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**Multiscalar black holes with contingent primary hair: Mechanics and stability**Salvatore Mignemi<sup>1,2,\*</sup> and David L. Wiltshire<sup>3,4,†,‡</sup><sup>1</sup>*Dipartimento di Matematica, Università di Cagliari, viale Merello 92, 09123 Cagliari, Italy*<sup>2</sup>*INFN, Sezione di Cagliari, Cittadella Universitaria, 09042 Monserrato, Italy*<sup>3</sup>*Department of Physics and Mathematical Physics, University of Adelaide, Adelaide, S.A. 5005, Australia*<sup>4</sup>*Department of Physics and Astronomy, University of Canterbury, Private Bag 4800, Christchurch, New Zealand*

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We generalize a class of magnetically charged black holes nonminimally coupled to two scalar fields previously found by one of us to the case of multiple scalar fields. The black holes possess a novel type of primary scalar hair, which we call a *contingent primary hair*: although the solutions possess degrees of freedom which are not completely determined by the other charges of the theory, the charges necessarily vanish in the absence of the magnetic monopole. Only one constraint relates the black hole mass to the magnetic charge and scalar charges of the theory. We obtain a Smarr-type thermodynamic relation, and the first law of black hole thermodynamics for the system. We further explicitly show in the two-scalar-field case that, contrary to the case of many other hairy black holes, the black hole solutions are stable to radial perturbations.

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**I. INTRODUCTION**

In classical general relativity, no-hair theorems [1] impose strong constraints on the possibility of obtaining black hole solutions of the Einstein equations coupled to nontrivial scalar fields. A crucial ingredient for their proof is that the scalars be minimally coupled to gravity and other fields. When this condition is relaxed new possibilities emerge for evading the no-hair theorems.

One of the first investigations was undertaken by Bekenstein [2], who attempted to show that a nontrivial black hole solution exists for a conformally coupled scalar field. Although the scalar field diverges at the horizon, Bekenstein argued that this solution nevertheless admits a physical interpretation. Unfortunately, a more detailed analysis of the problem casts serious doubt on this [3]. We will therefore take the view that scalar fields should be regular at the horizon for genuine black hole solutions.

Nontrivial scalar hair is possible, however, when black holes are coupled to scalar fields with nonlinear self-interactions. Such solutions were first found in the case of gravity coupled to the Skyrme model [4], and subsequently for the Einstein-Yang-Mills-Higgs model [5], and other generalizations. The large literature on this topic has been recently reviewed in Ref. [6]. What appears to be characteristic of the class of black holes with nonlinear self-interactions is that the scalar fields fall off very rapidly at spatial infinity—e.g., exponentially fast—and hence the asymptotic scalar charges vanish. The

scalar hair is characterized instead by nontrivial charges on the horizon.

A third possibility is that of minimally coupled scalar fields with potentials,  $V(\phi)$ , which violate the dominant energy condition (DEC) [7–9]. In single scalar field models, the DEC is equivalent to  $V(\phi) \geq 0$ , and thus one must choose potentials for which  $V(\phi) < 0$  for some field values. Examples of asymptotically flat black hole solutions have been found for cases in which  $V(\phi) < 0$  everywhere [8], and for cases in which  $V(\phi)$  possesses at least one global minimum at negative values [7,9]. Both analytic [7,8] and numerical [7,9] examples are known.

A final possibility is to consider black holes with scalar fields which are nonminimally coupled to gauge fields. Such models have been extensively investigated because they arise naturally in Kaluza-Klein theories and in the effective low-energy limit of string theory, where the dilaton plays a nontrivial role. In all these models one can find black hole solutions with nonzero scalar charges at spatial infinity [10,11]. An analogous phenomenon can be shown to take place also in the pure dilaton-gravity sector of effective string theories, when one takes into account the coupling of the dilaton to gravity via Gauss-Bonnet terms [12].

For this class of nonminimally coupled models with nonzero asymptotic scalar charges, however, the scalar charges in question are *not* independent parameters, but in all cases are a given function of the other asymptotic charges which characterize the solution, namely, the Arnowitt-Deser-Misner (ADM) mass and the electric and magnetic charges, etc.,. As a result such scalar charges have been called *secondary hair* by the authors of Refs. [13,14], to distinguish them from the theoretical possibility of a *primary hair*, namely, an asymptotic scalar charge which is completely independent of the other charges.

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In light of all the known solutions [6] one may therefore conjecture [14] that a weaker form of the no-hair property still holds, namely, that for theories which satisfy the DEC black hole solutions can be classified by a small number of parameters, which in the case of asymptotic charges include only conserved charges such as mass, angular momentum and gauge charges, but no asymptotic scalar charges.

Even this no-hair conjecture does not appear to be valid in general, however. In a recent paper by one of us [15] the properties of magnetically charged black holes coupled to a dilaton,  $\Phi$ , and an additional modulus field,  $\Psi$ , according to the field equations generated by the 4-dimensional action [16]

$$S = \frac{1}{4} \int d^4x \sqrt{-g} \{ -\mathcal{R} + 2\partial^\mu \Phi \partial_\mu \Phi + 2\partial^\mu \Psi \partial_\mu \Psi - [e^{-2\Phi} + (\lambda_2)^2 e^{-2q\Psi}] F_{\mu\nu} F^{\mu\nu} \}, \quad (1)$$

were studied using techniques from the general theory of dynamical systems, which have been previously applied to static spherically symmetric solutions of gravity coupled to scalar fields in a number of contexts [17]. In Ref. [15] it was shown that the regular black hole solutions were parametrized by an additional degree of freedom in addition to the mass,  $M$ , and magnetic charge,  $Q$ , and it was conjectured that this degree of freedom could be considered to be a “primary scalar hair”.

An important issue in this context is that of stability of the hairy solutions. The physical relevance of the solutions would in fact be spoiled by the presence of instabilities. Instabilities are known to be present in the case of Einstein-Yang-Mills models [18,19] and the DEC-violating solutions for which the stability issue has been most thoroughly studied [9]. Dilaton black holes with a single scalar do not appear to suffer from instability problems [20,21]; however, they possess only secondary hair.

