Decay constants, light quark masses, and quark mass bounds from light quark pseudoscalar sum rules

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The flavor ud and us pseudoscalar correlators are investigated using families of finite energy sum rules (FESR's) known to be very accurately satisfied in the isovector vector channel. It is shown that the combination of constraints provided by the full set of these sum rules is sufficiently strong to allow determination of both the light quark mass combinations m_u+m_d, m_s+m_u and the decay constants of the first excited pseudoscalar mesons in these channels. The resulting masses and decay constants are also shown to produce well-satisfied Borel transformed sum rules, thus providing nontrivial constraints on the treatment of direct instanton effects in the FESR analysis. The values of m_u+m_d and m_s+m_u obtained are in good agreement with the values implied by recent hadronic τ decay analyses and the ratios obtained from ChPT. New light quark mass bounds based on FESR's involving weight functions which strongly suppress spectral contributions from the excited resonance region are also presented.

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I. INTRODUCTION

The divergence of the flavor ij axial vector current in QCD is related to the corresponding pseudoscalar density by the Ward identity

$$\partial_\mu A^\mu_{ij}=(m_i+m_j)\bar{q}_i i\gamma_5 q_j . \tag{1}$$

As has been long recognized, this fact, together with the analyticity of the correlator, $\Pi_{ij}(q^2)$, defined by

$$\Pi_{ij}(q^2)=i\int d^4x e^{iq\cdot x}\langle 0|T(\partial_\mu A^\mu_{ij}(x)\partial_\nu A^{\nu\,\dagger}_{ij}(0))|0\rangle$$
$$\equiv(m_i+m_j)^2\hat{\Pi}_{ij}(q^2), \tag{2}$$

allows one to write down sum rules which relate the light quark mass combinations m_i+m_j to the decay constants of the flavor ij pseudoscalar mesons [1]. These sum rules, which include the basic unsubtracted dispersion relation (involving Π''_{ij}, and/or its derivatives) [1–6], the Borel transformed version of this relation [3,5,7–14], and finite energy sum rules [10,15–20], have been used to either place bounds on m_u+m_d and m_s+m_u, or estimate their values.

The basic forms of these relations are, for the unsubtracted dispersion relation (DR), the corresponding Borel sum rule (BSR) [7], and finite energy sum rules (FESR's),

$$\Pi''_{ij}(Q^2)=2\int_0^\infty ds\,\frac{\rho_{ij}(s)}{(s+Q^2)^3} \tag{3}$$

$$M^6\mathcal{B}[\Pi''_{ij}](M^2)=\int_0^\infty ds e^{-s/M^2}\rho_{ij}(s)$$
$$\simeq\int_0^{s_0} ds e^{-s/M^2}\rho_{ij}(s)$$
$$+\int_{s_0}^\infty ds e^{-s/M^2}\rho^{OPE}_{ij}(s) \tag{4}$$

$$\frac{-1}{2\pi i}\oint_{|s|=s_0} ds w(s)\Pi_{ij}(s)$$
$$=\int_0^{s_0} ds w(s)\rho_{ij}(s), \tag{5}$$

respectively, with ρ_{ij} the spectral function of Π_{ij}, s_0 in line 2 of Eq. (4) the ''continuum threshold'' [beyond which ρ_{ij} is approximated by its operator product expansion (OPE) form], M the Borel mass, and $w(s)$ in Eq. (5) any function analytic in the region of the contour. $\mathcal{B}[\Pi''_{ij}](M^2)$ in Eq. (4) is the Borel transform of the OPE representation of $\Pi''_{ij}(Q^2)$ [7].

The LHS of either Eq. (3) or (4) can be evaluated using the OPE provided the relevant scale (Q or M) is large compared to the QCD scale. For the FESR case, the condition that s_0 be similarly large is necessary, but not sufficient, to allow reliable evaluation of the LHS using the OPE. The reason is that, except at extremely large s_0, the OPE is expected to break down over some portion of the circle, $|s|=s_0$, sufficiently near the timelike real axis [21]. In the flavor ud vector channel, where the spectral function has been determined very accurately from hadronic τ decay data [22,23], one can, in fact, verify this breakdown: FESR's involving the weights $w(s)=s^k$ with $k=0,1,2,3$, which do not suppress contributions from the region near the timelike real

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axis, are typically rather poorly satisfied at scales 2 $\mathrm{GeV}^2 < s_0 < m_\tau^2$ [24]. At these scales, however, this breakdown turns out to be very closely localized to the vicinity of the timelike axis: as soon as one restricts one's attention to weights with even a single zero at $s=s_0$, the corresponding FESR's are very accurately satisfied over this whole range of s_0 [24]. Thus, for the ''intermediate'' scales 2 $\mathrm{GeV}^2 < s_0 < 4$ GeV^2 which will be of interest to us, we must also include, as a condition for the reliability of the OPE representation of the LHS of Eq. (5), the further requirement that $w(s_0)=0$. We will refer to FESR's satisfying this criterion as ''pinch-weighted'' FESR's (PFESR's) in what follows.

In the region below $s\sim 4$ GeV^2, where the resonances in the channels of interest ($ij=ud,us$) are well-separated, the spectral function will be dominated by contributions from the flavor ij pseudoscalar resonances, P. In the convention where $f_\pi=92.4$ MeV and $f_K=113.0$ MeV [25], the corresponding contribution to ρ_{ij}, ignoring interference, is

$$[\rho_{ij}(s)]_P = 2 f_P^2 m_P^4 B(s) \tag{6}$$

where $B(s)=\delta(s)$ in the narrow width approximation, with the standard Breit-Wigner generalization to finite width,

$$B(s)=\frac{1}{\pi}\frac{\Gamma_P m_P}{[(s-m_P^2)^2+\Gamma_P^2 m_P^2]}. \tag{7}$$

Experimentally, both f_π and f_K are very accurately known, while the higher resonance [$\pi(1300)$ and $\pi(1800)$ for $ij=ud$ and $K(1460)$ and $K(1830)$ for $ij=us$] decay constants are unknown at present.[1] The positivity of $\rho_{ij}(s)$, together with the fact that the weights appearing in the spectral integrals of Eqs. (3) and (4) are >0, implies that the π (or K) pole contributions provide lower bounds to these integrals. The same is true for Eq. (5) as long as the weight $w(s)$ employed is positive for $0<s<s_0$. These lower bounds allow one to obtain corresponding lower bounds for m_u+m_d and m_s+m_u [1]. To actually determine m_u+m_d and m_s+m_u, rather than just set bounds on them, however, one must at present provide theoretical input for the higher resonance contributions. These contributions cannot be expected to be negligible since the $f_P^2 m_P^4$ factors for all P are formally of the same order in the chiral expansion. In fact, in existing analyses, the higher resonance contributions are typically larger than the π (or K) pole contributions—as an example, the $\pi(1300)$ and $\pi(1800)$ contributions to the s^0-weighted FESR used to determine m_u+m_d in Refs. [10,20] are a factor of $\sim 2-3$ times the π pole contribution.

Two approaches to constraining the higher resonance contributions exist in the literature. In the first, additional sum rules have been used to provide an estimate of the decay constant of the first excited resonance [3,4,9,13,15–18]. In the second, resonance dominance has been assumed to be a good approximation, even in the 3π (or $K\pi\pi$) threshold region, and known chiral perturbation theory (ChPT) expressions for the threshold values of the spectral functions used to normalize sums-of-Breit-Wigner *Ansätze* for the higher resonance contributions. Since the thresholds are typically several resonance widths (or more) removed from the resonance masses, the peak normalizations (the features of the resonance contributions to which the sum rule determinations of the m_i+m_j are dominantly sensitive) will be ambiguous in this approach, depending, for example, on the treatment of the s dependence of the ''off-shell width.'' Potential dangers of this threshold normalization approach have been discussed in Refs. [26,27]. The situation in the us scalar channel, where the near-threshold behavior of the spectral function is significantly constrained by known $K\pi$ $I=1/2$ s-wave phase shifts, is particularly instructive. As shown in Ref. [27], the near-threshold spectral function implied, through unitarity, by the $K\pi$ phases and the resulting Omnes representation of the timelike scalar $K\pi$ *cannot* be well represented by the tail of a Breit-Wigner resonance form; a significant background component, interfering constructively with the resonance contribution in the threshold region, is required. The near-threshold normalization of the resonance contribution is, therefore, significantly reduced, producing a corresponding reduction in the value of the spectral function at the $K_0^*(1430)$ resonance peak. This reduction is very significant numerically: the $K_0^*(1430)$ peak value of the us scalar spectral function obtained in Ref. [27] (albeit with some additional assumptions about the high-s behavior of the $K\pi$ phase and the form of the Omnes representation) is a factor of ~ 3 smaller than that obtained, using the threshold-resonance-dominance assumption (TRDA), in Ref. [11]. Even if one questions the additional assumptions which go into the precise numerical value of the reduction in this case, one should bear in mind that the TRDA *Ansatz* for the us scalar channel was shown to correspond to a value of the slope of the timelike $K\pi$ form factor at threshold incompatible with that known from ChPT [27]. Further evidence of the potential problems of the TRDA approach are provided by the results of Ref. [24]. In Ref. [24], the TRDA *Ansätze* of Refs. [10,20] for the ud pseudoscalar channel and of Refs. [11,28] for the us scalar channel were tested using families of PFESR's in which the OPE scales used were the same as those employed in the earlier analyses. *If* the TRDA spectral *Ansatz* for a given channel is a good representation of the physical spectral function in that channel, and if the scale of the original analysis was such that the OPE representation could be reliably employed, then PFESR's constructed using the same spectral ansatz for the same correlator should also be well satisfied. It turns out that, in both the ud pseudoscalar and us scalar channels, the TRDA ansatz produces a very poor match between the OPE and spectral integral sides of the various PFESR's [24]. In contrast, the match corresponding to the us scalar spectral function of Ref. [27] is quite reasonable [24].

In view of the above observations, we do not employ the TRDA *Ansatz* for the excited pseudoscalar contributions, but

[1] $f_{\pi(1300)}$ and $f_{K(1460)}$ could, in principle, be determined using data from hadronic τ decay, but this would require disentangling these contributions from spin 1 resonance contributions in the same region. Neither the ud nor us spin decomposition for the excited resonance region has been performed to date.

instead constrain these contributions, in analogy to the treatment of the isovector vector and scalar channels in Ref. [29], by analyzing simultaneously two continuous families of PFESR's, corresponding to the weights $w_N^A(y)=(1-y)(1+Ay)$ and $w_D^A(y)=(1-y)^2(1+Ay)$, where $y\equiv s/s_0$. As we will show, the set of these constraints is sufficiently strong to allow determination of not only the excited resonance decay constants, but also the light quark mass combinations. The input required for this analysis, and the related BSR analysis, is outlined briefly in the next section. Our final results, together with a discussion of existing quark mass analyses, are provided in Sec. III while Sec. IV contains our conclusions.

II. INPUT FOR THE PFESR AND BSR ANALYSES

In this paper we perform both PFESR and BSR analyses of the flavor $ij=ud$ and us pseudoscalar correlators. The general BSR and PFESR forms are given in Eqs. (4) and (5). Lower bounds for the quark mass combinations m_i+m_j are obtained by neglecting spectral contributions other than those associated with the π (or K) pole on the RHS's of these equations, and employing the relevant OPE (which depends on m_i+m_j) on the LHS. In order to extend this analysis and obtain an actual determination of m_i+m_j, rather than just a lower bound on it, it is necessary to use the sum rule constraints to simultaneously determine the resonance region contributions to the relevant spectral function *and* m_i+m_j. This is possible only because the corresponding pole contributions to the spectral function are very accurately known, and to the extent that it is possible to construct a reasonable spectral *Ansatz* describing the resonance region. In the remainder of this section we discuss the input required on both the spectral and OPE sides of the BSR's and PFESR's employed in our analysis.

