2005

Optimal Dividend Strategies: Some Economic Interpretations for the Constant Barrier Case

Maite Marmol
University of Barcelona, mmarmoi@ub.edu

M. Merce Claramunt
University of Barcelona, mmclaramunt@ub.edu

Antonio Alegre
University of Barcelona, aalegre@ub.edu

Optimal Dividend Strategies: Some Economic Interpretations for the Constant Barrier Case

Maite Mármol,* M. Mercè Claramunt,[†] and Antonio Alegre[‡]

Abstract[§]

We consider the surplus process of a non-life insurance portfolio with a dividend component represented by a constant dividend barrier strategy. The optimal dividend barrier is known when individual claim amounts follow an exponential distribution. This result for the optimal dividend barrier is used to develop combinations of the levels of the insurer's initial surplus and of the barrier which, under certain economic and financial criteria, can be regarded as optimal.

Key words and phrases: *optimal dividend strategy, constant barrier, surplus process with dividends, solvency*

*Maite Mármol, Ph.D., is a professor, licensed actuary, and a member of the Catalan Actuarial Association. Dr. Mármol's research interests are in the areas of financial mathematics and risk theory.

Dr. Mármol's address is: Department of Economical, Financial and Actuarial Mathematics, University of Barcelona, Avda. Diagonal, 690. 08034 Barcelona, SPAIN. E-mail: *mmarmol@ub.edu*

[†]M. Mercè Claramunt, Ph.D., is a professor, licensed actuary, and a member of the Catalan Actuarial Association. Dr. Claramunt's research interest are in the areas of solvency in insurance, risk theory, and rate-making.

Dr. Claramunt's address is: Department of Economical, Financial and Actuarial Mathematics, University of Barcelona, Avda. Diagonal, 690. 08034 Barcelona, SPAIN. E-mail: *mmclaramunt@ub.edu*

[‡]Antonio Alegre, Ph.D. is Dean of the Faculty of Economics at the University of Barcelona, licensed actuary, and a member of the Catalan Actuarial Association.

Dr. Alegre's address is: Department of Economical, Financial and Actuarial Mathematics, University of Barcelona, Avda. Diagonal, 690. 08034 Barcelona, SPAIN. E-mail: *aalegre@ub.edu*

[§]The authors wish to thank the anonymous referees for their helpful comments and suggestions that have improved this paper.

1   Introduction

In the classical compound Poisson model of risk theory, an insurance company's surplus can increase without bounds. This is unrealistic, because the company could reinvest its excess surplus in search of even bigger returns or could simply pay them out as dividends to its shareholders. Thus, to make the classic model more realistic, we should include dividend payments.

The question of how much and when to make dividend payments first was studied by De Finneti (1957). He found that the optimal strategy to maximize the expected sum of the discounted dividends must be a barrier strategy, and he showed how the optimal level of the barrier can be determined. Bühlmann (1970, p. 164) proved that the introduction of a constant barrier in the classical model leads to certain ruin.

The problem of finding the optimal dividend-payment strategy assuming a constant barrier was discussed extensively by several other authors. Gerber (1972, 1979) and Bühlmann (1970) analyze the problem in the context of the classical risk model. The random variable representing the present value of dividends also has been analyzed in the discrete case: Claramunt, Mármol, and Alegre (2003) obtain a general solution of its expectation, and Dickson and Waters (2004) obtain higher order moments in the discrete and continuous case. Recently, authors have modified the risk process by considering a Brownian motion risk model; see, for example, Asmussen and Taskar (1997), Paulsen and Gjessing (1997) who include a stochastic interest on reserves, and Gerber and Shiu (2004) who obtain the moments of the present value of dividends. Other forms of the barriers have been considered, for example a linear barrier was studied by Gerber (1981) and Siegl and Tichy (1999) while a non-linear dividend barrier was first introduced by Alegre, Claramunt, and Mármol (2001) and generalized by Albrecher and Kainhofer (2002).

In this paper we study optimal dividend strategies for a non-life insurance portfolio under a compound Poisson model with a constant barrier. We provide combinations of the levels of the initial surplus and levels of dividend barriers that, under certain economic and financial criteria, can be regarded as optimal. For simplicity we assume the individual claim amounts are independent and identically distributed exponential variables, which makes our analysis easier. The analysis of optimal dividend strategies with other individual claim amount distributions can be performed with simulations or discrete risk models.

The paper is organized as follows: Section 2 gives the main characteristics of the model with a constant barrier. Section 3 contains an

analysis of the function of the expected present value of dividends, and in Section 4 the optimal combinations are proposed.