It is the aim of the present paper to clarify and extend the results of Ref. [15] by determining various relations satisfied by the charges, and discussing the stability of the solutions. We will find, in particular, that although the scalar degree of freedom cannot be considered to be a primary hair in the strictest sense, since it must necessarily vanish if the electromagnetic field vanishes, the regular static spherically symmetric black hole solutions are not completely specified by their mass and magnetic charge. The solutions therefore are potentially of considerable physical interest as the only known static spherically symmetric solutions with nontrivial scalar charges at spatial infinity which are not completely determined by the other asymptotic charges of the theory, and therefore provide a counterexample to some forms of the no-hair conjecture. We will show that the solutions are stable, adding significance to their interpretation.

Intuitively, one might attempt to understand the physical basis of the no-hair conjecture as arising from the fact there is no scalar equivalent of Gauss’s law to give a conserved charge. The presence of the electromagnetic field supports the scalar charge in the case of models with a secondary scalar hair since the scalar coupling enters into the equivalent of Gauss’s law. In the solutions described here the scalar charges are also supported by the presence of the electromagnetic field. However, they are not entirely determined by it.

It is quite consistent with past usage to adopt the terminology “primary hair” for the additional degrees of freedom which arise in the model [15]. However, the fact that the scalar charges are not entirely independent of the electromagnetic field suggests a new terminology might be appropriate, as a way of capturing the finer distinctions that the present model has revealed. We will therefore refer to the scalar charges which could theoretically exist in the absence of additional nonzero gauge charges as *elementary primary hair*, as would be consistent with the original aims of the first “no-hair” theorems [1]. The additional degrees of freedom which arise in the present model could by contrast be deemed to be a *contingent primary hair*.

## II. MULTISCALAR BLACK HOLE SOLUTIONS

Rather than restricting our attention to solutions of the field equations obtained from varying the action (1), we will instead consider the somewhat more general action for  $N$  scalar fields coupled to a single  $U(1)$  Abelian gauge field according to

$$S = \frac{1}{4} \int d^4x \sqrt{-g} \left\{ -\mathcal{R} + 2 \sum_{a=1}^N \partial^\mu \phi^a \partial_\mu \phi^a - \sum_{a=1}^N (\lambda_a)^2 e^{-2g_a \phi^a} F_{\mu\nu} F^{\mu\nu} \right\}, \quad (2)$$

where  $g_a \neq 0$ , since the problem is not significantly more complicated. Rather than using the coordinates which were exploited in [15], we will take static spherically symmetric solutions with coordinates

$$ds^2 = -e^{2U} dt^2 + e^{-2U} dr^2 + R^2 d\Omega_2^2, \quad (3)$$

where  $U = U(r)$  and  $R = R(r)$ , and  $d\Omega_2^2$  is the standard round metric on a 2-sphere. We take  $R > 0$  without loss of generality. We are primarily interested in black hole solutions with at least one regular horizon which are asymptotically flat as  $R \rightarrow \infty$ .

If we assume that the gauge field is given by a magnetic monopole configuration with components

$$F_{\hat{\theta}_1 \hat{\theta}_2} = \frac{Q}{R^2} \epsilon_{\hat{\theta}_1 \hat{\theta}_2} \quad (4)$$

in an orthonormal frame then the Maxwell-type equa-

tion,

$$\sum_{a=1}^N (\lambda_a)^2 \partial_\mu [e^{-2g_a \phi^a} \sqrt{-g} F^{\mu\nu}] = 0, \quad (5)$$

is satisfied identically, and the remaining field equations obtained from variation of the action (2) take the form

$$[R^2 e^{2U} \phi^{a\prime}]' = -\frac{g_a}{R^2} Q_a^2 e^{-2g_a \phi^a}, \quad (6)$$

$$R^2 (e^{2U})'' + 2RR'(e^{2U})' = \frac{2}{R^2} \sum_{a=1}^N Q_a^2 e^{-2g_a \phi^a}, \quad (7)$$

$$\frac{R''}{R} = -\sum_{a=1}^N \phi^{a\prime 2}, \quad (8)$$

$$[e^{2U} (R^2)']' = 2 - \frac{2}{R^2} \sum_{a=1}^N Q_a^2 e^{-2g_a \phi^a} \quad (9)$$

where

$$Q_a \equiv \lambda_a Q \quad (10)$$

and a prime denotes  $d/dr$ . By virtue of the Bianchi identity one of Eqs. (6)–(9) can be derived from the others.

One should note that in contrast to many simpler models [10,11,22] there is no simple duality relation between magnetic and electric solutions in the theory, on account of the multiscalar exponential coupling term which multiplies the electromagnetic part of the action (2). Thus we cannot simply read off the properties of solutions with an electric field in place of the monopole ansatz (4). It might be tempting to think of the action (2) as a special case of an alternative model in which each scalar field is associated with an independent  $U(1)$  gauge field. However, (2) is not a simple truncation of such a model, since such an increase of the number of  $U(1)$  fields would lead to  $N$  independent Maxwell-type equations, each with its own single scalar coupling, rather than a single equation of the form (5). Thus the present model can be expected to display differences in comparison to models in which moduli and several independent  $U(1)$  fields are present, allowing for the choice of duality-preserving combinations [22,23].

### A. Nature of horizons

A straightforward proof by contradiction of the type used in Refs. [24] may be used to establish that the solutions of (6)–(9) possess at most one regular horizon. We first note that the sum of Eqs. (7) and (9) gives

$$[e^{2U} R^2]'' = 2 \quad (11)$$

which may be integrated to yield

$$e^{2U} R^2 = r^2 - \alpha r + \beta, \quad (12)$$

for arbitrary constants  $\alpha$  and  $\beta$ . We assume that (12) possesses at least one real zero in order that there may exist at least one horizon and thus rewrite (12) as

$$e^{2U} R^2 = (r - r_-)(r - r_+), \quad (13)$$

where  $\alpha = r_+ + r_-$  and  $\beta = r_- r_+$ , and we may assume without loss of generality that  $r_- \leq r_+$ . We substitute (13) into (6) to obtain

$$[(r - r_-)(r - r_+) \phi^{a\prime}]' = -\frac{g_a}{R^2} Q_a^2 e^{-2g_a \phi^a}, \quad (14)$$

