Decoherence, entropy and thermal radiance using influence functionals

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This thesis forms part of a
Doctor of Philosophy
done in the
Department of Physics and Mathematical Physics,
University of Adelaide

Adelaide, July, 1996

Contents

Abstract

This thesis is concerned with the action of an environment on a system.

We first deal with the fundamentals of decoherence by describing a model for the interaction of two particles. We use standard quantum mechanics to derive the density matrix and its behaviour in time and space; this allows us to investigate what the model can tell us about any mechanism which might be involved.

The remainder of the thesis looks at scalar fields, and uses the influence functional technique to derive the density matrix resulting from the evolution of a gaussian density matrix, which itself describes the initial state of a field mode coupled to a bath (of oscillators).

First we cover the background to influence functionals and squeezed systems. We then consider a squeezed system, in particular calculating the entropy production. We show that influence functionals can reproduce and also extend earlier results that have been obtained by others in several different ways. These earlier results were all derived by calling on various ad hoc prescriptions for decohering a system; our approach is more systematic in that we do not force any coarse graining to occur. In doing so we place the system-environment interaction on a more secure footing.

Next we extend our formalism to consider the evolution of primordial fluctuations in the early universe, using new inflation as a starting point. We are able to follow the evolution of density fluctuations and their conjugate momenta to determine whether the general model of a scalar field mode outside the horizon coupled to an environment (composed of short wavelength modes which are still within the horizon) can adequately produce the perturbations needed to act as seeds for galaxy formation.

Lastly we use the influence functional technique to study particle creation in two types of spacetime: those which have a horizon and those which don't. Traditionally, particle creation has been associated with the presence of a horizon and its geometrical effect of distorting field modes. We show that in our language the nature of particle creation need not actually refer to the existence of a horizon; this goes some way towards tying together the seemingly disparate viewpoints of statistical mechanics and the geometry of relativity.

Declaration

This thesis does not contain work which has been accepted for any other award in any university. Nor does it contain work which has previously appeared elsewhere, except where referenced within the text. It is available to be photocopied and lent from the University Library.

Don Koks,
July 1996

Acknowledgements

Various people helped to bring this work to completion.

May I thank my supervisor and cosupervisor respectively, Paul Davies of Adelaide University and Andrew Matacz of Adelaide and lately Sydney University, for their guidance throughout the work; and also Bei-Lok Hu of the University of Maryland with whom I collaborated in the work of chapter 6.

This work was helped along considerably by the in depth discussions I have had in particular with Scott Foster and Andy Rawlinson, as well as helpful suggestions by many people along the way, for which my thanks go to Armin Ardekani, Lindsay Dodd, Adrian Flitney, Angas Hurst, Vivek Iyer, Alex Kalloniatis, Jim McCarthy, Steve Poletti, Alpan Raval of Maryland, Jie Ruan, Michael Simpson, Peter Szekeres, Jason Twamley, and David Wiltshire.

I am also grateful to the Australian Government, which provided generous assistance through a Commonwealth Scholarship.

Chapter 1

Introduction and statistical background

Quantum mechanics traditionally is concerned with isolated particles evolving in some potential, and it allows us to explore what characteristics they can be found to have as the result of a measurement.

The conventional interpretation of quantum mechanics does not ask what happens during this measurement process, merely what its results are. Of course the act of measurement itself has long been seen as problematic, as typified by the Schrödinger's cat paradox.

The resolution of such paradoxes—at least within the conventional framework—appears to lie with the identification of an "environment" which can act on a "system" of interest whether or not an observer is present to consciously make a measurement.

This view, that the environment cannot be neglected in the evolution of a system, is the subject of this thesis. The question of what "really" happens when a wavefunction collapses is not addressed; instead we focus on investigating to what extent an environment can influence a system in such a way that classical behaviour ultimately emerges from a quantum treatment. This investigation has important consequences for ideas of the universe's early evolution. For example, consider a system interacting with a detector. Somehow as a consequence of the interaction, the density matrix of the system becomes diagonal, which implies some sort of classical behaviour emerging. But if we consider the system-detector to be a new, bigger system, then *its* density matrix does not become diagonal. If we introduce another detector to interact with this new system, the same description holds: by introducing ever more environments which interact with the first system to become ever larger systems, we are effectively pushing the off diagonal terms further and further away.

In the original contexts where the density matrix played a role, this continual sweeping under the rug gave no problems. But in the context of the universe as a whole we clearly are running out of environments to soak up the extra terms. This thesis aims to develop some techniques of dealing with the system-environment interaction at both small levels and cosmological ones, with applications to investigating the emergence of classicality both in a simple system and in the early universe, and an alternative view of the production of thermal radiance for various types of motion and spacetimes.

It has two rather separate parts. The first comprises just one chapter on the DeWitt model of decoherence. This model considers a light particle incident on a more massive one, and it then goes on to consider the case of many light particles incident. DeWitt wished to use this to follow a relatively simple route from a quantum system to a classical one. Initially we studied his model in some detail to see what we could infer from it. Because DeWitt's explanation of the model was somewhat sketchy, our aim in this work was to calculate the maths fairly precisely, trying not to leave any stones unturned along the way.

Unfortunately, doing so defeated the whole purpose of discussing such a simple model in the first place. We made the model more realistic—and introduced the all-important concept of time—by modelling the particles not as plane waves but as wave packets. Not unnaturally, at this point a lot of numerical evaluation was needed to get some feel for the increasingly more complex density matrices involved, and current computer speeds are not up to allowing us to proceed at a worthwhile pace. It seems that on a closer inspection, the model didn't live up to its initial promise of simple results. However we did establish a new uncertainty principle in the plane wave case.

The second part of this thesis comprises the more technical, but hopefully more fruitful, models of system-environment interactions modelled by influence functionals. The influence functional technique, now something over thirty years old, is a formalism for evolving a reduced density matrix, and as with so many tools in physics it finds its easiest application to systems and environments composed of oscillators.

Any system with a quadratic lagrangian (which is always what we consider) has an evolution operator that can be described in quantum optical language using Bogoliubov coefficients. Besides giving the influence functional formulae in a more elegant format, chapter 3 (the introductory chapter on influence functionals) gives a simple but very useful theorem which states that the sum of the Bogoliubov coefficients of such a system satisfies its classical equation of motion. This theorem is used throughout this thesis, because such a sum figures prominently in the formalism. It has been noted before, for example in [1] and in a more limited form (for the static case only) in [2].

In chapters 4 and 5 we use influence functionals to evolve reduced density matrices. The main effort in these chapters concerns calculating coefficients in the matrix propagator. These coefficients contain divergent integrals, and when treating these in two different ways we found a contradiction in the results. In chapter 4, the results found by using a simple approach to regularising the integrals differ from the next chapter's more sophisticated approach. We have not been able to resolve this contradiction.

In the last chapter of this thesis we deal with the subject of thermal radiance as seen by a variety of different observers, such as in the familiar Davies-Unruh effect. Although we don't use influence functionals in this chapter, the sum of the Bogoliubovs again makes its appearance. We concentrate on using the formalism as an alternative way to consider the noise and dissipation created in the surrounding vacuum by say an accelerating observer, and show that it gives the usual result that these are identical to the noise and dissipation which an inertial observer would see in a thermal bath. We also show how the formalism can be used in examples of more complicated motion which were previously thought to be related to the existence of a horizon. In the influence functional treatment the idea of a horizon is no longer really necessary at all.

Many of the long calculations for the influence functionals have been put into appendices. Also in the appendices we have listed the various lagrangians for the scalar fields used in this thesis.

A note about notation: there are three system-environment models discussed in this thesis, each with different labels for the system and environment oscillators. I have chosen not to use a uniform notation for all three, since this would make them unrecognisable from other work which has been done in each of the three areas. Notation which has been used is as follows:

	chapter 2	chapter 4	chapter 5
system:	X	x	q
environment:	x	q	r

Statistical background

The idea of looking for the effects of an environment without having to include it in every question we might ask about some system of interest, is important to quantum statistical mechanics, and plays a major role in this thesis. For example it is thought to play a key role in the quantum to classical transition, and in such areas as entropy growth for an evolving system.

In practice we can "hide" the environment as follows. Suppose our system is labelled by x, with the environment (or bath) labelled by q. Consider finding the expected value of a system observable $A(x)$. The system plus environment is described by a density matrix

ρ. Elementary theory enables us to trace over both q, x:

$$\langle A \rangle = \text{tr } \rho A = \int \langle q\, x | \rho A | q\, x \rangle \; dq\, dx \tag{1.1}$$

We wish to make use of the fact that A is independent of the bath, q. So write

$$\langle A \rangle = \int \langle q\, x | \rho | q'\, x' \rangle \langle q'\, x' | A | q\, x \rangle \; dq\, dq'\, dx\, dx' \tag{1.2}$$

The term involving A can be split into its constituent bra-kets, since it doesn't depend on q. So we have

$$\begin{aligned} \langle A \rangle &= \int \langle q\, x | \rho | q'\, x' \rangle \langle x' | A | x \rangle \delta(q' - q) \; dq'\, dq\, dx\, dx' \\ &= \underbrace{\int \langle q\, x | \rho | q\, x' \rangle dq}_{\equiv \langle x | \rho_r | x' \rangle} \; \langle x' | A | x \rangle \; dx\, dx' \\ &= \int \langle x | \rho_r | x' \rangle \; \langle x' | A | x \rangle \; dx\, dx' \\ &= \text{tr } \rho_r A \end{aligned} \tag{1.3}$$

where this last trace is now only over x, the system variable. This defines a new quantity, the reduced density matrix, and armed with this we are in a position to treat the system in the usual way without needing to refer to the bath.

Although the density matrix is central to the work which follows, it certainly does not give an unambiguous description of the system. We can see this by describing two very different mixed ensembles, both of which have the same density matrix. Each ensemble comprises say three large sets of atoms. Suppose they are produced from an oven:

(1) In the first case, they leave the oven without our measuring their spin. Write a density matrix description of them in the z-basis: it will be just a weighted sum of $|\uparrow\rangle\langle\uparrow|$ and $|\downarrow\rangle\langle\downarrow|$, and since we have made no measurements, the weightings must both be $1/2$, so that the density matrix can be written:

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \tag{1.4}$$

We now could choose to send them through a Stern-Gerlach apparatus which sorts them into spin up and down. We might measure the populations of the three sets as follows:

$$\begin{gathered} 60\% \uparrow, 40\% \downarrow \\ 54\% \uparrow, 46\% \downarrow \\ 50\% \uparrow, 50\% \downarrow \end{gathered}$$

Obviously the percentage of each spin has some nonzero variance. Even for a large number of sets, the mean will tend toward 50%, but the variance will not tend to zero.

(2) For the second ensemble, we again have three sets each containing a large number of atoms, except that now after exiting the oven, the atoms have been prepared into equal up and down spins by a preliminary measurement made by someone else, before *we* get to measure them. Then their density matrix is again (1.4), but now on passing the atoms through our Stern-Gerlach apparatus we measure

$$\begin{gathered} 50\% \uparrow, 50\% \downarrow \\ 50\% \uparrow, 50\% \downarrow \\ 50\% \uparrow, 50\% \downarrow \end{gathered}$$

i.e. the mean is 50% and the variance is exactly zero. So these two ensembles are distinguishable even though they have identical density matrices. We do not treat this point in this thesis; we implicitly assume that the reduced density matrix we are using contains all the information about our system and its past interaction with the environment. However we should bear in mind that issues such as how fundamental the density matrix is, are still a matter of some debate within the various theories of quantum measurement.

Chapter 2

A simple model for decoherence

In this chapter we explore a model originally proposed by DeWitt [3], who suggested it as a simple means of exploring decoherence using elementary quantum mechanics. We examine his derivations and conclusions, but ultimately decide that the model is not as useful or simple as it might at first appear.

The basic model is that of one light particle (the environment) incident on an infinitely massive one (the system), in one dimension. The light particle is modelled by plane waves and the interaction potential is assumed to be a delta function. After analysing this we then introduce a large number of incident light particles, and ask how they affect the position and momentum of the heavy particle.

DeWitt originally calculated the density operator of the heavy mass, and used it to conclude that if the mass was bombarded by a large number of light particles of incommensurate momenta, then the density operator would become diagonal. He proposed that this was a simple mechanism for the emergence of classical behaviour for the heavy mass. We wish first of all to follow his argument, but with all details included.

Suppose the light particle has mass m and position x, while the heavy one has mass M and position X. We take as our hamiltonian for the assumed point interaction:

$$H = \frac{-\hbar^2}{2M}\frac{\partial^2}{\partial X^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + g\delta(x - X) \qquad (2.1)$$

where g denotes a coupling constant.

Let the state vector of M be $|\psi\rangle$ while that of the whole system is $|\Psi\rangle$. DeWitt originally assumed that the heavy particle is much more massive than the light one, and used this to give the following solution to the Schrödinger equation:

$$\Psi(x, X, t) = \frac{\psi(X,t)}{\sqrt{L}}\left\{\theta(X - x)\left[e^{i\frac{p}{\hbar}x} + Re^{-i\frac{p}{\hbar}(x-2X)}\right] + \theta(x - X)Te^{i\frac{p}{\hbar}x}\right\}e^{-\frac{i}{\hbar}\frac{p^2}{2m}t} \qquad (2.2)$$

where L is the length of a box used for normalisation, R and T are reflection and transmission coefficients: $|R|^2 + |T|^2 = 1$.

We first wish to derive this result and follow DeWitt's reasoning. To this end we will carefully define the problem and solve the Schrödinger equation.

2.1 Deriving DeWitt's result

The first thing apparent about (2.2) is that it appears to have been derived in a box, and yet for $X < x$ there are no left-travelling waves. This is an inconsistency: we expect that for both periodic and non-periodic boundary conditions we should always have waves moving in both directions.

Also DeWitt's argument depends upon one particle's being much more massive than the other, and among other things we wish to investigate the extent to which this condition must hold. So we solve the problem initially without assuming that $M \gg m$.

We will change to coordinates which make the equation separable, and the form of the potential suggests that the best transformation will be to the centre of mass; so define:

$$y = x - X \quad , \quad Y = \frac{mx + MX}{m + M} \tag{2.3}$$

together with the usual reduced mass μ, and write

$$\Psi(y, Y, t) = \Phi(Y, t)\phi(y, t) \tag{2.4}$$

so that Schrödinger' equation becomes

$$\left[\underbrace{\frac{-\hbar^2}{2\mu}\frac{\partial^2}{\partial y^2} + V(y)}_{\equiv H_y} \quad + \quad \underbrace{\frac{-\hbar^2}{2(M+m)}\frac{\partial^2}{\partial Y^2}}_{\equiv H_Y}\right]\Psi = i\hbar\frac{\partial \Psi}{\partial t} \tag{2.5}$$

If we follow the usual separation of variables approach, then we obtain an expression which equates a function of Φ with a function of ϕ, in which case each expression can be set equal to some function of time only, say $f(t)$. Then rearranging the equations gives

$$\begin{aligned} \Big(H_y - f(t)\Big)\phi &= i\hbar\frac{\partial \phi}{\partial t} \\ \Big(H_Y + f(t)\Big)\Phi &= i\hbar\frac{\partial \Phi}{\partial t} \end{aligned} \tag{2.6}$$

So $f(t)$ acts like a potential, and inasmuch as such a time dependence wasn't considered by DeWitt, we set it equal to zero. The two functions $\Phi(Y)$ and $\phi(y)$ each satisfy their own separate Schrödinger equations, and the one for Φ is easily solved to give plane wave solutions with the centre of mass energy:

$$\Phi(Y, t) = e^{iY\sqrt{2(M+m)E_{CM}}/\hbar - iE_{CM}t/\hbar} \tag{2.7}$$

A more general solution Φ_{gen} can be made by summing over these modes. We will later choose Φ_{gen} to be a gaussian wavepacket.

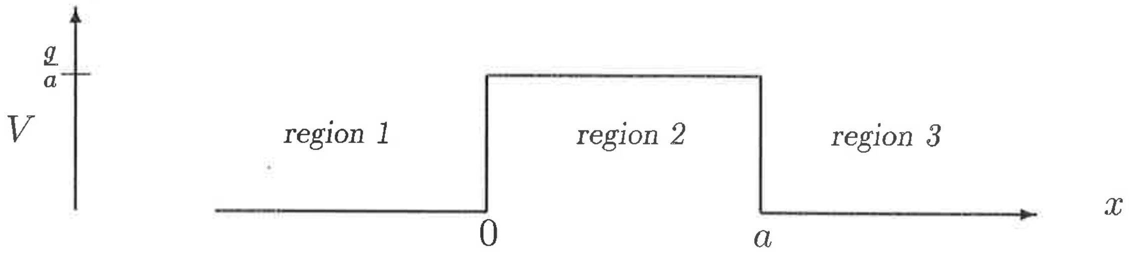
The wavefunction for the relative motion, ϕ, is expressed in terms of plane waves of energy E_{rel}. Separating the y and t variables yields

$$\phi(y,t) = \chi(y)e^{-i\frac{E_{rel}}{\hbar}t} \tag{2.8}$$

with

$$\left[\frac{-\hbar^2}{2\mu}\frac{d^2}{d^2y} + V(y)\right]\chi(y) = E_{rel}\chi \tag{2.9}$$

Suppose we obtain the delta function limit by solving the more general problem of a square barrier potential. Because this potential is a function of the relative positions of m and M, we can consider it as follows:



We will let $a \to 0$ to get the delta function limit. In order to try to reproduce DeWitt's solution we have not explicitly placed the interaction in a box, or rather, the box has a length which tends to infinity. The unnormalised wavefunction for the separation of the particles is:

$$\begin{array}{lll} \chi_1(y) = e^{iky} + Re^{-iky} & y < 0 \\ \chi_2(y) = Ae^{\gamma y} + Be^{-\gamma y} & 0 \le y \le a \\ \chi_3(y) = Te^{iky} & a < y \end{array} \tag{2.10}$$

where

$$\hbar k \equiv \sqrt{2\mu E_{rel}} \quad , \quad \hbar\gamma \equiv \sqrt{2\mu\left(\frac{g}{a} - E_{rel}\right)} \tag{2.11}$$

Matching the wavefunctions and derivatives at $y = 0$ and a yields

$$\begin{aligned} R &= \frac{\frac{k^2+\gamma^2}{ik\gamma}\,\mathrm{sh}\,\gamma a}{\frac{k^2-\gamma^2}{ik\gamma}\,\mathrm{sh}\,\gamma a + 2\,\mathrm{ch}\,\gamma a} \\ \begin{Bmatrix} A \\ B \end{Bmatrix} &= \frac{\left(1 \pm \frac{ik}{\gamma}\right)e^{\mp\gamma a}}{\frac{k^2-\gamma^2}{ik\gamma}\,\mathrm{sh}\,\gamma a + 2\,\mathrm{ch}\,\gamma a} \\ T &= \frac{2e^{-ika}}{\frac{k^2-\gamma^2}{ik\gamma}\,\mathrm{sh}\,\gamma a + 2\,\mathrm{ch}\,\gamma a} \end{aligned} \tag{2.12}$$

R and T are the usual reflection and transmission coefficients, with $|R|^2 + |T|^2 = 1$. We can introduce a normalisation $\mathcal{N}/\sqrt{L}$ to write:

$$\begin{aligned} \chi(y) &= \frac{\mathcal{N}}{\sqrt{L}} \Big\{\theta(-y)\left[e^{iky} + Re^{-iky}\right] \\ &\quad + \theta(y)\theta(a-y)\left[Ae^{\gamma y} + Be^{-\gamma y}\right] \\ &\quad + \theta(y-a)Te^{iky}\Big\} \end{aligned} \tag{2.13}$$

Finally the total wavefunction is

$$\Psi(x, X, t) = \Phi_{gen}\left(\frac{mx + MX}{m+M}, t\right) \chi(x - X) e^{-i\frac{E_{rel}}{\hbar}t} \tag{2.14}$$

We can show this leads to DeWitt's result. Set the width a of the top hat equal to zero and take out a factor of e^{-ikX} to match DeWitt's notation:

$$\begin{aligned} \Psi(x, X, t) &= \frac{\mathcal{N}e^{-ikX}}{\sqrt{L}} \Phi_{gen}\left(\frac{mx + MX}{m+M}, t\right) \Big\{\theta(X - x)\left[e^{ikx} + Re^{-ik(x-2X)}\right] \\ &\quad + \theta(x - X)Te^{ikx}\Big\} e^{-i\frac{E_{rel}}{\hbar}t} \end{aligned} \tag{2.15}$$

DeWitt took $M \gg m$, so that

$$Y \to X \quad , \quad k \to \sqrt{2mE_{rel}} = p/\hbar \tag{2.16}$$

where p is the momentum of the light particle in the original X, x frame. Hence the wavefunction becomes:

$$\Psi(x, X, t) = \frac{\mathcal{N}e^{-i\frac{p}{\hbar}X}}{\sqrt{L}} \Phi_{gen}(X, t) \Big\{\theta(X - x)\left[e^{i\frac{p}{\hbar}x} + Re^{-i\frac{p}{\hbar}(x-2X)}\right] + \theta(x - X)Te^{i\frac{p}{\hbar}x}\Big\} e^{-\frac{i}{\hbar}\frac{p^2}{2m}t} \tag{2.17}$$

If we put

$$\psi(X, t) \equiv \mathcal{N}e^{-i\frac{p}{\hbar}X}\Phi_{gen}(X, t) \tag{2.18}$$

then this solution reduces to (2.2), which was given by DeWitt.

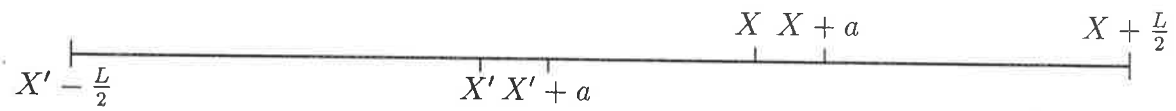
This factoring out of a wavefunction for the heavy particle depends on our taking $M \gg m$. The density matrix for the whole system is just $|\Psi\rangle\langle\Psi|$. We can calculate a reduced density matrix (called simply ρ with no subscript r since this is the only density matrix we will deal with) by tracing out m:

$$\begin{aligned} \rho(X, X', t) &= \int \langle x, X, t|\Psi\rangle\langle\Psi|x, X', t\rangle \, dx \\ &= \int \Phi_{gen}\left(\frac{mx + MX}{m+M}, t\right) \Phi^*_{gen}\left(\frac{mx + MX'}{m+M}, t\right) \chi(x - X)\chi^*(x - X') \, dx \end{aligned} \tag{2.19}$$

If Φ_{gen} is chosen to be plane waves, then the $\Phi_{gen}\Phi^*_{gen}$ in the last equation has no x-dependence, and merely multiplies the rest of the integral by a function of $X - X'$. Also, if $M \to \infty$, Φ_{gen} becomes a function of X only, and again it plays no part in the integral. This is the case considered by DeWitt, and he concludes that this fact is crucial to the decoherence process happening. In section 2.1.3 we'll investigate the consequences of assuming that the masses can be quite arbitrary, and will take (2.14) to be the correct solution, calculating the reduced density matrix for the mass M. But for now we keep $M \gg m$ but take $a \neq 0$, and calculate the reduced density matrix. In that case

$$\begin{aligned}
\rho(X, X', t) &= \Phi_{gen}(X,t)\Phi^*_{gen}(X',t) \int \chi(x-X)\chi^*(x-X')\, dx \\
&= \Phi_{gen}(X,t)\Phi^*_{gen}(X',t) \times \\
\lim_{L\to\infty} \frac{|\mathcal{N}|^2}{L} \int &\Big\{\theta(X-x)\left[e^{ik(x-X)} + Re^{ik(X-x)}\right] \\
&+ \theta(x-X)\theta(X+a-x)\left[Ae^{\gamma(x-X)} + Be^{\gamma(X-x)}\right] \\
&+ \theta(x-X-a)Te^{ik(x-X)}\Big\} \cdot \\
&\Big\{\theta(X'-x)\left[e^{ik(X'-x)} + R^* e^{ik(x-X')}\right] \\
&+ \theta(x-X')\theta(X'+a-x)\left[A^* e^{\gamma^*(x-X')} + B^* e^{\gamma^*(X'-x)}\right] \\
&+ \theta(x-X'-a)T^* e^{ik(X'-x)}\Big\}\, dx \qquad (2.20)
\end{aligned}$$

We assume $X' < X$, and first take the width of the hat a to be such that the ordering of variables appearing in the integrals is as follows:



When $L \to \infty$ the only surviving terms in the integral will be those with L in the limits of the integral, i.e.

$$\begin{aligned}
\rho(X,X',t) = &\Phi_{gen}(X,t)\Phi^*_{gen}(X',t)\times \\
&\lim_{L\to\infty} \frac{|\mathcal{N}|^2}{L} \left\{\int_{X'-L/2}^{X'} \left[e^{ik(x-X)} + Re^{ik(X-x)}\right]\left[e^{ik(X'-x)} + R^* e^{ik(x-X')}\right] dx \right. \\
&\left. + \int_{X+a}^{X+L/2} |T|^2 e^{ik(x-X)}\, e^{ik(X'-x)}\, dx \right\} \\
= &\Phi_{gen}(X,t)\Phi^*_{gen}(X',t)|\mathcal{N}|^2 \left[\cos k(X-X') - i|T|^2 \sin k(X-X')\right] \qquad (2.21)
\end{aligned}$$

In the limit of zero coupling ($g = 0$), $T = 1$, so that

$$\rho(X,X',t) = \Phi_{gen}(X,t)\Phi^*_{gen}(X',t)\, |\mathcal{N}|^2\, e^{-ik(X-X')} = \psi(X,t)\psi^*(X',t) \qquad (2.22)$$

as expected. Substituting this into (2.21), we can then recover DeWitt's case by setting $a \to 0$ as follows: firstly, with $a \to 0$, we have $\gamma a \to 0$, while $\gamma^2 a \to \frac{2mg}{\hbar^2}$. Hence, with

$$\alpha \equiv \frac{mg}{\hbar^2 k} \tag{2.23}$$

we have

$$R \to \frac{-i\alpha}{1+i\alpha} \quad , \quad T \to \frac{1}{1+i\alpha} \tag{2.24}$$

So finally

$$\rho(X, X', t) = \psi(X, t)\, \psi^*(X', t)\, (1+\alpha^2)^{-1} \left[1 + \alpha^2 e^{ik(X-X')} \cos k(X - X')\right] \tag{2.25}$$

which was arrived at by DeWitt, who points out that this density matrix can be modified to describe a localised state. Two things must be introduced: first, bombard M with a large number of identical masses m. In that case the "environmental modulation function"

$$E_k(X - X') \equiv (1+\alpha^2)^{-1} \left[1 + \alpha^2 e^{ik(X-X')} \cos k(X - X')\right] \tag{2.26}$$

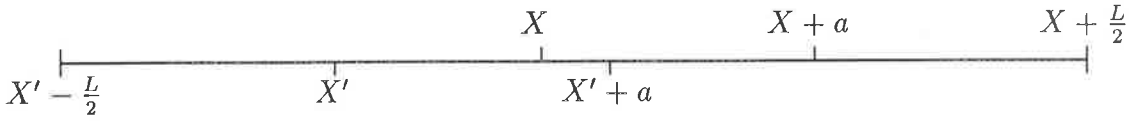
in the reduced density matrix (2.25) will become raised to some large power. A plot of the $|E_k|$ versus $X - X'$ will have sharp peaks, separated by $X - X' = \pi/k$. Second, bombard M with further large numbers of light particles, but make sure that there are at least two groups of incommensurate momenta. This has the effect of multiplying terms like (2.26)—one for each different value of momentum, so that the new environmental modulation function becomes

$$E_{k_1}^{N_1}(X - X')\, E_{k_2}^{N_2}(X - X') \, \ldots \tag{2.27}$$

When the modulus of this is plotted versus $X - X'$ we now find that, for $N_1, N_2 \ldots$ large and $k_1, k_2 \ldots$ incommensurate, that all peaks have been removed except for the one at $X - X' = 0$; that is, the reduced density matrix has become diagonal. DeWitt describes this as pointing to a localisation occurring. Actually, what has happened is that the density matrix has become diagonal. This doesn't mean the large mass has become localised; for this to happen we would require all but one of the diagonal elements to go to zero. What we are really dealing with here is decoherence in position, and this is a first step in the direction of classicality.

Does a similar thing happen for a top hat as opposed to a delta function interaction? Again plot the modulus of (2.27) versus $X - X'$, where each modulation factor now comes from (2.21) for $k_1, k_2 \ldots$, while α is given by (2.23) and T is calculated in (2.12). It's easy to verify that exactly the same thing happens: decoherence in position still occurs for a top hat potential.

Suppose we take the width a of the hat to be larger:



Then as before the only surviving terms in the integral, eqn (2.20), will be those with L in their limits; this doesn't change the final expression for ρ, eqn (2.21). Note we've assumed $X' < X$. If $X' > X$ we use the fact that ρ is hermitian: $\rho(X, X') = \rho^*(X', X)$.

A typical plot of $|E_k^N|$-vs-$X - X'$ for some large N is shown in figure 2.1. In section 2.1.3 we relax the requirement that M be much greater than m. But first we use the theory developed so far to introduce a new result not noted by DeWitt.

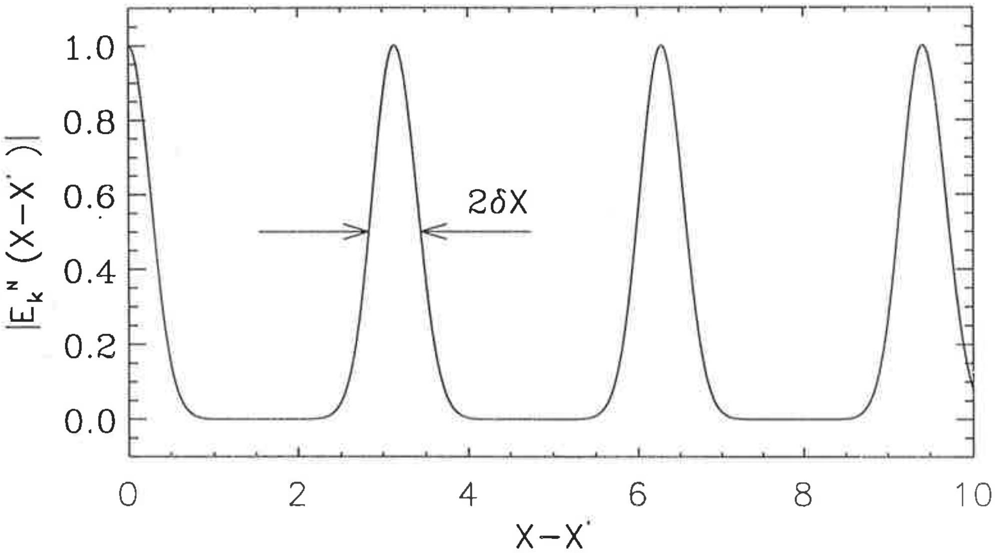


Figure 2.1: Modulation function for a large number of incident particles of one momentum.

2.1.1 A new uncertainty principle

Refer to figure 2.1. We define δX to be some measure of a peak's width (say, where the first inflection point of the $|E_k^N(u)|$-vs-u curve occurs), and focus on the spread in momentum of the mass M by defining a new quantity ΔP_c:

$$\underbrace{\Delta P^2}_{\text{large mass + interaction}} \equiv \underbrace{\Delta \overline{P}^2}_{\text{large mass without interaction}} + \underbrace{\Delta P_c^2}_{\text{correction}} \tag{2.28}$$

Armed with these definitions we here prove the important result of this section:

$$\Delta P_c \, \delta X \ \simeq \hbar \tag{2.29}$$

To do this we need to calculate ΔP^2:

$$\begin{aligned} \Delta P^2 &= \langle P^2 \rangle - \langle P \rangle^2 \\ &= \text{tr } \rho P^2 - (\text{tr } \rho P)^2 \end{aligned}$$

Now we can for example calculate $\langle P \rangle$ by: (dropping the t in the following expressions)

$$\begin{aligned} \langle P \rangle = \text{tr } \rho P &= \int dX \ \langle X|\rho P|X \rangle \\ &= \int dX \ dX' \ \langle X|\rho|X' \rangle \left[-i\hbar \frac{\partial}{\partial X'} \delta(X' - X) \right] \\ &= \int dX \ dX' \ \langle X|\psi\rangle\langle\psi|X'\rangle \ E_k(X - X') \left[-i\hbar \frac{\partial}{\partial X'} \delta(X' - X) \right] \end{aligned} \tag{2.30}$$

However it turns out to be simpler to write the trace in the following way:

$$\langle P \rangle = \int dX \ dX' \ \delta(X - X') \ \langle \psi | X' \rangle \left(-i\hbar \frac{\partial}{\partial X} \right) [\langle X|\psi\rangle \ E_k(X - X')] \tag{2.31}$$

Then using the fact that $E_k(0) = 1$, we have

$$\begin{aligned} \langle P \rangle &= -i\hbar \int dX \ dX' \ \delta(X - X') \ \psi^*(X') \frac{\partial}{\partial X} [\psi(X) \ E_k(X - X')] \\ &= \underbrace{-i\hbar \int dX \ \psi^*(X) \ [\psi'(X)}_{\langle \overline{P} \rangle, \text{ i.e. } \langle P \rangle \text{ for no interaction}} + \psi(X) \ E_k'(0)] \\ &= \langle \overline{P} \rangle - i\hbar \ E_k'(0) \end{aligned} \tag{2.32}$$

Also,

$$\begin{aligned} \langle P^2 \rangle &= -\hbar^2 \int dX \ dX' \ \delta(X - X') \ \psi^*(X') \frac{\partial^2}{\partial X^2} [\psi(X) \ E_k(X - X')] \\ &= -\hbar^2 \int dX \ dX' \ \delta(X - X') \ \psi^*(X') \frac{\partial^2}{\partial X^2} [\psi''(X) \ E_k(X - X') + \ 2\psi'(X) \ E_k'(X - X') \\ &\qquad + \psi(X) \ E_k''(X - X')] \\ &= \underbrace{-\hbar^2 \int dX \ \psi^*(X) \ [\psi''(X)}_{\langle \overline{P}^2 \rangle, \text{ i.e. } \langle P^2 \rangle \text{ for no interaction}} + 2\psi'(X) \ E_k'(0) + \ \psi(X) \ E_k''(0)] \\ &= \langle \overline{P}^2 \rangle - 2i\hbar \ E_k'(0) \ \langle \overline{P} \rangle - \hbar^2 E_k''(0) \end{aligned}$$

So

$$\begin{aligned}
\Delta P^2 &= \langle P^2\rangle - \langle P\rangle^2 \\
&= \langle \overline{P}^2\rangle - 2i\hbar\, E_k'(0)\,\langle\overline{P}\rangle - \hbar^2\, E_k''(0) \\
&\quad - \langle\overline{P}\rangle^2 + 2i\hbar\, E_k'(0)\,\langle\overline{P}\rangle + \hbar^2\, E_k'^2(0) \\
&= \underbrace{\langle \overline{P}^2\rangle - \langle\overline{P}\rangle^2}_{\Delta\overline{P}^2} + \underbrace{\hbar^2\,\left[E_k'^2(0) - E_k''(0)\right]}_{\equiv \Delta P_c^2}
\end{aligned}$$

Thus

$$\Delta P_c^2 = \hbar^2\left[E_k'^2(0) - E_k''(0)\right] \tag{2.33}$$

For simplicity, consider the case of N particles of mass m, each with momentum $\hbar k$, bombarding the mass M. Then as previously discussed,

$$\rho(X, X', t) = \psi(X,t)\psi^*(X',t)E_k(X - X') \tag{2.34}$$

with $(u \equiv X - X')$

$$E_k(u) = \left[e^{iku}\left(\cos ku - i|T|^2 \sin ku\right)\right]^N \tag{2.35}$$

Hence

$$E_k'(0) = -iNk|R|^2 \quad , \quad E_k''(0) = -N(N-1)k^2|R|^4 - 2Nk^2|R|^2 \tag{2.36}$$

and we obtain

$$\Delta P_c^2 = N\hbar^2k^2\left(1 - |T|^4\right) \tag{2.37}$$

We now calculate δX, where we have defined it to be the peak half width at the first inflection point of $|E_k^N(u)|$-vs-u. With

$$f(u) \equiv |E_k(u)| = \left[\cos^2 ku + |T|^4 \sin^2 ku\right]^{\frac{N}{2}} \tag{2.38}$$

then

$$\begin{aligned}
f''(u) &= Nk^2\left[\cos^2 ku + |T|^4\sin^2 ku\right]^{\frac{N}{2}-2}\left(|T|^4 - 1\right) \times \\
&\quad \left[\frac{1}{2}\left(\frac{N}{2} - 1\right)\left(|T|^4 - 1\right)\sin^2 2ku + \left[\cos^2 ku + |T|^4 \sin^2 ku\right]\cos 2ku\right]
\end{aligned} \tag{2.39}$$

and this is required to equal zero for $u \equiv \delta X$. Since we are dealing with a strongly peaked modulation function, ku will be much less than one, so replace the sin and cos by their leading order approximations. Then since $|T| \neq 1$ we require the square brackets in the above expression to equal zero. In that case, taking the reflection to be nonnegligible we obtain

$$\delta X^2 \simeq \frac{1}{k^2 N\left(1 - |T|^4\right)} \tag{2.40}$$

Finally then

$$\Delta P_c\, \delta X \simeq \hbar \tag{2.41}$$

2.1.2 Creating a transmission resonance

As is well known, when the energy of the bombarding particle is chosen so that an integral number of half-wavelengths fits inside the square barrier, the reflection coefficient R becomes zero. In this case it appears that the DeWitt style decoherence no longer happens. In this section we analyse this in more detail.

