PUBLISHED VERSION

1. Hawes, Frederick T.; Sizer, Tom; Williams, Anthony Gordon
Renormalized strong-coupling quenched QED in four dimensions Physical Review D, 1997; 55(6):3866-3872

© 1997 American Physical Society

http://link.aps.org/doi/10.1103/PhysRevD.55.3866

PERMISSIONS

http://publish.aps.org/authors/transfer-of-copyright-agreement

“The author(s), and in the case of a Work Made For Hire, as defined in the U.S. Copyright Act, 17 U.S.C.

§101, the employer named [below], shall have the following rights (the “Author Rights”):

[...]

3. The right to use all or part of the Article, including the APS-prepared version without revision or modification, on the author(s)’ web home page or employer’s website and to make copies of all or part of the Article, including the APS-prepared version without revision or modification, for the author(s)’ and/or the employer’s use for educational or research purposes.”

12th April 2013

http://hdl.handle.net/2440/10918

PHYSICAL REVIEW D VOLUME 55, NUMBER 6 15 MARCH 1997

Renormalized strong-coupling quenched QED in four dimensions

Frederick T. Hawes
Department of Physics and SCRI, Florida State University, Tallahassee, Florida 32306-3016

Tom Sizer
Department of Physics and Mathematical Physics, University of Adelaide, 5005, Australia

Anthony G. Williams
Department of Physics and Mathematical Physics, and Institute for Theoretical Physics,
University of Adelaide, 5005, Australia

(Received 9 August 1996)

We study renormalized quenched strong-coupling QED in four dimensions in an arbitrary covariant gauge. Above the critical coupling leading to dynamical chiral symmetry breaking, we show that there is no finite chiral limit. This behavior is found to be independent of the detailed choice of photon-fermion proper vertex in the Dyson-Schwinger equation formalism, provided that the vertex is consistent with the Ward-Takahashi identity and multiplicative renormalizability. We show that the finite solutions previously reported lie in an unphysical regime of the theory with multiple solutions and ultraviolet oscillations in the mass functions. This study is consistent with the assertion that in four dimensions strong coupling QED does not have a continuum limit in the conventional sense. [S0556-2821(97)03606-0]

PACS number(s): 12.20.Ds, 11.30.Qc, 11.30.Rd

I. INTRODUCTION

A useful approach to studying the mechanism of dynamical chiral symmetry breaking (DCSB) is through the Dyson-Schwinger equation (DSE) formalism [1–3]. The infinite set of coupled DSE's must always be truncated at some point, but we can still make progress by closing off the tower of equations with a suitable *Ansatz* consistent with all appropriate symmetries of the theory and having the correct perturbative limit. While not a complete first principles treatment of a theory, this approach is nonetheless a useful tool, since it does allow Lorentz covariance to be maintained as well as allowing for the infrared (IR) and ultraviolet (UV) limits to be numerically taken in a straightforward way. In addition, model-independent results following from symmetry principles alone can still be obtained and numerically verified in a rigorous way as we will see. On the other hand, lattice gauge theory (LGT) studies [4] are a first principles, approximation-free approach, but they still present a significant computational challenge when attempting to verify ultraviolet and infrared limits [1,2].

The Abelian nature of quantum electrodynamics (QED) in many ways makes it a much simpler system to study than a non-Abelian theory such as quantum chromodynamics (QCD). For this reason it has been the subject of many nonperturbative studies, which have as their long-term goal a detailed understanding of nonperturbative QCD. On the other hand, strong-coupling four-dimensional QED (QED_4) is widely anticipated to behave unconventionally in the continuum limit and for this reason is a theory of considerable interest in its own right.

In any well-behaved theory with dynamical chiral symmetry breaking, there are (at least) two solutions: (i) One of these has a mass function which vanishes identically in the absence of a nonzero bare mass and exists for all values of the coupling, and (ii) the other, which exists only above critical coupling, corresponds to the presence of dynamical chiral symmetry breaking. Below critical coupling the former chiral-symmetric solution is stable, whereas above critical coupling the latter is the stable solution. This is just the scenario found in studies of the Nambu–Jona-Lasinio model, for example (see, e.g., Ref. [5] and references therein).