2 The Constant Barrier Model

2.1 The Modified Surplus Process

In the classical model of risk theory, the surplus at a given time t, $U(t)$, is defined as

$$U(t) = u + ct - S(t)$$

for $t > 0$ with $U(0) = u$ being the insurer's initial surplus. The term $S(t)$ represents the aggregate claims in $(0, t)$ modeled as a time homogeneous compound Poisson process with rate λ and μ the expected claim amount. The rate at which the premiums are received is $c = \lambda\mu(1 + \theta)$, where $\theta > 0$ is the security loading.

The imposition of a constant dividend barrier $b \geq u$ modifies the behavior of the surplus process because, when the surplus reaches the level b, all premium incomes are paid out as dividends to shareholders, and the modified surplus process remains at level b until the occurrence of the next claim. The modified surplus process, $\bar{U}(t)$ is given by

$$\bar{U}(t) = U(t) - D(t) \tag{1}$$

where $D(t)$ is the aggregate of dividend payments in the interval $(0, t]$, i.e., for infinitesimally small dt,

$$D(t + dt) - D(t) = \begin{cases} 0 & \text{if } U(t) < b \\ c\,dt & \text{if } U(t) \geq b. \end{cases} \tag{2}$$

Ruin is said to occur at time T if $\bar{U}(T) < 0$ and $\bar{U}(t) \geq 0$ for $t < T$ with the understanding that $T = \infty$ if $\bar{U}(t) \geq 0$ for all $t > 0$.

Figure 1 shows a typical sample path of $U(t)$, $\bar{U}(t)$, and $D(t)$ where T_i for $i = 1, 2, \ldots$ denotes the time of occurrence of the ith claim. Notice that whenever the surplus $U(t)$ reaches b, dividends are paid out to the shareholders with intensity c, and the surplus remains on the barrier until the next claim occurs.

Let $W(u, b)$ denote the expected present value of the discounted aggregate dividend payments up to the moment of ruin T, i.e.,

$$W(u, b) = E\left[\int_0^T e^{-\delta t} dD(t)\right]$$
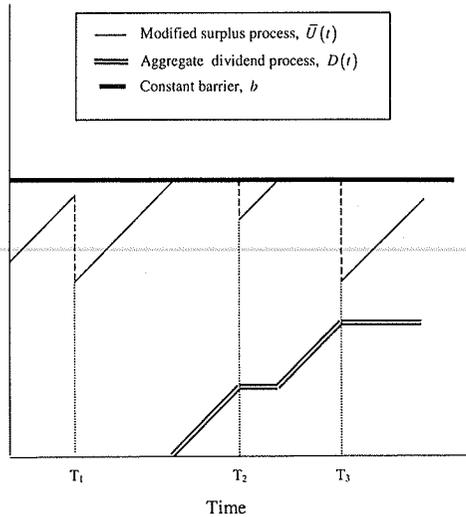
Figure 1: Typical Sample Path of $U(t)$, $\bar{U}(t)$, and $D(t)$

where $\delta \geq 0$ is the force of interest.[1] Bühlmann (1970, p. 173), assuming an exponential distribution for the individual claim amount, obtained an expression for $W(u,b)$. Without loss of generality, we assume the claim size unit is scaled so that the expected claim size is 1. Bühlmann's result can be rewritten as,

$$W(u,b) = \frac{\dfrac{1+r_1}{1+r_2}e^{r_1 u} - e^{r_2 u}}{\dfrac{1+r_1}{1+r_2}r_1 e^{r_1 b} - r_2 e^{r_2 b}} \qquad (3)$$

---

[1]Note that the surplus process is not discounted in order to obtain a tractable model. This is consistent with an economy where the rate of inflation is equal to the rate of return on investment income.

where $r_1 > r_2$ are the roots of

$$\lambda(1 + \theta)r^2 - (\delta - \lambda\theta)r - \delta = 0.$$

It is easy to demonstrate $r_1 > 0$ and $r_2 < 0$ and that the following relationships hold between the two roots:

1. $(\delta - \lambda\theta) > 0$ implies $-1 < r_2 < 0 < r_1 < \infty$ and $|r_2| < |r_1|$;

2. $(\delta - \lambda\theta) = 0$ implies $-1 < r_2 < 0 < r_1 < 1$ and $|r_2| = |r_1|$; and

3. $(\delta - \lambda\theta) < 0$ implies $-1 < r_2 < 0 < r_1 < 1$ and $|r_2| > |r_1|$.

Note that $(\lambda\theta - \delta)$ is the difference between the income rate from the security loading and the force of interest used to discount the dividends.