If $R = 0$ then

$$\frac{k^2 + \gamma^2}{ik\gamma} \operatorname{sh} \gamma a = 0 \tag{2.42}$$

This expression is not satisfied for $\gamma = 0$ since both k and a are nonzero. We also rule out $k^2 + \gamma^2 = 0$, since this implies $a \to \infty$ or $g = 0$, both uninteresting cases. So set $\operatorname{sh} \gamma a = 0$ and solve for complex γ: we write

$$\phi_2(y) = Ae^{i(-i\gamma)y} + Be^{-i(-i\gamma)y} \tag{2.43}$$

and note that when $E_{rel} > V_0 \equiv g/a$, ϕ_2 will be oscillatory with an associated wavelength

$$\lambda = \frac{2\pi}{-i\gamma} \tag{2.44}$$

Now, $\operatorname{sh} \gamma a = 0 \Rightarrow \sin -i\gamma a = 0$, or

$$-i\gamma a = n\pi \quad n \in \mathcal{N} \tag{2.45}$$

This is just the half-wavelength resonance condition. The relative energy in this resonant case becomes

$$E_n \equiv V_0 + \frac{n^2\pi^2\hbar^2}{2ma^2} \tag{2.46}$$

There are two competing influences here. The first is that by increasing the number N of bombarding particles we can decohere the position of M fully. However by making the relative energy equal to a resonant energy, we are setting $|T| = 1$ in (2.21), which sets $E_k(X - X') = 1$ and implies that no N, however large, will give decoherence. Which effect is the stronger here? Putting

$$\varepsilon_n \equiv E_{rel} - E_n \tag{2.47}$$

we can find out by calculating $|E_k(u)|$ as a function of N and ε_n. Firstly,

$$E_k(u) = e^{iNku} \left[\cos ku - i|T|^2 \sin ku\right]^N \tag{2.48}$$

Set $\Gamma \equiv -i\gamma$ and $\beta_n \equiv \frac{n^2\pi^2\hbar^2}{2ma^2}$, so that

$$\hbar^2 k^2 = 2m\left(\varepsilon_n + V_0 + \beta_n\right) \tag{2.49}$$

$$\hbar^2\Gamma^2 = 2m\left(\varepsilon_n + \beta_n\right) \tag{2.50}$$

This means

$$\frac{1}{|T|^2} = \left(\frac{k^2+\Gamma^2}{2k\Gamma}\right)^2 \sin^2\Gamma a + \cos^2\Gamma a \tag{2.51}$$

and with

$$\Gamma a = n\pi\left(1 + \frac{\varepsilon_n}{2\beta_n}\right) \tag{2.52}$$

$$\sin\Gamma a \simeq \frac{(-1)^n\varepsilon_n}{2\beta_n} \tag{2.53}$$

$$\cos\Gamma a \simeq (-1)^n \tag{2.54}$$

we can write $|T|^2$ as a series in ε_n, keeping terms to second order. Eventually we obtain

$$|T|^2 = 1 - \frac{(V_0+2\beta_n)^2\varepsilon_n^2}{16\beta_n^3(V_0+\beta_n)} + O(\varepsilon_n^3) \tag{2.55}$$

Substituting this into the expression (2.48) and taking the modulus gives

$$|E_k(u)| \simeq 1 - \frac{(V_0+2\beta_n)^2}{16\beta_n^3(V_0+\beta_n)} \sin^2 ku\; N\,\varepsilon_n^2 \tag{2.56}$$

So we see that the relative energy is the stronger competing factor, in that as we adjust this energy closer to a resonance value, the number N of incident particles required to offset the cohering effect increases dramatically.

2.1.3 Keeping M finite

When the mass M is not infinite, we can no longer replace the centre of mass coordinate Y with the large mass coordinate X. We need to evaluate (2.19), and one way to proceed is to choose some form for $|\Phi_{gen}|^2$. Suppose then that $|\Phi_{gen}(Y,0)|^2$ is a normalised gaussian, so that $\Phi_{gen}(Y,0)$ is a wavepacket centred around some wavenumber K_0. Then

$$\Phi_{gen}(Y,0) = \sqrt{\frac{1}{\sigma\sqrt{2\pi}}}\; e^{\frac{-Y^2}{4\sigma^2}+iK_0Y} \tag{2.57}$$

If we now express $\Phi_{gen}(Y,0)$ as a sum of plane waves and evolve each in time in the usual way, we obtain

$$\begin{aligned}\Phi_{gen}(Y,t) &= \sqrt{\frac{1}{\sigma\sqrt{2\pi}}}\;\frac{\sigma}{\sqrt{\pi}}\int e^{iKY-\frac{i\hbar K^2 t}{2(m+M)}-(K-K_0)^2\sigma^2}\,dK \\ &= \sigma^{\frac{1}{2}}(2\pi)^{-\frac{1}{4}}\left[\sigma^2+\frac{i\hbar t}{2(m+M)}\right]^{-\frac{1}{2}} e^{iK_0\left(Y-\frac{\hbar K_0}{2(m+M)}t\right)}\; e^{\frac{-\left(Y-\frac{\hbar K_0}{m+M}t\right)^2}{4\left(\sigma^2+\frac{i\hbar t}{2(m+M)}\right)}}\end{aligned}$$

This is of course a wavepacket travelling with group velocity $\frac{\hbar K_0}{m+M}$. We can now calculate $\Phi_{gen}\Phi^*_{gen}$. For what follows, define:

$$\begin{aligned}
a_1 &\equiv \frac{\hbar t}{2(m+M)} \\
a_2 &\equiv \frac{m}{m+M} \\
a_3 &\equiv \frac{M}{m+M} \\
a_4 &\equiv \sigma^2 + i a_1 \\
a_5 &\equiv \frac{\sigma^2 a_2^2}{2|a_4|^2} \quad . \\
b_1 &\equiv \frac{\sigma}{\sqrt{2\pi}|a_4|} e^{\frac{-a_3^2 X^2}{4a_4} + \frac{iK_0\sigma^2 a_3 X}{a_4} - \frac{a_3^2 X'^2}{4a_4^*} - \frac{iK_0\sigma^2 a_3 X'}{a_4^*} - \frac{2K_0^2\sigma^2 a_1^2}{|a_4|^2}} \\
b_2 &\equiv -\frac{a_2 a_3 X}{2a_4} - \frac{a_2 a_3 X'}{2a_4^*} + \frac{2K_0\sigma^2 a_1 a_2}{|a_4|^2}
\end{aligned}$$

The X, X' dependence is contained within b_1 and b_2 (and when the arguments are not explicitly inserted it's implied that they are X, X').
Then

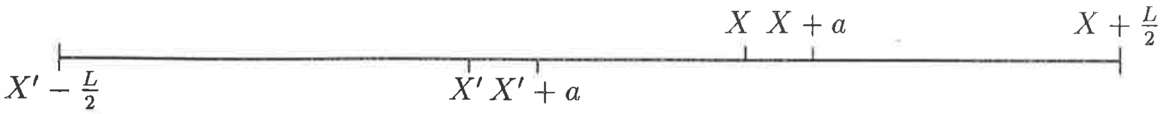
$$\Phi_{gen}\left(\frac{mx+MX}{m+M}, t\right)\Phi^*_{gen}\left(\frac{mx+MX'}{m+M}, t\right) = b_1\, e^{-a_5 x^2 + b_2 x} \tag{2.58}$$

Note that even though this is a gaussian in x, we cannot invoke its vanishing at $x = \pm\infty$ to do away with the $1/\sqrt{L}$ in the normalisation in (2.13). If we attempt this, setting $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Psi|^2\, dx\, dX = 1$ and solving for $\mathcal{N}$, the integration over X is still divergent; so we have no choice but to retain the $1/\sqrt{L}$ factor in (2.13).

The expression (2.19) for the reduced density matrix becomes

$$\begin{aligned}
\rho(X, X', t) &= \lim_{L\to\infty} \frac{|\mathcal{N}|^2}{L} b_1 \int e^{-a_5 x^2 + b_2 x} \times \\
&\Big\{\theta(X - x)\left[e^{ik(x-X)} + Re^{ik(X-x)}\right] \\
&+ \theta(x - X)\theta(X + a - x)\left[Ae^{\gamma(x-X)} + Be^{\gamma(X-x)}\right] \\
&+ \theta(x - X - a)Te^{ik(x-X)}\Big\} \times \\
&\Big\{\theta(X' - x)\left[e^{ik(X'-x)} + R^* e^{ik(x-X')}\right] \\
&+ \theta(x - X')\theta(X' + a - x)\left[A^* e^{\gamma^*(x-X')} + B^* e^{\gamma^*(X'-x)}\right] \\
&+ \theta(x - X' - a)T^* e^{ik(X'-x)}\Big\}\, dx
\end{aligned} \tag{2.59}$$

We assume $X' < X$, and first take the width of the hat a to be such that the ordering of variables appearing in the integrals is as follows:



Again, $L \to \infty$ means most terms in the integral are immaterial, except those with L in the integral limits, i.e.

$$\begin{aligned}
\rho(X,X',t) &= \lim_{L\to\infty} \frac{|\mathcal{N}|^2}{L} b_1(X,X') \times \\
&\left\{ \int_{X'-L/2}^{X'} e^{-a_5x^2+b_2x} \left[e^{ik(x-X)} + Re^{ik(X-x)}\right] \left[e^{ik(X'-x)} + R^* e^{ik(x-X')}\right] dx \right. \\
&\left. + \int_{X+a}^{X+L/2} |T|^2 e^{-a_5x^2+b_2x}\, e^{ik(x-X)}\, e^{ik(X'-x)}\, dx \right\} \qquad (2.60)
\end{aligned}$$

This is just a sum of gaussian integrals. If we let $L \to \infty$ in the integrals, they don't diverge, and hence don't provide any factor of L to cancel the $|\mathcal{N}|^2/L$ term. So despite the fact that we have already discarded some terms to get the previous expression, for now we will keep L finite and denote this by a subscript on ρ, obtaining

$$\begin{aligned}
\rho_L(X,X',t) &= \frac{|\mathcal{N}|^2}{L}\frac{1}{2a_2} e^{\frac{-a_3^2(X-X')^2}{8\sigma^2} + ia_3K_0(X-X')} \times \\
&\left\{ \left[e^{-ik(X-X')} + |R|^2 e^{ik(X-X')}\right] \left[\operatorname{erf}\left(\sqrt{a_5}\,X' - \frac{b_2}{2\sqrt{a_5}}\right) - \operatorname{erf}\left(\sqrt{a_5}\left(X'-\frac{L}{2}\right) - \frac{b_2}{2\sqrt{a_5}}\right)\right] \right. \\
&+ Re^{ik(X+X')} e^{\frac{-ib_2k-k^2}{a_5}} \left[\operatorname{erf}\left(\sqrt{a_5}\,X' - \frac{b_2-2ik}{2\sqrt{a_5}}\right) - \operatorname{erf}\left(\sqrt{a_5}\left(X'-\frac{L}{2}\right) - \frac{b_2-2ik}{2\sqrt{a_5}}\right)\right] \\
&+ R^* e^{-ik(X+X')} e^{\frac{ib_2k-k^2}{a_5}} \left[\operatorname{erf}\left(\sqrt{a_5}\,X' - \frac{b_2+2ik}{2\sqrt{a_5}}\right) - \operatorname{erf}\left(\sqrt{a_5}\left(X'-\frac{L}{2}\right) - \frac{b_2+2ik}{2\sqrt{a_5}}\right)\right] \\
&\left. + |T|^2 e^{-ik(X-X')} \left[\operatorname{erf}\left(\sqrt{a_5}\left(X+\frac{L}{2}\right) - \frac{b_2}{2\sqrt{a_5}}\right) - \operatorname{erf}\left(\sqrt{a_5}\,(X+a) - \frac{b_2}{2\sqrt{a_5}}\right)\right] \right\}
\end{aligned}$$

(2.61)

and we have $\rho \equiv \lim_{L\to\infty} \rho_L$. If we now take the width a of the hat to be larger so that the order of X, X' is as shown on page 13, the only surviving terms in eqn (2.59) will be those with L in their limits, and the expression for ρ is unchanged.

Showing a correspondence to DeWitt's result

Here we show that in the limit $M \to \infty$ with $a = 0$ our expression for $\rho(X,X',t)$ will reduce to DeWitt's corresponding quantities, as it should. Consider

$$\lim_{M\to\infty} \rho_L = \frac{e^{ik(X-X')}}{L} e^{\frac{-(X-X')^2}{8\sigma^2} + iK_0(X-X')} \times$$

$$\lim_{M\to\infty} \frac{M}{2m} \left\{ \left[e^{-ik(X-X')} + |R|^2 e^{ik(X-X')}\right] \left[\text{erf } \frac{X+X'}{\sqrt{8}\sigma} - \text{erf } \left(\frac{X+X'}{\sqrt{8}\sigma} - \frac{Lm}{\sqrt{8}\sigma M}\right)\right] \right.$$

$$\left. + |T|^2 e^{-ik(X-X')} \left[\text{erf } \left(\frac{X+X'}{\sqrt{8}\sigma} + \frac{Lm}{\sqrt{8}\sigma M}\right) - \text{erf } \frac{X+X'}{\sqrt{8}\sigma}\right]\right\} \quad (2.62)$$

The first error function limit is calculated as follows:

$$\begin{aligned} & \lim_{M\to\infty} \frac{M}{2m} \left[\text{erf } \frac{X+X'}{\sqrt{8}\sigma} - \text{erf } \left(\frac{X+X'}{\sqrt{8}\sigma} - \frac{Lm}{\sqrt{8}\sigma M}\right)\right] \\ = & \lim_{M\to\infty} \frac{M}{2m} \frac{2}{\sqrt{\pi}} \int_{\frac{X+X'}{\sqrt{8}\sigma} - \frac{Lm}{\sqrt{8}\sigma M}}^{\frac{X+X'}{\sqrt{8}\sigma}} e^{-x^2}\, dx \\ = & \lim_{M\to\infty} \frac{M}{2m} \frac{2}{\sqrt{\pi}} e^{-\frac{(X+X')^2}{8\sigma^2}} \frac{Lm}{\sqrt{8}\sigma M} \\ = & \frac{L}{2\sigma\sqrt{2\pi}} e^{-\frac{(X+X')^2}{8\sigma^2}} \end{aligned} \quad (2.63)$$

Similarly the second has the same value. The L's then cancel, and after a few steps we have

$$\lim_{M\to\infty} \rho_L = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{X^2+X'^2}{4\sigma^2} + iK_0(X-X')} (1+\alpha^2)^{-1} \left[1 + \alpha^2 e^{ik(X-X')} \cos k(X-X')\right] \quad (2.64)$$

Finally, this is just DeWitt's result with the appropriate $\psi(X,t)$ (and the L dependence has vanished).

2.1.4 Characteristic shape of the reduced density matrix

Although the full density matrix (2.61) is too unwieldy for any useful analytical work, we can plot its evolution for typical parameters with $L \to \infty$, to see how necessary the requirement is for $m \ll M$ (set $\mathcal{N} = 1$ without loss of generality). A typical plot of $|\rho(X,X')|$ is shown in figure 2.2 (note that to keep computing times manageable, the grid size chosen for this plot was insufficient to portray the detail of the inner peaks, as is evident from the cross section of this plot shown in figure 2.3). As expected $|\rho|$ is symmetrical about the main diagonal, $X - X' = 0$. Because of this, we can take a cross section in the orthogonal direction, namely $X + X' = \alpha$ for some constant α (which we have taken to be 70 for plotting purposes). This plot is shown in figure 2.3. It corresponds to DeWitt's modulation function, except that now we have included the gaussian density matrix due to $\psi(X,t)$ in (2.2) (and this acts to suppress all but the central ridges). There is no reason to exclude this, as we must deal with the entire density matrix to see how it will decohere. Further plotting of $|\rho(X,X')|$ for various times shows that the width of the modulation of the central ridges is constant through time. This is presumably due to our modelling the incident particle by plane waves, which by definition fill all of space

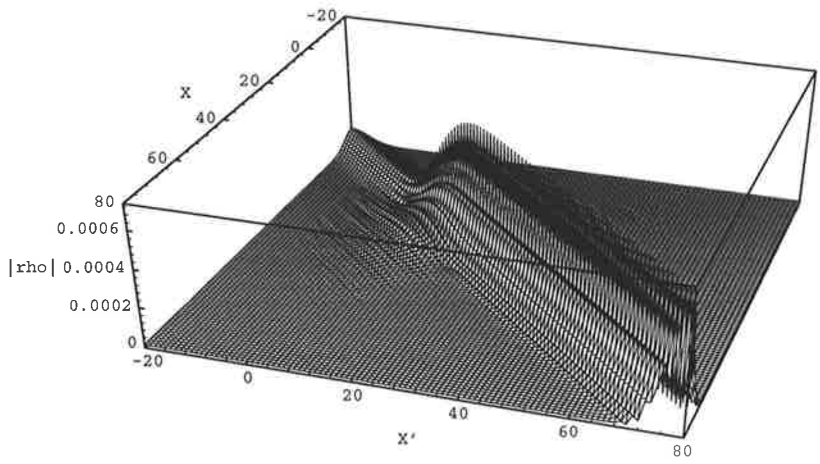


Figure 2.2: $|\rho(X, X')|$ for $t = 100$, top hat width $= 1$, equal masses

and time. Clearly, if we arrange for a large number of incident particles, then ρ is raised to a large power which leads to the central ridge being accentuated; that is, decoherence occurs.

Characteristic width of $|\rho|$

DeWitt states that in his model, localisation (which we suggest should be read as position decoherence) is not expected to be sharper than the width of the potential. In that case he can achieve decoherence with many particles only because he is using a delta function potential. Here we have used a square barrier, so we can test his idea for the case of one

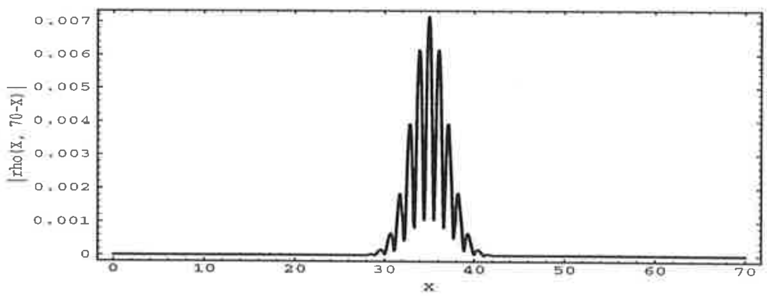


Figure 2.3: Cross section of previous $|\rho(X, X')|$ from $(X, X') = (0, 70)$ to $(70, 0)$.

incident particle by examining the width of $|\rho(X, X')|$.

Since we are taking a cross section of $|\rho(X, X')|$ along the line $X + X' = \alpha$, we consider

$$|\rho(X, \alpha - X, t)| = \frac{|\mathcal{N}|^2}{L} \frac{1}{2a_2} e^{\frac{-a_3^2(2X-\alpha)^2}{8\sigma^2}} \times |\textit{terms in braces in eqn (2.61)}| \tag{2.65}$$

where the terms in the braces have been modulated by

$$e^{\frac{-a_3^2(2X-\alpha)^2}{8\sigma^2}} \tag{2.66}$$

This is a gaussian centred around $\alpha/2$ (as we expect) with a half width (i.e. a standard deviation) of σ/a_3, which is just $\sigma(1+m/M)$. Note that the width of the modulation—the measure of decoherence—doesn't depend at all on the width of the potential, a. However decoherence is maximised (i.e. the modulation width is minimised) when $m/M \to 0$, which is not unlike DeWitt's assertion that in his model, localisation (i.e. decoherence) is a consequence of $m/M \to 0$. Certainly by choosing a small enough σ, we can have decoherence better than the square barrier width, but this can mean starting out with a well localised centre of mass of the system.

2.2 Conclusion

We started this chapter by explaining the model proposed by DeWitt who used it to try to understand decoherence in a simple way. DeWitt's result is not as realistic as it might at first appear, based as it is around modelling the incident particles by plane waves which by definition have existed for all time. As such it contains no information on the evolution of the systems, and can tell us nothing about the decoherence time. However, we did use this simple model to derive a new uncertainty principle. We then began a more rigorous treatment based on wavepackets, which unfortunately soon gets held up by the long computer time needed for numerical work. Although it seems that we can decohere the large mass arbitrarily finely in position, to do so requires an initial wavepacket for the centre of mass whose width is small. We still have no time evolution information because the small mass has still only been modelled as plane waves, but modelling it as a wavepacket increases the model's complexity significantly.

At least in the way we have pursued it, the model's initial attraction of allowing simple tests on decoherence ideas quickly becomes complicated when we make it more rigorous. In the remaining chapters of this thesis we consider the effect of an environment by modelling it as oscillators using the influence functional technique. This technique could possibly be developed for application to a bath of particles—as opposed to the more usual oscillators it more easily adapts itself to—and that forms a direction which further work on DeWitt's model could take.

Chapter 3

Influence functional theory

The influence functional formalism was first introduced by Feynman and Vernon [4] as a way of deducing the influence of an environment on some system of interest. It was later applied by Caldeira and Leggett [5] to the high temperature limit of a model where both system and environment are composed of static oscillators. Initially, a prerequisite for the use of the formalism was that the system and bath were initially uncorrelated—an assumption we use throughout this thesis—although it was later developed for more general cases of initial correlation [6].

The language of influence functionals is different to that of field theory, but nevertheless has points of contact with it (as indicated in this thesis), and in fact it can be shown [7] to be formally equivalent to the Schwinger-Keldysh closed time path formalism. Hu et al. [8, 9] used the influence functional approach to derive the master equation for a static oscillator coupled to a bath of static oscillators, and this was later extended to include time dependence in both system and bath [2]. Most work in this area has concentrated on a bilinear system-bath coupling, while a perturbation approach for other couplings [9] has also been developed using field theory formalism.

In this chapter we further develop the work of [2], by considering a squeezed system coupled bilinearly to a static bath, but now with a time dependent coupling constant. Including such a time dependence will be necessary for our work in chapters 4 and 5. We also lay out the groundwork for how we will go about calculating such quantities as entropy, fluctuations and coherence, which form the focus of those chapters.

3.1 Propagation of the density matrix using influence functionals

The primary thing we wish to consider is the evolution of the reduced density matrix of our system. For this we use the Feynman-Vernon influence functional method. This has been discussed at length in [2]; we describe it here in order to establish the notation, and just state its main results without deriving them.

Again consider our system described by x which interacts with its environment q through some interaction. The combined action is

$$S[x,q] = S[x] + S_E[q] + S_{int}[x,q] \tag{3.1}$$

We require the reduced density matrix of the system at time t. This is found by tracing out the environment:

$$\rho_r(x\,x'\,t) = \int_{-\infty}^{\infty} dq\; \rho(x\,q\,x'\,q\,t) \tag{3.2}$$

The full density matrix $\rho(x\,q\,x'\,q\,t)$ evolves unitarily. Suppose we expand it using completeness relations and then path integrals:

$$\begin{aligned}
\rho(x\,q\,x'\,q\,t) &= \langle x\,q\,t|\rho|x'\,q\,t\rangle \\
&= \int dx_i\,dq_i \int dx'_i\,dq'_i\; \langle x\,q\,t|x_i\,q_i\,0\rangle\langle x_i\,q_i\,0|\rho|x'_i\,q'_i\,0\rangle\langle x'_i\,q'_i\,0|x'\,q\,t\rangle \\
&= \int dx_i\,dq_i \int dx'_i\,dq'_i \int_{x_i}^{x} Dx \int_{q_i}^{q} Dq\; e^{iS[x,q]}\rho(x_i\,q_i\,x'_i\,q'_i\,0) \int_{x'_i}^{x'} Dx' \int_{q'_i}^{q} Dq'\; e^{-iS[x',q']} \\
&\equiv \int dx_i\,dq_i \int dx'_i\,dq'_i\; J(x\,q\,x'\,q\,t|x_i\,q_i\,x'_i\,q'_i\,0)\; \rho(x_i\,q_i\,x'_i\,q'_i\,0)
\end{aligned} \tag{3.3}$$

where J is seen to be an evolution operator for the entire system plus bath. Now to allow further calculation we make the assumption that the system and bath are initially uncorrelated, i.e.

$$\rho(x_i\,q_i\,x'_i\,q'_i\,0) = \rho_{sys}(x_i\,x'_i\,0)\; \rho_E(q_i\,q'_i\,0) \tag{3.4}$$

In this case we are able to rearrange the order of integration to write the reduced density matrix in the following way:

$$\rho_r(x\,x'\,t) = \int dx_i\,dx'_i\; J_r(x\,x'\,t|x_i\,x'_i\,0)\; \rho_{sys}(x_i\,x'_i\,0) \tag{3.5}$$

where the evolution operator for the reduced density matrix is defined by

$$J_r(x\,x'\,t|x_i\,x'_i\,0) \equiv \int_{x_i}^{x} Dx \int_{x'_i}^{x'} Dx'\; e^{iS[x]-iS[x']}\; F[x,x'] \tag{3.6}$$

and $F[x,x']$ is the so-called influence functional:

$$F[x,x'] = \int dq\,dq_i\,dq'_i\; \rho_E(q_i\,q'_i\,0) \int_{q_i}^{q} Dq\; e^{iS_E[q]+iS_{int}[x,q]} \int_{q'_i}^{q} Dq'\; e^{-iS_E[q']-iS_{int}[x',q']} \tag{3.7}$$

We can also write the influence functional in a basis-independent form as follows. First we write the path integrals as propagators

$$F[x,x'] = \int dq\, dq_i\, dq_i'\, \rho_E(q_i\, q_i'\, 0)\ \langle q|U(t)|q_i\rangle\ \langle q_i'|U'^{\dagger}(t)|q\rangle \tag{3.8}$$

where $U(t), U'(t)$ are the propagators for $S_E[q]+S_{int}[x,q]$ and $S_E[q]+S_{int}[x',q]$ respectively. Then upon integrating over q, q_i and writing the remaining integral as a trace, we obtain:

$$F[x,x'] = \mathrm{tr}\ U(t)\, \rho_E(0)\, U'^{\dagger}(t) \tag{3.9}$$

This form allows the influence functional to be found. The calculation is done elsewhere [2] and here we just list the result: if we use sum and difference coordinates (which we do throughout this thesis) defined by

$$\Sigma \equiv (x+x')/2 \quad , \quad \Delta \equiv x - x' \tag{3.10}$$

then the influence functional can be written in terms of two new quantities, the "dissipation" $\mu(s,s')$ and "noise" $\nu(s,s')$:

$$F[x,x'] = \exp \frac{-1}{\hbar} \int_0^t ds \int_0^s ds'\ \Delta(s) \big[\nu(s,s')\, \Delta(s') + i\mu(s,s')\, 2\Sigma(s')\big] \tag{3.11}$$

Thus the influence of the environment is completely invested in the dissipation and noise.

3.2 Squeezed states and their relevance to density matrices

A density matrix supplies information about a certain quantity, say position. We can of course convert to the corresponding matrix as a function of the conjugate variable, in this case momentum. As the system evolves, we can plot the uncertainties in phase space, in which case the resulting ellipse will flatten out with time, its axes of symmetry singling out two new variables which will be some linear combinations of x and p. This process is naturally enough called squeezing. We describe some of its formalism here, for what we will need throughout this thesis.

Suppose we start with the general oscillator Hamiltonian

$$H(t) = f(t)\frac{a^2}{2} + f^*(t)\frac{a^{\dagger 2}}{2} + h(t)(a^{\dagger}a + 1/2) + d(t)a + d^*(t)a^{\dagger} + g(t) \tag{3.12}$$

The propagator for this has been calculated in [2] and is

$$U(t,t_i) = S(r,\phi)R(\theta)D(p)e^{w-|p|^2/2} \tag{3.13}$$

where p, w are defined in terms of the coefficients appearing in H, and

$$\begin{aligned} D(p) &= \exp(-p^* a - \text{h.c.}) \\ R(\theta) &= \exp -i\theta(a^\dagger a + 1/2) \\ S(r,\phi) &= \exp(re^{-2i\phi}a^2/2 - \text{h.c.}) \end{aligned} \tag{3.14}$$

There is a large body of theory concerning the action of these three operators, and here we note just a few points to explain the formalism. First, suppose we start with a simple harmonic oscillator with lagrangian

$$L = \frac{M}{2}\left(\dot{x}^2 - \Omega^2 x^2\right) \tag{3.15}$$

If we construct a gaussian state in the position basis, with initially the same width σ_0 as that of the ground state of such an oscillator, displaced by some arbitrary amount and with a phase proportional to x, we find this to be an eigenstate of the lowering operator, and is called a coherent state. Suppose we locate the point $(\langle x\rangle, \langle p\rangle)$ in phase space and draw an ellipse about this point, the lengths of whose axes being the uncertainties $\Delta x^2, \Delta p^2$. Then as the oscillator evolves this uncertainty ellipse revolves about the origin with angular speed Ω.

A squeezed state is again such a state, but with an arbitrary initial width σ. We find that as the oscillator evolves the uncertainty ellipse again revolves about the origin, but its axes change length and it can also rotate about its own centre.

It turns out that the squeeze parameter r is related to the width of such a state:

$$r = \ln\frac{\sigma_0}{\sigma} \quad , \quad \sigma_0 \equiv \sqrt{\frac{\hbar}{2M\Omega}} \tag{3.16}$$

Hence a coherent state has $r = 0$, or zero squeezing. A gaussian that initially has a width smaller than σ_0 will have some $r > 0$, and so is squeezed (hence the name). We can generate a squeezed state by applying $S(r,\phi)$ to the ground state of the simple oscillator. Consider the new operator

$$b = U^\dagger a U \equiv \alpha\, a + \beta^*\, a^\dagger \tag{3.17}$$

where it turns out that

$$\begin{aligned} \alpha &= e^{-i\theta}\,\text{ch}\, r \\ \beta &= -e^{-i(\theta+2\phi)}\,\text{sh}\, r \end{aligned} \tag{3.18}$$

Going from a to b is then just a Bogoliubov transformation, and so α, β become Bogoliubov coefficients for our system. Their equations of motion are

$$\begin{aligned} \dot{\alpha} &= -ih\alpha - if^*\beta \\ \dot{\beta} &= if\alpha + ih\beta \end{aligned} \tag{3.19}$$

$$\alpha(t_i) = 1 \quad , \quad \beta(t_i) = 0$$

where f, h as defined in the hamiltonian (3.12) are calculated from the general system lagrangian. This lagrangian has time dependent mass and frequency, and we will also allow it to have a time dependent cross term for some $\mathcal{E}$:

$$L = \frac{M(t)}{2}\left(\dot{x}^2 + 2\mathcal{E}(t)\dot{x}x - \Omega^2(t)x^2\right) \tag{3.20}$$

Then f, h are given by [2]

$$\begin{aligned} f &= \frac{1}{2}\left[\frac{M}{\kappa}(\Omega^2 + \mathcal{E}^2) - \frac{\kappa}{M} + 2i\mathcal{E}\right] \\ h &= \frac{1}{2}\left[\frac{M}{\kappa}(\Omega^2 + \mathcal{E}^2) + \frac{\kappa}{M}\right] \end{aligned} \tag{3.21}$$

and κ is an arbitrary positive constant that can be chosen to simplify the relevant equations.

In the next section we shall find that the quantity of much importance to our work turns out to be the sum of the Bogoliubovs, $X \equiv \alpha + \beta$. It follows from (3.19) that X satisfies the classical equation of motion for the system:

$$\ddot{X} + \frac{\dot{M}}{M}\dot{X} + \left(\Omega^2 + \dot{\mathcal{E}} + \frac{\dot{M}\mathcal{E}}{M}\right)X = 0 \tag{3.22}$$

with initial conditions

$$X(t_i) = 1 \quad ; \quad \dot{X}(t_i) = \frac{-i\kappa}{M(t_i)} - \mathcal{E}(t_i) \tag{3.23}$$

This is an important result, reducing as it does the usual task of finding the Bogoliubov coefficients α, β from two coupled first order differential equations to that of solving one second order equation for X.

3.3 Evolution of the reduced density matrix

Suppose now that we work within the context of quantum brownian motion, using the notation of [2]. That is, our system is modelled by an oscillator with time dependent mass, cross term and natural frequency. This interacts bilinearly with an environment modelled in the same way, the total lagrangian being

$$\begin{aligned} S[x,\boldsymbol{q}] &= S[x] + S_E[\boldsymbol{q}] + S_{\text{int}}[x,\boldsymbol{q}] \\ &= \int_{t_i}^{t} ds \left\{ \frac{M(s)}{2}\left(\dot{x}^2 + 2\mathcal{E}(s)x\dot{x} - \Omega^2(s)x^2\right) \right. \\ &\quad \left. + \sum_n \left[\frac{m_n(s)}{2}\left(\dot{q}_n^2 + 2\varepsilon_n(s)q_n\dot{q}_n - \omega_n^2(s)q_n^2\right)\right] + \sum_n \left[-c(s)xq_n\right] \right\} \end{aligned} \tag{3.24}$$

where the particle and the bath oscillators have coordinates x and q_n respectively.

We wish to start with some initial system density matrix $\rho_{sys}(x_i\, x_i'\, 0)$ and evolve it using (3.5). As described in [2], J_r is calculated using the standard path integral approach. Using the sum and difference coordinates defined in (3.10), the classical paths followed by the system, Σ_{cl}, Δ_{cl}, can be written in terms of more elementary functions u, v:

$$\begin{aligned} \Sigma_{cl}(s) &= \Sigma_{cl}(t_i)u_1(s) + \Sigma_{cl}(t)u_2(s) \\ \Delta_{cl}(s) &= \Delta_{cl}(t_i)v_1(s) + \Delta_{cl}(t)v_2(s) \end{aligned} \tag{3.25}$$

Then it can be shown that the superpropagator J_r is equal to

$$\begin{aligned} J_r(x, x', t | x_i, x_i', t_i) &= \frac{|b_2|}{2\pi\hbar} \exp\left[\frac{i}{\hbar}(b_1\Sigma\Delta - b_2\Sigma\Delta_i + b_3\Sigma_i\Delta - b_4\Sigma_i\Delta_i) \right. \\ &\quad \left. - \frac{1}{\hbar}\left(a_{11}\Delta_i^2 + a_{12}\Delta_i\Delta + a_{22}\Delta^2\right)\right] \end{aligned} \tag{3.26}$$

The functions $b_1 \rightarrow b_4$ can be expressed as

$$\begin{aligned} b_1(t, t_i) &= M(t)\dot{u}_2(t) + M(t)\mathcal{E}(t) \\ b_2(t, t_i) &= M(t_i)\dot{u}_2(t_i) \\ b_3(t, t_i) &= M(t)\dot{u}_1(t) \\ b_4(t, t_i) &= M(t_i)\dot{u}_1(t_i) + M(t_i)\mathcal{E}(t_i) \end{aligned} \tag{3.27}$$

while the functions a_{ij} are defined by

$$a_{ij}(t, t_i) = \frac{1}{1 + \delta_{ij}} \int_{t_i}^{t} ds \int_{t_i}^{t} ds'\; v_i(s)\; \nu(s, s')\; v_j(s') \tag{3.28}$$

The functions $u_1 \rightarrow v_2$ are solutions to the following equations (dropping subscripts on u, v):

$$\ddot{u}(s) + \frac{\dot{M}}{M}\dot{u} + \left(\Omega^2 + \dot{\mathcal{E}} + \frac{\dot{M}}{M}\mathcal{E}\right) u + \frac{2}{M(s)} \int_{t_i}^{s} ds'\; \mu(s, s')\; u(s') = 0 \tag{3.29}$$

$$\ddot{v}(s) + \frac{\dot{M}}{M}\dot{v} + \left(\Omega^2 + \dot{\mathcal{E}} + \frac{\dot{M}}{M}\mathcal{E}\right) v - \frac{2}{M(s)} \int_{s}^{t} ds'\; \mu(s, s')\; v(s') = 0 \tag{3.30}$$

subject to the boundary conditions

$$\begin{aligned} &u_1(t_i) = v_1(t_i) = 1 \quad , \quad u_1(t) = v_1(t) = 0 \\ &u_2(t_i) = v_2(t_i) = 0 \quad , \quad u_2(t) = v_2(t) = 1 \end{aligned} \tag{3.31}$$

3.3.1 Calculating the superpropagator J_r: ohmic environment

To proceed further we need explicit expressions for $a_{11} \rightarrow b_4$. These are expressed in terms of $u_1 \rightarrow v_2$, which in turn come from solving (3.29, 3.30). To solve these equations we need to know the dissipation μ of the environment.

The noise and dissipation can be calculated from [2, eqns 2.18, 2.19]. We choose the bath oscillators to be simple harmonic, that is, static with no cross term, since this turns out to correspond to the simplest form of dissipation—local—as shown in appendix A. For such an environment the dissipation and noise can be shown to be

$$\begin{aligned}\mu(s,s') &= \int_0^\infty d\omega\ I(\omega,s,s')\mathrm{Im}\left[X(s)X^*(s')\right] \\ \nu(s,s') &= \int_0^\infty d\omega\ I(\omega,s,s')\coth\frac{\omega}{2T}\mathrm{Re}\left[X(s)X^*(s')\right]\end{aligned} \tag{3.32}$$

where by T we will always mean $k_BT/\hbar$; X is the sum of the Bogoliubov coefficients for the bath oscillators and I is the "spectral density", a function defined by

$$I(\omega,s,s') = \frac{c(s)c(s')}{2\kappa}\sum_n \delta(\omega-\omega_n) \tag{3.33}$$

which encodes information of the action of the environment on the system. In general the spectral density can be described by some function of ω^j, where j is set by the particular environment being modelled. The case of $j=1$, a so-called "ohmic" environment, is a borderline between the super-ohmic case ($j>1$)—which models weak damping—and the subohmic case ($j<1$) modelling strong damping. We can in effect consider both damping extremes by taking an ohmic environment together with some strength γ_0 which can be altered from zero, for a free system, up to higher strengths.

Also, by considering the continuum limit of the coupling constant, it can be shown that this constant's independence of n also leads to an ohmic environment; so we will only consider spectral densities of the following form:

$$I(\omega,s,s') = \frac{2\gamma_0}{\pi}\omega\ c(s)c(s') \tag{3.34}$$

For a general lagrangian the sum of the Bogoliubovs X will be complicated; however we have simplified our calculations by taking the bath to be composed of unsqueezed (i.e. coherent) static oscillators with unit mass. For this type of bath the dissipation and noise are calculated in appendix A for an arbitrary bath temperature; we use the integral form of the noise as being easier to work with:

$$\begin{aligned}\mu(s,s') &= 2\gamma_0\, c(s)c(s')\,\delta'(s-s') \\ \nu(s,s') &= \frac{2\gamma_0}{\pi}c(s)c(s')\int_0^\infty \omega\coth\frac{\omega}{2T}\cos\omega(s-s')\ d\omega\end{aligned} \tag{3.35}$$

In the high temperature limit the noise becomes white, that is it tends toward a delta function as shown in that appendix.

Calculating $u_1 \to v_2$

Now we are in a position to solve (3.29, 3.30) for $u_1 \to v_2$. First consider (3.29). We treat the integral of a delta function and its derivative in the following way: use a smooth step function (i.e. $\theta(0) \equiv 1/2$) to write ($x_1 > x_0$)

$$\int_{x_0}^{x_1} f(x)\delta(x-a)\,dx \equiv f(a)\,\theta(x_1-a)\,\theta(a-x_0) \tag{3.36}$$

$$\int_{x_0}^{x_1} f(x)\delta'(x-a)\,dx \equiv -f'(a)\,\theta(x_1-a)\,\theta(a-x_0) \tag{3.37}$$

These relations can easily be proved by checking the five cases individually, of $a < x_0$, $a = x_0$, $x_0 < a < x_1$ etc. Note that treating the delta function in this 'smoothed' way eliminates the need for the frequency renormalisation in [10]. This smoothing essentially just defines $\int_0^\infty \delta(x)dx = 1/2$ (see e.g. [11] for a discussion of this).

