Chiral symmetry breaking in dimensionally regularized nonperturbative quenched QED

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In this paper we study dynamical chiral symmetry breaking in dimensionally regularized quenched QED within the context of Dyson-Schwinger equations. In $D<4$ dimensions the theory has solutions which exhibit chiral symmetry breaking for all values of the coupling. To begin with, we study this phenomenon both numerically and, with some approximations, analytically within the rainbow approximation in the Landau gauge. In particular, we discuss how to extract the critical coupling $\alpha_c=\pi/3$ relevant in 4 dimensions from the D dimensional theory. We further present analytic results for the chirally symmetric solution obtained with the Curtis-Pennington vertex as well as numerical results for solutions exhibiting chiral symmetry breaking. For these we demonstrate that, using dimensional regularization, the extraction of the critical coupling relevant for this vertex is feasible. Initial results for this critical coupling are in agreement with cut-off based work within the currently achievable numerical precision. [S0556-2821(99)06916-7]

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I. INTRODUCTION

It is fairly well established that quantum electrodynamics (QED), and in particular quenched QED, breaks chiral symmetry for sufficiently large couplings. This phenomenon has been observed both in lattice simulations [1] as well as various studies based on the use of Dyson-Schwinger equations [2–4]. These latter calculations have generally relied on the use of a cut-off in Euclidean momentum in order to regulate divergent integrals, a procedure which breaks the gauge invariance of the theory.

On the other hand, continuation of gauge theories to $D<4$ dimensions has long been used as an efficient way to regularize perturbation theory without violating gauge invariance. In nonperturbative calculations, however, the use of this method of regularization is rarely used [5]. Within the context of the Dyson-Schwinger equations (DSEs) only a few publications [6,7] have employed dimensional regularization instead of the usual momentum cut-off.

It is the purpose of the present paper to study dynamical chiral symmetry breaking and the chiral limit within dimensionally regularized quenched QED. We are motivated to do this by the wish to avoid some gauge ambiguities occurring in cut-off based work, which we discuss in Sec. II. In that section we also outline some general results which one expects to be valid for $D<4$, independently of the particular vertex which one uses as an input to the DSEs. Having done this we proceed, in Sec. III, with a study of chiral symmetry breaking in the popular, but gauge non-covariant, rainbow approximation. Just as in cut-off regularized work, the rainbow approximation provides a very good qualitative guide to what to expect for more realistic vertices and has the considerable advantage that, with certain additional approximations, one may obtain analytical results. We check numerically that the additional approximations made are in fact quite justified. Indeed, it is very fortunate that it is possible to obtain this analytic insight into the pattern of chiral symmetry breaking in D dimensions as it provides us with a well defined procedure for extracting the critical coupling of the 4 dimensional theory with more complicated vertices. We proceed to the Curtis-Pennington (CP) vertex in Sec. IV. There we derive, for solutions which do not break chiral symmetry, an integral representation for the exact wave function renormalization function $\mathcal{Z}$ in D dimensions. We also provide an approximate, but explicit, expression for this quantity. The latter is quite useful, in the ultraviolet region, even if dynamical chiral symmetry breaking takes place as it provides a welcome check for the numerical investigation of chiral symmetry breaking with the CP vertex with which we conclude that section. Finally, in Sec. V, we summarize our results and conclude.

II. MOTIVATION AND GENERAL CONSIDERATIONS

Although chiral symmetry breaking appears to be universally observed independently of the precise nature of the vertex used in DSE studies, it has also been recognized for a long time that the critical couplings with almost all[1] of these vertices show a gauge dependence which should not be present for a physical quantity. With a bare vertex this is not surprising as this vertex Ansatz breaks the Ward-Takahashi identity. However, even with the Curtis-Pennington (CP) vertex, which does not violate this identity and additionally is constrained by the requirement of perturbative multiplicative renormalizability, a residual gauge dependence remains [11,12].

Apart from possible deficiencies of the vertex, which we do not investigate in this paper, the use of cut-off regularization explicitly breaks the gauge symmetry even as the cut-off is taken to infinity. This is well known in perturbation theory

[1]Some vertex *Ansätze* exist which lead to critical couplings which are strictly gauge independent [8–10]. However, these involve either vertices which have unphysical singularities or ensure gauge independence of the critical coupling by explicit construction.

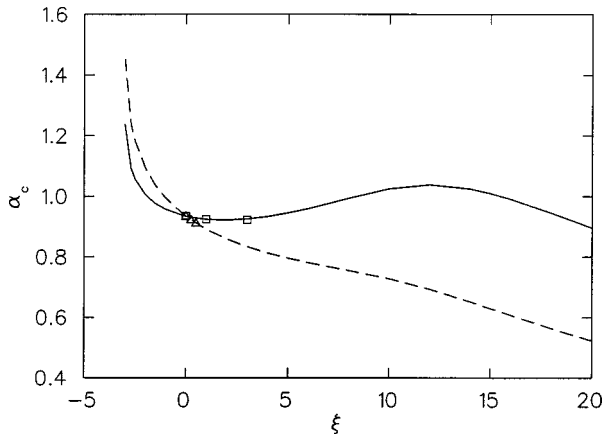

FIG. 1. The critical coupling for the CP vertex. The solid line is taken from the bifurcation analysis carried out in Ref. [12], which agrees with the numerical results (open squares) of Ref. [11]. The dashed line corresponds to the bifurcation analysis carried out with the ''gauge violating term'' removed (as suggested in Ref. [14]) and agrees with the numerical results (open triangles) of Ref. [15].

(see, for example, the discussion of the axial anomaly in Sect. 19.2 of Ref. [13]) and was pointed out by Roberts and collaborators [14] in the present context. The latter authors proposed a prescription for dealing with this ambiguity which ensures that the regularization does not violate the Ward-Takahashi identity.

As may be observed in Fig. 1, this ambiguity has a strong effect on the value of the critical coupling of the theory. The two curves in that figure correspond to the critical coupling α_c of Ref. [12] as well as the coupling α_c' one obtains by following the prescription of Roberts, *et al.* It is straightforward to show, following the analysis of Ref. [12], that these couplings are related through

$$\alpha_c' = \frac{\alpha_c}{1+\frac{\xi\alpha_c}{8\pi}}. \tag{1}$$

Also plotted in this figure are previously published numerical results [11,15] obtained with both of the above prescriptions. Note that, curiously, the critical couplings obtained with the prescription of Ref. [14] (i.e. the calculation which restores the Ward-Takahashi identity) exhibits a stronger gauge dependence, at least for the range of gauge parameters shown in Fig. 1.

Gauge ambiguities such as the one outlined above are absent if one does not break the gauge invariance of the theory through the regularization procedure. Hence, we now turn to dimensionally regularized (quenched) QED. The Minkowski space fermion propagator $S(p)$ is defined in the usual way through the dimensionless wave function renormalization function $\mathcal{Z}(p^2)$ and the dimensionful mass function $M(p^2)$, i.e.,

$$S(p) \equiv \frac{\mathcal{Z}(p^2)}{\not{p} - M(p^2)}. \tag{2}$$

The dependence of $\mathcal{Z}$ and M on the dimensionality of the space is not explicitly indicated here. Furthermore, note that to a large extent we shall be dealing only with the regularized theory without imposing a renormalization procedure, as renormalization [15,16] is inessential to our discussion.

In addition to the above, we shall consider the theory without explicit chiral symmetry breaking (i.e. zero bare mass). This theory would not contain a mass scale were it not for the usual arbitrary scale (which we denote by ν) introduced in $D=4-2\epsilon$ dimensions which provides the connection between the *dimensionful* coupling α_D and the usual *dimensionless* coupling constant $\alpha=e^2/4\pi$:

$$\alpha_D = \alpha \nu^{2\epsilon}. \tag{3}$$

As ν is *the only* mass scale in the problem, and as the coupling always appears in the above combination with this scale, on dimensional grounds alone the mass function must be of the form

$$M(p^2) = \nu \alpha^{1/2\epsilon} \tilde{M}\left(\frac{p^2}{\nu^2 \alpha^{1/\epsilon}}, \epsilon \right) \tag{4}$$

where $\tilde{M}$ is a dimensionless function and in particular

$$M(0) = \nu \alpha^{1/2\epsilon} \tilde{M}(0,\epsilon). \tag{5}$$

Moreover, as ϵ goes to zero the ν dependence on the right hand side must disappear and hence the dynamical mass $M(0)$ is either zero (i.e. no symmetry breaking) or goes to infinity in this limit. This situation is analogous to what happens in cut-off regularized theory, where the scale parameter is the cut-off itself and the mass is proportional it.

