Determination of the pion-nucleon coupling constant and scattering lengths

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(Received 6 October 2000; revised manuscript received 4 February 2002; published 24 July 2002)

We critically evaluate the isovector Goldberger-Miyazawa-Oehme (GMO) sum rule for forward πN scattering using the recent precision measurements of $\pi^- p$ and $\pi^- d$ scattering lengths from pionic atoms. We deduce the charged-pion-nucleon coupling constant, with careful attention to systematic and statistical uncertainties. This determination gives, directly from data, $g_c^2(\mathrm{GMO})/4\pi = 14.11 \pm 0.05(\mathrm{statistical}) \pm 0.19(\mathrm{systematic})$ or $f_c^2/4\pi = 0.0783(11)$. This value is intermediate between that of indirect methods and the direct determination from backward *np* differential scattering cross sections. We also use the pionic atom data to deduce the coherent symmetric and antisymmetric sums of the pion-proton and pion-neutron scattering lengths with high precision, namely, $(a_{\pi^- \mathrm{p}} + a_{\pi^- n})/2 = [-12 \pm 2(\mathrm{statistical}) \pm 8(\mathrm{systematic})] \times 10^{-4}\ m_\pi^{-1}$ and $(a_{\pi^- \mathrm{p}} - a_{\pi^- n})/2 = [895 \pm 3(\mathrm{statistical}) \pm 13\ (\mathrm{systematic})] \times 10^{-4}\ m_\pi^{-1}$. For the need of the present analysis, we improve the theoretical description of the pion-deuteron scattering length.

DOI: 10.1103/PhysRevC.66.014005 PACS number(s): 13.75.Gx, 11.55.Hx, 13.85.Lg, 25.80.Dj

I. INTRODUCTION

The pion-nucleon (πNN) coupling constant is of fundamental importance in both nuclear and particle physics. In nuclei it sets the scale of the interaction, together with the pion mass. In particle physics it is of great importance for the Goldberger-Treiman relation [1], one of the most important tests of chiral symmetry. Its experimental error is the main obstacle in the accurate discussion of the corrections to this relation as predicted from chiral symmetry breaking (see, for example, the discussion on p. 1086 of Ref. [2]). An accurate test requires a knowledge of the πNN coupling constant to a precision of about 1%, so as to match the experimental precision of the other quantities in the Goldberger-Treiman relation.

The present situation is summarized in Table I with uncertainties as quoted by the authors. In the 1980's, the πNN coupling constant was believed to be well known. In particular, Koch and Pietarinen [3] determined a value of the charged-pion coupling constant $g_c^2/4\pi = 14.28(18)$ from $\pi^\pm p$ scattering data, while Kroll [4] found the neutral-pion coupling constant $g_0^2/4\pi = 14.52(40)$ from a pp forward dispersion relation. This was put in question in the early 1990's, when the Nijmegen group published a series of papers [5–7] where they reported smaller values on the basis of energy-dependent partial-wave analyses (PWA) of NN scattering data. They obtained $g_0^2/4\pi = 13.47(11)$ and $g_c^2/4\pi = 13.58(5)$. Similarly low values with $g^2/4\pi$ about 13.7 have also been found by the Virginia Tech group [8–11] from an analysis of both $\pi^\pm N$ and NN data. Using a similar PWA method in the πp sector, Timmermans [12] found a value of 13.45(14). These more recent analyses often suffer from the drawback that they rely on the joint analysis of large data bases from many experiments with some of the data rejected according to various criteria. The statistical accuracy is high, but the systematic uncertainty is not clear. Exceptions are the Goldberger-Miyazawa-Oehme (GMO) sum rule [13] used by several groups [9,14,15] and the forward scattering sum rule for pp scattering [4], which, in principle, depend directly on physical observables. However, the dominant systematic uncertainties are not discussed and the uncertainties in the isovector scattering length used as input are large. In the case of Ref. [14] we have corrected their result as given in Table I to account for an erroneous input value according to the Erratum of Ref. [10]. Another direct determination is based on the extrapolation of experimental precision data on single-energy backward differential np cross sections to the pion pole [2,16]. This allows a systematic discussion of statistical and systematic uncertainties, but the uncertainty is so far larger than what can be achieved at present with the use of the GMO sum rule. The extrapolation method gives 14.52(26), a value significantly larger than those deduced by indirect methods. A review of the situation of the πNN coupling constant up to 1997 is found in Ref. [6]. The problems regarding its determination from np data have recently been discussed in a dedicated workshop [17–21] as well as in a recent conference working group [22].

To resolve these discrepancies it is desirable to have an independent precision determination, directly linked to mea-

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TABLE I. Some deduced values for the πNN coupling constant. The quoted uncertainty are those quoted by the authors and usually do not include systematic uncertainties.

Source	Year	System	$g^2_{\pi NN}/4\pi$
Karlsruhe-Helsinki [3]	1980	πp	14.28(18)[a]
Kroll [4]	1981	pp	14.52(40)[a]
Nijmegen [6]	1993	pp,np	13.58 (5)[a]
VPI [8]	1994	pp,np	13.70
Nijmegen [7]	1997	pp,np	13.54 (5)[a]
Timmermans [12]	1997	$\pi^+ p$	13.45(14)[a]
VPI [9]	1994	GMO, πp	13.75(15)[a]
Uppsala [2]	1998	$np \to pn$	14.52(26)
Pavan *et al.* [11]	1999	πp	13.73 (9)
Schröder *et al.*, corrected [14,10]	1999	GMO, $\pi^\pm p$	13.77(18)
Present work	2001	GMO, $\pi^\pm p$	14.11(20)

[a]Statistical uncertainty only.

sured quantities with quantifiable systematic and statistical errors. The systematic errors should be clearly identified, such that they can be explicitly discussed and they should be presented in an improvable form. The purpose of the present article is to demonstrate that recent experimental advances make the GMO relation suitable for this purpose. The GMO is a forward dispersion relation that expresses the charged coupling constant $g_c^2/4\pi$ in terms of the isovector πN scattering length (70% contribution) and a weighted integral J^- of the difference between the charged-pion total cross sections (30% contribution). This relation has been repeatedly evaluated in the past [9,10,14,15,23,24]. Since, until recently, there was little information on the scattering lengths available from direct data, these evaluations necessarily relied on scattering lengths extrapolated from semiphenomenological πN phase-shift analyses, using data from a range of energies above threshold. At the high precision needed, the systematic errors in the extrapolated scattering lengths are unclear and have, to our knowledge, not been estimated. The experimental situation has changed recently. The $\pi^- p$ and $\pi^- d$ scattering lengths can, to high precision, be deduced from recent experiments on pionic atoms. As a result, all the major ingredients in the GMO relation can now be discussed as experimentally derived quantities with transparent sources of uncertainty. Further, the approach can be improved by the observation that isospin conservation, which was previously assumed, can be replaced by the weaker assumption of charge symmetry. This avoids the possibility of perturbations from the rather important violation of isospin symmetry expected to be associated with the $\pi^0 p$ and $\pi^0 n$ scattering lengths [25,26]. The GMO relation can now be completely evaluated on the basis of data closely linked to direct experiments and it then determines the charged-pion coupling. We will develop this aspect below and also give a discussion of uncertainties in the dispersion integral.

The paper is organized as follows. In Sec. II we give a brief review of the GMO sum rule, reorganize it in the most efficient way for the present purpose, and discuss the magnitudes of the main contributions. Section III presents the information on the $\pi^- p$ and $\pi^- d$ scattering lengths deduced from data on pionic atoms. We draw the reader's attention to the most critical theoretical point in the present procedure for their extraction. Details on expressions used for the electromagnetic corrections to the experimental $\pi^- d$ scattering lengths are given in Appendix A. In Sec. IV we analyze and improve the theoretical approach to the πd scattering length with particular attention to a number of smaller terms. We use this understanding to deduce the most accurate values yet for the πN scattering lengths from the experimental data. Practical expressions for the theoretical $\pi^- d$ scattering length for separable scattering amplitudes are given in Appendix B. In Sec. V we analyze the uncertainties from different sources in the cross section integral J^-. In Sec. VI we summarize the conclusions about the scattering lengths and give the GMO sum rule result for the πNN coupling constant $g_c^2/4\pi$, with an explicit indication of systematic and statistical errors in each contributing term, presented in a form that can readily be improved or criticized.

II. THE GMO SUM RULE

The GMO sum rule for charged-pion–nucleon scattering is a very general forward dispersion relation, which assumes only analyticity and crossing symmetry. Contrary to the usual approach to its evaluation [9,10,14,15,23,24], it is not necessary to assume isospin symmetry [for a discussion of the GMO relation assuming isospin symmetry see Eq. (A.6.49) in Ref. [23]]. It takes the following form:

$$f_c^2/4\pi = [1-(m_\pi/2M)^2]\left[(1+m_\pi/M)\frac{m_\pi}{4}(a_{\pi^- p}-a_{\pi^+ p}) - \frac{m_\pi^2}{8\pi^2}\int_0^\infty \frac{\sigma^T_{\pi^- p}(k')-\sigma^T_{\pi^+ p}(k')}{\sqrt{k'^2+m_\pi^2}}dk'\right]. \quad (1)$$

Here m_π is the charged-pion mass and M the proton mass with the neutron-proton mass difference neglected, $a_{\pi^\pm p}$ the $\pi^\pm p$ scattering lengths, $\sigma^T_{\pi^\pm p}$ the total $\pi^\pm$ proton cross section, and k the pion laboratory momentum. The relation gives the charged-pion coupling constant $f_c^2/4\pi = (m_\pi/2M)^2 g_c^2/4\pi$ explicitly in terms of the charged-pion scattering lengths and total cross sections, all directly measurable. In writing Eq. (1) it has been tacitly assumed that Coulomb barrier corrections have been made to sufficient precision both in the extraction of the scattering lengths from pionic atoms and, in particular, in the determination of the total cross sections. We will discuss these issues as well as the effect of mass differences and isospin violation further below.

It is convenient to write the expression (1) in a simplified form with numerical coefficients

$$g_c^2/4\pi = -4.50\times J^- + 103.3\times\left(\frac{a_{\pi^- p}-a_{\pi^+ p}}{2}\right). \quad (2)$$

Throughout this paper the scattering lengths are in units of m_π^{-1} and J^-, given in mb, corresponds to

$$J^- = \frac{1}{4\pi^2}\int_0^\infty \frac{\sigma^T_{\pi^- p}(k') - \sigma^T_{\pi^+ p}(k')}{\sqrt{k'^2 + m_\pi^2}}\, dk'. \qquad (3)$$

Everything is in principle measurable to good precision. The relevant scattering lengths in Eq. (1) can be obtained to high precision using the $\pi^- d$ scattering length as a constraint as will be discussed below.

So as to obtain a robust evaluation of the coupling constant in the present context, we rearrange relation (2) in such a way that the most important experimental contributions are explicitly and separately identifiable:

$$g_c^2/4\pi = -4.50\times J^- + 103.3\times a_{\pi^- p} - 103.3\left(\frac{a_{\pi^- p} + a_{\pi^+ p}}{2}\right). \qquad (4)$$

For orientation, and as an initial basis for discussion, we use as a preliminary value $J^- = -1.077(47)$ mb from Koch [27] and the experimental $\pi^- p$ scattering length 0.0883(8) m_π^{-1} [14]. This gives the following relation, to be improved later: $g_c^2/4\pi = 4.85(22) + 9.12(8) - 103.3\times(a_{\pi^- p} + a_{\pi^+ p})/2 = 13.97(23) - 103.3\times(a_{\pi^- p} + a_{\pi^+ p})/2$. We stress that this is not our final result [our best estimate of these terms is given in Eqs. (20), (21), and (23) below]. Here the last term is a small quantity. If we use the old Koch-Pietarinen value [3] for $(a_{\pi^- p} + a_{\pi^+ p})/2 = a^+ = -83(38)\times 10^{-4}\ m_\pi^{-1}$ we will find $g_c^2/4\pi = 14.83(45)$, while the SM99 solution [28,29] with $a^+ = 20\times 10^{-4}\ m_\pi^{-1}$ will lead to $g_c^2/4\pi = 13.76$. A value for the coupling constant of the order of 13.6 would require either a relatively large positive magnitude for the isoscalar scattering length and/or a substantially less negative value for the cross section integral J^-. It is thus extremely important to obtain an accurate number for the small isoscalar amplitude. This quantity can be evaluated with small statistical and systematic uncertainties from the experimental $\pi^- d$ scattering length, assuming the validity of charge symmetry, i.e., that the scattering lengths $a_{\pi^+ p}$ and $a_{\pi^- n}$ are equal. This approximation is expected to be excellent, since the recent estimate of the isospin violation effect in this amplitude, mainly due to virtual photon effects [26], suggests that this leads to an increase of the coupling constant by only 0.2%. The cross section integral J^- is at present becoming the largest source of error. Uncertainties from the small $\pi^- d$ term will not have a major impact on the result. We now turn to a critical discussion of the different contributions.

