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# Heavy-quark axial charges to nonleading order

S. D. Bass,[1] R. J. Crewther,[2,3] F. M. Steffens,[4] and A. W. Thomas[2]
[1]Institut für Theoretische Physik, Universität Innsbruck, Technikerstrasse 25, A 6020 Innsbruck, Austria
[2]Department of Physics and Mathematical Physics and Centre for the Subatomic Structure of Matter (CSSM), University of Adelaide, Adelaide SA 5005, Australia
[3]Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, South Road, Durham DH1 3LE, United Kingdom
[4]Instituto de Fisica Teorica, Rua Pamplona 145, 01405-900 Sao Paulo SP, Brazil

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We combine Witten's renormalization group with the matching conditions of Bernreuther and Wetzel to calculate at next-to-leading order the complete heavy-quark contribution to the neutral-current axial-charge measurable in neutrino-proton elastic scattering. Our results are manifestly renormalization group invariant.

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This paper announces results for the next-to-leading-order (NLO) heavy-quark corrections to the axial charge $g_A^{(Z)}$ for protons to couple to the weak neutral current

$$J_{\mu 5}^{Z}=\frac{1}{2}\left\{\sum_{q=u,c,t}-\sum_{q=d,s,b}\right\}\bar{q}\gamma_\mu\gamma_5 q. \tag{1}$$

The calculation is performed by decoupling heavy quarks $h=t,b,c$ sequentially, i.e. one at a time. An extension to simultaneous decoupling of $t,b,c$ quarks is foreshadowed in our concluding remarks.

The charge $g_A^{(Z)}$ receives contributions from both light $u,d,s$ and heavy $c,b,t$ quarks,

$$2g_A^{(Z)}=(\Delta u-\Delta d-\Delta s)+(\Delta c-\Delta b+\Delta t) \tag{2}$$

where $\Delta q$ refers to expectation value $\langle p,s|\bar{q}\gamma_\mu\gamma_5 q|p,s\rangle=2m_p s_\mu \Delta q$ for a proton of spin $s_\mu$ and mass $m_p$. It governs parity-violating effects due to $Z^0$ exchange at low energies in elastic $\nu p$ and $\bar{\nu}p$ scattering [1,2] or in light atoms [3,4]. A definitive measurement of $\nu p$ elastic scattering may be possible using the MiniBooNE setup at Fermilab [5].

Once heavy-quark corrections [2,6,7] have been taken into account, $g_A^{(Z)}$ is related (modulo the issue of $\delta$-function terms at $x=0$ [8]) to the flavor-singlet axial charge, defined scale invariantly and extracted from polarized deep inelastic scattering:

$$g_A^{(0)}|_{\mathrm{inv}}=0.2-0.35. \tag{3}$$

The small value of this quantity has inspired vast experimental and theoretical activity to understand the spin structure of the proton [9]. As a result, new experiments are being planned to map out the spin-flavor structure of the proton. These include polarized proton-proton collisions at the BNL Relativistic Heavy Ion Collider (RHIC) [10], semi-inclusive polarized deep inelastic scattering, and polarized $ep$ collider studies [11]. Full NLO analyses are essential for a consistent interpretation of these experiments.

Many techniques for decoupling a single heavy quark are available. We rely on Witten's method [12], where the renormalization scheme is mass independent and improved Callan-Symanzik equations [13] can be exploited. In such schemes, the decoupling of heavy particles required by the Appelquist-Carrazone theorem [14] is not manifest. However, correct decoupling is ensured by applying the matching conditions of Bernreuther and Wetzel [15]; these relate coupling constant, mass and operator normalizations before and after the decoupling of a heavy quark. The advantages of this approach are its rigor and the fact that the final results are expressed in terms of renormalization group (RG) invariants. These invariants are Witten-style running couplings $\tilde{\alpha}_h$, one for each heavy quark $h=t,b,c$, and axial charges for nucleons in the residual theory with three light flavors.

