedback and Variability of Pension Contributions and Fund Levels." *Insurance: Mathematics and Economics* 13 (1993): 271-285.