III. THE EXPERIMENTAL $\pi^- p$ AND $\pi^- d$ SCATTERING LENGTHS

The $\pi^- p$ scattering length contributes the bulk of the GMO relation and must be very accurately controlled. It is deduced from the energy shift in pionic hydrogen, which (to about 2%) is proportional to the scattering length. The highly accurate value from PSI [14,30] has an uncertainty dominated by systematics in the analysis. The accuracy in the procedure for extracting the scattering length, with a number of small corrections of electromagnetic origin, has been discussed in detail by Sigg *et al.* [31]. The corrections include those for the finite nucleon and pion size as well as the feedback of the strong interaction shift on the long-ranged vacuum polarization. These can all be calculated to a precision more than an order of magnitude better than the present experimental error. They also include the effect of the proton e.m. polarizability. The crucial step in the analysis is the modeling of the hadronic interaction. Sigg *et al.* have simulated this by using a short-ranged potential for each of the isospin states with the strength tuned to the corresponding free scattering length in the absence of the open π^0 channel. They then introduce the open channel via coupled Klein-Gordon equations and explore the correction for different interaction ranges, with values near 0.7 m_π^{-1} that are considered realistic. The correction and uncertainty are mainly associated with the conversion between charged and neutral pions due to the available phase space. The final theoretical uncertainty is given as 0.5%, larger than the statistical uncertainty of 0.2%.

We have examined the procedure and agree with the quoted electromagnetic corrections and their precision, provided the hadronic interaction is tuned to correctly reproduce the experimental energy shift. The treatment of the corrections in the hadronic part, however could be improved, although it is convincing to a level of a few %.

Lipartia *et al.* have demonstrated that chiral effective field theory (EFT) gives the same result as the potential approach at least to next to leading order [32,33] if the physical amplitude is reproduced. This result is similar to the invariance of the leading order e.m. correction due to gauge invariance in a energy-dependent potential description [34]. It is thus reasonable to simulate the range dependence of the πN *s*-wave amplitude using potentials, provided the low-energy expansion of the *s*-wave scattering amplitude f_0 is correctly reproduced to order q^2. This latter approach automatically includes the wave function modification by the extended charge distribution, an effect of higher order in the EFT approach, but which gives here the largest numerical correction. However, the procedure in Ref. [31] does not respect the empirical values for the ''range'' terms, which leads to a larger uncertainty than the one quoted for their correction. The negative sign of the correction term and its approximate magnitude of -1% is basically correct. To account for the present inconsistency with the range expansion and using the numerical range of variation of Sigg *et al.*, the theoretical uncertainty must be increased from ± 0.5 to $\pm 1.0\,\%$, i.e., the overall systematic uncertainty in the scattering length taken in quadrature is increased from $\pm 6\times 10^{-4} m_\pi^{-1}$ to $\pm 10\times 10^{-4} m_\pi^{-1}$. We have not attempted to correct the deduced scattering length of Ref. [35], since this should be investigated specifically [36,37]. Range corrections to the $\pi^- p$ width are not relevant at present accuracy.

The isospin breaking in the $\pi^- p$ amplitude has been dimensionally estimated in chiral EFT theory [33]. Such effects are modeled in the potential approach as well. The estimate in EFT in next to leading order appears to be a considerable overestimate owing partly to higher order compensations. The main uncertainty in the estimate of Ref. [33] is absent in the difference between the $\pi^\pm p$ amplitudes,

which is the quantity relevant to the GMO relation for the πNN coupling constant.

The experimental $\pi^- d$ scattering length is derived from the energy shift in the $\pi^- d$ atom in close analogy to the case of the $\pi^- p$ scattering length. The deuteron electromagnetic corrections can in practice be calculated using a deuteron charge distribution, that correctly reproduces the experimental deuteron charge radius. Further, the deuteron is simpler in so far as the the correction for the open π^0 channel is negligible. The electromagnetic corrections produced to the strong πd amplitude should be included, however. The main one originates in the energy dependence, similar to the case of the proton. This small, repulsive contribution to the energy shift can be estimated to leading order from our approach in Ref. [34], Eqs. (3)–(5) and it is mainly produced by the leading order isoscalar range term (see Appendix A). The estimated change in the deduced scattering length is $-4m_\pi b^+ e\langle V_C^d(r)\rangle$, where the Coulomb potential from the extended deuteron charge distribution is averaged over the deuteron matter distribution. Note that there are no cancellations in the range terms, contrary to the massive cancellation of the πN scattering lengths in the single scattering term. Numerically, the empirical value for the range terms are $b^+ = -0.044(7) m_\pi^{-3}$; $b^- = 0.013(6) m_\pi^{-3}$ [23]. Any modern deuteron density distribution gives $e\langle V_C^d(r)\rangle = 0.86$ MeV and a correction of $12\times 10^{-4} m_\pi^{-1}$. An alternative estimate is obtained from the gauge correction to the $\pi^- n$ amplitude due to the Coulomb field of the proton, treated as a static spectator. Using the empirical πN range parameters this gives a contribution $-2m_\pi(b^+ - b^-)e\langle V_C^p(r)\rangle = 6\times 10^{-4} m_\pi^{-1}$ with $e\langle V_C^p(r)\rangle = 0.66$ MeV. A related estimate in a leading order chiral approach gives a correction $7.5\times 10^{-4} m_\pi^{-1}$ [38], but it is based only on the isovector term and does not include the constraints of the phenomenological range expansion. In the absence of correlations between the nucleons, the isovector range term does not contribute to leading order and it is further suppressed by its empirical weakness. We adopt the average of the first two estimates of $9\times 10^{-4} m_\pi^{-1}$ for this correction with an uncertainty of $5\times 10^{-4} m_\pi^{-1}$. This is well inside the present uncertainty in the theoretical deuteron scattering length (see Table IV) and has little influence on the present investigation.

In summary, we have adopted the following scattering lengths deduced from the data on the $\pi^- p$ atom [14,30] and the $\pi^- d$ atom [39] with the modifications described above. The transition amplitude $a_{\pi^- p\to\pi^0 n}$ is the one obtained from the width of the 1*s* state of the $\pi^- p$ atom [14,31]:

$$a_{\pi^- p\to\pi^- p} = [883 \pm 2(\text{statistical}) \pm 10(\text{systematic})] \times 10^{-4}\ m_\pi^{-1}, \tag{5}$$

$$a_{\pi^- p\to\pi^0 n} = 1280(60)\times 10^{-4}\ m_\pi^{-1}, \tag{6}$$

$$a_{\pi^- d} = [-252 \pm 5(\text{statistical}) \pm 5(\text{systematic}) + i63(7)] \times 10^{-4}\ m_\pi^{-1}. \tag{7}$$

We recall that the following relations hold, if isospin symmetry is assumed to be valid: $a_{\pi^- p} \equiv a_{\pi^- p\to\pi^- p} = a^+ + a^-$; $a_{\pi^- p\to\pi^0 n} = -\sqrt{2}a^-$, where $a^\pm$ are the symmetric and antisymmetric scattering lengths $a^\pm = \frac{1}{2}(a_{\pi^- p} \pm a_{\pi^+ p})$, respectively.

IV. THE THEORETICAL $\pi^- d$ SCATTERING LENGTH

The part of the GMO relation, Eq. (4), that it has not been possible to determine accurately up to now is the term proportional to the coherent, symmetric combination of the scattering lengths $(a_{\pi^- p} + a_{\pi^+ p})/2$. Assuming isospin symmetry, this is the isoscalar scattering length a^+. It follows from recent measurements of the hadronic energy shift and width of the pionic hydrogen atom [14] that this gives a directly determined value $a^+ = -22(43)\times 10^{-4}\ m_\pi^{-1}$. However, the accuracy of this direct determination is not sufficient for our present purpose. It is very difficult to determine a^+ with precision, directly from the coherent sum of the individual $\pi^- p$ and $\pi^+ p$ scattering lengths, because these cancel to a few percent. On the other hand, assuming only charge symmetry, this quantity is identical to the coherent scattering length for a negative pion on the neutron and proton, $(a_{\pi^- p} + a_{\pi^- n})/2$, which is the leading contribution to the accurately known $a_{\pi^- d}$ scattering length. The accuracy of this approximation is indicated by a recent estimate of the isospin violation effect in the amplitude ratio $R_4 = -0.008(1)$ [26] such that

$$a_{\pi^+ p} - a_{\pi^- n} = R_4\ a_{\pi^- n} = 3\times 10^{-4}\ m_\pi^{-1}. \tag{8}$$

Provided the remaining contributions can be reliably calculated, it is then possible to deduce the relevant coherent combination directly from the deuteron data with only minor assumptions concerning isospin symmetry. The situation is exceptionally favorable for the application of multiple scattering methods. The deuteron is a very loosely bound system and its wave function is accurately known. The nucleons have very little overlap and, consequently, the poorly controlled short range contribution is small. The particular case of the πd scattering length is even a textbook example of multiple scattering (see p. 111 in Ref. [40]), since the expansion parameters are small. The situation has been explored in detail, both within multiple scattering theory and using a three-body Faddeev approach, since it provides a clear-cut testing ground for methods [41–45].

In the static (fixed scattering centers) approximation the leading structure and scale of the pion-deuteron scattering length is set by the coherent single scattering term *S* and the dominant *s*-wave double scattering term *D* which is proportional to the inverse deuteron radius $\langle 1/r\rangle$ (p. 111 in Ref. [40]):

$$a_{\pi^- d}^{\text{static}} = S + D\cdots, \tag{9}$$

$$S = \frac{(1 + m_\pi/M)}{(1 + m_\pi/M_d)}(a_{\pi^- p} + a_{\pi^- n}), \tag{10}$$

$$D=2\frac{(1+m_\pi/M)^2}{(1+m_\pi/M_d)}\left[\left(\frac{a_{\pi^-p}+a_{\pi^-n}}{2}\right)^2-2\left(\frac{a_{\pi^-p}-a_{\pi^-n}}{2}\right)^2\right]\times\langle 1/r\rangle, \tag{11}$$

where M_d is the deuteron mass.

The static double scattering term represents about 90% of the experimental scattering length. It is in practice well defined numerically with a small error from the uncertainty in $\langle 1/r\rangle$. It has typically the value

$$D=-254(4)\times 10^{-4}\ m_\pi^{-1}, \tag{12}$$

where we have used the central values of the scattering lengths from Eqs. (20), (21). We will use this well defined static limit with point interactions as the starting point with respect to which various corrections will be introduced.

A. Previous approaches to a^+ from the deuteron data

Recently Baru and Kudryatsev (BK) [45] have investigated the πd scattering length using state-of-the-art multiple scattering methods. We will use the updated and unpublished version of their investigation [46] as the theoretical yardstick for the following discussion. We have numerically reproduced their findings to the same numerical precision, under the same assumptions. This approach is, however, still incomplete and contains, we believe, one erroneous term. As a consequence, the close agreement of their quoted value $a^+=-15(9)\times 10^{-4}\ m_\pi^{-1}$ with our final result for a^+ is only a fortuitous numerical coincidence without any special significance. It cannot be used as such. In the following we discuss the input parameters, corrections and systematics, and introduce substantial theoretical improvements. The classical three-body approach to the problem is still that of Afnan and Thomas and of Mizutani and Koltun, using separable interactions [41,42]. This approach gives the best picture of the dispersive effects due to absorption and supports the conclusions of the heavy cancellation of unitarity corrections in the multiple scattering approach. The approach, however, has not been updated in its overall accuracy to match the present high experimental precision and cannot be used directly.