We find that, when first $t$, then $b$, and finally $c$ are decoupled from Eq. (2), the full NLO result is

$$2g_A^{(Z)}=(\Delta u-\Delta d-\Delta s)_{\mathrm{inv}}+\mathcal{P}(\Delta u+\Delta d+\Delta s)_{\mathrm{inv}}+O(m_{t,b,c}^{-1}) \tag{4}$$

where $\mathcal{P}$ is a polynomial in the running couplings $\tilde{\alpha}_h$,

$$\mathcal{P}=\frac{6}{23\pi}(\tilde{\alpha}_b-\tilde{\alpha}_t)\left\{1+\frac{125663}{82800\pi}\tilde{\alpha}_b+\frac{6167}{3312\pi}\tilde{\alpha}_t-\frac{22}{75\pi}\tilde{\alpha}_c\right\}-\frac{6}{27\pi}\tilde{\alpha}_c-\frac{181}{648\pi^2}\tilde{\alpha}_c^2+O(\tilde{\alpha}_{t,b,c}^3) \tag{5}$$

and $(\Delta q)_{\mathrm{inv}}$ denotes the scale-invariant version of $\Delta q$ defined in the following way.

Let $\alpha_f=g_f^2/4\pi$ and $\beta_f(\alpha_f)$ be the gluon coupling and beta function for $\overline{\mathrm{MS}}$ renormalized quantum chromodynamics (QCD) with $f$ flavors and $N_c=3$ colors, and let $\gamma_f(\alpha_f)$ be the gamma function for the singlet current

$$(\bar{u}\gamma_\mu\gamma_5 u+\bar{d}\gamma_\mu\gamma_5 d+\cdots)_f=\sum_{k=1}^{f}(\bar{q}_k\gamma_\mu\gamma_5 q_k)_f. \tag{6}$$

A scale-invariant current $(S_{\mu 5})_f$ is obtained when Eq. (6) is multiplied by

$$E_f(\alpha_f)=\exp\int_0^{\alpha_f}dx\,\frac{\gamma_f(x)}{\beta_f(x)}. \tag{7}$$

 

Up to $O(m_h^{-1})$ corrections, the invariant singlet charge (3) is given by

$$g_A^{(0)}|_{\mathrm{inv}} = E_3(\alpha_3)(\Delta u + \Delta d + \Delta s)_3 = (\Delta u + \Delta d + \Delta s)_{\mathrm{inv}}. \tag{8}$$

Flavor-dependent, scale-invariant axial charges $\Delta q|_{\mathrm{inv}}$ such as

$$\Delta s|_{\mathrm{inv}} = \tfrac{1}{3}\,(g_A^{(0)}|_{\mathrm{inv}} - g_A^{(8)}) \tag{9}$$

can then be obtained from linear combinations of Eq. (8) and

$$g_A^{(3)} = \Delta u - \Delta d = (\Delta u - \Delta d)_{\mathrm{inv}}$$

$$g_A^{(8)} = \Delta u + \Delta d - 2\Delta s = (\Delta u + \Delta d - 2\Delta s)_{\mathrm{inv}}. \tag{10}$$

Here $g_A^{(3)} = 1.267 \pm 0.004$ is the isotriplet axial charge measured in neutron beta-decay, and $g_A^{(8)} = 0.58 \pm 0.03$ is the octet charge measured independently in hyperon beta decay. Taking $\tilde{\alpha}_t = 0.1$, $\tilde{\alpha}_b = 0.2$ and $\tilde{\alpha}_c = 0.35$ in Eq. (5), we find a small heavy-quark correction factor $\mathcal{P} = -0.02$, with LO terms dominant.

Our results extend and make more precise the well known work of Collins, Wilczek and Zee [6] and Kaplan and Manohar [2], where heavy-quark effective theory was used to estimate $g_A^{(Z)}$ in leading order (LO) for sequential decoupling of $t,b$ and $t,b,c$ respectively. Our analysis is also influenced by a discussion of [6] by Chetyrkin and Kühn [16], who considered some aspects of NLO decoupling of the $t$ quark from the neutral current and in particular, the requirement that the result be scale invariant. Related work has been done on heavy-quark production in polarized deep inelastic scattering using the QCD parton model [17] and in high-energy polarized $\gamma p$ and $pp$ at NLO [18].