2.2 Some Properties of $W(u, b)$

By considering the other parameters $(\lambda, \theta, \delta, u)$ as fixed, let us find the b^* that maximizes $W(u, b)$. Bühlmann (1970) minimized the denominator of equation (3) to give

$$b^* = \frac{1}{r_2 - r_1} \ln\left(\frac{r_1^2(1 + r_1)}{r_2^2(1 + r_2)}\right) \qquad -\infty < b^* < \infty. \qquad (4)$$

We can observe that b^* doesn't depend on u, so b^* can be less than u and even be negative. When u exceeds b^*, the optimal level of the barrier is $b = u$ (Dickson and Waters, 2004, p. 63).

If $u \le b^*$, then b^* is a maximum point of $W(u, b)$. Interestingly, as b increases to b^* the time that it takes for the surplus to reach the barrier and dividend payments to begin is lengthened; however, $W(u, b)$ increases because the time to ruin is increased thereby allowing dividend payments to be made over a longer period. When b gets beyond b^*, the dividend payments made in the distant future have less impact on the expected present value of the dividends due to the presence of the discount rate.

It is easy to see that

$$\frac{\partial}{\partial u}W(u, b) = \frac{-\frac{1 + r_1}{1 + r_2}r_1 e^{r_1 u} + r_2 e^{r_2 u}}{-\frac{1 + r_1}{1 + r_2}r_1 e^{r_1 b} + r_2 e^{r_2 b}} > 0.$$

When $0 \le u \le b$, let u^* denote the optimum value of the initial surplus that maximizes the expected present value of the dividend payments

for a given barrier value b, i.e., is $u^* = b$. Thus we must explore the function $W(u^*, b) \equiv W(b, b)$. For convenience we use the notation $W(k) = W(k, k)$, i.e.,

$$W(k) = \frac{-\dfrac{1 + r_1}{1 + r_2} e^{r_1 k} + e^{r_2 k}}{-\dfrac{1 + r_1}{1 + r_2} r_1 e^{r_1 k} + r_2 e^{r_2 k}}.$$

Note that

$$W(0) = \frac{r_2 - r_1}{r_2 (1 + r_2) - r_1 (1 + r_1)} > 0$$

while

$$W'(k) = \frac{\dfrac{1 + r_1}{1 + r_2} e^{(r_1 + r_2)k} (r_1 - r_2)^2}{\left(-\dfrac{1 + r_1}{1 + r_2} r_1 e^{r_1 k} + r_2 e^{r_2 k} \right)^2} > 0,$$

i.e., $W(k)$ is monotonically increasing. The upper bound of $W(k)$ is easily seen to be $W(\infty) = 1/r_1$.

Next we will establish that $W(k)$ has a point of inflection. Let

$$h(k) = -\frac{1 + r_1}{1 + r_2} e^{r_1 k} + e^{r_2 k}$$

so that $W(k) = h(k)/h'(k)$. Differentiating $W(k)$ twice with respect to k and equating this derivative to zero gives

$$[h'(k)]^2 h''(k) + h(k) h'(k) h'''(k) - 2h(k) [h''(k)]^2 = 0. \quad (5)$$

The solution to equation (5) is $k = k_i$ where

$$k_i = \frac{1}{r_2 - r_1} \ln \left[\left(\frac{-r_1}{r_2} \right) \left(\frac{1 + r_1}{1 + r_2} \right) \right]. \quad (6)$$

We see that $k = k_i$ is a point of inflection because $W''(k_i) = 0$ and $W'''(k_i) \neq 0$. Thus we have just established the following proposition:

Proposition 1. *For $k > 0$,*

1. *$W(k)$ is positive monotonically increasing;*

2. *$\lim_{k \to \infty} W(k) = 1/r_1$; and*

3. *$W(k)$ has a point of inflection at k_i given in equation (6).*

From equations (6) and (4),

$$k_i - b^* = \frac{1}{r_2 - r_1} \left(\ln \frac{-r_2}{r_1} \right), \tag{7}$$

which leads to the following results:

1. $(\delta - \lambda\theta) > 0$ implies $b^* < k_i < 0$;

2. $(\delta - \lambda\theta) = 0$ implies $b^* = k_i < 0$; and

3. $(\delta - \lambda\theta) < 0$ implies $-\infty < k_i < \infty$ and $k_i < b^*$.

3   Criteria for Choosing k

We now investigate three criteria for choosing k based on: (i) percentiles, (ii) the maximum marginal increase, and (iii) recouping the initial investment.