A rather different approach is that of Beane *et al.* [47], based on the nuclear chiral perturbation approach of Weinberg [48] and using phenomenological deuteron wave functions. This approach makes a systematic expansion in the pion four-momentum, using effective parameters; at present the calculations have been made to $O(q^3)$. The result has the same general structure as the static limit of multiple scattering. Several physical effects discussed in the following are not yet included in this order, such as the Fermi motion term and the dispersive correction from pion absorption. They conclude that $a^+=-30(5)\times 10^{-4}\ m_\pi^{-1}$ to $O(q^3)$, where the uncertainty represents only the experimental uncertainty in the deuteron scattering length. The systematic uncertainty from the omitted higher order terms is most likely nearly one order of magnitude larger.

B. The inverse deuteron radius

The inverse deuteron radius appearing in Eq. (11) must be evaluated from wave functions. It is essential that the asymptotic normalization be accurately consistent with the experimental np effective range and that the wave functions correspond to an energy-independent interaction. The Paris [49] and Bonn2 [50] wave functions satisfy these criteria and give $\langle 1/r\rangle_{\rm Paris}=0.449\ {\rm fm}^{-1}$ and $\langle 1/r\rangle_{\rm Bonn2}=0.463\ {\rm fm}^{-1}$ with asymptotic normalizations $A_S({\rm Paris})=0.8869\ {\rm fm}^{-1/2}$ and $A_S({\rm Bonn2})=0.8863\ {\rm fm}^{-1/2}$, respectively, but they differ relatively importantly in the dominant tensor interaction. The resulting uncertainty in the inverse radius is small, since the most contributions come typically from distances of about 2 fm and little inside 1 fm. We have conservatively used the average of these model values $\langle 1/r\rangle=0.456(7)\ {\rm fm}^{-1}=0.645(10)\ m_\pi$; the uncertainty given is set by their difference. We note that the inverse radius, $0.520\ {\rm fm}^{-1}$, of the Hulthén wave function [44], which is often used for explorations of various effects, is nearly 15% larger than these values and should not be used in quantitative studies. The uncertainty in the πd scattering length from the inverse radius is about five times smaller than its present overall theoretical uncertainty. The effect on the coupling constant is well over one magnitude less than our stated overall systematic uncertainty in the coupling constant.

C. Effects of the non-locality of the πN s-wave interaction

The simplest approximation to the double scattering term of Eq. (11) assumes that the πN scattering is pointlike. Such an approximation is appropriate if the two scatterers are well separated, as is the case for the bulk of the contributions in the case of the deuteron as a consequence of its loose binding. The rather small nonlocal correction must, however, be controlled in sign and magnitude at the level of precision aimed for here. However, it is not necessary to describe this effect very accurately. The nonlocal effects enter mainly in the description of the isovector πN s-wave interaction, which is well known to be closely associated with ρ-meson exchange and which heavily dominates the double scattering term. For calculational convenience it has been conventional to model the nonlocality of the scattering amplitude in terms of a separable form $v(k)v(k')$, with a monopole form factor $v(k)=c^2/(c^2+k^2)$. Since the initial and final pion are at rest with momentum 0 and the intermediate pion has momentum $\mathbf{q}$, this means that in momentum space the static pion propagator changes from q^{-2} to $v(q)^2q^{-2}$. In coordinate space this corresponds to a change of the expectation value $\langle 1/r\rangle$ by

$$\delta\langle 1/r\rangle=-\left\langle\left|\frac{1+cr/2}{r}\exp(-cr)\right|\right\rangle. \tag{13}$$

We list in Table II the values of $\delta\langle 1/r\rangle$ and the corresponding contribution to the deuteron scattering length for different values of c as well as the contribution to the scattering length for standard values of the πN scattering lengths.

We conservatively consider that plausible values for the parameter c lie in the interval $3.5\leqslant c\leqslant 5\ m_\pi$. This is a wide

TABLE II. Corrections to $\langle 1/r \rangle$ and to the πd scattering length for different cutoff values and wave functions. The πN scattering lengths are from Eqs. (20) and (21).

Model	Paris [49]		Bonn2 [50]	
$\langle 1/r \rangle$	0.449 fm^{-1}		0.463 fm^{-1}	
c	$\delta\langle 1/r \rangle$	$\delta a_{\pi d}$	$\delta\langle 1/r \rangle$	$\delta a_{\pi d}$
$[m_\pi]$	$[10^{-3}$ fm$^{-1}]$	$[10^{-4}\ m_\pi^{-1}]$	$[10^{-3}$ fm$^{-1}]$	$[10^{-4}\ m_\pi^{-1}]$
3.0	−50	28	−60	34
3.5	−37	21	−46	26
4.0	−28	16	−36	20
4.5	−21	12	−29	16
5.0	−16	9	−23	13

range, which should adequately cover any model dependence of the result. These values have been obtained using two extremes of strong form factors for the double scattering term. One choice is to consider each of the scatterings to be associated with a monopole form factor. Since the isovector scattering strongly dominates the double scattering, the natural cutoff parameter is the ρ-meson mass. This would give the same correction as quoted in Table II for $c=5\ m_\pi$. Another choice is include in addition a strong form factor of typical ρ-meson range for both the pion and the nucleon. The effective overall form factor in each of the pion scatterings is then a dipole form factor with the ρ-meson mass, corresponding to $c=3.5\ m_\pi$. It should be observed that the typical modification of $\langle 1/r \rangle$ is a negative contribution by 4 to 8 % corresponding to a positive contribution to $\delta a_{\pi^- d}$ of 9 to $20\times10^{-4}\ m_\pi^{-1}$. We choose the mean of these two approaches as a typical value with the spread setting the scale of the uncertainty, but note that in doing so we may somewhat underestimate the non-local effect, such that our final value of $g_c^2/4\pi$ may be somewhat too low.

We found that the results reported by BK in Ref. [45], Table 3, for the realistic Bonn1 and Bonn2 wave functions did not include the form factor (contrary to the statement in the paper), which the authors confirm. We have received their corrected and extended results [46] for the Bonn1 potential. Note that at the present level of precision it is important to use potentials fully consistently. The Bonn1 potential is energy dependent; as a consequence, orthonormality can only be respected in matrix elements calculated using this potential if nontrivial weight factors are introduced in the integrands. To eliminate this uncertainty we use here the similar, but energy independent, Bonn2 potential. BK consider without arguments cutoff values $c=2.5$, 3, and $3.5\ m_\pi$ in the form factor; this gives positive contributions to the scattering length as compared to the pointlike static approximation of 36, 27, and $22\times10^{-4}\ m_\pi^{-1}$, respectively. There are good physical reasons to believe that ρ-meson exchange sets the scale for the dominant isovector amplitude with a larger value for the effective c. To be conservative we take $c=3.5$ and $c=5\ m_\pi$ for the cutoff as limits for this systematic correction from the non-localities and use the central value of these two extremes as the correction. Our correction is smaller than the one found by BK. Nonlocality is one of the largest theoretical sources of systematic uncertainty in corrections to the pointlike static approximation.

D. Corrections to the static approximation

The nature of the leading nonstatic corrections and the reasons why the static expression (fixed scattering centers) still remains an excellent approximation are well understood. At first sight, even the single scattering amplitudes have rather important nonstatic modifications, representing about 30% of the total πd scattering length. Such corrections are systematically generated by the multiple scattering description in which physical amplitudes are used, thus guaranteeing the correct behavior of the scattered wave at large distances. The emphasis is thus not on the near-zone behavior of the scattering as in pseudopotential or effective Lagrangian approaches. In a situation such as the present one, this leads to a systematic cancellation of unitary binding corrections between single scattering and double scattering terms, when these are introduced consistently. This phenomenon was first demonstrated in the present context for an analytically soluble model by Fäldt [44]. It has been numerically investigated by BK [45] using a Hulthén wave function and a separable amplitude with a dipole form factor and a cutoff parameter $3\ m_\pi$. They conclude that the amplitude increases by only $10\times10^{-4}\ m_\pi^{-1}$, when the nonstatic term is included. This is only twice the experimental uncertainty and less than the uncertainty from the form factor. Fäldt evaluated the joint contribution of the nonstatic and the form factor terms using a dipole form factor with $c=3.6\ m_\pi$ with a Hulthén wave function [44]. The overall contribution corresponds to $34\times10^{-4}\ m_\pi^{-1}$. The comparison with our independent evaluation of pure form factor corrections indicates that the nonstatic term in this case is about $8\times10^{-4}\ m_\pi^{-1}$. A detailed calculation of this correction is complicated. Wycech informed us that he is in the process of reevaluating the nonstatic contributions using a Faddeev approach and separable interactions. At the present moment he has only results using an interaction that reproduces the Hulthén wave function; this gives $+12\times10^{-4}\ m_\pi^{-1}$, in excellent agreement with the previous results [51]. Following BK we have adopted a value $11(6)\times10^{-4}\ m_\pi^{-1}$, where the liberal uncertainty reflects the lack of verification of the value of nonstatic effects using high quality deuteron wave functions.

E. Fermi motion

Another well defined correction originates in the nucleon Fermi motion. In the case of s-wave scattering, such contributions cancel systematically to high precision with other binding terms [44]. In addition, the single scattering term from the πN p-wave scattering produces a small, attractive and physically well understood contribution, which can be reliably evaluated as a leading order effect originating in the nucleon momentum distribution and the spin-isospin averaged p-wave threshold scattering amplitude $c_0=0.208(3)\ m_\pi^{-3}$ [40]:

TABLE III. Estimates of the contribution a(Fermi) to $a_{\pi^- d}$ from single p-wave scattering as a result of Fermi motion according to Eq. (14) for various deuteron wave functions, different cutoff values, and separated into S- and D-state contributions. The last row gives $\langle p^2\rangle$ and the kinetic energy $\langle p^2\rangle/M$.

Model	Paris [49]			Bonn2 [50]		
c $[m_\pi]$	S state [in units of 10^{-4}	D state	Total m_π^{-1}]	S state [in units of 10^{-4}	D state	Total m_π^{-1}]
3	39.6	27.9	67.6	36.7	16.7	53.4
4	39.8	28.1	67.8	36.8	16.9	53.6
5	39.8	28.1	68.0	36.8	16.9	53.7
∞	39.9	28.3	68.2	36.8	17.0	53.9
$\langle p^2\rangle$ $[m_\pi^2]$	0.533	0.378	0.912	0.492	0.228	0.720
$\langle p^2\rangle/M$ [MeV]	11.1	7.9	19.0	10.3	4.7	15.0

$$a(\text{Fermi}) = 2c_0 \frac{m_\pi^2(1+m_\pi/M)}{(M+m_\pi)^2(1+m_\pi/M_d)} \left\langle p^2 v^2 \left(\frac{m_\pi}{M+m_\pi} p \right) \right\rangle. \tag{14}$$

We have calculated this expectation value for two high quality deuteron wave functions. The results are given in Table III. The form factors are manifestly of no importance. The relatively large difference between the Paris potential and the Bonn2 potential arises because of the D-state component, which generates contributions 10 times more effectively than the S-state one. The difference in the correction in the two cases is thus almost entirely a consequence of the well known difference in the D-state probability ($P_D = 5.7$ vs 4.3 %) for the two wave functions. The normalized momentum distributions for the S- and D-wave components, respectively, are very similar in the two models. We therefore treat its effect as a true model dependence. We take the spread in the values of the Fermi motion corrections as a measure of a systematic theoretical uncertainty, although physical arguments for the higher value of P_D exist [52]. Consequently, in the following evaluation, we use the value $a(\text{Fermi}) = 61(7)\times 10^{-4}\ m_\pi^{-1}$. This is consistent with, but somewhat larger than the value 50 to $53\times 10^{-4}\ m_\pi^{-1}$ found by BK based on the Bonn1 and 2 wave functions. This uncertainty in a correction term is about 3 times less than the present overall theoretical uncertainty in the scattering length and is not a significant source of uncertainty, as expected. The lower value of the Bonn potentials would lead to a 0.3% increase of the coupling constant.

F. Dispersion contribution

A small repulsive contribution, not described by multiple scattering, is produced by the dispersive term from the absorption reaction $\pi^- d \to nn$. This quantity has been repeatedly calculated using Faddeev approaches [41–43]. It typically has a theoretical uncertainty of 20% of its numerical value $-56(14)\times 10^{-4}\ m_\pi^{-1}$ [43]. The dispersive contribution is a theoretically calculated correction; a more detailed study of this term is highly desirable. The uncertainties reflect the model dependence of the approach.