The plan of this paper is as follows. First is a brief review of Witten's application of improved Callan-Symanzik equations [13] to the decoupling of a heavy quark in mass-independent renormalization schemes. Next, we combine it with matching conditions [15] to deal with next-to-leading-order (NLO) calculations involving axial-vector currents. Following is then a direct derivation of Eq. (5) from Eq. (1) for the neutral current. Our concluding remarks indicate the result of extending Eq. (5) to simultaneous decoupling of $t,b,c$—done not only for numerical reasons, but also to check that the $t,b$ contributions cancel for $m_t = m_b$.

We begin by considering mass-independent schemes, such as the modified minimal subtraction scheme $(\overline{\mathrm{MS}})$, where renormalized masses behave like coupling constants. This key property is exploited in Witten's method.

Let $\mu$ be the scale used to define dimensional regularization and renormalization. Then the $\overline{\mathrm{MS}}$ scale is

$$\bar{\mu} = \mu\sqrt{4\pi}e^{-\gamma/2}, \quad \gamma = 0.5772\ldots. \tag{11}$$

We choose the same scale $\bar{\mu}$ irrespective of the number of flavors $f$ being considered, and so hold $\bar{\mu}$ fixed as the heavy quarks (masses $m_h$) decouple:

$$F \to f \text{ flavors}, \quad m_h \to \infty.$$

Also held fixed in this limit are the coupling $\alpha_f$ and light-quark masses $m_{lf}$ of the *residual* $f$-flavor theory, and all momenta $\mathbf{p}$. Feynman diagrams for amplitudes

$$\mathcal{A}_F = \mathcal{A}_F(\mathbf{p}, \bar{\mu}, \alpha_F, m_{lF}, m_h) \tag{12}$$

give rise to power series in $m_h^{-1}$ modified by polynomials in $\ln(m_h/\bar{\mu})$. We consider just the leading power $\tilde{\mathcal{A}}_F$:

$$\mathcal{A}_F = \tilde{\mathcal{A}}_F\{1 + O(1/m_h)\}. \tag{13}$$

As $m_h$ tends to infinity, logarithms in $\tilde{\mathcal{A}}_F$ can be produced by any 1PI (one-particle irreducible) subgraph which contains at least one heavy-quark propagator and whose divergence by power counting is at least logarithmic. The effect is equivalent to shrinking all contributing 1PI parts of each diagram to a point. This means [14] that the $F$-flavor amplitudes $\tilde{\mathcal{A}}_F$ are the same as amplitudes $\mathcal{A}_f$ in the residual $f$-flavor theory, apart from $m_h$-dependent renormalizations of the coupling constant, light masses, and amplitudes:

$$\tilde{\mathcal{A}}_F(\mathbf{p}, \bar{\mu}, \alpha_F, m_{lF}, m_h) = \sum_{\mathcal{A}'} \mathcal{Z}_{\mathcal{A}\mathcal{A}'}(\alpha_F, m_h/\bar{\mu})\mathcal{A}'_f(\mathbf{p}, \bar{\mu}, \alpha_f, m_{lf}) \tag{14}$$

$$\alpha_f = \alpha_f(\alpha_F, m_h/\bar{\mu}), \quad m_{lf} = m_{lF}D(\alpha_F, m_h/\bar{\mu}). \tag{15}$$

Eventually, we will have to invert Eq. (15), i.e. use $\alpha_f$ and $m_{lf}$ as dependent variables instead of $\alpha_F$ and $m_{lF}$, because we hold $\alpha_f$ and $m_{lf}$ fixed as $m_h \to \infty$.