Let us now suppose that,  $r_+ \neq r_-$ , both values  $r = r_\pm$  correspond to regular horizons, and that  $g_a > 0$  for any one of the scalar fields. Since the scalar is smooth at the horizon, if we evaluate (14) at  $r = r_+$  we obtain

$$(r_+ - r_-) \phi^{a\prime}(r_+) = -\frac{g_a}{R^2(r_+)} Q_a^2 e^{-2g_a \phi^a(r_+)} \quad (15)$$

from which it follows that  $\phi^{a\prime}(r_+) < 0$ . Similarly, if we evaluate (14) at  $r = r_-$  we see that  $\phi^{a\prime}(r_-) > 0$ . Now given that  $\phi^a(r)$  is assumed to be smooth, it follows that it must have a maximum at an intermediate value  $r = r_0$  such that  $r_- < r_0 < r_+$ . However, if we evaluate (14) at  $r = r_0$  we obtain

$$(r_0 - r_-)(r_0 - r_+) \phi^{a\prime\prime}(r_0) = -\frac{g_a}{R^2(r_0)} Q_a^2 e^{-2g_a \phi^a(r_0)} \quad (16)$$

from which it follows that  $\phi^{a\prime\prime}(r_0) > 0$ , giving a minimum, which is a contradiction. Thus if  $\phi^a$  is regular at  $r = r_+$  it cannot also be regular at  $r = r_-$ . The point  $r = r_-$  should thus correspond to a curvature singularity.

Similarly, if we assume that  $g_a < 0$ , then all the signs in the above arguments are reversed but we still obtain contradiction. It therefore follows that the solutions can at most possess one regular horizon, at  $r = r_+$ .

It is also useful to note that if  $g_a > 0$  then  $\phi^a$  must be monotonically decreasing on the domain of outer communications of regular black hole solutions, since if it reached a minimum at a finite value  $r_0 > r_+$  then at such a point Eq. (15) would once again be true, but now with the implication that  $\phi^{a\prime\prime}(r_0) < 0$ , again giving a contradiction. Likewise if  $g_a < 0$  then  $\phi^a$  is monotonically increasing for  $r \geq r_+$ .

The function  $R(r)$  must be monotonically increasing in the domain of outer communications for either sign of  $g_a$ , since by (8)  $R'' < 0$  at finite  $r$  for any solutions with nontrivial scalars. This leaves a global maximum as the only possible turning point for the function  $R(r)$ , but such a choice would be inconsistent with asymptotic flatness, since by (13)  $R \sim r$  as  $r \rightarrow \infty$ , given  $e^{2U} \rightarrow 1$ .

### B. Constraints on charges

An additional first integral of the field equations may be extracted as follows: take the difference of (7) and (9), eliminate terms involving  $(e^{2U})''$  using (13), and eliminate terms involving  $e^{-2g_a\phi^a}$  using (6). Integrating the resulting equation one obtains

$$R^2 e^{2U} \sum_{a=1}^N \frac{\phi^{aI}}{g_a} = RR' e^{2U} - r + c \quad (17)$$

where  $c$  is an arbitrary constant. In order to obtain solutions which are regular at the outer horizon, it is in fact necessary to choose  $c = r_+$ . With this choice, and again using (13), (17) may be integrated to yield

$$A_0 \exp\left[\sum_{a=1}^N \frac{\phi^a}{g_a}\right] = \frac{R}{r - r_-}, \quad (18)$$

where  $A_0$  is an arbitrary constant. From (18) we see that  $r = r_-$  will correspond to a singularity, as expected from above.

Unfortunately, it does not appear to be possible to obtain an analytic solution in closed form to the remaining field equations. Since one of the  $N$  scalar Eqs. (6) can be eliminated with the use of (18), and since the function  $e^{2U}$  is given in terms of the function  $R$  according to (13), we are left with one first order ODE for  $R$ , namely

$$\begin{aligned} & \left(\frac{R'^2}{R^2} + \sum_{a=1}^N \phi^{aI2}\right)(r - r_+)(r - r_-) - 2\frac{R'}{R}(r - M) + 1 \\ &= \frac{1}{R^2} \sum_{a=1}^N Q_a^2 e^{-2g_a\phi^a} \end{aligned} \quad (19)$$

coupled nonlinearly to  $N - 1$  independent second order ODEs (14) for the scalars. This is equivalent therefore to a system of  $2N - 1$  first order ODEs, which can be solved numerically.

Much useful analytic information about the solutions can be obtained in relation to the values of the ADM mass,  $M$ , and the  $N$  scalar charges,  $\Sigma_a$ , which correspond to the  $O(r^{-1})$  terms in the asymptotic expansions at spatial infinity,

$$e^{2U} = 1 - \frac{2M}{r} + \frac{u_2}{r^2} + \dots, \quad (20)$$

$$\phi^a = \phi_\infty^a + \frac{\Sigma_a}{r} + \frac{\phi_2^a}{r^2} + \dots, \quad (21)$$

$$R^2 = r^2 \left(1 + \frac{R_1}{r} + \frac{R_2}{r^2} + \dots\right). \quad (22)$$

The  $O(r)$  coefficient,  $R_1$ , of the function  $R^2$  is a gauge parameter whose value fixes the choice of origin of the radial coordinate,  $r$ . We will use this gauge freedom to set

$$R_1 = 0.$$

Expanding Eq. (18) at spatial infinity by use of (21) and (22) it follows from the leading order term that the constant  $A_0$  is related to the moduli vacuum charges,  $\phi_\infty^a$ , according to

$$A_0 = \exp\left[-\sum_{a=1}^N \frac{\phi_\infty^a}{g_a}\right]. \quad (23)$$

Furthermore, if we also make use of (20) it follows from the next to leading order terms in (13) and (18) that the following relations hold between the constants  $r_\pm$  and the asymptotic charges

$$r_\pm = M \pm \left(M - \sum_{a=1}^N \frac{\Sigma_a}{g_a}\right). \quad (24)$$

The constraint that  $r_+ \geq r_-$  then yields the inequality

$$\sum_{a=1}^N \frac{\Sigma_a}{g_a} \leq M, \quad (25)$$

which is saturated for the extreme solutions for which the horizon is degenerate with the inner singularity.

With definitions of the asymptotic charges in hand we can now integrate various field equations between the horizon,  $r = r_+$ , and spatial infinity to obtain constraints on the charges. If the scalar Eqs. (6) are integrated on this interval, for example, we find that

$$\Sigma_a = g_a Q_a^2 \int_{r_+}^{\infty} dr \frac{e^{-2g_a\phi^a}}{R^2}. \quad (26)$$

We note that for solutions which are regular at the horizon, given that the integrand of (26) is positive it follows that

$$\frac{\Sigma_a}{g_a} \geq 0 \quad (27)$$

for each scalar charge, and furthermore  $\Sigma_a = 0$  if and only if  $Q = 0$ . Thus the charges  $\Sigma_a$  do not constitute an elementary primary scalar hair according to the definition adopted in the Introduction.

