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Risk-Sensitive Filtering and Smoothing for Continuous-Time Markov Processes

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Abstract—We consider risk sensitive filtering and smoothing for a dynamical system whose output is a vector process in \mathbb{R}^2 . The components of the observation process are a Markov process observed through a Brownian motion and a Markov process observed through a Poisson process. Risk-sensitive filters for the robust estimation of an indirectly observed Markov state processes are given. These filters are stochastic partial differential equations for which robust discretizations are obtained. Computer simulations are given which demonstrate the benefits of risk sensitive filtering.

Index Terms—Change of measure, martingales, risk-sensitive filtering.

I. INTRODUCTION

IN this paper, we consider robust nonlinear filtering and smoothing in the presence of model uncertainties. To derive filters robust to model uncertainties, we use the so-called risk-sensitive criterion [1]. The dynamical models considered involve indirect observations of a continuous-time Markov process. For a general model, we consider a vector observation process, where the two components of this vector are different observations of the same Markov state process. The two observation processes are: i) the Markov process observed through a Brownian motion, ii) the Markov process observed through a Poisson process. This particular model has also been considered in [2].

The paper is organized as follows. In Section II, the signal models for the state process and the observation process are defined. In Section III, we give a definition of risk-sensitive filtering and present risk-sensitive filters for our stochastic model. In this section, we derive versions of our risk-sensitive filters which are robust to time discretization. We obtain these versions by adapting the transformation techniques introduced by Clark [3]. In Section IV, we provide a corresponding robust smoothing algorithm. Finally, in Section VI, we present a simulation study

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to show the performance benefits in a counting process observation scenario.

II. SIGNAL MODELS

In this section, we describe dynamics for an indirectly observed state process and dynamics for two scalar-valued observation processes, each influenced by the same state process. All signal models are defined on the fixed probability space (Ω, \mathcal{F}, P) .

A. State Process

For our state variable, consider a Markov process in continuous time whose state space is a countable finite set. Our representation for this process follows that used in [4], whereby one can, without loss of generality, take the state space to be the set $\mathcal{L} \triangleq \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, whose elements \mathbf{e}_i are column vectors in \mathbb{R}^n ; \mathbf{e}_i has unity in the i th position and zero elsewhere. The essential point of this representation is the state process dynamics can then be written down in a semimartingale form (see [4])

$$X_t = X_0 + \int_0^t AX_u du + M_t. \quad (2.1)$$

Here M is a martingale with respect to the sigma field generated by the process X and A is an $n \times n$ rate matrix for X . Suppose that our Markov chain in fact takes real values $\lambda^1, \dots, \lambda^n$. Write $\boldsymbol{\lambda} \triangleq (\lambda^1, \dots, \lambda^n)'$. Then with the underlying Markov chain X taking values in \mathcal{L} , the related real-valued Markov process is just $\langle X, \boldsymbol{\lambda} \rangle$. We wish to consider the case where $\langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle$ are possible rate parameters of a Poisson process.

B. Observation Processes

We suppose that the Markov process X of Section II-A is not directly observed; instead, there are two scalar-valued observation processes y and N . The vector-valued observation process $Y = \{Y_t, 0 \leq t\}$ is a process taking values in \mathbb{R}^2 , where

$$Y \triangleq \begin{bmatrix} y \\ N \end{bmatrix}. \quad (2.2)$$

Case 1, the y -Component of Y : The process y is an observation of X through a Brownian motion, with dynamics

$$y_t = \int_0^t \langle X_u, \mathbf{g} \rangle du + w_t. \quad (2.3)$$

Here, $\mathbf{g} \triangleq (g^1, g^2, \dots, g^n)'$ is a vector of the drift coefficients and w_t is a standard Brownian motion, independent of the process X .

Case 2, the N-Component of Y: The process N is a Markov modulated Poisson process whose intensity process depends upon the process X . Its dynamics are

$$N_t = \int_0^t \langle X_u, \boldsymbol{\lambda} \rangle du + L_t. \quad (2.4)$$

Here L is a $(\mathcal{G}_t = \sigma\{Y_u, X_u; u \leq t\}, P)$ -martingale.

The two main problems we consider in this paper are; the robust estimation of the filter probabilities $P(X_t = e_i | \mathcal{Y}_t)$, for $i = 1, 2, \dots, n$ and the robust estimation of the smoother probabilities $P(X_t = e_i | \mathcal{Y}_T)$, for $t \leq T$ and $i = 1, 2, \dots, n$. Here the term robust is taken in the risk sensitive sense [1], [5].

C. Reference Probability

A reference probability P^\dagger is introduced under which the two components of the observation process Y_t are, respectively, a standard Brownian motion and a standard Poisson process. That is, under the measure P^\dagger , the dynamics have the form

$$P^\dagger \begin{cases} dX_t = AX_t dt + dM_t \\ dY_t = \begin{bmatrix} dw_t \\ dN_t \end{bmatrix}. \end{cases} \quad (2.5)$$

Of course, under the P , the dynamics have the form

$$P \begin{cases} dX_t = AX_t dt + dM_t \\ dY_t = \begin{bmatrix} \langle X_t, \mathbf{g} \rangle dt + dw_t \\ \langle X_t, \boldsymbol{\lambda} \rangle dt + dm_t \end{bmatrix}. \end{cases} \quad (2.6)$$

III. RISK-SENSITIVE FILTERING

A. Risk-Sensitive Estimation

Risk-sensitive filtering provides a type of robust filtering. The term “robust” is a relative one with a variety of possible meanings. For example, an estimation scheme could be robust against assumptions of independence, or robust against the parameters of a certain distribution. One general definition of robustness is “robustness signifies insensitivity against small deviations from assumptions,” [6]. In this paper, we consider robustness against uncertainties in the parameters of a probabilistic model for a stochastic dynamical system. We assume that there always exists a true and fixed, yet possibly unknown, probabilistic model for the system of interest. We denote the distribution of this true model by P_0 . Since we declare that the model being used may not necessarily correspond to the true model, we denote the design model by P_d , where d is a label corresponding to the design probability model.

Suppose we are interested in the estimation of a scalar-valued function of the Markov process X ; let this function be denoted by ξ . Further, suppose that the function $\xi(\cdot)$ is \mathcal{Y}_t -measurable.