In previous work [6] we first introduced a numerical renormalization procedure and applied it to QED_4 with a quenched photon propagator using the Dyson-Schwinger formalism. The regularization used in this work was a momentum cutoff scheme. This is a direct application of the standard renormalization procedure to the nonlinear self-consistent framework needed to study dynamical chiral symmetry breaking. This initial work, in the Landau gauge, was recently generalized to arbitrary covariant gauges [7]. The central result of these two works was to demonstrate that the numerical renormalization procedure works extremely well and allows the continuum limit ($\Lambda\to\infty$) to be taken numerically, while giving rise to stable finite solutions for the renormalized fermion propagator.

In this article we investigate the chiral limit in renormalized quenched strong-coupling QED_4, using a photon-fermion vertex that satisfies the Ward-Takahashi identity (WTI) and makes the fermion DSE multiplicatively renormalizable. We find that for couplings above the chiral critical coupling, keeping the bare mass $m_0(\Lambda)\equiv 0$ as the cutoff is relaxed results in either a dynamical mass function which is identically zero everywhere or a dynamical mass function which diverges with the cutoff. The finite solutions described in previous articles [6,7] showed damped oscillations in the dynamical mass functions at large p^2, which suggested that they were unphysical. Further, we show here that for a given supercritical coupling and the same bare mass $m_0(\Lambda)$, it is possible to have multiple solutions corresponding to different

© 1997 The American Physical Society

renormalized masses $m(\mu)$. We conclude that quenched strong-coupling QED in four dimensions does not have a chiral limit in the conventional sense above the chiral phase transition.

In Sec. II we briefly summarize the renormalized Dyson-Schwinger equation formalism, the numerical renormalization procedure, and a particular fermion-photon vertex *Ansatz* that we use to illustrate our general arguments numerically. In Sec. III we demonstrate the scaling of the DSE solutions with zero bare mass for supercritical coupling and the existence of multiple solutions with the same bare mass; we also present a general argument to show that a vertex which is consistent with the WTI and multiplicative renormalizability leads to a diverging mass function above critical coupling in the continuum limit. We discuss these results and their relevance to QCD and to unquenched QED_4 in Sec. IV. For further details and references and an expanded discussion we refer the reader to Refs. [1,6,7].

II. FORMALISM

Dynamical chiral symmetry breaking (DCSB) occurs when the fermion propagator develops a nonzero scalar self-energy in the absence of an explicit chiral-symmetry-breaking (ECSB) fermion mass. We will refer to coupling constants strong enough to induce DCSB as supercritical and those weaker as subcritical. We write the fermion propagator as

$$S(p)=\frac{Z(p^2)}{\not p-M(p^2)}=\frac{1}{A(p^2)\not p-B(p^2)}, \tag{1}$$

where we refer to $A(p^2)\equiv 1/Z(p^2)$ as the finite momentum-dependent fermion renormalization and where $M(p^2)\equiv B(p^2)/A(p^2)$ is the fermion mass function. In the massless theory [i.e., in the absence of an ECSB bare fermion mass $m_0(\Lambda)$] by definition DCSB occurs when $M(p^2)\neq 0$.

With the exception of Refs. [6,7], most studies have neglected the issue of the subtractive renormalization of the DSE for the fermion propagator. Typically these studies have assumed an initially massless theory and have renormalized at the ultraviolet cutoff of the loop integration, taking $Z_1=Z_2=1$. Where a nonzero bare mass has been used, it has simply been added to the scalar term in the propagator. Although there were earlier formal discussions of renormalization [2,8–11], the subtractive renormalization program had not previously been implemented.

We will concentrate our discussion on quenched strong-coupling QED_4, where here the term ‘‘quenched’’ means that the bare photon propagator is used in the fermion self-energy DSE, so that $Z_3=1$ and there is no renormalization of the electron charge. It should be carefully noted that this is a slightly different usage from that found in lattice gauge theory studies, since in DSE studies with a quenched photon propagator virtual fermion loops may still be present in the proper fermion-photon vertex.