Note that $\tilde{M}(0,\epsilon)$ is not dependent on α. This implies immediately that there can be no non-zero critical coupling in $D \neq 4$ dimensions: if $M(0)$ is non-zero for some coupling α then it must be non-zero for all couplings.

Given these general considerations (which are of course independent of the particular Ansatz for the vertex) it behooves one to ask how this situation can be reconciled with a critical coupling α_c of order 1 in four dimensions. In order to see how this might arise, we shall extract a convenient numerical factor out of $\tilde{M}$ and suggestively re-write the dynamical mass as

$$M(0) = \nu \left(\frac{\alpha}{\alpha_c} \right)^{1/2\epsilon} \bar{M}(0,\epsilon). \tag{6}$$

At present there is no difference in content between Eq. (5) and Eq. (6). However, if we now *define* α_c by demanding that the behavior of $M(0)$ is dominated by the factor $(\alpha/\alpha_c)^{1/2\epsilon}$ as ϵ goes to zero, which is equivalent to demanding that

$$[\bar{M}(0,\epsilon)]^\epsilon \xrightarrow[\epsilon \to 0]{} 1, \tag{7}$$

then the intent becomes clear: even though $M(0)$ may be nonzero for all couplings in $D<4$ dimensions, in the limit that ϵ goes to zero we obtain

$$M(0) \xrightarrow[\epsilon\to 0]{} 0, \quad \alpha<\alpha_c \tag{8}$$

$$M(0) \xrightarrow[\epsilon\to 0]{} \infty, \quad \alpha>\alpha_c .$$

Note that in the above we have not addressed the issue of whether or not there actually is a $\bar{M}(0,\epsilon)$ with the property of Eq. (7). In fact, the numerical and analytical work in the following sections is largely concerned with finding this function and hence determining whether or not chiral symmetry is indeed broken for $D<4$.[2] Notwithstanding this, as one knows from cut-off based work that there actually is a non-zero critical coupling for $D=4$, one can at this stage already come to the conclusion that $\bar{M}(0,\epsilon)$ exists and hence that quenched QED in $D<4$ dimensions has a chiral symmetry breaking solution for all couplings. Furthermore, even though the dimensionless coupling α does not get modified by vacuum polarization effects in the quenched approximation, the *effective* dimensionless coupling $\bar{\alpha}(q^2)$ in $D<4$ dimensions nevertheless runs as a function of q^2:

$$\bar{\alpha}(q^2)=\alpha(\mu^2)\left(\frac{\mu^2}{q^2}\right)^{\epsilon}, \tag{9}$$

where μ is the renormalization scale (for details see, for example, the discussion of the renormalization group equations in Chapter 3.2 of Ref. [17]). In the infrared the effective coupling increases without bound, suggesting not only that the theory is likely to break chiral symmetry for any coupling (for $D\neq 4$) but also that it may actually be confining. Although we will not pursue this further in this paper, this feature of quenched QED bears some similarity to QCD so that it may be of interest to investigate the theory without taking the limit $\epsilon\to 0$. In this connection, note also that in the ultraviolet the running coupling $\bar{\alpha}(q^2)$ vanishes; i.e., in $D\neq 4$ dimensions quenched QED exhibits ''asymptotic freedom''.

In summary, as the trivial solution $M(p^2)=0$ always exists as well, we see that in $D<4$ dimensions the trivial and symmetry breaking solutions bifurcate at $\alpha=0$ while for $D=4$ the point of bifurcation is at $\alpha=\alpha_c$; i.e., there is a discontinuous change in the point of bifurcation. As D approaches four (i.e. as ϵ approaches 0) the generated mass $M(0)$ decreases (grows) roughly like $(\alpha/\alpha_c)^{1/2\epsilon}$ for $\alpha\lesssim\alpha_c$ ($\gtrsim\alpha_c$), respectively, becoming an infinite step function at $\alpha=\alpha_c$ when ϵ goes to zero.

Finally, it is of interest to speculate as to how the above analysis might be carried over to other field theories, in particular full QED and QCD. Indeed, it is clear that Eq. (5) [as well as Eq. (6) in theories which have a non-zero critical coupling], which is based purely on a dimensional argument, will remain valid in any theory which does not contain an explicit mass scale. However, it should be kept in mind that the coupling occuring on the right hand side of these equations is the *unrenormalized* coupling α. In full QED or QCD, as opposed to quenched QED, this is not a convenient quantity as the bare coupling will itself be a function of ϵ if the renormalized coupling is kept fixed as the regulator is removed (indeed, in lattice studies of QCD the unrenormalized coupling goes to zero as the lattice spacing is taken to zero; see Section 9.2 of Ref. [18]). Hence Eqs. (5) and (6), although valid, loose their utility.

Nevertheless, it is possible to write down an equivalent expression in terms of the dimensionless renormalized coupling $\alpha^R(\mu^2)$. We may use the relationship between $\alpha^R(\mu^2)$, α and ν in order to eliminate the scale ν. Furthermore, as in D dimensions the dependence on the renormalized coupling enters through the dimensionful coupling $\alpha_D^R \equiv \alpha^R\mu^{2\epsilon}$, the appropriate equation for the unrenormalized mass $M(0)$ becomes

$$M(0)=\mu(\alpha^R(\mu^2))^{1/2\epsilon}\tilde{M}(0,\epsilon). \tag{10}$$

In addition, should the theory under consideration have a non-zero critical coupling (as has been observed in some Dyson-Schwinger calculations of full QED, with a small number of fermion flavors N_f; see Ref. [4]), then Eq. (6) and the subsequent discussion remains applicable as long as the coupling α is replaced by the renormalized coupling $\alpha^R(\mu^2)$. In particular, it would be possible to explore the dependence of the critical coupling on N_f by investigating the small ϵ behavior of the relevant $\tilde{M}(0,\epsilon)$. On the other hand, if the four-dimensional theory breaks chiral symmetry for all couplings (e.g. QCD) only Eq. (5) remains valid and there is no essential difference between $D=4$ and $D<4$. In this case the limit $\epsilon\to 0$ could be smooth.

III. THE RAINBOW APPROXIMATION

Let us now consider an explicit vertex. To begin with, we consider the rainbow approximation to the Euclidean mass function of quenched QED with zero bare mass in Landau gauge. It is given by

$$M(p^2)=(e\nu^{\epsilon})^2(3-2\epsilon)\int\frac{d^Dk}{(2\pi)^D}\frac{M(k^2)}{k^2+M^2(k^2)}\frac{1}{(p-k)^2}. \tag{11}$$

Note that the Dirac part of the self-energy is equal to zero in the Landau gauge in rainbow approximation even in $D<4$ dimensions and hence that $\mathcal{Z}(p^2)=1$ for all p^2.

It is of course possible to find the solution to Eq. (11) numerically—indeed we shall do so—however it is far more instructive to first try to make some reasonable approximations in order to be able to analyze it analytically. First, as the angular integrals involved in D-dimensional integration are standard (see, for example, Refs. [7] and [17]) we may reduce Eq. (11) to a one-dimensional integral, namely

[2]The reader will note that as neither $\tilde{M}(0,\epsilon)$ nor $\bar{M}(0,\epsilon)$ are functions of the coupling α, the value of α_c can be determined independently of the strength α of the self-interactions in $D<4$ dimensions.

$$M(p^2)=\alpha\nu^{2\epsilon}c_\epsilon\int_0^\infty \frac{dk^2(k^2)^{1-\epsilon}M(k^2)}{k^2+M^2(k^2)} \times\left[\frac{1}{p^2}F\left(1,\epsilon;2-\epsilon;\frac{k^2}{p^2}\right)\theta(p^2-k^2) +\frac{1}{k^2}F\left(1,\epsilon;2-\epsilon;\frac{p^2}{k^2}\right)\theta(k^2-p^2)\right], \quad (12)$$

where

$$c_\epsilon=\frac{3-2\epsilon}{(4\pi)^{1-\epsilon}\Gamma(2-\epsilon)} \quad \left(c_0=\frac{3}{4\pi}\right). \quad (13)$$

Note that for $D=4$ the mass function in Eq. (12) reduces to the standard one in QED_4.