A. Input to the PFESR and BSR spectral integral sides

We take as our spectral *Ansatz* for the ud pseudoscalar channel the expression

$$\rho_{ud}(s)=2f_\pi^2 m_\pi^4\delta(s-m_\pi^2)+2f_1^2m_1^4B_1(s)+2f_2^2m_2^4B_2(s), \tag{8}$$

where $m_{1,2}$ are the Partial Data Group 2000 (PDG2000) [25] masses of the $\pi(1300)$ and $\pi(1800)$, $f_{1,2}$ are their (as yet undetermined) decay constants, and $B_{1,2}(s)$ are the standard Breit-Wigner forms. We have employed PDG2000 values for all resonance widths. The corresponding expression for $\rho_{us}(s)$ is obtained by the replacements $\pi\to K$, $\pi(1300)\to K(1460)$ and $\pi(1800)\to K(1830)$. In order that this *Ansatz* provide a good representation of the spectrum over the whole range required in the PFESR spectral integrals, s_0 cannot be taken much greater than m_2^2; if it is, an unphysical ''gap,'' with little spectral strength, will be present in the integration region. We therefore require s_0 to remain less than about $(m_2+\Gamma_2)^2\simeq 4$ GeV^2. To create a good analysis window in s_0 without at the same time sacrificing good convergence of the integrated $D=0$ OPE series, we also take $s_0>3$ GeV^2. Note that, since m_2^2 lies in the lower half of the resulting analysis window, $3\ \text{GeV}^2<s_0<4\ \text{GeV}^2$, it is necessary to include the second resonance in the spectral *Ansatz*. Although the structure of the PFESR and BSR weights are such that the second resonance contribution is weighted less strongly than the first resonance contribution to the spectral integral, the former is typically not negligible (especially for the BSR analysis, and for s_0 in the upper part of the PFESR analysis window).

It is worth stressing that it is necessary to use the finite width representation of the resonance contributions (rather than the simpler narrow width approximation) in order to minimize uncertainties in the determination of the decay constants f_1 and f_2. The reason is that, by using the experimental widths (together with their experimental errors), one to some extent incorporates a partial representation of continuum contributions. There are two pieces of evidence to suggest that this approximation is actually rather accurate, at least for the purposes of PFESR determinations of resonance decay constants. The first concerns the isovector vector channel. In this channel, if one ignores the experimental spectral data and instead uses the PFESR OPE integrals to fit the decay constants of a spectral *Ansatz* consisting of a sum of Breit-Wigner resonance contributions in which one employs PDG2000 values for the resonance masses and widths, one obtains a value of the ρ decay constant in agreement with the experimental value to better than the experimental error [29]. The second piece of evidence concerns a channel in which there is known to be considerable non-resonant background, namely the flavor us scalar channel. As noted already in Sec. I, a spectral *Ansatz* for this channel was constructed by Colangelo, De Fazio, Nardulli, and Paver [27] (CFNP) using unitarity and the Omnes relation for the timelike scalar $K\pi$ form factor. BSR's and PFESR's based on this *Ansatz* show a good match between OPE and spectral integral sides, once m_s has been fixed [27,24]. An important feature of the spectral *Ansatz*, for our purposes, is that it displays a significant continuum $K\pi$ contribution above threshold not well represented by the tail of a $K_0^*(1430)$ resonance contribution. This is the type of situation where one might expect a sum-of-resonances approximation to the spectral function to produce a theoretical systematic error in the determination of any resonance decay constants extracted by fitting to the values of the PFESR (or BSR) OPE integrals. However, if one takes the OPE to be fixed (using the value of m_s corresponding to the CFNP *Ansatz*), and fits the resonance decay constants of the less realistic sum-of-resonances *Ansatz* to the PFESR OPE integrals (using PDG2000 input for the resonance masses, widths and errors) one finds that the output $K_0^*(1430)$ decay constant matches that of the CFNP *Ansatz* to better than 3%. Since the uncertainties in the decay constants produced by the use of the sum-of-resonances form is, in both of the examples discussed above, less than $\sim 3\%$, and since such an uncertainty would have negligible impact on the total errors on our decay constant determinations below, we will neglect this error in what follows.

The ability to avoid unphysical spectral gaps represents a potential advantage of the PFESR framework over its BSR counterpart. For BSR's, the continuum threshold, s_0, is usu-

ally set by requiring an optimal stability window with respect to the Borel mass, M. Taking the $ij=us$ analysis of Ref. [12] as an example, and considering the case, $\Lambda_{QCD}=380$ MeV, which most closely corresponds to the current experimental determination of $\alpha_s(m_\tau^2)$, the stability window is optimized for s_0 between 6 and 8 GeV^2 [12]. The resulting spectral *Ansatz*, therefore, has a gap with very little spectral strength from about 5 to 6 or 8 GeV^2. It is also worth noting that, after Borel transformation, the scale relevant to the running coupling in the OPE is $\mu=M$. For the correlators of interest to us the convergence of the transformed $D=0$ series becomes good only for M^2 greater than about 2 GeV^2. Even if one is willing to tolerate a spectral gap by allowing $s_0 \sim 6$ GeV^2, this means that s_0/M^2 will be $\sim 1-2$ over much of any putative stability window in M. Such a condition signals non-trivial contributions from the ''continuum'' region, where only a relatively crude approximation to the spectral function is being employed. This leaves only a small range of M having both good OPE convergence and acceptably small continuum contributions (say less than $\sim 30\%$ of the $D=0$ OPE term). With such a small range of M, the BSR constraints are not sufficiently strong to allow a simultaneous determination of the quark masses and excited resonance decay constants. In the case of PFESR's, empirical evidence from the isovector vector channel suggests that contributions analogous to the less reliable continuum BSR contributions (i.e., those contributions from the region of the contour $|s|=s_0$ near the timelike real axis, where the OPE is expected to break down) are strongly suppressed by the restriction to weights satisfying $w(s_0)=0$.

B. Input to the PFESR and BSR OPE sides

The OPE representation of $\Pi''_{ij}(Q^2)$ is known up to dimension $D=6$, with the dominant $D=0$ perturbative contribution known to 4-loop order [11,28]. The $D=0$ term is given by [28]

$$[\Pi''_{ij}(Q^2)]_{D=0}=\frac{3}{8\pi^2}\frac{(\overline{m}_i+\overline{m}_j)^2}{Q^2}\left(1+\frac{11}{3}\overline{a}+14.1793\overline{a}^2+77.3683\overline{a}^3\right), \tag{9}$$

where $\overline{a}\equiv a(Q^2)=\alpha_s(Q^2)/\pi$, $\overline{m}_k\equiv m_k(Q^2)$, with $\alpha_s(Q^2)$ and $m(Q^2)$ the running coupling and running mass at scale $\mu^2=Q^2$ in the modified minimal subtraction ($\overline{\rm MS}$) scheme. The $D=2$ term involves quark mass corrections to the leading $D=0$ result. For $ij=ud$ it is numerically negligible, while for $ij=us$ it is given by [28]

$$[\Pi''_{us}(Q^2)]_{D=2}=-\frac{3}{4\pi^2}\frac{(\overline{m}_s+\overline{m}_u)^2\overline{m}_s^2}{Q^4}\left(1+\frac{28}{3}\overline{a}+\left[\frac{8557}{72}-\frac{77}{3}\zeta(3)\right]\overline{a}^2\right). \tag{10}$$

In writing Eq. (10), we have dropped terms involving $m_{u,d}$, except in the overall prefactor $(\overline{m}_s+\overline{m}_u)^2$. The $D=4$ ud contributions are [11]

$$[\Pi''_{ud}(Q^2)]_{D=4}=\frac{(\overline{m}_u+\overline{m}_d)^2}{Q^6}\left(\frac{1}{4}\Omega_4+\frac{4}{9}\overline{a}\Omega_3^{ss}-\left[1+\frac{26}{3}\overline{a}\right]\times 2\hat{m}\langle\overline{u}u\rangle-\frac{3}{28\pi^2}\overline{m}_s^4\right), \tag{11}$$

where Ω_4 and Ω_3^{ss} are the RG invariant modifications of $\langle aG^2\rangle$ and $\langle m_s\overline{s}s\rangle$ defined in Ref. [11], $\hat{m}=(m_u+m_d)/2$, and we have dropped numerically negligible terms of $O(\hat{m}^4)$; the $D=4$ us contributions are, similarly, [11]

$$[\Pi''_{us}(Q^2)]_{D=4}=\frac{(\overline{m}_s+\overline{m}_u)^2}{Q^6}\left(\frac{1}{4}\Omega_4+\left[1+\frac{64}{9}\overline{a}\right]\Omega_3^{ss}-2\langle m_s\overline{u}u\rangle\left[1+\frac{23}{3}\overline{a}\right]-\frac{3}{7\pi^2}\left[\frac{1}{\overline{a}}+\frac{155}{24}\right]\overline{m}_s^4\right), \tag{12}$$

where we have again dropped terms suppressed by powers of $\hat{m}/m_s$ relative to those shown, except in the overall $(\overline{m}_s+\overline{m}_u)^2$ prefactor. Finally, the $D=6$ contributions are [11]

$$[\Pi''_{ij}(Q^2)]_{D=6}=\frac{(\overline{m}_s+\overline{m}_u)^2}{Q^8}\left(-3[\langle m_i g\overline{q}_j\sigma\cdot Gq_j+m_j g\overline{q}_i\sigma\cdot Gq_i\rangle]-\frac{32}{9}\pi^2 a\rho_{VSA}[\langle\overline{q}_iq_i\rangle^2+\langle\overline{q}_jq_j\rangle^2-9\langle\overline{q}_iq_i\rangle\langle\overline{q}_jq_j\rangle]\right), \tag{13}$$

where ρ_{VSA} describes the deviation of the four-quark condensates from their vacuum saturation values.

In writing down the theoretical representation of Π_{ij} for use on the LHS's of our BSR's and PFESR's, one must bear in mind that, in scalar and pseudoscalar channels, potentially important contributions from direct instantons exist which are not incorporated in the standard OPE representation of Π_{ij} [30]. Such contributions are, in fact, needed to produce a Borel transform, $\mathcal{B}[\hat{\Pi}_{ud}](M^2)$, which behaves correctly (i.e., is independent of M) in the chiral limit [30–33]. The instanton liquid model (ILM) [34] provides a tractable framework for estimating such contributions. In the ILM, an average density (related to the value of the gluon condensate) and fixed average size are employed for the instanton distribution. Phenomenological constraints require the average instanton size, ρ_I to be $\simeq 1/0.6$ GeV [31,32,34]. Instanton contributions to $\mathcal{B}[\Pi_{ud}](M^2)$ then exceed one-loop perturbative contributions below $M^2\sim 1$ GeV^2, but drop to less

than $\sim 15\%$ of this contribution for $M^2 \sim 2$ GeV^2 [32].[2] Direct instanton contributions have been neglected in recent treatments of the ud and us pseudoscalar channels, apart from the BSR ud analyses of Refs. [13,14], both of which employed the ILM. The numerical impact of the neglect of these contributions should be small for BSR analyses at scales $M^2 > 2$ GeV^2 since the Borel transform is known to rather strongly suppress ILM contributions with increasing scale.[3] This is, however, *not* true of FESR analyses, for which ILM contributions fall off, relative to the $D=0$ perturbative contributions, much more slowly with increasing s_0.

In what follows, we will use the ILM to estimate direct instanton contributions to the w_N^A and w_D^A PFESR's. ILM contributions to PFESR's corresponding to polynomial weights can be evaluated using the result [36]

$$\frac{-1}{2\pi i}\oint_{|s|=s_0} ds\, s^k [\hat{\Pi}_{ij}(s)]_{ILM} = \frac{-3\eta_{ij}}{4\pi}\int_0^{s_0} ds\, s^{k+1} J_1(\rho_I\sqrt{s}) Y_1(\rho_I\sqrt{s}), \quad (14)$$

where $\eta_{ud} \equiv 1$, η_{us} is an $SU(3)$-breaking factor whose value in the ILM is ~ 0.6 [32], and the result is relevant to scales ~ 1 GeV^2.