At first sight one might be tempted to assume that Eqs. (26) provide constraints on all  $N$  scalar charges, and that we are therefore dealing with a system with purely secondary scalar hair. However, this is *not* in fact the case since the lower limit of integration,  $r_+$ , already depends on the ADM mass and scalar charges according to (24).

The nature of the relations (26) is more readily understood if we rewrite them in terms of functions  $\tilde{\phi}^a \equiv \phi^a - \phi_\infty^a$ ,  $a = 1 \dots N$ , which have the leading order behavior  $\tilde{\phi}^a(r) \sim \Sigma_a/r + \dots$  at spatial infinity. We then see that Eqs. (26) can be rewritten as

$$\phi_\infty^a = \frac{1}{2g_a} \ln \left[ \frac{g_a Q_a^2}{\Sigma_a} \int_{r_+}^{\infty} dr \frac{e^{-2g_a\tilde{\phi}^a}}{R^2} \right] \quad (28)$$

for  $\Sigma_a \neq 0$ . Since the bounds of integration are independent of the moduli vacuum charges, we see that the relations (26) or (28) are constraints which determine the  $\phi_\infty^a$  in terms of  $Q$ ,  $M$  and  $\Sigma_a$ .

Let us turn to the question of whether there exist any constraints on the scalar charges  $\Sigma_a$ . In fact, there appears to be only one additional constraint on the asymptotic charges. This may be determined by noting that if one multiplies each of the scalar Eqs. (6) by  $2R^2 e^{2U} \phi^{a1}$  and then takes the sum of the resulting  $N$  equations plus  $\frac{1}{2} R^2 (e^{2U})'$  times the difference of Eqs. (9) and (11), one obtains the expression

$$\begin{aligned} \frac{d}{dr} \left\{ \frac{1}{4} [R^2 (e^{2U})']^2 + \sum_{a=1}^N [(R^2 e^{2U} \phi^{a1})^2] \right\} \\ = \frac{d}{dr} \left\{ e^{2U} \sum_{a=1}^N Q_a^2 e^{-2g_a \phi^a} \right\} \end{aligned} \quad (29)$$

We may integrate Eq. (29) from  $r = r_+$  to spatial infinity, and use (13) and (24) to obtain the following constraint on the charges

$$\sum_{a=1}^N \Sigma_a^2 + 2M \sum_{a=1}^N \frac{\Sigma_a}{g_a} - \left( \sum_{a=1}^N \frac{\Sigma_a}{g_a} \right)^2 = \bar{Q}^2, \quad (30)$$

where

$$\bar{Q}^2 \equiv \sum_{a=1}^N \bar{Q}_a^2 \quad (31)$$

with

$$\bar{Q}_a \equiv e^{-g_a \phi_\infty^a} Q_a = \lambda_a e^{-g_a \phi_\infty^a} Q. \quad (32)$$

The quantity  $\bar{Q}$  can be thought of as the magnetic monopole charge normalized by the weighted sum of the moduli vacuum charges. On account of the possibility of different vacuum moduli charges, the individual scalars can effectively “see” different magnetic monopole charges,  $\bar{Q}_a$ .

Using Eq. (30) we obtain the following expression equivalent to (24)

$$r_\pm = M \pm \left[ M^2 + \sum_{a=1}^N \Sigma_a^2 - \bar{Q}^2 \right]^{1/2}, \quad (33)$$

and  $\bar{Q}^2$  is bounded above according to

$$\bar{Q}^2 \leq M^2 + \sum_{a=1}^N \Sigma_a^2. \quad (34)$$

The constraint (30) reduces the number of independent scalar charges to  $N - 1$ . Do any further constraints remain to be found? In the  $N = 2$  case of two scalar fields this cannot be the case, given the numerical results of Fig. 1 and the results of the dynamical systems analysis of Ref. [15]: any further constraints would mean we no

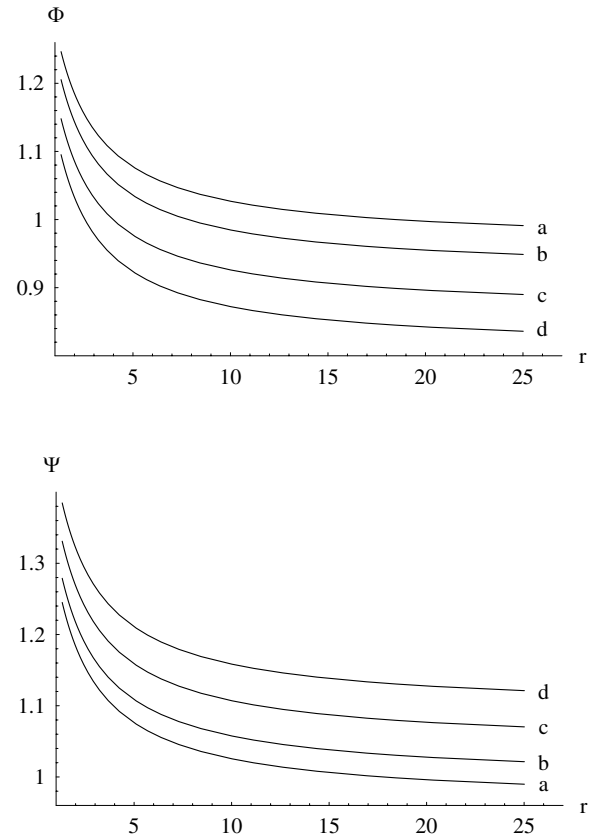


FIG. 1. The scalars  $\Phi$  and  $\Psi$ , solutions of the model (1) with  $\lambda_2 = 1$ ,  $q = 1$ , plotted for fixed values of  $r_+$  and  $r_-$ , and variable third independent parameter. Case (a) corresponds to  $\Sigma_\Phi = \Sigma_\Psi$ , and the other cases are ordered according to decreasing ratio  $\Sigma_\Phi/\Sigma_\Psi$ .