The DSE for the renormalized fermion propagator, in an arbitrary covariant gauge, is

$$S^{-1}(p)=Z_2(\mu,\Lambda)[\not p-m_0(\Lambda)]-iZ_1(\mu,\Lambda)e^2 \times\int^{\Lambda}\frac{d^4k}{(2\pi)^4}\gamma^{\mu}S(k)\Gamma^{\nu}(k,p)D_{\mu\nu}(q); \tag{2}$$

here, $q=k-p$ is the photon momentum, μ is the renormalization point, and Λ is a regularizing parameter (taken here to be an ultraviolet momentum cutoff). We write $m_0(\Lambda)$ for the regularization-parameter-dependent bare mass. The renormalized charge is e (as opposed to the bare charge e_0), and the general form for the renormalized photon propagator is

$$D^{\mu\nu}(q)=\left\{\left(-g^{\mu\nu}+\frac{q^{\mu}q^{\nu}}{q^2}\right)\frac{1}{1+\Pi(q^2)}-\xi\frac{q^{\mu}q^{\nu}}{q^2}\right\}\frac{1}{q^2}, \tag{3}$$

with ξ the renormalized covariant gauge parameter and $\xi_0\equiv Z_3(\mu,\Lambda)\xi$ the corresponding bare one. In the quenched approximation, we have for the coupling strength and gauge parameter, respectively, $\alpha\equiv e^2/4\pi=\alpha_0\equiv e_0^2/4\pi$ and $\xi=\xi_0$, and for the photon propagator we have

$$D^{\mu\nu}(q)\rightarrow D_0^{\mu\nu}(q)=\left\{\left(-g^{\mu\nu}+\frac{q^{\mu}q^{\nu}}{q^2}\right)-\xi\frac{q^{\mu}q^{\nu}}{q^2}\right\}\frac{1}{q^2}. \tag{4}$$

The requirement of gauge invariance in QED leads to a set of identities referred to as the Ward-Takahashi identities (WTI's). The WTI for the fermion-photon vertex is

$$q_{\mu}\Gamma^{\mu}(k,p)=S^{-1}(k)-S^{-1}(p), \tag{5}$$

where $q=k-p$. This is a generalization of the original, differential Ward identity, which expresses the effect of inserting a zero-momentum photon vertex into the fermion propagator:

$$\frac{\partial S^{-1}(p)}{\partial p_{\nu}}=\Gamma^{\nu}(p,p). \tag{6}$$

The Ward identity, Eq. (6), follows immediately from the WTI of Eq. (5). In general, for nonvanishing photon momentum q, only the longitudinal component of the proper vertex is constrained; i.e., the WTI provides no information on $\Gamma_{\mathrm{T}}^{\nu}(k,p)\equiv\mathcal{T}^{\mu\nu}\Gamma_{\nu}(p,k)$ for $q\neq 0$. [We use the notation $\mathcal{T}^{\mu\nu}\equiv g^{\mu\nu}-(q^{\mu}q^{\nu}/q^2)$ and $\mathcal{L}^{\mu\nu}\equiv(q^{\mu}q^{\nu}/q^2)$ for the transverse and longitudinal projectors, respectively.] In particular, the WTI guarantees the equality of the propagator and vertex renormalization constants, $Z_2\equiv Z_1$ (at least in any sensible choice of subtraction scheme [1]). The WTI can be shown to be satisfied order by order in perturbation theory and can also be derived nonperturbatively.

As discussed in [1,12], this can be thought of as just one of a set of six general requirements on the vertex: (i) The vertex must satisfy the WTI; (ii) it should contain no kinematic singularities; (iii) it should transform under charge conjugation (C), parity inversion (P), and time reversal (T) in the same way as the bare vertex, e.g.,

$$C^{-1}\Gamma_{\mu}(k,p)C=-\Gamma_{\mu}^{T}(-p,-k) \tag{7}$$

(where the superscript T indicates the transpose); (iv) it should reduce to the bare vertex in the weak-coupling limit; (v) it should ensure multiplicative renormalizability of the DSE in Eq. (2); (vi) the transverse part of the vertex should be specified to ensure gauge-covariance of the DSE.

Ball and Chiu [13] have given a description of the most general fermion-photon vertex that satisfies the WTI; it consists of a longitudinally constrained (i.e., ''Ball-Chiu'') part $\Gamma^\mu_{\rm BC}$ which is a minimal solution of the WTI with no artificial kinematic singularities, and a basis set of eight transverse vectors $T^\mu_i(k,p)$, which span the hyperplane specified by $\mathcal{L}_{\mu\nu}T^\nu_i(k,p)=0$ [i.e., $q_\nu T^\nu_i(k,p)=0$], where $q\equiv k-p$. The minimal longitudinally constrained part of the vertex will be referred to as the Ball-Chiu vertex and is given by

$$\Gamma^\mu_{\rm BC}(k,p)=\frac{1}{2}[A(k^2)+A(p^2)]\gamma^\mu+\frac{(k+p)^\mu}{k^2-p^2}\times\left\{[A(k^2)-A(p^2)]\frac{\not{k}+\not{p}}{2}-[B(k^2)-B(p^2)]\right\}. \qquad (8)$$