One should bear in mind that phenomenological support for the ILM exists primarily for those scales (~ 1 GeV^2) where instanton contributions are numerically important in pseudoscalar BSR's, and that this scale is significantly lower than that (~ 3–4 GeV^2) relevant to our PFESR analysis. It is, therefore, useful to have an independent test of our use of the ILM representation of instanton effects. In this regard, one can take advantage of the much stronger suppression of ILM contributions in the BSR framework. The basic idea is as follows. One first determines the excited meson decay constants for the channel of interest, using the PFESR framework. These values then determine the $s < s_0$ part of the spectral *Ansatz* for a BSR treatment of the same channel. (The spectral function for $s > s_0$ is, as usual, approximated using the continuum *Ansatz*; we fix the continuum threshold, s_0, following standard practice, by optimizing the stability of the output, in this case, the quark mass combination, $m_i + m_j$, with respect to M^2.) For $M^2 \sim 2$ GeV^2, where (1) convergence of the Borel transformed $D=0$ series is still reasonable and (2) continuum contributions are still relatively small (not yet exceeding $\sim 30\%$ of perturbative contributions), the resulting BSR should then allow determination of the only remaining unknown, $m_i + m_j$, with good accuracy. The ILM contributions play little role on the OPE +ILM side of the BSR's at these scales, but are important for the PFESR's, and hence for the values of the resonance decay constants used as input to the BSR's. If the ILM representation of direct instanton effects is reasonable at the scale of the PFESR analysis, the PFESR and BSR determinations of $m_i + m_j$, which will then have been obtained using the same excited resonance decay constants, should be compatible within their mutual errors. Since the continuum approximation for the spectral function is a relatively crude one, and the stability criterion for choosing s_0 typically leaves a gap in the BSR spectral model, there are uncertainties in the BSR analysis beyond those associated with the uncertainties in the OPE input, which are shared by the PFESR and BSR analyses. In order to get a rough estimate of these additional uncertainties we allow s_0 to vary in an interval of size 1 GeV^2, i.e., by ± 0.5 GeV^2 about the value corresponding to optimal stability, and assign a $\pm 20\%$ error to the size of continuum spectral contributions. Since the s_0 values we obtain are > 3.7 GeV^2, we consider the latter estimate sufficiently conservative.[4] The uncertainties on $m_i + m_j$ induced by use of the continuum approximation are then not large, particularly in the region near $M^2 = 2$ GeV^2, where the BSR continuum contributions are less than $\sim 30\%$ of the $D=0$ OPE term. The BSR/PFESR cross-check is, as a result, most reliable at these scales.[5]

Numerical values of the input required on the OPE +ILM side of the sum rules are as follows: $\rho_I = 1/(0.6\ \text{GeV})$ [32,34], $\alpha_s(m_\tau^2) = 0.334 \pm .022$ [22,23], $\langle \alpha_s G^2 \rangle = (0.07 \pm 0.01)$ GeV^4 [37], $(m_u + m_d)\langle \bar{u}u \rangle = -f_\pi^2 m_\pi^2$ (the GMOR relation),[6] $0.7 < \langle \bar{s}s \rangle / \langle \bar{u}u \rangle \equiv r_c < 1$ [11,28]; $\langle g\bar{q}\sigma Fq \rangle = (0.8 \pm 0.2\ \text{GeV}^2)\langle \bar{q}q \rangle$ [39] and $\rho_{VSA} = 5 \pm 5$ (i.e., allowing, to be conservative, up to an order of magnitude deviation from vacuum saturation for the four-quark condensates). The $D = 0,2$ and 4 contributions to the OPE integral have been evaluated using contour-improvement [40,41], which is known to improve convergence and reduce residual scale dependence [41]. For this purpose, we employ the analytic solutions for the running coupling and running mass obtained using the known 4-loop-truncated versions of the β [42] and γ [43] functions, with the value of $\alpha_s(m_\tau^2)$ noted above as input.

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[2]The combination of 2-, 3- and 4-loop contributions roughly doubles the Borel transformed 1-loop $D=0$ contribution at $M^2 = 2$ GeV^2, hence further suppressing the ratio of ILM to perturbative contributions.

[3]For example, the bound obtained in Ref. [14] is raised by $< 5\%$ if ILM contributions are turned off [35].

[4]For the analogous cases of the ud vector and axial vector channels, where the hadronic spectral functions are known experimentally from hadronic τ decay data, the *maximum* deviation of the actual spectral function from its 4-loop OPE continuum approximation is less than $\sim 1/3$ of the OPE version in the interval 2 GeV^2 $< s < m_\tau^2$ [22]. Note that these scales are smaller than those for which we will be employing the continuum approximation, and that we are concerned with the *average*, rather than maximum, deviation in the range $s > s_0$.

[5]The ratio of the continuum to the $D=0$ OPE contribution grows relatively rapidly with M^2. For the ud case, for example, it has already reached $\sim 50\%$ by $M^2 = 3$ GeV^2 and $\sim 65\%$ by $M^2 = 4$ GeV^2.

[6]Deviations from the Gell-Mann–Oakes–Renner (GMOR) relation have recently been shown to be at most 6% [38]. The resulting error on the m_s analysis is completely negligible.

III. RESULTS AND DISCUSSION

In this section we present our results. We first discuss the quark mass bounds, obtained by neglecting the resonance spectral contributions proportional to $f_{1,2}^2$ and then discuss the more complicated analysis in which both m_i+m_j and $f_{1,2}$ are obtained simultaneously. Finally we provide a detailed discussion of the relation of our results to other recent sum rule determinations of the light quark masses.

A. Quark mass bounds

Bounds for the light quark masses based on the known values of the π or K pole contributions and the positivity of the spectral function, whether obtained using the dispersion formulation, BSR's or FESR's, all depend on the scale employed in the OPE. Since, at the scales for which the resulting bounds are of phenomenological interest, the $O(a^2)$ and $O(a^3)$ terms in the integrated $D=0$ OPE series are not numerically negligible, earlier versions of these bounds, based on two-loop and three-loop forms of the $D=0$ part of $\Pi_{ud,us}$, are superceded by the work of Ref. [6] (LRT), which employed the 4-loop OPE expression. The bounds of LRT are based on the dispersion relation for Π_{ij}'', and the higher derivative moments thereof. Restricting our attention to the results in LRT corresponding most closely to the experimental value $\alpha_s(m_\tau^2)=0.334$, i.e., $\Lambda_{QCD}^{(3)}=380$ MeV, the most stringent bounds arise from what in LRT is called the "quadratic inequality" [6]. These bounds decrease with increasing OPE scale, Q^2, and, for $Q^2=4$ GeV^2, yield (from Figs. 2 and 3 of LRT)

$$[m_s+m_u](\mu=2\ \text{GeV})>105\ \text{MeV}$$

$$[m_u+m_d](\mu=2\ \text{GeV})>8.1\ \text{MeV}. \qquad (15)$$

Normally one would expect the convergence of the 4-loop $D=0$ OPE series to be quite good at scales as large as $Q^2=4$ GeV^2. In this case, however, the denominator appearing on the RHS of the quadratic bound [see Eq. (19) of LRT], which has the form

$$[3\mathcal{F}_0^{QCD}\mathcal{F}_2^{QCD}-2(\mathcal{F}_1^{QCD})]=1+\frac{25}{3}\overline{a}+61.79\overline{a^2}+517.15\overline{a^3}+\cdots, \qquad (16)$$

is very slowly converging, behaving as $1+0.83+0.61+0.51$ at $Q^2=4$ GeV^2. The bounds in Eq. (15) are thus likely to have a significant residual uncertainty associated with the truncation at $O(a^3)$.[7] The behavior of the $D=0$ series is in fact much better for the zeroth moment LRT bound. At the lowest scale shown in Fig. 1 of LRT ($Q=1.4$ GeV), the behavior of the $D=0$ series is $1+0.45+0.22+0.15$, already quite well-converged. The corresponding bound on m_s+m_u which, reading from Fig. 1, is

$$[m_s+m_u](\mu=2\ \text{GeV})>80\ \text{MeV}, \qquad (17)$$

thus seems to us to be subject to significantly less truncation-induced uncertainty. Although the zeroth moment bound for m_u+m_d is not quoted in LRT, the result of Eq. (17), together with the result $R\equiv 2m_s/(m_u+m_d)=24.4\pm 1.5$ determined from ChPT [44], would imply

$$[m_u+m_d](\mu=2\ \text{GeV})>6.6\ \text{MeV}. \qquad (18)$$

The result of Eq. (18) is in good agreement with the bound obtained by the same authors [6] from the study of the ud scalar channel[8] using constraints on the timelike scalar-isoscalar $\pi\pi$ form factor from ChPT and $\pi\pi$ phase shift data in the region $4m_\pi^2<s<(500\ \text{MeV})^2$ (see Fig. 4 of LRT),

$$[m_u+m_d](\mu=2\ \text{GeV})>6.8\ \text{MeV}. \qquad (19)$$

An analogous bound for m_s was obtained from a treatment of the us scalar correlator employing ChPT constraints for the timelike scalar $K\pi$ form factor [46]. Taking the case from that reference corresponding to the plausible assumption that the one-loop ChPT expression for the $K\pi$ form factor is accurate to the 0.5–1% level in the region $0<s<m_K^2-m_\pi^2$, the resulting bound is

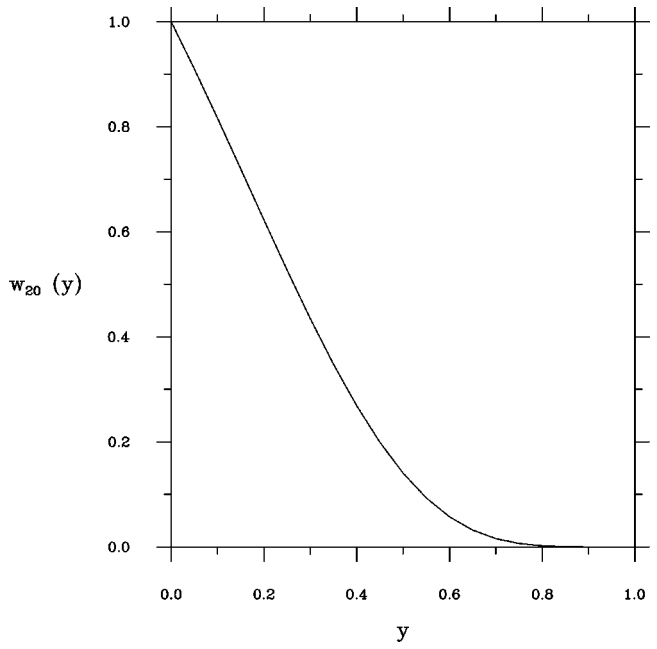

FIG. 1. The behavior of the weight $w_{20}(y)$ in the PFESR integration region.

[7] If one wished to work, e.g., at a scale such that the $O(a^3)$ term in Eq. (16) were less than $\sim$20% of the leading term, one would need to go to $Q^2\sim 9$ GeV^2, at which scale the bounds on $[m_s+m_u](\mu=2$ GeV) and $[m_u+m_d](\mu=2$ GeV) would be reduced to ~ 60 and ~ 3.4 MeV, respectively.

[8] The $D=0$ OPE series corresponding to this bound has the same (good) convergence behavior as that given for the zeroth moment bound above.

$$m_s(\mu=2\ \text{GeV})>65\ \text{MeV}, \tag{20}$$

which is less stringent than that in Eq. (17). Other recent bounds are (1) that obtained in Ref. [45] by combining the upper bound on $\langle\bar{q}q\rangle(1\ \text{GeV})$ allowed by the analysis of the $D\to K^* l\nu_l$ vector form factor with the (assumed to be well-satisfied) GMOR relation

$$[m_u+m_d](2\ \text{GeV})>6.8\ \text{MeV}, \tag{21}$$

and (2) that obtained in Ref. [14] using BSR's and Hölder inequalities at scales $\sim m_\tau$

$$[m_u+m_d](2\ \text{GeV})>4.2\ \text{MeV}. \tag{22}$$

Note that the latter bound was obtained including ILM contributions on the OPE side of the sum rule; the bound is $\sim 5\%$ higher if ILM contributions are turned off [35]. All other bounds noted above were obtained neglecting direct instanton contributions. This neglect should have little impact on dispersive bounds such as that of Eq. (17) since, if one uses the ILM to estimate these effects, the lower bound of Eq. (17) is reduced by only 3 MeV.