Hence (3.29) together with (3.35) becomes (with u being either u_1 or u_2)

$$\ddot{u}(s) + \left(\frac{\dot{M}}{M} + \frac{2\gamma_0 c^2}{M}\right)\dot{u} + \left(\Omega^2 + \frac{\dot{M}\mathcal{E}}{M} + \dot{\mathcal{E}} + \frac{2\gamma_0 c\dot{c}}{M}\right)u = 0 \tag{3.38}$$

Now define $\tilde{u}$ by

$$\tilde{u} \equiv u \exp\left[\gamma_0 \int_{t_i}^{s} \frac{c^2(s')}{M(s')}ds'\right] \tag{3.39}$$

in which case it follows that

$$\ddot{\tilde{u}} + \frac{\dot{M}}{M}\dot{\tilde{u}} + \left(\Omega^2 + \frac{\dot{M}\mathcal{E}}{M} + \dot{\mathcal{E}} - \frac{\gamma_0^2 c^4}{M^2}\right)\tilde{u} = 0 \tag{3.40}$$

Comparing with (3.22), we recognise this as just the equation of motion of an oscillator with mass M, cross term $\mathcal{E}$ and an effective frequency

$$\Omega_{\mathit{eff}}^2 \equiv \Omega^2 - \frac{\gamma_0^2 c^4}{M^2} \tag{3.41}$$

So, we are in a position to describe our system in terms of an equivalent system. Hence we know a solution for $\tilde{u}(s)$—it is the sum X of the Bogoliubov coefficients for this new system. So we write (with g_1, g_2 constants to be determined)

$$u(s) = \exp\left[-\gamma_0 \int_{t_i}^{s} \frac{c^2}{M}ds'\right]\left[g_1 X(s) + g_2 X^*(s)\right] \tag{3.42}$$

By including the boundary conditions for u_1 and u_2 we obtain

$$\begin{aligned} u_1(s) &= \exp\left[-\gamma_0 \int_{t_i}^{s} \frac{c^2}{M}ds'\right] \frac{\operatorname{Im}\left[X(t)X^*(s)\right]}{\operatorname{Im} X(t)} \\ u_2(s) &= \exp\left[\gamma_0 \int_{s}^{t} \frac{c^2}{M}ds'\right] \frac{\operatorname{Im} X(s)}{\operatorname{Im} X(t)} \end{aligned} \tag{3.43}$$

This tying in of the propagator formalism to the language of squeezed states (such as Bogoliubov coefficients) will be very useful in the next chapter where we relate the entropy of a field mode to its squeeze parameter r.

In the same way that we solved (3.29), eqn (3.30) becomes

$$\ddot{v}(s)+\left(\frac{\dot{M}}{M}-\frac{2\gamma_0 c^2}{M}\right)\dot{v}+\left(\Omega^2+\frac{\dot{M}\mathcal{E}}{M}+\dot{\mathcal{E}}-\frac{2\gamma_0 c\dot{c}}{M}\right)v=0 \tag{3.44}$$

Now write

$$\tilde{v}\equiv v\exp\left[-\gamma_0\int_{t_i}^{s}\frac{c^2}{M}ds'\right] \tag{3.45}$$

and just as for the case of u we have

$$\ddot{\tilde{v}}+\frac{\dot{M}}{M}\dot{\tilde{v}}+\left(\Omega^2+\frac{\dot{M}\mathcal{E}}{M}+\dot{\mathcal{E}}-\frac{\gamma_0^2c^4}{M^2}\right)\tilde{v}=0 \tag{3.46}$$

So now v_1 and v_2 can also be written as combinations of X and X^*. Including the boundary conditions we eventually obtain

$$\begin{aligned}
v_1(s) &= \exp\left[\gamma_0\int_{t_i}^{s}\frac{c^2}{M}ds'\right]\frac{\text{Im }[X(t)X^*(s)]}{\text{Im }X(t)}\\
v_2(s) &= \exp\left[-\gamma_0\int_{s}^{t}\frac{c^2}{M}ds'\right]\frac{\text{Im }X(s)}{\text{Im }X(t)}
\end{aligned} \tag{3.47}$$

Calculating $a_{11}\rightarrow b_4$

To facilitate our calculations we introduce dimensionless parameters for time

$$\begin{aligned}
z &\equiv \kappa t \quad , \quad \zeta\equiv\kappa s\\
X(z) &\equiv X(t) \ \text{ etc.}
\end{aligned} \tag{3.48}$$

and a carat will denote division by κ, e.g. $\hat{\gamma}_0=\gamma_0/\kappa$. Note that t is the lagrangian time, which isn't necessarily cosmic.

Now we are able to calculate the propagator. Making use of (3.28, 3.27) we obtain

$$\begin{aligned}
a_{11}(z,z_i) &= \frac{1}{2\kappa^2}\int_{z_i}^{z}d\zeta\int_{z_i}^{z}d\zeta'\; e^{\hat{\gamma}_0\int_{z_i}^{\zeta}\frac{c^2}{M}d\zeta''}\frac{\text{Im }[X(z)X^*(\zeta)]}{\text{Im }X(z)}\;\nu(\zeta,\zeta')\;e^{\hat{\gamma}_0\int_{z_i}^{\zeta'}\frac{c^2}{M}d\zeta''}\frac{\text{Im }[X(z)X^*(\zeta')]}{\text{Im }X(z)}\\
a_{12} &= \frac{1}{\kappa^2}\int_{z_i}^{z}d\zeta\int_{z_i}^{z}d\zeta'\; e^{\hat{\gamma}_0\int_{z_i}^{\zeta}\frac{c^2}{M}d\zeta''}\frac{\text{Im }[X(z)X^*(\zeta)]}{\text{Im }X(z)}\;\nu(\zeta,\zeta')\;e^{-\hat{\gamma}_0\int_{\zeta'}^{z}\frac{c^2}{M}d\zeta''}\frac{\text{Im }X(\zeta')}{\text{Im }X(z)}\\
a_{22} &= \frac{1}{2\kappa^2}\int_{z_i}^{z}d\zeta\int_{z_i}^{z}d\zeta'\; e^{-\hat{\gamma}_0\int_{\zeta}^{z}\frac{c^2}{M}d\zeta''}\frac{\text{Im }X(\zeta)}{\text{Im }X(z)}\;\nu(\zeta,\zeta')\;e^{-\hat{\gamma}_0\int_{\zeta'}^{z}\frac{c^2}{M}d\zeta''}\frac{\text{Im }X(\zeta')}{\text{Im }X(z)}
\end{aligned}$$

$$
\begin{aligned}
b_1(z,z_i) &= -\hat{\gamma}_0\,\kappa c^2(z) + \kappa M(z)\frac{\text{Im } X'(z)}{\text{Im } X(z)} + M(z)\mathcal{E}(z) \\
b_{\{{2 \atop 3}\}} &= \frac{\mp\kappa\, e^{\pm\hat{\gamma}_0 \int_{z_i}^{z} \frac{c^2}{M} d\zeta}}{\text{Im } X(z)} \\
b_4 &= -\hat{\gamma}_0\,\kappa c^2(z_i) + \kappa\,\frac{\text{Re } X(z)}{\text{Im } X(z)} + M(z_i)\mathcal{E}(z_i)
\end{aligned} \tag{3.49}
$$

3.3.2 Initial and final state

The systems we will deal with are initially in the vacuum state, so that their density matrix is gaussian. So we start with an arbitrary gaussian reduced density matrix and propagate it. The initial matrix is

$$
\rho_r(x_i\; x_i'\; t_i) \propto e^{-\xi x_i^2 + \chi x_i x_i' - \xi^* x_i'^2} \tag{3.50}
$$

This is propagated by using (3.5, 3.26) to give

$$
\rho(x,x',t) = N e^{-A\Delta^2 - 2iB\Delta\Sigma - 4C\Sigma^2} \tag{3.51}
$$

where we have used the same A, B and C notation of [12], and with ξ_r, ξ_i the real and imaginary parts of ξ:

$$
\begin{aligned}
N &= 2\sqrt{C/\pi} \\
A &= a_{22} + \frac{1}{D}\left\{[(2\xi_r + \chi)/4 + a_{11}]\, b_3^2 + (2\xi_i + b_4)\, a_{12}\, b_3 - (2\xi_r - \chi)a_{12}^2\right\} \\
B &= -b_1/2 + \frac{1}{D}[(\xi_i + b_4/2)\, b_2\, b_3 - (2\xi_r - \chi)a_{12}\, b_2] \\
C &= \frac{1}{4D}(2\xi_r - \chi)\, b_2^2 \\
D &= 4|\xi|^2 - \chi^2 + 4\,(2\xi_r - \chi)a_{11} + 4\,\xi_i\, b_4 + b_4^2
\end{aligned} \tag{3.52}
$$

These expressions form the basis of the calculations in chapters 4 and 5. The quantity we are focusing on is the reduced density matrix, (3.51), using the expressions in (3.52). These in turn use (3.49), which in their turn depend on our obtaining X, the sum of the Bogoliubov coefficients for the effective oscillator. The diagram shows the flow of logic, together with the means for producing the Bogoliubov coefficients and the squeeze parameter r, as described in the next chapter.

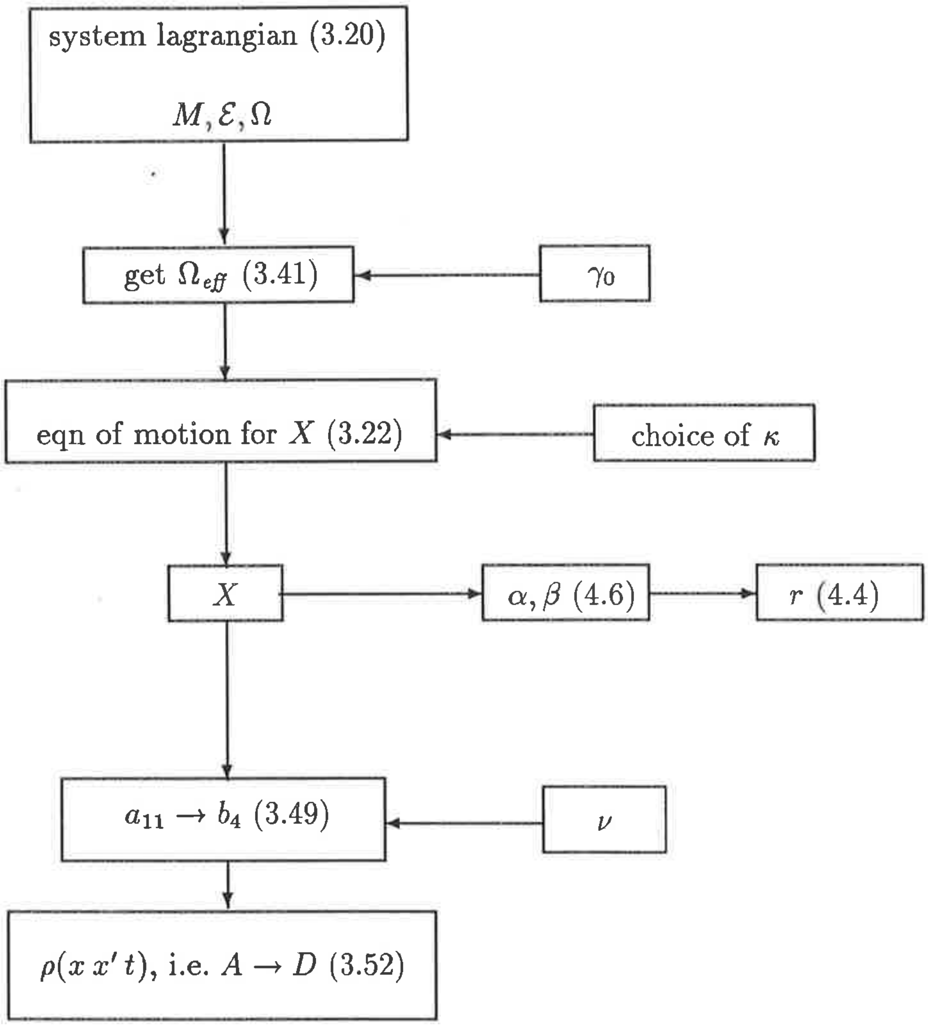


Figure 3.1: The general scheme we will follow to implement the influence functional approach.

Chapter 4

Entropy generation in squeezed systems

One of the issues of interest in studies of quantum cosmology is to what extent the universe retains information about its early evolution. The natural loss of information with time, familiar from statistical mechanics, reflects in some way the viewpoint of a particular observer, and as such might be thought to be very much observer dependent [13]. So we expect the entropy of some system to be dependent on its coupling to a measuring device. Conversely, we can consider the entropy of the measuring device itself as it couples to the system. Suppose we have some system which has been decomposed into two subsystems. Then using the definition of entropy below, it can be shown [14] that between the entropies S_1, S_2 of the two subsystems, and that of the total system, S_{12}, a triangle inequality holds:

$$|S_1 - S_2| \leq S_{12} \leq S_1 + S_2 \tag{4.1}$$

In particular, if the total system is closed and so in a pure state, then it has zero entropy, so that the two subsystems necessarily have equal entropies. Hence, asking for the entropy growth of a system is equivalent to asking for the entropy growth of the environment it couples to, if the overall larger system will be in a pure state.

The entropy of a system is usually defined as the logarithm of the number of accessible states it can occupy. This is useful for a system containing a finite number of degrees of freedom, but for systems such as fields, with an infinite number of degrees of freedom, we need to resort to other means. The von Neumann definition of quantum entropy uses the density matrix of a system:

$$S \equiv -\mathrm{tr}\ \rho \ln \rho \tag{4.2}$$

As it stands, the universe presumably evolves as a closed system in a pure state, so that $\rho^2 = \rho$ and hence the entropy is always zero. This global idea of entropy is not especially useful. More appropriate is an attempt to relate entropy growth to an arrow of time. Penrose conjectured that we may be able to assign entropy to the gravitational field by integrating the square of the Weyl tensor. Thus, the universe's evolution toward greater gravitational clumping would be accompanied by a growth in the entropy of the gravitational field, since the squared Weyl tensor increases regardless of whether the universe is expanding or contracting.

As the universe evolves, inertial observers must continually update the particle numbers they assign to Fock states, so that particles are being created by the fields that are present. Hu and others [16, 17] have given a description of the entropy growth of a system of particles in curved spacetime based on the von Neumann entropy of the density matrix of the system. If we distinguish a system of interest from the rest of the universe, which then acts as its environment, then we can calculate the entropy from (4.2) by using the reduced density matrix of the system. Calculating this quantity is generally only possible if we can diagonalise ρ_{red}; that is, if we can find a basis in which its off-diagonal elements are zero.

This vanishing of off-diagonal elements is generally equated with decoherence and a quantum to classical transition, at least in that particular basis. Other approaches to the idea of decoherence have also been proposed. Brandenberger, Mukhanov and Prokopec [18] introduced a decoherence procedure which relies on the squeeze formalism, first introduced in its full context by Grishchuk and Sidorov [19]. They average over the squeeze angles appearing in the probability functional (which plays a similar role in their formalism to the reduced density matrix). This coarse graining procedure has the effect that all off-diagonal elements go to zero.

A different decoherence procedure was used by Gasperini and Giovannini [20] who considered a squeezed vacuum in terms of new variables for which the fluctuations are maximised and minimised, and then neglecting information about the subfluctuant variable. Kruczenski, Oxman and Zaldarriaga [21] also use a procedure of setting off-diagonal elements to zero before calculating the von Neumann entropy. Likewise, in [12], Matacz considered the density matrix of a squeezed vacuum, decohering it by setting off diagonals to zero in the coherent state basis.

All of these approaches give an entropy of $S = 2r$ per mode in the high squeeze limit ($r \to \infty$). However, all of these methods are nondynamical schemes which decohere a system in some chosen basis by hand: simply setting the off diagonal elements to zero in that basis. What is needed is a rigorous dynamical approach to entropy generation using nonequilibrium statistical mechanics. The purpose of this chapter is to place the problem of calculating the entropy of a field in an open system framework, using influence functionals to evolve the system, and calculating the reduced density matrix which will then yield the von Neumann entropy in the usual way via (4.2). We then hope to see to what extent the seemingly very robust result of $S = 2r$ is true.

Our field will be initially in a vacuum state, corresponding to a gaussian density matrix as modelled in the last chapter, with examples residing in both flat and de Sitter spaces. We will calculate the entropy as a function of the amount of squeezing the system would have undergone in the absence of an environment.

Our plan is first to find an expression for the squeeze parameter r as a function of late time, for a given lagrangian; we then wish to calculate the entropy as a function of late time, which allows us to express the entropy as a function of r. Unfortunately, since many of the expressions resulting from our formalism have no closed form, we cannot calculate the entropy for a general squeezed system. Instead we'll concentrate on two examples of squeezed systems that are both solvable and relevant to cosmology. However in the discussion section we do point out that one might expect a contribution of $2r$ to the entropy quite generally, but that there may be other contributions of a similar size that need to be calculated on a case by case basis.

4.1 Calculating r from the system lagrangian

For an inverted oscillator, i.e. one with $\Omega^2 < 0$, at late times r is expected to blow up. In that case we can calculate it from (3.18) as follows.

$$|\alpha| \to |\beta| \to e^r/2 \tag{4.3}$$

so that

$$r \to \ln(2|\alpha|) \tag{4.4}$$

Rather than use (3.19) to calculate α, once we have X we can extract α from it. This is done by writing, from (3.19),

$$\begin{aligned} X &= \alpha + \beta \\ \partial X/\partial t &= i(f-h)\alpha + i(h-f^*)\beta \end{aligned} \tag{4.5}$$

which can be solved for α, β using (3.21):

$$\left\{ {\alpha \atop \beta} \right\} = \frac{1}{2}\left(1 \pm \frac{i\mathcal{E}M}{\kappa}\right) X \pm \frac{iM}{2\kappa}\frac{\partial X}{\partial t} \tag{4.6}$$

In terms of squeeze notation, we can follow the behaviour of r, ϕ, θ by writing (3.19) in squeeze language, with $f = |f|e^{i\varepsilon}$:

$$\begin{aligned} \dot{r} &= |f|\sin(2\phi + \varepsilon) \\ \dot{\phi} &= -h + |f|\coth 2r \cos(2\phi + \varepsilon) \\ \dot{\theta} &= h - |f| \operatorname{th} r \cos(2\phi + \varepsilon) \end{aligned} \tag{4.7}$$

These equations are useful for numerical work. They also tell us of the existence of constant, and so possibly attractor, solutions for ϕ, θ. If we set $r \to \infty$ then the equations for ϕ, θ become

$$\dot{\theta} = -\dot{\phi} = h - |f| \cos(2\phi + \varepsilon) \tag{4.8}$$

1. Suppose there exist some θ and ϕ such that $\dot{\theta} = \dot{\phi} = 0$. Then $h = |f| \cos(2\phi + \varepsilon)$, so that $|h| \leq |f|$. Thus, since h is real, we have $h^2 \leq |f|^2$, and from (3.21) this inequality is true if and only if $\Omega^2 \leq 0$.

2. Conversely suppose $\Omega^2 \leq 0$. Then by the previous argument, $|h| \leq |f|$, or $-1 \leq h/|f| \leq 1$. Thus there must exist some ϕ such that $\cos(2\phi + \varepsilon) = h/|f|$. From (4.8) we see that for this value of ϕ, $\dot{\theta} = \dot{\phi} = 0$.

In other words, there will exist constant solutions for ϕ, θ if and only if $\Omega^2 \leq 0$ (the oscillator is "inverted"). Of course, this doesn't reveal whether these constant solutions are attractors. Numerically solving (4.7) with $\Omega^2 \leq 0$, for various $\mathcal{E}$, Ω and κ, shows that ϕ, θ apparently do always quickly tend toward constants, always accompanied by one of $r \to \pm\infty$.

As a final comment, we note that it's common to eliminate the cross term in the action by adding a surface term:

$$\begin{aligned} L &\to \frac{M}{2}\left(\dot{x}^2 + 2\mathcal{E}\dot{x}x - \Omega^2 x^2\right) - \frac{1}{2}\frac{d}{dt}(M\mathcal{E}x^2) \\ &= \frac{M}{2}\left[\dot{x}^2 - \left(\Omega^2 + \frac{\dot{M}\mathcal{E}}{M} + \dot{\mathcal{E}}\right)x^2\right] \end{aligned} \tag{4.9}$$

Although this leaves the classical equation of motion unchanged, it will change the squeeze parameters. Throughout this thesis we leave the cross term in our lagrangians.

4.2 Entropy from the reduced density matrix

The entropy of a field mode has been calculated in [22]. It can be derived from the final density matrix by using (4.2), and is

$$S = \frac{-1}{w}[w \ln w + (1-w)\ln(1-w)] \simeq 1 - \ln w \quad \text{if } w \to 0 \tag{4.10}$$

where

$$w \equiv \frac{2\sqrt{C/A}}{1 + \sqrt{C/A}} \tag{4.11}$$

The linear entropy is often more useful to work with owing to its simplicity:

$$S_{lin} \equiv -\mathrm{tr}\,\rho^2 = -\sqrt{C/A} \tag{4.12}$$

and $S = 0 \to \infty$ is equivalent to $S_{lin} = -1 \to 0$, both strictly increasing. Then if $S_{lin} \to 0$ we have

$$S \to -\ln|S_{lin}| + 1 - \ln 2 \quad , \quad \text{i.e. } S_{lin} \to -e^{1-S}/2 \tag{4.13}$$

As an example, suppose we have a system in an initially pure gaussian state ($\chi = 0$), so that noise and dissipation are absent: $\gamma_0 = 0$. In this case, from (3.35, 3.49) we have

$$a_{11} = a_{12} = a_{22} = 0 \tag{4.14}$$

so that (3.52) gives $C/A = 1$ and hence from (4.10) $S = 0$ as expected.

4.3 Examples

4.3.1 Uncertainty and entropy of a static oscillator

We demonstrate how the previous results are used in the simplest case, by calculating the entropy of a static oscillator coupled to a thermal bath of static oscillators, with a static ohmic coupling. In this case from section 3.3.1 we have local dissipation [i.e. $\mu \propto \delta'(\Delta)$], and if we demand $T \to \infty$ then the noise becomes white [$\nu \propto \delta(\Delta)$]. To evaluate S, we need A and C; in turn for these we need $a_{11} \to b_4$. These are calculated from (3.49)[1]. For the static oscillator with unit mass choose the following lagrangian:

$$L = \frac{1}{2}\left(\dot{x}^2 - k^2x^2\right) \tag{4.15}$$

From (3.41) with $M = c = 1$ the effective frequency is

$$\Omega^2_{eff} = k^2 - \gamma_0^2 \equiv \kappa^2 \tag{4.16}$$

Then the equation of motion for X is, from (3.22) with $\Omega \to \Omega_{eff}$

$$\ddot{X} + \kappa^2 X = 0 \tag{4.17}$$

$$X(0) = 1 \quad , \quad \dot{X}(0) = -i\kappa \tag{4.18}$$

which leads to

$$X(z) = e^{-iz} \tag{4.19}$$

with $z = \kappa t$. Then

$$\frac{\text{Im}\,[X(z)X^*(\zeta)]}{\text{Im}\,X(z)} = \frac{\sin(z-\zeta)}{\sin z} \quad , \quad \frac{\text{Im}\,X(\zeta)]}{\text{Im}\,X(z)} = \frac{\sin\zeta}{\sin z} \tag{4.20}$$

[1]Various notations exist describing these results; see for example [12, 23, 24]. To compare with [24, eqn 2.2.7] is a matter of carefully transcribing the notation; key things to note are that $X \equiv \Sigma$, $Y \equiv -\Delta$; here we have taken $x_0 = p_0 = 0$; [24, eqn 2.2.6c] should have an a_{11} in place of the a_{22}; the b_i's in [2] are written explicitly in [24] via [2, 3.11]; [2, a_{12}] equals [24, $a_{12} + a_{21}$]; [2, γ_0] equals [24, $\gamma_0/2$].

$$e^{\hat{\gamma}_0 \int_{z_i}^{\zeta} \frac{c^2}{M} d\zeta''} = e^{\hat{\gamma}_0 \zeta} \quad , \quad e^{-\hat{\gamma}_0 \int_{\zeta}^{z} \frac{c^2}{M} d\zeta''} = e^{-\hat{\gamma}_0 (z-\zeta)} \tag{4.21}$$

with noise for $T \to \infty$ being white:

$$\nu(\zeta, \zeta') = 4\kappa\gamma_0 T \delta(\zeta - \zeta') \tag{4.22}$$

Then $a_{11} \to b_4$ follow:

$$\begin{aligned}
a_{11} &= \frac{T}{\sin^2 z} \cdot \frac{e^{2\hat{\gamma}_0 z} - 1 - \hat{\gamma}_0 \sin 2z - \hat{\gamma}_0^2 (1 - \cos 2z)}{2(1 + \hat{\gamma}_0^2)} \\
a_{12} &= \frac{2T}{\sin^2 z} \cdot \frac{-\cos z \text{ sh } \hat{\gamma}_0 z + \hat{\gamma}_0 \sin z \text{ ch } \hat{\gamma}_0 z}{1 + \hat{\gamma}_0^2} \\
a_{22} &= \frac{T}{\sin^2 z} \cdot \frac{-e^{-2\hat{\gamma}_0 z} + 1 - \hat{\gamma}_0 \sin 2z + \hat{\gamma}_0^2 (1 - \cos 2z)}{2(1 + \hat{\gamma}_0^2)}
\end{aligned}$$

$$b_{\{^1_4\}} = \kappa(-\hat{\gamma}_0 \pm \cot z) \quad , \quad b_{\{^2_3\}} = \frac{\pm\kappa \, e^{\pm\hat{\gamma}_0 z}}{\sin z} \tag{4.23}$$

For thermal equilibrium, the standard statistical mechanics result for the entropy at high temperature is

$$S \to 1 + \ln \frac{T}{k} \tag{4.24}$$

Can we show this using our formalism? First, we take our oscillator ground state as

$$\psi(x, 0) \propto \exp \frac{-x^2}{4\sigma^2} \tag{4.25}$$

so that its density matrix is

$$\rho(x \, x' \, 0) \propto \exp \frac{-x^2 - x'^2}{4\sigma^2} \tag{4.26}$$

and in (3.50) we have

$$\xi = \frac{1}{4\sigma^2} \quad , \quad \chi = 0 \tag{4.27}$$

The reduced density matrix evolves into (3.51), with

$$\begin{aligned}
A &= a_{22} + \frac{1}{D} \left\{ \left[\frac{1}{8\sigma^2} + a_{11} \right] b_3^2 + a_{12} b_3 b_4 - \frac{a_{12}^2}{2\sigma^2} \right\} \\
B &= \frac{-b_1}{2} + \frac{b_2}{2D} \left\{ b_3 b_4 - \frac{a_{12}}{\sigma^2} \right\} \\
C &= \frac{b_2^2}{8D\sigma^2} \\
D &= \frac{1}{4\sigma^4} + \frac{2a_{11}}{\sigma^2} + b_4^2
\end{aligned} \tag{4.28}$$

It's by no means trivial to show that the entropy calculated using these expressions does indeed tend toward (4.24), and in particular the $\csc z$ terms in the a_{ij}'s and b_i's mean their values can diverge depending on the time. But this divergence cancels out when physical quantities are measured, as we can see by verifying numerically that our entropy really does tend toward the usual asymptotic value at late times. First we can plot S-vs-z for say $\sigma = 1, k = 1, \gamma_0 = 0.1, T = 10^5$, as shown in figure 4.1. Note that for these numbers,

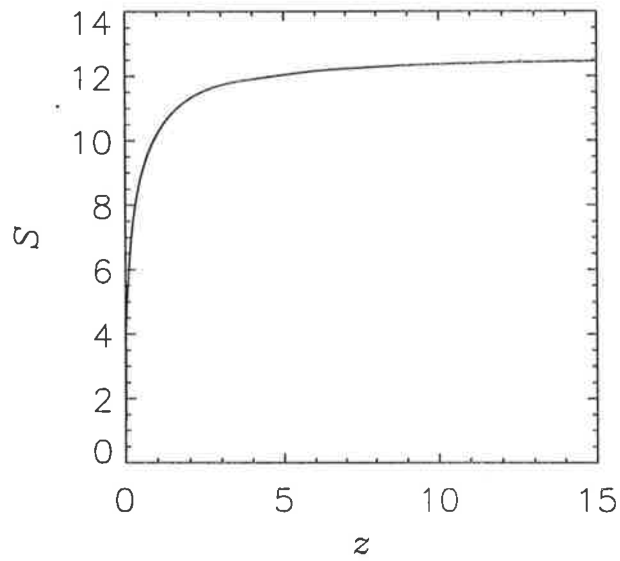


Figure 4.1: Entropy growth over time.

(4.24) gives $S \rightarrow 12.513$ as $z \rightarrow \infty$, as compared with $S \rightarrow 12.514$ numerically at $z = 100$, a result indicated by the figure. The relaxation time, defined to be

$$\frac{1}{2\gamma_0} = 5 \tag{4.29}$$

is apparent in the figure as a characteristic time over which the entropy climbs to its final value, while the decoherence time scale [10]

$$\frac{1}{4M\gamma_0 T\sigma^2} = 2.5 \times 10^{-5} \tag{4.30}$$

is too small to be noticeable.

Coherent state as the state of least entropy

We now use our entropy expression to investigate the claim that for large times the state of least entropy for the static oscillator is the coherent one, at least for white noise and local dissipation. This was shown in [23] in the small γ_0 limit by using a Wigner function approach.

Using our expression for the entropy S, we can plot S versus the initial squeeze parameter r for various times in fig. 4.2. We have chosen $k = 10, \gamma_0 = 0.1$. The squeeze parameter r is related to σ the width of the gaussian wavefunction by

$$r \equiv \ln \frac{\sigma_0}{\sigma} \quad ; \quad \sigma_0 \equiv \sqrt{\frac{1}{2\kappa}} \tag{4.31}$$

or in other words

$$\sigma = \frac{e^{-r}}{\sqrt{2\kappa}} \tag{4.32}$$

Note that at early times (e.g. $z = 0.001$), the entropy is minimised for high initial

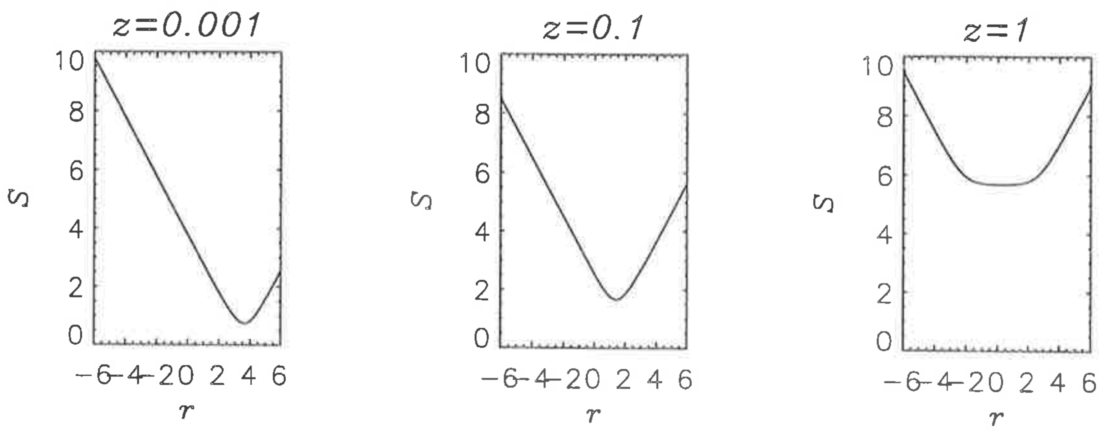


Figure 4.2: Entropy at various times.

squeezing, as noted in [23, fig. 1]; this is not unreasonable since such a highly squeezed state will spread with time, becoming indistinguishable at later times from states which started out being less highly squeezed. At late times the entropy is minimised by starting with small or zero squeezing, i.e. an initially coherent state is the one which minimises entropy at late times. Thus our approach agrees with [23], and may be more useful in that it allows us to directly calculate the entropy at all times.

4.3.2 Static inverted oscillator

The static inverted oscillator is the simplest squeezed system, and as such forms a good testing ground for the formalism developed so far. It also forms a model for the zero mode of the inflaton field in New Inflation. Its lagrangian is:

$$L(t) = \frac{1}{2}[\dot{x}^2 + k^2 x^2] \tag{4.33}$$

Suppose this is coupled to the usual environment of harmonic oscillators in a thermal state, with coupling constant $c(s) = 1$. Then the equivalent oscillator we consider has unit mass, no cross term and frequency

$$\Omega_{eff}^2 = -k^2 - \gamma_0^2 \equiv -\kappa^2 \tag{4.34}$$

so that from (3.22) the sum of its Bogoliubov coefficients is (taking $t_i = 0$)

$$X(t) = \operatorname{ch} z - i \operatorname{sh} z \tag{4.35}$$

Hence from (4.6) we have

$$\alpha = \operatorname{ch} z \quad , \quad \beta = -i \operatorname{sh} z \tag{4.36}$$

so that from (4.4) at late times ($z \to \infty$)

$$r \longrightarrow z \tag{4.37}$$

To investigate the dependence of the entropy on the various quantities in the propagator coefficients, we calculate these coefficients first for white noise analytically; we then calculate them numerically for zero temperature.

The b_i's are independent of the temperature, and using (3.49) they are found to be

$$b_{\{^1_4\}} = \kappa(\pm \coth z - \hat{\gamma}_0) \quad , \quad b_{\{^2_3\}} = \frac{\pm \kappa e^{\pm \hat{\gamma}_0 z}}{\operatorname{sh} z} \tag{4.38}$$

High temperature

White noise is given by $\nu(s, s') = 4\gamma_0 T\ \delta(s - s')$, or $\nu(\zeta, \zeta') = 4\hat{\gamma}_0 \kappa^2 T \delta(\zeta - \zeta')$; the relevant quantities are inserted into (3.49) with the a_{ij}'s then becoming

$$\begin{aligned}
a_{11} &= \frac{T}{2\hat{k}^2 \operatorname{sh}^2 z}\left[\hat{k}^2 + e^{2\hat{\gamma}_0 z} - \hat{\gamma}_0 \operatorname{sh} 2z - \hat{\gamma}_0^2 \operatorname{ch} 2z\right] \\
a_{12} &= \frac{T e^{-\hat{\gamma}_0 z}}{\hat{k}^2 \operatorname{sh}^2 z}\left[\left(1 - e^{2\hat{\gamma}_0 z}\right) \operatorname{ch} z + \left(1 + e^{2\hat{\gamma}_0 z}\right) \hat{\gamma}_0 \operatorname{sh} z\right] \\
a_{22} &= \frac{T e^{-2\hat{\gamma}_0 z}}{2\hat{k}^2 \operatorname{sh}^2 z}\left[-\hat{k}^2 e^{2\hat{\gamma}_0 z} - 1 + \hat{\gamma}_0 e^{2\hat{\gamma}_0 z}\left(\hat{\gamma}_0 \operatorname{ch} 2z - \operatorname{sh} 2z\right)\right]
\end{aligned} \tag{4.39}$$

Note that $\hat{\gamma}_0 = \gamma_0/\kappa < 1$; however if we assume small dissipation ($\hat{\gamma}_0 \ll 1$) we can write down large time limits of these quantities:

$$\begin{aligned}
&a_{11} \longrightarrow \frac{T\hat{\gamma}_0}{1 - \hat{\gamma}_0} \quad , \quad a_{12} \longrightarrow \frac{2T e^{-(1-\hat{\gamma}_0)z}}{1 + \hat{\gamma}_0} \quad , \quad a_{22} \longrightarrow \frac{T\hat{\gamma}_0}{1 + \hat{\gamma}_0} \\
&b_{\{^1_4\}} \longrightarrow \kappa(\pm 1 - \hat{\gamma}_0) \quad , \quad b_{\{^2_3\}} \longrightarrow \pm 2\kappa\, e^{-(1 \mp \hat{\gamma}_0)z}
\end{aligned} \tag{4.40}$$

We can now calculate large time limits of the density matrix coefficients from (3.52):

$$A \to a_{22} \quad , \quad B \to -b_1/2 \quad , \quad C \to \frac{b_2^2}{16a_{11}} \tag{4.41}$$

These coefficients are independent of the initial conditions, which might be expected since the dissipation is acting to damp out any late time dependence on these initial conditions. So we have

$$S_{lin} = -\sqrt{\frac{C}{A}} \to \frac{-\kappa^2 e^{-z}}{2\gamma_0 T} \tag{4.42}$$

so that from (4.13, 4.37)

$$S \to r + 1 + \ln \frac{T\gamma_0}{\kappa^2} \tag{4.43}$$

Zero temperature

We are interested in the case $T = 0$, since eliminating temperature has more potential to shed light on the action of the environment due to quantum effects only. In this case, (A.20) gives [with $\rho(x) \equiv P(1/x)$]

$$\nu(s, s') = \frac{2\gamma_0}{\pi}\rho'(s - s') \quad \text{i.e.} \quad \nu(\zeta, \zeta') = \frac{2\gamma_0 \kappa^2}{\pi}\rho'(\zeta - \zeta') \tag{4.44}$$

We could now use this principal part prescription to evaluate the a_{ij}'s. This approach is not particularly straightforward and we do not pursue it here. However, if we write the noise in its primitive form as the usual integral over frequency then we can leave this frequency integration until last after the time integrations have been done. Note that this is a more naive approach than the one we will follow in chapter 5, but we show it here to investigate what value it might have.

