Curci-Ferrari mass and the Neuberger problem

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Abstract

We study the massive Curci-Ferrari model as a starting point for defining BRST quantisation for Yang-Mills theory on the lattice. In particular, we elucidate this proposal in light of topological approaches to gauge-fixing and study the case of a simple one-link Abelian model.

Key words: BRST, gauge-fixing, lattice, Gribov copies
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BRST symmetry has proven an invaluable tool in the perturbative quantisation of gauge theories [1] so that its elevation to the non-perturbative level is clearly desirable. For one, BRST methods are crucial in the formulation of Schwinger-Dyson equations (SDEs) in covariant gauges, which seem genuinely non-perturbative, have been subject to study using various truncations at this level[2], and whose results are now subject to comparisons with corresponding computations in Landau gauge from lattice gauge theory (see, for example, [3]). However, it has been demonstrated that a standard formulation of BRST symmetry invoked for lattice fields rigorously forces the partition function and path integrals of BRST invariant operators with a BRST invariant measure to vanish identically [4]. Rather than defining the configuration space in terms of some subset with no Gribov copies [5,6], one sums them all with alternating sign of the Faddeev-Popov determinant (only for small fields about $A_\mu = 0$ is this positive and thus the Jacobian of a change of variables) and a complete cancellation takes place giving the nonsensical result

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0/0 for lattice expectation values. This corresponds to attempting to resolve unity in the Faddeev-Popov trick via the partition function of a topological quantum field theory (TQFT) whose fields are the $SU(n)$ group elements in the background of an external gauge or link field [7]. Such integrals of BRST invariant observables are topological invariants, the Euler character for $SU(n)$ for the partition function, which all vanish giving the zeroes of the Neuberger problem. Equivariant gauge-fixing [8,9] evades this no-go theorem through a sequential gauge-fixing via coset space decomposition of $SU(n)$, as in maximal Abelian gauge, so that the submanifolds of $SU(n)$ implicit in this decomposition have non-zero Euler character. This formulation is however distant from covariant gauge Schwinger-Dyson equations. Significantly though, there are quartic ghost couplings signalling that the damping term in the action for scalar Nakanishi-Lautrup auxiliary fields B is not BRST exact, $B^2 \neq s(\text{something})$. This also signals the break down of the Neuberger argument.

Quartic ghost couplings also arise in generalisations of the BRST and anti-BRST symmetry of Landau gauge through the so-called Curci-Ferrari (CF) "gauges" [10,11,12,13]. They also allow for a massive vector field while retaining BRST- (though not gauge-)invariance. But nilpotency of the (anti-)BRST algebra and thus unitarity are lost to be recovered, along with the original YM theory, in the massless limit. Nonetheless, this limit gives a theory which is local, covariant, BRST invariant, perturbatively renormalisable and close in spirit to Landau gauge. We will elucidate these details in light of the topological approach to gauge-fixing and the Neuberger problem in the following, showing how these "bugs" possibly become features which may enable a non-perturbative definition of BRST.

We assume antihermitian $SU(n)$ generators T^a so that ghost components C^a are hermitian in order that $C(x) = C^a(x)T^a = -C^\dagger$. On the lattice ghosts live on lattice sites C_i. In the continuum we deal with $SU(n)$ gauge fields $A_\mu(x) = A^a_\mu(x)T^a$ while on the lattice we denote the link fields as U_{ij} for the link from lattice site i to j, $U_{ij} = P\exp(\int_{x_i}^{x_j} dz\cdot A)$. The covariant derivative in the adjoint representation is $D_\mu\cdot = \partial_\mu \cdot + [A_\mu, \cdot]$. We also require a Nakanishi-Lautrup auxiliary $SU(n)$-algebra valued hermitian field $B(x)$ which lives on sites for the lattice theory, B_i. Note that the antihermiticity of the generators mean that the components of B are imaginary. Commutators and anti-commutators of variables will be separately indicated by $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$.