Note that since neither $\mathcal{L}_{\mu\nu}\Gamma^\nu_{\rm BC}(k,p)$ nor $\mathcal{T}_{\mu\nu}\Gamma^\nu_{\rm BC}(k,p)$ vanish identically, the Ball-Chiu vertex has both longitudinal and transverse components. The transverse vectors can be conveniently written as [14]

$$T^\mu_1(k,p)=p^\mu(k\cdot q)-k^\mu(p\cdot q), \qquad (9)$$

$$T^\mu_2(k,p)=[p^\mu(k\cdot q)-k^\mu(p\cdot q)](\not{k}+\not{p}), \qquad (10)$$

$$T^\mu_3(k,p)=q^2\gamma^\mu-q^\mu\not{q}, \qquad (11)$$

$$T^\mu_4(k,p)=q^2[\gamma^\mu(\not{p}+\not{k})-p^\mu-k^\mu]-2i(p-k)^\mu k^\lambda p^\nu\sigma_{\lambda\nu}, \qquad (12)$$

$$T^\mu_5(k,p)=-iq_\nu\sigma^{\nu\mu}, \qquad (13)$$

$$T^\mu_6(k,p)=\gamma^\mu(p^2-k^2)+(p+k)^\mu\not{q}, \qquad (14)$$

$$T^\mu_7(k,p)=\frac{1}{2}(p^2-k^2)[\gamma^\mu(\not{p}+\not{k})-p^\mu-k^\mu]-i(k+p)^\mu k^\lambda p^\nu\sigma_{\lambda\nu}, \qquad (15)$$

$$T^\mu_8(k,p)=i\gamma^\mu k^\nu p^\lambda\sigma_{\nu\lambda}+k^\mu\not{p}-p^\mu\not{k}, \qquad (16)$$

where we use the conventions $g^{\mu\nu}={\rm diag}(1,-1,-1,-1)$, $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$, and $\sigma^{\mu\nu}\equiv(i/2)[\gamma^\mu,\gamma^\nu]$. A general vertex is then written as

$$\Gamma^\mu(k,p)=\Gamma^\mu_{\rm BC}(k,p)+\sum_{i=1}^{8}\tau_i(k^2,p^2,q^2)T^\mu_i(k,p), \qquad (17)$$

where the τ_i are functions that must be chosen to give the correct C, P, and T invariance properties.

As previously mentioned, the renormalization procedure is entirely standard. One first determines a finite, *regularized* self-energy, which depends on both a regularization parameter and the renormalization point. One then performs a subtraction at the renormalization point, in order to define the renormalization parameters Z_1, Z_2, and Z_3 which give the full (renormalized) theory in terms of the regularized calculation. Consider the regularized self-energy $\Sigma'(\mu,\Lambda;p)$, leading to the DSE for the renormalized fermion propagator,

$$S^{-1}(p)=Z_2(\mu,\Lambda)[\not{p}-m_0(\Lambda)]-\Sigma'(\mu,\Lambda;p)=\not{p}-m(\mu)-\tilde{\Sigma}(\mu;p)=A(p^2)\not{p}-B(p^2), \qquad (18)$$

where $\tilde{\Sigma}(\mu;p)$ denotes the *renormalized* self-energy and the *regularized* self-energy is given by $[q\equiv(k-p)]$

$$\Sigma'(\mu,\Lambda;p)=iZ_1(\mu,\Lambda)e^2\int^\Lambda\frac{d^4k}{(2\pi)^4}\gamma^\lambda S(\mu;k)\times\Gamma^\nu(\mu;k,p)D_{\lambda\nu}(\mu;q). \qquad (19)$$

Here $D^{\lambda\nu}(\mu;q)$ and $\Gamma^\nu(\mu;k,p)$ denote the renormalized photon propagator and photon-fermion proper vertex, respectively. As suggested by the notation (i.e., the omission of the Λ dependence) renormalized quantities must become independent of the regularization parameter as the regularization is removed (i.e., as $\Lambda\to\infty$) in a renormalizable theory. The self-energies are decomposed into Dirac and scalar parts:

$$\Sigma'(\mu,\Lambda;p)=\Sigma'_d(\mu,\Lambda;p^2)\not{p}+\Sigma'_s(\mu,\Lambda;p^2) \qquad (20)$$