So we refer to (3.49, A.13), swapping the limits of integration to write

$$\begin{aligned} a_{11} &= \frac{\gamma_0}{\pi \operatorname{sh}^2 z}\int_0^{\hat{\omega}_{\max}} d\hat{\omega}\, \hat{\omega} \coth \frac{\hat{\omega}\kappa}{2T} \int_0^z d\zeta \int_0^z d\zeta' e^{\hat{\gamma}_0(\zeta+\zeta')} \operatorname{sh}(z-\zeta) \operatorname{sh}(z-\zeta') \cos\hat{\omega}(\zeta - \zeta') \\ &= \frac{\gamma_0}{2\pi \operatorname{sh}^2 z}\int_0^{\hat{\omega}_{\max}} d\hat{\omega}\, \hat{\omega} \coth \frac{\hat{\omega}\kappa}{2T}\, I_{11} \end{aligned} \tag{4.45}$$

where

$$\begin{aligned} I_{11} \equiv\ & \left\{\hat{k}^2 - \hat{\omega}^2 + 2e^{2\hat{\gamma}_0 z} + \left(1 + \hat{\gamma}_0^2 + \hat{\omega}^2\right) \operatorname{ch} 2z \right. \\ & \left. - 4e^{\hat{\gamma}_0 z}[\cos\hat{\omega}z\ (\operatorname{ch} z + \hat{\gamma}_0 \operatorname{sh} z) + \hat{\omega}\sin\hat{\omega}z \operatorname{sh} z] + 2\hat{\gamma}_0 \operatorname{sh} 2z\right\} \Big/ \\ & \left[\hat{k}^4 + 2\hat{\omega}^2\left(1 + \hat{\gamma}_0^2\right) + \hat{\omega}^4\right] \end{aligned} \tag{4.46}$$

Similarly

$$a_{12} = \frac{\gamma_0 e^{-\hat{\gamma}_0 z}}{\pi \operatorname{sh}^2 z}\int_0^{\hat{\omega}_{\max}} d\hat{\omega}\, \hat{\omega} \coth \frac{\hat{\omega}\kappa}{2T}\, I_{12} \tag{4.47}$$

where

$$
\begin{aligned}
I_{12} \equiv & \left\{-2 \operatorname{ch} z\left(1+e^{2\hat{\gamma}_0 z}\right)-2\hat{\gamma}_0 \operatorname{sh} z\left(1-e^{2\hat{\gamma}_0 z}\right)\right. \\
& +e^{\hat{\gamma}_0 z} \cos \hat{\omega} z\left[3+\hat{\gamma}_0^2+\hat{\omega}^2+\left(\hat{k}^2-\hat{\omega}^2\right) \operatorname{ch} 2z\right]+2\hat{\omega} e^{\hat{\gamma}_0 z} \sin \hat{\omega} z \operatorname{sh} 2z\Big\} / \\
& \left[\hat{k}^4+2\hat{\omega}^2\left(1+\hat{\gamma}_0^2\right)+\hat{\omega}^4\right]
\end{aligned} \tag{4.48}
$$

and

$$
a_{22}=\frac{\gamma_0 e^{-2\hat{\gamma}_0 z}}{2\pi \operatorname{sh}^2 z} \int_0^{\hat{\omega}_{\max}} d\hat{\omega}\, \hat{\omega} \coth \frac{\hat{\omega}\kappa}{2T} I_{22} \tag{4.49}
$$

where

$$
\begin{aligned}
I_{22} \equiv & \left\{2+e^{2\hat{\gamma}_0 z}\left[\hat{k}^2-\hat{\omega}^2+\left(1+\hat{\gamma}_0^2+\hat{\omega}^2\right) \operatorname{ch} 2z-2\hat{\gamma}_0 \operatorname{sh} 2z\right]\right. \\
& +4e^{\hat{\gamma}_0 z}\left[\cos \hat{\omega} z\left(-\operatorname{ch} z+\hat{\gamma}_0 \operatorname{sh} z\right)-\hat{\omega} \sin \hat{\omega} z \operatorname{sh} z\right]\Big\} / \\
& \left[\hat{k}^4+2\hat{\omega}^2\left(1+\hat{\gamma}_0^2\right)+\hat{\omega}^4\right]
\end{aligned} \tag{4.50}
$$

With $T=0$ the coth term is set to one. Then in all cases a_{ij} starts at zero at $z=0$; for low dissipation a_{11}, a_{22} quickly climb to similar constant values while a_{12} climbs briefly but then rapidly decreases to zero. This behaviour quantitatively matches the large time limits of the white noise a_{ij}'s in (4.40), even though the two calculations were done quite differently. The asymptotic value of a_{11} increases in even steps as we increase $\hat{\omega}_{\max}$ exponentially. So we can make a_{11} arbitrarily large by taking a large enough cutoff, so that it will always dominate D.

In that case, with $\hat{\gamma}_0 \ll 1$ we have at late times, using the b_i's in (4.40)

$$
A \rightarrow a_{22} \quad , \quad B \rightarrow -b_1/2 \quad , \quad C \rightarrow \frac{b_2^2}{16 a_{11}} \tag{4.51}
$$

Again the coefficients are independent of the initial conditions. Since b_2 is unchanged from the high temperature case and a_{11}, a_{22} tend toward constants, we now can say

$$
S_{lin} \rightarrow \frac{-\kappa e^{-z}}{2\sqrt{a_{11} a_{22}}} \tag{4.52}
$$

and so again from (4.13, 4.37)

$$
S \rightarrow r+1+\ln \frac{\sqrt{a_{11} a_{22}}}{\kappa} \tag{4.53}
$$

4.3.3 Scalar field in de Sitter space

In chapter 5 we will consider a phenomenological model for inflation. This model evolves the density matrix for a massless scalar field minimally coupled to gravity in a de Sitter spacetime. Such a field is chosen since it models the fluctuations in inflationary cosmology, as well as being a generally solvable squeezed system. It has lagrangian

$$L(\eta) = \frac{1}{2}\left[\dot{x}^2 + \frac{2}{\eta}\dot{x}x - \left(k^2 - \frac{1}{\eta^2}\right)x^2\right] \tag{4.54}$$

which arises from (B.13). We also use a spectral density of the form

$$I(\omega, \eta, \eta') = \frac{2\gamma_0}{\pi H}\frac{\omega}{\sqrt{\eta\eta'}} \tag{4.55}$$

so that $c(\eta) = 1/\sqrt{-H\eta}$. This form of spectral density will be justified in chapter 5, although for now we note that it does not make the equation of motion for X any harder to solve than if we had used a static coupling. Since γ_0/H is dimensionless we rewrite it as c [not to be confused with $c(\eta)$]. Incorporating the bath gives the equivalent oscillator with $M = 1, \mathcal{E} = 1/\eta$ and frequency, from (3.41),

$$\Omega_{eff}^2 = k^2 - \frac{1 + c^2}{\eta^2} \tag{4.56}$$

Also we choose $\kappa = k$ to simplify the equation of motion. With $z = k\eta$ we can write this together with its initial conditions from (3.20, 3.22, 3.23) as

$$X''(z) + \left(1 - \frac{2 + c^2}{z^2}\right)X = 0$$

$$X(z_i) = 1 \quad , \quad X'(z_i) = -i - 1/z_i \tag{4.57}$$

where $z < 0$. The solution of this equation can be constructed using Bessel functions whose index is a function of c; however since we are interested in small c we take the solution to be approximately that of the same equation but with c set to zero. This simplifies things greatly:

$$X(z) = \left(1 + \frac{i}{2z_i}\right)f(z) + \frac{i}{z_i}f^*(z) \tag{4.58}$$

where

$$f(z) \equiv \left(1 - \frac{i}{z}\right)e^{i(z_i - z)} \tag{4.59}$$

We can further simplify X by using a very early initial time, setting $z_i \to -\infty$. We also disregard the phase in the resulting expression for X, since this is not expected to make any difference to physical quantities. In this case we obtain a new function which we rename X:

$$X(z) \rightsquigarrow \left(1 - \frac{i}{z}\right)e^{-iz} \tag{4.60}$$

The Bogoliubov coefficients can now be found from (4.6):

$$\alpha = \left(1 - \frac{i}{2z}\right) e^{-iz} \quad , \quad \beta = \frac{-i}{2z} e^{-iz} \tag{4.61}$$

and so from (4.4) at late times

$$r \to -\ln|z| \tag{4.62}$$

This result was also obtained in [12] using a different formalism.

First we calculate the b_i's. Since we are only interested in late times we can work to leading order in z (although with hindsight we include some next higher order terms which will be needed later). Using (3.49) we find

$$\begin{aligned} b_1 &= ck/z + kz + O(z^3) \\ b_{\{{2 \atop 3}\}} &= \mp k|z|^{1\mp c}|z_i|^{\pm c} \\ b_4 &= (c+1)k/z_i + kz^3/3 + O(z^5) \end{aligned} \tag{4.63}$$

and for the a_{ij}'s we need the following expressions, calculated from (4.60):

$$\begin{aligned} \frac{\text{Im } [X(z)X^*(\zeta)]}{\text{Im } X(z)} &\simeq \frac{(1 - z/\zeta)\cos(\zeta - z) - (z + 1/\zeta)\sin(\zeta - z)}{\cos z + z \sin z} \\ \frac{\text{Im } [X(\zeta)]}{\text{Im } X(z)} &\simeq \frac{\frac{\cos\zeta}{\zeta} + \sin\zeta}{\frac{\cos z}{z} + \sin z} \end{aligned} \tag{4.64}$$

$$\exp\left(\hat{\gamma}_0 \int_{z_i}^{\zeta} \frac{c^2(\zeta'')}{M} d\zeta''\right) = (\zeta/z_i)^{-c} \quad ; \quad \exp\left(-\hat{\gamma}_0 \int_{\zeta}^{z} \frac{c^2(\zeta'')}{M} d\zeta''\right) = (z/\zeta)^{c} \tag{4.65}$$

High temperature

Write

$$\begin{aligned} \nu &= 4cc^2(s)T\delta(s - s') \\ &= \frac{-4ck^2T}{\zeta}\delta(\zeta - \zeta') \end{aligned} \tag{4.66}$$

We calculate a_{11} here and leave the details of a_{12}, a_{22} to appendix C. First, (3.49) gives

$$\begin{aligned} a_{11} &= \frac{1}{2k^2}\int_{z_i}^{z} d\zeta \int_{z_i}^{z} d\zeta' \left(\frac{\zeta}{z_i}\right)^{-c} \frac{\text{Im } [X(z)X^*(\zeta)]}{\text{Im } X(z)} \frac{4ck^2T}{-\zeta} \delta(\zeta - \zeta') \left(\frac{\zeta'}{z_i}\right)^{-c} \frac{\text{Im } [X(z)X^*(\zeta')]}{\text{Im } X(z)} \\ &= 2cT \int_{z_i}^{z} d\zeta \left(\frac{\zeta}{z_i}\right)^{-2c} \left(\frac{\text{Im } [X(z)X^*(\zeta)]}{\text{Im } X(z)}\right)^2 \frac{1}{-\zeta} \end{aligned} \tag{4.67}$$

We wish to investigate the dependence of the a_{ij}'s on z as $z \to 0$, and so we now separate each integral into a sum of two parts. The first is gotten by integrating in to some constant λ close to z, while the second integral contains the z upper limit:

$$a_{11} = 2cT\left[\int_{z_i}^{\lambda} + \int_{\lambda}^{z}\right] d\zeta \left(\frac{\zeta}{z_i}\right)^{-2c} \left(\frac{\text{Im } [X(z)X^*(\zeta)]}{\text{Im } X(z)}\right)^2 \frac{1}{-\zeta} \tag{4.68}$$

It's only necessary to work to leading order in z. We need the following expressions: when only $z \approx 0$ we have the z dependence in the integrands as

$$\begin{aligned}\frac{\text{Im }[X(z)X^*(\zeta)]}{\text{Im }X(z)} &= \cos\zeta - \sin\zeta/\zeta + O(z^2) \equiv f_1(\zeta) + O(z^2) \\ \frac{\text{Im }[X(\zeta)]}{\text{Im }X(z)} &\simeq z(\cos\zeta/\zeta + \sin\zeta) \equiv z f_2(\zeta) \end{aligned} \tag{4.69}$$

while if both $z, \zeta \approx 0$ then to leading order

$$\begin{aligned}\frac{\text{Im }[X(z)X^*(\zeta)]}{\text{Im }X(z)} &\simeq (-\zeta^2 + z^3/\zeta)/3 \\ \frac{\text{Im }[X(\zeta)]}{\text{Im }X(z)} &\simeq z/\zeta \end{aligned} \tag{4.70}$$

We are now in a position to write

$$\begin{aligned} a_{11} &\propto cT\left[\int_{z_i}^{\lambda} d\zeta\ |\zeta|^{-2c-1} f_1^2(\zeta) + \int_{\lambda}^{z} d\zeta\ |\zeta|^{-2c-1}(-\zeta^2 + z^3/\zeta)^2/9\right] \\ &= cT\left(O(1) + O|z|^{-2c+5}\right) \\ &= cT\ O(1) \end{aligned} \tag{4.71}$$

since we have taken c to be small. A similar approach gives the following results for a_{12}, a_{22} (details can be found in appendix C):

$$\begin{aligned} a_{12} &= cT\ O|z|^{c+1} \\ a_{22} &= cT\ O(1) \end{aligned} \tag{4.72}$$

Since T is large, a_{11} dominates D while a_{22} dominates A; so we have

$$A \rightarrow a_{22} \quad , \quad B \rightarrow -b_1/2 \quad , \quad C \rightarrow \frac{b_2^2}{16a_{11}} \tag{4.73}$$

These of course have the same form as for the static oscillator case, although it's by no means clear whether such a fact could have been deduced from the general expressions for the a_{ij}'s. We now have

$$S_{lin} \rightarrow \frac{-|b_2|}{4\sqrt{a_{11}a_{22}}} = O|z|^{1-c} \tag{4.74}$$

and using (4.13, 4.62) we can write

$$S \rightarrow (1-c)r + \text{constant} \tag{4.75}$$

Finite temperature

Here we leave the frequency integration until last as was done for the static oscillator. The integrals can then be done in the same way as in the last section, although some subtleties are present in this case (details are in appendix C). We finally obtain

$$\begin{aligned} a_{11} &= ck\, O(1) \\ a_{12} &= ck\, O|z|^{1/2} \\ a_{22} &= ck\, O(z) \end{aligned} \tag{4.76}$$

Again since we integrate over $\hat{\omega}$, a_{11} will be large and so dominate D. Again we'll have

$$A \rightarrow a_{22} - \frac{a_{12}^2}{4a_{11}} \quad , \quad B \rightarrow -b_1/2 \quad , \quad C \rightarrow \frac{b_2^2}{16a_{11}} \tag{4.77}$$

and so

$$S_{lin} \rightarrow O|z|^{1/2-c} \tag{4.78}$$

and so with (4.13, 4.62) we have

$$S \rightarrow (1/2 - c)r + \text{constant} \tag{4.79}$$

4.4 Discussion

In this chapter we have calculated the entropy of a static inverted oscillator and that of a scalar field mode in de Sitter space.

Bearing in mind that our results should be doubled since each of our modes was split into sine and cosine components (see appendix B), we might expect a result of $S = r$ if we are to agree with previous work described in the introduction to this chapter.

For the static inverted oscillator, in both temperature regimes for low coupling we obtain $S \rightarrow (1 - \hat{\gamma}_0)r$ + constant. In the de Sitter case, the high temperature result is $S \rightarrow (1 - c/2)r$ + constant. Thus these three examples certainly do confirm the ad hoc approaches to calculating entropy that have been used by others. However at lower temperatures the de Sitter entropy is $S \rightarrow (1/2 - c)r$ + constant. This last result requires us to look more closely at A and C which together give the entropy. First write from (4.12, 4.13), and neglecting the added constants which are always implied:

$$S \rightarrow \frac{1}{2}\ln A - \frac{1}{2}\ln C$$

When the system-environment coupling is small, all of the above cases give $-1/2 \ln C \rightarrow r$. We suspect this might be true for more general examples, if the dominant contribution to C always comes from b_2 in the late time limit. But the behaviour of A needs to be examined. For the static inverted oscillator and high temperature de Sitter examples $1/2 \ln A \to$ constant, while the finite temperature de Sitter case stands out in that there we find $1/2 \ln A \to -r/2$. In all cases, A and C have similar dependence on $a_{11} \to b_4$. A appears to have a strong dependence on the coupling, unlike C which is relatively immune to changes in the coupling.

It may well be that the standard result of $S \to r$ expresses the C dependence only and somehow misses any contribution of A. Remember that this result is normally derived using a coarse graining which sets all off-diagonal elements to zero in the reduced density matrix. It may well be that in doing so, these ad hoc approaches are discarding information about the system which should in fact be kept. It appears that A might give a contribution which can only be evaluated case by case. The finite temperature de Sitter result shows that this contribution can be important, and so the ad hoc result will not be reliable as it stands.

Chapter 5

Decoherence and fluctuations during inflation

The 1992 detection by COBE of angular variations in the cosmic microwave background radiation [26] gave new impetus to the idea that modern galactic structure might have been caused by primordial density fluctuations of a quantum scalar field.

By this time, the essence of the theory of inflation originally proposed by Guth [27] in 1980 had become widely accepted. This scenario asserts that the universe went through a de Sitter phase in its very early expansion. This inflationary phase is driven by a quantum scalar field with a potential $V(\Phi)$ which can assume different forms satisfying the "slow roll" conditions. Inflationary scenarios fall into two categories. The first includes Old and New Inflation, in which the scalar field (the "inflaton") is assumed to be in thermal equilibrium with the rest of the universe [28]. The universe obeys the standard hot big bang cosmology both before and after inflation. The second scenario is Chaotic Inflation, where the inflaton is assumed to be only very weakly coupled to other fields. This relaxing of the coupling strength allows one to choose initial conditions that are far from equilibrium. The standard hot big bang cosmology now only applies after the reheat stage of inflation.

Along with its solutions to the well known problems of Big Bang cosmology, inflation predicts an amplification of quantum fluctuations as the universe expands. It does this as follows. Divide the universe's early history into two main epochs: inflation, during which the scale factor is say $a = e^{Ht}$, followed by reheating, where $a \propto t^n$ where $n = 1/2$ in the initially radiation dominated phase, followed by $n = 2/3$ for later matter domination.

The linearised quantum fluctuations of the inflaton, thought to be responsible for generating density fluctuations, can be shown to be described by a free, massless minimally coupled scalar field. This field comprises modes of wavevector $\boldsymbol{k}$ whose physical wavelength is just the scale factor times the coordinate wavelength, i.e. $2\pi a/k$. Suppose we plot this physical wavelength versus cosmic time, and on the same graph superimpose a plot of horizon size, $a/a_{,t}$. The resulting plot is shown in figure 5.1. The horizon size is

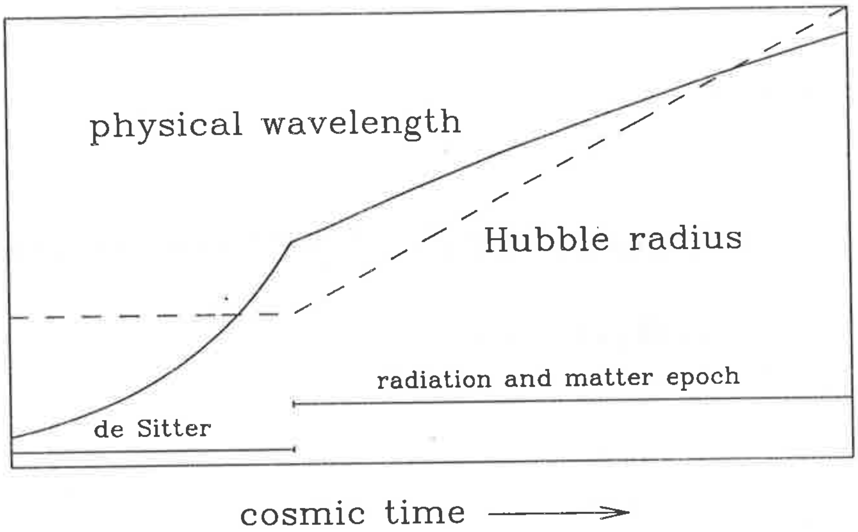


Figure 5.1: Physical wavelength and horizon size. Both axes are linear.

constant during the de Sitter phase, but increases with time in the radiation and matter dominated phases. At some time the physical wavelength will be equal to the horizon size. For modes that ultimately give rise to features typical in scale of today's observable universe, this time is about ten Hubble times after inflation begins (i.e. a cosmic time of $10/H$). Because inflation is thought to continue for at least 60 Hubble times, it's apparent that modes of interest to us today really will become longer than the horizon before inflation ends. (See [29] for a summary of the COBE data in relation to what modes we need to consider).

What is the significance of this period for which the wavelength is greater than the horizon size? To answer this question, consider the lagrangian for a free massless, minimally coupled scalar field in conformal time, equation (B.20) (now with $\dot{} = d/d\eta$):

$$L(\eta) = \frac{1}{2}\left[\dot{q}^2 - 2\frac{\dot{a}}{a}\dot{q}q - \left(k^2 - \frac{\dot{a}^2}{a^2}\right)q^2\right] \tag{5.1}$$

From (3.20, 3.22) the equation of motion for q is

$$\ddot{q} + \left(k^2 - \frac{\ddot{a}}{a}\right) = 0 \tag{5.2}$$

which becomes, in de Sitter space,

$$\ddot{q} + \left(k^2 - \frac{2}{\eta^2}\right)q = 0 \tag{5.3}$$

Now at early times such that $k\eta < -\sqrt{2}$, the solution to this equation will be sinusoidal. However for times later than this the oscillations stop, and instead the solution becomes amplified; and it's approximately just at this time that the wavelength becomes larger than the horizon size (which from setting $2\pi a/k$ equal to $a/a_{,t}$ for de Sitter, occurs at $k\eta = -2\pi$).

So, the scenario is that at $k\eta \approx -1$, a mode "leaves the horizon", and as it does its amplitude grows; this corresponds to particle creation. Quantum fluctuations in the scalar field modes which reenter the horizon today have been shown able to perturb the CMBR through the Sachs-Wolfe effect [30].

There have been many studies of the free quantum scalar field in a de Sitter phase (see for example [31]), in the hope of shedding light on how density perturbations might be created. In Old and New Inflation the inflaton is assumed to be in thermal equilibrium with the rest of the universe, and this implies a coupling to an external environment, so that the field is not free at all. Yet despite this there has been little work on the dynamics of fluctuation creation for such a coupling.

One such model was investigated by Cornwall and Bruinsma [32], who represented the zero mode of New Inflation by an inverted oscillator bilinearly coupled to a thermal bath. This bath was comprised of modes of another scalar field, conformally coupled to a background de Sitter spacetime. Although they did not attempt to study the bath's effect on the scalar field perturbations, they did consider it plausible that the bath might introduce a damping effect which would affect the amplitude of the generated perturbations. In principle this could lead to an easing of the fine tuning problem that has plagued inflationary theories.

In this chapter we wish to investigate the effect such a thermal bath might have on the scalar field fluctuations. To this end we study a massless, minimally coupled scalar field of wavevector $\boldsymbol{k}$, bilinearly coupled to a conformally coupled scalar field bath. This model of the bath is similar to the one used by Cornwall and Bruinsma, but with two modifications: the addition of a phenomenological damping term in the system lagrangian, and a time dependent system-bath coupling. Our main aim will be to calculate the statistical properties of the quantum fluctuations of our system, and to compare these with the standard results for the free system. Any significant deviation from the standard results would have implications for our current understanding of the how a quantum scalar field might generate density fluctuations. The results we derive will also be useful in further investigating the entropy issue of the last chapter, as well as showing any quantum to classical transition that might occur as a result of the system's coupled evolution.

First we state our lagrangian, which includes a phenomenological term whose origin we justify. We then start with a vacuum, and define two linear combinations of the mode amplitude and momentum: these are the super- and subfluctuants as used extensively by [20]. We calculate the fluctuations in these new variables as the system interacts with its environment. The concept of a coherence length is defined, and we calculate this for the super- and subfluctuants, as well as the entropy produced by the system's evolution.

5.1 The model

Our system will comprise a fluctuation mode of the inflation field, Q, evolving in a Robertson-Walker universe with metric

$$ds^2 = dt^2 - a^2(t) d\boldsymbol{x}^2 \tag{5.4}$$

In the following calculations we will always take:

$$\begin{aligned} {}_{,s} &\equiv d/ds \quad \text{cosmic time} \\ {}^{\cdot} &\equiv d/d\tau \quad \text{conformal time} \end{aligned} \tag{5.5}$$

while a prime is used to denote an alternative variable, such as q, q', and also for a derivative with respect to z (where z will be written explicitly). We will work in conformal time $\eta = \int dt/a(t)$ and the scaled system variable $q = aQ$. The system plus bath that we will study has the action

$$\begin{aligned} S[q, \boldsymbol{r}] &= S[q] + S_E[\boldsymbol{r}] + S_{\text{int}}[q, \boldsymbol{r}] \\ &= \int_{\eta_i}^{\eta} d\tau \left\{ \frac{1}{2} \left[\dot{q}^2 - 2\frac{\dot{a}}{a}\dot{q}q - \left(k^2 - \frac{\dot{a}^2}{a^2} - 2\gamma_0 (1+p) a^{2p-1} \dot{a} \right) q^2 \right] \right. \\ &\quad \left. + \sum_k \left[\frac{1}{2} \left(\dot{r}_k^2 - k^2 r_k^2 \right) \right] + \sum_k \left[-a^p(\tau) q \, r_k \right] \right\} \end{aligned} \tag{5.6}$$

The system action is the same as (5.1) but with an extra term in the potential. This will be necessary to obtain the appropriate semiclassical equation of motion for the system. The bath action is that of a conformally coupled massless scalar field in an FRW universe, i.e. (B.14), with the field rescaled analogously to q. This type of bath was considered by Cornwall and Bruinsma who investigated the zero mode of the inflaton, and who also used an influence functional approach to investigate viscosity and diffusion during inflation. However because they added a different surface term to their bath action (see appendix B), this action was time dependent, leading to a more complicated open system than ours. Our interaction is more general than Cornwall and Bruinsma's because we consider a time dependent coupling; we will see shortly that such a coupling is necessary for a more realistic model.

The motivation for including the extra potential term in the system action is as follows. Suppose we write down (3.38):

$$\ddot{u}+\left(\frac{\dot{M}}{M}+\frac{2\gamma_0 c^2}{M}\right)\dot{u}+\left(\Omega^2+\frac{\dot{M}\mathcal{E}}{M}+\dot{\mathcal{E}}+\frac{2\gamma_0 c\dot{c}}{M}\right)u=0 \tag{5.7}$$

Referring to (3.10, 3.25), we see that since the above is an equation of motion for u, it's also an equation of motion for Σ. But setting $q=q'$ in (3.10) gives $\Sigma=q$; hence this is an equation of motion for q as well. The relevant parameters are, from (5.6),

$$M=1 \quad , \quad \mathcal{E}=-\dot{a}/a \quad , \quad \Omega^2=k^2-\dot{a}^2/a^2-2\gamma_0(1+p)a^{2p-1}\dot{a} \quad , \quad c=a^p \tag{5.8}$$

Then (5.7) becomes

$$\ddot{q}+2\gamma_0 a^{2p}\dot{q}+(k^2-\ddot{a}/a-2\gamma_0 a^{2p-1}\dot{a})q=0 \tag{5.9}$$

Now convert this to Q, t, to get

$$Q_{,tt}+(3H+2\gamma_0 a^{2p-1})Q_{,t}+k^2/a^2\, Q=0 \tag{5.10}$$

This is the effective semiclassical equation of motion of our system. For the choice $p=1/2$ it describes the usual classical equation of motion along with an additional constant damping term proportional to γ_0. Our system action was chosen in order to generate this simple damped effective semiclassical dynamics for the system. The choice of $p=0$, as used by Cornwall and Bruinsma, would be inappropriate since it leads to a damping coefficient which decays in time. Kolb and Turner have argued for a constant damping [33], and we have included this by setting $p=1/2$.

5.2 Quantum to classical transition

The description of cosmological particle production in terms of squeezing language, which we will use here, was first introduced by Grishchuk and Sidorov [19]. The dynamics of our system in (5.6) can be parametrised in terms of the language of squeezed states by defining the squeeze parameter and angle, r, ϕ and the rotation angle θ, as discussed in section 3.3.1. The phase space these refer to is defined by our variables q and $p \equiv \partial/\partial\dot{q}$.

A clearer picture of the open and closed dynamics of such a system can be obtained if we rotate the phase space axes so that the density matrix can be expressed in terms of so called super- and subfluctuant variables. (Alternatively, we are rotating the Wigner function in phase space so as to eliminate the cross term there). Call these variables u, v, expressed as real linear combinations of q, p. We fix the linear combinations such that one variable (u, the superfluctuant) grows exponentially while the other decays exponentially.

In the case of no coupling to the environment we proceed by expressing $\langle u^2\rangle, \langle v^2\rangle$ in terms of $\langle q^2\rangle, \langle qp+pq\rangle, \langle p^2\rangle$, and then substituting for these the standard squeezed state results [12]. This enables us to write

$$\langle u^2\rangle = \frac{\kappa e^{2r}}{2} \quad , \quad \langle v^2\rangle = \frac{e^{-2r}}{2\kappa} \tag{5.11}$$

These relations fix u, v in terms of q, p, and we now use the same transformation for the case of nonzero dissipation:

$$\begin{aligned} u &= -\kappa \sin\phi\, q + \cos\phi\, p \\ v &= \cos\phi\, q + \frac{\sin\phi}{\kappa} p \end{aligned} \tag{5.12}$$

What we wish to do is take a density matrix in position, (3.51), and write it in the u, v basis. Consider first of all calculating $\rho(u, u')$:

$$\rho(u,u') = \int \langle u|q\rangle\, \rho(q,q')\, \langle q'|u'\rangle\; dq\, dq' \tag{5.13}$$

We need $\langle u|q\rangle$. This can be found by solving the p.d.e which follows by quantising (5.12) and applying both sides to $\langle q|u\rangle$:

$$u\langle q|u\rangle = (-\kappa \sin\phi\; q - i\cos\phi\; \partial_q)\, \langle q|u\rangle \tag{5.14}$$

which has solution

$$\langle q|u\rangle = f(u) \exp \frac{i}{\cos\phi} \left[\frac{\kappa \sin\phi\, q^2}{2} + qu\right] \tag{5.15}$$

with $f(u)$ to be determined. We determine this by redoing this calculation with the roles of q and u interchanged; since $[v, u] = i$, we have

$$q\langle u|q\rangle = \left(\frac{-\sin\phi\, u}{\kappa} + i\cos\phi\partial_u\right) \langle u|q\rangle \tag{5.16}$$

Solving this determines $f(u)$ and allows us to finally write (up to a phase)

$$\langle q|u\rangle = \frac{1}{\sqrt{2\pi\cos\phi}} \exp \frac{i}{\cos\phi} \left[\frac{\kappa\sin\phi\, q^2}{2} + qu + \frac{\sin\phi\, u^2}{2\kappa}\right] \tag{5.17}$$

Similarly we find

$$\langle q|v\rangle = \sqrt{\frac{\kappa}{2\pi\sin\phi}} \exp \frac{i\kappa}{\sin\phi} \left[\frac{-\cos\phi\, q^2}{2} + qv - \frac{\cos\phi\, v^2}{2}\right] \tag{5.18}$$

Now, suppose we start with a gaussian density matrix as in (3.51). We can then easily change bases using (5.13, 5.17, 5.18) to get, with

$$\gamma \equiv \frac{\kappa}{2}\cot\phi \quad , \quad \sigma \equiv \frac{\sin^2\phi}{\kappa^2}\left[4AC + (B-\gamma)^2\right] \tag{5.19}$$

$$\lambda \equiv \frac{4AC + (4\gamma\sigma + B - \gamma)^2}{4\sigma^2} \tag{5.20}$$

$$\begin{aligned} \rho(u,u') &= \sqrt{\frac{C}{\pi\sigma\lambda}} \exp \frac{-1}{4\sigma\lambda} \left[A\Delta_u^2 + 2i(4\gamma\sigma + B - \gamma)\Delta_u\Sigma_u + 4C\Sigma_u^2\right] \\ \rho(v,v') &= \sqrt{\frac{C}{\pi\sigma}} \exp \frac{-1}{4\sigma} \left[A\Delta_v^2 - 2i(4\gamma\sigma + B - \gamma)\Delta_u\Sigma_u + 4C\Sigma_v^2\right] \end{aligned} \tag{5.21}$$

where we have used sum and difference variables, e.g. $\Sigma_u \equiv (u + u')/2,\ \Delta_u \equiv u - u'$.

We can show that in the absence of a bath, these matrices reduce to the expected ones for a squeezed vacuum. First, in the q-representation the density matrix of a squeezed vacuum is known to be [25]

$$\rho(q,q') \propto \frac{-\kappa}{2} \frac{1 + e^{2i\phi}\tanh r}{1 - e^{2i\phi}\tanh r} \left(q^2 + q'^2\right) \tag{5.22}$$

If we write $\rho(q,q')$ in terms of sum and difference coordinates and compare with the definitions of A, B, C in (3.51), we find

$$\begin{aligned} A = C &= \frac{\kappa}{4} \frac{1 - \tanh^2 r}{1 - 2\cos 2\phi \tanh r + \tanh^2 r} \\ B &= \frac{\kappa \sin 2\phi \tanh r}{1 - 2\cos 2\phi \tanh r + \tanh^2 r} \end{aligned} \tag{5.23}$$

Substituting these into (5.21) gives (after much computation)

$$\begin{aligned} \rho(u,u') &= \frac{e^{-r}}{\sqrt{\pi\kappa}} \exp \frac{-e^{-2r}}{2\kappa} \left(u^2 + u'^2\right) \\ \rho(v,v') &= \sqrt{\frac{\kappa}{\pi}} e^{r} \exp \frac{-\kappa e^{2r}}{2} \left(v^2 + v'^2\right) \end{aligned} \tag{5.24}$$

These are the expected results, as can be seen by the fact that with p, q replaced by u, v respectively, they are produced when ϕ is set to zero in $\rho(p,p')$ and $\rho(q,q')$.

Measures of fluctuations and coherence

Returning to the general case of dissipation, the fluctuations in u and v are calculated from the density matrices:

$$\begin{aligned} \Delta u^2 &= \langle u^2 \rangle - \langle u \rangle^2 = \int u^2 \rho(u,u)\, du - \left[\int u\, \rho(u,u)\, du\right]^2 = \frac{\sigma\lambda}{2C} \\ \Delta v^2 &= \frac{\sigma}{2C} \end{aligned} \tag{5.25}$$

and both of these are just equal to 1/2 divided by the coefficient of $-\Sigma^2$ in their density matrix.

As a measure of coherence we note that a large coefficient of $-\Delta^2$ means that the density matrix is strongly peaked along its diagonal, i.e. there is very little coherence in the system. A measure of coherence was defined in [34] as a squared coherence length L^2, equal to 1/8 divided by the coefficient of $-\Delta^2$, so that a large L^2 means a high degree of coherence in the system. With this definition of L^2, (5.21) gives

$$L_u^2 = \frac{\sigma\lambda}{2A}, \quad L_v^2 = \frac{\sigma}{2A} \tag{5.26}$$

We can also relate the coherence lengths and fluctuations to the entropy of the system (see section 4.2 for definitions). We can write

$$\frac{L_u^2}{\Delta u^2} = \frac{L_v^2}{\Delta v^2} = S_{lin}^2 = \frac{C}{A} \tag{5.27}$$

(Note that the linear entropy is negative by definition so that it will increase with S, so that as S_{lin} increases, S_{lin}^2 decreases). Also the uncertainty relation for u, v becomes, from (5.19, 5.20, 5.25):

$$\Delta u^2 \Delta v^2 = \frac{1}{S_{lin}^2}\left[\frac{1}{4} + \frac{(4\gamma\sigma + B - \gamma)^2}{16AC}\right] \tag{5.28}$$

For the free field the last term in the square brackets is zero while $S_{lin} = -1$ (since $S = 0$), so that $\Delta u \Delta v = 1/2$.

5.3 Calculation of fluctuations and energy

We wish to calculate the fluctuations in amplitude and momentum of a mode of the minimally coupled massless system inflaton, of wavenumber k. These can also be used to calculate the average energy of a mode. We take $P \equiv Q_{,t}$ [12], in which case the following hold:

$$\Delta Q^2 = \Delta q^2/a^2 \quad , \quad \Delta P^2 = \Delta p^2/a^4 \tag{5.29}$$

We first calculate ΔQ^2, starting from

$$\Delta q^2 \equiv \langle q^2 \rangle - \langle q \rangle^2 \tag{5.30}$$

We start with an arbitrary gaussian initial density matrix (3.50) and propagate it to get (3.51), calculating the expectation values via

$$\langle q^n \rangle = \int q^n \rho(q, q, t)\, dq \tag{5.31}$$

Then the n^{th} moment of the mode amplitude is

$$\langle q^n \rangle = \int_{-\infty}^{\infty} q^n N e^{-4Cq^2}\, dq \tag{5.32}$$

so that we obtain

$$\Delta Q^2 = \frac{1}{8Ca^2} \tag{5.33}$$

Momentum fluctuations are calculated from $\rho(p,p')$, which itself comes from the amplitude in the usual way:

$$\begin{aligned}
\rho(p,p') &= \int\int dq\, dq'\, \langle p|q\rangle\, \rho(q,q')\, \langle q'|p'\rangle \\
&= \frac{1}{2\pi}\int\int dq\, dq'\, e^{-iqp+iq'p'}\, N e^{-A(q-q')^2-iB(q^2-q'^2)-C(q+q')^2} \\
&= \frac{N}{2\sqrt{B^2+4AC}} \exp\frac{-1}{4(B^2+4AC)}\left[A\Delta_p^2 - 2iB\Delta_p\Sigma_p + 4C\Sigma_p^2\right] \qquad (5.34)
\end{aligned}$$

Then using $\langle p^n\rangle = \int p^n \rho(p,p,t)\, dp$ we calculate

$$\Delta P^2 = \left(\frac{B^2}{2C} + 2A\right)\frac{1}{a^4} \tag{5.35}$$

Fluctuations with no environment

We can check that the above expressions reduce to the well known scale invariant ones for a coherent de Sitter vacuum as follows. From (5.22), take $\gamma_0 = 0$ with $\xi = k/2$, $\chi = 0$ in (3.50). With no noise the a_{ij}'s reduce to zero. The sum X of the Bogoliubovs was calculated in section 4.3.3 to be

$$X(z) = \left(1 - \frac{i}{z}\right)e^{-iz} \tag{5.36}$$

so that to leading order the b_i's become:

$$b_1 = b_2 = kz \ \ , \ \ b_3 = -kz \ \ , \ \ b_4 = k/z_i + kz^3/3 \tag{5.37}$$

Then using (3.52) we have in the small z limit

$$A = C = k^3\eta^2/4 \ \ , \ \ B \simeq -k^2\eta/2 \tag{5.38}$$

finally giving

$$\Delta P^2 = \frac{kH^4\eta^4}{2} \ \ , \ \ \Delta Q^2 = \frac{H^2}{2k^3} \tag{5.39}$$

as expected; these are the well-known expressions.

We can easily derive the fluctuations in u, v from (5.11) by calculating the squeeze parameter r. We simply insert (5.36) into (4.6), using (4.3) to finally give

$$e^{2r} \to 1/z^2 \tag{5.40}$$

so that the fluctuations are

$$\Delta u^2 = \frac{k}{2z^2} \ \ , \ \ \Delta v^2 = \frac{z^2}{2k} \tag{5.41}$$

again as expected: the superfluctuant u has increasing fluctuations as $z \to 0$ while those of the subfluctuant v are decreasing, and $\Delta u \Delta v = 1/2$.