longer had a primary hair in contradiction with the results derived there. We will argue that no further constraints exist for  $N > 2$  either. In particular, if further constraints exist on the charges then it would be possible to extract them from the field equations. If we consider the field equations at spatial infinity, then solving order by order in inverse powers of  $r$  no constraints on the charges  $\Sigma_a$  are found, though we do find that all coefficients of terms  $O(r^{-n})$ ,  $n \geq 2$ , in the asymptotic series (20)–(22) are completely determined. At the next order, for example,

$$u_2 = \bar{Q}^2, \quad (35)$$

$$\phi_2^a = M \Sigma_a - \frac{1}{2} g_a \bar{Q}_a^2, \quad (36)$$

$$R_2 = - \sum_{a=1}^N \Sigma_a^2. \quad (37)$$

Solutions with the asymptotic expansions (20)–(22) include many which correspond to naked singularities rather than black holes. The requirement that solutions also have a regular horizon leads to the further con-

straints (28) and (30) found upon integrating the independent field equations from  $r = r_+$  to spatial infinity, as above. However, we can obtain no more than one constraint for each independent field Eq. (14) and (19), and the relations (28) and (30), which give one constraint for each equation, exhaust the possibilities. Thus we find that there are  $N - 1$  independent parameters among the  $N$  scalar charges,  $\Sigma_a$ .

### C. Thermodynamic quantities

Even in the absence of complete analytic solutions, some thermodynamic relations may be obtained, given that the black hole temperature and entropy are defined at the horizon,  $r_+$ , which is related to the mass by (24). In particular, let us evaluate the derivative of (13) at the horizon. In terms of the surface gravity,  $\kappa = \frac{1}{2}(e^{2U})'|_{r=r_+}$ , the horizon area  $\mathcal{A}_{\mathcal{H}} = 4\pi R^2(r_+)$ , and substituting for  $r_+$  from (24) we then find

$$M = \frac{\kappa \mathcal{A}_{\mathcal{H}}}{4\pi} + \sum_{a=1}^N \frac{\Sigma_a}{g_a}. \quad (38)$$

We now define a magnetostatic potential,  $\chi(r)$ , according to

$$-\partial_r \chi = \left[ \sum_{a=1}^N (\lambda_a)^2 e^{-2g_a \phi^a} \right]^* F_{tr} \quad (39)$$

$$= \left[ \sum_{a=1}^N (\lambda_a)^2 e^{-2g_a \phi^a} \right] \frac{Q}{R^2}. \quad (40)$$

We integrate (40) from  $r = r_+$  to  $r = \infty$ , and use (26) to find that the magnetostatic potential of the horizon is given by

$$\chi_{\mathcal{H}} \equiv \chi(r_+) = \frac{1}{Q} \sum_{a=1}^N \frac{\Sigma_a}{g_a}. \quad (41)$$

It then follows that (38) is equivalent to the Smarr-type relation

$$M = \frac{\kappa \mathcal{A}_{\mathcal{H}}}{4\pi} + Q \chi_{\mathcal{H}}. \quad (42)$$

For completeness, we also note that according to (24), the mass  $M$  is a homogeneous function of degree  $\frac{1}{2}$  in  $\mathcal{A}_{\mathcal{H}}$ , and of degree one in each  $\Sigma_a$ . The appropriate first law of black hole mechanics for the system is therefore

$$dM = T dS + \sum_{a=1}^N \frac{d\Sigma_a}{g_a} \quad (43)$$

where we have identified the temperature,  $T = \kappa/(2\pi)$ , and entropy,  $S = \frac{1}{4} \mathcal{A}_{\mathcal{H}}$ , in the usual fashion. There is no independent variation  $dQ$  in (43), since  $Q$  is related to the scalar charges via the constraint (30). Indeed, for adiabatic variations

$$dQ = \frac{1}{\chi_{\mathcal{H}}} \sum_{a=1}^N \frac{d\Sigma_a}{g_a} \quad (44)$$

Interestingly enough, given the form of Eq. (24) for  $r_+$ , there is no further contribution to the first law from independent variations of the vacuum moduli charges,  $\phi_\infty^a$ , as there is in the case of other theories such as those of Refs. [25]. In view of the relations (28) this is to be expected.

### III. LINEAR STABILITY

In many cases hairy black holes are unstable. In this section we examine whether this is the case also for our solutions. It turns out that our solutions are stable against linear radial perturbations. The general perturbations of the solutions in the case of a single scalar field (dilaton) has been studied in great detail in [20], using the methods of Ref. [26], and there stability has been proved. The calculations were, however, already very involved in that relatively simple case, and hence, following most of the literature on the subject [18,19], we prefer to limit ourselves to the study of radial perturbations. Moreover, we shall consider only the case of two scalar fields.

We consider the action (1), and for simplicity we put  $q = 1$ ,  $\lambda_2 = 1$ , since considering the more general set of coupling parameters in the  $N = 2$  case of Eq. (2) does not affect our conclusions.

For the discussion of stability, it is convenient to use coordinates in which the metric takes the form

$$ds^2 = -e^{\Gamma(R,t)} dt^2 + e^{\Lambda(R,t)} dR^2 + R^2 d\Omega_2^2, \quad (45)$$

with

$$\Phi = \Phi(t, R), \quad \Psi = \Psi(t, R), \quad (46)$$

and the magnetic field is given in an orthonormal basis by

$$F_{\hat{\theta}_1 \hat{\theta}_2} = Q \epsilon_{\hat{\theta}_1 \hat{\theta}_2} \quad (47)$$