Definition 1: Suppose c and h are convex scalar “cost” functions with $c(\xi, \hat{\xi}) = 0$ and $h(\xi, \hat{\xi}) = 0$, iff $\xi = \hat{\xi}$. For example, $c(\xi, \hat{\xi}) = \|\xi - \hat{\xi}\|^2$. Suppose further that there exists a process $\{\hat{\xi}_t^\mu, 0 \leq t\}$ which is continuous on the right with limits on the left (that is, CORLLOL) and which satisfies

$$\hat{\xi}_t^\mu = \underset{\hat{\xi} \in \mathcal{Y}_t}{\operatorname{argmin}} E_d \left[\exp \mu \left\{ \int_0^t c(\xi_u, \hat{\xi}_u) du + h(\xi_t, \hat{\xi}_t) \right\} \right]. \quad (3.7)$$

Then $\hat{\xi}_t^\mu$ is a risk sensitive estimate of ξ_t .

Notation: The expectation $E_d[\cdot]$ in (3.7) denotes expectation under the design distribution P_d , and the parameter $\mu > 0$ is the so-called risk-sensitive parameter. Quantitatively, the risk-sensitive parameter determines the degree of “risk.”

Risk-sensitive estimators defined by (3.7) enjoy an estimation error which is upper-bounded. This is in stark contrast to estimators defined by the minimum mean-square error criterion for which no such bound exists. The upper error bound corresponding to (3.7) is determined in [1] and is

$$\mathbf{E}_0 \left[\int_0^t c(\xi_u, \hat{\xi}_u^\mu) du \right] \leq \frac{\log J_t(\hat{\xi}^\mu) + R_t(P_0 | P_d)}{\mu}. \quad (3.8)$$

Here

$$J_t(\hat{\xi}^\mu) = \mathbf{E}_d \left[\exp \mu \left\{ \int_0^t c(\xi_u, \hat{\xi}_u^\mu) du + h(\xi_t, \hat{\xi}_t^\mu) \right\} \right] \quad (3.9)$$

and $R_t(P_0 | P_d)$ denotes the relative entropy between the two probability models P_0 and P_d . This result (due to Boel *et al.* [1]) is an important theoretical contribution, as it established a “precise” meaning for risk-sensitive estimation. The bound defined by (3.8) tells us that the error in risk-sensitive estimation has an upper bound given by the sum of two terms. The first term coincides with a cost if the model were known precisely, while the second term is a measure of distance between the true probability and the design probability models for the system.

B. Filtering Equations

Suppose $c(\cdot)$ is a cost function, as in Definition 1 and

$$Z_t \triangleq \operatorname{diag}\{z_t\}, \quad (3.10)$$

where

$$z_t \triangleq \exp \left(\mu \int_0^t c(\xi_u, \hat{\xi}_u^\mu) du \right).$$

Consider the state variable X_t with state space \mathcal{L} .

The risk-sensitive unnormalized measure of X_t , given the history \mathcal{Y}_t , is defined by

$$q_t^\mu \triangleq E^\dagger[X_t \Lambda_t Z_t | \mathcal{Y}_t]. \quad (3.11)$$

The expectation in this definition is with respect to the reference measure P^\dagger and Λ_t denotes the Radon–Nikodym derivative

$$\Lambda_t = \frac{dP}{dP^\dagger} \Big|_{\mathcal{G}_t}. \quad (3.12)$$

Write

$$c_i(\hat{\xi}_t^\mu) \triangleq c(\mathbf{e}_i, \hat{\xi}_t^\mu). \quad (3.13)$$

Theorem 1: The process q^μ satisfies the stochastic integral equation

$$\begin{aligned} q_t^\mu &= q_0^\mu + \int_0^t \left[A + \mu \times \operatorname{diag} \left\{ c_i(\hat{\xi}_u^\mu) \right\} \right] q_u^\mu du \\ &\quad + \int_0^t \operatorname{diag} \{ \langle \mathbf{g}, \mathbf{e}_i \rangle \} q_u^\mu dy_u \\ &\quad + \int_0^t \operatorname{diag} \{ \langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle - 1 \} q_{u-}^\mu (dN_u - du). \end{aligned} \quad (3.14)$$

A proof of Theorem 1 is given in the Appendix.

Corollary 1: The process q^μ , defined by (3.14), is an unnormalized measure-valued process. The corresponding normalized measure-valued process $\Pi^\mu = \frac{q^\mu}{\langle q^\mu, \mathbf{1} \rangle}$ has dynamics

$$\begin{aligned} \Pi_t^\mu &= \Pi_0^\mu + \int_0^t \left[A + \mu \times \text{diag} \left\{ c_i \left(\hat{\xi}_u^\mu \right) \right\} \right] \Pi_u^\mu du \\ &+ \int_0^t \text{diag} \{ \langle \mathbf{g}, \mathbf{e}_i \rangle \} \Pi_u^\mu - \Pi_{u-}^\mu \langle \Pi_u^0, \mathbf{g} \rangle \} d\nu_u^{(1)} \\ &+ \int_0^t \left\{ \frac{\text{diag} \{ \langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle \} \Pi_{u-}^\mu - \Pi_{u-}^\mu \langle \Pi_u^0, \boldsymbol{\lambda} \rangle}{\langle \Pi_u^0, \boldsymbol{\lambda} \rangle} \right\} d\nu_u^{(2)}. \end{aligned} \quad (3.15)$$

Here

$$d\nu_t^{(1)} = dy_t - \langle \Pi_t^0, \mathbf{g} \rangle dt \quad (3.16)$$

$$d\nu_t^{(2)} = dN_t - \langle \Pi_t^0, \boldsymbol{\lambda} \rangle dt \quad (3.17)$$

and Π_t^0 denotes risk neutral expectation, for example, $\langle \Pi_t^0, \mathbf{g} \rangle = E[\langle X_t, \mathbf{g} \rangle \mid \mathcal{Y}_t]$.

A proof of Corollary 1 is given in the Appendix.

Remark 1: The corresponding risk-neutral form of (3.14), and (3.15), are recovered by considering the limit, $\lim_{\mu \rightarrow 0} \frac{J_t(\hat{\xi}_t^\mu)}{\mu}$.