An alternate approach to using the positivity of ρ_{ij} to obtain quark mass bounds is to employ PFESR's with weights satisfying $w(s)\geqslant 0$ in the region $0<s<s_0$. A potential advantage of this approach is the freedom to choose weights which strongly suppress contributions from the excited resonance region. Strong suppression of this type should lead to bounds which are ''close'' to the actual mass values. One can arrange such strong suppression by choosing $w(y)=(1-y)^N p(y)$ with N sufficiently large. Here $p(y)$ is a ''residual polynomial'' which has to be chosen in such a way as to (1) keep the coefficients in $w(y)$ small (thus avoiding the growth of unknown higher D contributions)[9] and (2) retain good convergence of the integrated $D=0$ OPE series. The construction of such weights was considered in a different context previously [47]. Here we consider quark mass bounds based on PFESR's employing the three weights of this type constructed in Ref. [47]. It turns out that both the $D=0$ OPE convergence and the stringency of the resulting bounds is best for the case of the weight called $w_{20}(y)$ in Ref. [47], so we present results only for this case. The behavior of $w_{20}(y)$ in the integration region $(0<y<1)$ is shown in Fig. 1. (Its explicit form may be found in Ref. [47].) For $s_0=4\ \text{GeV}^2$, the contour-improved $D=0$ OPE series for the w_{20} PFESR, truncated at $O(a^3)$, converges quite reasonably, behaving as $\sim 1+0.55+0.28+0.19$. Moreover, since, for example, if we define $y_{K(1460)}\equiv m^2_{K(1460)}/4\ \text{GeV}^2$, $w_{20}(y_{K(1460)})=0.11$, there will be nearly an order of magnitude suppression of excited resonance contributions, relative to the K contribution, in the us channel. Unfortunately, the $D=0$ convergence deteriorates if one tries to go to lower s_0, where this suppression would be much stronger. Ignoring possible direct instanton contributions, one obtains

$$m_s(2\ \text{GeV})>93\ \text{MeV}. \tag{23}$$

The convergence is obviously not sufficiently rapid that one should rule out values of the bound a further ~ 5 or so MeV lower. The analogous bound for m_u+m_d is

$$[m_u+m_d](2\ \text{GeV})>6.6\ \text{MeV}. \tag{24}$$

These bounds should be compared only to those bounds listed above which also neglect possible instanton effects. As expected, the rather strong suppression of excited resonance contributions relative to K pole term produces a bound on m_s, Eq. (23), which is more stringent than the zeroth moment LRT bound. The m_u+m_d bound of Eq. (24), however, remains comparable to the LRT m_u+m_d bound, though still having the advantage that one would expect it to represent a better approximation to the true value. If one now incorporates an estimate of direct instanton effects using the ILM, the bounds of Eqs. (23) and (24) are reduced to

$$m_s(2\ \text{GeV})>84\ \text{MeV} \tag{25}$$

and

$$[m_u+m_d](2\ \text{GeV})>5.7\ \text{MeV}. \tag{26}$$

The bound of Eq. (25) remains slightly more stringent than that of Eq. (17). A more stringent bound on m_u+m_d,

$$[m_u+m_d](2\ \text{GeV})>6.9\ \text{MeV}, \tag{27}$$

can be obtained using Eq. (25) in combination with the mass ratios obtained from ChPT [44].

To go beyond these bounds, we must attempt to also determine the excited resonance decay constants as part of the PFESR analysis. This extension of the analysis is described in the next section.

B. Quark masses and excited meson decay constants

To simultaneously extract m_i+m_j and the corresponding excited pseudoscalar decay constants, we have performed a combined analysis of PFESR's based on the weight families $w_N^A(y)$ and $w_D^A(y)$. Our s_0 analysis window was $3\ \text{GeV}^2 \leqslant s_0 \leqslant 4\ \text{GeV}^2$. For these s_0, the $D=0$ OPE series converges well for all $A\geqslant 0$, and the spectral *Ansatz* should be of the correct qualitative form. Larger values of A correspond to larger relative contributions from the excited resonance region, and hence are useful for constraining the unknown resonance decay constants. To explore sensitivities to the choice of analysis regions, we have also considered the alternate ranges $3.6\ \text{GeV}^2 \leqslant s_0 \leqslant 4\ \text{GeV}^2$ and $2\leqslant A\leqslant 6$, as well as considering separate $w_N^A(y)$ and $w_D^A(y)$ analyses (thus checking the mutual consistency of the PFESR's corresponding to the two weight families). The only significant

[9]Without this constraint, working with high powers of the factor $(1-y)$ typically produces polynomials with large coefficients for the higher degree y^k terms. Since y^k terms with large k are associated with OPE contributions of large dimension, which are poorly constrained phenomenologically, large y^k coefficients signal potentially large, and essentially unknown, non-perturbative contributions [48,49], and hence must be avoided.

impact of uncertainties in the experimental input for the resonance parameters is that occurring in the ud analysis, associated with the $\pi(1300)$ width; this is a consequence of the rather wide range, $200<\Gamma(\pi(1300))<600$ MeV given in the PDG2000 compilation. In what follows, we quote errors from this source separately, labelling them with the subscript "Γ." Uncertainties associated with changes in the s_0 and A analysis windows and weight family choice are added in quadrature and denoted by the subscript "method." Finally, those errors denoted by the subscript "theory" are obtained by combining in quadrature errors associated with uncertainties in the OPE input parameters ρ_{VSA}, $\langle\alpha_s G^2\rangle$, $\alpha_s(m_\tau^2)$ and r_c and our estimate of the error due to truncation of the dominant $D=0$ OPE contribution at 4-loop order. The latter is obtained by evaluating the $O(a^4)$ contribution that would result if we assumed continued geometric growth of the coefficients, i.e., the presence of an additional term $\sim 422\overline{a^4}$ in the polynomial factor of Eq. (9).[10] It turns out that, when ILM contributions are included, the s_0 dependence of the theoretical side is such that the contribution of the second resonance on the spectral side must be relatively small for both channels. As a result, the corresponding decay constant can be determined with only limited accuracy. When quoting results for the second decay constant in this case, we will, therefore, display only the range of values allowed by the combined [i.e., "theory," "method" and (for the ud channel only) "Γ"] errors. The analysis of the ud channel has been described briefly already in Ref. [51].

The results obtained from the analysis, when ILM contributions are included on the theoretical side of the PFESR's, are as follows. For the ud channel we have

$$[m_u+m_d](2\ \text{GeV})=7.8\pm 0.8_\Gamma\pm 0.5_{theory}\pm 0.4_{method}\ \text{MeV} \tag{28}$$

$$f_{\pi(1300)}=2.20\pm 0.39_\Gamma\pm 0.18_{theory}\pm 0.18_{method}\ \text{MeV} \tag{29}$$

$$0<f_{\pi(1800)}<0.37\ \text{MeV}, \tag{30}$$

and for the us channel

$$m_s(2\ \text{GeV})=100\pm 4_{theory}\pm 5_{method}\ \text{MeV} \tag{31}$$

$$f_{K(1460)}=21.4\pm 1.6_{theory}\pm 2.3_{method}\ \text{MeV} \tag{32}$$

$$0<f_{K(1830)}<8.9\ \text{MeV}. \tag{33}$$

Note that the "theory" errors do not yet include an estimate of the error associated with the crudeness of the ILM representation of direct instanton effects. We will return to this point below. From Eqs. (28) and (29), we see that the uncertainty in the $\pi(1300)$ width is, in fact, the dominant source of error in the determination of both m_u+m_d and $f_{\pi(1300)}$.

[10]In view of the discussion in Sec. 5 of Ref. [50], this estimate is likely to be a very conservative one.

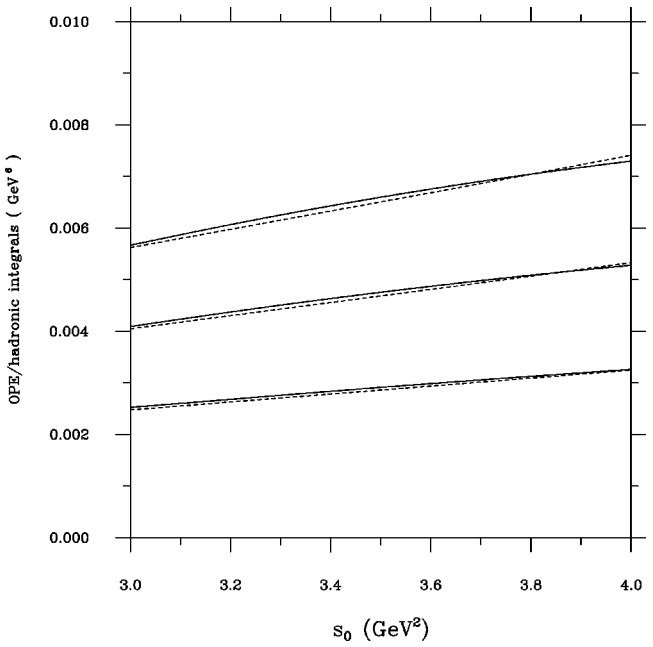

FIG. 2. The OPE+ILM versus hadronic (spectral integral) sides of the us w_N^A family of PFESR's, for m_s+m_u, $f_{K(1460)}$ and $f_{K(1830)}$ given by the central values of Eqs. (31), (32) and (33). The solid lines are the hadronic integrals, the dashed lines the corresponding OPE integrals. The lower, middle and upper lines in each case correspond to $A=0$, 2 and 4, respectively.

To get a feel for the relative size of the various contributions to the "theory" error we note that, for the ud case, the errors in $[m_u+m_d](2\ \text{GeV})$ due to the uncertainties noted above on the input parameters ρ_{VSA}, $\langle\alpha_s G^2\rangle$, $\alpha_s(m_\tau^2)$ and truncation at $O(a^3)$ are ± 0.25, ± 0.05, ± 0.28 and ± 0.25 MeV, respectively. The corresponding contributions to the errors on $m_s(2\ \text{GeV})$ are ± 1.5, ± 0.4, ± 2.3 and ± 3.1 MeV, respectively, with a further contributions of ± 0.2 MeV due to the range of r_c employed in this case. The agreement between the OPE and spectral integral sides of the various PFESR's corresponding to the results above is very good. The fit quality for the us channel is displayed, for the w_N^A and w_D^A families, in Figs. 2 and 3, respectively. The analogous w_N^A and w_D^A fits for the ud channel are shown in Figs. 4 and 5 of Ref. [51], respectively. The ratio $R=25.6\pm 2.6$ implied by the above results is in good agreement with the value, 24.4 ± 1.5 obtained from ChPT in Ref. [44].