In these coordinates, the field equations read

$$\begin{aligned} \Phi'' + \left( \frac{\Gamma' - \Lambda'}{2} + \frac{2}{R} \right) \Phi' - e^{\Lambda - \Gamma} \left( \ddot{\Phi} + \frac{\dot{\Lambda} - \dot{\Gamma}}{2} \dot{\Phi} \right) \\ = -\frac{Q^2}{R^4} e^{\Lambda - 2\Phi}, \end{aligned} \quad (48)$$

$$\begin{aligned} \Psi'' + \left( \frac{\Gamma' - \Lambda'}{2} + \frac{2}{R} \right) \Psi' - e^{\Lambda - \Gamma} \left( \ddot{\Psi} + \frac{\dot{\Lambda} - \dot{\Gamma}}{2} \dot{\Psi} \right) \\ = -\frac{Q^2}{R^4} e^{\Lambda - 2\Psi}, \end{aligned} \quad (49)$$

$$\begin{aligned} \Lambda' = R[\Phi'^2 + \Psi'^2 + e^{\Lambda - \Gamma}(\dot{\Phi}^2 + \dot{\Psi}^2)] + \frac{1 - e^\Lambda}{R} \\ + \frac{Q^2}{R^3} (e^{\Lambda - 2\Phi} + e^{\Lambda - 2\Psi}), \end{aligned} \quad (50)$$



$$\begin{aligned}\Gamma' &= R[\Phi'^2 + \Psi'^2 + e^{\Lambda-\Gamma}(\dot{\Phi}^2 + \dot{\Psi}^2)] \\ &\quad - \frac{1 - e^\Lambda}{R} - \frac{Q^2}{R^3}(e^{\Lambda-2\Phi} + e^{\Lambda-2\Psi}),\end{aligned}\quad (51)$$

$$\dot{\Lambda} = 2R(\dot{\Phi}\Phi' + \dot{\Psi}\Psi'),\quad (52)$$

$$\begin{aligned}\Gamma'' + \left(\frac{\Gamma'}{2} + \frac{1}{R}\right)(\Gamma' - \Lambda') - e^{\Lambda-\Gamma}\left(\ddot{\Lambda} + \frac{\dot{\Lambda} - \dot{\Gamma}}{2}\right)\dot{\Lambda} \\ = 2[e^{\Lambda-\Gamma}(\dot{\Phi}^2 + \dot{\Psi}^2) - (\Phi'^2 + \Psi'^2)] \\ + \frac{2Q^2}{R^4}(e^{\Lambda-2\Phi} + e^{\Lambda-2\Psi}),\end{aligned}\quad (53)$$

where the prime and the dot denote differentiation with respect to  $R$  and  $t$ , respectively.

We perturb the field equations by time-dependent linear perturbations of the form

$$\begin{aligned}\Gamma(R, t) &= \Gamma(R) + \delta\Gamma(R, t)e^{i\omega t}, \\ \Lambda(R, t) &= \Lambda(R) + \delta\Lambda(R, t)e^{i\omega t}, \\ \Phi(R, t) &= \Phi(R) + \delta\Phi(R, t)e^{i\omega t}, \\ \Psi(R, t) &= \Psi(R) + \delta\Psi(R, t)e^{i\omega t},\end{aligned}$$

where the perturbations are assumed small and the functions  $\Gamma(R)$ ,  $\Lambda(R)$ ,  $\Phi(R)$  and  $\Psi(R)$  denote the time-independent unperturbed solutions of the field equations. We did not perturb the Maxwell field since the electromagnetic Bianchi identities imply that the monopole-like solution (47) must be independent of the radial coordinate.

The perturbed equations then read

$$\begin{aligned}\delta\Phi'' + \left(\frac{\Gamma' - \Lambda'}{2} + \frac{2}{R}\right)\delta\Phi' + \frac{\Phi'}{2}(\delta\Gamma' - \delta\Lambda') - e^{\Lambda-\Gamma}\delta\ddot{\Phi} \\ = -\frac{Q^2}{R^4}e^{\Lambda-2\Phi}(\delta\Lambda - 2\delta\Phi),\end{aligned}\quad (54)$$

$$\begin{aligned}\delta\Psi'' + \left(\frac{\Gamma' - \Lambda'}{2} + \frac{2}{R}\right)\delta\Psi' + \frac{\Psi'}{2}(\delta\Gamma' - \delta\Lambda') - e^{\Lambda-\Gamma}\delta\ddot{\Psi} \\ = -\frac{Q^2}{R^4}e^{\Lambda-2\Psi}(\delta\Lambda - 2\delta\Psi),\end{aligned}\quad (55)$$

$$\begin{aligned}\delta\Lambda' - 2R(\Phi\delta\Phi' + \Psi\delta\Psi') + \frac{e^\Lambda}{R}\delta\Lambda \\ = \frac{Q^2}{R^3}[e^{\Lambda-2\Phi}(\delta\Lambda - 2\delta\Phi) + e^{\Lambda-2\Psi}(\delta\Lambda - 2\delta\Psi)],\end{aligned}\quad (56)$$

$$\begin{aligned}\delta\Gamma' - 2R(\Phi\delta\Phi' + \Psi\delta\Psi') - \frac{e^\Lambda}{R}\delta\Lambda \\ = -\frac{Q^2}{R^3}[e^{\Lambda-2\Phi}(\delta\Lambda - 2\delta\Phi) + e^{\Lambda-2\Psi}(\delta\Lambda - 2\delta\Psi)],\end{aligned}\quad (57)$$

$$\delta\dot{\Lambda} = 2R(\Phi'\delta\dot{\Phi} + \Psi'\delta\dot{\Psi}),\quad (58)$$

$$\begin{aligned}\delta\Gamma'' + \left(\Gamma' - \frac{\Lambda'}{2} + \frac{1}{R}\right)\delta\Gamma' - \left(\frac{\Gamma'}{2} + \frac{1}{R}\right)\delta\Lambda' - e^{\Lambda-\Gamma}\delta\ddot{\Lambda} \\ = -4(\Phi'\delta\Phi' + \Psi'\delta\Psi') + \frac{2Q^2}{R^4}[e^{\Lambda-2\Phi}(\delta\Lambda - 2\delta\Phi) \\ + e^{\Lambda-2\Psi}(\delta\Lambda - 2\delta\Psi)].\end{aligned}\quad (59)$$

