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Scaling of fat-link irrelevant-clover fermions

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Hadron masses are calculated in quenched lattice QCD on a variety of lattices in order to probe the scaling behavior of the Fat-Link Irrelevant Clover (FLIC) fermion action, a fat-link clover fermion action in which the purely irrelevant operators of the fermion action are constructed using APE-smeared links. The scaling analysis indicates FLIC fermions provide a new form of nonperturbative $\mathcal{O}(a)$ improvement where near-continuum results are obtained at finite lattice spacing.

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Discretization of the continuum action of QCD for use on a space-time lattice grid with finite lattice spacing a introduces errors of order a^n. For Wilson fermions $n = 1$ due to the introduction of an irrelevant energy dimension-five lattice-Laplacian operator designed to remove fermion doublers from the naive lattice theory by giving the doublers a mass proportional to a^{-1}. While Wilson fermions are computationally inexpensive, the approach to the continuum limit is slow. Observables are spoiled by large $\mathcal{O}(a)$ discretization errors and quantitative extrapolations to the continuum limit must be performed from simulations performed at small lattice spacings typically less than 0.1 fm. This necessitates the use of very large four-dimensional lattices in order to provide reasonable physical simulation volumes exceeding 2 fm on a side.

Since the cost of simulations increases as a^{-n} with n exceeding 5, considerable savings have been achieved by designing lattice fermion actions free of $\mathcal{O}(a)$ discretization errors. A systematic approach [1] to achieving $\mathcal{O}(a)$ improvement of the lattice fermion action in general [2] is to consider all possible gauge invariant, local dimension-five operators, respecting the symmetries of QCD:

$$\begin{aligned}
\mathcal{O}_1 &= -\frac{igaC_{\rm SW}r}{4}\bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi,\\
\mathcal{O}_2 &= c_2 a\{\bar{\psi}D_\mu D_\mu\psi + \bar{\psi}\overleftarrow{D}_\mu\overleftarrow{D}_\mu\psi\},\\
\mathcal{O}_3 &= \frac{b_g a m_q}{2}\mathrm{tr}\{F_{\mu\nu}F_{\mu\nu}\},\\
\mathcal{O}_4 &= c_4 m_q\{\bar{\psi}\gamma_\mu D_\mu\psi - \bar{\psi}\overleftarrow{D}_\mu\gamma_\mu\psi\},\\
\mathcal{O}_5 &= -b_m a m_q^2\bar{\psi}\psi.
\end{aligned} \tag{1}$$

Operator $\mathcal{O}_1$ is a new local operator in the lattice fermion action and must be included. On the other hand, $\mathcal{O}_3$ and $\mathcal{O}_5$ of Eq. (1) act to simply renormalize the coefficients of existing terms in the lattice action, removing $\mathcal{O}(am_q)$ terms from the relation between bare and renormalized quantities [3].

The key observation to efficient $\mathcal{O}(a)$ improvement is that the $\mathcal{O}(a)$ improvement afforded by two-link terms of the fermion action [4] may be incorporated to $\mathcal{O}(a)$ into the standard Wilson fermion action complemented by $\mathcal{O}_1$ through the following transformation of the fermion fields

$$\begin{aligned}
\psi \to \psi' &= (1 + b_q r a m_q)(1 - c_q r a \not{D})\psi,\\
\bar{\psi} \to \bar{\psi}' &= (1 + b_q r a m_q)\bar{\psi}(1 + c_q r a \overleftarrow{\not{D}}),
\end{aligned} \tag{2}$$

where ψ' represents the physical fermion field recovered in the continuum limit, while ψ is the lattice fermion field used in the numerical simulations. At tree level, $b_q = c_q = 1/4$ correctly incorporates the $\mathcal{O}(a)$ corrections of $\mathcal{O}_2$ and $\mathcal{O}_4$ into the fermion action.