If one repeats the PFESR analysis, but now with the ILM contributions set to zero, one finds, for the ud case,

$$[m_u+m_d](2\ \text{GeV})=9.9\pm 1.2_\Gamma\pm 1.0_{theory}\pm 0.5_{method}\ \text{MeV} \tag{34}$$

$$f_{\pi(1300)}=2.41\pm 0.50_\Gamma\pm 0.21_{theory}\pm 0.27_{method}\ \text{MeV} \tag{35}$$

$$f_{\pi(1800)}=1.36\pm 0.16_\Gamma\pm 0.09_{theory}\pm 0.11_{method}\ \text{MeV}, \tag{36}$$

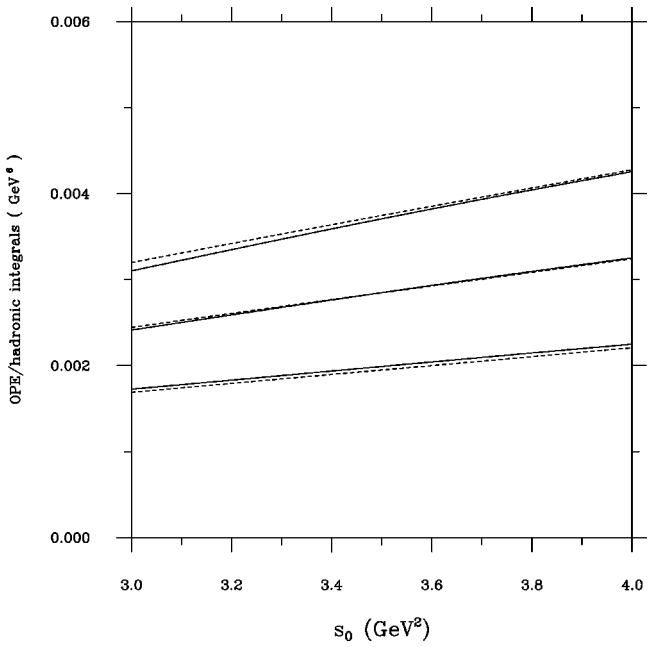

FIG. 3. The OPE+ILM versus hadronic (spectral integral) sides of the us w_D^A family of PFESR's for m_s+m_u, $f_{K(1460)}$ and $f_{K(1830)}$ given by the central values of Eqs. (31), (32) and (33). The identification of OPE and hadronic integrals, and the cases $A=0,2,4$ is as for Fig. 2 above.

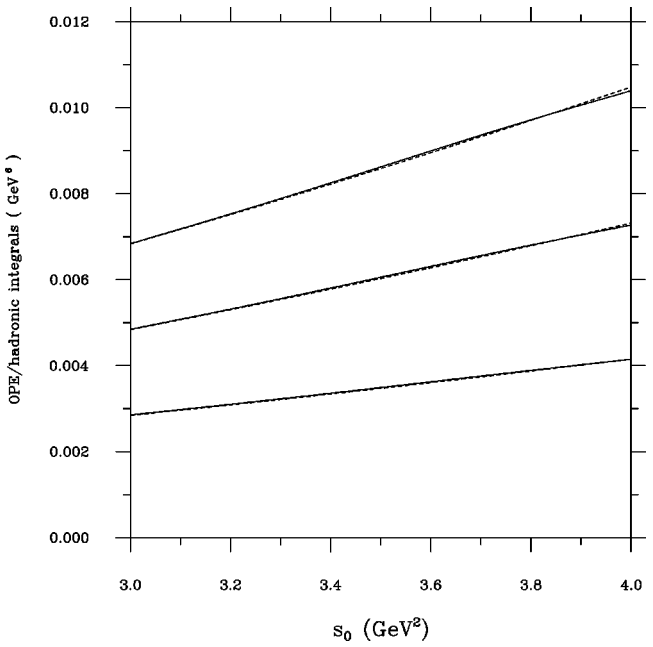

FIG. 4. The OPE versus hadronic (spectral integral) sides of the us w_N^A family of PFESR's for m_s+m_u, $f_{K(1460)}$ and $f_{K(1830)}$ given by the central values of Eqs. (37), (38) and (39), i.e., in the absence of ILM contributions. The identification of OPE and hadronic integrals, and the cases $A=0,2,4$ is as for Fig. 2 above.

and, for the us case,

$$m_s(2\ \text{GeV}) = 116 \pm 7_{theory} \pm 3_{method}\ \text{MeV} \tag{37}$$

$$f_{K(1460)} = 22.9 \pm 2.1_{theory} \pm 1.2_{method}\ \text{MeV} \tag{38}$$

$$f_{K(1830)} = 14.5 \pm 1.5_{theory} \pm 0.4_{method}\ \text{MeV}. \tag{39}$$

The corresponding OPE versus spectral integral match is again excellent. This is illustrated for the us case, for the w_N^A family of PFESR's, in Fig. 4. (The agreement for the corresponding w_D^A PFESR's as well as that for the ud case is not shown explicitly, but is, in fact, of equal quality to that for the us w_N^A family.) The resulting mass ratio, $R=23.3\pm 2.8$, is also in good agreement with that obtained from ChPT. We thus see that, while the PFESR fit provides a good determination of m_i+m_j *and* the resonance decay constants *once the form of the theoretical side of the sum rule (i.e., whether including or excluding ILM contributions) has been fixed*, it does not, by itself, provide any additional evidence as to whether inclusion or exclusion of these contributions is favored. While inclusion of ILM effects is, of course, indicated by arguments external to the PFESR analysis, the PFESR analysis itself shows only that, in the absence of these contributions, significantly larger values of the relevant quark mass combination and second resonance decay constant are required in both the ud and us channels.

We now turn to the BSR analyses of the ud and us channels, which should provide additional constraints on the ILM modeling of instanton effects in the PFESR analyses. Expressions for the Borel transforms of the OPE side of the sum rules can be found in Refs. [11,12,28], and that for the Borel transform of the ILM contributions in Ref. [13]. We take central values for all OPE input, and employ the corresponding central values for the excited resonance decay constants, determined above, as input to the BSR analysis. To facilitate the BSR-PFESR comparison, we quote only those errors present in the BSR analysis which do *not* also enter the PFESR analysis, namely those associated with (1) the ± 0.5 GeV^2 variation of the continuum threshold parameter s_0 about its optimal stability value and (2) the assumed 20% uncertainty in the size of the continuum spectral contribution. (Additional errors, associated with uncertainties in the values of the OPE input parameters, are common to both analyses, and the corresponding errors, as a result, are strongly correlated between the PFESR and BSR treatments.) To be conservative, we take, as our estimates for these errors, the maximum change in the value extracted for m_i+m_j in our BSR analysis window (see below) produced by the stated variations in s_0 and the magnitude of the continuum contribution. These two sources of error have been combined in quadrature in quoting results below. A conventional rule-of-thumb is that the BSR analysis window should be restricted to M^2 values for which the perturbative continuum contribution is less than $\sim 50\%$ of the OPE contribution (for a discussion see, for example, Ref. [52]). Since, for the ud case, this corresponds to M^2 less than ~ 3 GeV^2, we work with a BSR analysis window $2\ \text{GeV}^2 \leqslant M^2 \leqslant 3\ \text{GeV}^2$.

The dependence of $[m_i+m_j](2\ \text{GeV})$ on M^2 in the extended range $2\ \text{GeV}^2 \leqslant M^2 \leqslant 4\ \text{GeV}^2$ resulting from the BSR ILM analyses is shown in Fig. 5 for $ij=us$. (The analo-

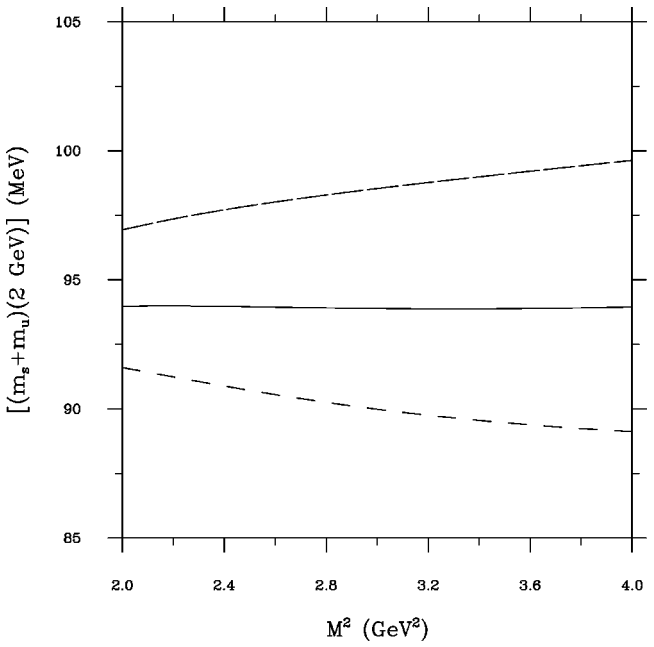

FIG. 5. The value of $[m_s+m_u](2\ \mathrm{GeV})$, as a function of the square of the Borel mass, M^2, extracted from the BSR analysis of the us pseudoscalar correlator described in the text. The solid line corresponds to $s_0=4.22\ \mathrm{GeV}^2$, which produces optimal stability for m_s+m_u with respect to M^2 in the window $2\ \mathrm{GeV}^2 \leqslant M^2 \leqslant 3\ \mathrm{GeV}^2$. The lower (short) dashed line corresponds to $s_0=4.72\ \mathrm{GeV}^2$ and the upper (long) dashed line to $s_0=3.72\ \mathrm{GeV}^2$.

gous result for $ij=ud$ is shown in Fig. 6 of Ref. [51].) The solid line corresponds to the optimal stability value of s_0, the upper and lower lines to values $0.5\ \mathrm{GeV}^2$ lower and higher, respectively. The quark mass values obtained from this analysis are

$$[m_u+m_d](2\ \mathrm{GeV})=7.5\pm 0.9\ \mathrm{MeV} \tag{40}$$

$$m_s(2\ \mathrm{GeV})=91\pm 9\ \mathrm{MeV}. \tag{41}$$

These results are to be compared to the central PFESR values of Eqs. (28) and (31) above. The consistency of the two determinations is excellent for the ud channel, but only marginally acceptable for the us channel. The consistency of the central PFESR and BSR us determinations can be improved by allowing somewhat larger values of η_{us}. For example, $\eta_{us}=0.8$ produces a central value $m_s(2\ \mathrm{GeV})=97$ MeV, with a corresponding central BSR determination 89 ± 9 MeV, while $\eta_{us}=1$ corresponds to $m_s(2\ \mathrm{GeV})=92$ MeV (PFESR) and 87 ± 10 MeV. In view of the size of the BSR errors, such improvement cannot be taken as physically meaningful; this exercise does, however, indicate that errors comparable in size to the difference of the PFESR and BSR central values, associated with the crudeness of the ILM representation of instanton effects, may still be present in the PFESR results. We will therefore include an additional ILM-induced error, to be discussed in more detail below, in the final version of our errors for the light quark masses.

For the case in which ILM contributions are included, the BSR results, corresponding to central values of all input, and the corresponding central values of the resonance decay constants, are

$$[m_u+m_d](2\ \mathrm{GeV})=8.8\pm 0.6\ \mathrm{MeV}$$

$$m_s(2\ \mathrm{GeV})=100\pm 6\ \mathrm{MeV}, \tag{42}$$

which are to be compared to the central values of Eqs. (34) and (37). The PFESR determinations in both cases lie significantly outside the range allowed by the BSR error.

Consistency between PFESR and BSR analyses thus favors inclusion of the ILM contributions. To see that the level of inconsistency between the PFESR and BSR results in the absence of ILM contributions is, in fact, significant, the following exercise is useful. Rather than optimizing the PFESR analysis by varying simultaneously m_i+m_j, f_1 and f_2, we may, for each value of m_i+m_j, find the values of f_1 and f_2 which produce the best OPE versus spectral integral match. We then use these values of f_1 and f_2, as usual, as input to the corresponding BSR analysis and look for those values of m_i+m_j for which the PFESR input value is compatible with the BSR output value, within the additional errors of the BSR analysis.

For the ud case, in the absence of ILM contributions, this compatibility is obtained only for $[m_u+m_d](2\ \mathrm{GeV})$ less than 8.1 MeV (PFESR value)/7.6 MeV (BSR value). Taking the ''marginal'' case, corresponding to the PFESR value $[m_u+m_d](2\ \mathrm{GeV})=8.1$ MeV to be specific, one finds that, although the quality of the OPE+ILM versus spectral integral match is significantly worse than that for the fully optimized fit above, it is perhaps still acceptable (see Fig. 6 for the fit quality for the w_D case; the quality is comparable, though marginally better, for the w_N case). Thus, in this case, although the inclusion of ILM contributions is favored, we do not consider it possible to rule out their absence. Note, however, that the analysis, in the absence of ILM contributions, is only self-consistent for values of m_u+m_d compatible with those obtained from the analysis including ILM contributions. The value of $f_{\pi(1300)}$ obtained in this case, 1.74 MeV, also turns out to be compatible, within errors, with that given by Eq. (29).