G. *sp* interference

This is the name given by BK to a term originating in pion p-wave scattering on one of the nucleons due to Galilean invariance [45]. Such Galilean terms generate s-wave scattering contributions even for pion scattering on free nucleons. In the present situation the relevant spin-averaged on-shell scattering volume for charge exchange of a p-wave pion is well known and the corresponding scattering amplitude on-the-mass-shell depends on the pion momentum in a well defined way. The Galilean correction for nucleon motion involves going off the mass shell and usually depends on the description. BK advocate that a contribution of about $42\times 10^{-4}\ m_\pi^{-1}$ originates from p-wave scattering due to the momentum of the intermediate pion when expressed in the πN c.m. system. However, in the present situation the contribution is almost entirely generated by the isovector πN Born term and it can be evaluated exactly. From the expressions given in Höhler's reference book, Eq. (A.8.2) [23], one finds that this term is proportional to

$$\nu^2 - \frac{(k^2+k'^2-t)}{2} = \nu^2 - q\cdot q'. \tag{15}$$

Here, ν is (to order M^{-2}) the Breit frame pion energy, which is proportional to the scalar product of the average four vectors of the nucleons (p and p') and pions (q and q'), respectively:

$$\nu = \frac{1}{M}\frac{(p+p')}{2}\frac{(q+q')}{2} = \frac{(q_0+q_0')}{2} - \frac{1}{M}\frac{(\mathbf{p}+\mathbf{p}')}{2}\frac{(\mathbf{q}+\mathbf{q}')}{2} \tag{16}$$

[Eq. (A.1.6) in Ref. [23]]. Thus, neglecting terms of order M^{-2}, the pion pole term is proportional to

$$\frac{(q_0-q_0')^2}{4} - \frac{(q_0+q_0')}{M}\frac{(\mathbf{p}+\mathbf{p}')}{2}\frac{(\mathbf{q}+\mathbf{q}')}{2} + \mathbf{q}\cdot\mathbf{q}'. \tag{17}$$

In the double scattering term, the contribution comes from nucleon 1 with initial (final) momentum $\mathbf{p}$ ($\mathbf{p}-\mathbf{q}'$) and with the initial (intermediate) pion momentum $\mathbf{0}$ ($\mathbf{q}'$), respectively, while for nucleon 2 the initial (final) nucleon momentum is $-\mathbf{p}$ ($-\mathbf{p}+\mathbf{q}'$) with intermediate (final) pion momentum $\mathbf{q}'$ ($\mathbf{0}$), respectively; the pion energies, q_0 and q_0', are unchanged in this term. The sum of these two contributions are

$$\frac{q_0}{M}\frac{\mathbf{q}'^2}{2} - \frac{q_0}{M}\frac{\mathbf{q}'^2}{2} = 0. \tag{18}$$

On the other hand, BK make the choice of Galilean invariance for the incoming and outgoing πN systems calculated separately in the primary amplitude and find in the same limit $0 + q_0\mathbf{q}'^2/M$ in Eq. (18). Instead the exact pole

term corresponds, to order M^{-2}, to a Galilean invariant expression using the *average* velocity of the initial and final nucleons, contrary to the BK assumption. In other words, the pole term is proportional to the scalar product of the pion momenta $\mathbf{q}_B \cdot \mathbf{q}'_B$ in the nucleon Breit frame. We have therefore suppressed this term in the BK multiple scattering expansion.

We note in passing that, even if the Galilean contributions were of the type proposed by BK, their importance would most likely be strongly suppressed. The reason is that these terms generate a δ-function interaction in the absence of form factors. We therefore suspect that *NN* correlations would largely suppress such contributions, in analogy with the Ericson-Ericson-Lorenz-Lorentz effect for *p*-wave π propagation in the nuclear medium (p. 140ff in Ref. [40]).

H. Isospin and mass difference corrections

In the above expressions, we assumed that isospin holds for the calculation of double scattering and that charge symmetry holds for the single scattering. We now quantify the effect of these approximations. BK have investigated the consequence of the physical mass difference between π^- and π^0 and between the neutron and the proton in the multiple scattering. They find an increase of the scattering length by about $3.5\times10^{-4}\ m_\pi^{-1}$. The smallness of this term is in part due to a systematic compensation of single and double scattering contributions in analogy to the compensation of unitarity corrections to single and double scattering terms. As an alternative approach we use the recent estimates of the violation of isospin symmetry from light quark mass differences and virtual photon effects in the πN scattering lengths [26]. We maintain only the effects of violations in the amplitudes in the double scattering term in view of the systematic cancellation between single scattering and propagator modifications in the double scattering term. This leads to an increase of the scattering amplitude by $3.5\times10^{-4}\ m_\pi^{-1}$, numerically identical to the previous estimate. It is not clear whether these approaches represent the same physics and this point should be further investigated. However, both results indicate that the effects are small in the present context, although they will become of importance in the future. In view of its smallness and since it is not at present established experimentally, we have not included this correction, which is within experimental uncertainties. It has, however, been included as an uncertainty in our estimate of systematic errors.

I. Higher order multiple scattering corrections

In the present case the multiple scattering expansion is rapidly convergent beyond the double scattering term. In the fixed scattering approximation with separable interactions, these higher order terms can be summed exactly to all orders. BK calculated these terms approximately, assuming pointlike scatterers. We have verified these calculations and reproduce their results. They have since improved the evaluation of this small term, using form factors and find a stable contribution to the scattering length of the order of $+6\times10^{-4}\ m_\pi^{-1}$ [45,46]. Our independent evaluation also gives very stable values, but somewhat smaller, in the range of 3 to 4 $\times10^{-4}\ m_\pi^{-1}$ for the form factors considered. We have used the value $4(1)\times10^{-4}\ m_\pi^{-1}$ for this correction. The effect is much smaller than other uncertainties, for example, those due to form factors.

J. Inverse pion photo-production

Another small electromagnetic correction comes from the physical *s*-wave photoproduction process $\pi^- p \to \gamma n$ acting on one nucleon followed by the inverse reaction on the other one. This double scattering process has nearly the same structure as the corresponding *s*-wave charge exchange process $\pi^- p \to \pi^0 n$ in Eq. (11), but for the fact that the intermediate photon now has momentum $k_\gamma = m_\pi$ in the static limit, such that

$$\mathrm{Re}\, D_\gamma = -2/3 \frac{(1+m_\pi/M)^2}{(1+m_\pi/M_d)} [E_{0+}(\gamma n \to \pi^- p)]^2 \left\langle \frac{\cos(k_\gamma r)}{r} \right\rangle . \tag{19}$$

Here the photoproduction amplitude $E_{0+}(\gamma n \to \pi^- p) = -31.4\times10^{-3}\ m_\pi^{-1}$ (Table 8.3 in Ref. [40]). This small charge dependent term is of order $-2\times10^{-4}\ m_\pi^{-1}$, which is a magnitude less than the overall theoretical uncertainty; see also Ref. [38].

K. Double *p*-wave scattering

A small correction results from the *p*-wave scattering due to nucleon motion at both vertices. This effect has been estimated by BK for an analytically soluble deuteron model with Gaussian wave functions. They find a contribution of about $-3\times10^{-4}\ m_\pi^{-1}$. We have included this small effect.

L. Scattering on virtual pions

Finally, one may envisage a contribution from the scattering of the pion on a virtually exchanged pion in the deuteron. However, we are dealing with an isoscalar system, and such a contribution is proportional to virtual isoscalar $\pi\pi$ *s*-wave scattering and should be very small, from a chiral perspective. In particular, since the deuteron is such a loosely bound system, one expects this term to be small. Robilotta and Wilkin showed that large cancellations in a consistent treatment give only $-5\times10^{-4}\ m_\pi^{-1}$ [53]. This is confirmed by a recent chiral estimate of -8 to $-6\times10^{-4}\ m_\pi^{-1}$ [47]. We adopt a contribution of $(-6\pm2)\times10^{-4}\ m_\pi^{-1}$ from this effect.

M. Results for the πN scattering lengths

The different contributions from the previous subsections are summarized in Table IV, using the final parameters from Eqs. (20) and (21) whenever appropriate. Consequently, the present energy shift in the $\pi^- d$ atom leads to the following value for the coherent scattering length from a proton and a neutron:

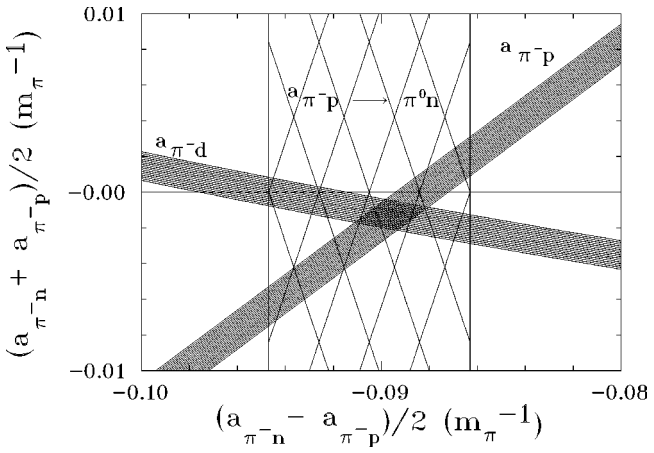

FIG. 1. Graphical determination of the πN scattering lengths $(a_{\pi^- p}+a_{\pi^- n})/2 \simeq a^+$ and $(a_{\pi^- p}-a_{\pi^- n})/2 \simeq a^-$ from the constraints imposed by the pionic atom scattering lengths.

$$\frac{a_{\pi^- p}+a_{\pi^- n}}{2} = [-12 \pm 2(\text{statistical}) \pm 8(\text{systematic})] \times 10^{-4}\ m_\pi^{-1}. \tag{20}$$

The high accuracy is a direct consequence of the very strong constraint provided by the $\pi^- d$ atom level shift. The usual determination via phase shift analysis is difficult, since it requires differences between large numbers. In the limit of isospin symmetry, this quantity is the isoscalar scattering length a^+. The main systematic error in Eq. (20) comes from the uncertainty in the dispersive correction term and, to a lesser degree, from the form factor or nonlocality in the deuteron double scattering term. The small corrections for isospin violation in the double scattering term and for charge symmetry breaking in the single scattering on the deuteron are well within the stated uncertainties and have no substantial influence on the result.

Combining the information from the experimental $\pi^- p$ and $\pi^- d$ scattering lengths with the constraints of the theoretical analysis (20), we obtain a substantially improved determination in the difference $(a_{\pi^- p}-a_{\pi^- n})/2$ (this quantity is, in the limit of isospin symmetry, identical to the isovector scattering length a^-):

$$\frac{a_{\pi^- p}-a_{\pi^- n}}{2} = [895 \pm 3(\text{statistical}) \pm 13(\text{systematic})] \times 10^{-4}\ m_\pi^{-1}. \tag{21}$$

A graphical determination of these πN scattering lengths is shown in Fig. 1, which also emphasizes that this is a substantial improvement on determinations using only data from pionic hydrogen. The results are in excellent agreement with the central values deduced from the pionic hydrogen shift and width by the experimental PSI group, since it follows from Eqs. (7) and (8) of Ref. [14] that $a^+=(-22 \pm 43)\times 10^{-4}\ m_\pi^{-1}$; $a^-=(905 \pm 42)\times 10^{-4}\ m_\pi^{-1}$. The PSI group[1] also used the constraint from the pionic deuterium shift, assuming the old calculation of Ref. [43] to be accurate enough and found $a^+=(+16 \pm 13)\times 10^{-4}\ m_\pi^{-1}$; $a^-=(868 \pm 14)\times 10^{-4}\ m_\pi^{-1}$.

From our evaluation here, we have achieved quantitative control of the dominant contribution to the GMO relation from the scattering lengths to about 1% or better in $g_c^2/4\pi$. It is interesting to compare our results with the extrapolations of scattering amplitudes to threshold as given in Refs. [55,56]. They find the value $a_{\pi^+ p \to \pi^+ p}=(-770 \pm 30) \times 10^{-4}\ m_\pi^{-1}$. This corresponds to $a^+=(57 \pm 15) \times 10^{-4}\ m_\pi^{-1}$ assuming isospin symmetry invariance and using the experimental value for $a_{\pi^- p \to \pi^- p}$ from pionic hydrogen. On the other hand, the charge symmetric scattering length $a_{\pi^- n \to \pi^- n}=(-917 \pm 18)\times 10^{-4}\ m_\pi^{-1}$ follows from Eqs. (20) and (21) and within charge symmetry the two values should be identical. According to Eq. (8) the estimated effect of charge symmetry breaking in effective chiral theory is $a_{\pi^+ p}-a_{\pi^- n}=3\times 10^{-4}\ m_\pi^{-1}$. The above values give, instead, $(147 \pm 35)\times 10^{-4}\ m_\pi^{-1}$, 50 times larger than the expected value. Thus, unless charge symmetry is unexpectedly badly broken, the scattering length of Refs. [55,56] based on scattering experiments is implausible and should be rejected.