In summary, $\mathcal{O}_1$, the "clover" term, is the only dimension-five operator explicitly required to complement the Wilson action to obtain $\mathcal{O}(a)$ improvement. This particular action is known as the Sheikholeslami-Wohlert fermion [2] action

$$S_{\rm SW} = S_{\rm W} - \frac{igaC_{\rm SW}r}{4}\bar{\psi}(x)\sigma_{\mu\nu}F_{\mu\nu}\psi(x), \tag{3}$$

where $S_{\rm W}$ is the standard Wilson action [5], and $C_{\rm SW}$ is the clover coefficient which can be tuned to remove $\mathcal{O}(a)$ artifacts to all orders in the gauge coupling constant g.

While this action has been known for some time, the difficulty has been in accurately determining the renormalization of the improvement coefficients, $C_{\rm SW}$, b_q, c_q, etc., in the interacting quantum field theory. At the lattice spacings typically considered in today's lattice simulations, the tree-level values can differ by a factor of 2 from the renormalized values. While mean-field improved estimates of the renormalized coefficients provide substantial corrections, they are not sufficiently accurate to remove $\mathcal{O}(a)$ errors to all orders in the gauge coupling constant g [6]. Nonperturbative (NP) $\mathcal{O}(a)$ improvement [7] tunes $C_{\rm SW}$ to all powers in g^2 and displays excellent scaling [6] as discussed further in the following.

In a previous paper [8], we introduced a new form of $\mathcal{O}(a)$ fermion-action improvement in which the renormalization of improvement coefficients is addressed in a very different manner. Central to the approach is the observation that the fermion doublers of the naive theory are removed by the Wilson term at tree level. In place of applying techniques to estimate the renormalization of the improvement coefficients induced by the gauge fields of QCD, techniques are applied to modify the gauge fields to suppress renormalizations such that tree-level knowledge of improvement coefficients is adequate.

There are many accepted methods for removing short-distance fluctuations from gauge field configurations including APE smearing, HYP smearing, and their variants. The central feature of FLIC fermions is to construct the fermion action using two sets of gauge fields. In the lattice operators providing the relevant dimension-four operators of the continuum action, one works with the untouched gauge fields generated via Monte Carlo methods, while the smoothed gauge fields are introduced only in the purely irrelevant lattice operators having dimension-five or more. We refer to this action as the Fat-Link Irrelevant Clover (FLIC) fermion action.

The motivation behind constructing the FLIC action is to benefit from the reduced exceptional configuration problem of fat-link actions [9], while retaining short-distance quark interactions in the relevant operators of the fermion action. The expectation is that improvement in the condition number of the FLIC fermion matrix will allow rapid calculations of fermion propagators and efficient access to the chiral limit of full QCD [10].

In this paper we present the first comprehensive scaling analysis of FLIC fermions where four different lattice spacings are considered on five lattices each sampled by 200 configurations. The scaling analysis shows convincingly that the new FLIC fermion action removes $\mathcal{O}(a)$ lattice artifacts from the Wilson fermion action, on a level comparable to that of the nonperturbative improved clover action.

The simulations are performed using an $\mathcal{O}(a^2)$–mean-field improved Luscher-Weisz plaquette plus rectangle gauge action [11] on $12^3 \times 24$, $16^3 \times 32$ and $20^3 \times 40$ lattices with lattice spacings of 0.093, 0.122, 0.134 and 0.165 fm determined from a string tension analysis incorporating the lattice coulomb potential [12] with $\sqrt{\sigma} = 440$ MeV. Initial studies of FLIC, mean-field improved clover and Wilson quark actions were made using 50 configurations. The scaling analysis of FLIC fermions presented here is performed with a total of 200 configurations at each lattice spacing and volume. Gauge configurations are generated using the Cabibbo-Marinari pseudo-heat-bath algorithm with three diagonal SU(2) subgroups looped over twice. Simulations are performed using a parallel algorithm with appropriate link partitioning [13], and the error analysis is performed by a third-order, single-elimination jackknife, with the χ^2 per degree of freedom ($N_{\rm DF}$) obtained via covariance matrix fits.