In $D\neq 4$ dimensions the hypergeometric functions in Eq. (12) preclude a solution in closed form. However, note that these hypergeometric functions have a power expansion in ϵ so that for small ϵ one is not likely to go too far wrong by just replacing these by their $\epsilon=0$ (i.e. $D=4$) limit. After all, the reason for choosing dimensional regularization in the first place is in order to regulate the integral, and this is achieved by the factor of $k^{-2\epsilon}$, not the hypergeometric functions. In addition, this approximation also corresponds to just replacing the hypergeometric functions by their IR and UV limits, so that one might expect that even for larger ϵ that the approximation is not too bad in these regions.[3]

Making this replacement, i.e.

$$M(p^2)=\alpha\nu^{2\epsilon}c_\epsilon\left[\frac{1}{p^2}\int_0^{p^2}\frac{dk^2(k^2)^{1-\epsilon}M(k^2)}{k^2+M^2(k^2)} +\int_{p^2}^\infty\frac{dk^2(k^2)^{-\epsilon}M(k^2)}{k^2+M^2(k^2)}\right], \quad (14)$$

allows us to convert Eq. (12) into a differential equation, namely

$$[p^4M'(p^2)]'+\alpha\nu^{2\epsilon}c_\epsilon\frac{(p^2)^{1-\epsilon}}{p^2+M^2(p^2)}M(p^2)=0, \quad (15)$$

with the boundary conditions

$$p^4M'(p^2)|_{p^2=0}=0, \quad [p^2M(p^2)]'|_{p^2=\infty}=0. \quad (16)$$

Unfortunately, the differential equation (15) still has no solutions in terms of known special functions. Since the mass function in the denominator of Eq. (14) serves primarily as an infrared regulator we shall make one last approximation and replace it by an infrared cut-off for the integral, which can be taken as a fixed value of $M^2(k^2)$ in the infrared region (for convenience we shall call this value the ''dynamical mass'' m). This simplifies the problem sufficiently to allow the derivation of an analytical solution.

In terms of the dimensionless variables $x=p^2/\nu^2$, $y=k^2/\nu^2$ and $a=m^2/\nu^2$ the linearized equation becomes

$$M(x)=\alpha c_\epsilon\left[\frac{1}{x}\int_a^x\frac{dyy^{1-\epsilon}M(y)}{y}+\int_x^\infty\frac{dyy^{-\epsilon}M(y)}{y}\right]; \quad (17)$$

for simplicity, we do not explicitly differentiate between $M(x)$ and $M(p^2)$. This may be written in differential form as

$$[x^2M'(x)]'+\alpha c_\epsilon x^{-\epsilon}M(x)=0, \quad (18)$$

with the boundary conditions

$$M'(x)|_{x=a}=0, \quad [xM(x)]'|_{x=\infty}=0. \quad (19)$$

This differential equation has solutions in terms of Bessel functions

$$M(x)=x^{-1/2}\left[C_1J_\lambda\left(\frac{\sqrt{4\alpha c_\epsilon}}{\epsilon x^{\epsilon/2}}\right)+C_2J_{-\lambda}\left(\frac{\sqrt{4\alpha c_\epsilon}}{\epsilon x^{\epsilon/2}}\right)\right], \quad (20)$$

where we have defined $\lambda=1/\epsilon$ in order to avoid cumbersome indices on the Bessel functions. The ultraviolet boundary condition Eq. (19) gives $C_2=0$ while the infrared boundary condition leads to

$$C_1\left[J_\lambda\left(\frac{\sqrt{4\alpha c_\epsilon}}{\epsilon x^{\epsilon/2}}\right)+\frac{\sqrt{4\alpha c_\epsilon}}{x^{\epsilon/2}}J'_\lambda\left(\frac{\sqrt{4\alpha c_\epsilon}}{\epsilon x^{\epsilon/2}}\right)\right]_{x=a}=0. \quad (21)$$

This equation may be simplified using the relation among Bessel functions

$$zJ'_\lambda(z)+\lambda J_\lambda(z)=zJ_{\lambda-1}(z), \quad (22)$$

and becomes

$$C_1\left[\frac{\sqrt{4\alpha c_\epsilon}}{\epsilon x^{\epsilon/2}}J_{\lambda-1}\left(\frac{\sqrt{4\alpha c_\epsilon}}{\epsilon x^{\epsilon/2}}\right)\right]_{x=a}=0. \quad (23)$$

Clearly this equation is satisfied by $C_1=0$, which corresponds to the trivial chirally symmetric solution $M(x)=0$. However, for values of a which are such that the argument of the Bessel function in Eq. (23) corresponds to one of its zeroes, the equation is also satisfied for $C_1\neq 0$, i.e. for these values of a there exist solutions with dynamically broken chiral symmetry. If we define $j_{\lambda-1,1}=\sqrt{4\alpha c_\epsilon}/\epsilon a^{\epsilon/2}$ to be the smallest positive zero of Eq. (23), the dynamical mass for this solution becomes

$$m=\nu a^{1/2}=\nu\alpha^{1/2\epsilon}\left(\frac{\sqrt{4c_\epsilon}}{\epsilon j_{\lambda-1,1}}\right)^{1/\epsilon}. \quad (24)$$

[3]It is however possible to show that a linearized version of Eq. (11) always has symmetry breaking solutions even without making this approximation of the angular integrals. We indicate how this may be done in Appendix A.

Note that for this solution the normalization C_1 is not fixed by Eq. (17) as this equation is linear in $M(x)$. Later on we shall fix C_1 by demanding that $M(a)=m$, however there is no compelling reason to do this and one might alternatively fix the normalization in such a way as to approximate the true (numerical) solutions of Eq. (11) as well as possible. Finally, note that, as expected, a dynamical symmetry breaking solution exists for any value of the coupling and that the expression for the dynamical mass is in agreement with the general form expected from dimensional considerations [i.e. Eq. (5)].

In order to extract α_c, we need to look at the behavior of m as ϵ goes to zero (i.e. $\lambda\to\infty$). This may be done by noting that the positive roots of the Bessel function J_λ have the following asymptotic behavior (see, for example, Eq. 9.5.22 in Ref. [19]):

$$j_{\lambda,s}\sim\lambda z(\zeta)+\sum_{k=1}^{\infty}\frac{f_k(\zeta)}{\lambda^{2k-1}},\quad \zeta=\lambda^{-2/3}a_s, \tag{25}$$

where a_s is the sth negative zero of Airy function $\mathrm{Ai}(z)$, and $z(\zeta)$ is determined $[z(\zeta)>1]$ from the equation

$$\frac{2}{3}(-\zeta)^{3/2}=\sqrt{z^2-1}-\arccos\frac{1}{z}. \tag{26}$$

For large λ the variable ζ is small and so it is valid to expand z around 1. Writing $z=1+\delta$ we obtain

$$\arccos\frac{1}{1+\delta}\sim\sqrt{2\delta}-\frac{5\sqrt{2}}{12}\delta^{3/2}, \tag{27}$$

and so $\delta\simeq-\zeta/2^{1/3}$ yielding, to leading order,

$$z=1-\frac{a_1}{2^{1/3}\lambda^{2/3}}. \tag{28}$$

If we define

$$\gamma=-\frac{a_1}{2^{1/3}}\sim 1.855757 \tag{29}$$

then the leading terms in the expansion of $j_{\lambda-1,1}$ are

$$j_{\lambda-1,1}\sim\lambda+\gamma\lambda^{1/3}-1+O(\lambda^{-1/3}). \tag{30}$$

Also, the coefficient c_ϵ appearing in Eq. (24) may be expanded

$$c_\epsilon\underset{\epsilon\to 0}{\sim}\frac{3}{4\pi}(1+d\epsilon),\quad d=\ln(4\pi)+\frac{1}{3}+\psi(1) \tag{31}$$

so that for small ϵ the dynamical mass becomes

$$m\sim\nu\alpha^{1/2\epsilon}\frac{\left[\frac{3}{\pi}(1+d\epsilon)\right]^{1/2\epsilon}}{(1+\gamma\epsilon^{2/3}-\epsilon)^{1/\epsilon}}\sim\nu\left(\frac{\alpha}{\pi/3}\right)^{1/2\epsilon}e^{1+d/2-\gamma\epsilon^{-1/3}}. \tag{32}$$

Note that the behavior of the first term (for ϵ going to zero) dominates over the exponential function, as required in Eq. (7). Hence the critical coupling in four dimensions is given by $\pi/3$, as expected from cut-off based work [2,3].