[and similarly for the renormalized quantity $\tilde{\Sigma}(\mu,p)$]. By imposing the renormalization boundary condition

$$S^{-1}(p)|_{p^2=\mu^2}=\not{p}-m(\mu), \qquad (21)$$

one gets the relations

$$\tilde{\Sigma}_{d,s}(\mu;p^2)=\Sigma'_{d,s}(\mu,\Lambda;p^2)-\Sigma'_{d,s}(\mu,\Lambda;\mu^2) \qquad (22)$$

for the self-energy,

$$Z_2(\mu,\Lambda)=1+\Sigma'_d(\mu,\Lambda;\mu^2) \qquad (23)$$

for the renormalization constant, and

$$m_0(\Lambda)=[m(\mu)-\Sigma'_s(\mu,\Lambda;\mu^2)]/Z_2(\mu,\Lambda) \qquad (24)$$

for the bare mass. A mass renormalization constant can be defined as

$$Z_m(\mu,\Lambda)=m_0(\Lambda)/m(\mu), \qquad (25)$$

i.e., as the ratio of the bare to renormalized mass. It should be noted that this does not correspond to the usual mass renormalization constant which one has in mind in QCD, where it typically is taken to refer to the ratio of bare to *current* quark mass, i.e., after removing that part of the quark mass due to dynamical chiral symmetry breaking. The vertex renormalization $Z_1(\mu,\Lambda)$ is identical to $Z_2(\mu,\Lambda)$ as long as the vertex *Ansatz* satisfies the Ward identity; this is how it is recovered for multiplication into $\Sigma'(\mu,\Lambda;p)$ in Eq. (19).

The chiral limit occurs by definition when the bare mass is set to zero and the regularization is removed, i.e., maintaining $m_0(\Lambda)=0$ while taking the limit $\Lambda\to\infty$. Explicit chiral symmetry breaking (ECSB) occurs when the bare mass $m_0(\Lambda)$ is not zero. Dynamical mass generation or dynamical chiral symmetry breaking (DCSB) is said to have occurred when $M(p^2)\neq 0$ in the absence of ECSB. As the coupling strength increases from zero there is a transition to a DCSB phase at the critical coupling strength α_c. Concisely, the absence of ECSB means that $m_0(\Lambda)=0$ and the absence of both ECSB and DCSB (i.e., $\alpha<\alpha_c$) means that $M(p^2)$, $m(\mu)$, and $m_0(\Lambda)$ simultaneously vanish. [Recall that in the notation that we use here, $M(p^2)\equiv B(p^2)/A(p^2)$ and $m(\mu)\equiv M(\mu^2)$.] This is the same definition of the chiral limit that is used in nonperturbative studies of QCD; see, e.g., Refs. [1–4] and references therein. Obviously, any limiting procedure where we take $m_0(\Lambda)\to 0$ sufficiently rapidly as $\Lambda\to\infty$ will also lead to the chiral limit.

Let us temporarily indicate explicitly the choice of renormalization point by a μ dependence of the renormalized quantities, i.e., $A(\mu;p^2)\equiv 1/Z(\mu;p^2)$, $M(\mu;p^2)\equiv B(\mu;p^2)/A(\mu;p^2)$, etc. Note that Eq. (18) implies that

$$\begin{aligned}A(\mu;p^2)&=Z_2(\mu,\Lambda)-\Sigma'_d(\mu,\Lambda;p^2)=1-\tilde{\Sigma}_d(\mu,\Lambda;p^2),\\ B(\mu;p^2)&=Z_2(\mu,\Lambda)m_0(\Lambda)+\Sigma'_s(\mu,\Lambda;p^2)\\ &=m(\mu)+\tilde{\Sigma}_s(\mu,\Lambda;p^2).\end{aligned}\tag{26}$$