For any number of flavors $f$ (including $F$), let

$$\mathcal{D}_f = \bar{\mu}\frac{\partial}{\partial\bar{\mu}} + \beta_f(\alpha_f)\frac{\partial}{\partial\alpha_f} + \delta_f(\alpha_f)\sum_{k=1}^{f} m_{kf}\frac{\partial}{\partial m_{kf}} \tag{16}$$

be the corresponding Callan-Symanzik operator. Then the amplitude $\mathcal{A}_F$ and hence its leading power $\tilde{\mathcal{A}}_F$ both satisfy an $F$-flavor improved Callan-Symanzik equation:

$$\{\mathcal{D}_F + \gamma_F(\alpha_F)\}\tilde{\mathcal{A}}_F = 0. \tag{17}$$

In general, both $\gamma_F$ and $\mathcal{Z} = (\mathcal{Z}_{\mathcal{A}\mathcal{A}'})$ are matrices.

If we substitute Eq. (14) in Eq. (17) and change variables,

$$\mathcal{D}_F = \bar{\mu}\frac{\partial}{\partial\bar{\mu}} + (\mathcal{D}_F\alpha_f)\frac{\partial}{\partial\alpha_f} + \sum_{k=1}^{f}(\mathcal{D}_F m_{kf})\frac{\partial}{\partial m_{kf}} \tag{18}$$

the result is an improved Callan-Symanzik equation for each residual amplitude,

$$\{\mathcal{D}_f + \gamma_f(\alpha_f)\}\mathcal{A}_f = 0 \tag{19}$$

where the functions [12,15]

$$\beta_f(\alpha_f) = \mathcal{D}_F\alpha_f \tag{20}$$

$$\delta_f(\alpha_f)=\mathcal{D}_F \ln m_l \tag{21}$$

$$\gamma_f(\alpha_f)=\mathcal{Z}^{-1}[\gamma_F(\alpha_F)+\mathcal{D}_F]\mathcal{Z} \tag{22}$$

depend on $\alpha_f$ *alone*. The lack of $m_l$ dependence of the renormalization factors in Eqs. (14) and (15) ensures mass-independent renormalization for the residual theory.

Although these equations hold for any $f<F$, their practical application is straightforward only when heavy quarks are decoupled one at a time. So we set $F=f+1$, where just one quark $h$ is heavy. Then it is convenient to introduce a running coupling [12]

$$\tilde{\alpha}_h=\tilde{\alpha}_h(\alpha_F,\ln(m_h/\overline{\mu})) \tag{23}$$

associated with the $\overline{\mathrm{MS}}_F$ renormalized mass $m_h$:

$$\ln(m_h/\overline{\mu})=\int_{\alpha_F}^{\tilde{\alpha}_h} dx[1-\delta_F(x)]/\beta_F(x). \tag{24}$$

It satisfies the constraints

$$\tilde{\alpha}_h(\alpha_F,0)=\alpha_F,\tilde{\alpha}_h(\alpha_F,\infty)=0 \tag{25}$$

the latter being a consequence of the asymptotic freedom of the $F$-flavor theory ($F\leqslant 16$). Also, Eqs. (16), (20) and (24) imply that $\tilde{\alpha}_h$ is renormalization group (RG) invariant:

$$\mathcal{D}_F\tilde{\alpha}_h=0. \tag{26}$$

Witten's solution of Eq. (22) for the matrix $\mathcal{Z}$ is

$$\mathcal{Z}(\alpha_F,m_h/\mu)=\exp\left\{\int_{\alpha_F}^{\tilde{\alpha}_h} dx\frac{\gamma_F(x)}{\beta_F(x)}\right\}_{\mathrm{ord}}\mathcal{Z}(\tilde{\alpha}_h,1)$$
$$\times\exp-\left\{\int_{\alpha_f}^{\alpha_f(\tilde{\alpha}_h,1)} dx\frac{\gamma_f(x)}{\beta_f(x)}\right\}_{\mathrm{ord}} \tag{27}$$

where "ord" indicates $x$-ordering of matrix integrands in the exponentials. Note that it is the *relative* scaling between the initial and residual theories which matters.