Returning now to the mass function itself, we may substitute the expression for the dynamical mass, i.e., Eq. (24), together with our choice of normalization condition

$$M(p^2=m^2)=m, \tag{33}$$

into Eq. (11) in order to eliminate C_1. One obtains

$$M(p)=\frac{m^2}{|p|}\frac{J_\lambda\left[j_{\lambda-1,1}\cdot\left(\frac{m}{|p|}\right)^\epsilon\right]}{J_\lambda[j_{\lambda-1,1}]}. \tag{34}$$

Note that the explicit dependence on ν (and hence α) has been completely replaced by m in this expression.

So far we have taken α independent of the regularization. As we have seen this leads to a dynamically generated mass which becomes infinite as the regulator is removed. Fomin, *et al.* [2] examined (within cut-off regularized QED) a different limit, namely one where the mass m is kept constant while the cut-off is removed. In our case this limit necessitates that the coupling α is dependent on ϵ through

$$\alpha\simeq\frac{\pi}{3}(1+2\gamma\epsilon^{2/3}) \tag{35}$$

[see Eq. (32); note that α_c is approached from above]. The limit may be taken analytically in Eq. (34) by making use of the known asymptotic behavior of the Bessel functions (see Eq. 9.3.23 of Ref. [19]), i.e.

$$J_\lambda(\lambda+\lambda^{1/3}z)\sim\left(\frac{2}{\lambda}\right)^{1/3}\mathrm{Ai}(-2^{1/3}z), \tag{36}$$

as well as the asymptotic expansion of $j_{\lambda-1,1}$ in Eq. (30). One obtains

$$M(p)=\frac{m^2}{p}\left(\ln\frac{p}{m}+1\right), \tag{37}$$

which agrees with the result in Ref. [2].

To conclude this section, we analyze the validity of the approximations made by solving Eq. (11) numerically and comparing it to the Bessel function solution in Eq. (34). In Fig. 2a we have plotted the mass function (divided by ν) as a function of the dimensionless momentum x for a moderately large coupling ($\alpha=0.6$) and $\epsilon=0.03$. The solid curve corresponds to the exact numerical result [Eq. (11)] while the dashed line is a plot of Eq. (34) for these parameters. As can be seen, the approximation is not too bad and could actually be made significantly better by adopting a different normalization condition to that in Eq. (33). However, no further insight is gained by doing this and we shall not pursue it further.

One might naively think that most of the difference between the Bessel function and the exact numerical solution comes from the linearization of Eq. (11)—i.e., the approxi-

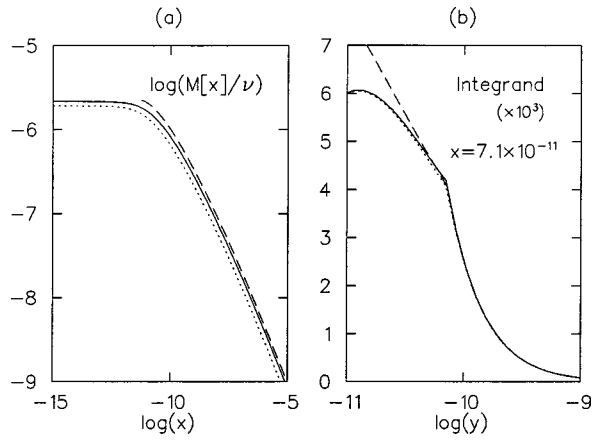

FIG. 2. The mass function for rainbow QED for $\alpha=0.6$ and $\epsilon=0.03$ as a function of $x\equiv p^2/\nu^2$. The dynamical mass is specified by $m/\nu=2.24\times10^{-6}$, where ν is the scale introduced by dimensional regularization. The solid line corresponds to the exact numerical solution of Eq. (11), the dashed line is the Bessel function of Eq. (34) and the dotted line is the solution of the Dyson-Schwinger equation with the hypergeometric function replaced by unity [Eq. (14)]. In (a) the actual mass function is shown, while in (b) we show the integrand at a particular value of x.

mation made by going from Eq. (14) to Eq. (17), as the only approximation made prior to this is to replace the hypergeometric functions by unity, which is expected to be good to order ϵ (i.e. in this case, 3%). This turns out to be not the case; the dotted curve in Fig. 2a corresponds to the (numerical) solution of Eq. (14). Not only is the difference to the true solution essentially an order of magnitude larger than expected (about 30%—note that Fig. 2a is a log-log plot), it is actually of opposite sign to the equivalent difference for the Bessel function. In other words, the validity of the two approximations is roughly of the same order of magnitude and they tend to compensate.

Why are the quantitative differences rather larger than expected? On the level of the integrands the approximations are actually quite good. In Fig. 2b we show the integrands of Eqs. (11), (17) and Eq. (14) for a value of x in the infrared ($x\approx7.1\times10^{-11}$). Clearly the replacement of the hypergeometric functions by unity is indeed an excellent approximation, as is the linearization performed in Eq. (17) (except in the infrared, as expected. Note that when estimating the contribution to the integral from different y one should take into account that the x-axis in Fig. 2b is logarithmic). The real source of the ''relatively large'' differences observed for the integrals in Fig. 2a is the fact that these are integral equations for the function $M(x)$—small differences in the integrands do not necessarily guarantee small differences in $M(x)$. To illustrate this point, consider a hypothetical ''approximation'' to Eq. (17) in which we just scale the integrands by a constant factor $1+\epsilon$ and ask the question how much this affects the solution $M(x)$. For $x=0$ the answer is rather simple: the hypothetical approximation just corresponds to a rescaling of α by $1+\epsilon$ and as $M(0)$ scales like $\alpha^{1/2\epsilon}$ we find that the solution has increased by a factor $(1+\epsilon)^{1/2\epsilon}$. In other words, even in the limit $\epsilon\to0$ there remains a remnant of the ''approximation,'' namely a rescaling of $M(0)$ by a factor $e^{1/2}\approx1.6$!

IV. THE CURTIS-PENNINGTON VERTEX

We shall now leave the rainbow approximation and turn to the CP vertex. The expressions for the scalar and Dirac self-energies for this vertex, using dimensional regularization and in an arbitrary gauge, have already been given in Ref. [7]. Before we discuss chiral symmetry breaking for this vertex we shall first examine the chirally symmetric phase. We remind the reader that in this phase in four dimensions the wave function renormalization has a very simple form for this vertex [20], namely

$$\mathcal{Z}(x,\mu^2)|_{M(x)=0}=\left(\frac{x}{\mu^2}\right)^{\xi\alpha/4\pi}, \tag{38}$$

where the renormalized Dirac propagator is given by

$$S(p)=\frac{\mathcal{Z}(x,\mu^2)}{\not p}. \tag{39}$$

Here ξ is the gauge parameter and μ^2 is the (dimensionless) renormalization scale. This power behavior of $\mathcal{Z}(x)$ is in fact demanded by multiplicative renormalizability [21] as well as gauge covariance [14]. We shall derive the form of this self-energy in $D<4$ dimensions, which will provide a very useful check on the numerical results even if $M(x)\neq0$ as long as $x\gg[M(x)/\nu]^2$.