The renormalization point boundary condition in Eq. (21) then leads to $\tilde{\Sigma}(\mu,\Lambda;\mu^2)=0$ or, equivalently, to the two boundary conditions $A(\mu;\mu^2)=1$ and $M(\mu;\mu^2)=B(\mu;\mu^2)=m(\mu)$. From Eq. (26) we have

$$\begin{aligned}&[A(\mu;p^2)/Z_2(\mu,\Lambda)]=1-[\Sigma'_d(\mu,\Lambda;p^2)/Z_2(\mu,\Lambda)],\\ &[B(\mu;p^2)/Z_2(\mu,\Lambda)]=m_0(\Lambda)+[\Sigma'_s(\mu,\Lambda;p^2)/Z_2(\mu,\Lambda)].\end{aligned}\tag{27}$$

The renormalization group is the set of renormalization point transformations which by definition leave the bare quantities of the theory unchanged. Hence, since $m_0(\Lambda)$ is renormalization point independent, it is clear from Eq. (27) that $A(\mu;p^2)/Z_2(\mu,\Lambda)$, $B(\mu;p^2)/Z_2(\mu,\Lambda)$, $\Sigma'_d(\mu,\Lambda;p^2)/Z_2(\mu,\Lambda)$, and $\Sigma'_s(\mu,\Lambda;p^2)/Z_2(\mu,\Lambda)$ are renormalization point independent. Hence $S(\mu;p)Z_2(\mu,\Lambda)$ is renormalization point independent. In shorthand form we can express this as $S(\mu;p)\propto 1/Z_2(\mu,\Lambda)$ under a renormalization point transformation. Hence, the choice of renormalization point is equivalent to the choice of scale for the functions A and B. Similarly, in the general unquenched case [15] under a renormalization point transformation we have in addition $D^{\sigma\nu}(\mu;q)\propto\xi(\mu)\propto 1/Z_3(\mu,\Lambda)$, $e(\mu)\propto Z_2(\mu,\Lambda)\sqrt{Z_3(\mu,\Lambda)}/Z_1(\mu,\Lambda)$, and $\Gamma^\nu(\mu;q,p)\propto Z_1(\mu,\Lambda)$. It is straightforward to verify the consistency of these renormalization point transformations. For example, from Eq. (19) we see that these scaling properties ensure that $\Sigma'\propto Z_2$ and hence from Eq. (18) that $S(p)\propto 1/Z_2$ as it should. Thus, since $Z_1=Z_2$, multiplicative renormalizability will automatically follow if we ensure that $\Gamma\to c\Gamma$ as $A(p^2)\to cA(p^2)$ and $B(p^2)\to cB(p^2)$. This behavior is automatic for the Ball-Chiu part of the proper vertex $\Gamma^\nu_{\rm BC}(\mu;p,k)$ as can be seen from Eq. (8). This consistency is unaffected by considering the quenched photon propagator case where $Z_3=1$. Clearly in order to ensure multiplicative renormalizability for an arbitrary vertex it is necessary to choose the functions τ_i to scale in the same way, i.e., $\tau_i\propto c\tau_i$. This is the precise statement of the restriction that multiplicative renormalizability imposes on the proper photon-fermion vertex.

From the above arguments we see that under a renormalization point transformation we must have, *for all* p^2,

$$\begin{aligned}&M(\mu';p^2)=M(\mu;p^2)\equiv M(p^2),\\ &\frac{A(\mu';p^2)}{A(\mu;p^2)}=\frac{Z_2(\mu',\Lambda)}{Z_2(\mu,\Lambda)}=A(\mu';\mu^2)=\frac{1}{A(\mu;\mu'^2)},\end{aligned}\tag{28}$$

from which it follows for the fermion propagator that $S(\mu';p)/S(\mu;p)=Z_2(\mu,\Lambda)/Z_2(\mu',\Lambda)$ in the usual way. The behavior in Eq. (28) is explicitly tested for our numerical solutions. It is clear from Eq. (28) that having a solution at one renormalization point (μ) completely determines the solution at any other renormalization point (μ') without the need for any further calculation.

Example vertex choice

Our results and conclusions regarding the chiral limit are general and do not depend on any specific vertex choice, i.e., any specific choice for the functions τ_i. We only require that the vertex satisfy the Ward-Takahashi identity and that it be consistent with multipicative renormalizability. However, in order to discuss the renormalized finite solutions it is necessary to present detailed numerical calculations. For this purpose we will use as an example the modified treatment of the Curtis-Pennington vertex introduced in Ref. [7].