C. State Estimation

In the previous subsection, we defined a stochastic integral equation satisfied by the conditional distribution q_t^μ . We now use this distribution to write down a risk-sensitive state estimator for the function ξ_t . In Definition 1, the risk-sensitive estimate $\hat{\xi}_t^\mu$ was defined as an estimate which minimized an expectation involving an exponential cost. It was shown in [1] that one can identify a *version* of the minimum risk-sensitive estimator for ξ_t which attains the minimum defined by (3.7). The equation for the risk-sensitive estimate $\hat{\xi}_t^\mu$ is

$$\hat{\xi}_t^\mu \triangleq \underset{\hat{\xi} \in \mathbb{R}^n}{\text{argmin}} \sum_{i=1}^n \exp(\mu \times c(\xi^i, \hat{\xi})) \frac{q_t^\mu}{\langle q_t^\mu, \mathbf{1} \rangle} \quad (3.18)$$

where $\mathbf{1}$ denotes the vector in \mathbb{R}^n whose elements are each unity. Taken together, the stochastic differential (3.14) and the state estimator (3.18) comprise a risk-sensitive filter. To implement such a filter, one needs to first compute the density q_t^μ at time t , then, using this density, determine the estimate $\hat{\xi}_t^\mu$ defined by (3.18).

D. Transformed Filtering Equations

Notation: Let ϕ_t^i be a scalar-valued stochastic process defined by

$$\phi_t^i \triangleq \exp \left(\langle \mathbf{g}, \mathbf{e}_i \rangle y_t - \frac{1}{2} \langle \mathbf{g}, \mathbf{e}_i \rangle^2 t \right). \quad (3.19)$$

Let Φ_t be a matrix exponential, defined by

$$\Phi_t \triangleq \text{diag} \{ \phi_t^1, \phi_t^2, \dots, \phi_t^n \}. \quad (3.20)$$

Notation: Let γ_t^i be a scalar-valued stochastic process defined by

$$\gamma_t^i \triangleq \exp((1 - \langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle)t) \langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle^{N_t} \quad (3.21)$$

where $\langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle$ is the i th Poisson rate and N_t is the observation process defined by the model (2.4). Let Γ_t be a matrix exponential defined by

$$\Gamma_t \triangleq \text{diag} \{ \gamma_t^1, \gamma_t^2, \dots, \gamma_t^n \}. \quad (3.22)$$

Equation (3.14) can be solved by variation of constants. Consider the two subsidiary equations for the diagonal matrices Φ_t and Γ_t

$$d\Phi_t = \text{diag} \{ \langle \mathbf{g}, \mathbf{e}_i \rangle \} \Phi_t dy_t, \quad \Phi_0 = I \quad (3.23)$$

$$d\Gamma_t = \text{diag} \{ \langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle - 1 \} \Gamma_t (dN_t - dt), \quad \Gamma_0 = I \quad (3.24)$$

where I is the identity matrix in \mathbb{R}^n . The matrices Φ_t and Γ_t are diagonal so we are considering the following scalar equations for each diagonal entry:

$$d\phi_t^i = \langle \mathbf{g}, \mathbf{e}_i \rangle \phi_t^i dy_t, \quad \phi_0^{(i)} = 1 \quad (3.25)$$

$$d\gamma_t^i = (\langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle - 1) \gamma_t^i (dN_t - dt), \quad \gamma_0^{(i)} = 1. \quad (3.26)$$

Therefore,

$$\phi_t^i = \exp \left\{ \langle \mathbf{g}, \mathbf{e}_i \rangle y_t - \frac{1}{2} \langle \mathbf{g}, \mathbf{e}_i \rangle^2 t \right\} \quad (3.27)$$

$$\gamma_t^i = ((\langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle - 1)t) \langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle^{N_t}. \quad (3.28)$$

Consequently, Φ_t and Γ_t are nonsingular and

$$\Phi_t^{-1} = \text{diag} \{ 1 / (\phi_t^i) \} \quad (3.29)$$

$$\Gamma_t^{-1} = \text{diag} \{ 1 / (\gamma_t^i) \}. \quad (3.30)$$

Consider the solution of \bar{q}_t^μ of the ordinary differential equation

$$\frac{d\bar{q}_t^\mu}{dt} = \Phi_t^{-1} \Gamma_t^{-1} \left[A + \mu \times \text{diag} \left\{ c_i \left(\hat{\xi}_t^\mu \right) \right\} \right] \Phi_t \Gamma_t \bar{q}_t^\mu \quad (3.31)$$

with initial condition $\bar{q}_0^\mu = q_0^\mu$.

Theorem 2: The process $q^\mu \triangleq \Phi \Gamma \bar{q}^\mu$ is a unique solution of the stochastic integral equation

$$\begin{aligned} q_t^\mu &= q_0^\mu + \int_0^t \left[A + \mu \times \text{diag} \left\{ c_i \left(\hat{\xi}_u^\mu \right) \right\} \right] q_u^\mu du \\ &+ \int_0^t \text{diag} \{ \langle \mathbf{g}, \mathbf{e}_i \rangle \} q_{u-}^\mu dy_u \\ &+ \int_0^t \text{diag} \{ \langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle - 1 \} q_{u-}^\mu (dN_u - du). \end{aligned} \quad (3.32)$$

Proof of Theorem 2: Note that the matrices Φ_t , Φ_t^{-1} , Γ_t , and Γ_t^{-1} are all diagonal and, therefore, commute. Setting $\tilde{q}_t^\mu \triangleq \Phi_t \Gamma_t \bar{q}_t^\mu$, we see that

$$\begin{aligned} d\tilde{q}_t^\mu &= \Phi_t \Gamma_t d(\bar{q}_t^\mu) + d(\Phi_t \Gamma_t) \bar{q}_t^\mu + \Phi_t d(\Gamma_t) \bar{q}_t^\mu \\ &= \Phi_t \Gamma_t \Phi_t^{-1} \Gamma_t^{-1} [A + \mu \times \text{diag} \{ c_i(\hat{\xi}_t) \}] \Phi_t \Gamma_t \bar{q}_t^\mu \\ &+ \Phi_t \Gamma_t \text{diag} \{ \langle \mathbf{g}, \mathbf{e}_i \rangle \} \bar{q}_t^\mu dy_t \\ &+ \Phi_t \Gamma_t \text{diag} \{ \langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle - 1 \} \bar{q}_t^\mu (dN_t - dt) \quad (3.33) \\ &= [A + \mu \times \text{diag} \{ c_i(\hat{\xi}_t) \}] \tilde{q}_t^\mu \\ &+ \text{diag} \{ \langle \mathbf{g}, \mathbf{e}_i \rangle \} \tilde{q}_t^\mu dy_t \\ &+ \text{diag} \{ \langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle - 1 \} \tilde{q}_t^\mu (dN_t - dt). \end{aligned}$$

Therefore, \tilde{q}^μ is a solution of (3.14). Since solutions of this equation are unique, the result follows. \square

The importance of Theorem 2 is that it identifies a process \bar{q}^μ which satisfies an ordinary differential equation. From this

process, the process q_t^μ can always be determined. Further, to implement equations such as (3.14) on a digital computer, a discretization of continuous time must be used. Discretising (3.31) provides a robust discretization. Here the term “robust” refers to the sensitivity of \bar{q}_t^μ with respect to variations in the model parameters $\{A, \mathbf{g}, \boldsymbol{\lambda}\}$.