A. $\mathcal{Z}(p^2)$ in the chirally symmetric phase

In the chirally symmetric phase, the unrenormalized $\mathcal{Z}(x)$ corresponding to the CP vertex in D dimensions is given by

$$\mathcal{Z}(x)=1+\frac{\alpha}{4\pi}\frac{(4\pi)^\epsilon}{\Gamma(2-\epsilon)}\xi\int_0^\infty dy\,\frac{y^{-\epsilon}}{x-y}\mathcal{Z}(y)\times\left[(1-\epsilon)\left(1-I_1^D\left(\frac{y}{x}\right)\right)+\frac{y}{y+x}I_1^D\left(\frac{y}{x}\right)\right]. \tag{40}$$

This equation may be obtained from Eq. (A6) of Ref. [7] by setting $b(y)$ equal to zero in that equation and by using Eq. (A8) of the same reference in order to eliminate the terms with coefficient $a^2(y)$. The angular integral $I_1(w)$ is defined to be

$$I_1^D(w)=(1+w)_2F_1(1,\epsilon;2-\epsilon;w),\quad 0\leqslant w\leqslant1 \tag{41}$$

$$I_1^D(w)=I_1^D(w^{-1}),\quad w\geqslant1. \tag{42}$$

In four dimensions the solution to Eq. (40) is given by a $\mathcal{Z}(x)$ having a simple power behavior while for $D<4$ this is clearly no longer the case. Nevertheless, it is possible to derive an integral representation of the solution of Eq. (40) by making use of the gauge covariance of this equation. We do so in Appendix B, with the result

$$\mathcal{Z}(x)=x^{\epsilon/2}2^{1-\epsilon}\Gamma(2-\epsilon)\int_0^\infty du\, u^{\epsilon-1}e^{-ru^2\epsilon}J_{2-\epsilon}(\sqrt{x}u), \tag{43}$$

where r is defined in Eq. (B4). Although this result is exact it is somewhat cumbersome to evaluate numerically because, for $\epsilon\to 0$, the oscillations in the integrand become increasingly important. For this reason we shall approximate the integrand in Eq. (40) by its IR and UV limits, as we did for the rainbow approximation (as before, this approximation is good to order ϵ). Using

$$I_1^D(w)=1+\frac{2}{2-\epsilon}w+O[w^2] \tag{44}$$

this approximation yields

$$\mathcal{Z}(x)=1+\frac{\alpha}{4\pi}\frac{(4\pi)^\epsilon}{\Gamma(2-\epsilon)}\xi\left[\frac{\epsilon}{2-\epsilon}\int_0^x dy\frac{y^{1-\epsilon}}{x^2}\mathcal{Z}(y)-\int_x^\infty dy\, y^{-\epsilon-1}\mathcal{Z}(y)\right]. \tag{45}$$

This may be converted to the differential equation

$$\mathcal{Z}''(x)+\frac{3}{x}\mathcal{Z}'(x)=\frac{\tilde{c}}{x^{1+\epsilon}}\left[\mathcal{Z}'(x)+2\frac{1-\epsilon}{x}\mathcal{Z}(x)\right] \tag{46}$$

where $\tilde{c}$ is defined to be

$$\tilde{c}=\frac{\alpha}{2\pi}\frac{(4\pi)^\epsilon}{\Gamma(3-\epsilon)}\xi \tag{47}$$

and the appropriate boundary conditions are

$$x^{2-\epsilon}\mathcal{Z}(x)|_{x=0}=0,\quad \mathcal{Z}(x)|_{x=\infty}=1. \tag{48}$$

[The IR boundary condition arises from the requirement that the integral in Eq. (45) needs to converge at its lower limit.] In order to solve Eq. (46), it is convenient to change variables to

$$z=\frac{\tilde{c}}{2-\epsilon}x^{-\epsilon}, \tag{49}$$

and to define

$$a=\frac{2}{\epsilon}-1 \tag{50}$$

so that the differential equation becomes

$$z\mathcal{Z}''-a(1-z)\mathcal{Z}'-a(a-1)\mathcal{Z}=0, \tag{51}$$

while the boundary conditions now are

$$z^{-a}\mathcal{Z}|_{z=\infty}=0,\quad \mathcal{Z}|_{z=0}=1. \tag{52}$$

This equation is essentially Kummer's equation (see Eq. 13.1.1 of Ref. [19]; we use the notation of that reference in the following). Its general solution may be expressed in terms of confluent hypergeometric functions, i.e.

$$\begin{aligned}\mathcal{Z}&=z^{a+1}e^{-az}[\tilde{C}_1M(a,a+2;az)+\tilde{C}_2U(a,a+2;az)]\\&=e^{-az}\{C_1[\gamma(a+1,-az)+az\,\gamma(a,-az)]+C_2[1+z]\}.\end{aligned} \tag{53}$$

The UV boundary condition is fulfilled if $C_2=1$ while C_1 is not fixed by the boundary conditions. Although Eq. (45) is solved by Eq. (53) for arbitrary C_1 we shall concentrate on the solution with $C_1=0$. The reason for this is that the solution to the unapproximated integral [Eq. (43)] vanishes at $x=0$ (see Appendix B) while the term multiplying C_1 in Eq. (53) diverges like $x^{2\epsilon-2}$ and is therefore unlikely to provide a good approximation to Eq. (40). Hence we obtain

$$\mathcal{Z}(x)=\left[1+\frac{\tilde{c}}{2-\epsilon}x^{-\epsilon}\right]\exp\left(-\frac{\tilde{c}}{\epsilon}x^{-\epsilon}\right). \tag{54}$$

Finally, the renormalized function $\mathcal{Z}(x,\mu^2)$ may be obtained from this by demanding that $\mathcal{Z}(\mu^2,\mu^2)=1$ so that the renormalized wave function renormalization becomes

$$\mathcal{Z}(x,\mu^2)=\frac{1+\frac{\tilde{c}}{2-\epsilon}x^{-\epsilon}}{1+\frac{\tilde{c}}{2-\epsilon}\mu^{-2\epsilon}}\frac{\exp\left(-\frac{\tilde{c}}{\epsilon}x^{-\epsilon}\right)}{\exp\left(-\frac{\tilde{c}}{\epsilon}\mu^{-2\epsilon}\right)}. \tag{55}$$

Only in the limit $D\to 4$ does this reduce to the usual power behaved function found in cut-off based work [Eq. (38)] while for $D<4$ it vanishes non-analytically at $x=0$. On the other hand, note that the solution to Eq. (40)—for finite ϵ—only goes to zero linearly in x. For the purpose of this paper this difference in the analytic behavior in the infrared does not concern us as for solutions which break chiral symmetry the infrared region is regulated by $M^2(x)$ so that we do not expect the chirally symmetric $\mathcal{Z}$ to be a good approximation in this region in any case.

B. Chiral symmetric breaking for the CP vertex

We shall now examine dynamical chiral symmetry breaking for the CP vertex in the absence of any explicit symmetry breaking by a nonzero bare mass, as before. Even for solutions exhibiting dynamical symmetry breaking, it is to be expected that the analytic result derived for $\mathcal{Z}(x)|_{M(x)=0}$ [Eq. (55)] remains valid as long as x is large compared to $(M(x)/\nu)^2$ and ϵ is sufficiently small. That this is indeed the case is illustrated in Fig. 3, where we show a typical example of $\mathcal{Z}^{-1}(x)$ for a solution which breaks chiral symmetry. In this figure, as well as in the rest of this section, we shall be dealing with the renormalized $\mathcal{Z}(x)$ and $M(x)$ instead of the unrenormalized quantities in the previous sections. This makes no essential difference to the physics of chiral symmetry breaking, although it of course effects the absolute scale of $\mathcal{Z}(x)$. For a discussion of the renormalization of the dimensionally regularized theory we refer the reader to Ref. [7].

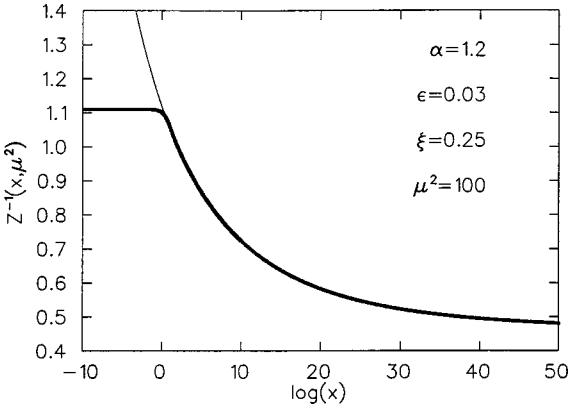

FIG. 3. A typical (inverse) wave function renormalization function $\mathcal{Z}^{-1}(x,\mu^2)$ corresponding to a chiral symmetry breaking solution. Note that the mass function $M(x)$ is of the same order as $\mathcal{Z}^{-1}(x,\mu^2)$ itself. Nevertheless, the analytical chirally symmetric solution of Sec. IV A (thin line) provides an excellent approximation (better than one part in a thousand for $x>\mu^2$) for $x>M(x)$.