Also, $A = C$ and so the linear entropy is -1, again as expected since this implies the usual entropy is zero. Hence from (5.27) we have coherence lengths of

$$L_u^2 = \frac{k}{2z^2} \quad , \quad L_v^2 = \frac{z^2}{2k} \tag{5.42}$$

The average energy per mode is obtained as follows. The energy density of the field is the $\eta\eta^{\text{th}}$ component of the stress-energy tensor:

$$T_{\eta\eta} = \frac{1}{2}\left[(\phi_{,\eta})^2 + \sum_i (\phi_{,i})^2\right] \tag{5.43}$$

and since we have expanded the field modes in terms of Q we can write the energy per mode as

$$\frac{1}{2}\left[(\partial_\eta Q)^2 + k^2 Q^2\right] \tag{5.44}$$

But $\partial_\eta Q = a\,\partial_t Q \equiv aP$, so that finally the mean energy is

$$\begin{aligned} \langle\text{energy}\rangle/\text{mode} &= \frac{1}{2}\left(a^2\langle P^2\rangle + k^2\langle Q^2\rangle\right) \\ &= \frac{1}{2}\left(a^2\Delta P^2 + k^2\Delta Q^2\right) \end{aligned} \tag{5.45}$$

5.3.1 de Sitter expansion with environment

We now turn our attention to the real case of interest. The spectral density is, from (3.34, 5.6) with $p = 1/2$:

$$\begin{aligned} I(\omega, \eta, \eta') &= \frac{2\gamma_0}{\pi}\omega\, a^{1/2}(\eta) a^{1/2}(\eta') \\ &= \frac{2c\,\omega}{\pi\sqrt{\eta\eta'}} \quad , \quad c \equiv \gamma_0/H \end{aligned} \tag{5.46}$$

Again choose $\kappa = k$ so that

$$z = k\eta = \frac{-k}{H}e^{-Ht} \tag{5.47}$$

To calculate X, use (5.6, 3.22, 3.23, 3.41) to give

$$\begin{aligned} &X''(z) + \left(1 - \frac{2 + 3c + c^2}{z^2}\right) X = 0 \\ &X(z_i) = 1 \\ &X'(z_i) = -i - 1/z_i \simeq -i \end{aligned} \tag{5.48}$$

which has solution

$$X(z) = a\sqrt{-z}\, J|z| + b\sqrt{-z}\, Y|z| \tag{5.49}$$

where by $J|z|, Y|z|$ we mean $J_\nu(-z), Y_\nu(-z)$ since z is always negative, and

$$\begin{aligned}
\nu &= 3/2 + c \\
a &= \frac{\pi}{2}\sqrt{|z_i|}(c_1 - ic_3) \\
b &= \frac{\pi}{2}\sqrt{|z_i|}(c_2 + ic_4) \\
c_1 &= \frac{-Y|z_i|}{2z_i} + Y'|z_i| \quad , \quad c_2 = \frac{J|z_i|}{2z_i} - J'|z_i| \\
c_3 &= Y|z_i| \quad , \quad c_4 = J|z_i| \qquad (5.50)
\end{aligned}$$

We wish to calculate $a_{11} \to b_4$ for $z \to 0$, i.e. long after exit where the physical wavelength is much bigger than the horizon. First we consider what choices can or must be made for some of the parameters.

Time scale of interest

We are interested in evolving the field mode of a coherent vacuum, and for such a field the initial density matrix coefficients are $\xi = k/2, \chi = 0$. Calculating the times of interest involves switching back and forth between cosmic and conformal time (multiplied by k): t and z. The initial time is

$$t_i = 0 \quad ; \quad z_i = -k/H \equiv -e^x \qquad (5.51)$$

Modes able to be probed by COBE [29] are those with coordinate wavelengths in the range $10^{-6}/H \to 10^{-3}/H$, so that we can take x in the range $7 \to 14$.

Horizon exit occurs when the Bessel solutions to (5.48) stop oscillating, that is, approximately when their arguments equal -1. So write $z_{cross} = -1$, or $t_{cross} = x/H$. Now, it's known that inflation must occur for perhaps 60 Hubble times, so that the final time of interest is

$$t \approx 60/H \quad ; \quad z \approx -10^{-20} \qquad (5.52)$$

so that we have constructed the time scale in both cosmic and conformal times. We will keep t and z arbitrary, although bearing in mind that they're of the above order of magnitude.

Calculation of a_{ij}'s and b_i's

As usual these are calculated from (3.49). The b_i's are straightforward; by considering large argument forms of the Bessel functions [see (D.36)] we are led to define

$$\left\{ \begin{matrix} (\sin) \\ (\cos) \end{matrix} \right\} \equiv \left\{ \begin{matrix} -\sin \\ -\cos \end{matrix} \right\} (|z_i| - \pi c/2) \qquad (5.53)$$

and so use (3.49) to write

$$\begin{aligned} b_1/k &= \frac{c+3/2}{z} + \frac{c_3 J'|z| - c_4 Y'|z|}{-c_3 J|z| + c_4 Y|z|} \\ b_{\{^2_3\}}/k &= \frac{\mp(z/z_i)^{\mp c}}{\sqrt{\pi|z|/2}\left(-(\sin)J|z| + (\cos)Y|z|\right)} \\ b_4/k &= \frac{c+1}{z_i} + \frac{c_1 J|z| + c_2 Y|z|}{-c_3 J|z| + c_4 Y|z|} \end{aligned} \tag{5.54}$$

When $|z| \ll 1$ we can write, with α_1 defined in (D.39):

$$b_1 \simeq \frac{kz}{1+2c} \quad , \quad b_2 \simeq e^{cx}\,\alpha_1\,kz \quad , \quad b_3 \simeq -e^{-cx}\,\alpha_1\,kz^{1+2c} \quad , \quad b_4 \simeq k\left[-\nu e^{-x} + \frac{(\sin)}{(\cos)}\right] \tag{5.55}$$

where in simplifying the expression for b_4 we have assumed $(\cos) \not\to 0$.

The a_{ij}'s are much more involved—we have left all details for appendix D, here simply quoting results for typical values of x, c:

$$a_{11} \simeq 10^{12}\,\alpha_1^2\,e^{-x}\,ck \quad , \quad a_{12} \simeq 10^4\,\alpha_1^2\,ck\,|z| \quad , \quad a_{22} \simeq 0.01\,\alpha_1^2\,ck\,|z|^{-1} \tag{5.56}$$

There are two regimes of interest to us: values of c for which a_{11} either does or does not dominate D in (3.52). Taking $\xi = k/2, \chi = 0$ allows us to write, from (3.52):

$$\begin{aligned} D &= k^2 + 4ka_{11} + b_4^2 \\ &\simeq k^2 + \frac{10^7 ck^2}{(\cos)^2} + k^2\left[-\nu e^{-x} + \frac{(\sin)}{(\cos)}\right]^2 \end{aligned} \tag{5.57}$$

and a_{11} will dominate this expression when the second term is much larger than the third, which occurs when c is larger than some critical value, $c_{crit} \simeq e^x/10^{12}$. As x increases from 7 to 14, c_{crit} increases exponentially from 10^{-10} to 10^{-7}. So we define two regimes of weak and strong coupling.

5.3.2 Weak coupling regime: $c \ll c_{crit}$

This is the case for which a_{11} doesn't dominate D. We wish to calculate the fluctuations and compare the results with the free field case, in order to establish whether a small but nonzero c gives different results to the free field case ($c = 0$).

By writing the expressions for $A \to D$ from (3.52) and making use of (5.55, 5.56) with

$$D \simeq k^2 + b_4^2 \simeq k^2/(\cos)^2 \tag{5.58}$$

we can compare the various terms and see which are dominant. This allows us to write

$$A \simeq a_{22} \simeq 0.01\,\alpha_1^2\,ck\,|z|^{-1} \quad , \quad B \simeq \frac{-b_1}{2} \simeq \frac{-kz}{2(1+2c)} \quad , \quad C = \frac{kb_2^2}{4D} \simeq e^{2cx}\,kz^2 \tag{5.59}$$

At this point refer to (5.19, 5.20, 5.25, 5.26, 5.33, 5.35) to calculate the fluctuations and coherence lengths. We first need γ, and for this we need the squeeze angle ϕ. This is calculated as follows. From (3.18) we infer that for large squeezing,

$$2\phi + \pi = \arg \frac{\alpha}{\beta} \tag{5.60}$$

Then using (4.6) we write

$$\frac{\alpha}{\beta} = \frac{\left(1 + \frac{i}{z}\right) X(z) + iX'(z)}{\left(1 - \frac{i}{z}\right) X(z) - iX'(z)} \tag{5.61}$$

From (5.49, D.37) we have (working to higher order with the benefit of hindsight)

$$X(z) \propto |z|^{-c-1} \left(1 + \frac{z^2}{4(\nu - 1)}\right) \tag{5.62}$$

Hence we can extract α, β using (4.6); taking their ratio allows us to conclude that to leading order,

$$\phi + \frac{\pi}{2} = \arg \left[-z + i\left(c + \frac{z^2(c-2)}{4c+2}\right)\right] \tag{5.63}$$

If c is large enough to dominate the imaginary part of this, we'll obtain

$$\phi = z/c \tag{5.64}$$

while if this condition isn't met, or in particular if $c = 0$, then

$$\phi = \frac{-\pi}{2} + \frac{z(2-c)}{4c+2} \tag{5.65}$$

so that for the free case ($c = 0$) we have[1]

$$\phi = z - \pi/2 \tag{5.68}$$

Even for the weak coupling case we will take c large enough so that (5.64) holds, in which case we can now proceed to calculate γ, σ, λ. For all the calculations we will assume that (cos) $\not\to 0$—allowing for this not to be necessarily so complicates the discussion

[1]Given ϕ for these cases, we can relate u, v to q, p via (5.12) as follows. For the free case

$$u = kq + zp \quad , \quad v = zq + p/k \tag{5.66}$$

while for $c \gg |z|$

$$u = \frac{-kz}{c}q + p \quad , \quad v = q + \frac{z}{kc}p \tag{5.67}$$

considerably, which we will not do here—and simply quote the results; we will also replace e^x by the equivalent k/H:

$$\phi = \frac{z}{c} \quad , \quad \gamma = \frac{ck}{2z} \quad , \quad \sigma = \frac{1}{4} \quad , \quad \lambda \simeq \frac{0.01 e^{2cx}\, ck^2\, z}{(\cos)^2} \tag{5.69}$$

so arriving at

$$\Delta u^2 \simeq \frac{0.01\, ck}{(\cos)^2 |z|} \quad , \quad \Delta v^2 \simeq \frac{1}{(k/H)^{2c} k z^2} \tag{5.70}$$

$$L_u^2 \simeq (k/H)^{2c} k z^2 \quad , \quad L_v^2 \simeq \frac{10\,(\cos)^2 |z|}{ck} \tag{5.71}$$

The fluctuations are growing in both variables while the coherence lengths decrease. Momentum and energy fluctuations are

$$\Delta P^2 \simeq \frac{0.02\, cH^4 |z|^3}{(\cos)^2 k^3} \quad , \quad \Delta Q^2 \simeq \frac{H^2}{(k/H)^{2c} k^3} \tag{5.72}$$

and for the contributions to the average energy per mode we have

$$a^2 \Delta P^2 \simeq \frac{0.02\, cH^2 |z|}{(\cos)^2 k} \quad , \quad k^2 \Delta Q^2 \simeq \frac{H^2}{(k/H)^{2c} k} \tag{5.73}$$

It can be seen that for small c, the momentum fluctuations are heavily dependent on c, unlike the amplitude fluctuations.

Entropy for weak coupling

The entropy as a function of the squeeze parameter for the free system is of interest, and can be calculated as follows. From (5.27, 5.59) we have

$$S_{lin} = -(k/H)^c (\cos)\, |z|^{3/2} \tag{5.74}$$

We now need r as a function of z for the free case. Equation (5.40) leads to

$$r \to -\ln |z| \tag{5.75}$$

so that (4.13, 5.74) together yield for each mode polarisation,

$$S \to 3r/2 + \text{constant} \tag{5.76}$$

5.3.3 Strong coupling regime: $c \gg c_{crit}$

In this regime a_{11} dominates D so that

$$D \simeq 4ka_{11} \simeq 10^7 ck^2/(\cos)^2 \tag{5.77}$$

In that case from (5.55, 5.56) we obtain

$$A \simeq a_{22} \simeq 0.01\,\alpha_1^2\, ck|z|^{-1} \quad , \quad B \simeq \frac{-b_1}{2} \simeq \frac{-kz}{2(1+2c)} \quad , \quad C = \frac{kb_2^2}{4D} \simeq 10^{-8}\, e^{2cx} kz^2 \tag{5.78}$$

which are very similar to the weak coupling results. Then ϕ, γ, σ are the same as for the weak coupling case while λ is different:

$$\lambda \simeq \frac{0.01 e^{2cx}\, ck^2\, z}{(\cos)^2} \tag{5.79}$$

In the same way the fluctuations and coherence lengths are also similar:

$$\Delta u^2 \simeq \frac{10^{-3} ck}{(\cos)^2 |z|} \quad , \quad \Delta v^2 \simeq \frac{10^7\, c}{(k/H)^{2c} kz^2} \tag{5.80}$$

$$L_u^2 \simeq \frac{10^{-9}\, (k/H)^{2c} kz^2}{c} \quad , \quad L_v^2 \simeq \frac{10 (\cos)^2\, |z|}{ck} \tag{5.81}$$

Momentum and energy fluctuations are

$$\Delta P^2 \simeq \frac{0.01\, cH^4 |z|^3}{(\cos)^2 k^3} \quad , \quad \Delta Q^2 \simeq \frac{10^7\, cH^2}{(k/H)^{2c} k^3} \tag{5.82}$$

while the contributions to the average energy per mode are

$$a^2 \Delta P^2 \simeq \frac{0.01\, cH^2 |z|}{(\cos)^2 k} \quad , \quad k^2 \Delta Q^2 \simeq \frac{10^7\, cH^2}{(k/H)^{2c} k} \tag{5.83}$$

and the entropy is again $S \to 3/2\, r +$ constant. For both weak and strong coupling regimes, we have found the super- and subfluctuants growing (i.e. these become seeds for galaxy formation) while their coherence lengths are decreasing, from which it appears that our model is becoming classical.

5.4 Discussion

Several results have been derived in this chapter.

- The entropy was calculated using a more sophisticated approach than that of the last chapter, insofar as we related the noise to the symmetric 2-point function for the conformally coupled bath field, which allowed us to use Vilenkin and Ford's regularised value of the 2-point function.

  We found that for both strong and weak couplings $S \to 3r/2 +$ constant, and (referring to the discussion of the last chapter) $-1/2 \ln C \to r$ while $1/2 \ln A \to r/2$. This differs from the result $S \to r/2$ calculated in the last chapter. There, in expressions (4.45, 4.47, 4.49) we assumed a static cutoff in the $\hat{\omega}$ integrations which was a first attempt at regularising the otherwise divergent integrals. On the other hand, in the current chapter we have done the calculation quite differently, assuming a time dependent cutoff to the integrations. In this case we've again obtained $-1/2 \ln C \rightarrow r$, but now we find $1/2 \ln A \rightarrow +r/2$. So apparently our choices of how to evaluate the a_{ij} integrals may be naive in some ways; it appears that the calculation of A is completely sensitive to the regularisation approach used, while that of C is not.

- For both couplings, the coherence lengths of both super- and subfluctuant variables are tending to zero with time [see (5.71, 5.81), and note that these assume the coupling is nonzero, so that we can't just set $c = 0$ to recover the free case there]. Compare this result with that for the free field, (5.42), which shows the superfluctuant coherence length L_u to be increasing, while the subfluctuant length L_v decreases. The environment-induced decay of coherence lengths in both variables is an indication of decoherence occurring.

- Also in both coupling regimes the uncertainties in both super- and subfluctuants are increasing with time (5.70, 5.80); compare this with the free case (5.41) in which the superfluctuant uncertainties grow while the subfluctuant's are suppressed. We interpret these environment-induced fluctuation growths as seeds for structure formation.

- For both couplings, fluctuations in the amplitude Q have similar forms to the free field (5.39, 5.72, 5.82), whereas the P fluctuations, while tending to zero as z^4 in the free case, tend to zero as z^3 in both coupled cases.

  Note that even for weak coupling, the P contributions to energy, $a^2 \Delta P^2$, go to zero at different rates in free and coupled cases (z^2 and z respectively), showing that even a weakly coupled bath can have a strong effect on observables long after Hubble crossing. This point is relevant to the gauge invariant theory of perturbations [35] in which P is the most important variable. This theory is equivalent to ours in the massless, minimally coupled case, and stresses the importance of the Bardeen variable, which is the same as our P (up to a numerical factor).

  The amplitude contributions to energy, $k^2 \Delta Q^2$, are constant for both free and coupled cases, so that they dominate the momentum ones both in the standard free theory and in the coupled theory.

Chapter 6

Thermal radiance for near-exponential dynamics

This chapter concerns work done towards the building up of a unified picture of the production of thermal radiance in different spacetimes.

Traditionally, the notion of particle creation has been linked with the concepts of field theory, and in particular the idea of modes being distorted either by a source, or by the spacetime geometry itself [36]. These distortions are quantified by our examining the relationship between field modes in the far past and far future, described by the Bogoliubov transformation. Modes which represent the vacuum in the far past will in general contain particles when viewed from a different basis in the far future.

Particle creation has been viewed within a variety of spacetimes. As early as the late fifties particle creation in an expanding universe was discussed by Takahashi and Umezawa [37] and later by Parker [38]. In the early seventies Hawking considered black hole particle creation [39] while Fulling, Davies and Unruh [40, 46] analysed the spectrum seen by a uniformly accelerated detector.

Previously it was thought that the production of thermal radiation went hand in hand with the existence of an event horizon [41], and certainly the previous examples all possess this. For example, in the black hole case, the geometric approach considers the modes around a collapsing star as being embedded in the collapsing geometry, and as such takes the event horizon as having a global significance. For accelerated detectors, it was previously assumed that the entropy increase associated with the detector's seeing thermal radiation was due to its being unable to access information outside the Rindler wedge. However it was later shown that accelerated frames not possessing event horizons *can* be constructed containing radiation [42], and similarly cosmological examples have been found [43, 44].

The work of this chapter is a continuation of earlier work by Hu et al. [2, 45] and is aimed at further developing this idea, by approaching particle creation from a stochastic viewpoint. Our basic viewpoint is that vacuum fluctuations of a quantum field mode are amplified by an interaction with a bath of oscillators. For each system we consider—accelerating mirror, collapsing spherical mass and cosmological examples—we calculate the noise and dissipation of the scalar field environment. By comparing these expressions with the corresponding ones for a thermal bath of static oscillators, we are able to extract the temperature that such a bath would need to have in order to reproduce the noise and dissipation seen by the fields in our examples. This temperature, together with the effective spectral density (which we also calculate, but which is not central to our discussion) completely characterises the effect of the environment on the system of interest.

Such a characterisation enables known results to be rederived, such as the Hawking temperature of a Schwarzschild black hole, or the Davies-Unruh temperature measured by an accelerated observer in a Minkowski vacuum [2]. But it goes further, in that because it doesn't assume the existence of a horizon (together with the geometrical interpretation that this attracts), it can be extended to more arbitrary cases of spacetime or detector motion. Thus our method differs from traditional approaches which assume equilibrium thermodynamics and stress the mode distortions.

We note that cases which give thermal spectra involve an exponential scale transformation, and we show that by perturbing this transformation the resulting perturbed spectrum can be derived. Note that in principle we can treat arbitrary motions and are not restricted to perturbations around the usual exponential ones. But in practice arbitrary motion is technically difficult to treat, and does not cast any new light on the principle of the way we are treating the problem.

The disparate perturbed cases of accelerating mirrors, black holes or expanding spacetimes can all be characterised by a parameter h which measures the departure from the straightforward exponential transformation. This parameter then appears in the perturbed spectrum.

6.1 Moving mirror in Minkowski space

We first treat the motion of a mirror following a trajectory $x = z(t)$ in a zero temperature scalar field, which in turn is coupled to a detector moving at constant velocity. We can calculate the noise and dissipation produced by the field, and write these together from (3.32):

$$\zeta \equiv \nu + i\mu = \int_0^\infty dk\, I(k,s,s')X(s)X^*(s') \tag{6.1}$$

Given this form, we can equate it with the standard form

$$\zeta = \int_0^\infty dk\ I_{\textit{eff}}(k,\Sigma)\ \left[C(k,\Sigma)\cos k\Delta - i\sin k\Delta\right] \tag{6.2}$$

where

$$\Sigma = (t+t')/2 \quad , \quad \Delta = t - t'$$

This is the form of ζ which, with the function C replaced by $\coth\frac{k}{2T}$, would describe a thermal bath of static oscillators each in a coherent state. We will show that the unknown function C does indeed have the form of a coth, and can then deduce the temperature of the radiation seen by the detector. Here $I_{\textit{eff}}(k,\Sigma)$ is the effective spectral density, also to be determined. We can always write ζ in this way since ν is even in Δ while μ is odd. By equating the real and imaginary parts of the two forms of ζ and Fourier inverting, we obtain

$$\begin{aligned} I_{\textit{eff}}C &= \frac{1}{\pi}\int_{-\infty}^\infty d\Delta\ \cos k\Delta\ \nu(\Sigma,\Delta) \\ I_{\textit{eff}} &= \frac{-1}{\pi}\int_{-\infty}^\infty d\Delta\ \sin k\Delta\ \mu(\Sigma,\Delta) \end{aligned} \tag{6.3}$$

These expressions will be used throughout this chapter to calculate C for the various cases of induced radiance that we consider.

6.1.1 Thermal Radiance

We start by obtaining the field $\phi(t,x)$, so must solve

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0 \tag{6.4}$$

subject to the condition that the field vanishes on the mirror:

$$\phi\Big(t, z(t)\Big) = 0 \tag{6.5}$$

For a general mirror path this equation is difficult to solve; however we can exploit the invariance of the wave equation under a conformal transformation to change to simpler coordinates. We follow the treatment of [46, 47]. If the mirror is static for $t < 0$, we can introduce a transformation between the null coordinates u, v and new coordinates $\overline{u}, \overline{v}$ defined through some function f by

$$u = f(\overline{u}) \quad , \quad v = \overline{v} \tag{6.6}$$

where

$$u = t - x \quad , \quad v = t + x \tag{6.7}$$

together with new coordinates $\bar{t}, \bar{x}$ derived as might be expected:

$$\bar{u} = \bar{t} - \bar{x} \quad , \quad \bar{v} = \bar{t} + \bar{x} \tag{6.8}$$

These coordinates will be chosen so that the mirror trajectory is just $\bar{x} = 0$. To do this, we relate the two sets of coordinates as follows:

$$\begin{aligned} t &= \frac{1}{2}[\,\bar{v} + f(\bar{u})] \\ x &= \frac{1}{2}[\,\bar{v} - f(\bar{u})] \end{aligned} \tag{6.9}$$

On the mirror path, setting $\bar{x} = 0$ means that the trajectory can be written as

$$\frac{1}{2}[\,\bar{t} - f(\bar{t})] = z\left(\frac{1}{2}[\,\bar{t} + f(\bar{t})]\right) \tag{6.10}$$

which allows f to be implicitly determined. In the new coordinates the wave equation is unchanged, however it now has a time independent boundary condition, meaning the mirror is static, while the detector moves along some more complicated path. Thus the wave equation with boundary condition can easily be solved to give

$$\phi(\bar{t}, \bar{x}) = \int_0^\infty (2\pi k)^{-1/2} \sin k\bar{x}\; e^{-ik\bar{t}}\, dk \tag{6.11}$$

where the mode functions are orthonormal in the Klein-Gordon inner product. The barred coordinates are especially useful since they just give us the usual vacuum influence functional; alternatively this can be seen by calculating $X(\bar{t})$. To do this we make use of a result which was stated in chapter 3, specifically (3.22, 3.23): for a quadratic lagrangian the sum of the Bogoliubov coefficients satisfies the classical equation of motion. In this chapter we always take $\kappa = k$.

In the present case the time dependent modes of the field are just exponentials. That is, they can be described by oscillators with unit mass and frequency k. So $X(\bar{t})$ is a solution to this oscillator equation, and by satisfying the initial conditions $X(0) = 1, X'(0) = -ik$ we obtain

$$X(\bar{t}) = e^{-ik\bar{t}} \tag{6.12}$$

If we now write

$$\zeta(\bar{t}, \bar{t}') = \int_0^\infty dk\; I(k, \bar{t}, \bar{t}')\; X(\bar{t}) X^*(\bar{t}') \tag{6.13}$$

we have the usual zero temperature vacuum form of the influence functional.

The spectral density of the field is determined by the path of the detector, and the coupling which we take to be a delta function. With c denoting the coupling of the detector's internal variable ρ to the field ϕ, we write

$$L_{int} = \int -c\,\rho\,\phi(\bar{t}, \bar{x})\;\delta(\bar{r} - \bar{x})\; d\bar{x}$$

$$
\begin{aligned}
&= -c\,\rho\,\phi(\bar{t},\bar{r}) \\
&= \int -c\,\rho\,q_k(\bar{t})\ \sin k\bar{r}\ dk \\
&\equiv \int -c_k\,\rho\,q_k\ dk
\end{aligned} \tag{6.14}
$$

then we have

$$
\begin{aligned}
I(k,\bar{t},\bar{t}') &= \int \frac{dk_n}{2k_n}\delta(k-k_n)c^2 \sin k_n\bar{r}(\bar{t}) \sin k_n\bar{r}(\bar{t}') \\
&= \frac{c^2}{2k} \sin k\,\bar{r}(\bar{t}) \sin k\,\bar{r}(\bar{t}')
\end{aligned} \tag{6.15}
$$

We now need an expression for the detector path, $\bar{r}$. Since it's inertial, we have $r(t) = r_* + wt$ in the usual coordinates, with r_*, w constants. Convert to barred coordinates via

$$
\begin{aligned}
\bar{t} &= \frac{1}{2}[t + x + f^{-1}(t-x)] \\
\bar{x} &= \frac{1}{2}[t + x - f^{-1}(t-x)]
\end{aligned} \tag{6.16}
$$

in which case the detector path is written by identifying $r, \bar{r}$ with $x, \bar{x}$ in (6.16):

$$
\bar{r}(t) = \frac{1}{2}\left[t + r(t) - f^{-1}\big(t - r(t)\big)\right] \tag{6.17}
$$

In order to find f, we need to specify a mirror path. A convenient choice of path is the following:

$$
z(t) = -t - Ae^{-2\kappa t} + B \tag{6.18}
$$

for A, B, κ positive (note that this κ is used in [47] but is different to the one we have used previously, which we just write now as k). This path provides a horizon in the sense that there is a last ingoing ray which the mirror will reflect; all later rays never catch up with the mirror and so are not reflected. It's this aspect which enables the moving mirror to emulate a black hole. Note that a variety of mirror paths are possible, but we will pick one with sufficient generality but simple enough to admit an analytic solution to the equations that follow.

Equation (6.10) can now be solved to give

$$
f(\bar{t}) = -\bar{t} - \frac{1}{\kappa} \ln \frac{B - \bar{t}}{A} \tag{6.19}
$$

which can be inverted in the large time limit to give

$$
f^{-1}(u) \simeq B - Ae^{-\kappa(B+u)} \tag{6.20}
$$

The detector path becomes

$$
\bar{r} = \frac{1}{2}\left[r_* + (1+w)t - B + Ae^{-\kappa[B - r_* + (1-w)t]}\right] \tag{6.21}
$$

For use in the black hole calculations that we do presently, we rewrite the field modes in terms of the null coordinates. By using (6.6) the modes in (6.11) can be written as

$$\phi_k \propto e^{-ikv} - e^{-ikf^{-1}(u)} \tag{6.22}$$

Then with $f^{-1}(u)$ calculated above, we write for the c, d constants

$$\phi_k \propto e^{-ikv} - e^{ik\left(ce^{-\kappa u}+d\right)} \tag{6.23}$$

For black holes (as described in the next section) this will be found to be describing the radial modes of infalling matter (excepting the r^{-1} factor due to the spherical geometry). This fact enables us to extend these results very quickly to the black hole case. It also justifies the choice of (6.18) for the mirror path.

We are now in a position to calculate ζ. With

$$\overline{r} \equiv \overline{r}(\overline{t}) \quad , \quad \overline{r}' \equiv \overline{r}(\overline{t}') \quad , \quad \overline{\Delta} \equiv \overline{t} - \overline{t}' \quad , \quad \overline{R}_{\pm} \equiv \overline{r} \pm \overline{r}' \tag{6.24}$$

we write (6.15) as exponentials, and substitute it together with (6.12) into (6.13), arriving at

$$\zeta = \frac{-c^2}{8} \int_0^\infty \frac{dk}{k} \left[e^{ik(\overline{R}_+ - \overline{\Delta})} - e^{ik(\overline{R}_- - \overline{\Delta})} - e^{ik(-\overline{R}_- - \overline{\Delta})} + e^{ik(-\overline{R}_+ - \overline{\Delta})}\right] \tag{6.25}$$

What is $\int_0^\infty e^{ikx}/k \; dk$? We can write

$$\frac{d}{dx} \int_0^\infty \frac{e^{ikx}}{k} dk = i \int_0^\infty e^{ikx} dk = \frac{-1}{x + i\varepsilon} \tag{6.26}$$

and then integrate to obtain

$$\int_0^\infty \frac{e^{ikx}}{k} dk = -\ln(x + i\varepsilon) + constant \tag{6.27}$$

Actually this constant is formally infinite, but will cancel in the expressions to follow. Equation (6.25) now becomes

$$\zeta = \frac{-c^2}{8} \ln \frac{(-\overline{R}_- + \overline{T})\,(\overline{R}_- + \overline{T})}{(-\overline{R}_+ + \overline{T})\,(\overline{R}_+ + \overline{T})} \equiv \frac{-c^2}{8} \overline{\zeta} \tag{6.28}$$

where $\overline{T} \equiv \overline{\Delta} - i\varepsilon$ and we have changed all signs in the logarithm's argument: with this change, the logarithm can be seen to be the same as the two point function for the in-vacuum in [47], a fact we make use of shortly[1]. This correspondence is important since it establishes a connection between the influence functional theory we are using here, and the conventional field theory approach based on the in and out states.

[1]This can be seen as follows. Consider the first term in the argument, $-\overline{R} + \overline{\Delta}$. This equals $-\overline{r} + \overline{t} + \overline{r}' - \overline{t}'$, which from (6.16) and (6.17) is just $f^{-1}(t - r(t)) - f^{-1}(t' - r(t'))$. In the notation of [47] this is written $p(u) - p(u')$. The other terms follow similarly.

Given this form for ζ, we now proceed to calculate I_{eff} and C from (6.3). We can make use of a result found in [47] by first writing the Fourier transforms as complex exponentials involving ζ, which is possible since $\nu(\Delta)$ is even and so contributes nothing to the sine transform:

$$\begin{aligned} I_{eff} &= \frac{-1}{\pi}\int_{-\infty}^{\infty} d\Delta\ \mu \sin k\Delta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\Delta\ \left[e^{ik\Delta} - e^{-ik\Delta}\right]\zeta \\ &= \frac{-c^2}{16\pi}\int_{-\infty}^{\infty} d\Delta\ \left[e^{ik\Delta}\overline{\zeta} - e^{-ik\Delta}\overline{\zeta}\right] \end{aligned} \tag{6.29}$$

Substituting the identity [47]

$$\int_{-\infty}^{\infty} d\Delta\ e^{-ik\Delta}\ \overline{\zeta} = \frac{-4\pi}{k\left(e^{\frac{k}{T}} - 1\right)} \quad \text{where} \quad T \equiv \frac{\kappa}{2\pi}\sqrt{\frac{1-w}{1+w}} \tag{6.30}$$

into (6.29), we get the effective spectral density

$$I_{eff} = \frac{c^2}{4k} \tag{6.31}$$

Similar steps allow us to use (6.3) to calculate C. Writing

$$I_{eff}C = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\Delta\ \left[e^{ik\Delta} + e^{-ik\Delta}\right]\zeta \tag{6.32}$$

and again using (6.30) we arrive at

$$C = \coth\frac{k}{2T} \tag{6.33}$$

So a thermal spectrum is observed, redshifted by the Doppler factor involving w as in (6.30). This is the influence functional derivation, in contrast to the usual approach of dealing with two point functions for 'in' Fock states.

A different mirror path

As a further example we will consider a more specific mirror path, again static for $t < 0$, but one which reduces to the above path for large t. This is

$$z(t) = \frac{-1}{\kappa}\ln\ \text{ch}\,\kappa t \tag{6.34}$$

Following our previous calculation, we can use (6.28) to calculate ζ. For this we need $\overline{r}$, and hence f^{-1}. Again, by solving (6.10) this is found to be (for all $t > 0$)

$$f^{-1}(u) = \frac{1}{\kappa}\ln\left(2 - e^{-\kappa u}\right) \tag{6.35}$$

So from (6.17) the detector path is

$$\overline{r}(t) = \frac{1}{2}\left[r_* + (1+w)t - \frac{1}{\kappa}\ln\left(2 - e^{-\kappa[-r_*+(1-w)t]}\right)\right] \tag{6.36}$$

This can be used in (6.28) to calculate ζ, and hence the spectrum; however in this case the calculation is harder since the Fourier transform might not exist in closed form.

6.1.2 Perturbed Mirror Trajectory

We now wish to calculate the particle creation from a mirror trajectory deviating from that which produces a thermal spectrum (which we call a 'thermal mirror trajectory'). Consider the perturbed trajectory

$$z(t) = -t - Ae^{-2\int_0^t \kappa(\tau)d\tau} + B \tag{6.37}$$

and introduce a dimensionless parameter measuring the deviation from the thermal mirror trajectory (6.18):

$$h = \frac{\dot{\kappa}}{\kappa^2} \tag{6.38}$$

We expand $\kappa(t)$ to first order in t:

$$\kappa(t) \simeq \kappa(0) + h(0)\kappa^2(0)\, t \tag{6.39}$$

provided that this expansion is only taken as valid for $t \ll \left(h(0)\kappa(0)\right)^{-1}$. In that case, we are essentially taking h as roughly constant, so that we can replace $h(0)$ by h in what follows. Substituting (6.39) into the above expression for z, we obtain (with $\kappa_0 \equiv \kappa(0)$)

$$z(t) \simeq -t - Ae^{-2\kappa_0 t - h\kappa_0^2 t^2} + B \tag{6.40}$$

Following the same route as for the thermal case, we find f by solving (6.10). It is inverted in the large time limit to give

$$f^{-1}(u) \simeq B - Ae^{-\kappa_0(B+u) - \frac{h\kappa_0^2}{4}(B+u)^2} \tag{6.41}$$

To calculate ζ we again use (6.12) and (6.15) in (6.13). If we indicate the dependence of $\overline{r}$ on h then we can write to first order in h:

$$\begin{aligned} \overline{r}(h,t) &\simeq \overline{r}(0,t) - \frac{h\kappa_0^2 A}{8} e^{-\kappa_0[B+t-p(t)]}[B+t-p(t)]^2 \\ &\equiv \overline{r}_0 + \delta\overline{r} \end{aligned} \tag{6.42}$$

where $\overline{r}_0$ is just the unperturbed $\overline{r}$ in (6.21). Note that although the form of $\overline{r}$ has changed, because we're obliged to work in the new coordinates as dictated by the perturbed mirror path, the physical path of the detector remains inertial.

Next we calculate the perturbations to the noise and dissipation, $\delta\zeta$,

$$\delta\zeta = -\frac{c^2}{8}\delta\overline{\zeta} \tag{6.43}$$

by referring to (6.28). To first order

$$\begin{aligned}\overline{\zeta}(\overline{r}_0 + \delta\overline{r}, \overline{r}'_0 + \delta\overline{r}') &= \overline{\zeta}(\overline{r}_0, \overline{r}'_0) + \frac{4\overline{r}'_0(\overline{T}^2 + \overline{r}_0^2 - \overline{r}_0'^2)\delta\overline{r} + 4\overline{r}_0(\overline{T}^2 - \overline{r}_0^2 + \overline{r}_0'^2)\delta\overline{r}'}{(\overline{T}^2 - \overline{r}_0^2 - \overline{r}_0'^2)^2 - 4\overline{r}_0^2\,\overline{r}_0'^2} \\ &\equiv \overline{\zeta} + \delta\overline{\zeta}\end{aligned} \tag{6.44}$$

Also the change in the spectrum of fluctuations is given by

$$\begin{Bmatrix}\delta(I_{\mathit{eff}}C) \\ \delta I_{\mathit{eff}}\end{Bmatrix} = \frac{-c^2}{8\pi}\int_{-\infty}^{\infty} d\Delta \begin{Bmatrix}\cos k\Delta \\ i\sin k\Delta\end{Bmatrix}\delta\overline{\zeta} \tag{6.45}$$

The fluctuations are seen to be proportional to h.

Note that these integrals are by no means trivially done, and in particular we cannot just set $\varepsilon = 0$, since this would immediately make both I_{eff} and C pure imaginary.

Alternative perturbation

We can follow a slightly different approach which will make the previous calculation more applicable to the black hole case in the next section. First, we rewrite (6.40) by taking $h \ll 1$ and expanding the exponential:

$$\begin{aligned} z(t) &\simeq \underbrace{-t - Ae^{-2\kappa_0 t} + B}_{} + \underbrace{hAe^{-2\kappa_0 t}\kappa_0^2\, t^2}_{} \\ &\equiv \qquad z_0(t) \qquad\quad + \qquad \delta z_0(t)\end{aligned} \tag{6.46}$$

where $z_0(t)$ is just our unperturbed mirror trajectory. We wish now to calculate the new $\overline{r}$. For this we need the new f^{-1} [see (6.17)]. So we solve (6.10) for $\overline{t}$ in terms of f, whereby the resulting function is by definition f^{-1}. We write

$$\overline{t} = f^{-1}(f(\overline{t})) \equiv f_0^{-1}(f) + \delta(f_0^{-1})(f) \tag{6.47}$$

where from (6.20)

$$f_0^{-1}(u) \equiv . B - Ae^{-\kappa(B+u)} \tag{6.48}$$

To solve (6.10) we expand z_0 to first order in $\delta(f_0^{-1})$ obtaining

$$\delta(f_0^{-1})(u) \simeq \frac{2\delta z_0\left(\frac{1}{2}(f_0^{-1}(u) + u)\right)}{1 - z'_0\left(\frac{1}{2}(f_0^{-1}(u) + u)\right)} \tag{6.49}$$

From (6.17) the detector path can now be written in the new coordinates as

$$\overline{r} \equiv \overline{r}_0 + \delta\overline{r} = \frac{1}{2}\left[t + r - f_0^{-1}(t - r) - \delta(f_0^{-1})(t - r)\right] \tag{6.50}$$

or

$$\delta\overline{r} = \frac{-1}{2}\delta(f_0^{-1})(t-r) \tag{6.51}$$

Now that we have obtained $\delta\overline{r}$, we can substitute it together with $\overline{r}_0$ from (6.21) into Eq (6.44), which then gives us the corrections to the spectrum as derived in the last section.