Remark 2: The obvious special cases are obtained when either $\boldsymbol{\lambda} = \mathbf{0}$ or $\mathbf{g} = \mathbf{0}$.

E. The Risk Sensitive Duncan–Mortensen–Zakai Equation

We now write down a robust (to discretization) form of the Duncan–Mortensen–Zakai equation for the risk-sensitive filter of Theorem 2. The interesting point is that the observation process appears as a parameter, and to update the estimate at the next time point only the new observation is used.

From Theorem 2, the transformed process \bar{q}^μ satisfies the linear ordinary differential equation

$$\frac{d\bar{q}_t^\mu}{dt} = \Phi_t^{-1} \Gamma_t^{-1} \left[A + \mu \times \text{diag} \left\{ c_i \left(\hat{\xi}_t^\mu \right) \right\} \right] \Phi_t \Gamma_t \bar{q}_t^\mu. \quad (3.34)$$

To discretise this equation, we consider a regular partition in time

$$0 = t_0 < t_1 < \dots < t_{k-1} < t_k \dots \quad (3.35)$$

where $t_k - t_{k-1}$ is a constant time step, denoted by Δ . We make the following approximation of (3.34) between the sampling instants t_{k-1} and t_k :

$$\begin{aligned} \bar{q}_{t_k}^\mu &= \bar{q}_{t_{k-1}}^\mu + \int_{t_{k-1}}^{t_k} \Phi_u^{-1} \Gamma_u^{-1} \\ &\quad \times \left\{ A + \mu \times \text{diag} \left\{ c_i \left(\hat{\xi}_u^\mu \right) \right\} \right\} \Phi_u \Gamma_u \bar{q}_u^\mu du \\ &\approx \bar{q}_{t_{k-1}}^\mu + \Phi_{t_{k-1}}^{-1} \Gamma_{t_{k-1}}^{-1} \\ &\quad \times \left\{ A + \mu \times \text{diag} \left\{ c_i \left(\hat{\xi}_{t_{k-1}}^\mu \right) \right\} \right\} \Phi_{t_{k-1}} \Gamma_{t_{k-1}} \bar{q}_{t_{k-1}}^\mu \Delta. \end{aligned} \quad (3.36)$$

For the values $\{t_n', t_n'', t_n'''\}$ in the subinterval $(t_{n-1}, t_n]$ we choose $t_n' = t_n'' = t_n''' = t_{n-1}$. This choice leads to an explicit approximation scheme

$$\begin{aligned} \bar{q}_{t_k}^\mu &\approx \left[\mathbf{I} + \Delta \Phi_{t_{k-1}}^{-1} \Gamma_{t_{k-1}}^{-1} \left\{ A + \mu \times \text{diag} \left\{ c_i \left(\hat{\xi}_{t_{k-1}}^\mu \right) \right\} \right\} \right] \\ &\quad \times \Phi_{t_{k-1}} \Gamma_{t_{k-1}} \bar{q}_{t_{k-1}}^\mu. \end{aligned} \quad (3.37)$$

Finally, multiplying both sides of (3.37) by $\Phi_{t_k} \Gamma_{t_k}$ we determine a robust version of the risk-sensitive Duncan–Mortensen–Zakai equation for $q_{t_k}^\mu$

$$\begin{aligned} q_{t_k}^\mu &= \Phi_{t_k} \Phi_{t_{k-1}}^{-1} \Gamma_{t_k} \Gamma_{t_{k-1}}^{-1} \\ &\quad \times \left[\mathbf{I} + \Delta \left\{ A + \mu \times \text{diag} \left\{ c_i \left(\hat{\xi}_{t_{k-1}}^\mu \right) \right\} \right\} \right] q_{t_{k-1}}^\mu. \end{aligned} \quad (3.38)$$

Note that the stochastic processes $\{y_t, 0 \leq t\}$ and $\{N_t, 0 \leq t\}$ now appear as *parameters* in the matrix product $\Phi_{t_k} \Phi_{t_{k-1}}^{-1} \Gamma_{t_k} \Gamma_{t_{k-1}}^{-1}$ of (3.38) rather than as stochastic integrators in the martingale terms of (3.14).

IV. RISK-SENSITIVE SMOOTHERS

For smoothed state estimates, we wish to evaluate the expectation $E^\dagger[X_t | \mathcal{Y}_T]$, where $0 \leq t \leq T$.

By a version of Bayes' rule

$$E[X_t | \mathcal{Y}_T] = \frac{E^\dagger[\Lambda_{0,T} X_t | \mathcal{Y}_t]}{E^\dagger[\Lambda_{0,T} | \mathcal{Y}_t]}. \quad (4.39)$$

The Radon–Nikodym derivative in the right member of (4.39) can be written as a product due to its exponential form, that is, we may write, $\Lambda_{0,T} = \Lambda_{0,t} \Lambda_{t,T}$. This representation proves convenient in establishing smoother dynamics.

Notation: The Radon–Nikodym derivatives $\Lambda_{0,t}$ and $\Lambda_{t,T}$ are defined, respectively, by

$$\begin{aligned} \Lambda_{0,t} &= \exp \left(\int_0^t \langle X_u, \mathbf{g} \rangle du - \frac{1}{2} \int_0^t \langle X_u, \mathbf{g} \rangle^2 du \right) \\ &\quad \times \prod_{0 < u \leq t} \langle X_u, \boldsymbol{\lambda} \rangle^{\Delta N_u} \\ &\quad \times \exp \left(\int_0^t (1 - \langle X_u, \boldsymbol{\lambda} \rangle) du \right) \end{aligned} \quad (4.40)$$

$$\Lambda_{t,T} = \exp \left(\int_t^T \langle X_u, \mathbf{g} \rangle du - \frac{1}{2} \int_t^T \langle X_u, \mathbf{g} \rangle^2 du \right) \quad (4.41)$$

$$\begin{aligned} &\quad \times \prod_{t < u \leq T} \langle X_u, \boldsymbol{\lambda} \rangle^{\Delta N_u} \\ &\quad \times \exp \left(\int_t^T (1 - \langle X_u, \boldsymbol{\lambda} \rangle) du \right). \end{aligned} \quad (4.42)$$

Using the derivatives above and the version of Bayes' rule at (4.39), it is routine to compute robust smoother dynamics. Examples of similar calculations for risk-neutral dynamics are given in [7]. The following theorem is therefore stated without proof.