Fat links [9,14] are created by averaging or smearing links on the lattice with their nearest transverse neighbors in a gauge covariant manner (APE smearing). The smearing procedure [15] replaces a link, $U_\mu(x)$, with a sum of the link and α times its staples

$$U_\mu(x) \to U'_\mu(x) = (1-\alpha)U_\mu(x) + \frac{\alpha}{6}\sum_{\substack{\nu=1\\ \nu\neq\mu}}^{4}[U_\nu(x)U_\mu(x+\nu a)U_\nu^\dagger(x+\mu a) + U_\nu^\dagger(x-\nu a)U_\mu(x-\nu a)U_\nu(x-\nu a+\mu a)], \tag{4}$$

followed by projection back to SU(3). We select the unitary matrix $U_\mu^{\rm FL}$ which maximizes

$$\mathrm{Re}\,\mathrm{tr}(U_\mu^{\rm FL} U_\mu'^\dagger), \tag{5}$$

by iterating over the three diagonal SU(2) subgroups of SU(3). Performing eight iterations over these subgroups gives gauge invariance up to seven significant figures. The combined procedure of smearing and projection is repeated to create a fat link.

The mean-field improved FLIC action now becomes

$$S_{\rm SW}^{\rm FL} = S_{\rm W}^{\rm FL} - \frac{igC_{\rm SW}\kappa r}{2(u_0^{\rm FL})^4}\bar\psi(x)\sigma_{\mu\nu}F_{\mu\nu}\psi(x), \tag{6}$$

where $F_{\mu\nu}$ is constructed using fat links, $u_0^{\rm FL}$ is the mean link calculated with fat links, and where the mean-field improved Fat-Link Irrelevant Wilson action is

$$\begin{aligned} S_{\rm W}^{\rm FL} = \sum_x \bar\psi(x)\psi(x) + \kappa\sum_{x,\mu}\bar\psi(x)\bigg[\gamma_\mu\bigg(\frac{U_\mu(x)}{u_0}\psi(x+\hat\mu) \\ - \frac{U_\mu^\dagger(x-\hat\mu)}{u_0}\psi(x-\hat\mu)\bigg) \\ - r\bigg(\frac{U_\mu^{\rm FL}(x)}{u_0^{\rm FL}}\psi(x+\hat\mu) + \frac{U_\mu^{\rm FL\dagger}(x-\hat\mu)}{u_0^{\rm FL}}\psi(x-\hat\mu)\bigg)\bigg], \end{aligned} \tag{7}$$

with $\kappa = 1/(2m + 8r)$. We take the standard value $r = 1$. Our notation uses the Pauli (Sakurai) representation of the Dirac γ-matrices defined in Appendix B of Sakurai [16]. In particular, the γ matrices are Hermitian and $\sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/(2i)$.

As reported in Ref. [8], the mean-field improvement parameter for the fat links is very close to 1. Hence, the mean-field improved coefficient for $C_{\rm SW}$ is expected to be accurate. A significant advantage of the fat-link irrelevant operator approach is that one can now use highly improved definitions of $F_{\mu\nu}$ which give impressive near-integer results for the topological charge [17].

In particular, we employ the 3-loop $\mathcal{O}(a^4)$-improved definition of $F_{\mu\nu}$ in which the standard clover sum of four 1×1 loops lying in the μ, ν plane is combined with 2×2 and 3×3 loop clovers. Bilson-Thompson *et al.* [17] find

$$gF_{\mu\nu} = \frac{1}{8i}\left[\left(\frac{3}{2}W^{1\times1} - \frac{3}{20u_0^4}W^{2\times2} + \frac{1}{90u_0^8}W^{3\times3}\right) - \text{H.c.}\right]_{\text{traceless}} \qquad (8)$$

where $W^{n\times n}$ is the clover sum of four $n \times n$ loops and $F_{\mu\nu}$ is made traceless by subtracting $1/3$ of the trace from each diagonal element of the 3×3 color matrix. This definition reproduces the continuum limit with $\mathcal{O}(a^6)$ errors. On approximately self-dual configurations, this operator produces integer topological charge to better than four parts in 10^4. We have also considered a 5-loop improved $F_{\mu\nu}$ which agrees with the 3-loop version to better than four parts in 10^4 [17].