Equation (58) can be immediately integrated. With suitable boundary conditions it yields

$$\delta\Lambda = 2R(\Phi'\delta\Phi + \Psi'\delta\Psi).\quad (60)$$

The problem of stability can then be reduced to the study of the perturbation of the scalar fields  $\Phi$  and  $\Psi$  [18,19]. After long manipulations of the perturbed equations, one can obtain a coupled system of second order linear equations for  $\delta\Phi$  and  $\delta\Psi$ :

$$\begin{aligned}\delta\Phi'' + \left(\frac{\Gamma' - \Lambda'}{2} + \frac{2}{R}\right)\delta\Phi' + A(R)\delta\Phi + C(R)\delta\Psi \\ = e^{\Lambda-\Gamma}\delta\ddot{\Phi},\end{aligned}\quad (61)$$

$$\begin{aligned}\delta\Psi'' + \left(\frac{\Gamma' - \Lambda'}{2} + \frac{2}{R}\right)\delta\Psi' + C(R)\delta\Phi + B(R)\delta\Psi \\ = e^{\Lambda-\Gamma}\delta\ddot{\Psi},\end{aligned}\quad (62)$$

where

$$\begin{aligned}A(R) &= -2R\left[2\Phi'\Phi'' + \left(\frac{\Gamma' - \Lambda'}{2} + \frac{3}{R}\right)\Phi'^2\right] \\ &\quad + 2\left[\Phi'' + \left(\frac{\Gamma' - \Lambda'}{2} + \frac{2}{R}\right)\Phi'\right],\end{aligned}\quad (63)$$

$$\begin{aligned}B(R) &= -2R\left[2\Psi'\Psi'' + \left(\frac{\Gamma' - \Lambda'}{2} + \frac{3}{R}\right)\Psi'^2\right] \\ &\quad + 2\left[\Psi'' + \left(\frac{\Gamma' - \Lambda'}{2} + \frac{2}{R}\right)\Psi'\right],\end{aligned}\quad (64)$$

$$C(R) = -2R\left[\Phi'\Psi'' + \Phi''\Psi' + \left(\frac{\Gamma' - \Lambda'}{2} + \frac{3}{R}\right)\Phi'\Psi'\right].\quad (65)$$

In Schwarzschild coordinates (45) the previous equations are not regular at the horizon. Therefore, it necessary to define new ‘‘tortoise’’ coordinates, given by [19]

$$R^* = \int e^{(\Gamma-\Lambda)/2} dR.$$

Defining new fields

$$u = R\delta\Phi, \quad v = R\delta\Psi,$$

and using the explicit time-dependence of the perturbative modes, one can finally put the stability Eqs. (61) and (62) in the Schrödinger form

$$\frac{d^2\mathbf{u}}{dR^{*2}} + \omega^2\mathbf{u} = \mathbf{A}\mathbf{u}, \quad (66)$$

where  $\mathbf{u}$  is the vector of components  $(u, v)$  and  $\mathbf{A}$  is a symmetric matrix with entries

$$A_{11} = -\left(A - \frac{\Gamma' - \Lambda'}{2R}\right)e^{\Gamma-\Lambda}, \quad A_{12} = A_{21} = -Ce^{\Gamma-\Lambda},$$

$$A_{22} = -\left(B - \frac{\Gamma' - \Lambda'}{2R}\right)e^{\Gamma-\Lambda}.$$

The matrix  $\mathbf{A}$  can be diagonalized, and has eigenvalues

$$V_{1,2} = \frac{1}{2}[(A_{11} + A_{22}) \pm \sqrt{(A_{11} - A_{22})^2 + 4A_{12}^2}].$$

The classical solutions are stable under linear perturbations if the potentials  $V_{1,2}$  are everywhere positive. This can be proved by generalizing the arguments of Chandrasekhar [26]. In fact, (66) can be written as

$$\frac{\partial^2\mathbf{u}}{\partial t^2} - \frac{\partial^2\mathbf{u}}{\partial R^{*2}} + \mathbf{A}\mathbf{u} = 0. \quad (67)$$

Multiplying (67) by  $\partial\mathbf{u}^\dagger/\partial t$  and integrating over  $R^*$ , one gets

$$\int \left( \frac{\partial\mathbf{u}^\dagger}{\partial t} \frac{\partial^2\mathbf{u}}{\partial t^2} - \frac{\partial\mathbf{u}^\dagger}{\partial t} \frac{\partial^2\mathbf{u}}{\partial R^{*2}} + \frac{\partial\mathbf{u}^\dagger}{\partial t} \mathbf{A}\mathbf{u} \right) dR^*.$$

After integrating the second term by parts, and adding the complex conjugate equation, one obtains the energy integral

$$\int \left( \left| \frac{\partial\mathbf{u}}{\partial t} \right|^2 + \left| \frac{\partial\mathbf{u}}{\partial R^*} \right|^2 + \mathbf{u}^\dagger \mathbf{A}\mathbf{u} \right) dR^* = \text{const}. \quad (68)$$

If the last two terms in the integral are bounded and positive definite, it follows that the integral of  $|\partial\mathbf{u}/\partial t|^2$  is also bounded, ruling out any exponential growth of the perturbations. In our case, it is sufficient to show that  $\mathbf{u}^\dagger \mathbf{A}\mathbf{u}$  is positive. This can be easily checked by diagonalizing the matrix  $\mathbf{A}$ . If the eigenvalues  $V_{1,2}$  are non-negative functions, then  $\mathbf{u}^\dagger \mathbf{A}\mathbf{u}$  is clearly positive.

This can be readily checked for the exact solutions. In the other cases, one has of course to resort to numerical calculations. Let us consider for example the solution with  $\Phi = \Psi$  [15]. This can be written as

$$ds^2 = -\frac{(r-r_+)(r-r_-)^{1/3}}{r^{4/3}} dt^2 + \frac{r^{4/3} dr^2}{(r-r_+)(r-r_-)^{4/3}} + r^{2/3}(r-r_-)^{4/3} d\Omega^2, \quad (69)$$

$$e^{-2\Phi} = e^{-2\Psi} = \left(1 - \frac{r_-}{r}\right)^{2/3}. \quad (70)$$

In terms of the coordinate  $R$  such that the metric takes the form (45), one has

$$e^\Gamma = \frac{(a^2 - 2a\Delta + \Delta^2)^{1/3}(a - b\Delta + \Delta^2)}{(a^2 + \Delta + \Delta^2)^{4/3}}, \quad (71)$$

$$e^\Lambda = \frac{R(a^2 - \Delta^2)^2(a^2 + a\Delta + \Delta^2)^{1/3}}{(R^3 + 4a^3)\Delta^2(a^2 - 2a\Delta + \Delta^2)^{1/3}(a^2 - b\Delta + \Delta^2)}, \quad (72)$$

$$e^{-2\Phi} = e^{-2\Psi} = \left(\frac{a^2 - 2a\Delta + \Delta^2}{a^2 + a\Delta + \Delta^2}\right)^{2/3}, \quad (73)$$