While the extrapolation [55,56] leads to important differences, it cannot, of course, be completely ruled out that other, more constrained, extrapolations from πN scattering data could lead to scattering lengths slightly different from the ones found here. The origin would then most likely be due either to isospin violation in the scattering data or, alternatively, to some unexpected modification of the least controlled part of our deuteron terms, such as the absorption contribution. In the dispersion-relation-constrained extrapolation advocated by Pavan *et al.* [11] they give $a^+=+20 \times 10^{-4}\ m_\pi^{-1}$ to be compared with $(-12 \pm 8)\times 10^{-4}\ m_\pi^{-1}$ above. Interpreted as a modification of the dispersive term due to deuteron absorption, it would require an increase by a factor of 2 in this term in order to make the results compatible, which appears an implausibly large modification. We believe our result to be the preferable one, since it is a more

[1]After the submission of the present paper, the PSI group has published a revised analysis [54] based on the BK treatment [45] and assuming strict isospin symmetry. They quote $a^+ \equiv b_0 = -0.0001^{+0.0009}_{-0.0021} m_\pi^{-1}$; $a^- \equiv -b_1 = 0.0885^{+0.0010}_{-0.0021} m_\pi^{-1}$. Their systematic errors are not well controlled. First, BK explicitly omit the large dispersive correction, which contributes a term of the order of $0.0030 m_\pi^{-1}$ to a^+. Second, the sp interference contribution is negligible as we discuss in detail Sec. IV G, while it is derived in BK from an erroneous assumption with a value similar to that of the dispersive correction. The statement based on their Ref. [54] that the sp interference term partly could contain part of the absorption term is incorrect. In addition, the dominant contribution to their theoretical error appears to be based on a confusion about the form factor correction. They introduce twice the BK form factor effect, counting it as well as an (inexistent) off-energy shell correction of the deuteron wave function. This leads to an overestimate of the lower systematic uncertainty from this source (double counting).

TABLE IV. Typical contributions to $a_{\pi d}$ scattering length in units of $10^{-4}\ m_\pi^{-1}$.

Contributions	Present work	BK [45]
$a_{\pi^- d}$(double scattering; static)	−254 (4)[a]	−252
Fermi motion	60 (7)	50
dispersion correction	−56 (14)	not included
isospin violation	3.5	3.5
$(\pi^- p, \gamma n)$ double scattering	−2	not considered
form factor	17 (9)	29 (7)
higher orders	4 (1)	6
sp interference	small	−44
nonstatic effects	11 (6)	10
p-wave double scattering [45]	−3	−3
virtual pion scattering [47,53]	−7 (2)	not considered
total$=a_{\pi^- d}-1.07\times(a_{\pi^- p}+a_{\pi^- n})$	−227 (20)	−198
$a_{\pi d}$(experimental) [39]	−252 (7)	

[a]The uncertainty from the πN scattering lengths would typically contribute ± 6 units to this term.

direct determination and fully consistent. The margin for modifications of our theoretical analysis is small.

V. EVALUATION OF THE CROSS SECTION INTEGRAL J^- FROM DATA

The cross section integral represents only one third of the total contribution to the GMO relation. This means that an uncertainty of (say) 3% in the integral would give only 1% uncertainty in the coupling constant. At the present precision, and in spite of this insensitivity, this has now become one of the main sources of uncertainty in the determination of the coupling constant. Since total cross sections tend to be inherently accurate, the evaluation can be performed with precision, but for the high-energy region. There exists a vast amount of high quality data up to very high energies (beyond 240 GeV/c) and, in the dominant region below 1 GeV/c, there are detailed results from partial wave analyses. The only previous evaluation with a detailed discussion and clearly stated sources of errors known to us is an unpublished study of 1985 by Koch, which gives $J^-=-1.077(47)$ mb [27]. Later evaluations find values within this band of errors, but the uncertainties are not discussed. In

TABLE V. Values of J^- from the literature. Quoted errors include both statistical and systematic uncertainties.

Source	J^- mb
Koch 1985 [27]	−1.077 (47)
Workman *et al.* 1992; KH [10]	−1.056
Workman *et al.* 1992; VPI [10]	−1.072
Arndt *et al.* 1995 [24]	−1.050
Gibbs *et al.* 1998 [15]	−1.051 (5)[a]
Present work	−1.083 (32)

[a]Statistical error only.

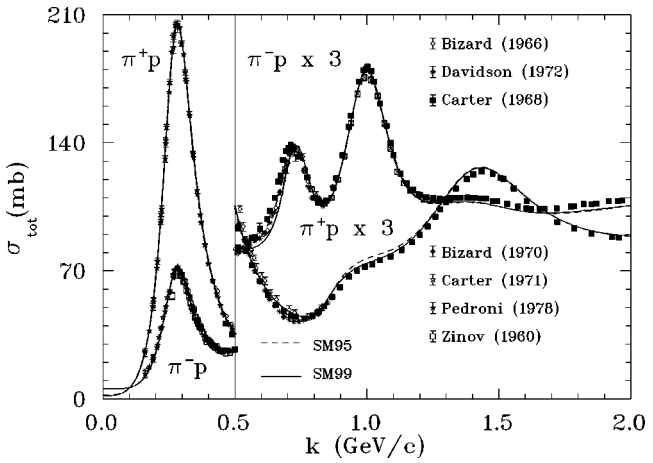

FIG. 2. The experimental total $\pi^+ p$ and $\pi^- p$ cross sections below 2 GeV/c [58–64] compared with the SM95 [57] and SM99 [28] PWA hadronic solutions, where Coulomb barrier effects have not been taken into account.

1992 Workman *et al.* [10] gave the values −1.056 and −1.072 mb based on the Karlsruhe-Helsinki and VPI πN amplitudes of the time, respectively. In 1995 the VPI group gave the value −1.05 mb [24]. Gibbs *et al.* give a similar value $J^-=-1.051$ mb [15]. In this case the dominant contribution below 2 GeV (−1.308 mb) was evaluated using the SM95 phase-shift analysis [57] for the πN cross sections. These values are summarized in Table V.

In view of the importance of obtaining a clear picture of the origin of present uncertainties, we have reexamined this problem in spite of the approximate consensus. The πN total cross sections below 2 GeV/c [58–64] are shown in Fig. 2 and compared with the SM95 [57] and SM99 [28] PWA hadronic solutions. The typical shape of the integrand J^- is seen in Fig. 3. As might be expected, the main contributions come from the region of the Δ resonance and just above. It would be false, however, to believe that this is the region that

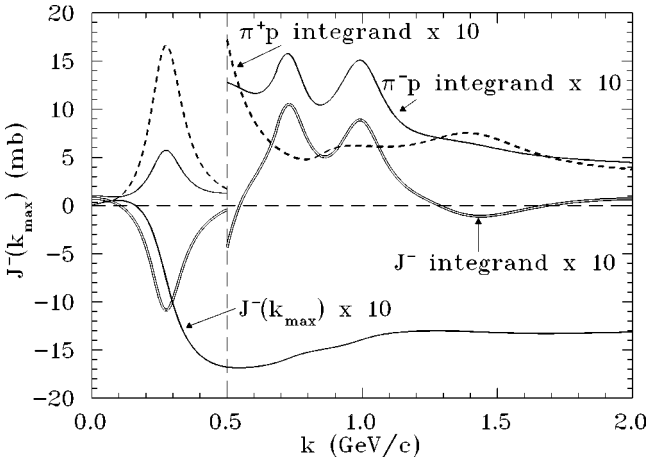

FIG. 3. The separate integrands for $\pi^\pm p$, as well as for their difference as a function of laboratory momentum k, together with the cumulative value of the integral $J^-(k_{\max})$ integrated from threshold to $k=k_{\max}$. The curves are based on the SM99 solution [28]. The integrands are in units of mb GeV/c.

TABLE VI. Evaluation of J^- in the Δ resonance region and up to 2 GeV/c. Here "data" refers to "nuclear" experimental cross sections uncorrected for Coulomb penetration and "nuclear SM99" to the corresponding PWA cross sections; I_- and I_+ are the corresponding integrals for $\pi^- p$ and $\pi^+ p$, respectively, with $J^- = I_- - I_+$.

Input	k [GeV/c]	I_- [mb]	I_+ [mb]	J^- [mb]
Hadronic SM95 [57]	0.00 to 0.16	0.164	0.157	0.007
Hadronic SM99 [28]	0.00 to 0.16	0.162	0.152	0.011
Hadronic SM95 [57]	0.16 to 0.55	1.078	2.763	−1.685
Hadronic SM99 [28]	0.16 to 0.55	1.071	2.767	−1.696
Nuclear SM99 [28]	0.16 to 0.55	1.090	2.726	−1.636
Data [70]	0.16 to 0.55	1.101	2.753	−1.652
Hadronic SM95 [57]	0.55 to 1.20	0.800	0.414	0.386
Hadronic SM99 [28]	0.55 to 1.20	0.789	0.411	0.378
Nuclear SM99 [28]	0.55 to 1.20	0.804	0.400	0.404
Data [70]	0.55 to 1.20	0.816	0.400	0.416
Hadronic SM95 [57]	1.20 to 2.00	0.450	0.460	−0.010
Hadronic SM99 [28]	1.20 to 2.00	0.451	0.458	−0.007
Nuclear SM99 [28]	1.20 to 2.00	0.458	0.450	0.008
Data [70]	1.20 to 2.00	0.459	0.443	0.016

produces the main uncertainty of the integral. There are no strong cancellations in the difference between the total $\pi^\pm p$ cross sections in that region and the cross sections have been very carefully analyzed. Systematic uncertainties contributing 2–3 % or more to the total J^- are very unlikely indeed; if they occur, they will certainly have an important influence on other determinations of the coupling constant as well.

In the following we examine in detail the uncertainties arising from various energy regions with different characteristics (the numerical conclusions are summarized in Tables VI and VII). In Sec. V A, the threshold region below 160 MeV/c is dominated by the s- and p-wave threshold parameters (s-wave contribution of about +6%, p-wave one of about −6%). In Sec. V B, the Δ resonance region from 160 to 550 MeV/c, in which the major phase shifts are very accurately known (main contribution of about 155%). In Sec. V C, the resonance region from 0.55 to 2 GeV/c, which is partly dominated by higher resonances with mostly high quality data (about −33% contribution). In Sec. V D, the high-energy region and the asymptotic region from 2 GeV/c to ∞ (totally about −22% contribution); about half originates from the asymptotic region beyond 10 GeV/c, for which data are accurately described by asymptotic expressions.

The total cross sections in the integral J^- are the hadronic ones. The experimentally defined total cross sections differ from these due to the electromagnetic corrections. These are nearly model independent in the present context. They are proportionally more important in the difference between the cross sections, since the $\pi^+ p$ total cross sections are systematically reduced at all energies by the Coulomb repulsion between the particles and, conversely, the $\pi^- p$ ones are systematically increased by the attraction [65,66]. This effect gives a positive contribution to J^-; the coupling constant would be underestimated by about 3% neglecting such corrections. The dominant correction comes from the Δ resonance region (see Table VI). For total cross sections there is little sensitivity to the detailed procedure: the Nordita approach is frequently used [67] below 500 MeV/c. The Coulomb correction to the integrand at high energy, where the $\pi^\pm$ total cross sections are nearly equal, is approximately $(4\pi^2)^{-1} 2A_c \sigma^T(k)/k^2$ with $A_c \simeq 3.7$ MeV/c. For constant cross sections, the total correction above a momentum k_1 is then typically $0.007 k_1^{-1}$ mb, where k_1 is in units of GeV/c [65]. It therefore rapidly becomes negligible above a few GeV/c.