For the us case, in the absence of ILM contributions, compatibility is achieved only for $m_s(2\ \mathrm{GeV})$ less than 94 MeV (PFESR value)/89 MeV (BSR value). The ''best'' fit PFESR solution for such a value of m_s, however, represents an extremely poor quality OPE+ILM versus spectral integral match.[11] We thus find no acceptable, consistent spectral solution in the us case without the inclusion of ILM contributions. This, of course, also favors the inclusion of such contributions for the ud channel.

In view of these observations, we take as our final central values those obtained from the PFESR analysis with direct instanton contributions estimated using the ILM. Because of

[11]The *average* OPE versus spectral integral discrepancy over the s_0,A analysis window is, for example, 23% for the w_D PFESR family.

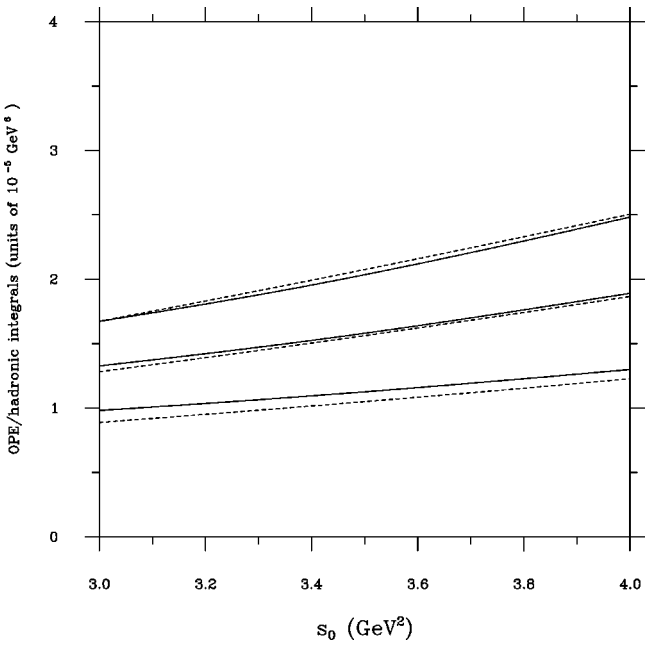

FIG. 6. The ud OPE-spectral integral match obtained for the w_D^A PFESR family using as PFESR input the value $[m_u+m_d](2\ \mathrm{GeV})=8.1$ MeV, the *largest* PFESR input for which PFESR and BSR values of m_u+m_d are consistent. All notation is as for the PFESR figures above. This largest "marginal" m_u+m_d value produces the best OPE-spectral integral match among those input values for which the PFESR input and BSR output values are consistent; the fit quality, moreover, deteriorates rapidly as one goes to lower values of the PFESR input.

the crudeness of the ILM, however, we need to include an estimate of the uncertainties associated with the use of the ILM in our final errors. We estimate this uncertainty as described in the following paragraph.

First we note that, for a fixed spectral *Ansatz* and fixed s_0, the shift in m_u+m_d (m_s+m_u) produced by including ILM contributions on the theoretical side of our PFESR's, averaged over the s_0 values in the analysis window and the full set of PFESR weights employed, is $\sim 8\%$ $(\sim 5\%)$. The much larger $(\sim 20\%)$ difference between the ILM and no-ILM best-fit values given in Eqs. (28) and (34) is *not* an accurate reflection of the relative size of OPE and ILM contributions to the theoretical PFESR integrals. Rather, the fact that ILM contributions decrease with increasing s_0, while the dominant $D=0$ OPE contributions increase with increasing s_0, means that the OPE+ILM sum is less strongly increasing with s_0 than is the OPE term alone. This behavior on the theoretical side of the PFESR's can only be matched on the spectral integral side if the ratio of the second to the first resonance spectral contributions is smaller in the OPE +ILM case. Since the decrease in the size of direct instanton contributions with increasing s_0 is an expected feature of such contributions, independent of the precise details of the ILM implementation, this type of qualitative shift in the structure of the resonance contributions to the spectral function is expected to be a general feature, so long as instanton contributions are not, in fact, negligible at the scales of our analysis. The incompatibility of the BSR and PFESR results obtained when one neglects direct instanton contributions in the PFESR analysis strongly suggests that they are, indeed, not negligible. We, therefore, take, as a (hopefully conservative) estimate of the ILM-induced uncertainty, the maximum of (i) the 8% (5%) average ILM contribution and (ii) the difference between the central values of the ILM PFESR and BSR extractions. In the flavor ud case, the difference of central values is only 0.3 MeV, so the ILM-induced error, ± 0.6 MeV, is determined by the average ILM contribution; in the us case, the average ILM contribution is only 5 MeV, so the error, ± 9 MeV, is determined by the difference of the central values given in Eqs. (31) and (41).

Including our additional estimate of ILM-induced error, and combining all sources of error in quadrature, our final results for the light quark masses thus become

$$[m_u+m_d](2\ \mathrm{GeV})=7.8\pm 1.2\ \mathrm{MeV} \tag{43}$$

$$m_s(2\ \mathrm{GeV})=100\pm 12\ \mathrm{MeV}. \tag{44}$$

Since the PFESR-BSR consistency is excellent for the ud channel, but marginal for the us channel, an alternate determination of m_s, using the result of Eq. (43) above in combination with the ChPT-determined mass ratio $R=24.4\pm 1.5$, might be preferable. The result of this determination,

$$m_s(2\ \mathrm{GeV})=95\pm 15\ \mathrm{MeV}, \tag{45}$$

is in good agreement with that of Eq. (44), with only slightly larger errors. Recall that *self-consistent* versions of the combined PFESR-BSR analysis in which direct instanton contributions are neglected, in fact, yield values for the light quark masses completely compatible with those of Eqs. (43) and (44). For the resonance decay constants, we note that, although the value of the second resonance decay constant is sensitive to whether or not one includes ILM contributions, that of the first resonance is largely insensitive to the presence or absence of ILM contributions, the central values differing by considerably less than the uncertainties on the individual determinations. We thus believe that, although the ILM may represent a relatively crude model for implementing direct instanton effects, the determination of the $\pi(1300)$ and $K(1460)$ decay constants given by Eqs. (29) and (32) should be reliable to within the stated errors. Combining these errors in quadrature we then have, for our final results,

$$f_{\pi(1300)}=2.20\pm 0.46\ \mathrm{MeV} \tag{46}$$

$$f_{K(1460)}=21.4\pm 2.8\ \mathrm{MeV}. \tag{47}$$

That these values differ by a factor of ~ 10 is compatible with the fact that the excited pseudoscalar decay constants vanish in the chiral limit, and hence are proportional to the relevant quark mass combination near that limit.

C. Discussion

Other recent sum rule analyses exist for the pseudoscalar ud [10,20] and us [11,12] channels. In addition, sum rule analyses of the us scalar correlator [11,27,28,53,54], and of flavor breaking in hadronic τ decay [47,48,55–58], have

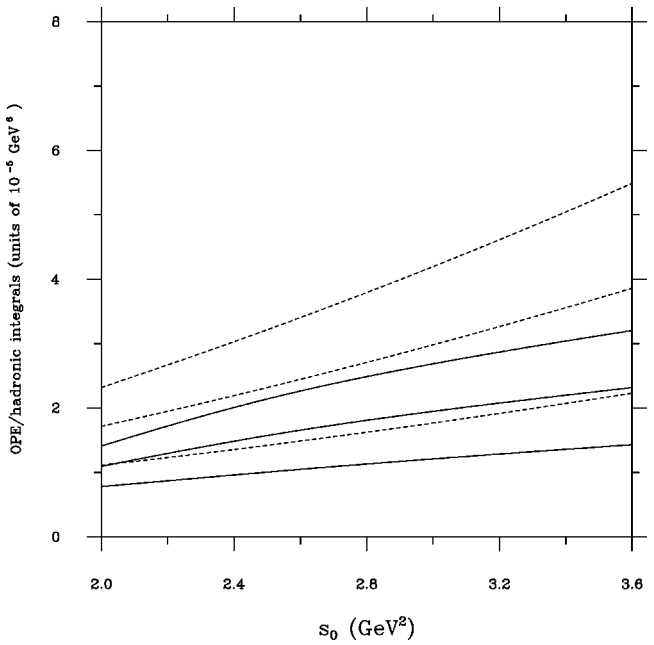

FIG. 7. The ud OPE-spectral integral match obtained for the w_N^A PFESR family using the central values of all OPE input, the quoted P98 value of $[m_u+m_d](1\ \text{GeV})$ and the P98 spectral *Ansatz*. All notation is as for the PFESR figures above.

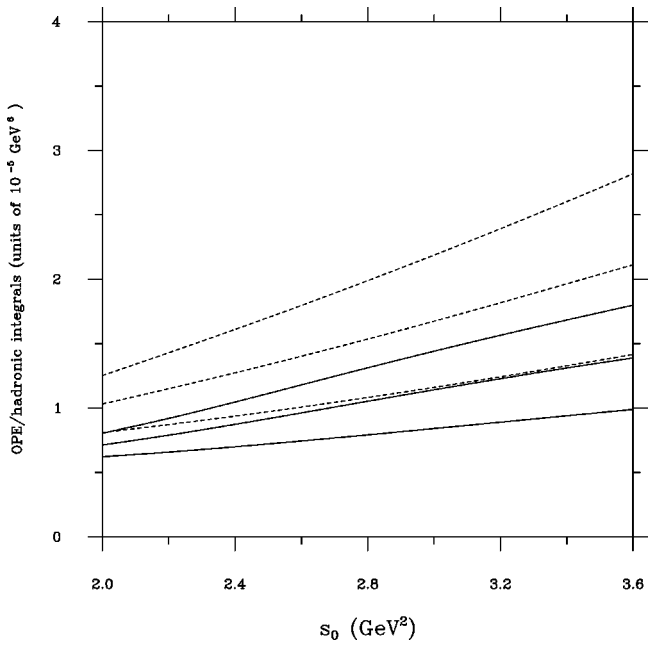

FIG. 8. The ud OPE-spectral integral match obtained for the w_D^A PFESR family using the central values of all OPE input, the quoted P98 value of $[m_u+m_d](1\ \text{GeV})$ and the P98 spectral *Ansatz*. All notation is as for the PFESR figures above.

been used to extract m_s. In this section we discuss the relation of our work to that of these earlier references.