For our NLO calculation, we need the formulas

$$\beta_f(x)=-\frac{x^2}{3\pi}\left(\frac{33}{2}-f\right)-\frac{x^3}{12\pi^2}(153-19f)+O(x^4)$$

$$\gamma_f(x)=\frac{x^2}{\pi^2}f+\frac{x^3}{36\pi^3}(177-2f)f+O(x^4)$$

$$\delta_f(x)=-\frac{2x}{\pi}+O(x^2) \tag{28}$$

where $\gamma_f$ refers to the $f$-flavor singlet current (6) and includes the three-loop term found by Larin [19] and Chetyrkin and Kühn [16].

Our matching procedure amounts to evaluating to NLO accuracy the quantities $\tilde{\alpha}_h$, $\alpha_f(\tilde{\alpha}_h,1)$ and $\mathcal{Z}(\tilde{\alpha}_h,1)$ in Eq. (27), such that the answers depend on $\alpha_f$ and not $\alpha_F$.

Bernreuther and Wetzel [15] applied the Appelquist-Carrazone decoupling theorem [14] to the gluon coupling constant $\alpha_Q^{\mathrm{MO}}$ renormalized at space-like momentum $Q$,

$$\alpha_Q^{\mathrm{MO}}\big|_{\text{with }h}=\alpha_Q^{\mathrm{MO}}\big|_{\text{no }h}+O(m_h^{-1}) \tag{29}$$

and compared calculations of $\alpha_Q^{\mathrm{MO}}$ in the $F=f+1$ and $f$-flavor $\overline{\mathrm{MS}}$ theories. This reduces to a determination of the leading power of the one-$h$-loop $\overline{\mathrm{MS}}_F$ gluon self-energy. The result is a matching condition

$$\alpha_F^{-1}-\alpha_f^{-1}=C_{\mathrm{LO}}\ln(m_h/\overline{\mu})+C_{\mathrm{NLO}}+O(\alpha_f,m_h^{-1}) \tag{30}$$

with $\alpha_f$-independent LO and NLO coefficients given by

$$C_{\mathrm{LO}}=\frac{1}{3\pi},\quad C_{\mathrm{NLO}}=0. \tag{31}$$

As a result, we find

$$\alpha_f(\tilde{\alpha}_h,0)=\tilde{\alpha}_h+O(\tilde{\alpha}_h^3)\underset{\mathrm{NLO}}{=}\tilde{\alpha}_h. \tag{32}$$

Bernreuther and Wetzel showed that it is possible to deduce *all* LO and NLO terms in Eq. (30) from Eq. (31) and $\beta_f$ and $\delta_f$ in Eq. (28). We have done the calculation explicitly:

$$\alpha_{f+1}^{-1}\underset{\mathrm{NLO}}{=}\alpha_f^{-1}+\frac{1}{3\pi}\ln\frac{m_h}{\overline{\mu}}+c_f\ln\left[1+\frac{\alpha_f}{3\pi}\ln\frac{m_h}{\overline{\mu}}\right]$$
$$+d_f\ln\left[1+\frac{\alpha_f}{3\pi}\left(\frac{33}{2}-f\right)\ln\frac{m_h}{\overline{\mu}}\right]$$
$$c_f=\frac{142-19f}{2\pi(31-2f)},\quad d_f=\frac{57+16f}{2\pi(33-2f)(31-2f)}. \tag{33}$$

From Eq. (24), we have also found $\tilde{\alpha}_h$ in NLO,

$$\tilde{\alpha}_h^{-1}\underset{\mathrm{NLO}}{=}\alpha_f^{-1}+\frac{1}{3\pi}\left(\frac{33}{2}-f\right)\ln\frac{\overline{m}_h}{\overline{\mu}}+\frac{153-19f}{2\pi(33-2f)}$$
$$\times\ln\left[1+\frac{\alpha_f}{3\pi}\left(\frac{33}{2}-f\right)\ln\frac{m_h}{\overline{\mu}}\right] \tag{34}$$

where $\overline{m}_h$ is Witten's RG invariant mass:

$$\overline{m}_h=m_h\exp\int_{\alpha_F}^{\tilde{\alpha}_h} dx\,\delta_F(x)/\beta_F(x). \tag{35}$$