For the continuum theory we have the BRST and anti-BRST algebras

$$sA_\mu = D_\mu C, \;\; \bar{s}A_\mu = D_\mu \bar{C}, \tag{1}$$

$$sC = C^2 = \frac{1}{2}\{C, C\}, \;\; \bar{s}\bar{C} = \bar{C}^2 = \frac{1}{2}\{\bar{C}, \bar{C}\} \tag{2}$$

$$s\bar{C} = B + \frac{1}{2}\{\bar{C}, C\},\ \bar{s}C = -B + \frac{1}{2}\{C, \bar{C}\}. \tag{3}$$

where for the lattice Eqs.(1) are replaced by

$$sU_{ij} = C_i U_{ij} - U_{ij} C_j, \quad \bar{s}U_{ij} = \bar{C}_i U_{ij} - U_{ij}\bar{C}_j. \tag{4}$$

With these, the gauge-fixing is symmetric under $C \to \bar{C},\ \bar{C} \to -C$. We have then $s\bar{C} + \bar{s}C - \{\bar{C}, C\} = 0$ which can be geometrically interpreted as the vanishing of a curvature in the extended space and so both ghosts and anti-ghosts are Maurer-Cartan one-forms. We still have some freedom in specifying the variations of the auxiliary field B. We choose

$$\begin{aligned} sB &= m^2 C - \frac{1}{2}[B, C] + \frac{1}{8}[\bar{C}, \{C, C\}] \\ \bar{s}B &= m^2 \bar{C} - \frac{1}{2}[B, \bar{C}] - \frac{1}{8}[C, \{\bar{C}, \bar{C}\}]. \end{aligned} \tag{5}$$

An $Sp(2)$ group with generators $\sigma^+, \sigma^0, \sigma^-$ specified by $\sigma^i A_\mu = \sigma^i B = 0$ and

$$\sigma^- C = \bar{C}, \sigma^0 C = C, \sigma^+ C = 0,\ \sigma^- \bar{C} = 0, \sigma^0 \bar{C} = -\bar{C}, \sigma^+ \bar{C} = C \tag{6}$$

enables us to replace nilpotency by the relations:

$$s^2 = m^2 \sigma^+, \bar{s}^2 = -m^2 \sigma^-, s\bar{s} + \bar{s}s = -m^2 \sigma^0. \tag{7}$$

The gauge-fixing of the Yang-Mills action can be achieved by the addition of a (anti-)BRST-invariant but not BRST-exact action, namely

$$S_{GF} = (s\bar{s} - m^2)W, \tag{8}$$

where for the continuum (respectively lattice) theory the most general choice for W is

$$W_{\rm cont} = \frac{1}{2}{\rm Tr}\int d^4x[(A_\mu)^2 - \xi\bar{C}C],\ W_{\rm lat} = \frac{1}{2}\sum_{ij}{\rm Tr}[{\rm Re}U_{ij} - \xi\bar{C}_i C_i] \tag{9}$$

the first terms of which one recognises as the continuum (lattice) functionals whose stationary points with respect to gauge transformations give the Landau gauge.

The detailed form of the continuum Lagrangian after implementing the algebra has been given elsewhere [10,12]. We merely highlight specific terms.

Firstly, after integration by parts, $\bar{s}\int d^4xA^2 = -2\int d^4x\bar{C}\partial_\mu A_\mu$. Acting with s then gives the ghost kinetic term and multiplier field term $B\partial_\mu A_\mu$, which are standard in covariant gauges. Secondly, the most complicated structures emerge from $s\bar{s}(\bar{C}C)$. The damping term for the multiplier field B^2 emerges from here, also as in standard covariant gauges. But one ghost mass term, three-point couplings as well as quartic couplings $(\bar{C}C)^2$ and permutations thereof appear. Finally the $m^2(A^2 - \xi\bar{C}C)$ generates both a gluon and ghost mass term.

Let us consider now some typical expectation value of a BRST invariant observable of the link fields $O[U]$ in the lattice theory corresponding to the massive Curci-Ferrari gauge:

$$\langle O[U]\rangle_{mCF} = \frac{\int DUD\phi e^{-S_{YM}[U]-S_{GF}[U,\phi]}O[U]}{\int DUD\phi e^{-S_{YM}[U]-S_{GF}[U,\phi]}} \tag{10}$$

where ϕ represents the auxiliary fields, $C, \bar{C}$ and B. We shall now examine the precise relationship between this expectation value and that for lattice YM theory which, in terms of link variables, is well-defined even without gauge-fixing. We can factor into numerator and denominator the finite (on the lattice) integration over the gauge group, $V_G = \int Dg < \infty$ and use the standard trick of exploiting the invariance of the measure, observable and YM action under gauge transformations to rewrite the expectation value as