An important consideration is the amount of smearing to apply to the gauge fields of the irrelevant operators. Since the aim is to remove perturbative renormalizations of the improvement coefficients, we monitor the 3-loop $\mathcal{O}(a^4)$-improved topological charge constructed with the 3-loop $\mathcal{O}(a^4)$-improved definition of $F_{\mu\nu}$ of Eq. (8) as a function of smearing sweep. The topological charge is known to have a large multiplicative renormalization [18] and serves as an ideal operator for monitoring the removal of perturbative physics under smearing. We find that the topological charge varies rapidly over the first few sweeps of smearing but then makes only small variations thereafter. We define the optimal number of sweeps to be the minimum number of sweeps required to reach the smoothly varying regime. It is interesting that our findings for optimal smearing coincide with that required to provide the optimal condition number of the FLIC fermion matrix in the negative mass regime relevant to overlap fermions [19].

Hadron masses are extracted from the Euclidean time dependence of the calculated two-point correlation functions using standard techniques [20]. To compare with the results of Ref. [6], we interpolate our results to a pseudoscalar to vector meson mass ratio of $m_\pi/m_\rho = 0.7$. The scaling behavior of the various fermion actions is illustrated in Fig. 1. Lattice volumes, spacings and hadron masses in units of the string tension are given in Table I.

Since the smearing radius [21] is proportional to the product of α and the number of smearing sweeps, n, we fix $\alpha = 0.7$ and vary n. For the fine lattices with $a^2\sigma \sim 0.075$ and 0.045, four smearing sweeps are performed. For the coarser lattices with $a^2\sigma \sim 0.09$ and 0.14 we perform

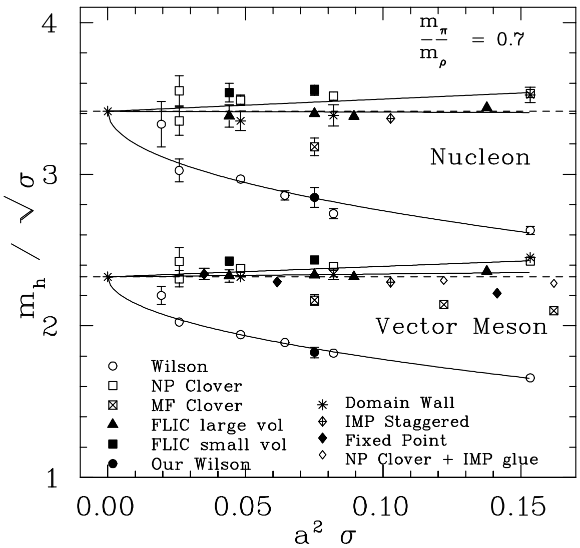

FIG. 1. Nucleon and vector meson masses for the Wilson, Mean-Field (MF) improved, NP-improved clover, domain wall, fixed point, improved staggered and FLIC actions obtained by interpolating simulation results to $m_\pi/m_\rho = 0.7$. Results from the current simulations are indicated by the solid symbols; those from earlier simulations by open or hatched symbols. The solid lines illustrate fits, constrained to have a common continuum limit, to FLIC, NP-improved clover and Wilson fermion-action results obtained on physically large lattice volumes. Further details are provided in the text.

six and eight smearing sweeps, respectively. The effective range of the smearing [22] within which interactions are suppressed is $\langle r^2 \rangle = a^2\alpha n/3$, providing an rms radius of 1.0, 1.2 and 1.4 lattice spacings for $n = 4$, six and eight smearing sweeps at $\alpha = 0.7$, respectively.

Actions with fat-link irrelevant operators perform extremely well, lying very near the horizontal dashed lines corresponding to continuum limit results at finite lattice spacing. For reference we have also calculated masses for the Wilson action at $a^2\sigma \simeq 0.075$ which agree with those of Ref. [6]. We also compare our results with the standard Mean-Field Improved Clover (MFIC) action. We mean-field improve as defined in Eqs. (6) and (7) but with thin links throughout. For this action, the standard 1-loop definition of $F_{\mu\nu}$ is used. For both the vector meson and the nucleon, the FLIC actions perform significantly better than the mean-field improved clover action.