If desired, $\ln(\overline{m}_h/\overline{\mu})$ can be eliminated by substituting

$$\ln\frac{\overline{m}_h}{\overline{\mu}}\underset{\mathrm{LO}}{=}\ln\frac{m_h}{\overline{\mu}}-\frac{12}{31-2f}\ln\left[1+\frac{\alpha_f}{3\pi}\left(\frac{31}{2}-f\right)\ln\frac{m_h}{\overline{\mu}}\right]. \tag{36}$$

Therefore the asymptotic formula for $\tilde{\alpha}_h$ as $m_h\to\infty$ is

$$\tilde{\alpha}_h \sim 3\pi \Bigg/ \left\{ \left( \frac{33}{2} - f \right) \ln \frac{m_h}{\bar{\mu}} + k_f \ln \ln \frac{m_h}{\bar{\mu}} + O(1) \right\}$$

$$k_f = \frac{3(153-19f)}{2(33-2f)} - \frac{6(33-2f)}{31-2f}. \qquad (37)$$

To find the matrix $\mathcal{Z}(\tilde{\alpha}_h,1)$ in NLO, we need a matching condition for the $\overline{\text{MS}}$ amplitude $\Gamma_{\mu 5}$ for $\bar{h}\gamma_\mu\gamma_5 h$ to couple to a light quark $l$. We have calculated the leading power due to the two-loop diagram  :

$$\Gamma_{\mu 5} = \left( \frac{\alpha_F}{\pi} \right)^2 \gamma_\mu \gamma_5 \left( \ln \frac{m_h}{\bar{\mu}} + \frac{1}{8} \right) + O(\alpha_F^3, m_h^{-1}). \qquad (38)$$

Consequently, there is a NLO term $\tilde{\alpha}_h^2/8\pi^2$ in $\mathcal{Z}(\tilde{\alpha}_h,1)$ for $\bar{h}\gamma_\mu\gamma_5 h$ to produce $\bar{l}\gamma_\mu\gamma_5 l$ as $m_h \to \infty$.

Now we consider the special case where heavy quarks are decoupled from the weak neutral axial current. Let us adopt the shorthand notation $q_f$ for $\overline{\text{MS}}$ currents $(\bar{q}\gamma_\mu\gamma_5 q)_f$ in the $f$-flavor theory, e.g. the neutral current $J_{\mu 5}^{(\text{Z})}$ and the scale-invariant singlet current $(S_{\mu 5})_f$:

$$J^{\text{Z}} = \tfrac{1}{2}(t-b+c-s+u-d)_6 \qquad (39)$$

$$S_f = E_f(\alpha_f)(u+d+s+\cdots)_f. \qquad (40)$$

We begin by decoupling the $t$ quark. Because of

$$(c-s+u-d)_6 = (c-s+u-d)_5 + O(1/m_t) \qquad (41)$$

we see that Eq. (27) is nontrivial only for

$$(t-b)_6 = \mathcal{Z}_{6\to 5}(u+d+s+c+b)_5 + \tfrac{1}{5}(u+d+s+c-4b)_5 + O(1/m_t). \qquad (42)$$

Since $(t-b)_6$ is scale invariant, we have $\gamma_F = 0$ in Eq. (27):

$$\mathcal{Z}_{6\to 5}(\alpha_6, m_t/\bar{\mu}) \underset{\text{NLO}}{=} \mathcal{Z}_{6\to 5}(\tilde{\alpha}_t,1)\exp - \int_{\alpha_5}^{\tilde{\alpha}_t} dx \frac{\gamma_5(x)}{\beta_5(x)}. \qquad (43)$$

The operator matching condition (38) corresponds to

$$t_6 = \frac{\alpha_6^2}{\pi^2} \left( \ln \frac{m_t}{\bar{\mu}} + \frac{1}{8} \right) (u+d+s+c+b)_5 + O(\alpha_6^3, m_t^{-1}) \qquad (44)$$

and so we conclude:

$$\mathcal{Z}_{6\to 5}(\tilde{\alpha}_t,1) = -\tfrac{1}{5} + (8\pi^2)^{-1}\tilde{\alpha}_t^2 + O(\tilde{\alpha}_t^3). \qquad (45)$$