$$\langle O[U]\rangle_{mCF} = \frac{\int DUe^{-S_{YM}[U]}O[U]\bar{Z}[U]}{\int DUe^{-S_{YM}[U]}\bar{Z}[U]} \tag{11}$$

where

$$\bar{Z}[U] = \int DgD\phi e^{-S_{GF}[U^g,\phi]} \tag{12}$$

represents the partition function of a field theory in the group g and auxiliary variables ϕ in the background of the link field U. *But this is not a topological quantum field theory* because the action of this theory is not based on a nilpotent algebra [14]. The consequence of this is that, unlike for TQFTs, this partition function depends on the background field, $\frac{\delta\bar{Z}[U]}{\delta U} \neq 0$. The proof can be sketched as follows. Since the measure of $\bar{Z}[U]$ is independent of the link field the variation with respect to U acts directly onto the exponential of the action of the theory, $S_{GF}[U^g,\phi]$, bringing the action into the measure. The variation $\delta/\delta U$ commutes with the operation $s\bar{s} - m^2$ so that we effectively have

$$\int DgD\phi(s\bar{s} - m^2)\frac{\delta W}{\delta U}e^{-S_{GF}[U^g,\phi]}. \tag{13}$$

But the integral of a BRST exact quantity with respect to an invariant measure still vanishes, despite the lack of nilpotency, thus

$$\frac{\delta\bar{Z}[U]}{\delta U} = m^2 \int DgD\phi \frac{\delta W}{\delta U} e^{-S_{GF}[U^g,\phi]} \tag{14}$$

which is not evidently constrained to vanish by any symmetry argument. Thus the partition function depends on the background link. Thus $\bar{Z}[U]$ cannot be factored out and cancelled except in the massless limit, so that YM expectation values can only be defined via the limit

$$\langle O[U]\rangle_{YM} = \lim_{m\to 0}\langle O[U]\rangle_{mCF}. \tag{15}$$

Unfortunately, the usual tricks of TQFT cannot help us in explicitly evaluating $\bar{Z}[U]$ here: because of explicit dependence on U neither a semiclassical limit can be taken (as in TQFT) nor can the trivial link $U = 1$ be chosen (which for the lattice theory would result in some spin model [8]). We argue however that for *generic* $m \neq 0$ the partition function $\bar{Z}[U]$ will be non-vanishing: the functional being introduced into the measure *a la* Faddeev-Popov trick is *orbit-dependent* and this is precisely what we require in order to lift the degeneracy between Gribov regions to avoid the Neuberger problem. We shall illustrate this below for the case of a simple one-link model.

At any rate, we can give a final formula for the expectation value of a gauge-invariant observable in Yang-Mills theory in terms of the present construction:

$$\langle O[U]\rangle_{YM} = \lim_{m\to 0}\frac{\int DUe^{-S_{YM}[U]}O[U]\bar{Z}[U]}{\int DUe^{-S_{YM}[U]}\bar{Z}[U]} \tag{16}$$

with $\bar{Z}[U]$ defined by Eq.(12). We can use the language of soft-meson theorems where the pion is a pseudo-Goldstone boson for massive quarks and thus describe $\bar{Z}[U]$ as the partition function of a *pseudo*-topological quantum field theory PTQFT.

We can now elucidate how the Neuberger problem is avoided. Neuberger [4] considers the integral

$$I_O(t) = \int D\phi e^{-S_0 - tsF} O[U,\phi]. \tag{17}$$

The measure $D\phi$ is BRST invariant as is the action S_0, which includes the Yang-Mills action. Expectation values in the theory are obtained for $t = 1$, namely $\langle O\rangle = I_O(1)/I_1(1)$. However S_0 must also contain damping terms for the scalar B field integrations in $D\phi$ since even on the lattice these field directions (unlike the link field U) are not compact. The damping term must be itself BRST invariant (upon an appropriate shift of the B-field – actually in the standard case it is even BRST exact, $B^2 = s(\text{something})$ but this is not so relevant here). Variation of $I_O(t)$ with respect to t brings sF into the measure, the integrals of which vanish. Thus $dI_O/dt = 0$ and $I_O(t)$ is t-independent. But for $t = 0$ one has an integrand containing no ghost fields which vanishes by the rules of Grassmann integration: $\int DC = 0$. Thus $I_O(0) = 0 = I_O(1)$ and all expectation values are of the form $0/0$. *This assumes I is well-defined at each step, which is only the case if S_0 contains the damping term for B.* For the massive-Curci-Ferrari case the structure of the action is different on two grounds: B^2 is not BRST exact, $B^2 \neq s(\text{something})$, but more importantly shifting B to $b = B + \frac{1}{2}\{\bar{C}, C\}$ gives