The comparison to the analytic result in Fig. 3 provides a very convenient check on the numerics. Another check is provided by plotting the logarithm of $M(0)$ against the logarithm of the coupling. According to Eq. (5) this should be a straight line with gradient $1/2\epsilon$. As can be seen in Fig. 4 not only does one observe chiral symmetry breaking down to couplings as small as $\alpha=0.15$, the expected linear behavior is confirmed to quite high precision.

Although the numerics in $D<4$ dimensions are clearly under control, the extraction of the critical coupling (appropriate in four dimensions) has proven to be numerically quite difficult. From the discussion in Secs. II and III, we anticipate that the logarithm of the dynamical mass has the general form

$$\ln\left(\frac{M(0)}{\nu}\right)=\frac{1}{2\epsilon}\ln\left(\frac{\alpha}{\alpha_c}\right)+\ln(\bar{M}(0,\epsilon)) \tag{56}$$

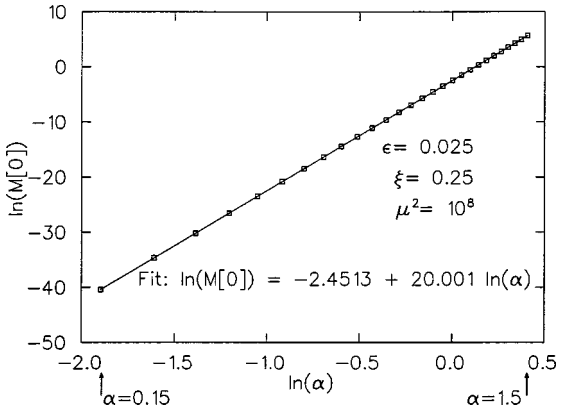

FIG. 4. The logarithm of the dynamical mass as a function of $\ln(\alpha)$ for $\epsilon=0.025$ (i.e. $1/2\epsilon=20$). The gauge parameter is fixed at $\xi=0.25$ and the renormalization point is $\mu^2=10^8$. The open squares are the numerical values while the solid line is a linear fit to these points. Note that the dependence on the coupling expected in Eq. (5) is reproduced to high precision.

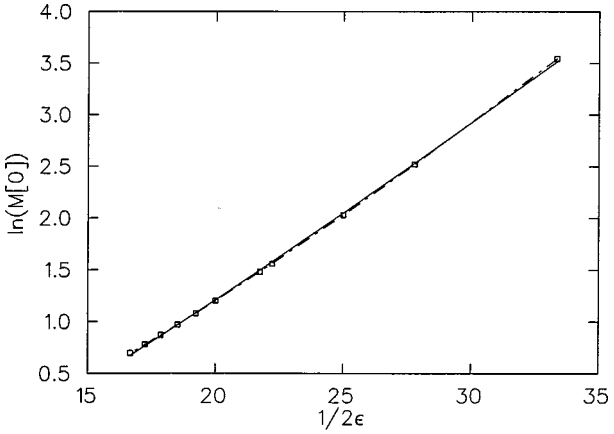

FIG. 5. The logarithm of the dynamical mass as a function of $1/2\epsilon$ for a coupling of $\alpha=1.20$. All other parameters are as in Fig. 4. The open squares are the numerical values while the two lines are fits (see main text).

where the last term is subleading as compared to the first as ϵ tends to zero. For sufficiently small ϵ, therefore, α_c is related to the gradient of $\ln(M(0))$ plotted against ϵ^{-1}.

In Fig. 5 we attempt to extract α_c in this way. The logarithm of $M(0)$ was evaluated for ϵ ranging from 0.03 down to $\epsilon=0.015$ for a fixed gauge $\xi=0.25$. The squares corresponds to a coupling constant $\alpha=1.2$, although some of the points at lower ϵ have actually been calculated at smaller α and then rescaled according to Eq. (5). At present we are unable, for these parameters, to decrease ϵ significantly further without a significant loss of numerical precision. (We also note in passing that it is quite difficult numerically to move away from small values of the gauge parameter; $\xi=20$, which, judging from Fig. 1, would not require a very high numerical accuracy for α_c, is unfortunately not an option.)

The two fits shown in Fig. 5 correspond to two different assumptions for the functional form of $\bar{M}(0,\epsilon)$, which is *a priori* unknown. The curves do indeed appear to be well approximated by a straight line, however we caution the reader that this does not allow an accurate determination of α_c as the gradient is essentially determined by the ''trivial'' dependence on $\log(\alpha)$ (more on this below). The solid line corresponds to the assumption that the leading term in $\bar{M}(0,\epsilon)$ has the same form as what we found in the rainbow approximation, i.e.

$$\ln\left(\frac{M(0)}{\nu}\right)=\frac{1}{2\epsilon}\ln\left(\frac{\alpha}{\alpha_c}\right)+c_1\left(\frac{1}{2\epsilon}\right)^{1/3}. \tag{57}$$

With this form the fit parameters α_c and c_1 are found to be

$$\alpha_c=0.966, \quad c_1=-1.15. \tag{58}$$

Indeed, the critical coupling is similar to what is found in cut-off based work (see Sec. II; in Ref. [15] the value was 0.9208 for $\xi=0.25$). At present it is difficult to make a more precise statement, let alone differentiate between the two curves plotted in Fig. 1, as α_c is quite strongly dependent on

the functional form assumed in Eq. (57). In fact, allowing an extra constant term on the right hand side of Eq. (57) reduces the critical coupling to 0.920 and the addition of yet a further term proportional to $\epsilon^{1/3}$ increases it again to 0.931. As these numbers appear to converge to something of the order of 0.92 or 0.93 one might think that α_c has been determined to this precision. However, it is not clear that the functional form suggested by the rainbow approximation should be taken quite this seriously. The dashed line in Fig. 5 corresponds to a fit where the power of ϵ of the subleading term has been left free, i.e.

$$\ln\left(\frac{M(0)}{\nu}\right)=\frac{1}{2\epsilon}\ln\left(\frac{\alpha}{\alpha_c}\right)+c_1\left(\frac{1}{2\epsilon}\right)^{c_2}. \tag{59}$$

The optimum fit assuming this form for $\ln(M(0))$ yields a power quite different to $\frac{1}{3}$ and a very much smaller α_c:

$$\alpha_c=0.825, \quad c_1=-0.801, \quad c_2=0.688. \tag{60}$$

To conclude this section, let us discuss why it is that the functional form of the subleading term $\bar{M}(0,\epsilon)$ appears to be rather important even if ϵ is already rather small. The reason for this is two-fold: most importantly, although the leading ϵ dependence of $\ln(M(0))$ is indeed ϵ^{-1}, the coefficient of this term (leaving out the trivial α dependence) is $\ln(\alpha_c)$. As α_c is rather close to 1 one therefore obtains a strong suppression of this leading term, increasing the relative importance of the subleading terms. In addition, it appears as if the numerical results favor a subleading term which is not as strongly suppressed (as a function of ϵ) as suggested by the rainbow approximation (i.e. the power of ϵ^{-1} of the subleading term appears to be closer to $\frac{2}{3}$ rather than $\frac{1}{3}$). This again increases the importance of the subleading terms.

V. CONCLUSIONS AND OUTLOOK

The primary purpose of this paper was to explore the phenomenon of dynamical chiral symmetry breaking through the use of Dyson-Schwinger equations with a regularization scheme which does not break the gauge covariance of the theory, namely dimensional regularization. It is necessary to do this as the cut-off based work leads to ambiguous results for the critical coupling of the theory precisely because of the lack of gauge covariance in those calculations. In particular, this should be kept in mind when using the expected gauge invariance of the critical coupling as a criterion for judging the suitability of a particular vertex.