For the ud pseudoscalar channel, Ref. [20] (P98) represents an update of Ref. [10]. (The latter employed 3-loop versions for the OPE $D=0$ contribution, the running mass and the running coupling, P98 the 4-loop versions.) We therefore restrict our discussion to the latter analysis. The resonance part of the P98 spectral function is of the TRDA form, but rescaled by an overall factor 1.5. Two points should be borne in mind regarding the value quoted for m_u+m_d in P98. The first is that the analysis is based on FESR's involving the weights $w(s)=1$ and s. For these weights, however, the corresponding vector isovector channel FESR's are *not* well-satisfied at the scales employed in P98. (The OPE side has a significantly weaker s_0 dependence than the spectral integral side, the latter being obtained, in this case, from experimental τ decay data [24].) The second point is that the ratio of quoted values for the running mass at scales 1 and 2 GeV, $m(1\ \text{GeV})/m(2\ \text{GeV})=1.31$ [20], differs from that, 1.38, obtained using 4-loop running with the central ALEPH determination of $\alpha_s(m_\tau)$ as input. The results of P98 thus correspond to a smaller value, $\alpha_s(m_\tau)=0.307$, the effect of which would be to produce a larger value of m_u+m_d. One would thus expect a poor match between the OPE and spectral integral sides of PFESR's employing the P98 spectral *Ansatz* and central m_u+m_d value in combination with current central values for the OPE input. This is confirmed by the results shown in Figs. 7 and 8, which correspond, respectively, to the output from the w_N^A and w_D^A PFESR weight family analyses, in our s_0,A analysis window, obtained using the P98 central value for m_u+m_d and the P98 spectral *Ansatz*. If one performs a re-analysis, still using the P98 spectral *Ansatz*, but now optimizing the value of m_u+m_d using the PFESR approach, one finds, using central values for all OPE input, and including ILM contributions,

$$[m_u+m_d](2\ \text{GeV})=6.8\ \text{MeV}. \tag{48}$$

The same analysis, without ILM contributions, similarly yields

$$[m_u+m_d](2\ \text{GeV})=7.3\ \text{MeV}. \tag{49}$$

Both of these values are, in fact, in agreement with those corresponding to the upper part of the s_0 range displayed in Fig. 2 of P98, though not with those for $s_0\sim 2\ \text{GeV}^2$. In both cases, however, the quality of the OPE($+$ILM) versus spectral integral match is much inferior to that obtained obtained using the solutions for m_u+m_d, $f_{\pi(1300)}$ and $f_{\pi(1800)}$ above. The optimized match is significantly better when ILM contributions are included than when they are not. However, in spite of optimization, the consistency between the w_N^A and w_D^A PFESR families is not good for the P98 spectral *Ansatz*: as shown in [51], the match for w_N^A is best where that for w_D^A is worst, and vice versa (see Figs. 2,3 in [51]).

For the us pseudoscalar channel, the BSR analyses of Refs. [11] (JM) and [12] (DPS) both employ a TRDA construction for the $K(1460)$ and $K(1830)$ contributions to the spectral *Ansatz*, but differ in their assumptions about the relative sizes of the two resonance decay constants: JM assume $f_2^2m_2^4/f_1^2m_1^4=0.25$, DPS that the spectral contributions of the two resonances at threshold are approximately equal (for PDG2000 values of the masses and widths, this corresponds to $f_2^2m_2^4/f_1^2m_1^4\simeq 1.8$). The two analyses also differ in their

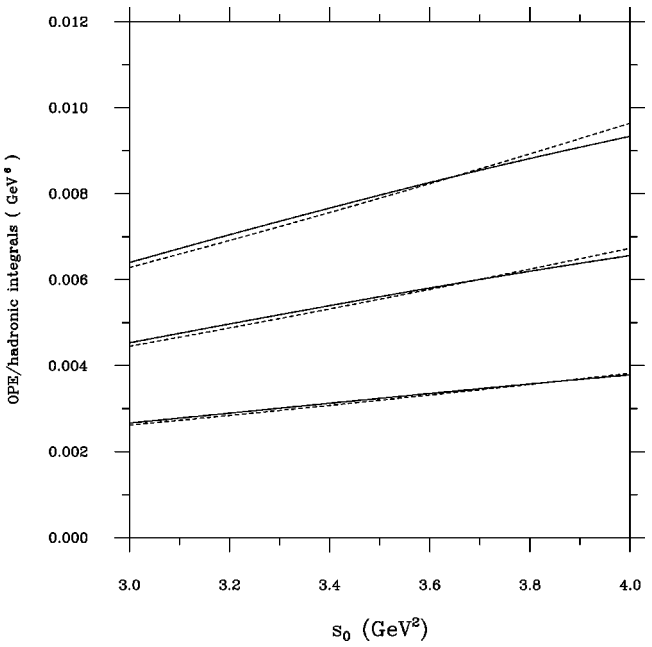

FIG. 9. The *us* pseudoscalar OPE-spectral integral match obtained for the w_N^A PFESR family using the JM spectral *Ansatz*, central values of all OPE input, and no ILM contributions, after optimization of m_s+m_u in a combined w_N^A, w_D^A PFESR analysis. All notation is as for the PFESR figures above.

treatment of the theoretical side, JM employing 3-loop expressions on the OPE side and DPS the 4-loop expressions which became available subsequent to the publication of the JM paper.

We have updated the JM BSR analysis to include 4-loop contributions to the running mass, coupling and $D=0$ OPE term. For OPE input we use the values employed in our analyses above. Including ILM contributions on the theoretical side of the BSR, we then find that the JM spectral *Ansatz*, ρ_{JM}, corresponds to

$$m_s(2\ \text{GeV}) = 96 \pm 7\ \text{MeV}. \tag{50}$$

Neglecting instanton contributions, as in JM, we obtain instead

$$m_s(2\ \text{GeV}) = 98 \pm 6\ \text{MeV}. \tag{51}$$

(The errors in these equations have the same meaning as those for the BSR analyses above.) If, however, we employ ρ_{JM}, as input, not to a BSR analysis, but to our usual PFESR analysis, we find for our central values

$$m_s(2\ \text{GeV}) = 107\ \text{MeV} \tag{52}$$

if ILM contributions are included, and

$$m_s(2\ \text{GeV}) = 111\ \text{MeV} \tag{53}$$

if they are not. The fit quality for the optimized match is rather poor when ILM contributions are included, but is quite good when they are not. The latter point is illustrated for the w_N^A family of PFESR's in Fig. 9 (the quality of the match for the w_D^A family, which is not shown, is even better). Despite the existence of both a good quality PFESR OPE or spectral integral match and an excellent BSR stability window, however, we see that the no-ILM PFESR and BSR m_s determinations based on ρ_{JM} are inconsistent, just as was the case for the determinations associated with the spectral *Ansatz* based on the values of $f_{K(1460)}$ and $f_{K(1830)}$ obtained from the no-ILM PFESR analysis. This is, in fact, not surprising, since the optimized PFESR spectral *Ansatz* turns out to be rather similar to ρ_{JM}, the $K(1460)$ decay constants of the two models, for example, differing by less than 6%.

In discussing the DPS analysis of the *us* pseudoscalar channel, one should bear in mind that the result quoted by DPS, $m_s(1\ \text{GeV}) = 155 \pm 25$ MeV corresponds to (1) an average of the values obtained using $\Lambda^{(3)}_{QCD} = 280$ MeV and 380 MeV, (2) an average over values associated with a range of s_0, and (3) neglect of m_u in the OPE prefactor.[12] Since the choice $\Lambda^{(3)}_{QCD} = 280$ MeV is not consistent with the ALEPH determination of $\alpha_s(m_\tau)$, we restrict our attention to the DPS results obtained using $\Lambda^{(3)}_{QCD} = 380$ MeV, which corresponds very closely to the central ALEPH determination. Restoring m_u in the overall OPE prefactor, and reading off from Fig. 2 of DPS, concentrating on the curve corresponding to $s_0 = 6\ \text{GeV}^2$, which displays the best stability of m_s with respect to M^2, the central DPS BSR determination becomes $m_s(2\ \text{GeV}) = 97$ MeV. Since the details of the spectral *Ansatz* employed are not fully specified in DPS, we are unable to quote errors equivalent to those of our BSR analyses above. If, however, we fix the ratio of decay constants in such a way as to ensure exact equality of the $K(1460)$ and $K(1830)$ contributions to the spectral function at physical threshold, and neglect ILM contributions, as in DPS, we find that, after performing our usual PFESR analysis, the resulting spectral *Ansatz*, run through a BSR analysis with $s_0 = 6\ \text{GeV}^2$, reproduces the DPS central value exactly. Estimating our BSR errors as for the analyses above, we then have, for our DPS-like BSR determination,

$$m_s(2\ \text{GeV}) = 97 \pm 6\ \text{MeV}. \tag{54}$$

The PFESR OPE versus spectral integral match corresponding to this BSR determination is reasonable (see Fig. 10 for the w_N^A family case; the fit quality for the w_D^A family is not shown, but is better than for the w_N^A family). The central no-ILM PFESR m_s value,

$$m_s(2\ \text{GeV}) = 109\ \text{MeV}, \tag{55}$$

however, is again inconsistent with the corresponding BSR value. The situation is not improved by including ILM contributions: re-doing the PFESR analysis, still with the constrained form of the spectral *Ansatz*, but now incorporating ILM contributions on the theoretical side, one finds a poor quality optimized OPE+ILM versus spectral integral match.

[12]Restoring m_u to the prefactor, using ChPT values for the quark mass ratios, and converting to the scale $\mu = 2$ GeV, the DPS result becomes $m_s(2\ \text{GeV}) = 109 \pm 18$ MeV.

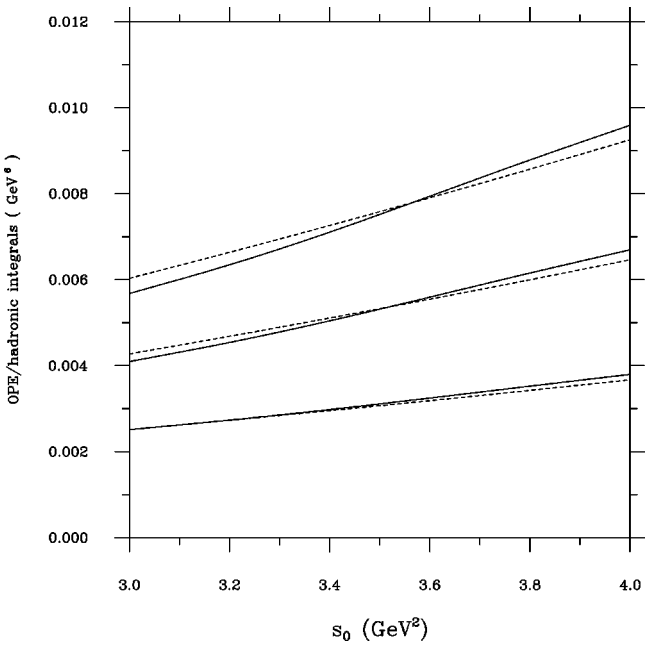

FIG. 10. The us pseudoscalar OPE-spectral integral match for the w_N^A PFESR family involving the spectral *Ansatz* obtained from a combined w_N^A, w_D^A PFESR analysis after imposing the DPS-like constraint on the ratio of $K(1460)$ and $K(1830)$ decay constants. The results correspond to central values of all OPE input, and to neglect of ILM contributions. All notation is as for the PFESR figures above.

We conclude this section with a reminder of the values obtained for m_s via sum rule analyses of other channels. Recent treatments of the correlator of $\partial_\mu(\bar{s}\gamma^\mu u)$ [27,53,54], for which the low-s part of the spectral function is constrained by $K\pi$ phases, yield values of $m_s(2\ \text{GeV})$ in the range 115 ± 25 MeV, compatible with either the ILM or no-ILM results above. Assumptions about the form of the Omnes representation of the timelike scalar $K\pi$ form factor, and the behavior of the $K\pi$ phase in the region $s>2.9\ \text{GeV}^2$, where experimental phase data does not exist, however, enter the construction of the spectral function used in those analyses, so that a significant theoretical systematic error is present, in addition to the errors quoted in Refs. [27,53,54]. A much cleaner approach, in principle, is the extraction of m_s via PFESR analyses of the flavor-breaking difference of ud and us vector-plus-axial-vector correlator sums. The hadronic spectral function required in this case is measurable in hadronic τ decay. There are two basic complications, first, that the OPE representation of the longitudinal contribution to the τ hadronic decay width is very badly behaved at those scales which are kinematically allowed [59–62] and, second, that, because of the rather strong cancellation in the ud-us spectral difference, the extracted value of m_s is quite sensitive to even $\sim1\%$ uncertainties in the value of $|V_{us}|$. The first problem can be handled by appropriate weight choices [47]. The second is numerically relevant because the central values of the determinations of $|V_{us}|$ based on (1) experimental K_{l3} data, $|V_{us}|=0.2196\pm0.0023$ and (2) Cabibbo-Kobayashi-Maskawa (CKM) unitarity, in combination with the experimental value of $|V_{ud}|$, $|V_{us}|=0.2225\pm0.0035$,[13] while consistent within errors, differ by $\sim1.3\%$. There has also been some confusion in the literature resulting from the use, in the various recent theoretical analyses, of three different sets of values for the weighted ud-us spectral differences, corresponding to three different values of B_{us}, the total $(V+A)$ branching fraction into strange hadronic states.[14] The strong ud-us cancellation makes the extracted value of m_s quite sensitive to the (apparently rather small) differences between these B_{us} values. The discrepancies between the various values of m_s reported in the literature, all of which are nominally based on the "same" (ALEPH) τ decay data, turn out to be almost entirely a reflection of this sensitivity. The situation is discussed in some detail in Ref. [64], where the various analyses have also been updated to reflect the current experimental situation (as reported in Ref. [58]). Once common input is employed, all hadronic τ determinations of m_s are in excellent agreement [64]. The dominant uncertainty remains that associated with $|V_{us}|$. Using central values of $|V_{ud}|$ and $|V_{us}|$ corresponding to either (1) the PDG2000 best independent individual determinations (CKMN) ($|V_{ud}|=0.9735$ and $|V_{us}|=0.2196$) or (2) the PDG2000 unitarity-constrained fit (CKMU) ($|V_{ud}|=0.9749$ and $|V_{us}|=0.2225$), one obtains

$$m_s(2\ \text{GeV})=101\pm18\ \text{MeV}\quad(\text{CKMN}),\tag{56}$$

and

$$m_s(2\ \text{GeV})=114\pm16\ \text{MeV}\quad(\text{CKMU})\tag{57}$$

respectively [64]. Either of these results is compatible with that obtained from the pseudoscalar channel analyses above.