Theorem 3: Suppose the process X satisfies dynamics given by (2.1) and a vector-valued process Y is observed, as is described by (2.2)–(2.4). For any t , $0 \leq t \leq T$, the risk-sensitive normalized conditional probability of X_t , given the observations \mathcal{Y}_T , is given by

$$\begin{aligned} p_t^\mu &= E[X_t | \mathcal{Y}_T] = \frac{\sum_{i=1}^n \langle \bar{q}_t^\mu, \mathbf{e}_i \rangle \langle \bar{v}_t^\mu, \mathbf{e}_i \rangle \mathbf{e}_i}{\langle \bar{q}_t^\mu, \bar{v}_t^\mu \rangle} \\ &= \frac{\sum_{i=1}^n \langle q_t^\mu, \mathbf{e}_i \rangle \langle v_t^\mu, \mathbf{e}_i \rangle \mathbf{e}_i}{\langle q_t^\mu, v_t^\mu \rangle}. \end{aligned} \quad (4.43)$$

Here \bar{q}_t^μ satisfies the forward linear ordinary differential equation

$$\frac{d\bar{q}_t^\mu}{dt} = \Phi_t^{-1} \Gamma_t^{-1} \left[A + \mu \times \text{diag} \left\{ c_i \left(\hat{\xi}_t^\mu \right) \right\} \right] \Phi_t \Gamma_t \bar{q}_t^\mu \quad (4.44)$$

and \bar{v}_t^μ satisfies the backward linear ordinary differential equation

$$\frac{d\bar{v}_t^\mu}{dt} = \Phi_t^{-1} \Gamma_t^{-1} \left[A^* + \mu \times \text{diag} \left\{ c_i \left(\hat{\xi}_t^\mu \right) \right\} \right] \Phi_t \Gamma_t \bar{v}_t^\mu. \quad (4.45)$$

The corresponding discrete-time recursions for the processes q^μ and v^μ are, respectively,

$$\begin{aligned} q_{t_k}^\mu &= \Phi_{t_k} \Phi_{t_{k-1}}^{-1} \Gamma_{t_k} \Gamma_{t_{k-1}}^{-1} \\ &\quad \times \left[\mathbf{I} + \Delta \left\{ A + \mu \times \text{diag} \left\{ c_i \left(\hat{\xi}_{t_{k-1}}^\mu \right) \right\} \right\} \right] q_{t_{k-1}}^\mu \end{aligned} \quad (4.46)$$

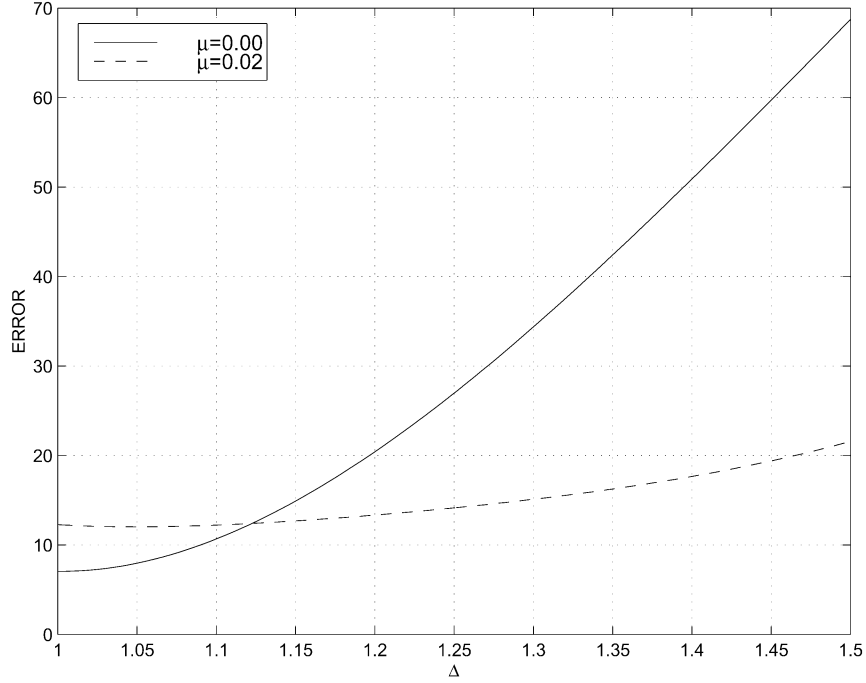


Fig. 1. Estimation error performance comparison for the risk-sensitive and the risk-neutral filters over a family of nominal models. In this study, each of the nominal models is a perturbation of the true model.

and

$$v_{t_{k-1}}^{\mu} = \left[\mathbf{I} + \Delta \left\{ A^* + \mu \times \text{diag} \left\{ c_i \left(\hat{\xi}_{t_{k-1}}^{\mu} \right) \right\} \right\} \right] \times \Phi_{t_k} \Phi_{t_{k-1}}^{-1} \Gamma_{t_k} \Gamma_{t_{k-1}}^{-1} v_{t_k}^{\mu}. \quad (4.47)$$

V. EXAMPLE

The example we consider here is a Markov process observed through a Poisson process where the stochastic system is the combination of (2.1) and (2.4).

A. Simulation Models

The True and Fixed Model: The true model used to simulate the observation process $\{N_t, 0 \leq t\}$ is a Poisson process with a jump stochastic rate function defined by

$$\lambda_t = \langle X_t, \lambda_0 \rangle \quad (5.48)$$

where $\lambda_0 = \{12, 27, 17, 32\}$. For the infinitesimal characteristics of the Markov process X_t we define the rate matrix by

$$A = \begin{bmatrix} -a_1 & \frac{1}{3}a_1 & \frac{1}{3}a_1 & \frac{1}{3}a_1 \\ \frac{1}{3}a_2 & -a_2 & \frac{1}{3}a_2 & \frac{1}{3}a_2 \\ \frac{1}{3}a_3 & \frac{1}{3}a_3 & -a_3 & \frac{1}{3}a_3 \\ \frac{1}{3}a_4 & \frac{1}{3}a_4 & \frac{1}{3}a_4 & -a_4 \end{bmatrix}. \quad (5.49)$$

The values taken by the process λ_t are ascribed expected sojourn times of: 20 s for the states $\{\lambda_0^{(1)}, \lambda_0^{(3)}\}$ and 30 s for states $\{\lambda_0^{(2)}, \lambda_0^{(4)}\}$. These sojourn time statistics correspond to $a_1 = a_3 = 0.05$ and $a_2 = a_4 = 0.0333$.