where  $a = r_-/3$ ,  $b = r_+ - r_-/3$ , and

$$\Delta = \sqrt[3]{R^3 + 2a^3 + \sqrt{R^3(R^3 + 4a^3)}}. \quad (74)$$

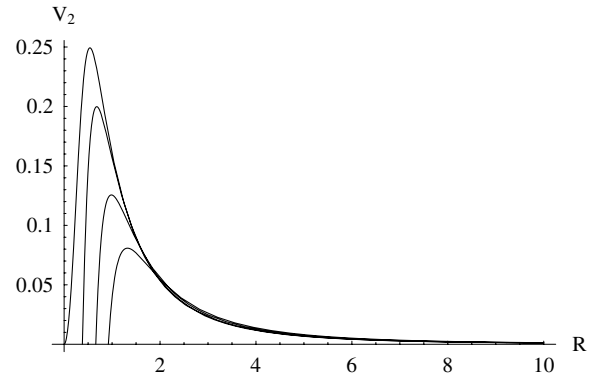
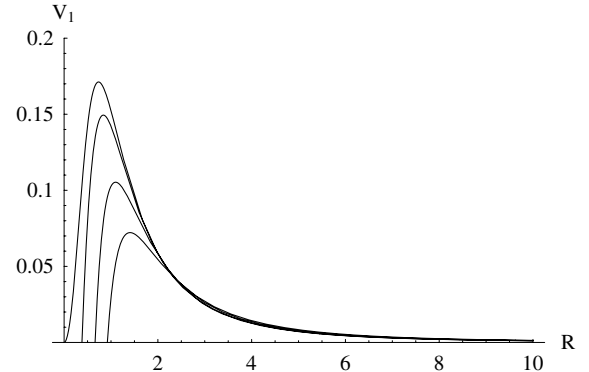


FIG. 2. The potentials  $V_1$  and  $V_2$  for the exact solution, with different values of  $b/a$ , and hence of the horizon. The extremal case corresponds to horizon at  $R = 0$ .

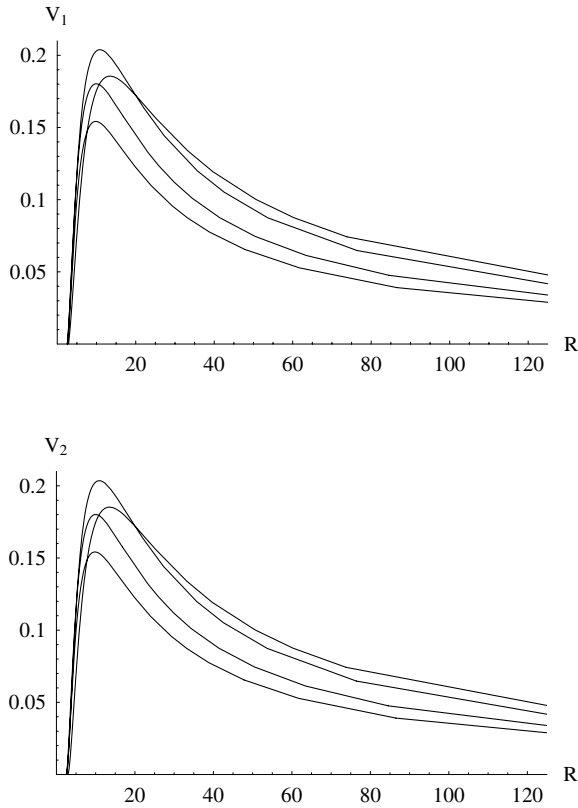


FIG. 3. The potentials  $V_1$  and  $V_2$  for numerical solutions with fixed values of  $r_+$  and  $r_-$ , and variable third independent parameter. The height of the peak increases with the value of the ratio between the scalar charges. The lowest one corresponds to  $\Sigma_\Phi = \Sigma_\Psi$ .

In these coordinates, the singularity is located at  $R = 0$ , and the horizon at  $R = (b + a)^{2/3}(b - 2a)^{1/3}$ .

One can now substitute the metric functions in (66). The analysis is greatly simplified by the fact that for this solution  $\Phi = \Psi$  and hence  $A(R) = B(R)$ . The equations readily separate into two independent equations for  $u + v$  and  $u - v$ , with potentials  $V_1 = A_{11} + A_{12}$  and  $V_2 = A_{11} - A_{12}$ , respectively.

The potentials are plotted for different values of the ratio  $r_+/r_-$  in Fig. 2. They vanish at the horizon and at infinity and are regular and positive in the interval. We can hence deduce the stability of the solution with  $\Phi = \Psi$ .

In the general case, numerical calculations show that the behavior of  $V_{1,2}$  is qualitatively the same as for exact solutions. Some examples are given in Fig. 3. We can conclude that all the classical solutions are stable against radial linear perturbations.

#### IV. DISCUSSION

We have generalized the solutions previously found in Ref. [15] for magnetically charged black holes nonminimally coupled to two scalar fields, to the case of multiple

scalar fields nonminimally coupled to a single magnetic monopole. Even though the complete analytic solutions have not been derived, we have succeeded in integrating enough of the field equations that constraints on the masses and charges can be derived. This analysis supports the claim made in Ref. [15] that the solutions possess a primary hair. In the case of  $N$  scalar fields there are  $N - 1$  independent parameters among the scalar charges.

We have further shown that in the case of two scalar fields, the black hole solutions are classically stable to radial perturbations, a feature which is absent in the case of a number of other hairy black hole solutions. We have no reason to expect that the case of multiple scalar fields as given in Sec. II would be any different.

Our present analysis indicates that the primary hair of the multiscalar black holes has quite novel features as compared to the case of other hairy black holes. In particular, while the black hole is characterized by new independent degrees of freedom which are defined at spatial infinity, these charges must necessarily vanish if the magnetic field is turned off. To distinguish such scalar hair from the case of *elementary primary scalar hair*, which would theoretically exist even in the absence of gauge charges, we identify this new form of hair as *contingent primary scalar hair*. This suggests a further refinement of the no-hair theorems by the statement: *In theories satisfying the DEC, black holes do not possess elementary primary scalar hair.*

Our analysis of the thermodynamic properties of the solutions has been limited to the derivation of a Smarr formula, and of the first law. A further step would be the thorough study of the thermodynamical properties of the solutions using explicit numerical solutions.

We also remark that solutions with properties analogous to those of the model studied in this paper have been obtained in the case of Gauss-Bonnet gravity nonminimally coupled to two scalar fields [27]. It would be interesting to investigate if our results can be extended also to that case.

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