6.2 Collapsing spherical mass

In this section we study thermal radiance from a collapsing spherical mass coupled to a scalar field by analogy with the moving mirror case just discussed. Outside the body the metric is conformal to flat space, and we write it as

$$ds^2 = C(r)du\,dv \tag{6.52}$$

where u, v are null coordinates defined as in the mirror case, except that now x is replaced by the Regge-Wheeler coordinate r^*:

$$r^* \equiv \int_{R_0}^{r} \frac{dr'}{C(r')} \tag{6.53}$$

with R_0 the radius of the body. That is,

$$\begin{Bmatrix} du \\ dv \end{Bmatrix} = dt \mp dr^* \tag{6.54}$$

The field equation outside the body, $\phi^{;\alpha}{}_{;\alpha} = 0$, is now solved in the large time limit [47] to give radial modes containing time dependence which are written

$$e^{-ikv} - e^{ik\left(ce^{-\kappa u}+d\right)} \tag{6.55}$$

where κ is defined by

$$\kappa = \frac{1}{2}C'(R_{horizon}) \tag{6.56}$$

These modes are identical to those of (6.23); this shows that the collapsing body case is equivalent to the mirror case already calculated. So by analogy with the previous calculation, we can define a function $f^{-1}(u)$ by writing the field modes as in (6.22), and similarly define coordinates $\overline{t}$ and $\overline{x}$ as in (6.6):

$$\begin{aligned} \overline{t} - \overline{x} &= f^{-1}(u) \\ \overline{t} + \overline{x} &= v \end{aligned} \tag{6.57}$$

Hence the radial modes are again $\sin k\overline{x}\ e^{-ik\overline{t}}$ and just as for the mirror, we obtain $X(\overline{t}) = e^{-ik\overline{t}}$.

We can now calculate ζ for a detector placed at constant r. Such a detector is also at constant r^*, or by analogy with the mirror case, constant x. This just corresponds to the mirror case when $w = 0$. So the mirror calculation carries over to here, and we obtain $T = \kappa/(2\pi)$.

For the case of a Schwarzschild hole, $C(r) = 1 - 2M/r, \kappa = 1/(4M)$ is the surface gravity of the star, and we arrive at Hawking's well known result

$$T = \frac{1}{8\pi M} \tag{6.58}$$

6.2.1 Perturbed Schwarzschild metric

Suppose we now perturb the Schwarzschild metric, say by the arrival of a gravitational wave. How does this affect the spectral density and temperature of the hole?

Perturbations to the Schwarzschild metric have been studied by many authors [48] for stability considerations [49] (a necessary condition for the existence of black holes in nature) and gravitational wave analysis [50]. We use the notation of [49], restricting our attention to radial modes only. In this case only the even parity modes survive, and we can write the perturbed metric using two new functions, $H_0(r), H_1(r)$, as

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left(1 - H_0 e^{-ikt}\right) dt^2 - 2H_1 e^{-ikt} dt\, dr - \left(1 - \frac{2M}{r}\right)^{-1}\left(1 + H_0 e^{-ikt}\right) dr^2 \tag{6.59}$$

Since we assume $H_0 \ll 1$ we can rewrite the metric in a more convenient form, with $C(t,r) \equiv \left(1 - \frac{2M}{r}\right)\left(1 - H_0 e^{-ikt}\right)$:

$$ds^2 = C(t,r)\left[dt - \left(1 + H_1 e^{-ikt}\right)\frac{dr}{C(t,r)}\right]\left[dt + \left(1 - H_1 e^{-ikt}\right)\frac{dr}{C(t,r)}\right] \tag{6.60}$$

Now in the static case we calculate the form of the wave modes resulting in (6.55), where we are using (6.54) to define u, v. These unperturbed expressions for du, dv are easily integrable. However in the perturbed case this is no longer true: if we define du, dv to be the two bracketed expressions above in analogy to (6.54) then they cannot be integrated to give u, v. This is because the expression

$$du = f(t,r)\, dt + g(t,r)\, dr \tag{6.61}$$

is only integrable if $\partial f/\partial r = \partial g/\partial t$, a requirement that fails for both bracketed expressions above. So what we must do is introduce a new function $F(t,r)$ which allows us to define u and v again in analogy to (6.54) while keeping the form of (6.52):

$$\begin{aligned} du &= F(t,r)\left[dt - \left(1 + H_1 e^{-ikt}\right)\frac{dr}{C(t,r)}\right] \\ dv &= \frac{1}{F(t,r)}\left[dt + \left(1 - H_1 e^{-ikt}\right)\frac{dr}{C(t,r)}\right] \end{aligned} \tag{6.62}$$

This function is in principle known by applying (6.61), although in our case the differential equations are probably not analytically solvable. We assume it's known and follow the method of [47] to obtain the scalar field modes. Writing the modes in the form

$$e^{-ikv} - e^{-ik\beta(\alpha(u)-2R_0)} \tag{6.63}$$

for some R_0, where $U = \alpha(u)$, $V = \beta^{-1}(v)$, we must obtain the functions α, β. To do this we first specify the two line elements: the first, outside the perturbed hole is

$$ds^2 = C(u,v)du\,dv, \tag{6.64}$$

while the second, inside the perturbed hole, is

$$ds^2 = D(U,V)dU\,dV, \tag{6.65}$$

where the surface of the hole is at $r = R$, and

$$\begin{aligned} U &\equiv \tau - (R - R_0) \\ V &\equiv \tau + (R - R_0) \end{aligned} \tag{6.66}$$

for some constant R_0. On the horizon we match up the two metrics and solve for $\dot{t}$ (where $\dot{} \equiv d/d\tau$). Then we have

$$\begin{Bmatrix} dU/du \\ dV/dv \end{Bmatrix} = \begin{Bmatrix} 1/F \\ F \end{Bmatrix} \frac{(1 \mp \dot{R})C}{\sqrt{\dot{R}^2 + CD(1-\dot{R}^2)} \mp \dot{R}} \tag{6.67}$$

The experience of an observer at late times is determined almost entirely by rays not far separated from the last ray to emerge from the hole. This being the case, we can concentrate on near-horizon approximations and solutions of the equations for α, β in (6.67).

Near the horizon (6.67) simplifies to

$$\begin{aligned} \frac{dU}{du} &\simeq \frac{1}{F}\frac{\dot{R}-1}{2\dot{R}}C \\ \frac{dV}{dv} &\simeq F\frac{\dot{R}-1}{2\dot{R}}D \end{aligned} \tag{6.68}$$

Hence by expanding C at the horizon to first order in $R - R_h$ (the radial distance from the horizon) and noting that C vanishes at the horizon, we can again define

$$\kappa \equiv \frac{1}{2}\frac{\partial C}{\partial r}\bigg|_{r=R_h} \tag{6.69}$$

and write, with $v \simeq B$ some constant near the horizon

$$\frac{dU}{du} \simeq \frac{-\kappa(U-U_h)}{F(u, v=B)} \tag{6.70}$$

So the first equation in (6.68) has the solution

$$U = c_1 e^{-\kappa \int \frac{du}{F(u,B)}} + U_h \tag{6.71}$$

for some constant c_1. The second equation in (6.68) is solved by the same argument as used in [47]. However we must include the function F evaluated at $u \to \infty$ and $v = B$. This gives

$$V = \frac{D\,(1+\sigma)\,F(\infty, B)}{2\sigma}(v - c_2) \tag{6.72}$$

with $\sigma \equiv -\dot{R}(\tau_h)$. Hence

$$\beta\big(\alpha(u) - 2R_0\big) = \frac{2\sigma c_1}{D(1+\sigma)F(\infty,B)} e^{-\kappa \int \frac{du}{F(u,B)}} + \frac{2\sigma U_h}{D(1+\sigma)F(\infty,B)} + c_2 \tag{6.73}$$

and in the static case with $F = 1$ this must reduce to $-ce^{-\kappa u} - d$.

We want now to connect this with the perturbed moving mirror case. We do this via the expressions for the modes. Since $f^{-1}(u) = f_0^{-1}(u) + \delta(f_0^{-1})(u)$, eqn (6.22) can be written as

$$e^{-ikv} - e^{-ik[-ce^{-\kappa u} - d + \delta(f_0^{-1})(u)]} \tag{6.74}$$

Now compare this to (6.63, 6.73). Without perturbations ($\delta(f_0^{-1}) = 0$) we have

$$\frac{2\sigma c_1 e^{-\kappa u}}{D(1+\sigma)} + \frac{2\sigma U_h}{D(1+\sigma)} + c_2 \equiv -ce^{-\kappa u} - d \tag{6.75}$$

For convenience write $F(u, B) = 1 + \varepsilon(u)$. The modes become

$$\exp(-ikv) - \exp -ik\left[\frac{2\sigma c_1}{D(1+\sigma)[1+\varepsilon(\infty)]} e^{-\kappa u} e^{\kappa \int \varepsilon(u)du} + \frac{2\sigma U_h}{D(1+\sigma)[1+\varepsilon(\infty)]} + c_2\right] \tag{6.76}$$

The bracketed part above can be written as

$$\underbrace{\frac{2\sigma c_1 e^{-\kappa u}}{D(1+\sigma)} + \frac{2\sigma U_h}{D(1+\sigma)} + c_2}_{= -ce^{-\kappa u} - d} + \underbrace{\frac{2\sigma c_1}{D(1+\sigma)} e^{-\kappa u}\left[-\varepsilon(\infty) + \kappa \int \varepsilon(u)du\right] - \frac{2\sigma U_h}{D(1+\sigma)}\varepsilon(\infty)}_{\textit{therefore} = \delta(f_0^{-1})(u)} \tag{6.77}$$

This $\delta(f_0^{-1})$ can now be used in (6.51) to give $\delta\overline{r}$ and hence the corrections to the spectrum and temperature of the hole.

Note that we haven't used the original perturbation parameter h here. The perturbations to the Schwarzschild metric were directly calculated from the Einstein field equations. However we can recover the h that these perturbations imply, by connecting $\delta(f_0^{-1})$ with the original δz_0 via (6.49). More specifically, starting with some perturbation h we have the following progression:

$$h \xrightarrow{(6.46)} \delta z_0 \xrightarrow{(6.49)} \delta(f_0^{-1}) \xrightarrow{(6.51)} \delta\overline{r} \xrightarrow{(6.43,6.44)} \textit{corrections to spectrum and temperature} \tag{6.78}$$

The route is admittedly quite convoluted. (Even in the perturbed Schwarzschild case, solving for the quantities introduced in [49], as well as the function $F(t,r)$, are difficult analytically or numerically. The limit involved in (6.45) is also not trivial). Here, we are content with just sketching out a pathway to obtain the spectrum from a perturbed metric, but leave the details, which may prove necessary in the consideration of backreaction problems.

6.3 A cosmological spacetime: the Parker metric

In [2] the radiation seen in de Sitter space was calculated as an example of the usefulness of the influence functional formalism. In this section we again use this approach to investigate particle creation in a different spacetime: that of [43], which reports a thermal radiance.

The metric is as follows:

$$ds^2 = a^6 d\eta^2 - \sum_i a^2 (dx^i)^2 \tag{6.79}$$

where

$$a^4(\eta) = 1 + e^{\rho\eta} \tag{6.80}$$

with ρ some parameter. We consider the action of a massless, minimally coupled real scalar field ϕ, which forms an environment acting upon a detector coupled to this field at some point in space. The field can be decomposed into a collection of oscillators of time-dependent frequency. Using the influence functional formalism, we can determine the effect of such an environment on the detector, which is also modelled by an oscillator.

To do this we calculate the noise and dissipation produced by the field. These are given as before:

$$\zeta = \int_0^\infty dk\; I(k,\tau,\tau')X(\tau)X^*(\tau') \tag{6.81}$$

Here we calculate X, the sum of the Bogoliubov coefficients for the bath. First we decompose the field into its modes; the lagrangian density is

$$\begin{aligned}\mathcal{L}(x) &= \frac{\sqrt{-g}}{2}\phi^{,\mu}\phi_{,\mu} \\ &= \frac{1}{2}\left[(\phi_{,\eta})^2 - a^4\sum_i(\phi_{,i})^2\right]\end{aligned} \tag{6.82}$$

In terms of normal modes the lagrangian becomes

$$L(\eta) = \sum_{k,\sigma=\pm}\frac{1}{2}\left[(q^\sigma_{\mathrm{k},\eta})^2 - a^4k^2(q^\sigma_{\mathrm{k}})^2\right] \tag{6.83}$$

We see then that the bath can be described by a set of oscillators with mass and frequency

$$m = 1 \quad , \quad \omega^2 = a^4 k^2 \tag{6.84}$$

Now as was already mentioned, X satisfies the classical equation of motion for an oscillator with the given parameters. So we need to solve

$$\begin{aligned} X''(\eta) + k^2(1 + e^{\rho\eta})X &= 0 \\ X(\eta_0) = 1 \quad , \quad X'(\eta_0) &= -ik \end{aligned} \tag{6.85}$$

The solutions are written in terms of Bessel functions:

$$X(\eta) = c_1 J_{\frac{2ik}{\rho}}\left(\frac{2k}{\rho}e^{\rho\eta/2}\right) + c_2 J_{\frac{-2ik}{\rho}}\left(\frac{2k}{\rho}e^{\rho\eta/2}\right) \tag{6.86}$$

To fix the constants c_1, c_2 consider that the initial time is $\eta_0 \to -\infty$; unfortunately the complex index Bessel functions oscillate infinitely often as their arguments approach zero, and so for now we leave η_0 unspecified. In that case we can calculate[2] c_1 and c_2; the final expression for X becomes, with

$$z \equiv \frac{2k}{\rho}e^{\rho\eta/2} \quad , \quad z_0 \equiv \frac{2k}{\rho}e^{\rho\eta_0/2}$$

and Bessel indices labelled by $\nu \equiv 2ik/\rho$:

$$X(\eta) = \frac{\pi k}{\rho}\text{csch}\frac{2\pi k}{\rho}\left\{ie^{\rho\eta_0/2}\begin{vmatrix} J_\nu(z) & J_{-\nu}(z) \\ J'_\nu(z_0) & J'_{-\nu}(z_0)\end{vmatrix} - \begin{vmatrix} J_\nu(z) & J_{-\nu}(z) \\ J_\nu(z_0) & J_{-\nu}(z_0)\end{vmatrix}\right\} \tag{6.87}$$

In the limiting case of $\eta \to \infty$ (with $\eta_0 \to -\infty$) we can use first order and asymptotic expressions for J and J' to write (which defines the phases θ_1, θ_2)

$$\begin{aligned} J_\nu\left(z_0\right) &\simeq \sqrt{\frac{\text{sh}\, 2\pi k/\rho}{2\pi k/\rho}}e^{i\theta_1(k)} \\ J'_\nu\left(z_0\right) &\simeq \frac{2k}{\rho z_0}\sqrt{\frac{\text{sh}\, 2\pi k/\rho}{2\pi k/\rho}}e^{i\theta_2(k)} \\ J_\nu\left(z\right) &\simeq \sqrt{\frac{2}{\pi \ln z}}\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \end{aligned} \tag{6.88}$$

In evaluating $X(\tau)X^*(\tau')$ we obtain various products of Bessel functions with their derivatives (note: $J_\nu^* = J_{-\nu}$); in particular we need

$$\begin{aligned} \theta_2 - \theta_1 &= \arg\Gamma\left(1 + \frac{2ik}{\rho}\right) - \arg\Gamma\left(\frac{2ik}{\rho}\right) \\ &= \arg\frac{2ik}{\rho} = \frac{\pi}{2} \text{ provided } k \neq 0 \end{aligned} \tag{6.89}$$

[2]To calculate these coefficients the wronskian of $J_{2ik/\rho}$ and $J_{-2ik/\rho}$ is needed; note that there is a misprint in Gradshteyn and Ryzhik §8.474: the relevant quantity should be $\frac{-2}{\pi z}\sin\nu\pi$.

Also, when calculating the Bessel products, there arise sines and cosines with argument $f(\tau) + f(\tau') \equiv 2k/\rho\left(e^{\rho\tau/2} + e^{\rho\tau'/2}\right)$; when $\eta \to \infty$ and we ultimately integrate over k, these terms won't contribute to the integral and so can be discarded. Changing to Σ, Δ variables we finally obtain

$$\zeta = e^{-\rho\Sigma/2}\int_0^\infty dk\ I(k,\tau,\tau')\left[\cos\frac{2k}{\rho}\left(e^{\rho\tau/2} - e^{\rho\tau'/2}\right)\coth\frac{2\pi k}{\rho} - i\sin\frac{2k}{\rho}\left(e^{\rho\tau/2} - e^{\rho\tau'/2}\right)\right] \tag{6.90}$$

We can now equate ζ with the standard form (6.2) and, as before, obtain the expressions for the effective spectral density and temperature of the radiation, (6.3). In order to use these we need to calculate the dissipation and noise, μ and ν.

We first evaluate μ as given by (6.90); substituting it into (6.3) will then give us the effective spectral density $I_{eff}(k,\Sigma)$. Define

$$\sigma \equiv \frac{2}{\rho}\left(e^{\rho\tau/2} - e^{\rho\tau'/2}\right) = \frac{4}{\rho}e^{\rho\Sigma/2}\,\mathrm{sh}\,\frac{\rho\Delta}{4} \tag{6.91}$$

To proceed, we need to specify a form for the spectral density. This has been calculated in [2], and in 3+1 dimensions it is

$$I(k,\tau,\tau') = \frac{c^2 k}{4\pi^2} \tag{6.92}$$

where c is the coupling strength of the detector to the field. Then from (6.90) we have

$$\mu = \frac{-c^2}{4\pi^2}e^{-\rho\Sigma/2}\int_0^\infty dk\ k\sin\sigma k \tag{6.93}$$

$$= \frac{c^2}{4\pi^2}e^{-\rho\Sigma/2}\ \pi\delta'(\sigma) \tag{6.94}$$

where the last result follows from (A.11). Substituting this form for μ into (6.3) gives the following result:

$$I_{eff}(k,\Sigma) = \frac{c^2 k}{4\pi^2}e^{-3\rho\Sigma/2} \tag{6.95}$$

Evaluating the noise kernel ν is a more complicated affair. From (6.90) we write

$$\begin{aligned}\nu &= \frac{c^2}{4\pi^2}e^{-\rho\Sigma/2}\int_0^\infty dk\ k\cos\sigma k\coth\frac{2\pi k}{\rho}\\ &= \frac{c^2}{4\pi^2}e^{-\rho\Sigma/2}\left[\frac{d}{d\sigma}P(1/\sigma) + \frac{1}{\sigma^2} - \frac{\rho^2}{16}\mathrm{csch}^2\frac{\rho\sigma}{4}\right]\end{aligned} \tag{6.96}$$

where again the last result has been calculated in appendix A. Upon substituting this into (6.3) and replacing I_{eff} with the appropriate result, we obtain

$$\begin{aligned}C(k,\Sigma) &= \frac{e^{\rho\Sigma}}{\pi k}\int_{-\infty}^\infty d\Delta\ \cos k\Delta\frac{d}{d\sigma}P(1/\sigma)\\ &+ \frac{\rho^2}{16\pi k}\int_{-\infty}^\infty d\Delta\ \cos k\Delta\left[\mathrm{csch}^2\frac{\rho\Delta}{4} - e^{\rho\Sigma}\mathrm{csch}^2\left(e^{\rho\Sigma/2}\,\mathrm{sh}\,\frac{\rho\Delta}{4}\right)\right]\end{aligned} \tag{6.97}$$

The first integral can be done by parts to get

$$\begin{aligned}\int_{-\infty}^{\infty} d\Delta \, \cos k\Delta \frac{d}{d\sigma} P(1/\sigma) &= \int_{-\infty}^{\infty} d\sigma \, \frac{d\Delta}{d\sigma} \cos k\Delta \frac{d}{d\sigma} P(1/\sigma) \\ &= -\mathrm{PV} \int_{-\infty}^{\infty} \frac{d\Delta}{\sigma} \frac{d}{d\Delta} \left[\frac{d\Delta}{d\sigma} \cos k\Delta \right] \\ &= \frac{4\pi k}{\rho} \coth \frac{2\pi k}{\rho} \end{aligned} \tag{6.98}$$

The second integral in (6.97) does not appear to be expressible in terms of known functions. Suppose we call it $B(k, \rho, \Sigma)$, and write

$$B(k, \rho, \Sigma) = 2 \int_0^{\infty} d\Delta \, \cos k\Delta \left[\mathrm{csch}^2 \frac{\rho\Delta}{4} - e^{\rho\Sigma} \mathrm{csch}^2 \left(e^{\rho\Sigma/2} \, \mathrm{sh} \, \frac{\rho\Delta}{4} \right) \right] \tag{6.99}$$

Then we have

$$C(k, \rho, \Sigma) = e^{\rho\Sigma} \left[\frac{4}{\rho} \coth \frac{2\pi k}{\rho} + \frac{\rho^2}{16\pi k} e^{-\rho\Sigma} B(k, \rho, \Sigma) \right] \tag{6.100}$$

We need to examine the second term in the last brackets. The function B tends to zero for large k (by Riemann-Lebesgue), and attains a maximum at $k = 0$ (since the cos term stops oscillating there). However, numerical work shows that this maximum value increases roughly with $e^{\rho\Sigma}$ which means that on first glance the second term in the brackets won't necessarily vanish at late times ($\Sigma \to \infty$).

So we need to examine the value of B at $k = 0$ more closely, to see precisely how it changes with Σ. To this end we can consider plotting a graph of $B(0, \rho, \Sigma)$-vs-$e^{\rho\Sigma}$ and analysing its concavity; i.e. with $x \equiv e^{\rho\Sigma}$ we need $\partial^2 B/\partial x^2$. Differentiating twice under the integral sign gives an integrand which is everywhere negative, and so we conclude that $\partial^2 B/\partial x^2 < 0$, which means that B as a function of x is everywhere concave down. But B increases with x, and thus $B/x \equiv e^{-\rho\Sigma} B \to 0$ as $\Sigma \to \infty$. In that case the second term in the brackets gives no contribution in the large time limit.

Finally, from (6.2) we can write ζ in a form which reveals the thermal nature of the detected radiation:

$$\zeta = \int_0^{\infty} dk \, I_{\mathit{eff}}(k, \Sigma) \left[\frac{4e^{\rho\Sigma}}{\rho} \coth \frac{2\pi k}{\rho} \cos k\Delta - i \sin k\Delta \right] \tag{6.101}$$

The temperature of the radiation is now found by equating the coth argument, $2\pi k/\rho$, with the coth argument in the noise term for static coherent oscillators, $k/(2T)$, giving:

$$T = \frac{\rho}{4\pi} \tag{6.102}$$

Parker metric with alternative time

Inspection of (6.90) suggests that an alternative time variable can be chosen:

$$t = \frac{2}{\rho} e^{\rho\eta/2} \tag{6.103}$$

The metric becomes

$$ds^2 = \frac{4a^6}{\rho^2 t^2} dt^2 - \sum_i a^2 (dx^i)^2 \tag{6.104}$$

with

$$a^4 = 1 + \frac{\rho^2 t^2}{4} \tag{6.105}$$

Again following the previous formalism, we arrive at a description of the environment in terms of oscillators, this time with time dependent mass and frequency:

$$m = \frac{\rho t}{2} \quad , \quad \omega^2 = \frac{4a^4 k^2}{\rho^2 t^2} \tag{6.106}$$

Solutions for X in this case follow through from (3.20, 3.22, 3.23) and are the same as before, and all the calculations carry through in much the same way. With now Σ and Δ defined as mean and differences of t and t' we again arrive at thermal forms for the noise and dissipation:

$$\zeta = e^{-\rho\Sigma/2} \int_0^\infty dk\ I(k, s, s') \left[\cos k\Delta \coth \frac{2\pi k}{\rho} - i \sin k\Delta \right] \tag{6.107}$$

and the detected temperature is the same as in (6.102).

6.3.1 Perturbing the Parker metric

We now want to perturb the Parker metric (5.1) by writing

$$a^4(\eta) = 1 + \exp\left[\int_0^\eta \rho(\tau)\ d\tau \right] \tag{6.108}$$

and then seeing whether we obtain a near-thermal spectrum characterised by the parameter

$$h = \frac{\dot{\rho}}{\rho^2} \tag{6.109}$$

If we expand ρ to first order in τ [with $\rho_0 \equiv \rho(0)$], and also assume $h\rho_0^2\eta^2 \ll 1$ (as well as the usual $h\rho_0\eta \ll 1$) we arrive at:

$$a^4(\eta) \simeq \underbrace{1 + e^{\rho_0\eta}}_{\equiv a_0^4(\eta)} + \underbrace{\frac{h}{2}\rho_0^2\eta^2 e^{\rho_0\eta}}_{\equiv \delta(a_0^4)(\eta)} \tag{6.110}$$

As before we need to solve for X from

$$X''(\eta) + k^2[1 + e^{\rho_0 \eta} + \delta(a_0^4)(\eta)]X = 0 \tag{6.111}$$

Since we are only perturbing our original X, which we now call X_0, we change this last equation slightly to make it resemble (6.85):

$$X''(\eta) + k^2(1 + e^{\rho_0 \eta})X = -k^2 \delta(a_0^4) X_0 \tag{6.112}$$

We can solve this last equation by the method of variation of parameters. First we postulate a solution

$$X(\eta) = \gamma_1(\eta) J_{\frac{2ik}{\rho_0}}\left(\frac{2k}{\rho_0} e^{\rho_0 \eta/2}\right) + \gamma_2(\eta) J_{\frac{-2ik}{\rho_0}}\left(\frac{2k}{\rho_0} e^{\rho_0 \eta/2}\right) \tag{6.113}$$

Variation of parameters then yields the following results [for c_1, c_2 approximately the same as in the X_0 case, i.e. (6.86, 6.87)]:

$$\begin{aligned}
\gamma_1(\eta) &\simeq \frac{i\pi k^2}{\rho_0} \operatorname{csch} \frac{2\pi k}{\rho_0} \int_0^\eta J_{\frac{-2ik}{\rho_0}}\left(\frac{2k}{\rho_0} e^{\rho_0 \tau/2}\right) \delta(a_0^4)(\tau)\, X_0(\tau)\, d\tau + c_1 \\
\gamma_2(\eta) &= \frac{-i\pi k^2}{\rho_0} \operatorname{csch} \frac{2\pi k}{\rho_0} \int_0^\eta J_{\frac{2ik}{\rho_0}}\left(\frac{2k}{\rho_0} e^{\rho_0 \tau/2}\right) \delta(a_0^4)(\tau)\, X_0(\tau)\, d\tau + c_2
\end{aligned} \tag{6.114}$$

From this one can in principle calculate $X(\tau)X^*(\tau')$ and hence $\zeta(\tau, \tau')$. We can immediately see from (6.110, 6.114) that the spectrum perturbation is again proportional to h. However, rather than pursuing the details for this model, it is more instructive to consider a similar perturbed model in an inflationary universe, which has more astrophysical and practical importance.

6.4 Inflationary Universe

6.4.1 Eternal versus Slow-roll Inflation

In this section we consider particle creation of a massless, zero temperature conformally coupled scalar field in a spatially flat FRW universe undergoing a near-exponential (inflationary) expansion. The example of de Sitter space which corresponds to the exact exponential case has been treated in [2]. Here we first solve for a general scale factor $a(t)$ using a slightly different language from [2]—our approach is that developed in this thesis. We then specialise to a spacetime (the Brandenberger-Kahn metric [51]) which has initial de Sitter behaviour but with scale factor tending toward a constant at late (cosmic) times. We can also define the parameter h which measures the departure from an exact exponential expansion.

The calculation has been done for arbitrary temperature in appendix F. Set $T = 0$ in (F.15) to get the following equation [where x, y are defined in that appendix, eqn (F.6)]:

$$C = \frac{-4}{\pi k} \int_0^\infty \left[\frac{d}{d\Delta} \frac{\cos k\Delta}{x} \right] \frac{d\Delta}{y} \tag{6.115}$$

This equation is the main tool of this section: it allows us to compute the spectrum corresponding to an arbitrary scale factor. For example, in the de Sitter case with $a = e^{Ht}$ we use

$$x = 2e^{H\Sigma} \operatorname{ch} \frac{H\Delta}{2} \quad ; \quad y = \frac{2e^{-H\Sigma}}{H} \operatorname{sh} \frac{H\Delta}{2} \tag{6.116}$$

which when substituted into (6.115) gives

$$C = \coth \frac{\pi k}{H} \tag{6.117}$$

So for this case we can infer the temperature seen to be

$$T = \frac{H}{2\pi} \tag{6.118}$$

as was calculated using a slightly different approach in [2].

As an aside, we note that from the above analysis for a general scale factor, the noise kernel is

$$\nu = \frac{-\varepsilon^2}{\pi^3} \int_0^\infty dk \cos k\Delta \int_0^\infty \left[\frac{d}{d\Delta} \frac{\cos k\Delta}{x} \right] \frac{d\Delta}{y} \tag{6.119}$$

with dissipation

$$\mu = \frac{\varepsilon^2 \delta'(\Delta)}{4\pi} \tag{6.120}$$

An often-used alternative to our principal part prescription is the introduction of a cutoff in the $\int_0^\infty$ expressions; unfortunately following this procedure doesn't lead to tractable integrals even for the relatively simple de Sitter case.

Note that in equation (6.115) for the temperature in the general case, we are essentially dealing with products of a and η, and it's therefore not surprising that for de Sitter expansion, where $a \propto 1/\eta$, that (6.115) can be done analytically. For other forms of a, even very simple ones, (6.115) becomes very complicated.

6.4.2 Near exponential expansion

In this section we consider the case of a near de Sitter universe with a scale factor composed of the usual de Sitter one together with a factor that decays exponentially. We show that the spectrum seen is near-thermal tending toward thermal at late times.

We start by considering the Hubble parameter to have a constant value (characterising de Sitter space) plus an exponentially decaying term:

$$H(t) = H_0 \left(1 + \alpha e^{-\beta H_0 t}\right) \tag{6.121}$$

and from this our aim is to calculate C using (6.115). The scale factor then follows:

$$a(t) = \exp\left(H_0 t - \frac{\alpha}{\beta} e^{-\beta H_0 t}\right) \tag{6.122}$$

We can define a parameter h which measures the departure from exact exponential expansion to be

$$h(t) \equiv \frac{\dot{H}(t)}{H(t)^2} \rightarrow -\alpha\beta e^{-\beta H_0 t} \quad \text{as } \beta t \rightarrow \infty \tag{6.123}$$

and as we might expect it decays exponentially at late times.

To proceed we indicate the de Sitter quantities by a subscript zero as well as writing

$$\widetilde{\Sigma} = H_0 \Sigma \quad ; \quad \widetilde{\Delta} = H_0 \Delta \tag{6.124}$$

so that from (6.116),

$$x_0 = 2e^{\widetilde{\Sigma}} \operatorname{ch} \frac{\widetilde{\Delta}}{2} \quad ; \quad y_0 = \frac{2e^{-\widetilde{\Sigma}}}{H} \operatorname{sh} \frac{\widetilde{\Delta}}{2} \tag{6.125}$$

We wish to perturb these by using the new scale factor. Suppose we write

$$\begin{aligned} x &= x_0\left(1 + f_1(\widetilde{\Sigma}, \widetilde{\Delta})\right) \\ y &= y_0\left(1 + f_2(\widetilde{\Sigma}, \widetilde{\Delta})\right) \end{aligned} \tag{6.126}$$

We first have

$$x = a(t) + a(t') = e^{H_0 t - \frac{\alpha}{\beta} e^{-\beta H_0 t}} + e^{H_0 t' - \frac{\alpha}{\beta} e^{-\beta H_0 t'}} \tag{6.127}$$

In the late time limit we can approximate this by

$$x \simeq x_0 - \frac{2\alpha}{\beta} e^{(1-\beta)\widetilde{\Sigma}} \operatorname{ch} \frac{(1-\beta)\widetilde{\Delta}}{2} \tag{6.128}$$

Then f_1 follows:

$$f_1 = \frac{-\alpha}{\beta} e^{-\beta\widetilde{\Sigma}} \frac{\operatorname{ch} \frac{(1-\beta)\widetilde{\Delta}}{2}}{\operatorname{ch} \frac{\widetilde{\Delta}}{2}} \tag{6.129}$$

Next we write

$$\begin{aligned} y = \eta(t) - \eta(t') &= \int_{t'}^{t} \frac{dt}{a(t)} \\ &= \int_{t'}^{t} \exp\left(-H_0 t + \frac{\alpha}{\beta} e^{-\beta H_0 t}\right) \end{aligned} \tag{6.130}$$

and by making the same late time approximation as for x we get

$$\begin{aligned} y &\simeq \int_{t'}^{t} e^{-H_0 t}\left(1+\frac{\alpha}{\beta}e^{-\beta H_0 t}\right) \\ &= y_0 + \frac{2\alpha\, e^{-(1+\beta)\tilde{\Sigma}}}{\beta(1+\beta)H_0}\,\mathrm{sh}\,\frac{(1+\beta)\tilde{\Delta}}{2} \end{aligned} \tag{6.131}$$

This leads to

$$f_2 = \frac{\alpha\, e^{-\beta\tilde{\Sigma}}}{\beta(1+\beta)}\frac{\mathrm{sh}\,\frac{(1+\beta)\tilde{\Delta}}{2}}{\mathrm{sh}\,\frac{\tilde{\Delta}}{2}} \tag{6.132}$$

Note that at late times f_1, f_2 tend to zero. In that case to calculate C we write (6.115) in the form

$$C \simeq \frac{-4}{\pi k}\int_0^\infty \frac{d}{d\Delta}\left[\frac{\cos k\Delta}{x_0}(1-f_1)\right]\frac{1-f_2}{y_0}\,d\Delta \tag{6.133}$$

and so write, to first order in f_1, f_2:

$$C \simeq \underbrace{\frac{-4}{\pi k}\int_0^\infty \frac{d\Delta}{y_0}\frac{d}{d\Delta}\frac{\cos k\Delta}{x_0}}_{=\coth \pi k/H_0} + \underbrace{\frac{4}{\pi k}\int_0^\infty \frac{d\Delta}{y_0}\left[f_2\frac{d}{d\Delta}\frac{\cos k\Delta}{x_0} + \frac{d}{d\Delta}\frac{f_1\cos k\Delta}{x_0}\right]}_{\equiv \Delta C,\ \text{the perturbation}} \tag{6.134}$$

Evaluating ΔC is lengthy but straightforward so we merely write the answer in terms of an integral:

$$\begin{aligned} \Delta C = \frac{H_0^2\alpha\, e^{-\beta\tilde{\Sigma}}}{2\pi k\beta}\int_0^\infty d\Delta\Bigg[&\frac{-2k}{H_0(1+\beta)}\frac{\mathrm{sh}\,\frac{(1+\beta)\tilde{\Delta}}{2}\sin k\Delta}{\mathrm{sh}^2\frac{\tilde{\Delta}}{2}\,\mathrm{ch}\,\frac{\tilde{\Delta}}{2}} - \frac{1}{1+\beta}\frac{\mathrm{sh}\,\frac{(1+\beta)\tilde{\Delta}}{2}\cos k\Delta}{\mathrm{ch}^2\frac{\tilde{\Delta}}{2}\,\mathrm{sh}\,\frac{\tilde{\Delta}}{2}} \\ &+\frac{2k}{H_0}\frac{\mathrm{ch}\,\frac{(1-\beta)\tilde{\Delta}}{2}\sin k\Delta}{\mathrm{ch}^2\frac{\tilde{\Delta}}{2}\,\mathrm{sh}\,\frac{\tilde{\Delta}}{2}} - \frac{(1-\beta)\,\mathrm{sh}\,\frac{(1-\beta)\tilde{\Delta}}{2}\cos k\Delta}{\mathrm{ch}^2\frac{\tilde{\Delta}}{2}\,\mathrm{sh}\,\frac{\tilde{\Delta}}{2}} + \frac{2\,\mathrm{ch}\,\frac{(1-\beta)\tilde{\Delta}}{2}\cos k\Delta}{\mathrm{ch}^3\frac{\tilde{\Delta}}{2}}\Bigg] \end{aligned} \tag{6.135}$$

The important point is that the factor $e^{-\beta\tilde{\Sigma}}$ ensures that this perturbation to the thermal spectrum dies off exponentially at late times.

6.4.3 Brandenberger-Kahn model

We are now in a position to derive the function $C(k, \Sigma)$ for the Brandenberger-Kahn model. In this case,

$$a(t) = e^{\frac{2H_0}{\alpha}\left(1-e^{-\alpha t/2}\right)} \tag{6.136}$$

with H_0, α constants. As t tends toward zero and infinity, $a(t)$ tends toward e^{Ht} and $e^{2H/\alpha}$ respectively. The Hubble expansion function is

$$H(t) \equiv \frac{\dot{a}}{a} = H_0 e^{-\alpha t/2} \tag{6.137}$$

and the parameter $h(t)$ which measures the departure from exact exponential expansion is

$$h \equiv \frac{\dot{H}(t)}{H(t)^2} = -\frac{\alpha}{2H_0}e^{\alpha t/2} = \frac{-\alpha}{2H_0} + O(\alpha^2 t^2) \tag{6.138}$$

We assume that $|\alpha t| \ll 1$. Equation (6.115) is much too difficult to evaluate analytically here, but we can get some insight by calculating it as a first order correction in h to the de Sitter case.

At this point, we also mention an alternative perturbation of de Sitter space, given by the scale factor

$$a(t) = e^{\int_0^t H(t)dt} \tag{6.139}$$

which describes a solution of the vacuum Einstein equations with a time-dependent cosmological constant $\Lambda(t) = 3H^2(t)$. One may expand $H(t)$ in a power series about $t = 0$. Defining h as in (6.138), this form of perturbation turns out to be identical to the Brandenberger-Kahn model to first order in h. We have to first order, from (6.138),

$$H(t) = H_0 + H_0^2 ht \tag{6.140}$$

We will therefore calculate the detector response for the Brandenberger-Kahn model only, keeping in mind its correspondence with this last model.

Again define f_1, f_2 as in (6.126). The corrections are then written as

$$\begin{aligned} f_1(\Delta) &= \frac{h}{2}\left[\tilde{\Sigma}^2 + \frac{\tilde{\Delta}^2}{4} + \tilde{\Sigma}\tilde{\Delta}\,\mathrm{th}\,\frac{\tilde{\Delta}}{2}\right] \\ f_2(\Delta) &= \frac{h}{2}\left[\left(\tilde{\Sigma}+1\right)\left(\tilde{\Delta}\coth\frac{\tilde{\Delta}}{2} - 2\right) - \left(\tilde{\Sigma}^2 + \frac{\tilde{\Delta}^2}{4}\right)\right] \end{aligned} \tag{6.141}$$

After some computation we obtain

$$C(k, \Sigma) = (1 + h\Gamma_1)\,\coth\frac{\pi k}{H_0} \tag{6.142}$$

a form which shows its approximately thermal nature, with

$$\Gamma_1 = \frac{-\tilde{\Sigma}\frac{\pi k}{H_0}}{\mathrm{sh}\,\frac{\pi k}{H_0}} + \frac{\tilde{\Sigma}}{\mathrm{ch}\,\frac{\pi k}{H_0}} + (\tilde{\Sigma}+1)\left[1 - \left(\mathrm{th}\,\frac{\pi k}{H_0}\right)\frac{2}{\pi}\int_0^\infty \frac{u \sin\frac{2ku}{H_0}}{\mathrm{sh}^2 u}\,du\right] \tag{6.143}$$

(As a function of k/H_0 the unevaluated integral looks much like $\tan^{-1}$, tending to $\pi/2$ as $k/H_0 \to \infty$.)