IV. CONCLUSIONS

We have determined m_u+m_d, m_s, and the decay constants of the $\pi(1300)$ and $K(1460)$ with good accuracy from a combined PFESR-BSR study of the ud and us pseudoscalar correlators. Our results show that it is important to require the consistency of the two different sum rule approaches. Indeed, we have seen that there exist *Ansätze* for the hadronic spectral functions which produce both extremely good BSR stability plateaus and high-quality PFESR OPE(+ILM) versus spectral integral matches, but for which the output quark mass combinations are inconsistent. This means that BSR or PFESR treatments, by themselves, do not provide sufficiently strong constraints to allow one to simultaneously constrain the unknown quark masses, unknown

[13]The central value and "errors" quoted here correspond to the mid-point and extent of the PDG2000 unitarity-constrained fit range.

[14]These three values, which are in the ratios 1:1.04:1.05 correspond to (1) the preliminary (1998) ALEPH analysis of strange decay modes [63], (2) the final (1999) version of this analysis [55], and (3) the recent update (2000) reported by Davier [58]. Larger values of B_{us} correspond to smaller values of the ud-us difference, and hence to lower values of m_s.

resonance decay constants and the theoretical modelling of direct instanton effects. The combination of the two approaches does, however, provide sufficiently strong constraints. The consistency of the combined analysis is particularly compelling for the ud case. The values obtained for the light quark masses are in excellent agreement with determinations from other sources, giving us further confidence in the reliability of the combined analysis. The corresponding determinations of the $\pi(1300)$ and $K(1460)$ decay constants are accurate to 20% and 10% respectively. The latter determination is relevant to future improvements in the extraction of m_s from hadronic τ decay data. While B factory data will dramatically reduce the errors on the experimental us vector-plus-axial-vector τ decay distribution, the ability to use this improvement to significantly reduce the errors on the corresponding determination of m_s will depend on one's ability to work with PFESR's involving weights for which the ud-us cancellation is significantly reduced, so that the errors resulting from the uncertainty in $|V_{us}|$ will, as a result, play a significantly reduced role. The existence of significant theoretical systematic uncertainties in versions of this analysis which include longitudinal OPE contributions [62] means that "non-inclusive" analyses (involving only the sum of spin 0 and 1 correlator components) will eventually be required. A knowledge of the decay constants of the excited strange pseudoscalar and scalar resonances allows a straightforward subtraction of the longitudinal contributions to the experimental distributions. In the absence of an experimental spin separation, sum rule determinations of the strange scalar and pseudoscalar resonance decay constants with an accuracy even a factor of three worse than that obtained above for the $K(1460)$ are already extremely useful as input for such non-inclusive analyses.

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[1] A.I. Vainshtein *et al.*, Sov. J. Nucl. Phys. **27**, 274 (1978).
[2] C. Becchi, S. Narison, E. de Rafael, and F.J. Yndurain, Z. Phys. C **8**, 335 (1981).
[3] J. Gasser and H. Leutwyler, Phys. Rep. **87**, 77 (1982).
[4] S. Narison, N. Paver, and D. Treleani, Nuovo Cimento A **74**, 347 (1983).
[5] C.A. Dominguez, Z. Phys. C **26**, 269 (1984).
[6] L. Lellouch, E. de Rafael and J. Taron, Phys. Lett. B **414**, 195 (1997).
[7] M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. **B147**, 385, (1979); **B147**, 448 (1979).
[8] S. Narison and E. de Rafael, Phys. Lett. **103B**, 57 (1981).
[9] S. Narison, Phys. Lett. B **216**, 191 (1989).
[10] J. Bijnens, J. Prades, and E. de Rafael, Phys. Lett. B **348**, 226 (1995).
[11] M. Jamin and M. Münz, Z. Phys. C **66**, 633 (1995).
[12] C.A. Dominguez, L. Pirovano, and K. Schilcher, Phys. Lett. B **425**, 193 (1998).
[13] V. Elias *et al.*, Phys. Lett. B **412**, 131 (1997).
[14] T.G. Steele, K. Kostuik, and J. Kwan, Phys. Lett. B **451**, 201 (1999); T.G. Steele, hep-ph/0010056.
[15] N. Hubschmid and S. Mallik, Nucl. Phys. **B193**, 368 (1981).
[16] T.N. Truong, Phys. Lett. **117B**, 109 (1982).
[17] A.L. Kataev, N.V. Krasnikov, and A.A. Pivovarov, Phys. Lett. **123B**, 93 (1983).
[18] S.G. Gorishny, A.L. Kataev, and S.A. Larin, Phys. Lett. **135B**, 457 (1984).
[19] C.A. Dominguez and E. de Rafael, Ann. Phys. (N.Y.) **174**, 372 (1987).
[20] J. Prades, Nucl. Phys. B (Proc. Suppl.) **64**, 253 (1998).
[21] E. Poggio, H. Quinn, and S. Weinberg, Phys. Rev. D **13**, 1958 (1976).
[22] ALEPH Collaboration, R. Barate *et al.*, Z. Phys. C **76**, 379 (1997); Eur. Phys. J. C **4**, 409 (1998).
[23] OPAL Collaboration, G. Abbiendi *et al.*, Phys. Lett. B **447**, 134 (1999); Eur. Phys. J. C **7**, 571 (1999).
[24] K. Maltman, Phys. Lett. B **440**, 367 (1998).
[25] Particle Data Group, D.E. Groom *et al.*, Eur. Phys. J. C **15**, 1 (2000).
[26] T. Bhattacharya, R. Gupta, and K. Maltman, Phys. Rev. D **57**, 5455 (1998).
[27] P. Colangelo, F. De Fazio, G. Nardulli, and N. Paver, Phys. Lett. B **408**, 340 (1997).
[28] K.G. Chetyrkin, D. Pirjol, and K. Schilcher, Phys. Lett. B **404**, 337 (1997).
[29] K. Maltman, Phys. Lett. B **462**, 14 (1999); Nucl. Phys. **A680**, 171 (2000).
[30] V.A. Novikov *et al.*, Nucl. Phys. **B191**, 301 (1981).
[31] E.V. Shuryak, Nucl. Phys. **B203**, 93 (1982); **B203**, 116 (1982).
[32] E.V. Shuryak, Nucl. Phys. **B214**, 237 (1983).
[33] A.E. Dorokhov and N.I. Kochelev, Z. Phys. C **46**, 281 (1990).
[34] E.V. Shuryak, Phys. Rep. **115**, 151 (1984); E.V. Shuryak and J.J.M. Verbaarschot, Nucl. Phys. **B410**, 55 (1993); E.V. Shuryak, Rev. Mod. Phys. **65**, 1 (1993); T. Schafer and E.V. Shuryak, *ibid.* **70**, 323 (1998).
[35] T.G. Steele (private communication).
[36] A.E. Dorokhov, S.V. Esaibegian, N.I. Kochelev, and N.G. Stefanis, J. Phys. G **23**, 643 (1997); V. Elias, F. Shi, and T.G. Steele, *ibid.* **24**, 267 (1998).
[37] S. Narison, Nucl. Phys. B (Proc. Suppl.) **54A**, 238 (1997).
[38] G. Colangelo, J. Gasser, and H. Leutwyler, Phys. Rev. Lett. **86**, 5008 (2001).
[39] A.A. Ovchinnikov and A.A. Pivovarov, Sov. J. Nucl. Phys. **48**, 721 (1988).
[40] A.A. Pivovarov, Sov. J. Nucl. Phys. **54**, 676 (1991); Z. Phys. C **53**, 461 (1992).

[41] F. Le Diberder and A. Pich, Phys. Lett. B **286**, 147 (1992); *ibid.* **289**, 165 (1992).
[42] T. van Ritbergen, J.A.M. Vermaseren, and S.A. Larin, Phys. Lett. B **400**, 379 (1997).
[43] K.G. Chetyrkin, Phys. Lett. B **404**, 161 (1997); T. Van Ritbergen, J.A.M. Vermaseren, and S.A. Larin, *ibid.* **405**, 327 (1997).
[44] H. Leutwyler, Phys. Lett. B **374**, 163 (1996); **378**, 313 (1996); Nucl. Phys. B (Proc. Suppl.) **94**, 108 (2001).
[45] H.G. Dosch and S. Narison, Phys. Lett. B **417**, 173 (1998).
[46] R.F. Lebed and K. Schilcher, Phys. Lett. B **430**, 341 (1998).
[47] J. Kambor and K. Maltman, Phys. Rev. D **62**, 093023 (2000); Nucl. Phys. B (Proc. Suppl.) **98**, 314 (2001).
[48] K.G. Chetyrkin, J.H. Kühn, and A.A. Pivovarov, Nucl. Phys. **B533**, 473 (1998).
[49] S. Groote, J.G. Körner, and A.A. Pivovarov, Phys. Rev. D **65**, 036001 (2002).
[50] A.L. Kataev, hep-ph/0102223.
[51] K. Maltman and J. Kambor, Phys. Lett. B **517**, 332 (2001).
[52] D.B. Leinweber, Ann. Phys. (N.Y.) **254**, 328 (1997).
[53] M. Jamin, Nucl. Phys. B (Proc. Suppl.) **64**, 250 (1998).
[54] K. Maltman, Phys. Lett. B **462**, 195 (1999).
[55] ALEPH Collaboration, R. Barate *et al.*, Eur. Phys. J. C **11**, 599 (1999).
[56] A. Pich and J. Prades, J. High Energy Phys. **10**, 004 (1999).
[57] J.G. Körner, F. Krajewski, and A.A. Pivovarov, Eur. Phys. J. C **20**, 259 (2001).
[58] M. Davier, S.M. Chen, A. Höcker, J. Prades, and A. Pich, Nucl. Phys. B (Proc. Suppl.) **98**, 319 (2001); A. Pich, *ibid.* **98**, 385 (2001); S. Chen *et al.*, Eur. Phys. J. C **22**, 31 (2001).
[59] K. Maltman, Phys. Rev. D **58**, 093015 (1998).
[60] A. Pich and J. Prades, J. High Energy Phys. **06**, 013 (1998).
[61] J.G. Körner, F. Krajewski, and A.A. Pivovarov, Eur. Phys. J. C **14**, 123 (2000).
[62] K. Maltman and J. Kambor, Phys. Rev. D **64**, 093014 (2001).
[63] S. Chen, Nucl. Phys. B (Proc. Suppl.) **64**, 265 (1998).
[64] R. Gupta and K. Maltman, hep-ph/0101132; ''Status Report on Lattice QCD and QCD Sum Rule Determinations of the Light Quark Masses'' (in preparation).