The Design Models: We consider a family of design models where each model is a continuous perturbation of the true model defined by

$$\lambda_d = \{12, 27/\Delta, 17, \Delta 32\} \quad (5.50)$$

where the scaling parameter varies in the interval $\Delta \in [1, 1.5]$ and $\lambda_d \rightarrow \lambda_0$ as $\Delta \rightarrow 1$.

B. Results

Fig. 1 shows the results of a comparison between the risk-sensitive and the risk-neutral filter over a family of design models. At each from a list of the values of $\Delta \in [1, 1.5]$, the risk-sensitive and the risk-neutral filters were given an observation realization generated by the true model, yet configured with parameters from a perturbed model according to the value of Δ . The error between the true and the estimated rate functions was calculated by

$$\text{Error} = 1/T \int_0^T (\lambda_u - \hat{\lambda}_u)^2 du \quad (5.51)$$

where $T = 3000$ s. Note that this is a time integral of the squared error, not an expectation of the squared error.

Error performance curves such as those in Fig. 1 provide an illustration of the possible benefits of risk-sensitive filtering. The robust filter used to generate the curve in Fig. 1 demonstrates an estimation error which degrades more gracefully than the corresponding risk-neutral filter. The comparison of the curves in Fig. 1 shows that a risk-sensitive filter can offer robustness against model uncertainties.

In Fig. 2, we show the error defined by (5.51) calculated as a function of the parameter μ for a fixed value of $\Delta = 1.5$. The minimum of this curve marks the *best* risk-sensitive parameter for this specific scenario. In Fig. 3, we show a family of error

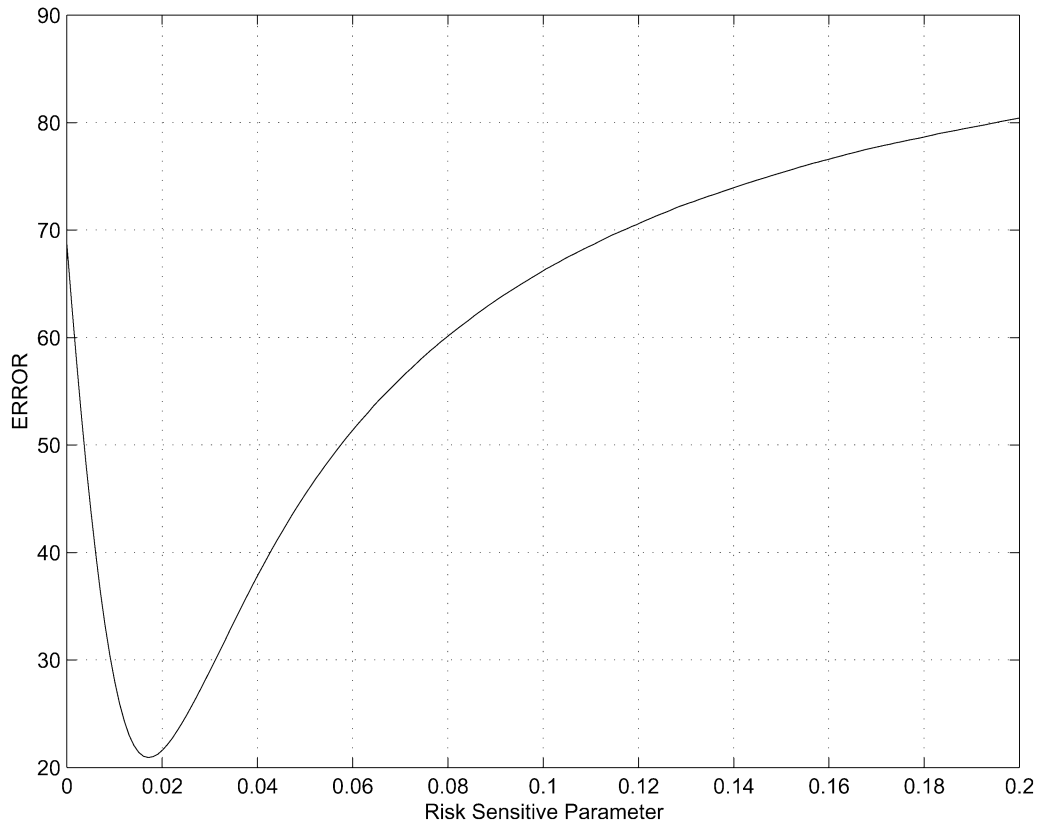


Fig. 2. Estimation error for the risk-sensitive filter as a function of the risk-sensitive parameter μ in the scenario $\Delta = 1.5$.

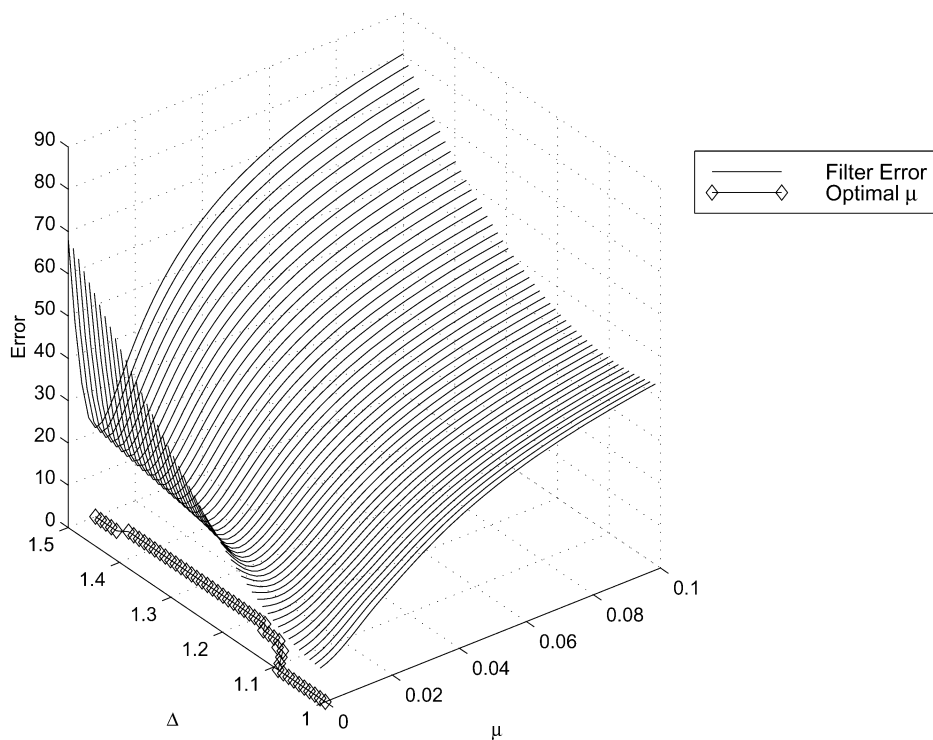


Fig. 3. Risk-sensitive filter error curves as a function of both the parameter μ and the number Δ . Here the number Δ quantifies the degree of difference between the true and design probability models. $\Delta = 1$ corresponds to a scenario where the true and design probability models coincide.