To begin with, we have shown on dimensional grounds alone and for an arbitrary vertex, that in $D<4$ dimensions either a symmetry breaking solution does not exist at all (in which case, however, it would also not exist in $D=4$ dimensions) or it exists for all nonzero values of the coupling (in which case a chiral symmetry breaking solution exists in $D=4$ for $\alpha>\alpha_c$). For Dyson-Schwinger analyses employing the rainbow and CP vertices we have shown that it is the second of these possibilities which is realized. For these symmetry breaking solutions the limit to $D=4$ is necessarily discontinuous and so the extraction of the critical coupling of the theory (in 4 dimensions) is not as simple as in cut-off regularized work.

We next turned to an examination of symmetry breaking in the rainbow approximation in Landau gauge, both analytically and numerically. Indeed, for this vertex one could rewrite the (linearized) Dyson-Schwinger equation as a Schrödinger equation in 4 dimensions and appeal to standard results from elementary quantum mechanics to explicitly show that the theory always breaks chiral symmetry if $D<4$. We also showed how the usual critical coupling $\alpha_c=\pi/3$ may be extracted from the dimensionally regularized work.

We concluded this work with an examination of the CP vertex. By making use of the gauge covariance of the theory we derived an exact integral expression for the wave function renormalization function $\mathcal{Z}(p^2)$ of the chirally symmetric solution. Furthermore we obtained a compact expression for this quantity which is an excellent approximation to the true $\mathcal{Z}(p^2)$ even for solutions which break the chiral symmetry. Finally, we extracted the critical coupling corresponding to this vertex and found that, within errors, it agrees with the standard cut-off results.

In the future, we plan to increase the numerical precision with which we can extract this critical coupling for the CP vertex by an order of magnitude or so. The factor limiting the precision at present is that when solving the propagator's Dyson-Schwinger equation with the CP vertex by iteration the rate of convergence decreases dramatically as ϵ is decreased below $\epsilon\approx 0.015$. If this increase in precision can be attained it will enable one to make a meaningful comparison with the cut-off based results shown in Fig. 1.

In addition to the above, it would be interesting to extend the work described in this paper to unquenched QED. Chiral symmetry breaking in this theory has been much studied in four dimensions (employing a cut-off as a regulator; for a review, and references, see Sec. 10.9 of Ref. [3]) as well as in three dimensions (see Ref. [22] as well as Sec. 3 of Ref. [4]). In particular, for $D=3$ it is known that (at least in the $N_f\to\infty$ limit, with fixed αN_f) the theory has both an ultraviolet fixed point at $\bar{\alpha}=0$ and an infrared fixed point at $\bar{\alpha}\sim 1$. These fixed points will survive in $4-2\epsilon$ dimensions, however in this case (within the same approximation) the value of the running coupling at the infrared fixed point will go to zero like $\bar{\alpha}\sim\epsilon$. The issue of how chiral symmetry breaking is manifested in dimensionally regularized unquenched QED, without a cutoff, will then become closely connected to the possible existence of additional fixed points in the theory when going beyond the large N_f approximation. If there are none, full QED would be trivial and uninteresting without the introduction (even in $D\neq 4$ dimensions) of a new dimensionful scale signifying the onset of physics outside the theory.

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APPENDIX A: CHIRAL SYMMETRY BREAKING IN RAINBOW APPROXIMATION

In this appendix we show that the linearized version of Eq. (11), i.e.

$$M(p^2)=(e\nu^{\epsilon})^2(3-2\epsilon)\int\frac{d^Dk}{(2\pi)^D}\frac{M(k^2)}{k^2+m^2}\frac{1}{(p-k)^2}, \tag{A1}$$

has symmetry breaking solutions for all values of the coupling. Our aim here is to convert this equation to a Schrödinger-like equation, which we do by introducing the function

$$\psi(r)=\int\frac{d^Dk}{(2\pi)^D}\frac{e^{ikr}M(k^2)}{k^2+m^2}. \tag{A2}$$

With this definition we have

$$(-\Box+m^2)\psi(r)=\int\frac{d^Dk}{(2\pi)^D}e^{ikr}M(k^2) \tag{A3}$$

where $\Box$ is the D-dimensional Laplacian and so

$$(-\Box+m^2)\psi(r)=e^2\nu^{2\epsilon}(3-2\epsilon)\int\frac{d^Dp}{(2\pi)^D}e^{ipr}\times\int\frac{d^Dk}{(2\pi)^D}\frac{M(k^2)}{k^2+m^2}\frac{1}{(p-k)^2}. \tag{A4}$$

After shifting the integration variable $(p\to p+k)$ the last equation can be written in the form of a Schrödinger-like equation

$$H\psi(r)=-m^2\psi(r), \tag{A5}$$

where $H=-\Box+V(r)$ is the Hamiltonian, $E=-m^2$ plays the role of an energy and the potential $V(r)$ given by

$$V(r)=-e^2\nu^{2\epsilon}(3-2\epsilon)\int\frac{d^Dp}{(2\pi)^D}\frac{e^{ipr}}{p^2}=-\frac{\eta}{r^{D-2}}, \tag{A6}$$

where

$$\eta=\frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}e^2\nu^{2\epsilon}(3-2\epsilon). \tag{A7}$$

For $D=3$ the coefficient η is $2\nu\alpha$ while near $D=4$ it is $(3/\pi)\alpha\nu^{2\epsilon}$. It is well known from any standard course of quantum mechanics (see, for example, Ref. [23]) that potentials behaving as $1/r^s$ at infinity, with $s<2$, always support bound states (actually, an infinite number of them). In the present case this can be seen by considering the Schrödinger equation (A5) for zero energy, i.e. $E=0$. The s-symmetric wave function then satisfies the equation

$$\psi''+\frac{D-1}{r}\psi'+\frac{\eta}{r^{D-2}}\psi=0. \tag{A8}$$

The solution finite at the origin $r=0$ is

$$\psi(r)=\text{const}\times r^{\epsilon-1}J_{1/\epsilon-1}\left(\frac{\sqrt{\eta}}{\epsilon}r^{\epsilon}\right). \tag{A9}$$

The Bessel function in Eq. (A9) has an infinite number of zeros, which means that there is an infinite number of states with $E<0$.

Returning now to Eq. (A5), we can estimate the lowest energy eigenvalue variationally by using

$$\psi(r)=Ce^{-\kappa r} \tag{A10}$$

as a trial wave function. Here C is related to κ by demanding that ψ is normalized, i.e.

$$|C|^2=\frac{(2\kappa)^D}{\Omega_D\Gamma(D)}, \tag{A11}$$

where Ω_D is the volume of a D-dimensional sphere. Calculating the expectation value of the ''Hamiltonian'' H on the trial wave function in Eq. (A10) we find

$$E_0(\kappa^2)=\langle\psi|H|\psi\rangle=\kappa^2\left[1-\frac{2^{D-2}}{\Gamma(D)}\kappa^{D-4}\eta\right]. \tag{A12}$$

The minimum of the ''ground state energy'' in Eq. (A12), $E_0(\kappa)$, is reached at

$$\kappa^{4-D}=(D-2)\frac{2^{D-3}}{\Gamma(D)}\eta \tag{A13}$$

(for $D=3$ the parameter κ is $\nu\alpha$ while near $D=4$ it is $\nu[\alpha/(\pi/2)]^{1/2\epsilon}$) and is given by the expression

$$(E_0)_{\text{var}}=-m^2=\kappa^2\left(1-\frac{1}{\frac{D}{2}-1}\right)=\kappa^2\frac{D-4}{D-2}, \tag{A14}$$

where the 1 is the contribution from the kinetic energy while the $(D/2-1)^{-1}$ corresponds to the potential energy. For $D>2$ the potential is attractive and for $2<D<4$ it is always larger than the kinetic energy, so for this case we get dynamical symmetry breaking for any value of α. For example, for $D=3$, one obtains $E_0=-\kappa^2=-\nu^2\alpha^2$ which coincides precisely with the ground-state energy of the hydrogen atom (not surprisingly, as we have used the ground-state hydrogen wave function as our trial function). In this case the dynamical mass is $m=\nu\alpha$.