As an illustration of contributions, the resulting fits to data [58–61,68,69] for the solution SM99 of Arndt *et al.* are shown in the range $0.5 \leqslant k_{lab} \leqslant 2$ GeV/c in Figs. 4 and 5.

The recent VPI/GWU partial wave amplitude (PWA) solution up to 2 GeV/c [28] is in good agreement with observations with a few exceptions. We will therefore use the hadronic cross sections deduced from this solution as a guide for the numerical contribution. We estimate its uncertainties below. We also give numbers from the earlier PWA solution SM95 [57] for comparison.

A. The threshold region

There are no direct measurements of total cross sections below 160 MeV/c, but the hadronic cross-section difference can be well reconstructed from other considerations. In this range the low-energy s- and p-wave parameters determine the result. The cross-section difference at threshold is

$$\sigma^T_{\pi^- p}(0) - \sigma^T_{\pi^+ p}(0) = 8\pi[(a_{\pi^- p})^2 - (a^+)^2], \qquad (22)$$

assuming isospin invariance and neglecting the mass differences. Here the first term is accurately known from the $\pi^- p$ atom, as previously discussed, and the second term is extremely small. With increasing energy the p-wave contributions of opposite sign, governed by the tail of the Δ resonance, take over and compensate the s-wave term beyond 100 MeV/c. These two terms contribute together +0.011 mb [28], but taken individually the s- and p-wave terms represent about 6% each of the total J^-. The uncertainty is dominated by the error in the rather small contribution from the s-wave range terms, while the accurate $\pi^- p$ scattering length is imposed in the SM99 analysis. The corresponding error in J^-, of about 0.5%, is not a major source of overall uncertainty and even if this uncertainty is underestimated this has little importance.

The 3.3 MeV mass difference in the $\pi^- p$ and $\pi^0 n$ thresholds breaks the isospin invariance leading to a potentially significant correction, in particular, since the $\pi^- p$ total cross section diverges at threshold due to the open $\pi^0 n$ channel. The smallness of the contributions from the threshold region hints at a small correction. We have investigated this effect using a simplified model based on the s- and p-wave low-energy parameters only. The dispersion re-

TABLE VII. Different contributions to J^- as function of the k range and of the input data. The first number in the parenthesis is the statistical error, while the second numbers correspond to the systematic uncertainty. The selected data correspond to the world data as given by PDG Tables, where we have suppressed all data with statistical and systematic errors larger than 1%. Lines labeled ''selected'' and ''data'' refer to ''nuclear'' cross sections.

i	Input	k(GeV/c)	I_-(mb)	I_+(mb)	J^-(mb)$=I_- - I_+$
1	SM95 [57]	0.00 to 0.16	0.164	0.157	0.007
2	SM99 [28]	''	0.163	0.152	0.011
3	Selected [70]	0.16 to 2.00	2.360 (2) (3)	3.596 (6) (1)	−1.237 (6)(4)
4	Data [63,62]	''	2.377 (3) (2)	3.596 (5) (2)	−1.219 (6)(4)
5	SM95 [57]	0.00 to 2.00	2.492	3.794	−1.302 (6)(20)
6	SM99 [28]	''	2.474	3.788	−1.314 (6)(20)
7	Selected [70]	2.00 to 4.03	0.560 (2) (3)	0.496 (1) (5)	0.064 (2) (7)
8	Data [60,75]	''	0.580 (1) (5)	0.518 (1) (5)	0.063 (1)(10)
9	Selected [70]	4.03 to 240	2.672 (4)(10)	2.539 (3)(12)	0.133 (5)(22)
10	Fit PDG94 [76]	''	2.645	2.489	0.155
11	Regge 94 [76]	240 to ∞			0.030 (5)
12	Regge 00 [80]	''			0.025 (4)
13	Regge 98 [70]	''			0.018 (3)
14	6+7+10+11	0 to ∞			−1.055(10)(31)
15	6+7+9+11	''			−1.087 (9)(31)
16	6+7+9+13	''			−1.099 (8)(31)
17	6+7+9+12	''			−1.092 (9)(31)

lation must now be evaluated using the imaginary part of the scattering amplitude $\mathrm{Im}\,F = 4\pi k\sigma^T$, which is well behaved at threshold, but which differs from zero below the physical $\pi^- p$ threshold. The correction occurs predominantly in the 6% s-wave term. The approximate modification up to the momentum $k_1 = 160$ MeV/c is of $O(-\kappa^2/2k_1^2) \simeq -0.02$, where $\kappa^2 \simeq 0.045\, m_\pi^2$ is the π^0 squared momentum at threshold. This represents a -0.1% contribution to the integral J^-, which is negligible compared with other uncertainties.

B. The Δ resonance region

This is the main contribution to the integral and it must be accurately evaluated. The resonant 33 wave dominates heavily and its behavior is strongly constrained by other ex-

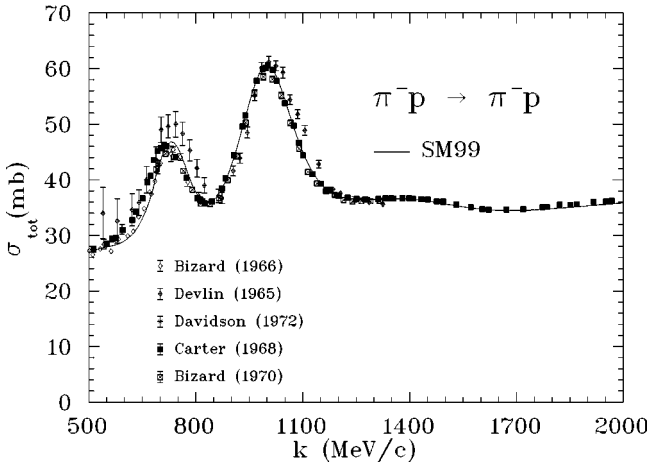

FIG. 4. The $\pi^- p$ experimental total cross sections in the region $0.5 \leqslant k \leqslant 2$ GeV/c [58–61,68] compared to SM99 [28] with Coulomb barrier effects accounted for.

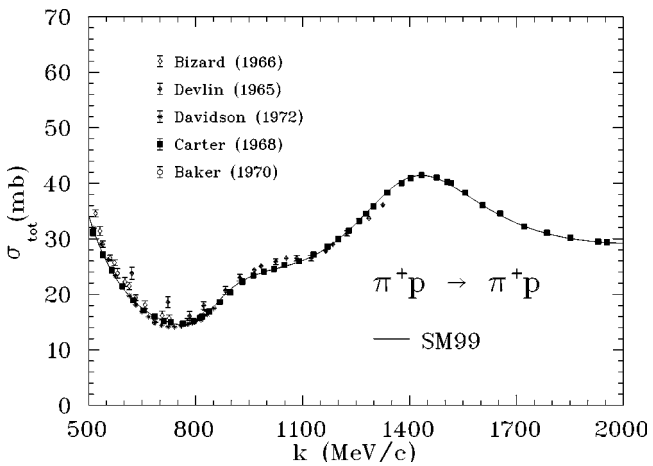

FIG. 5. The $\pi^+ p$ experimental total cross sections in the region $0.5 \leqslant k \leqslant 2$ GeV/c [58–60,68,69] compared to SM99 [28] with Coulomb barrier effects accounted for.

periments and theory. The main contribution comes from the $\pi^+ p$ cross section, which is approximately three times larger than the $\pi^- p$ one.

The systematic uncertainties are more important than the statistical ones. In order to judge their importance, we first evaluated this contribution directly from the experimental π^+ and π^- data sets taken separately, with statistical errors added in quadrature [70]. The result is $-1.652(6)$ mb. It differs by only 0.020 mb from the corresponding quantity evaluated from the phase shift solution SM99. Since the total cross sections have incoherent sums of the squared partial wave amplitudes the large, accurately known, phase shifts dominate. The phase shift solution incorporates strong additional constraints and eliminates minor inconsistencies in the data and is preferable to the raw data. The e.m. corrections to J^- come mainly from this region. They are only weakly model dependent and are included in SM99 using the Nordita procedure [67]. This correction amounts to 0.060 mb as seen in Table VI for the difference in J^- evaluated hadronic vs nuclear cross sections from the same phase shift solution. This well controlled correction increases the coupling constant by only 1.8% and it represents the main e.m. correction. Its uncertainty is at least about 4–5 times less, since it would otherwise seriously affect many conclusions derived from the dominant 33 amplitude. Its contribution to the overall uncertainty of the coupling is therefore nearly negligible.

A modern analysis such as SM99 favors the use of the experimental cross sections dominated by the data of Pedroni *et al.* [63]; the cross sections from Carter *et al.* [62], which dominated the analysis in the 1980's, would lead to a more negative value for J^- and, correspondingly, to a πNN coupling constant larger by about 1%. The hadronic SM99 total cross sections do not contain the inverse photoproduction cross section, which contributes 1 mb (1.5%) of the total nuclear $\pi^- p$ cross section at the resonance peak. This is a negligible source of uncertainty.

The Δ mass splitting may also affect the coupling constant deduced from determinations based on πN data. The empirical isovector mass splitting directly observed in the null experiment by Pedroni *et al.* [63] corresponds to $M_{\Delta^0} - M_{\Delta^+} = 1.38(6)$ MeV. To our knowledge, there exists no information on the isotensor splitting, and we neglect it. With respect to strict isospin symmetry and with the effective position of the Δ resonance that of the Δ^+, the correction to J^- is approximately $4(M_{\Delta^+} - M_{\Delta^0})/3(M_\Delta - M) \simeq -0.6\%$, which would *increase* the coupling constant $g_c^2/4\pi$ by 0.2%. The same conclusion follows from a study by Arndt [71] who used a mass splitting of 0.2 MeV and the same effective Δ position. The corresponding change in J^- is only 0.05%, which scaled to the observed mass splitting contributes 0.17% to $g^2/4\pi$. Consequently, the mass splitting does not significantly influence the value of $g^2/4\pi$.

C. The resonance region

This region from 0.55 to 2 GeV/c is, as a whole, well measured and is analyzed in SM99. The contributions to the integral are positive and rather important up to 1.2 GeV/c, partly compensating the contribution from the Δ resonance region. The remaining region contributes little, but is a minor source of uncertainties.

It is interesting to quantify the difference between a state of the art phase shift solution and data in more detail (see Table VI). In the region of 0.55 to 1.2 GeV/c the nuclear SM99 solution gives 0.012 mb less contribution to J^- (-1.1%) than the direct experimental cross sections, while from 1.2 to 2 GeV/c the corresponding contribution is 0.008 mb less (-0.7%). The overall e.m. corrections to the integral in this region 0.041 mb or 1.3% in $g^2/4\pi$ (see Table VI). The SM99 solution assumes a point charge distribution. Improved Coulomb corrections using an extended charge distribution [72] are being implemented in the phase shift solution SP02, but the results are little changed [73]. Assuming pessimistically that the correction is accurate only to 33%, the overall uncertainty from this source would still be only ± 0.014 mb in J^- or $\pm 0.4\%$ in the coupling constant. The main uncertainty is therefore not due to the e.m. correction, but to systematic differences between SM99 and data. It comes mainly from the region just above 550 MeV/c, as will now be discussed.

At the low energy end of the region between 550 and 1200 MeV/c, there are long-standing experimental problems of systematic nature with the total cross section data. Those of Davidson *et al.* [59] have an incorrect energy calibration, too low by about 10 MeV/c, and its 72 data points must either be recalibrated or eliminated from the analysis [11,23,74]. Similarly, the SM99 solution, driven by modern angular distributions, is systematically lower than the $\pi^- p$ data of Carter *et al.* [60] below 700 MeV/c, a region where data for experimental reasons, are less reliable than at higher energies. These points have been omitted from the PWA analysis [11,74] (see also Fig. 4). This discrepancy is larger than the e.m. corrections in the same energy region. Under the circumstances we have preferred to use the SM99 PWA solution as the best guide, but we use the difference with data as a liberal measure of the uncertainty. We therefore use the overall SM99 contribution from this region of 0.378 ± 0.020 mb.

D. The high-energy and asymptotic regions

There exists abundant experimental information on $\pi^\pm p$ cross sections to high precision from 2 GeV/c up to 350 GeV/c. The main uncertainty in J^- in this region is associated with the relatively slow convergence of the integral. At energies beyond 4 GeV/c there has been an important effort to measure and analyze cross sections, since the issue of the rate at which the $\pi^\pm p$ cross sections become asymptotically equal, is important theoretically for asymptotic theorems. The discussion below is summarized in Table VII.