$$s\text{Tr } b^2 = 2m^2\text{Tr}(Cb), \tag{18}$$

so that the damping term cannot be placed in S_0 but must be placed in the term multiplied by t. There are two ways to do this, but keeping as close as possible to Neuberger's original argument we can reassign $s\bar{s}W \rightarrow ts\bar{s}W$ such that under derivation with respect to t a BRST exact term comes down into the measure. However, either way we cannot consider the $t = 0$ limit as the functional integral then becomes undamped. This makes the Neuberger limit $t \rightarrow 0$ fail, and so the usual proof fails.

We now explicitly study this proposal in the context of the simple model introduced by Testa [15]. We consider an Abelian model with only two lattice sites, x_1, x_2 and thus only one degree of freedom, a link variable U which is parametrised through its phase $U = e^{iaA}$. A is compact, $A \in [-\frac{\pi}{a}, \frac{\pi}{a})$. A gauge transformation corresponds to shifting A by a difference $(\omega(x_2) - \omega(x_1))/a$ which is a fixed quantity for any function $\omega(x)$. There are no plaquettes so the action is zero. The model essentially only contains topological information. The BRST and anti-BRST algebras for this simple gauge field theory can be written

$$\begin{aligned}
sA &= C, \ \ \bar{s}A = \bar{C} \\
sC &= 0, \ \ \bar{s}\bar{C} = 0 \\
s\bar{C} &= i(B + \bar{C}C), \ \ \bar{s}C = i(-B + \bar{C}C) \\
sB &= -i(m^2 + B)C, \ \ \bar{s}B = -i(m^2 - B)\bar{C}.
\end{aligned} \tag{19}$$

Note that B is now a real field, ghosts are Maurer-Cartan one-forms and

$$s^2\bar{C} = m^2C, \ \ \bar{s}^2C = -m^2\bar{C}, \tag{20}$$

so nilpotency is lost. Using that $sV[A] = V'C$ and $\bar{s}V[A] = \bar{C}V'$ the gauge-fixing action for the massive Curci-Ferrari model here gives

$$
\begin{aligned}
S_{gf} &= (s\bar{s} - m^2)\left(V[A] - \xi\bar{C}C\right) \\
&= \bar{C}[-V'' + iV' + 2\xi B + 2m^2\xi]C + \xi B^2 + iBV' - m^2V
\end{aligned} \quad (21)
$$

where $V[A]$ is constrained only by the requirement of periodicity under $A \to A + 2\pi/a$. Now this action appears Gaussian in all fields because quartic terms $\bar{C}^2C^2$ all vanish since there is only one species of Grassmann field. Integration out of either ghosts or scalar field B will upset this. In this case we can shift B via $b = B + \bar{C}C$ and avoid this. The action then becomes

$$
\bar{C}[-V'' + 2m^2\xi]C + ibV' + \xi b^2 - m^2V \quad (22)
$$

and integration out of b will give the Gaussian $\sqrt{\pi}\exp(-\frac{1}{4\xi}(V'[A])^2)/\sqrt{\xi}$. The procedure is dependent on the gauge parameter ξ for the same reasons as $\bar{Z}$ is U-dependent, but we will consider the case closest to the Landau gauge, $\xi \to 0$ for which we obtain then the delta function $\delta[V']$ in the measure. Integrating out ghost fields gives for the partition function of the original theory:

$$
Z = \int_{-\pi/a}^{\pi/a} dA(-V''[A])\delta[V'[A]]e^{m^2V[A]} \quad (23)
$$

which should be compared with the partition function for the unfixed theory: $\int_{-\pi/a}^{\pi/a} dA = 2\pi/a$.

We see that only stationary points of $V[A]$ contribute. They are weighted by the second derivative which would otherwise correspond to the Faddeev-Popov determinant; both signs of V'' can appear. We crucially see the additional orbit-dependent weighting exponential in the mass consistent with our observations above. If there are many such stationary points - Gribov copies - all of them will be summed over. But since $V[A]$ is not a gauge-invariant functional of A, they will generally come with different weight unless they represent degenerate stationary points of $V[A]$. For $m = 0$ we recover the Neuberger pathology: critical points cancel according to the sign of the second derivative. The weight factor $e^{m^2V[A]}$ breaks this degeneracy so that the partition function will not vanish and the 0/0 problem disappears.