In the low frequency limit the departure from a thermal spectrum is, to $O(k^2)$:

$$h\Gamma_1 \simeq h\left[\tilde{\Sigma} + 1 - (\tilde{\Sigma} + 2/3)\left(\frac{\pi k}{H_0}\right)^2\right] \sim h\tilde{\Sigma} \tag{6.144}$$

Note that we stipulated that $|h\widetilde{\Sigma}| \sim |\alpha t| \ll 1$, so that $h\Gamma_1$ remains small as time passes. In the high frequency limit the departure is given by

$$h\Gamma_1 \to -2h\widetilde{\Sigma}\, e^{-\pi k/H_0} \left(\frac{\pi k}{H_0} - 1\right) \tag{6.145}$$

which again remains small, and is especially close to zero for high frequencies.

6.5 Conclusion

The influence functional approach to thermal radiance is useful in that it places the origin of the radiance in the excitations produced in the vacuum by a detector, and allows these excitations to be described in a statistical way via the noise and dissipation of the environment.

All of the variables which characterise the models we considered, whether mirror trajectories, spherical infalling modes or cosmological scale factors, have had one thing in common: they all contained an exponential growth in scale. This is what gives rise to the Planckian spectrum of thermal radiance, and by specifying the parameter h which measures any departure from an exact exponential transformation, we have found perturbatively that the departure from a Planckian spectrum is just proportional to h.

We reiterate that the statistical approach can quite adequately cover more general cases of scale transformation; but the calculations appear to be only straightforward for exponential ones—see for example the discussion on page 86 concerning the de Sitter case in equation (6.115).

Appendix A

Calculation of noise and dissipation for a static oscillator bath

We wish to derive the noise and dissipation corresponding to a bath of static oscillators with unit mass at arbitrary temperature. The spectral density has been chosen to be ohmic:

$$I(\omega, s, s') = \frac{2\gamma_0}{\pi}\omega\ c(s)c(s') \tag{A.1}$$

To calculate $X(s)$, note that each bath oscillator has lagrangian

$$L(t) = \frac{1}{2}[\dot{q}^2 - \omega^2 q^2] \tag{A.2}$$

and from (3.20, 3.22, 3.23) the sum of its Bogoliubov coefficients with $t_i = 0$ is

$$X(s) = e^{-i\omega s} \tag{A.3}$$

where we have chosen $\kappa = \omega$ in (3.23). Then from (3.32) the dissipation becomes the distribution

$$\mu(s, s') = \frac{-2\gamma_0}{\pi} c(s)c(s') \int_0^\infty \omega \sin\omega(s - s')\, d\omega \tag{A.4}$$

This integral tends to be seen as obscure, and we will first devote some explanation to calculating it.

Principal part formalism

To calculate the above integral, we first consider the meaning of

$$\int_0^\infty e^{ikx}\, dk \tag{A.5}$$

We can calculate this using a real variable approach as follows. First, we define

$$\begin{aligned}\int_0^\infty e^{ikx}dk &\equiv \lim_{\varepsilon\to 0^+}\int_0^\infty e^{i(x+i\varepsilon)k}dk \\ &= i\lim_{\varepsilon}\frac{1}{x+i\varepsilon} \\ &= \underbrace{\lim_{\varepsilon}\frac{\varepsilon}{x^2+\varepsilon^2}}_{\equiv L_1(x)} + i\underbrace{\lim_{\varepsilon}\frac{x}{x^2+\varepsilon^2}}_{\equiv L_2(x)} \qquad (A.6)\end{aligned}$$

Now we introduce a finite support test function $T(x)$ and integrate L_1 and L_2 with it in turn, freely swapping limits and integrals where necessary. First L_1:

$$\begin{aligned}\int_{-\infty}^{\infty} T(x)L_1(x)dx &= \lim_{\varepsilon}\varepsilon\int\frac{T(x)}{x^2+\varepsilon^2}\,dx \\ &= \lim_{\varepsilon}\left[T(x)\tan^{-1}\frac{x}{\varepsilon}\right]_{x=-\infty}^{\infty} - \lim_{\varepsilon}\int T'(x)\tan^{-1}\frac{x}{\varepsilon}\,dx \\ &= -\int_{-\infty}^{\infty} T'(x)\cdot\begin{Bmatrix}\pi/2 & x>0 \\ -\pi/2 & x<0\end{Bmatrix}dx \\ &= \pi\, T(0) \\ &\equiv \int_{-\infty}^{\infty} T(x)\,\pi\,\delta(x)\,dx \qquad (A.7)\end{aligned}$$

A similar approach gives L_2:

$$\begin{aligned}\int_{-\infty}^{\infty} T(x)L_2(x)dx &= \lim_{\varepsilon}\int_{-\infty}^{\infty}\frac{T(x)\,x}{x^2+\varepsilon^2}\,dx \\ &= \lim_{\varepsilon}\left[\int_{-\infty}^{-\eta}+\int_{-\eta}^{\eta}+\int_{\eta}^{\infty}\right]\frac{T(x)\,x}{x^2+\varepsilon^2}\,dx \\ &= \lim_{\varepsilon}\lim_{\eta\to 0^+}[\text{ditto}]\,\frac{T(x)\,x}{x^2+\varepsilon^2}\,dx \\ &= \lim_{\varepsilon}\lim_{\eta}\left[\int_{-\infty}^{-\eta}+\int_{\eta}^{\infty}\right]\frac{T(x)\,x}{x^2+\varepsilon^2}\,dx + \lim_{\varepsilon}\lim_{\eta}\int_{-\eta}^{\eta}\frac{T(x)\,x}{x^2+\varepsilon^2}\,dx \\ &= \mathrm{PV}\int_{-\infty}^{\infty}\frac{T(x)}{x}\,dx + \lim_{\varepsilon}T(0)\lim_{\eta}\int_{-\eta}^{\eta}\frac{x}{x^2+\varepsilon^2}\,dx \\ &= \mathrm{PV}\int_{-\infty}^{\infty}\frac{T(x)}{x}\,dx + \lim_{\varepsilon}T(0)\lim_{\eta}\left[\frac{1}{2}\ln(x^2+\varepsilon^2)\right]_{-\eta}^{\eta} \\ &= \mathrm{PV}\int_{-\infty}^{\infty}\frac{T(x)}{x}\,dx \quad . \\ &\equiv \int_{-\infty}^{\infty} T(x)\,P(1/x)\,dx \qquad (A.8)\end{aligned}$$

We finally summarise:

$$\int_0^\infty e^{ikx}\,dk = \pi\delta(x) + i\,P(1/x) \qquad (A.9)$$

Note that the principal part distribution has effectively been defined as

$$P(1/x) = \lim_{\varepsilon\to 0^+}\frac{x}{x^2+\varepsilon^2} \qquad (A.10)$$

and just as the delta function can be visualised as the limit of a sequence of, say, gaussians, so the principal part can be visualised as identical to $1/x$ for $x \neq 0$, while continuous at $x = 0$ and equal to zero there.

Having said this, any impulse to treat $P(1/x)$ just like $1/x$ can give wrong answers. In particular, does $d/dx\ P(1/x)$ equal $-P(1/x^2)$ as is claimed in [52]? We need only test this by integrating each of these with a test function, say e^{-x^2}, for $x = -\infty \to \infty$. For $d/dx\ P(1/x)$ the integral converges; for $-P(1/x^2)$ it doesn't. So these two distributions are certainly not the same.

μ and ν from (A.9)

Differentiating (A.9) w.r.t. x gives

$$\int_0^\infty \omega e^{i\omega x}\, d\omega = -i\pi\delta'(x) + \frac{d}{dx}P(1/x) \tag{A.11}$$

The dissipation then follows straight away:

$$\begin{aligned}\mu(s,s') &= \frac{-2\gamma_0}{\pi}\, c(s)c(s')\mathrm{Im}\int_0^\infty \omega e^{i\omega(s-s')}\, d\omega \\ &= 2\gamma_0\, c(s)c(s')\delta'(s-s') \end{aligned} \tag{A.12}$$

This form of the dissipation is called local; it is physically reasonable and justifies the choice of the spectral density, as well as simplifying subsequent calculations.

Calculating the noise is more difficult; from (3.32):

$$\begin{aligned}\nu(s,s') &= \int_0^\infty \coth\frac{\omega}{2T} I(\omega,s,s')\cos\omega(s-s')\, d\omega \\ &= \frac{2\gamma_0}{\pi}\, c(s)c(s')\int_0^\infty \omega\coth\frac{\omega}{2T}\cos\omega(s-s')\, d\omega \end{aligned} \tag{A.13}$$

This integral can be calculated by first expanding the coth term, with $\Delta \equiv s - s'$ for brevity:

$$\int_0^\infty \omega\coth\frac{\omega}{2T}\cos\omega\Delta\, d\omega = \underbrace{\int_0^\infty \omega\cos\omega\Delta\, d\omega}_{I_1} + \underbrace{2\int_0^\infty \frac{\omega\cos\omega\Delta}{e^{\omega/T}-1}\, d\omega}_{I_2} \tag{A.14}$$

Then

$$\begin{aligned} I_1 &= \mathrm{Re}\int_0^\infty \omega e^{i\omega\Delta}\, d\omega \\ &= \frac{d}{d\Delta}P(1/\Delta) \end{aligned} \tag{A.15}$$

which is a distribution identical to $-1/\Delta^2$ for $\Delta \neq 0$, but is continuous with an upward-pointing spike at $\Delta = 0$. Also, I_2 becomes

$$I_2 = \frac{1}{\Delta^2} - \pi^2 T^2 \mathrm{csch}^2\, \pi T \Delta \tag{A.16}$$

and so we arrive at the general form for the noise[1]:

$$\nu(s, s') = \frac{2\gamma_0}{\pi} c(s) c(s') \left[\frac{d}{d\Delta} P(1/\Delta) + \frac{1}{\Delta^2} - \pi^2 T^2 \mathrm{csch}^2\, \pi T \Delta \right] \tag{A.20}$$

This expression is nonlocal: that is, this is an example of the presence of weak damping without an associated Markov dynamics. Normally it's assumed that weak damping necessarily implies Markov dynamics [55]. Here we see that this needn't be so.

When $c(s) = 1$ we can show this expression for the noise becomes the usual delta in the limit of high T. First, we notice that as $T \to \infty$, $d/d\Delta\ P(1/\Delta) + 1/\Delta^2$ becomes localised at the origin, as does the csch^2 term. If we then integrate $\nu(\Delta) \equiv \nu(s, s')$ with a test function we get

$$\int_{-\infty}^{\infty} f(\Delta) \nu(\Delta)\, d\Delta \;\; = \;\; f(0) \int_{-\infty}^{\infty} \nu(\Delta)\, d\Delta$$

[1] In reference [53] Boyer calculates $I_1 = -1/\Delta^2$, contrary to the usual approach of using generalised functions, and concludes

$$\int_0^{\infty} d\omega\, \omega \coth \frac{\omega}{2T} \cos \omega \Delta = -\pi^2 T^2 \mathrm{csch}^2\, \pi T \Delta \tag{A.17}$$

This is manifestly wrong, since we expect I_1 to be a generalised function. Further, we can show it is wrong as follows: first, introduce a test function, say $e^{-\Delta^2}$, and integrate this with the noise by swapping the order of integration:

$$\begin{aligned} \int_{-\infty}^{\infty} d\Delta\, e^{-\Delta^2} \int_0^{\infty} d\omega\, \omega \coth \frac{\omega}{2T} \cos \omega \Delta &= \int_0^{\infty} d\omega\, \omega \coth \frac{\omega}{2T} \int_{-\infty}^{\infty} d\Delta\, e^{-\Delta^2} \cos \omega \Delta \\ &= \int_0^{\infty} d\omega\, \omega \coth \frac{\omega}{2T}\, \mathrm{Re} \int_{-\infty}^{\infty} d\Delta\, e^{-\Delta^2 + i\omega\Delta} \\ &= \sqrt{\pi} \int_0^{\infty} d\omega\, \omega \coth \frac{\omega}{2T} e^{\frac{-\omega^2}{4}} \end{aligned} \tag{A.18}$$

This is finite. In contrast, Boyer's result (A.17) gives .

$$\int_{-\infty}^{\infty} d\Delta\, e^{-\Delta^2} \int_0^{\infty} d\omega\, \omega \coth \frac{\omega}{2T} \cos \omega \Delta = -\pi^2 T^2 \int_{-\infty}^{\infty} d\Delta\, e^{-\Delta^2} \mathrm{csch}^2\, \pi T \Delta \tag{A.19}$$

the integrand of which is an even function, everywhere positive, and divergent at $\Delta = 0$ like $1/\Delta^2$; so the integral on the right hand side of this last equation must diverge, which we know is not true. Hence (A.17) is wrong.

Note that [54] has essentially arrived at our result (A.20), although there the principal part notation has not been taken advantage of, and instead the answer is expressed in terms of the limit in (A.10). This is not really useful, missing as it does the utility of the principal part formalism.

$$
\begin{aligned}
&= f(0)\frac{2\gamma_0}{\pi}\left[P(1/\Delta) - \frac{1}{\Delta} + \pi T \coth\, \pi T \Delta\right]_{-\infty}^{\infty} \\
&= 4\gamma_0 T\, f(0) \\
&= \int_{-\infty}^{\infty} f(\Delta)\, 4\gamma_0 T\, \delta(\Delta)\, d\Delta
\end{aligned}
$$

and thus for $T \to \infty$,

$$
\nu(\Delta) = 4\gamma_0 T\, \delta(\Delta) \tag{A.21}
$$

which is the well known form for white noise.

Evident here is a convergence subtlety that is not being addressed, since T appears in this expression for the noise. The usual approach taken in deriving the high temperature form offers some help here and is as follows. From (A.13) we write (with $c(s) = 1$ for brevity)

$$
\nu(s, s') = \frac{2\gamma_0}{\pi}\int_0^{\infty} \omega \coth \frac{\omega}{2T} \cos \omega\Delta \, d\omega \tag{A.22}
$$

Now strictly speaking this integral is only defined for some frequency cutoff, so we write

$$
\nu(s, s') = \frac{2\gamma_0}{\pi}\int_0^{\omega_{\max}} \omega \coth \frac{\omega}{2T} \cos \omega\Delta \, d\omega \tag{A.23}
$$

When $\omega_{\max} \ll T$ the argument of the coth will always be small, so we can approximate it for small argument and write

$$
\nu(s, s') \simeq \frac{4\gamma_0 T}{\pi}\int_0^{\omega_{\max}} \cos \omega\Delta \, d\omega \tag{A.24}
$$

Then as $\omega_{\max} \to \infty$ this tends toward $4\gamma_0 T \delta\Delta$. So this is the condition (at least a sufficient one) for which we can use the delta function as the high temperature noise:

$$
\omega_{\max} \ll T \tag{A.25}
$$

Although we have calculated the noise explicitly in (A.20), in fact it turns out that this form can be difficult to work with analytically. Actually, in appendix F we do make use of this explicit expression. However in chapters 4 and 5 we work with the original expression (A.13) as an integral over ω.

Appendix B

Calculating the relevant lagrangians

This appendix derives the relevant cosmological lagrangians we have used.

We always deal with a spatially flat FRW universe, containing a scalar field Φ of mass m, coupled to the curvature by ξ. The lagrangian density for this field is

$$\mathcal{L} = \frac{\sqrt{-g}}{2}\left[g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} - \left(m^2 + \xi R\right)\Phi^2\right] \tag{B.1}$$

We calculate the lagrangian first in conformal time ($' = d/d\eta$), then in cosmic time ($\dot{} = d/dt$).

Conformal time

The metric is

$$ds^2 = a^2(\eta)\left[d\eta^2 - \sum_i (dx^i)^2\right] \tag{B.2}$$

For the FRW universe, $R = 6a''/a^3$, and by rescaling the field with $\chi = a\Phi$ we obtain a new lagrangian density

$$\mathcal{L} = \frac{1}{2}\left[\chi'^2 - \sum \chi_{,i}^2 - 2\frac{a'}{a}\chi\chi' - \chi^2\left(m^2a^2 - \frac{a'^2}{a^2} + 6\xi\frac{a''}{a}\right)\right] \tag{B.3}$$

We wish to place the field in a box of side L, and express χ in terms of plane wave modes q. Fourier expanding gives

$$\chi(\eta, \mathbf{x}) = \sqrt{\frac{2}{L^3}} \sum_{k_x,k_y,k_z=-\infty}^{\infty} q_{\mathbf{k}}(\eta)e^{i\mathbf{k}\cdot\mathbf{x}} \tag{B.4}$$

where the normalisation will be justified later. We can split the field into two sets, one with positive k_z's and the other with negative (discarding $k_z = 0$ as we will take the continuum limit eventually) and write

$$\chi(\eta,\mathbf{x}) = \sqrt{\frac{2}{L^3}} \underbrace{\sum_{k_z>0} \sum_{k_x,k_y=-\infty}^{\infty}}_{\equiv \sum_{\mathbf{k}>0}} q_{\mathbf{k}}(\eta)e^{i\mathbf{k}\cdot\mathbf{x}} + q_{-\mathbf{k}}(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{B.5}$$

$$= \sqrt{\frac{2}{L^3}} \sum_{\mathbf{k}>0} (q_{\mathbf{k}} + q_{-\mathbf{k}}) \cos \mathbf{k}\cdot\mathbf{x} + i(q_{\mathbf{k}} - q_{-\mathbf{k}}) \sin \mathbf{k}\cdot\mathbf{x}$$

$$\equiv \sqrt{\frac{2}{L^3}} \sum_{\mathbf{k}>0} q_{\mathbf{k}}^{+} \cos \mathbf{k}\cdot\mathbf{x} + q_{\mathbf{k}}^{-} \sin \mathbf{k}\cdot\mathbf{x} \tag{B.6}$$

As an aside, given the normalisation we have chosen in (B.4) it follows that we can second quantise the modes by taking $\omega \equiv \sqrt{\mathbf{k}\cdot\mathbf{k} + m^2} \equiv \sqrt{k^2 + m^2}$ and

$$q_{\mathbf{k}} = \frac{e^{-i\omega\eta} a}{2\sqrt{\omega}} \quad , \quad q_{-\mathbf{k}} = \frac{e^{i\omega\eta} a^{\dagger}}{2\sqrt{\omega}} \tag{B.7}$$

Then (B.5) will become

$$\chi(\eta,\mathbf{x}) = \frac{1}{\sqrt{L^3}} \sum_{\mathbf{k}>0} \frac{e^{i\mathbf{k}\cdot\mathbf{x} - i\omega\eta} a}{\sqrt{2\omega}} + \text{h.c.} \tag{B.8}$$

which is the usual expression in terms of raising and lowering operators. This last expression is the starting point for appendix E.

Now, to calculate $L(\eta) = \int \mathcal{L}(x) d^3x$, we need expressions for $\int \chi^2 d^3x$ etc. Write χ in terms of real modes, (B.6), and use

$$\int_0^L \cos \mathbf{k}\cdot\mathbf{x} \ \cos \boldsymbol{\ell}\cdot\mathbf{x} \ d^3x = \int_0^L \sin \mathbf{k}\cdot\mathbf{x} \ \sin \boldsymbol{\ell}\cdot\mathbf{x} \ d^3x = \frac{L^3}{2} \delta_{\mathbf{k},\boldsymbol{\ell}}$$

$$\int_0^L \cos \mathbf{k}\cdot\mathbf{x} \ \sin \boldsymbol{\ell}\cdot\mathbf{x} \ d^3x = 0 \tag{B.9}$$

so that after some straightforward calculation we obtain

$$\int_0^L \chi^2 d^3x = \sum_{\mathbf{k}>0} q_{\mathbf{k}}^{+2} + q_{\mathbf{k}}^{-2} \equiv \sum q^2 \tag{B.10}$$

which motivates the original choice of normalisation in (B.4). Similarly,

$$\int_0^L \chi'^2 d^3x = \sum q'^2 \quad , \quad \int_0^L \sum_i \chi_{,i}^{\,2} \, d^3x = \sum k^2 q^2 \quad , \quad \int_0^L \chi\chi' d^3x = \sum q q' \tag{B.11}$$

Now it becomes a simple matter to use these to calculate the lagrangian from the density (B.3), and we merely state the result:

$$L(\eta) = \sum \frac{1}{2} \left[q'^2 - 2\frac{a'}{a} q q' - q^2 \left(k^2 + m^2 a^2 - \frac{a'^2}{a^2} + 6\xi \frac{a''}{a} \right) \right] \tag{B.12}$$

Inside the brackets we now add a surface term of $6\xi(q^2a'/a)'$ to eliminate the a'' term, which now leads to the final, new lagrangian:

$$L_{new}(\eta) = \sum \frac{1}{2}\left[q'^2 + 2(6\xi - 1)\frac{a'}{a}qq' - q^2\left(k^2 + m^2a^2 + (6\xi - 1)\frac{a'^2}{a^2}\right)\right] \tag{B.13}$$

This becomes, for a massless conformally coupled field:

$$L_{new}(\eta) = \sum \frac{1}{2}\left[q'^2 - k^2q^2\right] \tag{B.14}$$

and for a massless minimally coupled field:

$$L_{new}(\eta) = \sum \frac{1}{2}\left[q'^2 - 2\frac{a'}{a}qq' - q^2\left(k^2 - \frac{a'^2}{a^2}\right)\right] \tag{B.15}$$

which has the de Sitter space form:

$$L_{new}(\eta) = \sum \frac{1}{2}\left[q'^2 + \frac{2}{\eta}qq' - q^2\left(k^2 - \frac{1}{\eta^2}\right)\right] \tag{B.16}$$

Cosmic time

Here the metric is

$$ds^2 = dt^2 - a^2(t)\sum_i (dx^i)^2 \tag{B.17}$$

The calculation of course follows through in much the same way, except that the Ricci scalar is now $R = 6(\dot{a}^2/a^2 + \ddot{a}/a)$. The lagrangian density (B.1) becomes

$$\mathcal{L} = \frac{a^3}{2}\left[\dot{\Phi}^2 - \frac{1}{a^2}\sum_i \Phi_{,i}^2 - \left\{m^2 + 6\xi\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)\right\}\Phi^2\right] \tag{B.18}$$

We choose not to scale the Φ into a χ as before, since doing so turns out to produce a more complicated equation of motion (i.e. now we are just expanding Φ itself in plane waves). The lagrangian becomes

$$L(t) = \sum \frac{a^3}{2}\left[\dot{q}^2 - q^2\left\{\frac{k^2}{a^2} + m^2 + 6\xi\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)\right\}\right] \tag{B.19}$$

Now we add a surface term $6\xi/a^3\ d/dt\ (a^2\dot{a}q^2)$ within the brackets to remove the $\ddot{a}$:

$$L_{new}(t) = \sum \frac{a^3}{2}\left[\dot{q}^2 + 12\xi\frac{\dot{a}}{a}q\dot{q} - q^2\left(\frac{k^2}{a^2} + m^2 - 6\xi\frac{\dot{a}^2}{a^2}\right)\right] \tag{B.20}$$

These lagrangians are used in the body of this thesis, although there the "new" subscript has been omitted.

Appendix C

Calculation of a_{ij}'s in chapter 4

de Sitter with high temperature

Here we evaluate the a_{ij}'s leading to (4.72). We are using the following small z, ζ approximations:

$$\begin{aligned}
\frac{\text{Im}\,[X(z)X^*(\zeta)]}{\text{Im}\,X(z)} &\xrightarrow{z\to 0} \cos\zeta - \sin\zeta/\zeta + O(z^2) \equiv f_1(\zeta) + O(z^2) \\
&\xrightarrow{z,\zeta\to 0} (-\zeta^2 + z^3/\zeta)/3 \\
\frac{\text{Im}\,X(\zeta)}{\text{Im}\,X(z)} &\xrightarrow{z\to 0} z(\cos\zeta/\zeta + \sin\zeta) \equiv z f_2(\zeta) \\
&\xrightarrow{z,\zeta\to 0} z/\zeta
\end{aligned} \tag{C.1}$$

Firstly,

$$\begin{aligned}
a_{12} &= \frac{1}{k^2}\int_{z_i}^{z} d\zeta \int_{z_i}^{z} d\zeta' \left(\frac{\zeta}{z_i}\right)^{-c} \frac{\text{Im}\,[X(z)X^*(\zeta)]}{\text{Im}\,X(z)} \frac{4ck^2T}{-\zeta}\,\delta(\zeta-\zeta')\left(\frac{z}{\zeta'}\right)^{c} \frac{\text{Im}\,X(\zeta')}{\text{Im}\,X(z)} \\
&= 4cT\int_{z_i}^{z} d\zeta \left(\frac{\zeta}{z_i}\right)^{-c} \frac{\text{Im}\,[X(z)X^*(\zeta)]}{\text{Im}\,X(z)} \frac{1}{-\zeta}\left(\frac{z}{\zeta}\right)^{c} \frac{\text{Im}\,X(\zeta)}{\text{Im}\,X(z)} \\
&\propto cT|z|^c\left[\int_{z_i}^{\lambda} d\zeta\, |\zeta|^{-2c-1} f_1(\zeta)\, z\, f_2(\zeta) + \int_{\lambda}^{z} d\zeta\, |\zeta|^{-2c-1}(-\zeta^2 + z^3/\zeta)\,\frac{z}{3\zeta}\right] \\
&= cT\, O|z|^{c+1}
\end{aligned} \tag{C.2}$$

provided $c < 1/2$. Finally,

$$\begin{aligned}
a_{22} &= \frac{1}{2k^2}\int_{z_i}^{z} d\zeta \int_{z_i}^{z} d\zeta' \left(\frac{z}{\zeta}\right)^{c} \frac{\text{Im}\,X(\zeta)}{\text{Im}\,X(z)} \frac{4ck^2T}{-\zeta}\,\delta(\zeta-\zeta')\left(\frac{z}{\zeta'}\right)^{c} \frac{\text{Im}\,X(\zeta')}{\text{Im}\,X(z)} \\
&= 2cT\int_{z_i}^{z} d\zeta \left(\frac{z}{\zeta}\right)^{2c} \left(\frac{\text{Im}\,X(\zeta)}{\text{Im}\,X(z)}\right)^{2} \frac{1}{-\zeta} \\
&\propto cT|z|^{2c}\left[\int_{z_i}^{\lambda} d\zeta\, |\zeta|^{-2c-1} f_2^2(\zeta)\, z^2 + \int_{\lambda}^{z} d\zeta\, |\zeta|^{-2c-1} z^2/\zeta^2\right] \\
&= cT\, O(1)
\end{aligned} \tag{C.3}$$

de Sitter with finite temperature

We leave the frequency integration until last:

$$\begin{aligned}\nu &= \frac{2c}{\pi}\frac{1}{\sqrt{ss'}}\int_0^\infty \omega\coth\frac{\omega}{2T}\cos\omega(s-s')\,d\omega \\ &= \frac{2c}{\pi}\frac{k^3}{\sqrt{\zeta\zeta'}}\int_0^\infty d\hat\omega\,\hat\omega\coth\frac{\hat\omega}{2T}\cos\hat\omega(\zeta-\zeta')\end{aligned} \tag{C.4}$$

The a_{ij}'s are

$$\begin{aligned}a_{11} &= z_i^{2c}\frac{ck}{\pi}\int_0^\infty d\hat\omega\,\hat\omega\coth\frac{\hat\omega k}{2T}\times \\ &\underbrace{\int_{z_i}^z d\zeta\int_{z_i}^z d\zeta'\,|\zeta|^{-c-1/2}\frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}\,X(z)}\cos\hat\omega(\zeta-\zeta')\,|\zeta'|^{-c-1/2}\frac{\mathrm{Im}\,[X(z)X^*(\zeta')]}{\mathrm{Im}\,X(z)}}_{\equiv I_{11}} \\ a_{12} &= (z_iz)^c\frac{2ck}{\pi}\int_0^\infty d\hat\omega\,\hat\omega\coth\frac{\hat\omega k}{2T}\times \\ &\underbrace{\int_{z_i}^z d\zeta\int_{z_i}^z d\zeta'\,|\zeta|^{-c-1/2}\frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}\,X(z)}\cos\hat\omega(\zeta-\zeta')\,|\zeta'|^{-c-1/2}\frac{\mathrm{Im}\,X(\zeta')}{\mathrm{Im}\,X(z)}}_{\equiv I_{12}} \\ a_{22} &= z^{2c}\frac{ck}{\pi}\int_0^\infty d\hat\omega\,\hat\omega\coth\frac{\hat\omega k}{2T}\times \\ &\underbrace{\int_{z_i}^z d\zeta\int_{z_i}^z d\zeta'\,|\zeta|^{-c-1/2}\frac{\mathrm{Im}\,X(\zeta)}{\mathrm{Im}\,X(z)}\cos\hat\omega(\zeta-\zeta')\,|\zeta'|^{-c-1/2}\frac{\mathrm{Im}\,X(\zeta')}{\mathrm{Im}\,X(z)}}_{\equiv I_{22}}\end{aligned} \tag{C.5}$$

Using the expressions from (C.1, 4.65) the first of the inner integrals becomes

$$\begin{aligned}I_{11} &= \int_{z_i}^\lambda d\zeta\,|\zeta|^{-c-1/2}f_1(\zeta)\left[\int_{z_i}^\lambda d\zeta'\,\cos\hat\omega(\zeta-\zeta')\,|\zeta'|^{-c-1/2}f_1(\zeta')\right. \\ &\left.+\int_\lambda^z d\zeta'\,\cos\hat\omega\zeta\,|\zeta'|^{-c-1/2}(-\zeta'^2+z^3/\zeta')/3\right] \\ &+\int_\lambda^z d\zeta\,|\zeta|^{-c-1/2}(-\zeta^2+z^3/\zeta)/3\left[\int_{z_i}^\lambda d\zeta'\,\cos\hat\omega\zeta'\,|\zeta'|^{-c-1/2}f_1(\zeta')\right. \\ &\left.+\int_\lambda^z d\zeta'\,\cos\hat\omega(\zeta-\zeta')\,|\zeta'|^{-c-1/2}(-\zeta'^2+z^3/\zeta'/3)\right]\end{aligned} \tag{C.6}$$

We now have a difficulty. In order to get a reasonably useful analytic result, it will be an advantage to replace the $\cos\hat\omega(\zeta-\zeta')$ term in the fourth integral above by something simpler. We will have competition between $\hat\omega$ increasing in the frequency integral versus z decreasing in time. Suppose then we use a frequency cutoff ω_{max}. In that case we can approximate $\cos\hat\omega(\zeta-\zeta')$ for $\zeta,\zeta'\approx 0$ by choosing $\hat\omega_{max}$ such that $\cos\hat\omega(\zeta-\zeta')\approx 1$ in the fourth integral. This will be true provided

$$\hat\omega_{max} \ll -1/\lambda \tag{C.7}$$

However now we don't expect our result to necessarily agree with the high T result found in (4.71), since there we had taken $\hat{\omega}_{max} \to \infty$, which was made possible by the use of the delta function.

At this point we refer to the discussion of the high temperature limit in appendix A. There it is shown that the high temperature (delta function) regime is that for which $\omega_{max} \ll T$ and $\omega_{max} \to \infty$. This absence of a cutoff in the high temperature limit is usually not stressed, but it forms the most relevant fact here. In general we must impose a cutoff for all finite T values, otherwise the frequency integral is not well defined—unless $T \to \infty$. So we conclude that the regime for which our analysis is valid here is $T \lesssim \omega_{max}$.

With the last cosine set equal to 1 as before, these integrals are all $O(1)$ and therefore so is a_{11}. Next:

$$\begin{aligned} I_{12} &= \int_{z_i}^{\lambda} d\zeta \; |\zeta|^{-c-1/2} f_1(\zeta) \left[\int_{z_i}^{\lambda} d\zeta' \; \cos\hat{\omega}(\zeta-\zeta') \; |\zeta'|^{-c-1/2} f_2(\zeta') \, z \right. \\ &\quad \left. + \int_{\lambda}^{z} d\zeta' \; \cos\hat{\omega}\zeta \; |\zeta'|^{-c-1/2} z/\zeta' \right] \\ &\quad + \int_{\lambda}^{z} d\zeta \; |\zeta|^{-c-1/2} (-\zeta^2 + z^3/\zeta)/3 \left[\int_{z_i}^{\lambda} d\zeta' \; \cos\hat{\omega}\zeta' \; |\zeta'|^{-c-1/2} f_2(\zeta') \, z \right. \\ &\quad \left. + \int_{\lambda}^{z} d\zeta' \; \cos\hat{\omega}(\zeta-\zeta') \; |\zeta'|^{-c-1/2} z/\zeta' \right] \end{aligned} \tag{C.8}$$

Evaluating these integrals gives $I_{12} = O|z|^{-c+1/2}$ so that $a_{12} = O|z|^{1/2}$. Lastly,

$$\begin{aligned} I_{22} &= \int_{z_i}^{\lambda} d\zeta \; |\zeta|^{-c-1/2} f_2(\zeta) \, z \left[\int_{z_i}^{\lambda} d\zeta' \; \cos\hat{\omega}(\zeta-\zeta') \; |\zeta'|^{-c-1/2} f_2(\zeta') \, z \right. \\ &\quad \left. + \int_{\lambda}^{z} d\zeta' \; \cos\hat{\omega}\zeta \; |\zeta'|^{-c-1/2} z/\zeta' \right] \\ &\quad + \int_{\lambda}^{z} d\zeta \; |\zeta|^{-c-1/2} z/\zeta \left[\int_{z_i}^{\lambda} d\zeta' \; \cos\hat{\omega}\zeta' \; |\zeta'|^{-c-1/2} f_2(\zeta') \, z \right. \\ &\quad \left. + \int_{\lambda}^{z} d\zeta' \; \cos\hat{\omega}(\zeta-\zeta') \; |\zeta'|^{-c-1/2} z/\zeta' \right] \\ &= O|z|^{-2c+1} \end{aligned} \tag{C.9}$$

so that $a_{22} = O(z)$.

Appendix D

Calculation of a_{ij}'s in chapter 5

From (3.49) it's apparent that the first thing we need to consider is an expression for the noise. Comparing our bath in (5.6) with (3.24, 3.35) we find that our noise kernel is

$$\nu(\tau,\tau') = \frac{2\gamma_0}{\pi} a^{1/2}(\tau) a^{1/2}(\tau') \int_0^\infty \omega \coth \frac{\omega}{2T_i} \cos\omega(\tau-\tau')\, d\omega \tag{D.1}$$

where the initial temperature we use is

$$T_i = \sqrt{M_{Pl} H} \tag{D.2}$$

(where M_{Pl} is the Planck mass; this form for the temperature follows from one of the slow roll conditions for inflation:

$$H^2 \simeq \frac{V(\Phi)}{M_{Pl}^2} \tag{D.3}$$

together with the potential's being $V(\Phi) \sim T_i^4$ at the start of inflation).

The noise kernel is formally infinite, and so we must find a way of regularising it. As noted in section 5.1, the bath is equivalent to a massless, conformally coupled scalar field (call this field χ) in an FRW background. We show in appendix E that the above noise kernel is directly related to the two point function of this field. We can therefore use known results for the regularised two point function to deduce a consistent (regularised) expression for the noise kernel. For this we will make use of Vilenkin and Ford's result [56] for the two point function.

So, from (D.1, E.8) with $a^{1/2}(\tau) = 1/\sqrt{-H\tau}$ we have (with $c = \gamma_0/H$)

$$\nu(\tau,\tau') = \frac{8\pi c}{\sqrt{\tau\tau'}} \langle\chi\chi\rangle_{sym} \tag{D.4}$$

so that

$$\nu(\tau,\tau) = \frac{8\pi c}{|\tau|}\, 2\langle\chi^2\rangle_{ren} \tag{D.5}$$

Now, choose a cutoff ω_{max} in the noise such that $\nu(\tau, \tau)$ can be equated with Vilenkin and Ford's regularised $\langle\chi^2\rangle$:

$$\nu(\tau,\tau) \overset{(D.1)}{=} \frac{2c}{\pi|\tau|}\int_0^{\omega_{max}} \omega \coth\frac{\omega}{2T_i}\, d\omega \text{ required to } \overset{(D.5)}{=} \frac{8\pi c}{|\tau|}\, 2\langle\chi^2\rangle_{ren} \tag{D.6}$$

But

$$\int_0^{\omega_{max}} \omega \coth\frac{\omega}{2T_i}\, d\omega \simeq \frac{\omega_{max}^2}{2} + \frac{\pi^2 T_i^2}{3} \tag{D.7}$$

so that

$$\frac{\omega_{max}^2}{2} + \frac{\pi^2 T_i^2}{3} \simeq 8\pi^2\langle\chi^2\rangle_{ren} \tag{D.8}$$

Vilenkin and Ford deal with a scalar field ψ which just equals $1/a$ times our bath field which is responsible for the noise. We therefore have

$$\langle\chi^2\rangle_{ren} = a^2 \times \text{Vilenkin and Ford's } \langle\psi^2\rangle = \frac{T_i^2}{12} + \frac{H^2 a^2}{48\pi^2} \tag{D.9}$$

So (D.8) gives (with $\zeta \equiv k\tau$)

$$\omega_{max}^2 \simeq \frac{2\pi^2 T_i^2}{3} + \frac{H^2 a^2}{3} \tag{D.10}$$

At late times the second term will dominate; we ask for the time at which this begins to happen. All we need do here is to set the two terms equal to one another and solve for the time. Using a value of [29]

$$\frac{M_{Pl}}{H} \simeq 10^6 \tag{D.11}$$

we find that the terms are of equal size at around 8 Hubble times after inflation begins. Before this time the temperature term dominates, while afterwards the scale factor has grown large enough for the H^2a^2 to dominate the cutoff frequency. In that case we divide the time into two regimes, each with its own ω_{max}:

$$\omega_{max}^2 \simeq \begin{cases} \frac{2\pi^2 T_i^2}{3} \approx (2T_i)^2 & 0 \leq Ht \lesssim 8 \\ \frac{H^2 a^2}{3} = \frac{1}{3\tau\tau'}\,. & 8 < Ht \end{cases} \tag{D.12}$$

where in the late time regime we have replaced the result from (D.10), $1/\tau$, by a symmetrised version: $1/\sqrt{\tau\tau'}$. We now use this frequency cutoff in (D.1) to calculate the noise in each regime, and these will be labelled $\nu_{<8}$ and $\nu_{>8}$ respectively.

It's not clear that the noise kernel (D.1) describes a bath whose initial temperature is falling exponentially to the de Sitter temperature. This is because the noise has been expressed in conformal time. In appendix F we reexpress it in cosmic time, and show that it does have the expected behaviour.