TABLE I. Lattice parameters and results for the vector meson and nucleon masses interpolated to $m_\pi/m_\rho = 0.7$.

β	Volume	N_{configs}	$a\sqrt{\sigma}$	$m_v/\sqrt{\sigma}$	$m_N/\sqrt{\sigma}$	u_0
4.38	$16^3 \times 32$	200	0.371	2.360(20)	3.439(27)	0.8761
4.53	$20^3 \times 40$	200	0.299	2.324(15)	3.382(24)	0.8859
4.60	$12^3 \times 24$	200	0.274	2.434(26)	3.554(33)	0.8889
4.60	$16^3 \times 32$	200	0.274	2.336(22)	3.400(26)	0.8889
4.80	$16^3 \times 32$	200	0.210	2.427(23)	3.538(61)	0.8966

Finally, our FLIC results compare extremely well with a variety of improved actions found in the literature. In particular, in Fig. 1 we compare with results using domain wall [23], fixed point [24] and improved staggered fermions [25].

The two different volumes used at $a^2\sigma \sim 0.075$ reveal a small finite volume effect, which increases the mass for the smaller volumes at $a^2\sigma \sim 0.075$ and ~ 0.045. Examination of points from the small and large volumes separately indicates continued scaling toward the continuum limit. While the finite volume effect will produce a different continuum limit value, the slope of the points from the smaller and larger volumes agree, consistent with errors of $\mathcal{O}(a^2)$.

Focusing on simulation results from physical volumes with extents $\sim$2 fm and larger, we perform a simultaneous fit of the FLIC, NP-improved clover and Wilson fermion-action results. The fits are constrained to have a common continuum limit and assume errors are $\mathcal{O}(a^2)$ for FLIC and NP-improved clover actions and $\mathcal{O}(a)$ for the Wilson action. To obtain a data point at our fine lattice spacing, $a^2\sigma \sim 0.045$, we use the observed finite volume effect at $a^2\sigma \sim 0.075$ to correct the point at $a^2\sigma \sim 0.045$ as illustrated by the open triangles. The results from these fits are given in Table II. In light of the different gauge actions used in the analyses and the fact that the lattice volumes considered are not perfectly matched throughout all the simulations, an acceptable χ^2 per degree of freedom is obtained for both the nucleon and ρ-meson fits.

To assess the sensitivity of our results on the number of smearing sweeps, we perform a second calculation with only four smearing sweeps for our lattice having the coarsest lattice spacing. This result is indicated by the solid diamond symbol offset to the right for clarity. These two results at $a^2\sigma \sim 0.135$ reveal an insensitivity to the number of APE smearing sweeps used in constructing the irrelevant operators of the FLIC fermion action. This insensitivity suggests that one could define the FLIC action in terms of a fixed number of APE smearing sweeps independent of the lattice spacing. Upon taking the continuum limit the smearing radius would still tend to zero as required.

TABLE II. Fit parameters and $\chi^2_{\rm DF}$ for joint and separate fits of the FLIC, NP-improved clover and Wilson hadron masses We fit to an ansatz of the form $m_H/\sqrt{\sigma} = H_0 + H_1 a\sqrt{\sigma} + H_2 a^2\sigma$, where the hadron, H, can be the vector meson, V, or the nucleon, N.