The Percentile Criterion

From Proposition 1, there exists no value of k that maximizes $W(k)$. As $W(k)$ has an upper limit, $1/r_1$, which is independent of the initial surplus level and the barrier level, an obvious question is what is the value of k that allows us to achieve a specified percentage ($100\alpha\%$) of this limit? Let k_α denote this value, i.e., k_α satisfies $W(k_\alpha) = \alpha/r_1$. It can be proved that

$$k_\alpha = \frac{1}{r_2 - r_1} \ln \left(\frac{(1 - \alpha)}{r_1 - \alpha r_2} \times \frac{r_1^2 + r_1}{1 + r_2} \right). \tag{8}$$

In Table 1 we provide some numerical results:

The Maximum Marginal Increase Criterion

Proposition 1 states that the greater the value of the initial reserve and barrier k, the greater the expected present value of the dividends. It is costly, however, for companies to keep increasing the level of k because of the opportunity cost of tying up the company's capital in its surplus. So the question then becomes how large should k be?

Let $M(k)$ denote the marginal rate of increase in the expected present value of the aggregate dividends paid given a barrier at k and initial reserve k, i.e., $M(k) = W'(k)$. This criteria states that investors set k to

Table 1

k_α for Various Values of α

($\delta = 0.03$, $\lambda = 1$, $\theta = 0.5$)

α	k_α	$W(k_\alpha)$
0.1	0.262	1.839
0.2	1.343	3.677
0.3	2.258	5.516
0.4	3.101	7.354
0.5	3.930	9.193
0.6	4.799	11.032
0.7	5.778	12.870
0.8	7.002	14.709
0.9	8.882	16.547

maximize $M(k)$, i.e., k is such that $M'(k) = 0$ and $M''(k) < 0$. In other words the criteria to set $k = k_i$, the point of inflexion of the function $W(k)$, with $W'''(k_i) < 0$. The resulting expression for W is:

$$W(k_i) = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right). \tag{9}$$

In Section 2.2 we obtained the values of k_i and b^* according to $(\delta - \lambda\theta)$. If $k_i < b^*$, for $k_i, b^* > 0$, on the combination (k_i, k_i) we can raise the level of the barrier (which involves no extra effort in the initial investment), converting it into (k_i, b^*). We attain a combination that can be regard as optimal for the decision maker, with

$$W(k_i, b^*) = \frac{r_1 + r_2}{r_1 - r_2}\left[\frac{(r_2)^{r_1}}{(-r_1)^{r_2}}\right]^{\frac{1}{r_2 - r_1}}.$$

If $k_i \geq b^*$ we are in a situation in which $k_i, b^* < 0$, and therefore the optimal combination as a function of the values of k_i and b^* is meaningless. We then should focus on the value of k_α, which, fixing the percentage that we consider acceptable to obtain on the maximum of the expected present value of the dividends, α, leads us to choose $u = b = k_\alpha$ as the optimal combination.

Recoup the Initial Investment Criterion

Another way to choose k is for investors to require total recovery of their initial investment of k through future expected dividends, i.e.,

$$W(k) \geq k. \tag{10}$$

Let k_e satisfy the equality $W(k_e) = k_e$. We call k_e the efficiency threshold when the dividends are discounted at a rate δ. At k_e the insurer's rate of return, which we shall represent as $\hat{\theta}$, coincides with the rate δ. It is easy to prove the existence of a unique efficiency threshold, k_e, and that $k > W(k)$ for $k > k_e$ while $k < W(k)$ for $k < k_e$. It follows that the insurer's rate of return is less than δ for $k > k_e$ while it is greater than δ for $k < k_e$. Thus the investors will demand that the insurer set $k < k_e$.

Given that $k < k_e$, it is natural to ask whether there exists a k that maximizes $W(k) - k$. We refer to such a k as k^*, i.e.,

$$k^* = \sup_{0 \leq k \leq k_e} \{W(k) - k\}$$

It is easy to prove that $k^* = b^*$. Thus $k^* > 0$ only when $(\delta - \lambda\theta) < 0$. Under this condition, we therefore can affirm that the optimal value of the expected present value of the dividends according to this criteria is obtained for $k = k^* = b^*$ giving

$$W(k^*) = \frac{r_1 + r_2}{r_1 r_2} = \frac{\lambda\theta}{\delta} - 1.$$

For the case in which $(\delta - \lambda\theta) \geq 0$ leads to $k^* < 0$ and $k_i > k^*$, the maximum difference for $k \geq 0$ will be with a zero initial investment, which is meaningless from an economic standpoint.