Equation (43) is to be expanded about $\tilde{\alpha}_t \sim 0$ with $\alpha_5$ held fixed. In that limit, the exponential tends to the constant factor $E_5(\alpha_5)$ of Eq. (7). This factor combines with the singlet current in Eq. (42) to form the scale-invariant operator $S_5$, as required by $\text{RG}_{f=5}$ invariance. The full NLO result is then obtained by writing

$$(t-b)_6 \underset{\text{NLO}}{=} \mathcal{Z}_{6\to 5}(\tilde{\alpha}_t,1)\exp\left\{ -\int_0^{\tilde{\alpha}_t} dx \frac{\gamma_5(x)}{\beta_5(x)} \right\} S_5 + \tfrac{1}{5}(u+d+s+c-4b)_5 \qquad (46)$$

and expanding in $\tilde{\alpha}_t$, keeping all quadratic terms:

$$(t-b)_6 = \left\{ -\frac{1}{5} - \frac{6}{23}\frac{\tilde{\alpha}_t}{\pi}\left( 1 + \frac{6167}{3312}\frac{\tilde{\alpha}_t}{\pi} \right) + O(\tilde{\alpha}_t^3) \right\} S_5 + \tfrac{1}{5}(u+d+s+c-4b)_5 + O(1/m_t). \qquad (47)$$

Next we decouple the $b$ quark. Here, it is natural to define five-flavor quantities $\tilde{\alpha}_{b_5}$ and $\bar{m}_{b_5}$ analogous to the six-flavor running coupling $\tilde{\alpha}_t$ and mass $\bar{m}_t$ for the top quark:

$$\ln \frac{m_{b5}}{\bar{\mu}} = \int_{\alpha_5}^{\tilde{\alpha}_{b_5}} dx \frac{1-\delta_5(x)}{\beta_5(x)},$$

$$\ln \frac{\bar{m}_{b_5}}{m_{b_5}} = \int_{\alpha_5}^{\tilde{\alpha}_{b_5}} dx \frac{\delta_5(x)}{\beta_5(x)}. \qquad (48)$$

Equations (20) and (21) imply that $\tilde{\alpha}_{b_5}$ and $\bar{m}_{b_5}$ are both $\text{RG}_{f=5}$ and $\text{RG}_{f=6}$ invariant

$$\mathcal{D}_5\tilde{\alpha}_{b_5} = 0 = \mathcal{D}_6\tilde{\alpha}_{b_5}, \quad \mathcal{D}_5\bar{m}_{b_5} = 0 = \mathcal{D}_6\bar{m}_{b_5} \qquad (49)$$

and hence physically significant in the original six-flavor theory. So we write $\tilde{\alpha}_b$ and $\bar{m}_b$ for $\tilde{\alpha}_{b_5}$ and $\bar{m}_{b_5}$.

Consider decoupling the $b$ quark from Eq. (47). The NLO matching condition (38) becomes

$$b_5 = \frac{\alpha_5^2}{\pi^2}\left( \ln \frac{\bar{m}_{b_5}}{\bar{\mu}} + \frac{1}{8} \right)(u+d+s+c)_4 + O(\alpha_5^3, m_{b5}^{-1}) \qquad (50)$$

so the nonsinglet current in Eq. (47) can be written

$$(u+d+s+c-4b)_5 = \{1-(\tilde{\alpha}_b^2/2\pi^2)\}E_4^{-1}(\tilde{\alpha}_b)S_4 + O(\tilde{\alpha}_b^3, m_{b5}^{-1}). \qquad (51)$$