Let us see this more explicitly. For simplicity we work in units of lattice spacing now ($a = 1$) and choose $V[A] = \frac{1}{2}\sin^2 A$ so that $V'[A] = \frac{1}{2}\sin 2A$ and $V''[A] = \cos 2A$. The "gauge-fixed" configurations are thus

$$
A_{\text{fixed}} = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2} \quad (24)
$$

for which V'' has values $1, -1, 1, -1$ respectively while the original V has values $0, \frac{1}{2}, 0, \frac{1}{2}$. The partition function is trivially evaluated to be

$$Z = 2(e^{\frac{m^2}{2}} - 1) \approx m^2. \qquad (25)$$

Note that the constraint that observables O be BRST invariant in this case means that $dO[A]/dA = 0$, thus BRST invariant observables are just constants, $O = c$. All integrals of O are just multiples of the partition function, cZ. Thus the expectation value $\langle O[A] \rangle = c$ in the unfixed theory. Setting $m = 0$ before taking the ratio reintroduces the pathological $0/0$ result for the expectation value. Conversely, keeping m small but finite while taking the ratio for the expectation value allows m^2 factors to safely cancel in numerator and denominator, giving the correct result for the expectation value, namely the constant c.

Whether such an elegant cancellation in expectation values of observables can take place in general, or subtleties of the $m \to 0$ limit need to be taken into consideration is an open question. For example, the vanishing of the partition function at $m = 0$ can also be understood as the vanishing of the Witten index of the underlying TQFT with its supersymmetry. This leaves open the possibility of spontaneous breaking of BRST symmetry which would jeopardise the BRST cohomology construction of a physical state space. This aside, there is the question whether renormalisation effects can hinder the program. This is related to the other limits which must follow a lattice calculation: the continuum ($a \to 0$) and thermodynamic ($L \to \infty$) limits. Note that for a finite lattice, the lattice gauge group is a simple product of $SU(n)$ gauge groups per lattice site. Thus the original Neuberger zero corresponds to $0^{\text{no. of sites}}$ which goes over to a single zero in the $a \to 0$ limit, related to the remaining global gauge symmetry of the torus at finite volume [8]. In order to avoid the Neuberger problem reappearing in our proposal it is essential that the Curci-Ferrari mass remain non-zero for finite L but vanishing a, particularly in light of renormalisation of the mass. For sufficiently large lattice volume, there will be no L dependence in the appropriate (and only UV sensitive) renormalisation constants for m; for example [16,17] have computed these in continuum perturbation theory. Thus renormalisation will not introduce L-dependence into m. This suggests that it is safe to take $m \to 0$ either as fast as $L \to \infty$ or independently. But we stress that these considerations are only heuristic.

We reiterate that unlike approaches seeking to isolate the Gribov or fundamental modular region in the space of gauge fields, this approach takes all Gribov regions into account. There is the proposal that in the *infinite volume limit* configurations on the common boundary of Gribov and fundamental modular regions will dominate the ensemble averages of gauge invariant observables [18]. However such "dominant" fields will always be some subset of those configurations contributing at finite (L, a) thus representing no contradiction. This is also explicitly evident in the simple model.

Open work includes careful examination of the $m \to 0$ limit for the full theory.

We mention here that the violation of nilpotency of the BRST algebra results in negative norm states appearing in the physical Hilbert space [19,16] defined according to the Kugo-Ojima criterion [20]. This is one way of manifesting the loss of unitarity. These states do not belong to Kugo-Ojima quartets. In [19] one sees explicitly that in the massless limit they become zero norm ("daughter") states. Thus from the point of view of the Hilbert space, the massless limit is smooth, as it is also for perturbative Green's functions. However, whether the same can be said for non-perturbative Green's functions is an open challenging question. There also remain technical challenges to implementing quartic ghost couplings in the lattice framework. The introduction of additional auxiliary bosonic fields, as in Nambu–Jona-Lasinio models, may be a way forward in this problem.

To conclude, we have shown that the massive Curci-Ferrari model overcomes the Neuberger problem for elevating BRST symmetry to the non-perturbative level on a finite lattice. Beyond this first step, the verification that the massless, continuum, and thermodynamic limit of this procedure is the physically relevant theory faces a number of difficult challenges still.

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