The region $2 \leqslant k \leqslant 4.03$ GeV/c has been calculated using the Particle Data Group (PDG) 1998 tables [70] (see also [60,75]) and gives a moderate contribution of 0.064 mb, with a modest ± 0.007 mb systematic error. The statistical uncertainty is small. Beyond this region, cross section data with considerable systematic and statistical accuracy exist from

$4.03 \leqslant k \leqslant 370$ GeV/c and are listed in the PDG tables [70,76]. We first evaluated the contribution directly from the precision data. This gives 0.133 mb in the range $4.03 \leqslant k \leqslant 240$ GeV/c, with a small statistical error and a systematic error of about ± 0.022 mb or $\pm 1.8\%$ in J^-. In addition, the 1994 version of the PDG tables [76] also lists a fit to these data from 4.03 to 240 GeV/c (Table 33.3). Using the fitted expression, we have evaluated the contribution in the same interval as above using this expression and find 0.155 mb. This is 0.022 mb higher than the value of 0.133 mb by direct evaluation, but in good general agreement. This larger value has been used in several previous GMO evaluations [8,15]. We prefer the lower value as more transparently linked to the actual data.

Finally, there is a small, but not negligible, contribution from the very high-energy region from 240 GeV/c to ∞. We determine this from the Donnachie-Landshoff Regge fit to the data [77], which describes the observed cross section difference well at the highest energies. This fit is a sum of two Regge terms, one arising from Pomeron exchange and the second from lower-lying resonance exchange. It gives a contribution of 0.030 mb. Alternatively, one might consider using the three-term fit (one for the Pomeron and two for the Reggeons) in the 1998 PDG tables (Table 38.2), which gives 0.018 mb. This low value is not surprising, since the 1998 PDG parametrization gives a difference 27% lower than the PDG 1994 one in the region above 200 GeV/c, at variance with the data [78]. At lower energies, this parametrization agrees better with the data in the region of 100 GeV/c [78,79]. We have also used a recent high-energy fit based on a two-Pomeron pole expression fully compatible with universality, Regge factorization, weak Regge exchange degeneracy, and generalized vector dominance model [80]. This parametrization [see Eq. (13) and Table 1 of Ref. [80]] gives a contribution of 0.025 mb. The corresponding uncertainties, given in Table VII, come mainly from the 4% uncertainty in the Regge intercept. This spread of values according to the model considered for the fit introduces an additional systematic uncertainty of 0.006 mb from this high-energy region. The integrated Coulomb correction above 2 GeV/c is negligible, since it is only 0.003 mb using the estimate of Ref. [65] as given in the beginning of this section.

E. Summary of the results for J^-

The purpose of this section has been to establish the importance of different energy regions for the integral J^- and their contribution to the uncertainty. Since there exists total cross sections from state of the art partial wave analysis up to 2 GeV/c we first studied the accuracy to which such an approach describes contributions to J^- based on actual data in the region 0.16 to 2 GeV/c. To this end, we evaluated the contributions to J^- from data in different energy regions with no Coulomb corrections other than those introduced by the experimental authors (nuclear cross sections). The results are summarized in Tables VI and VII. The statistical uncertainty in the evaluation is small. The trapezoidal formula was used to integrate the data and the corresponding statistical errors were added quadratically. Within the different integration ranges, as given in the tables, the systematic error was calculated by varying the experimental results inside the interval defined by the quoted systematic error. The full systematic uncertainty was obtained by the quadratic sum of the error in each interval, since their origin is different. We then confronted these results with the analogous quantities obtained from the partial wave analysis SM99. The main deviations occurs in the experimentally difficult region 0.55–0.70 GeV/c. Since the partial wave solution incorporates additional experimental constraints, we consider it superior to the direct data in the crucial region and we base the further analysis on the PWA solution SM99.

We then examine the e.m. corrections. These are under theoretical control inside the PWA analysis. The uncertainties in these corrections are less important than the systematic difference between data and the PWA solution. The consequence of the Δ mass splitting is negligible.

The low energy region below 0.16 GeV/c contributes little to J^- and there are no experimental total cross sections in this region. It is strongly constrained by the $\pi^- p$ scattering length and the tail of the Δ resonance such that it can be well controlled without the necessity of e.m. corrections.

The systematic uncertainty has its origin principally in the region above 4 GeV/c. There is also a sizable systematic uncertainty that is due to the inconsistencies of the $\pi^- p$ data in the region 550–700 MeV/c, although we have probably overestimated this uncertainty. We find from Table VII, rows 15–17, that three different descriptions, based on the SM99 PWA below 2 GeV/c, give values in a rather narrow range; $-1.087 \pm 0.009 \pm 0.031$ mb, $-1.099 \pm 0.008 \pm 0.031$ mb, and $-1.092 \pm 0.009 \pm 0.031$ mb. The difference between these values is smaller than the estimated systematic uncertainty. We also give in row 14 the less negative result obtained with the older SM95 PWA below 2 GeV/c and the fit PDG94 in the momentum range from 4.03 to 240 GeV/c: $J^- = -1.053 \pm 0.010 \pm 0.031$ mb. We have chosen the average of these four values as characteristic of the integral. The systematic uncertainty provides an adequate band of possible values, so that

$$J^- = -1.083 \pm 0.009 \pm 0.031 \text{ mb}. \tag{23}$$

Our result for J^- is close to the unpublished value of Koch [27], $J^- = -1.077 \pm 0.047$ mb, which is the only previous explicitly documented and detailed evaluation known to us. The main difference in the input data with Koch is an updated evaluation of the contributions from the high-energy region and better control of e.m. corrections. We show also that the Δ mass splitting is unimportant and include an improved discussion of the threshold region using modern data. It is important to realize that the main uncertainty to J^- comes from the very high energy contribution. It is difficult to ascribe a major uncertainty to the Coulomb corrections. We note that the previous evaluations of J^- quoted in Table V without uncertainties stay within our range of errors.

VI. RESULTS

In conclusion, we summarize our work as follows. We first derived new values for the πN scattering lengths, using

the $\pi^- d$ atomic data analyzed in an improved theoretical approach. The statistical and systematic uncertainties contributions were thoroughly examined. The corresponding $\pi^- d$ scattering length gives a nearly direct determination of the small ''isoscalar'' $\pi^- N$ scattering length to good precision. From this constraint together with the $\pi^- p$ scattering length from pionic hydrogen we obtain a high precision value also for the isovector length. In fact when we examine the basic experimental input of the highly accurately quoted scattering length $a_{\pi^- p}$, deduced from the $\pi^- p$ atomic energy shift and quoted to high accuracy [30], we found that there are small inconsistencies in their current procedure at the level of $\pm 1\%$. This should be improved, since the precision is otherwise unsatisfactory for the determination of the πNN coupling constant. In addition, the experimental accuracy is now so high that systematics in the theoretical analysis of the $\pi^- d$ scattering length is the main source of uncertainty in the disentangling of the isospin components of the πN scattering length. The dominant limitation to higher accuracy is the dispersive contribution from the physical absorption process $\pi^- d \to nn$. A thorough modern reexamination of this contribution is highly desirable. Our analysis does not assume strict isospin symmetry, although we do not see any signs of violation at the present level of precision. We present the results, however, so that they can be directly used in discussions of the validity of this symmetry. The values we find using the empirical $\pi^- p$ and $\pi^- d$ scattering lengths from Sec. IV A, Eqs. (20) and (21), are

$$a^+ \simeq \frac{a_{\pi^- p} + a_{\pi^- n}}{2} = (-12 \pm 2 \pm 8) \times 10^{-4}\ m_\pi^{-1}, \quad (24)$$

$$a^- \simeq \frac{a_{\pi^- p} - a_{\pi^- n}}{2} = (895 \pm 3 \pm 13) \times 10^{-4}\ m_\pi^{-1}. \quad (25)$$

These values are based on theoretical improvements on previous work. [See also comments and footnote after Eq. (21).]

Our second conclusion concerns the charged πNN coupling constant, which can be derived from the GMO forward dispersion relation, using our new, accurate value for the symmetric πN scattering length (24). Use of Eq. (4) assuming charge symmetry and with input values from Eqs. (5), (24) as well as with $J^- = (-1.083 \pm 0.009 \pm 0.031)$ from Table V gives

$$g_c^2/4\pi = (4.87 \pm 0.04 \pm 0.14) + (9.12 \pm 0.02 \pm 0.10) + (0.12 \pm 0.02 \pm 0.08) = (14.11 \pm 0.05 \pm 0.19). \quad (26)$$

The uncorrelated statistical and systematic uncertainties have been added separately in quadrature. The main uncertainty is no longer dominated by the scattering lengths, but comes as much from the weighted integral J^- of the difference between the charged-pion total cross sections. Its dominant systematic uncertainty comes from the region above 4 GeV/c. Previous determinations using the GMO relation [10,14,15,24] will all give similar results, provided one uses the empirical scattering lengths, which are by now well established.[2]

The value, $g_c^2/4\pi = 14.11$, which we obtain for the coupling constant is intermediate between the low value of about 13.6 deduced from the large data banks of NN and πN scattering data using the PWA approach [7,12,8] and the high value of 14.52(26) from np charge exchange cross sections [2]. The uncertainties in the determination of the coupling constant using any method are dominated by systematics. Consequently, we have refrained from combining our result with those from other approaches. However, if the systematic error were to have Gaussian distributions, our result differs from that of Uppsala [2] by only 1.25 standard deviations (21% probability) and from that of Pavan *et al.* [11] by 1.7 standard deviations (8% probability). The PWA results have probably systematic errors far larger than the small statistical errors to judge from the corresponding situation using the data banks with dispersive constraints [11], but this is not quantified yet. The modification of the value of J^- required to accommodate a value of 13.6 is about 10%. The major part of such a modification would most likely come from the region above 2 GeV/c, which implies changes in the contributions from that region of the order of 50%. Such large changes appear unlikely to us.

We therefore conclude that the present evaluation of the GMO sum rule, with quantitatively controlled uncertainties in the input values for the πN isoscalar scattering length, as well as for the cross section integral J^-, does not readily support the conclusion of the indirect PWA determinations that the πNN coupling is close to 13.6. It should be noted that our value has consistently been evaluated in a conservative way, such that the parameters used in the evaluation systematically lead to a value for the coupling constant, which is somewhat on the low side.

The strongest support for a relatively low value of the coupling constant comes from the careful dispersive analysis

[2]After the submission of our paper, the PSI group evaluated the GMO relation from scattering lengths obtained using the BK corrections to the pion-deuteron scattering length [54]. We have discussed the problems of this determination in Sec. IV M, footnote 1. Their quoted value $g^2/4\pi = 13.89^{+0.23}_{-0.11}$ is consistent with our result, but the systematic uncertainties are not well controlled. In particular, there is a substantial additional systematic error of about 0.25 or more from the isoscalar scattering length. In addition, they use a value for J^- derived from an average of those given in Refs. [10,15,24] with the spread of values as the only uncertainty. Of these values, two (Refs. [10,24]) do not state any uncertainty at all, while Ref. [15] states the (small) statistical uncertainty only. In particular, the rather large uncertainty from the high energy region 4–240 GeV/c is neglected. Note that the Particle Data Group 1994 fit to the region 4–240 GeV/c gives a +0.02 mb higher contribution to J^- (Table VII, line 10) than the direct data of the 1998 version (Table VII, line 9). The latter corresponds to a 0.10 higher value of the coupling constant.

by Pavan *et al.* [11], based on the VPI/GWU PWA description of πN scattering. It selectively concentrates on pion-dominated amplitudes. They find a value of $13.73 \pm 0.01 \pm 0.08$, where the first uncertainty is statistical and the second systematic. The authors use a variety of dispersive methods and find $a^+ = +0.0020 \times 10^{-4}\ m_\pi^{-1}$. This value is small, but it has the opposite sign from ours. They evaluate the GMO relation as a consistency check and find a value of $g_c^2/4\pi = 13.75$, in agreement with their dispersion result. Since their evaluation is constrained by the experimental $\pi^- p$ scattering length and their value for the dispersive integral J^- is nearly the same as ours, which is based to a large extent on their PWA analysis, the difference with our result must be almost entirely ascribed to the difference in the value of a^+, a small quantity, which is difficult to calculate from scattering data. The origin of this difference is not known yet, but it might originate in the treatment of small electromagnetic corrections to the scattering data. The minor inconsistency in their analysis is of little importance for most of their discussions, but it becomes highly relevant in the present context.