Table 2 provides some numerical results as examples of the maximum marginal increase and the recouping of the initial investment criteria presented above. Using $\lambda = 1$, $\delta = 0.03$ and $\delta = 0.05$, and $\theta = 0.2$ and $\theta = 0.5$, we indicate the resulting values of the roots, b^*, the inflection point k_i, the efficiency threshold k_e and the expected present value of dividends for the combinations of u and b.

For example, assuming $\delta = 0.03$ and $\theta = 0.2$, under the maximum marginal increase criteria, we first choose $u = b = k_i = 1.417$, which gives $W(1.417) = 2.833$. Then, without any extra increase in the initial investment, we can raise the level of the barrier in order to increase the expected present value of dividends, $W(1.417, 3.923) = 3.088$. While under the recouping of the initial investment criteria, we have to choose

Table 2
An Example Using the Maximum Marginal
Increase and the Recouping of the Initial
Investment Criteria with $\lambda = 1$

	$\delta = 0.03$		$\delta = 0.05$	
	$\theta = 0.2$	$\theta = 0.5$	$\theta = 0.2$	$\theta = 0.5$
r_1	0.102	0.054	0.151	0.086
r_2	−0.244	−0.368	−0.276	−0.386
$b^* = k^*$	3.923	7.844	1.740	5.135
k_i	1.417	3.316	0.327	1.963
$W(k_i)$	2.833	7.833	1.500	4.500
$W(k_i, b^*)$	3.088	10.669	1.567	5.595
k_e	8.752	18.350	5.677	11.429
$W(k^*)$	5.667	15.667	3.000	9.000

$u = b^* = k^* = 3.923$ giving the expected present value of dividends as $W(3.923) = 5.667$.

4 Summary

We analyzed the expected present value of dividend payments under a constant dividend barrier, when the aggregate claim amount is assumed to follow a compound Poisson process and the individual claim amount has an exponential distribution. Under these assumptions, we provide some economic/financial criteria for deciding the optimal combination of the initial surplus and the level of the barrier.

An area for further research is to consider other distributions for the individual claim amount, using simulations or discrete approximations. Further research could be done with other models of the risk process such as the Erlang process or Brownian motion (Gerber and Shiu, 2004).

References

Albrecher, H. and Kainhofer, R. "Risk Theory with a Non-Linear Dividend Barrier." *Computing* 68, no. 4 (2002): 289–311.

Alegre, A., Claramunt, M.M., and Mármol, M. "Probabilidad de Ruina y Estrategias de Barrera Bajo un Proceso de Poisson Compuesto." *Cuadernos de Ciencias Economicas y Empresariales* 41 (2001): 75–92.

Asmussen, S. and Taskar, M. "Controlled Diffusion Models for Optimal Dividend Pay-Out." *Insurance: Mathematics and Economics* 20 (1997): 1–15.

Bühlmann, H. *Mathematical Methods in Risk Theory.* Heidelberg, Germany: Springer-Verlag, 1970.

Claramunt, M.M., Mármol, M., and Alegre, A. "A Note on the Expected Present Value of Dividends with a Constant Barrier on the Discrete Time Model." *Mitteilungen der Schweizerischen Aktuarvereinigung* no. 2 (2003): 149–159.

Dickson, D.C.M. and Waters, H.R. "Some Optimal Dividend Problems." *ASTIN Bulletin* 34, no. 1 (2004): 49–71.

de Finetti, B. "Su un Impostazione Alternativa Della Teoria Collectiva del Rischio." *Transactions XV. International Congress of Actuaries* 2 (1957): 433–443.

Gerber, H.U. "Games of Economic Survival with Discrete and Continuous Income Process." *Operations Research* 20, no. 1, (1972): 37–45.

Gerber, H.U. *An Introduction to Mathematical Risk Theory.* Philadelphia, PA: S.S. Huebner Foundation, University of Pennsylvania, 1979

Gerber, H.U. *On the Probability of Ruin in the Presence of a Linear Dividend Barrier.* Scandinavian Actuarial Journal (1981): 105–115.

Gerber, H.U. and Shiu, E.S.W. "Optimal Dividends: Analysis with Brownian Motion." *North American Actuarial Journal* 8, no. 1, (2004): 1–20.

Paulsen, J. and Gjessing, H.K. "Optimal Choice of Dividend Barriers for a Risk Process with Stochastic Return on Investments." *Insurance: Mathematics and Economics* 20 (1997): 215–223.

Siegl, T. and Tichy, F. "A Process with Stochastic Claim Frequency and a Linear Dividend Barrier." *Insurance: Mathematics and Economics* 24 (1999): 51–65.