	FLIC	NP clover	Wilson		
V_0	V_2	V_2	V_1	V_2	$\chi^2_{\rm DF}$
2.324(24)	0.18(23)	0.69(18)	−1.78(20)	0.17(43)	0.96
2.317(18)	0.24(18)	0.74(14)	−1.71(7)	0	0.87
2.320(25)	0.22(24)	0.72(19)			0.76
2.291(34)	0.47(32)				0.40
N_0	N_2	N_2	N_1	N_2	$\chi^2_{\rm DF}$
3.415(55)	−0.05(56)	0.81(41)	−2.18(45)	0.34(97)	1.88
3.402(39)	0.07(42)	0.90(31)	−2.04(15)	0	1.69
3.402(50)	0.08(51)	0.90(38)			1.52
3.335(53)	0.71(53)				0.39

In conclusion, the use of fat links in the irrelevant operators of the FLIC fermion action provides a new form of nonperturbative $\mathcal{O}(a)$ improvement without the need of nonperturbative fine-tuning. In addition, the $\mathcal{O}(a^2)$ errors are small for this action. FLIC fermions display nearly perfect scaling, providing near-continuum limit results at finite lattice spacing.

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[1] K. Symanzik, Nucl. Phys. **B226**, 187 (1983).
[2] B. Sheikholeslami and R. Wohlert, Nucl. Phys. **B259**, 572 (1985).
[3] C. Dawson *et al.*, Nucl. Phys. Proc. Suppl. **63**, 877 (1998).
[4] G. Heatlie *et al.*, Nucl. Phys. **B352**, 266 (1991).
[5] K. G. Wilson, in *New Phenomena in Subnuclear Physics*, edited by A. Zichichi (Plenum, New York, 1975), Pt. A, p. 69.
[6] R. G. Edwards, U. M. Heller, and T. R. Klassen, Phys. Rev. Lett. **80**, 3448 (1998).
[7] M. Luscher *et al.*, Nucl. Phys. **B478**, 365 (1996); M. Luscher *et al.*, Nucl. Phys. **B491**, 323 (1997); **B491**, 344 (1997).
[8] J. M. Zanotti *et al.*, Phys. Rev. D **65**, 074507 (2002); Nucl. Phys. Proc. Suppl. **109**, 101 (2002).
[9] MILC Collaboration, T. DeGrand *et al.*, hep-lat/9807002.
[10] W. Kamleh, D. B. Leinweber, and A. G. Williams, Phys. Rev. D **70**, 014502 (2004).
[11] M. Luscher and P. Weisz, Commun. Math. Phys. **97**, 59 (1985); **98**, 433(E) (1985).
[12] R. G. Edwards, U. M. Heller, and T. R. Klassen, Nucl. Phys. **B517**, 377 (1998).
[13] F. D. Bonnet, D. B. Leinweber, and A. G. Williams, J. Comput. Phys. **170**, 1 (2001).
[14] MILC Collaboration, T. DeGrand *et al.*, Phys. Rev. D **60**, 094501 (1999).

[15] M. Falcioni *et al.*, Nucl. Phys. **B251**, 624 (1985); M. Albanese *et al.*, Phys. Lett. B **192**, 163 (1987).
[16] J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, Reading, MA, 1982).
[17] S. Bilson-Thompson *et al.*, Nucl. Phys. Proc. Suppl. **109**, 116 (2002); S. O. Bilson-Thompson *et al.*, Ann. Phys. (N.Y.) **304**, 1 (2003).
[18] M. Campostrini *et al.*, Phys. Lett. B **212**, 206 (1988); B. Alles *et al.*, Nucl. Phys. **B494**, 281 (1997).
[19] W. Kamleh, D. H. Adams, D. B. Leinweber, and A. G. Williams, Phys. Rev. D **66**, 014501 (2002).
[20] J. M. Zanotti *et al.*, Phys. Rev. D **68**, 054506 (2003).
[21] F. D. R. Bonnet *et al.*, Phys. Rev. D **65**, 114510 (2002).
[22] C. W. Bernard and T. DeGrand, Nucl. Phys. Proc. Suppl. **83**, 845 (2000).
[23] T. Blum *et al.*, Phys. Rev. D **69**, 074502 (2004).
[24] S. Hauswirth, Ph.D. thesis, Bern University, hep-lat/0204015; C. Gattringer *et al.* Nucl. Phys. **B677**, 3 (2004).
[25] MILC Collaboration, C. W. Bernard *et al.*, Phys. Rev. D **61**, 111502 (2000); Phys. Rev. D **64**, 054506 (2001).