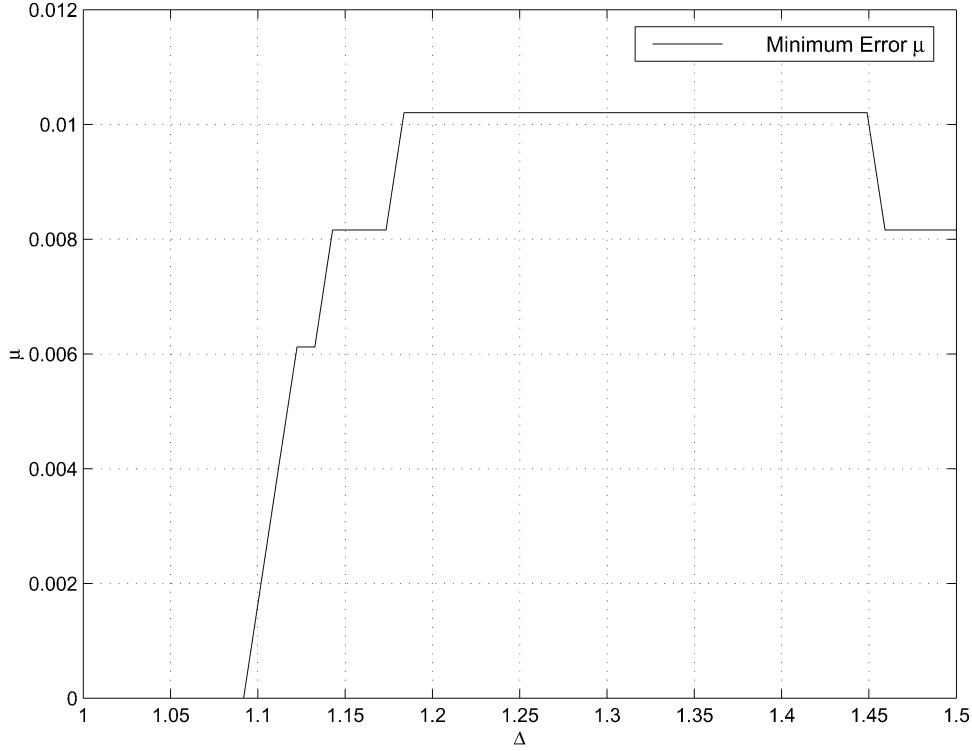


Fig. 4. The optimal value μ as a function of the number Δ . This curve shows the value of μ that is required to achieve the best performance for a given scenario depending upon Δ .

curves such as the single curve shown in Fig. 2. Each curve in Fig. 3 represents a particular scenario, where the degree of difference between the true and design probability models is determined by the number Δ . In simulation studies to generate families of curves such as those in Fig. 3, it was observed that the local curvature about the minima changed with Δ , such that choosing a “good” value for μ became increasingly difficult as the true and design models became more distinct. It should be further noted, that although the curves in Fig. 3 suggest the existence and uniqueness of optimal values for μ , these curves were determined via complete knowledge of the true Markov state process. Perceivably, one could estimate the location of the optimal μ if an equation for the state estimator error variance were available. In Fig. 4, the optimal μ (risk-sensitive parameter corresponding to the minima of the curves in Fig. 3) is plotted for values of Δ shown in Fig. 3.

VI. CONCLUSION

In this paper, we presented risk-sensitive filters and smoothers for an observation process whose components represent the observation of Markov process through a Brownian motion and the observation of a Markov process through a Poisson process. Using a gauge transformation, a robust (to discretization) version of the Duncan–Mortensen–Zakai equation corresponding to our risk-sensitive filter was obtained. A simulation study for an example of counting process observations was provided.

APPENDIX

A. Proof of Theorem 1

The proof of this Theorem has two parts. In the first part, we use the Ito product rule to determine the semimartingale $\Lambda_t X_t Z_t$. In the second part, we condition this semimartingale under the reference probability P^\dagger .

The semimartingale Λ_t satisfies the stochastic integral equation (see [8])

$$\Lambda_t = 1 + \int_0^t \Lambda_u \langle X_u, \mathbf{g} \rangle dy_u + \int_0^t \Lambda_{u-} (\langle X_u, \boldsymbol{\lambda} \rangle - 1) (dN_u - du). \quad (\text{A52})$$

First we note that

$$d\Lambda_t = \Lambda_t \langle X_t, \mathbf{g} \rangle dy_t + \Lambda_{t-} (\langle X_t, \boldsymbol{\lambda} \rangle - 1) (dN_t - dt), \quad (\text{A53})$$

$$dZ_t = \mu c \left(\xi_t, \hat{\xi}_t^\mu \right) Z_t dt, \quad (\text{A54})$$

$$dX_t = AX_t dt + dM_t, \quad (\text{A55})$$

then

$$\begin{aligned} \Lambda_t X_t Z_t &= X_0 + \int_0^t \Lambda_u AX_u Z_u du + \int_0^t \Lambda_u Z_u dM_u \\ &+ \int_0^t \Lambda_u X_u Z_u \langle X_u, \mathbf{g} \rangle dy_u \\ &+ \int_0^t \Lambda_{u-} X_u Z_u (\langle X_u, \boldsymbol{\lambda} \rangle - 1) (dN_u - du) \\ &+ \int_0^t \mu \Lambda_u X_u Z_u c \left(\xi_u, \hat{\xi}_u^\mu \right) du. \end{aligned} \quad (\text{A56})$$