For the singlet current $S_5$ in Eq. (47), we find

$$S_5 = E_5(\tilde{\alpha}_b)\left\{ 1 + \frac{\tilde{\alpha}_b^2}{8\pi^2} \right\} E_4^{-1}(\tilde{\alpha}_b)S_4 + O(\tilde{\alpha}_b^3, m_{b5}^{-1}) \qquad (52)$$

taking into account the definitions (7) and (40). Then we expand Eqs. (51) and (52) in $\tilde{\alpha}_b$, keeping quadratic terms:

$$(t-b)_6 = \frac{6}{23\pi}(\tilde{\alpha}_b - \tilde{\alpha}_t)\left\{ 1 + \frac{125663}{82800\pi}\tilde{\alpha}_b + \frac{6167}{3312\pi}\tilde{\alpha}_t \right\} S_4 + O(\tilde{\alpha}_{t,b}^3, m_{t,b}^{-1}). \qquad (53)$$

The same technique can be applied to decouple the $c$ quark from $S_4$ in Eq. (53) and $(c-s+u-d)_4$ [the result of decoupling $b$ from Eq. (41)]. That yields the final results (4) and (5) given in the introduction.

Notice that our results depend on two key features:

(i) Like previous workers in this area, we decouple heavy quarks sequentially, i.e. one at a time.

(ii) Our running couplings $\widetilde{\alpha}_t$, $\widetilde{\alpha}_b$ and $\widetilde{\alpha}_c$, which correspond to Witten's prescription [12], are all renormalization group invariant.

The restriction to sequential decoupling is numerically reasonable for the $t$ quark, but dubious for the $b$ and $c$ quarks, because it amounts to an assumption that $\ln(m_c/\overline{\mu})$ is negligible compared with $\ln(m_b/\overline{\mu})$. This inhibits detailed comparison of NLO results with data, which ought to be carried out with NLO accuracy [20].

There is also a theoretical issue here: one would like to check that, in the limit $m_t = m_b$, the $t$ and $b$ contributions cancel. However, that is outside the region of validity $\ln(m_t/\overline{\mu}) \gg \ln(m_b/\overline{\mu})$ for sequential decoupling.

For these reasons, we have extended our analysis to the case of simultaneous decoupling, where the mass logarithms are allowed to grow large together: $\ln(m_c/\overline{\mu}) \sim \ln(m_b/\overline{\mu}) \sim \ln(m_t/\overline{\mu}) \to$ large. This requires a considerable theoretical development of matching conditions and the renormalization group, which we will present separately. It involves the construction of running couplings $\alpha_t$, $\alpha_b$, $\alpha_c$ with the following properties: (i) They are renormalization group invariant; (ii) they are defined for $m_t \geqslant m_b \geqslant m_c$, and can have a nontrivial dependence on more than one heavy-quark mass; (iii) in the special case of sequential decoupling, they agree with $\widetilde{\alpha}_t$, $\widetilde{\alpha}_b$ and $\widetilde{\alpha}_c$ to NLO; and (iv) for the case of equal masses, they coincide, e.g.

$$\alpha_t = \alpha_b \quad \text{for} \quad m_t = m_b. \tag{54}$$

Then we find that the result for the simultaneous decoupling of the $t,b,c$ quarks from the neutral current is of the same form (4) as the sequential answer, but with the sequential running couplings in Eq. (5) replaced by our simultaneous couplings $\alpha_t$, $\alpha_b$, and $\alpha_c$:

$$\mathcal{P} = \frac{6}{23\pi}(\alpha_b - \alpha_t)\left\{1 + \frac{125663}{82800\pi}\alpha_b + \frac{6167}{3312\pi}\alpha_t - \frac{22}{75\pi}\alpha_c\right\} - \frac{6}{27\pi}\alpha_c - \frac{181}{648\pi^2}\alpha_c^2 + O(\alpha_{t,b,c}^3). \tag{55}$$

Notice the factorization of the terms depending on $\alpha_t$ and $\alpha_b$. Given Eq. (54), the factor $\alpha_b - \alpha_t$ ensures that all contributions from $b$ and $t$ quarks cancel (as they should) for $m_t = m_b$.

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[20] This includes matching conditions for the $b$ and $c$ masses, to be discussed elsewhere.