For D near 4, on the other hand, we obtain from Eq. (A14) that

$$m \simeq \nu(\epsilon)^{1/2}\left(\frac{\alpha}{\pi/2}\right)^{1/2\epsilon}. \tag{A15}$$

This is of the general form anticipated in Sec. II, with $\alpha_c = \pi/2$. Indeed, for $D=4$, the Schrödinger equation (A5) becomes an equation with the singular potential

$$V(r) = -\frac{\eta}{r^2}, \quad \eta = \frac{\alpha}{\pi/3}. \tag{A16}$$

Again, it is known from standard quantum mechanics [24] that the spectrum of bound states for such a potential depends on the strength η of the potential: it has an infinite number of bound states with $E<0$ if $\eta>1$ and bound states are absent if $\eta<1$. Thus, the true critical value for the coupling is expected to be $\alpha_c=\pi/3$ instead of the $\alpha_c=\pi/2$ obtained with the help of the variational method (which made use of the exponential *Ansatz* for the wave function and thus only gave an upper bound for the energy eigenvalue).

APPENDIX B: CHIRALLY SYMMETRIC QED FROM THE LANDAU-KHALATNIKOV TRANSFORMATION

Because the CP vertex in the chirally symmetric phase of QED is gauge-covariant [14] it is possible to derive an integral representation of the wave function renormalization function $\mathcal{Z}(x)$ [see Eq. (40)] from the Landau-Khalatnikov transformation [25]. This transformation relates the coordinate space propagator $\tilde{S}^{\xi}(u)$ in one gauge to the propagator in a different gauge. Specifically, with covariant gauge fixing, we have

$$\tilde{S}^{\xi}(u) = e^{4\pi\alpha\nu^{2\epsilon}[\Delta(0)-\Delta(u)]}\tilde{S}^{\xi=0}(u) \tag{B1}$$

where $\Delta(u)$ is essentially the Fourier transform of the gauge-dependent part of the photon propagator, i.e.

$$\Delta(u) = -\xi\int \frac{d^Dk}{(2\pi)^D}\frac{e^{-ik\cdot u}}{k^4}. \tag{B2}$$

Specifically, we obtain

$$\tilde{S}^{\xi}(u) = e^{-r(\nu u)^{2\epsilon}}\tilde{S}^{\xi=0}(u) \tag{B3}$$

where

$$r = -\frac{\alpha}{4\pi}\Gamma(-\epsilon)(\pi)^{\epsilon}\xi. \tag{B4}$$

Substituting the coordinate-space propagator in Landau gauge, i.e.

$$\tilde{S}^{\xi=0}(u) = \int \frac{d^Dp}{(2\pi)^D}\frac{e^{ip\cdot u}}{\not p} = \frac{i}{2\pi^{D/2}}\Gamma\left(\frac{D}{2}\right)\frac{\not u}{u^D}, \tag{B5}$$

and carrying out the inverse Fourier transform of Eq. (B3) one obtains the wave function renormalization function in an arbitrary gauge, namely

$$\begin{aligned}\mathcal{Z}(x) &= -\frac{i}{2\pi^{D/2}}\Gamma\left(\frac{D}{2}\right)\int d^Du e^{ip\cdot u}\frac{p\cdot u}{u^D}e^{-r(\nu u)^{2\epsilon}} \\ &= x^{\epsilon/2}2^{1-\epsilon}\Gamma(2-\epsilon)\int_0^{\infty} du\, u^{\epsilon-1}e^{-ru^{2\epsilon}}J_{2-\epsilon}(\sqrt{x}u).\end{aligned} \tag{B6}$$

Note that for small x this function vanishes:

$$\mathcal{Z}(x) = \frac{\Gamma\left(\frac{1}{\epsilon}\right)}{4\epsilon(2-\epsilon)}r^{-1/\epsilon}x + O(x^2). \tag{B7}$$

It may be checked explicitly that Eq. (B6) is indeed a solution to Eq. (40) for arbitrary D by making use of the expansion of Eq. (B6) around $x^{-\epsilon}=0$. To be more precise, consider the RHS of Eq. (40) upon insertion of the power y^{δ} in the place of $\mathcal{Z}(y)$. Note that the integral converges only if $\epsilon>\delta>\epsilon-2$. After some work the result is that the RHS of Eq. (40) becomes

$$1+\tilde{c}\frac{2-\epsilon}{2}x^{\delta-\epsilon}\left[-(1+\delta-\epsilon)\frac{\Gamma(2-\epsilon)}{\Gamma(\epsilon)}\times\sum_{n=-\infty}^{\infty}\frac{\Gamma(\epsilon+n)}{\Gamma(2-\epsilon+n)}\frac{1}{n-\delta+\epsilon}\right]. \tag{B8}$$

For $\epsilon<1$ this may simplified further by applying Dougall's formula (Eq. 1.4.1 in [26]) which, in this case, reduces to

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}\frac{\Gamma(\epsilon+n)}{\Gamma(2-\epsilon+n)}\frac{1}{n-\delta+\epsilon} \\ &\quad = \frac{\pi^2}{\sin(\pi\epsilon)\sin(\pi[\epsilon-\delta])}\frac{1}{\Gamma(1-\delta)\Gamma(2+\delta-2\epsilon)}.\end{aligned} \tag{B9}$$

Using this result, Eq. (B8) becomes

$$1-\tilde{c}\frac{2-\epsilon}{2}x^{\delta-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)\Gamma(\epsilon-\delta)\Gamma(2+\epsilon-\delta)}{\Gamma(2+\delta-2\epsilon)\Gamma(1-\delta)}. \tag{B10}$$

Note that, as opposed to the integral representation Eq. (40), this expression is defined for δ outside the range $\epsilon>\delta>\epsilon-2$ and so we may use it as an analytical continuation of the integral. Furthermore, note that this last expression vanishes for integer $\delta\geqslant 1$ hence we cannot obtain a simple power expansion around $x=0$ for $\mathcal{Z}(x)$ in this way.

On the other hand, an expansion in powers of $x^{-\epsilon}$ is possible. If we seek a solution of the form

$$\mathcal{Z}(x)=\sum_{n=0}^{\infty} c_n x^{-n\epsilon}, \tag{B11}$$

we may equate the coefficients of equal powers of $x^{-\epsilon}$ after inserting the series (B11) into both sides of Eq. (40). This way we obtain the recurrence relation for the coefficients $c_n(c_0=1)$ as

$$\frac{c_{n+1}}{c_n}=\frac{\tilde{c}}{2(n+1)}\Gamma(-\epsilon)\Gamma(3-\epsilon) \times\frac{\Gamma(1+\epsilon n+\epsilon)\Gamma(2-\epsilon-\epsilon n)}{\Gamma(2-2\epsilon-\epsilon n)\Gamma(1+\epsilon n)}. \tag{B12}$$

This may be solved leading to

$$c_n=\left[\frac{\tilde{c}}{2}\Gamma(-\epsilon)\Gamma(3-\epsilon)\right]^n\frac{\Gamma(2-\epsilon)\Gamma(1+n\epsilon)}{\Gamma(2-\epsilon-n\epsilon)n!}, \tag{B13}$$

so that finally we obtain

$$\mathcal{Z}(x)=\Gamma(2-\epsilon)\sum_{n=0}^{\infty}\frac{\Gamma(1+n\epsilon)}{\Gamma(2-\epsilon-n\epsilon)n!} \times\left[\frac{\tilde{c}\Gamma(-\epsilon)\Gamma(3-\epsilon)}{2}x^{-\epsilon}\right]^n \tag{B14}$$

as the series expansion of the solution to Eq. (40). The reader may check that this coincides precisely with the corresponding expansion of the solution obtained via the Landau-Khalatnikov transformations [Eq. (B6)]. The latter may be obtained by changing the variable of integration from u to $u/\sqrt{x}$, expanding the exponential in the integrand and making use of the standard integral

$$\int_0^{\infty} x^{\alpha-1}J_{\nu}(cx)dx=2^{\alpha-1}c^{-\alpha}\frac{\Gamma\left(\frac{\alpha+\nu}{2}\right)}{\Gamma\left(1+\frac{\nu-\alpha}{2}\right)}. \tag{B15}$$

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