It is interesting to examine the consequences of our analysis for the Goldberger-Treiman (GT) discrepancy [1]. Following the discussion in Ref. [2] the value for the coupling constant found here corresponds to a discrepancy of $\Delta_{\mathrm{GT}} = (3.6 \pm 1.0)\%$, with Δ_{GT} defined as

$$g_c(1-\Delta_{\mathrm{GT}}) = \mathrm{M}\, g_{\mathrm{A}}/f_\pi . \tag{27}$$

This corresponds to a πNN monopole form factor with a cutoff $\Lambda = 800 \pm 80$ MeV/c. There exists no direct experimental information on this form factor, which is inherently an off-mass-shell quantity. On the other hand, within the framework of PCAC, it is naturally expected to be similar to the axial form factor of the nucleon, a dipole with a 1 GeV/c cutoff. This expectation has been confirmed in many models, using a variety of approaches [81–85], beginning with Ref. [81]. Such values are fully consistent with our findings for the coupling constant. In contrast to these rather soft form factors, the deuteron properties, and in particular its quadrupole moment, require an effective cutoff of 1.3 GeV/c or more, since the tensor force otherwise becomes too weak [50,86]. It is, however, believed at present that this hard effective form factor is generated by the correlated exchange of an interacting $\pi\rho$ pair, which generates additional tensor strength, when explicitly accounted for: the true one-pion-exchange form factor is softer [87–90]. A low value for the coupling constant should therefore not be considered an advantage in resolving the Goldberger-Treiman discrepancy.

Additional support for a coupling constant $g^2/4\pi$ somewhat larger than 14 comes from the recent measurements by Raichle *et al.* of polarized np total cross sections [91]. From these, the pion-dominated ϵ_1 parameter can be determined. They find that it is systematically larger than the values in the phase-shift analysis PWA93 of the Nijmegen group [6]. If the discrepancy persists in other PWAs, this observation suggests, as a possible partial explanation, that the PWA coupling constant is too small. In any case, it points to an unexplained discrepancy with those PWA analyses on which the argument for a low coupling constant is based.

In order to facilitate future improvements on the present work, we have presented the various corrections in such a way that modifications of any individual contributions can be readily incorporated without the necessity of a complete reanalysis. We see three main areas in which the present work can be improved. First, theoretical investigations of the relation between the hadronic energy shift of the pionic atom and the scattering length should diminish the present uncertainty in the deduced $\pi^- p$ and $\pi^- d$ scattering lengths by a factor of at least 2. Second, the measurement to high precision of the width in pionic hydrogen should give a separation of the isospin components in the $\pi^- p$ scattering lengths to similar precision as that obtained from the deuteron data, but without invoking deuteron structure. Third, studies of the dispersion shift for threshold pion absorption on the deuteron should eliminate a major uncertainty in the theoretical treatment of the $\pi^- d$ scattering length. This would allow the πNN coupling constant to be determined to 1% precision.

ACKNOWLEDGMENTS

We are particularly indebted to R. L. Arndt, V. V. Baru, A. E. Kudryatsev, M. Pavan, and M. Sainio for providing us with unpublished material and to D. Bugg, M. Ericson, G. Fäldt, J. Gasser, W. R. Gibbs, G. Höhler, O. Krehl, S. Krewald, M. Lacombe, U.-G. Meissner, and S. Wycech for valuable information. This work was supported by grants from the Swedish National Research Council and from the Swedish Royal Academy of Sciences. Laboratoire de Physique Nucléaire et de Hautes Énergies is Unité de Recherche des Universités Paris 6 et Paris 7, associée au CNRS.

APPENDIX A: EXPRESSIONS USED FOR THE ELECTROMAGNETIC CORRECTIONS TO THE EXPERIMENTAL $\pi^- d$ SCATTERING LENGTH

We here give details on expressions we used in the evaluation of the electromagnetic corrections to the experimental $\pi^- d$ scattering length. One can write, with $q^2 = \omega^2 - m_\pi^2$, the low-energy πN amplitudes as

$$a_{\pi^- p}(\omega) = a^+ + a^- + (b^+ + b^-)q^2,$$

$$a_{\pi^- n}(\omega) = a^+ - a^- + (b^+ - b^-)q^2. \tag{A1}$$

Höhler [23] gives $b^- = (133 \pm 60) \times 10^{-4} m_\pi^{-3}$ and $b^+ = (-443 \pm 67) \times 10^{-4}\ m_\pi^{-3}$. The $\pi^- d$ single scattering term is

$$S = \lambda_1 [a_{\pi^- p}(\omega) + a_{\pi^- n}(\omega)] = 2\lambda_1 (a^+ + b^+ q^2), \tag{A2}$$

where

$$\lambda_1 = \frac{(1 + m_\pi/M)}{(1 + m_\pi/M_d)} = 1.0691. \tag{A3}$$

Minimal coupling corresponds to $\omega \to \omega - eV_C$, i.e.,

$$q^2 = \omega^2 - m_\pi^2 \to (\omega - eV_C)^2 - m_\pi^2 \simeq q^2 - 2e\omega V_C,$$

where the Coulomb field V_C originates from the extended deuteron charge distribution averaged over the deuteron matter distribution. One then has, for the single scattering term, the electromagnetic correction

$$\Delta S = -4\lambda_1 m_\pi b^+ e\langle V_C^d(r)\rangle, \tag{A4}$$

where, for point particles and in terms of the relative deuteron coordinate $\mathbf{r}$,

$$\begin{aligned} e\langle V_C^d(r)\rangle &= \alpha \times \left\langle \int d\mathbf{r}' \frac{\rho(r')}{(|\mathbf{r}-\mathbf{r}'|/2)} \right\rangle \\ &\equiv 2\alpha \langle 1/r\rangle_{ch} \\ &= 2\alpha \times \int\int d\mathbf{r} d\mathbf{r}' \rho(r)\rho(r') \times (1/|\mathbf{r}-\mathbf{r}'|) \\ &= 4\alpha \times \int_0^\infty dr[u(r)^2 + w(r)^2] \\ &\quad \times (1/r) \int_0^r dr'[u(r')^2 + w(r')^2]. \end{aligned} \tag{A5}$$

For both the Paris [49] and Bonn2 [50] deuteron wave functions $\langle 1/r\rangle_{ch} = 0.300$ fm^{-1} and $e\langle V_C^d(r)\rangle = 0.86$ MeV, then

$$\Delta S = (12 \pm 2) \times 10^{-4}\ m_\pi^{-1}.$$

The alternative evaluation, which gauges the $\pi^- n$ interaction with the Coulomb field from the static spectator proton, gives

$$\Delta S = -2\lambda_1 m_\pi (b^+ - b^-) e\langle V_C^p(r)\rangle \tag{A6}$$

with

$$e\langle V_C^p(r)\rangle = \alpha\langle 1/r\rangle = 0.66(1)\ \text{MeV}, \tag{A7}$$

using the average inverse deuteron radius of Paris and Bonn2 models, viz. $\langle 1/r\rangle_d = 0.456(7)$ fm^{-1}. One obtains,

$$\Delta S = (6 \pm 1) \times 10^{-4}\ m_\pi^{-1}.$$

APPENDIX B: PRACTICAL EXPRESSIONS FOR THE THEORETICAL $\pi^- d$ SCATTERING LENGTH FOR SEPARABLE SCATTERING INTERACTIONS

We give here full practical expressions for the theoretical $\pi^- d$ scattering length for separable scattering amplitudes with a dipole form factor $v^2(q) = (1 + q^2/c^2)^{-2}$:

$$\begin{aligned} a_{\pi^- d} &= \lambda_1 (a_{\pi^- p} + a_{\pi^- n}) \\ &\quad + \lambda_2 \left[\left(\frac{a_{\pi^- p} + a_{\pi^- n}}{2} \right)^2 - 2 \left(\frac{a_{\pi^- p} - a_{\pi^- n}}{2} \right)^2 \right] \langle f(r)/r \rangle \\ &\quad + a(\text{Fermi}) + a(\text{dispersion}) + \delta a, \end{aligned} \tag{B1}$$

where

$$\lambda_2 = 2\frac{(1 + m_\pi/M)^2}{(1 + m_\pi/M_d)} = 2.4560, \tag{B2}$$

$$f(r) = 1 - (1 + cr/2)\exp(-cr), \tag{B3}$$

and with the sum of small correction terms

$$\begin{aligned} \delta a &= \delta a(\text{multiple}) + \delta a(\text{isospin}) + \delta a(\text{nonstatic}) \\ &\quad + \delta a(\pi^- p, \gamma n) + \delta a(\text{double } p \text{ wave}) \\ &\quad + \delta a(\text{virtual pion}). \end{aligned} \tag{B4}$$

Assuming isospin symmetry in all terms but the leading order one, and emphasizing the accurate experimental knowledge of $a_{\pi^- p}$, we have

$$\begin{aligned} a_{\pi^- d} &= \lambda_1 (a_{\pi^- p} + a_{\pi^- n}) + \lambda_2 [a^{+2} - 2(a_{\pi^- p} - a^+)^2] \\ &\quad \times \langle f(r)/r \rangle + a(\text{Fermi}) + a(\text{dispersion}) + \delta a. \end{aligned} \tag{B5}$$

The correction for nucleon motion is, according to Eq. (14),

$$a(\text{Fermi}) = 2\left(\frac{m_\pi}{M + m_\pi} \right)^2 \lambda_1 c_0 \left\langle p^2 v^2 \left(\frac{m_\pi p}{M + m_\pi} \right) \right\rangle, \tag{B6}$$

where the form factor correction is negligible and $c_0 = 0.208(3)\ m_\pi^{-3}$ (p. 18 in Ref. [40]). The dispersion correction has been taken to be $a(\text{dispersion}) = -56(14) \times 10^{-4}\ m_\pi^{-1}$ [42]. The remaining values of the small terms are taken to be (see text) $\delta a(\text{nonstatic}) = 11(6) \times 10^{-4}\ m_\pi^{-1}$, $\delta a(\text{double } p \text{ wave}) = -3 \times 10^{-4}\ m_\pi^{-1}$ and $\delta a(\text{virtual pion}) = -7(2) \times 10^{-4}\ m_\pi^{-1}$. In addition it is desirable to control the convergence of the multiple scattering expansion explicitly. We have evaluated the higher order multiple scattering corrections from the expression given by Kolybasov and Kudryatsev for the sum of the multiple scattering series to all orders for pointlike scatterers, in the static approximation and neglecting binding and recoil corrections [92]. We have, however, generalized their expression to include the effect of separable form factors for each scattering

$$\begin{aligned} \delta a(\text{multiple}) = \left\langle \left[2\lambda_1 a^+ + \lambda_2 [a^{+2} - 2(a_{\pi^- p} - a^+)^2] \frac{f(r)}{r} \right] \right. \\ \left. \times [(1 - C)^{-1} - 1] \right\rangle, \end{aligned} \tag{B7}$$

where $C = (1 + m_\pi/M)^2 [a^{+2} - 2(a_{\pi^- p} - a^+)^2] f^2(r)/r^2$. In order to extract the value of $(a_{\pi^- p} + a_{\pi^+ p})/2$ from the ex-

perimental $a_{\pi^- d}$ and $a_{\pi^- p}$, we now observe that Eq. (B5) is quadratic in a^+ except for higher power terms from the small δa(multiple) of Eq. (B7). To check the self-consistency with δa(multiple) it should be solved iteratively. We have done this with the experimental values and the resulting a^+ is small (about $10^{-3}\ m_\pi^{-1}$). Equation (B5) can then be safely linearized for a fixed value of δa(multiple) and the consistency checked by iteration. We have

$$\frac{a_{\pi^- p}+a_{\pi^- n}}{2} = (2\lambda_1 + 4\lambda_2 a^{\exp}_{\pi^- p}\langle f(r)/r\rangle)^{-1} \times [a^{\exp}_{\pi^- d} + 2\lambda_2 a^{\exp\,2}_{\pi^- p}\langle f(r)/r\rangle - a(\text{Fermi}) - a(\text{dispersion}) - \delta a]. \quad \text{(B8)}$$

Two iterations are sufficient.

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