Before 8 Hubble times. Here $\omega_{max} \simeq 2T_i$, so we replace the coth term in (D.1) by $2T_i/\omega$, integrating up to $2T_i$ to get

$$\nu_{<8}(\zeta,\zeta') = \frac{4ck^2T_i}{\pi\sqrt{\zeta\zeta'}} \frac{\sin\left[\frac{2T_i}{k}(\zeta-\zeta')\right]}{\zeta-\zeta'} \tag{D.13}$$

In terms of cosmic time, this noise can be written in the following way. Working in terms of sum and difference coordinates of two cosmic times s, s', we define

$$S \equiv \frac{Hs + Hs'}{2} \quad ; \quad D \equiv Hs - Hs' \tag{D.14}$$

so that after some algebra we can write

$$\nu_{<8}(\zeta,\zeta') = \frac{2}{\pi}H^2 cT_i\, e^{2S} \frac{\sin\left[\frac{4T_i}{H}e^{-S}\,\mathrm{sh}\,\frac{D}{2}\right]}{\mathrm{sh}\,\frac{D}{2}} \tag{D.15}$$

For some fixed S, if we consider the noise as a function of D, we find that it oscillates more and more rapidly as D increases. So we model this noise by a top hat function of height

$$\nu_{<8}(D=0) = \frac{8}{\pi}cH\,T_i^2 e^S \tag{D.16}$$

and half width determined by the sine's argument equaling π, which leads to

$$D_{half\ width} = 2\ln\left[\frac{H\pi e^S}{4T_i} + \sqrt{\left(\frac{H\pi e^S}{4T_i}\right)^2 + 1}\right] \tag{D.17}$$

Over the period $S = 0 \to 8$, this half width increases from about 0.002 to 3 Hubble times.

After 8 Hubble times. In this regime $\omega_{max} \sim (3\tau\tau')^{-1/2}$, so set the coth equal to one in the noise and integrate to ω_{max}. Defining

$$\Sigma \equiv \frac{\tau+\tau'}{2} \quad ; \quad \Delta \equiv \tau - \tau' \tag{D.18}$$

we have

$$\nu_{>8}(\tau,\tau') = \frac{2c}{\pi\Delta\sqrt{\Sigma^2-\Delta^2/4}}\left[\frac{\sin\frac{\Delta}{\sqrt{3(\Sigma^2-\Delta^2/4)}}}{\sqrt{3(\Sigma^2-\Delta^2/4)}} + \frac{\cos\frac{\Delta}{\sqrt{3(\Sigma^2-\Delta^2/4)}} - 1}{\Delta}\right] \tag{D.19}$$

Again, switch to cosmic time S, D variables via

$$\begin{aligned} \Delta &= \tau - \tau' = \frac{-e^{-Hs}}{H} + \frac{e^{-Hs'}}{H} = \frac{2}{H}e^{-S}\sinh\frac{D}{2} \\ \Sigma &= (\tau+\tau')/2 = \frac{-e^{-S}}{H}\cosh\frac{D}{2} \end{aligned} \tag{D.20}$$

A plot of $\nu_{>8}(S,D)$-vs-D looks much like a sinc function, decaying after a characteristic time of about 3 independently of the value of S. So again we model the noise by a top hat of half width 3 Hubble times, and height given by the noise when $D=0$:

$$\nu_{>8} = \begin{cases} \frac{cH^3e^{3S}}{3\pi} & |D|<3 \\ 0 & 3<|D| \end{cases} \tag{D.21}$$

Switching to cosmic time sum and difference variables S, D, allows the noise to be modelled by a top hat, which would not be possible using conformal time sum and difference variables. However the price to be paid for this simplifying of the noise is that the integration becomes more complicated when written in S, D variables.

To calculate the a_{ij}'s we use (3.49, 4.65), so that the general expressions are

$$\begin{aligned} a_{11} &= \frac{z_i^{2c}}{2k^2}\int_{z_i}^{z} d\zeta \int_{z_i}^{z} d\zeta'\ f_1(\zeta)\ \nu(\zeta,\zeta')\ f_1(\zeta') \\ a_{12} &= \frac{(z_i z)^c}{k^2}\int_{z_i}^{z} d\zeta \int_{z_i}^{z} d\zeta'\ f_1(\zeta)\ \nu(\zeta,\zeta')\ f_2(\zeta') \\ a_{22} &= \frac{z^{2c}}{2k^2}\int_{z_i}^{z} d\zeta \int_{z_i}^{z} d\zeta'\ f_2(\zeta)\ \nu(\zeta,\zeta')\ f_2(\zeta') \end{aligned} \tag{D.22}$$

where f_1, f_2 are given by

$$f_1(\zeta) \equiv (-\zeta)^{-c}\frac{\mathrm{Im}\ [X(z)X^*(\zeta)]}{\mathrm{Im}\ X(z)} \quad ; \quad f_2(\zeta) \equiv (-\zeta)^{-c}\frac{\mathrm{Im}\ X(\zeta)}{\mathrm{Im}\ X(z)} \tag{D.23}$$

We will calculate a_{11} showing the steps of the calculation; a_{12} and a_{22} follow in exactly the same way, and so only the results of these will be shown.

Since we will be integrating over cosmic sum and difference coordinates S, D, we start by writing the region of integration in these variables. Note that although we're using S, D, we will continue to use ζ, ζ' as shorthand for the corresponding expressions [see (5.51) for the definition of x]:

$$\zeta = k\tau = \frac{-k}{H}e^{-Hs} = -e^{x-S-D/2} \quad ; \quad \zeta' = -e^{x-S+D/2} \tag{D.24}$$

Then

$$a_{11} = \frac{z_i^{2c}}{2k^2}\int_{z_i}^{z} d\zeta \int_{z_i}^{z} d\zeta'\ f_1(\zeta)\ \nu(\zeta,\zeta')\ f_1(\zeta') \tag{D.25}$$

$$= \frac{z_i^{2c}}{2k^2}\int_0^t ds \int_0^t ds'\ k^2 e^{-2S} f_1(\zeta)\ \nu(\zeta,\zeta')\ f_1(\zeta') \tag{D.26}$$

Since the domain of integration is a square in the s, s' plane, the new axes of S, D are at a 45° angle to the s, s' axes, breaking the integration into two regions:

$$a_{11} = \frac{z_i^{2c}}{2k^2}\frac{k^2}{H^2}\left[\int_0^{Ht/2} dS \int_{-2S}^{2S} dD + \int_{Ht/2}^{Ht} dS \int_{-(2Ht-2S)}^{2Ht-2S} dD\right] e^{-2S} f_1(\zeta)\ \nu(\zeta,\zeta')\ f_1(\zeta') \tag{D.27}$$

Introducing the two different noises further increases the number of integrals, since we must now consider two regimes for the times:

$$\begin{aligned} a_{11} &= \frac{z_i^{2c}}{2H^2}\left[\int_0^8 dS\, e^{-2S}\int_{-2S}^{2S} dD\ f_1(\zeta)\ \nu_{<8}(\zeta,\zeta')\ f_1(\zeta')\right. \\ &\quad + \int_8^{Ht/2} dS\, e^{-2S}\int_{-2S}^{2S} dD\ f_1(\zeta)\ \nu_{>8}(\zeta,\zeta')\ f_1(\zeta') \\ &\quad \left. + \int_{Ht/2}^{Ht} dS\, e^{-2S}\int_{-(2Ht-2S)}^{2Ht-2S} dD\ f_1(\zeta)\ \nu_{>8}(\zeta,\zeta')\ f_1(\zeta')\right] \end{aligned} \tag{D.28}$$

Now, consider the first integral, involving $\nu_{<8}$. Now for $S = 0 \to 7$ approximately, the half width of the $\nu_{<8}$ top hat is on average about 0.25, increasing to 3 when $S = 8$ [see (D.17)]. Compare this with the dynamical time scale of the system, which we define to be 1/frequency of system as determined by its equation of motion:

$$X''(\zeta) + \left(1 - \frac{2+3c+c^2}{\zeta^2}\right) X = 0 \tag{D.29}$$

so that the angular frequency is approximately

$$\sqrt{\frac{2+3c+c^2}{\zeta^2}} \simeq \frac{3/2+c}{|\zeta|} \tag{D.30}$$

which gives the time scale as approximately $|\zeta|$, for small c. In that case, since $\zeta = -k/He^{-Hs}$, we see that the system changes significantly over one Hubble time. So write the first integral as $\int_0^8 = \int_0^7 + \int_7^8$ and note that in the first region $(0 \to 7)$ the noise correlation time of about 0.25 is much less than the system time scale. Hence we take the limits of the first integral to be constant at $-0.25 \to 0.25$ (since this is the average half width of the top hat for $\nu_{<8}$). We are effectively writing $\nu_{<8}$ as

$$\nu_{<8} = \begin{cases} \frac{8}{\pi} cH\, T_i^2 e^S & |D| < 0.25 \\ 0 & 0.25 < |D| \end{cases} \tag{D.31}$$

The last two integrals in (D.28) can be rewritten as follows. Since both use $\nu_{>8}$ which has a half width of 3, we can combine them almost into one; but as S increases from $Ht/2$ to Ht, the D integration covers an ever shrinking domain until the top hat starts to become obscured, which happens when $2Ht - 2S = 3$, or $S = Ht - 3/2$. Taking this into account, a_{11} now becomes

$$\begin{aligned} a_{11} &= \frac{z_i^{2c}}{2H^2}\left[\int_0^7 dS\, e^{-2S}\int_{-0.25}^{0.25} dD\ f_1(\zeta)\ \nu_{<8}(\zeta,\zeta')\ f_1(\zeta')\right. \\ &\quad + \int_7^8 dS\, e^{-2S}\int_{-3}^{3} dD\ f_1(\zeta)\ \nu_{<8}(\zeta,\zeta')\ f_1(\zeta') \\ &\quad + \int_8^{Ht-3/2} dS\, e^{-2S}\int_{-3}^{3} dD\ f_1(\zeta)\ \nu_{>8}(\zeta,\zeta')\ f_1(\zeta') \\ &\quad \left. + \int_{Ht-3/2}^{Ht} dS\, e^{-2S}\int_{-(2Ht-2S)}^{2Ht-2S} dD\ f_1(\zeta)\ \nu_{>8}(\zeta,\zeta')\ f_1(\zeta')\right] \end{aligned} \tag{D.32}$$

Finally, the fourth integral in the above expression can be broken into two regions: $\int_8^{Ht-3/2} = \int_8^x + \int_x^{Ht-3/2}$. The reason we do this is because now in the second integral of this pair, we can use a small argument approximation to the Bessel functions (see section 5.3.1 for an explanation of the time scales involved). The expression for a_{11} which we will use can then be written

$$\begin{aligned} a_{11} &= \frac{z_i^{2c}}{2H^2}\left[\int_0^7 dS\ e^{-2S}\int_{-0.25}^{0.25} dD\ f_1(\zeta)\ \nu_{<8}(\zeta,\zeta')\ f_1(\zeta')\right. \\ &\quad + \int_7^8 dS\ e^{-2S}\int_{-3}^{3} dD\ f_1(\zeta)\ \nu_{<8}(\zeta,\zeta')\ f_1(\zeta') \\ &\quad + \int_8^x dS\ e^{-2S}\int_{-3}^{3} dD\ f_1(\zeta)\ \nu_{>8}(\zeta,\zeta')\ f_1(\zeta') \\ &\quad + \int_x^{Ht-3/2} dS\ e^{-2S}\int_{-3}^{3} dD\ f_1(\zeta)\ \nu_{>8}(\zeta,\zeta')\ f_1(\zeta') \\ &\quad \left. + \int_{Ht-3/2}^{Ht} dS\ e^{-2S}\int_{-(2Ht-2S)}^{2Ht-2S} dD\ f_1(\zeta)\ \nu_{>8}(\zeta,\zeta')\ f_1(\zeta')\right] \end{aligned} \tag{D.33}$$

The five integrals above can be characterised in the following way.

1. The first has a noise correlation time (0.25 Hubble times) much less than the system dynamical time (1 Hubble time), which allows us to extract the f_1 functions out of the D integration, setting $D = 0$ in them. Also, because $|\zeta|$ is large in this region of integration, we can replace the Bessels by their large argument forms which makes the integrations tractable.

2. The f_1's can't be extracted from the D integration for the second integral, since here the noise correlation time (~ 3) is greater than the system dynamical time. Also the Bessel arguments are approaching one, so that we can use neither the large nor the small argument forms of the Bessels. Hence this integral remains a double integral over the Bessels.

3. The third integral behaves just like the second, although of course for values of x less than 8 it won't arise.

4. For the fourth and fifth integrals we can invoke small argument forms of the Bessel functions ($|\zeta| \ll 1$, and of course for all the integrals we can take $|z| \ll 1$).

We now calculate the first, fourth and fifth integrals, and assume that the second and third have intermediate values. In fact, for all the a_{ij}'s the three calculable integrals are dominated by either the first or last one, so we will assume that integrals 2 and 3 can safely be neglected.

First integral in (D.33) Write this as

$$\begin{aligned} a_{11}^{(1)} &\equiv \frac{z_i^{2c}}{2H^2}\int_0^7 dS\ e^{-2S}\int_{-0.25}^{0.25} dD\ f_1(\zeta)\ \nu_{<8}(\zeta,\zeta')\ f_1(\zeta') \\ &\simeq \frac{z_i^{2c}}{2H^2}\int_0^7 dS\ e^{-2S} f_1(\zeta|_{D=0}) f_1(\zeta'|_{D=0})\ \int_{-0.25}^{0.25} dD\ \nu_{<8}(\zeta,\zeta') \end{aligned} \tag{D.34}$$

which can be done since we are using the fact that the system is not changing greatly over the noise correlation time. Hence

$$a_{11}^{(1)} \simeq \frac{e^{2cx}2cT_i^2}{H\pi}\int_0^7 dS\ e^{-S} f_1^2\left(-e^{x-S}\right) \tag{D.35}$$

To treat the Bessel functions in large and small argument limits we need the following expressions. For large arguments:

$$\begin{Bmatrix} J_\nu(y) \\ Y_\nu(y) \end{Bmatrix} \simeq \sqrt{\frac{2}{\pi y}}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(y - \pi\nu/2 - \pi/4) \tag{D.36}$$

while for small arguments

$$J_\nu(y) \simeq \frac{y^\nu}{2^\nu \nu!}\left(1 - \frac{y^2}{\nu+1}\right) \quad , \quad Y_\nu(y) \simeq \frac{-2^\nu\Gamma(\nu)}{\pi y^\nu}\left(1 + \frac{y^2}{4(\nu-1)}\right) \tag{D.37}$$

[We work only to leading order, but in fact the full expressions in (D.37) are needed to calculate b_1 in (5.55) and also for (5.62)].

The following are the forms of f_1, f_2 together with what is needed for subsequent calculations:

$$\begin{aligned} f_1(\zeta) \equiv |\zeta|^{-c}\frac{\text{Im}\ [X(z)X^*(\zeta)]}{\text{Im}\ X(z)} &= |\zeta|^{1/2-c}\left(\alpha_1 J_\nu|\zeta| - \alpha_2 Y_\nu|\zeta|\right) \\ &\xrightarrow{z\to 0} \alpha_1|\zeta|^{1/2-c}\left(J_\nu|\zeta| + \frac{\nu\pi}{2^{2\nu}\nu!^2}|z|^{2\nu}Y_\nu|\zeta|\right) \\ &\xrightarrow{\zeta,z\to 0} \frac{\alpha_1}{2^\nu\nu!}|\zeta|^{1/2-c}\left(|\zeta|^\nu - |z|^{2\nu}|\zeta|^{-\nu}\right) \\ f_2(\zeta) \equiv |\zeta|^{-c}\frac{\text{Im}\ X(\zeta)}{\text{Im}\ X(z)} &= |\zeta|^{1/2-c}\left(\alpha_3 J_\nu|\zeta| + \alpha_4 Y_\nu|\zeta|\right) \\ &\xrightarrow{z\to 0} \alpha_5|z|^{c+1}|\zeta|^{1/2-c}\left(J_\nu|\zeta| - Y_\nu|\zeta|/(\sin)\right) \\ &\xrightarrow{\zeta,z\to 0} |z|^{c+1}|\zeta|^{-2c-1}/(\cos) \end{aligned} \tag{D.38}$$

with

$$\begin{aligned} \alpha_1 &\equiv \sqrt{\frac{\pi}{2}}\frac{Y|z|}{(\cos)Y|z| - (\sin)J|z|} \xrightarrow{z\to 0} \sqrt{\frac{\pi}{2}}\frac{1}{(\cos)} \\ \alpha_2 &\equiv \sqrt{\frac{\pi}{2}}\frac{J|z|}{(\cos)Y|z| - (\sin)J|z|} \xrightarrow{z\to 0} \frac{-\alpha_1\nu\pi|z|^{2\nu}}{2^{2\nu}\nu!^2} \end{aligned}$$

$$\alpha_3 \equiv \frac{1}{\sqrt{|z|}} \frac{-(\sin)}{(\cos)Y|z| - (\sin)J|z|} \xrightarrow{z\to 0} \alpha_5 |z|^{c+1}$$
$$\alpha_4 \equiv \frac{1}{\sqrt{|z|}} \frac{(\cos)}{(\cos)Y|z| - (\sin)J|z|} \xrightarrow{z\to 0} -\alpha_5 |z|^{c+1}/(\sin)$$
$$\alpha_5 \equiv \frac{(\sin)\pi}{(\cos)\Gamma(\nu)2^\nu} \tag{D.39}$$

For $c \lesssim 1, z \ll 1$ we have

$$f_1\left(-e^{x-S}\right) \simeq -\alpha_1 \sqrt{2/\pi}\; e^{-c(x-S)} \cos e^{x-S} \tag{D.40}$$

so that we can write (D.35) as

$$a_{11}^{(1)} \simeq \frac{\alpha_1^2\, 4ck\; e^{-x}}{\pi^2} \left(\frac{T_i}{H}\right)^2 \int_0^7 dS\; e^{(2c-1)S} \cos^2 e^{x-S} \tag{D.41}$$

The integrand is a nonnegative, rapidly oscillating function modulated by $e^{(2c-1)S}$, so that we can estimate the integral as just half the area under this exponential, which for small c is approximately $(1-2c)^{-1}/2$; so that we finally arrive at

$$a_{11}^{(1)} \simeq \frac{\alpha_1^2\, 2ck\; e^{-x}}{\pi^2(1-2c)} \left(\frac{T_i}{H}\right)^2 \tag{D.42}$$

Fourth integral in (D.33) This is

$$a_{11}^{(4)} \equiv \frac{z_i^{2c}}{2H^2} \int_x^{Ht-3/2} dS\; e^{-2S} \int_{-3}^{3} dD\; f_1(\zeta)\; \nu_{>8}(\zeta,\zeta')\; f_1(\zeta') \tag{D.43}$$

where as usual we are using (D.24) as shorthand. We can use the small ζ, z expressions from (D.38) to simplify this as

$$a_{11}^{(4)} \simeq \frac{ck}{6\pi} \left(\frac{\alpha_1}{2^\nu \nu!}\right)^2 e^{(2c-1)x} \int_x^{Ht-3/2} dS\; e^S \int_{-3}^{3} dD\; (\zeta\zeta')^{1/2-c} \left(|\zeta|^\nu - |z|^{2\nu}|\zeta|^{-\nu}\right)\left(|\zeta'|^\nu - |z|^{2\nu}|\zeta'|^{-\nu}\right) \tag{D.44}$$

The integral over D is done first, yielding

$$\int_{-3}^{3} dD = 2e^{(x-S)(1-2c)} \left[3e^{(x-S)(3+2c)} - 2|z|^{3+2c}\, \mathrm{sh}\,(3\nu) + 3|z|^{6+4c} e^{-(x-S)(3+2c)}\right] \tag{D.45}$$

The S integration is now easily done. The resulting expression contains terms involving e^{Ht}, and it remains only to convert this to x, z via

$$e^{Ht} = e^x/|z| \tag{D.46}$$

to finally give

$$a_{11}^{(4)} \simeq \frac{ck}{3\pi} \left(\frac{\alpha_1}{2^\nu \nu!}\right)^2 e^{2cx} \left[1 - e^{9/2}|z|^3 + \frac{3|z|^{6+4c}}{3+4c}\left(|z|^{-3-4c} e^{-9/2-6c} - 1\right) - \frac{\mathrm{sh}\,(3\nu)}{c} |z|^{3+2c} \left(|z|^{-2c} e^{-3c} - 1\right)\right] \tag{D.47}$$

Clearly, for $z \ll 1$ the 1 in the brackets dominates, leading to

$$a_{11}^{(4)} \simeq \frac{ck}{3\pi}\left(\frac{\alpha_1}{2^\nu \nu!}\right)^2 e^{2cx} \tag{D.48}$$

Fifth integral in (D.33) The last integral to be done is

$$a_{11}^{(5)} \equiv \frac{z_i^{2c}}{2H^2}\int_{Ht-3/2}^{Ht} dS\; e^{-2S}\int_{-(2Ht-2S)}^{2Ht-2S} dD\; f_1(\zeta)\;\nu_{>8}(\zeta,\zeta')\; f_1(\zeta') \tag{D.49}$$

This is done in much the same way as $a_{11}^{(4)}$: on using small ζ, z expressions the integrand is written as in (D.44), and the D integration is then done to give

$$\begin{aligned}\int_{-3}^{3} dD \;&=\; 2e^{(x-S)(1-2c)}\Big[(2Ht-2S)\left(e^{(x-S)(3+2c)} + |z|^{6+4c}e^{-(x-S)(3+2c)}\right)\\ &\quad - \frac{2}{\nu}|z|^{3+2c}\,\mathrm{sh}\left[\nu(2Ht-2S)\right]\Big]\end{aligned} \tag{D.50}$$

Just as for $a_{11}^{(4)}$, this can be easily integrated over S, and resulting terms involving e^{Ht} can be replaced by x, z via (D.46). The final expression is

$$\begin{aligned}a_{11}^{(5)} \;&\simeq\; \frac{ck}{3\pi}\left(\frac{\alpha_1}{2^\nu \nu!}\right)^2 e^{2cx}|z|^3\Big[2/9 + 7/9\; e^{9/2} + \frac{2}{(3+4c)^2}\left[1 + e^{-9/2-6c}(-11/2-6c)\right]\\ &\quad + \frac{1}{9/4+3c}\left(1 - e^{-3c}\left[\,\mathrm{ch}\,3\nu + c/\nu\;\mathrm{sh}\,3\nu\right]\right)\Big]\\ &=\; ck\;\alpha_1^2\; O|z|^3\end{aligned} \tag{D.51}$$

Comparing (D.42, D.48, D.51) we see that a_{11} is dominated by the expressions of $a_{11}^{(1)}, a_{11}^{(4)}$ (and presumably $a_{11}^{(2)}, a_{11}^{(3)}$ as well), and will be a function of c with vanishing z dependence. We expect the contributions of $a_{11}^{(2)}, a_{11}^{(3)}$ to be secondary to that of $a_{11}^{(1)}$, since their regions of S-integration are smaller than that of $a_{11}^{(1)}$ while the Bessels are still oscillating there. In that case this function of c is determined by adding $a_{11}^{(1)}, a_{11}^{(4)}$. Write

$$a_{11}^{(1)} + a_{11}^{(4)} = \alpha_1^2\; ck\left[\frac{2e^{-x}}{\pi^2(1-2c)}\left(\frac{T_i}{H}\right)^2 + \frac{e^{2cx}}{3\pi 2^{2\nu}\nu!^2}\right] \tag{D.52}$$

Remember that we are considering x to be in the range $7 \to 14$. Taking typical parameters as $x = 10, c = 0.2, T_i/H = 10^6$, the first term in the brackets equals 10^7 while the second is 0.2. In fact if c is held at 0.2, the second term will only begin to dominate when $x \simeq 22$, which is far outside the bounds we are concerned with. Alternatively set $x = 14$, the most extreme value, and calculate a bound on c such that the first term dominates. We find that this will occur when $c \gtrsim 10^{-5}$ (and at the other extreme when $x = 7$ the condition becomes $c \gtrsim 10^{-3}$).

Hence we take the dominant contribution to a_{11} to be $a_{11}^{(1)}$ and write

$$a_{11} \simeq 10^{12}\, e^{-x} \alpha_1^2\, ck \tag{D.53}$$

where we have set the $1-2c$ term to be 0.6 since it depends only weakly on small c, while leaving the overall factor of c explicitly written. Note that the term containing α_1 has been left unevaluated; from (5.53, D.39) we have

$$\alpha_1 = \sqrt{\frac{\pi}{2}}\, \frac{Y|z|}{-\cos(e^x - \pi c/2)Y|z| + \sin(e^x - \pi c/2)J|z|} \stackrel{z\to 0}{\longrightarrow} \sqrt{\frac{\pi}{2}}\frac{1}{(\cos)} \tag{D.54}$$

This term is very sensitive to $e^x = z_i = k/H$. This might be an effect which has been introduced due to our choice of the system-environment being initially uncoupled. Or it might be an artifact of the normalisation carried out in (3.43), since several quantities are tending to zero in the expression for X in (5.49).

The remaining quantities, a_{12}, a_{22}, are handled in exactly the same way as we have shown for a_{11}. Here we just quote the results, with numerical parameters evaluated in the same way as was done with a_{11}.

$$\begin{aligned} a_{12} &\simeq 10^4\, \alpha_1^2\, ck\, |z| \\ a_{22} &\simeq 0.01\, \alpha_1^2\, ck\, |z|^{-1} \end{aligned} \tag{D.55}$$

Appendix E

Equivalence of two point function and noise

Since the two point function $\langle\chi(\eta,\mathbf{x})\chi(\eta',\mathbf{x})\rangle$ can be calculated by taking a trace with a thermal density matrix, we choose to work in the number basis. In that case, we can decompose the field χ into plane wave modes as was done in appendix B, eqn (B.8):

$$\chi(\eta,\mathbf{x}) = \frac{1}{\sqrt{L^3}}\sum_{\mathbf{k}>0}\frac{e^{i\mathbf{k}\cdot\mathbf{x}-i\omega\eta}a}{\sqrt{2\omega}} + \text{h.c.} \tag{E.1}$$

We can write

$$\begin{aligned}
\langle\chi(\eta,\mathbf{x})\chi(\eta',\mathbf{x})\rangle &= \text{tr}\ \chi(\eta,\mathbf{x})\chi(\eta',\mathbf{x})\rho_{th} \\
&= \sum_m \langle m|\frac{1}{L^3}\sum_{\mathbf{k}>0}\left(\frac{e^{i\mathbf{k}\cdot\mathbf{x}-i\omega\eta}a}{\sqrt{2\omega}} + \text{h.c.}\right)\frac{1}{L^3}\sum_{\mathbf{k}'>0}\left(\frac{e^{i\mathbf{k}'\cdot\mathbf{x}-i\omega'\eta}a}{\sqrt{2\omega'}} + \text{h.c.}\right) \\
&\quad\times\left(1-e^{-\omega/T}\right)\sum_n e^{-n\omega/T}|n\rangle\langle n|m\rangle
\end{aligned} \tag{E.2}$$

We sum over n and note that the $\mathbf{k},\mathbf{k}'$ cross terms will ultimately vanish (see e.g. [17]), allowing us to write (keeping only nonzero contributions from the $a, a^\dagger$):

$$\begin{aligned}
\langle\chi(\eta,\mathbf{x})\chi(\eta',\mathbf{x})\rangle &= \frac{1}{L^3}\sum_{m;\mathbf{k}>0}\left(1-e^{-\omega/T}\right)\frac{e^{-m\omega/T}}{2\omega}\left[e^{-i\omega(\eta-\eta')}\langle m|aa^\dagger|m\rangle + e^{i\omega(\eta-\eta')}\langle m|a^\dagger a|m\rangle\right] \\
&= \frac{1}{L^3}\sum_{\mathbf{k}>0}\frac{1}{\left(1-e^{-\omega/T}\right)2\omega}\left[e^{-i\omega(\eta-\eta')} + e^{i\omega(\eta-\eta')-\omega/T}\right]
\end{aligned} \tag{E.3}$$

The symmetrised two point function is then

$$\langle\chi\chi\rangle_{sym} \equiv \langle\chi(\eta,\mathbf{x})\chi(\eta',\mathbf{x})\rangle + \eta\leftrightarrow\eta' = \frac{1}{L^3}\sum_{\mathbf{k}>0}\frac{1}{\omega}\coth\frac{\omega}{2T}\cos\omega(\eta-\eta') \tag{E.4}$$

We now take the continuum limit, bearing in mind that from (B.5),

$$\sum_{\mathbf{k}>0} \equiv \sum_{k_z>0} \sum_{k_x,k_y=-\infty}^{\infty}$$

Since an integral number of wavelengths must fit into L, we must have $\Delta k_x = \Delta k_y = \Delta k_z = 2\pi/L$, in which case

$$\sum_{\mathbf{k}>0} = \left(\frac{L}{2\pi}\right)^3 \sum_{\mathbf{k}>0} \Delta^3 k \longrightarrow \left(\frac{L}{2\pi}\right)^3 \int_{k_z>0} \int_{k_x,k_y=-\infty}^{\infty} d^3k \tag{E.5}$$

If the field is massless then $\omega = |\mathbf{k}|$, and so $d^3k = \omega^2 \sin\theta \, d\omega \, d\theta \, d\phi$. Hence

$$\sum_{\mathbf{k}>0} \longrightarrow \frac{L^3}{4\pi^2} \int_0^{\infty} \omega^2 \, d\omega \tag{E.6}$$

which when substituted into (E.4) gives

$$\langle \chi\chi \rangle_{sym} = \frac{1}{4\pi^2} \int_0^{\infty} \omega \coth \frac{\omega}{2T} \cos\omega(\eta - \eta') \, d\omega \tag{E.7}$$

This is the main result of this appendix: it relates the noise produced by a massless scalar field to its two point function. This is another point of contact between the influence functional formalism and the more well known techniques of quantum field theory.

Finally, from (D.1) it follows that

$$\nu(\eta, \eta') = \frac{2\gamma_0}{\pi} \, a^{1/2}(\eta) a^{1/2}(\eta') \, 4\pi^2 \langle \chi\chi \rangle_{sym} \tag{E.8}$$

This expression is used as the starting point for the calculation in appendix D.

Appendix F

Effect due to nonzero temperature bath during inflation

This appendix calculates the function C as used in section 6.4, and also shows that the effects due to the bath temperature decrease exponentially as mentioned in appendix D (page 106).

We are working in cosmic time. The lagrangian for a massless, conformally coupled field is (B.20):

$$L_{new}(t) = \sum \frac{a^3}{2}\left[\dot{q}^2 + 2\frac{\dot{a}}{a}q\dot{q} - q^2\left(\frac{k^2}{a^2} - \frac{\dot{a}^2}{a^2}\right)\right] \tag{F.1}$$

Using (3.20, 3.22, 3.23) we can write the equation of motion of X as (with $\kappa = k$)

$$\ddot{X} + 3\frac{\dot{a}}{a}\dot{X} + \left(\frac{k^2}{a^2} + \frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)X = 0 \tag{F.2}$$

$$X(t_i) = 1 \quad ; \quad X'(t_i) = -ik - \dot{a}(t_i) \tag{F.3}$$

The solution is simply

$$X(t) = \frac{e^{-ik\eta}}{a(t)} \tag{F.4}$$

where η is the usual conformal time.

We wish to show how the radiation observed for this type of field depends on the temperature of the bath. Referring to [2], for a bath of oscillators in a thermal state the influence kernel is given by

$$\zeta = \int_0^\infty d\omega\; I(\omega, s, s')\left(\coth\frac{\hbar\omega}{2k_BT}\text{Re }[X(s)X^*(s')] + i\,\text{Im }[X(s)X^*(s')]\right) \tag{F.5}$$

We introduce for brevity:

$$\begin{aligned} x(\Sigma,\Delta) &\equiv a(t) + a(t') \\ y(\Sigma,\Delta) &\equiv \eta(t) - \eta(t') \end{aligned} \tag{F.6}$$

in which case it follows that

$$\zeta(s,s') = \frac{1}{a(s)a(s')}\int_0^\infty dk\; I(k,s,s')\left(\coth\frac{k}{2T}\cos ky - i\,\sin ky\right) \qquad (F.7)$$

$$\equiv \int_0^\infty dk\; I_{eff}(k,s,s')\,(C(k)\cos k\Delta - i\,\sin k\Delta) \qquad (F.8)$$

where $C, I_{eff}(k,\Sigma)$ are two functions to be determined. As explained in section 6.1, if C was replaced by $\coth k/(2T)$, ζ would describe a thermal bath of static oscillators. Our plan is to show that C indeed has the form of a coth. Here $I_{eff}(k,\Sigma)$ is the effective spectral density. Equating the real and imaginary parts of the two forms of ζ and Fourier inverting, leads to (6.3) which will be used to calculate C.

The spectral density of the scalar field is found by calculating (3.33) for the case of a point interaction between a system harmonic oscillator and the field; the short calculation is done in [2] and gives

$$I(k,s,s') = \frac{\varepsilon^2 k}{4\pi^2} \qquad (F.9)$$

In this case we can evaluate the integral in (F.7): see appendix A for details on how this is done. The imaginary and real parts of ζ are, respectively,

$$\mu = \frac{1}{a(s)a(s')}\frac{\varepsilon^2}{4\pi^2}\,\pi\delta'(y)$$

$$\nu = \frac{1}{a(s)a(s')}\frac{\varepsilon^2}{4\pi^2}\left[d/dy\;P(1/y) + 1/y^2 - \pi^2T^2\mathrm{csch}^2\pi Ty\right] \qquad (F.10)$$

Comparing this with (F.8) allows us to write

$$I_{eff} = \frac{-\varepsilon^2}{4\pi^2}\int_{-\infty}^{\infty} d\Delta\;\frac{\sin k\Delta}{a(s)a(s')}\,\delta'(y)$$

$$I_{eff}C = \frac{\varepsilon^2}{4\pi^3}\int_{-\infty}^{\infty} d\Delta\;\frac{\cos k\Delta}{a(s)a(s')}\left[d/dy\;P(1/y) + 1/y^2 - \pi^2T^2\mathrm{csch}^2\pi Ty\right] \qquad (F.11)$$

The first expression is easily evaluated by changing variables from Δ to y, using

$$\frac{d\Delta}{dy} = \frac{2a(t)a(t')}{a(t)+a(t')} \qquad (F.12)$$

A useful fact here is

$$\Delta = 0 \Longleftrightarrow y = 0 \quad ; \quad \Delta = \pm\infty \Longleftrightarrow y = \pm\infty \qquad (F.13)$$

in which case it follows that independently of the scale factor $a(t)$,

$$I_{eff} = \frac{\varepsilon^2 k}{4\pi^2} \qquad (F.14)$$

The second expression in (F.11) now becomes

$$C = \frac{2}{\pi k}\int_{-\infty}^{\infty} dy\ \frac{\cos k\Delta}{a(t)+a(t')}\left[d/dy\ P(1/y) + 1/y^2 - \pi^2 T^2 \mathrm{csch}^2 \pi T y\right] \tag{F.15}$$

We can integrate this by parts to write

$$C = \frac{4T}{k}\left.\frac{\cos k\Delta}{a(t)+a(t')}\right|_{\Delta\to\infty} - \frac{2}{\pi k}\int_{-\infty}^{\infty} d\Delta\ \left[\frac{d}{d\Delta}\frac{\cos k\Delta}{a(t)+a(t')}\right]\left[P(1/y) - 1/y + \pi T \coth \pi T y\right] \tag{F.16}$$

Suppose we now specialise to de Sitter space:

$$a(t) = e^{Ht} \quad ; \quad \eta(t) = -e^{-Ht}/H \tag{F.17}$$

for which we need

$$a(t) + a(t') = 2e^{H\Sigma}\cosh H\Delta/2 \quad ; \quad y = \frac{2e^{-H\Sigma}}{H}\sinh H\Delta/2 \tag{F.18}$$

The usual Hawking temperature is now easily recovered by setting $T = 0$, so that $-1/y + \pi T \coth \pi T y$ is set to zero in (F.16) to give

$$\begin{aligned} C &= \frac{-2}{\pi k}\,\mathrm{PV}\int_{-\infty}^{\infty}\frac{d\Delta}{y}\,\frac{d}{d\Delta}\frac{\cos k\Delta}{a(t)+a(t')} \\ &= \frac{-4}{\pi k}\int_{0}^{\infty}\frac{d\Delta}{y}\,\frac{d}{d\Delta}\frac{\cos k\Delta}{a(t)+a(t')} \\ &= \coth\frac{\pi k}{H} \end{aligned} \tag{F.19}$$

and by comparing this with (F.8) we immediately arrive at the well known effective temperature of $H/(2\pi)$.

If we keep T nonzero in de Sitter space then our analysis proceeds as follows, referring to (F.16). First, the principal part just gives the coth term as just calculated. The argument of the integral is even in Δ and well behaved at the origin, so we write

$$C = \coth\frac{\pi k}{H} - \frac{4}{\pi k}\int_0^{\infty} d\Delta\ \left[\frac{d}{d\Delta}\frac{\cos k\Delta}{2e^{H\Sigma}\cosh H\Delta/2}\right]\left[-1/y + \pi T\coth \pi T y\right] \tag{F.20}$$

We now concentrate on the integral ($\equiv J$) in this last equation, and show that it tends to zero exponentially for large times. First we write all Δ's explicitly:

$$J = \frac{1}{2}\int_0^{\infty} d\Delta\ \left[\frac{d}{d\Delta}\frac{\cos k\Delta}{\cosh H\Delta/2}\right]\left[\frac{-H}{2\sinh H\Delta/2} + e^{-H\Sigma}\pi T\coth\left(\frac{2\pi T}{H}e^{-H\Sigma}\sinh H\Delta/2\right)\right] \tag{F.21}$$

Suppose we now assume late times (large Σ) and split the integral into two intervals. For the first we assume Σ is large enough to allow a small argument expansion of the coth term ($\coth x \simeq 1/x + x/3$); for the second we assume Δ is too large to allow this, in which case we use a large argument approximation of the coth, i.e. $\coth x \simeq 1$. The coth can be expanded to first order as long as its argument is much less than one, and this is true provided that

$$\Delta \ll 2\Sigma + \frac{2}{H}\ln\frac{H}{\pi T} \equiv \Delta_0 \tag{F.22}$$

With these approximations the last integral becomes

$$\begin{aligned} J \simeq\ & \frac{1}{2}\int_0^{\Delta_0} d\Delta \left[\frac{-k\sin k\Delta}{\cosh H\Delta/2} - \frac{H\cos k\Delta \sinh H\Delta/2}{2\cosh^2 H\Delta/2}\right]\frac{2\pi^2T^2}{3H}e^{-2H\Sigma}\sinh H\Delta/2 \\ & + \frac{1}{2}\int_{\Delta_0}^{\infty} d\Delta \left[\frac{-k\sin k\Delta}{\cosh H\Delta/2} - \frac{H\cos k\Delta \sinh H\Delta/2}{2\cosh^2 H\Delta/2}\right]\frac{-H}{2\sinh H\Delta/2} \end{aligned} \tag{F.23}$$

Clearly the first integral goes to zero as $T^2e^{-2H\Sigma}/H$. For the second, if we approximate each sinh and cosh term by an exponential then a straightforward if slightly lengthy calculation shows that its modulus is always less than a term which goes to zero in exactly the same way. So we can write (F.20) as

$$C \simeq \coth\frac{\pi k}{H} + O\left(\frac{T^2e^{-2H\Sigma}}{kH}\right) \tag{F.24}$$

It's evident that for late times, the effects due to a nonzero temperature environment become washed out exponentially quickly. As might be expected, a slightly longer time is needed as T increases and H decreases.

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