By using Fubini's theorem ([8]) on (A56), noting that under the reference measure P^\dagger , both y_t and N_t have independent increments, and

$$E^\dagger \left[\int_0^t Z_u \Lambda_u \langle dM_u, X_u \rangle \mid \mathcal{Y}_t \right] = 0, \quad (\text{A57})$$

then the result follows. \square

B. Proof of Corollary 1

Proof: Note that $E^\dagger[\Lambda_t \mid \mathcal{Y}_t] = \langle q_t^0, \mathbf{1} \rangle = \widehat{\Lambda}_t$. To prove Corollary 1, we follow the method in [9] in which one uses the special semimartingale $\widehat{\Lambda}_t^{-1}$ and the Ito rule to write down the product $q_t^\mu \widehat{\Lambda}_t^{-1}$. This, by definition is the process Π_t^μ . The semimartingale decomposition for the process $\widehat{\Lambda}_t$ reads [8]

$$\begin{aligned} \widehat{\Lambda}_t &= 1 + \int_0^t \widehat{\Lambda}_{u-} \langle \Pi_u^0, \mathbf{g} \rangle dy_u \\ &\quad + \int_0^t \widehat{\Lambda}_{u-} (\langle \Pi_{u-}^0, \boldsymbol{\lambda} \rangle - 1) (dN_u - du). \end{aligned} \quad (\text{A58})$$

Note that the terms $\widehat{\Lambda}_t$, $\langle \Pi_t^0, \mathbf{g} \rangle$, and $\langle \Pi_t^0, \boldsymbol{\lambda} \rangle$ in (A58) are projections under *different* probability measures: where

$$\widehat{\Lambda}_t \triangleq E^\dagger[\Lambda_t \mid \mathcal{Y}_t], \quad (\text{A59})$$

$$\Pi_t^0 \triangleq E[X_t \mid \mathcal{Y}_t]. \quad (\text{A60})$$

Now

$$\begin{aligned} \widehat{\Lambda}_t^{-1} &= 1 - \int_0^t \widehat{\Lambda}_{u-}^{-1} \langle \Pi_u^0, \mathbf{g} \rangle^2 du - \int_0^t \widehat{\Lambda}_{u-}^{-1} \langle \Pi_u^0, \mathbf{g} \rangle dy_u \\ &\quad - \int_0^t \widehat{\Lambda}_{u-}^{-1} \left\{ \frac{\langle \Pi_u^0, \boldsymbol{\lambda} \rangle - 1}{\langle \Pi_u^0, \boldsymbol{\lambda} \rangle} \right\} (dN_u - \langle \Pi_u^0, \boldsymbol{\lambda} \rangle du). \end{aligned} \quad (\text{A61})$$

Therefore,

$$\begin{aligned} d(q_t^\mu \widehat{\Lambda}_t^{-1}) &= \widehat{\Lambda}_t^{-1} dq_t^\mu + q_t^\mu d\widehat{\Lambda}_t^{-1} + d \left[q_t^\mu \widehat{\Lambda}_t^{-1} \right] \\ &= \widehat{\Lambda}_t^{-1} \left\{ \left[A + \mu \times \text{diag} \left\{ c_i \left(\hat{\xi}_t^\mu \right) \right\} \right] q_t^\mu dt \right. \\ &\quad \left. + \text{diag} \{ \langle \mathbf{g}, \mathbf{e}_i \rangle \} q_t^\mu dy_t \right. \\ &\quad \left. \times \text{diag} \{ \langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle - 1 \} q_{t-}^\mu dQ_t \right\} \\ &\quad + q_t^\mu \left\{ -\widehat{\Lambda}_t^{-1} \langle \Pi_t^0, \mathbf{g} \rangle^2 dt - \widehat{\Lambda}_t^{-1} \langle \Pi_t^0, \mathbf{g} \rangle dy_t \right. \\ &\quad \left. - \widehat{\Lambda}_t^{-1} \left\{ \frac{\langle \Pi_{t-}^0, \boldsymbol{\lambda} \rangle - 1}{\langle \Pi_{t-}^0, \boldsymbol{\lambda} \rangle} \right\} \right. \\ &\quad \left. \times (dN_t - \langle \Pi_{t-}^0, \boldsymbol{\lambda} \rangle dt) \right\} \end{aligned}$$

$$\begin{aligned} &- \text{diag} \{ \langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle - 1 \} q_{t-}^\mu \widehat{\Lambda}_t^{-1} \\ &\times \left\{ \frac{\langle \Pi_{t-}^0, \boldsymbol{\lambda} \rangle - 1}{\langle \Pi_{t-}^0, \boldsymbol{\lambda} \rangle} \right\} dN_t \\ &- \widehat{\Lambda}_t^{-1} \langle \Pi_t^0, \mathbf{g} \rangle q_{t-}^\mu \text{diag} \{ \langle \mathbf{g}, \mathbf{e}_i \rangle \} dt. \end{aligned} \quad (\text{A62})$$

Writing $\widehat{\Lambda}_t^{-1} q_t^\mu = \Pi_t^\mu$ we see

$$\begin{aligned} d \left(q_t^\mu \widehat{\Lambda}_t^{-1} \right) &= \left[A + \mu \times \text{diag} \left\{ c_i \left(\hat{\xi}_t^\mu \right) \right\} \right] \Pi_t^\mu dt \\ &\quad + \text{diag} \{ \langle \mathbf{g}, \mathbf{e}_i \rangle \} \Pi_t^\mu dy_t \\ &\quad - \Pi_t^\mu \langle \Pi_t^0, \mathbf{g} \rangle dy_t \\ &\quad + \Pi_t^\mu \langle \Pi_t^0, \mathbf{g} \rangle^2 dt \\ &\quad - \text{diag} \{ \langle \mathbf{g}, \mathbf{e}_i \rangle \} \Pi_t^\mu \langle \Pi_t^0, \mathbf{g} \rangle dt \\ &\quad + \text{diag} \{ \langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle - 1 \} \Pi_{t-}^\mu (dN_t - dt) \\ &\quad - \Pi_{t-}^\mu \left\{ \frac{\langle \Pi_{t-}^0, \boldsymbol{\lambda} \rangle - 1}{\langle \Pi_{t-}^0, \boldsymbol{\lambda} \rangle} \right\} (dN_t - \langle \Pi_t^0, \boldsymbol{\lambda} \rangle dt) \\ &\quad - \text{diag} \{ \langle \boldsymbol{\lambda}, \mathbf{e}_i \rangle - 1 \} \\ &\quad \times \Pi_{t-}^\mu \left\{ \frac{\langle \Pi_{t-}^0, \boldsymbol{\lambda} \rangle - 1}{\langle \Pi_{t-}^0, \boldsymbol{\lambda} \rangle} \right\} dN_t. \end{aligned} \quad (\text{A63})$$

With simple rearrangement, Corollary 1 follows.

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