Curtis and Pennington published a series of articles [8–11] describing their specification of a particular transverse vertex term, in an attempt to produce gauge-covariant and multiplicatively renormalizable solutions to the DSE. In the framework of massless QED$_4$, they eliminated the four transverse vectors which are Dirac-even and must generate a scalar term. By requiring that the vertex $\Gamma^\mu(k,p)$ reduce to the leading logarithmic result for $k\gg p$ they were led to eliminate all the transverse basis vectors except T_6^μ, with a dynamic coefficient chosen to make the DSE multiplicatively renormalizable. This coefficient had the form

$$\tau_6(k^2,p^2,q^2)=-\tfrac{1}{2}[A(k^2)-A(p^2)]/d(k,p),\tag{29}$$

where $d(k,p)$ is a symmetric, singularity-free function of k and p, with the limiting behavior $\lim_{k^2\gg p^2}d(k,p)=k^2$. [Here, $A(p^2)\equiv 1/Z(p^2)$ is their $1/\mathcal{F}(p^2)$.] For purely massless QED, they found a suitable form $d(k,p)=(k^2-p^2)^2/(k^2+p^2)$. This was generalized to the case with a dynamical mass $M(p^2)$, to give

$$d(k,p)=\frac{(k^2-p^2)^2+[M^2(k^2)+M^2(p^2)]^2}{k^2+p^2}.\tag{30}$$

It is clear that this choice scales as described in our discussion above and is consistent with multiplicative renormalizability.

One refinement in our application of the C-P vertex [7] in the present work is associated with subtleties in the ultraviolet regularization scheme. Although there have been some exploratory studies of dimensional regularization for the DSE [16], this has not yet proved practical in nonperturbative field theory and momentum cutoffs for now remain the regularization scheme of choice in such studies. Naive imposition of a momentum cutoff destroys the gauge covariance of the DSE because the fermion self-energy integral contains terms, related to the vertex WTI, which should vanish but which are nonzero when integrated under cutoff regularization [17,18]. Based on these considerations a ''gauge-covariance-improved'' treatment of the C-P vertex was proposed, which consists of the replacement (in the quenched approximation) $\Sigma_d'(\mu,\Lambda;p^2)+Z_1(\mu,\Lambda)\alpha\xi/8\pi\to\Sigma_d'(\mu,\Lambda;p^2)$. Full details and the derivation of this modification can be found in Appendix A of Ref. [7]. This quenched approximation correction does not spoil the scaling needed for multiplicative renormalization since in the quenched approximation the correction term scales with $Z_2(\mu,\Lambda)$ as does Σ'. We have now fully specified the sample vertex that we use in our numerical calculations.

For the numerical calculations [1,6,7] the equations are separated into a Dirac-odd part describing the finite propagator renormalization $A(p^2)$ and a Dirac-even part for the scalar self-energy, by taking $\frac{1}{4}\mathrm{Tr}$ of the DSE multiplied by $\not{p}/p^2$ and 1, respectively. The equations are solved in Euclidean space and so the volume integrals $\int d^4k$ can be separated into angle integrals and an integral $\int dk^2$; the angle integrals are easy to perform analytically, yielding the two equations which will be solved numerically.

In order to obtain numerical solutions, the final Minkowski-space integral equations are first rotated to Euclidean space. They are then solved by iteration on a logarithmic grid from an initial guess. The solutions are confirmed to be independent of the initial guess and are solved with a wide range of cutoffs (Λ), renormalization points (μ), couplings (α), covariant gauge choices [$\xi(\mu)$], and renormalized masses [$m(\mu)$]. As has been reported in detail elsewhere [6,7], choosing the renormalized mass $m(\mu)$ and then solving for the bare mass $m_0(\Lambda)$ leads to extremely well behaved finite solutions for $A(p^2)$ and $M(p^2)$, which do not vary as we take the continuum limit ($\Lambda\to\infty$). In addition, the renormalization point transformation properties given in Eq. (28) have also been explicitly verified using our numerical solutions, which typically have an accuracy of better than 1 in 10^4.

An important outcome of these studies was the observation that all solutions which were well-behaved in the continuum limit contained decaying oscillations in the mass function $M(p^2)$ if one looked sufficiently far into the ultraviolet (in p^2). As a result, as $\Lambda\to\infty$ for any given solution the values for the bare mass also presented a decaying oscillatory behaviour [7]. Clearly, these oscillating solutions cannot be obtained from the chiral limit procedure [i.e., $m_0(\Lambda)=0$ and $\Lambda\to\infty$] and hence cannot be chirally symmetric solutions.

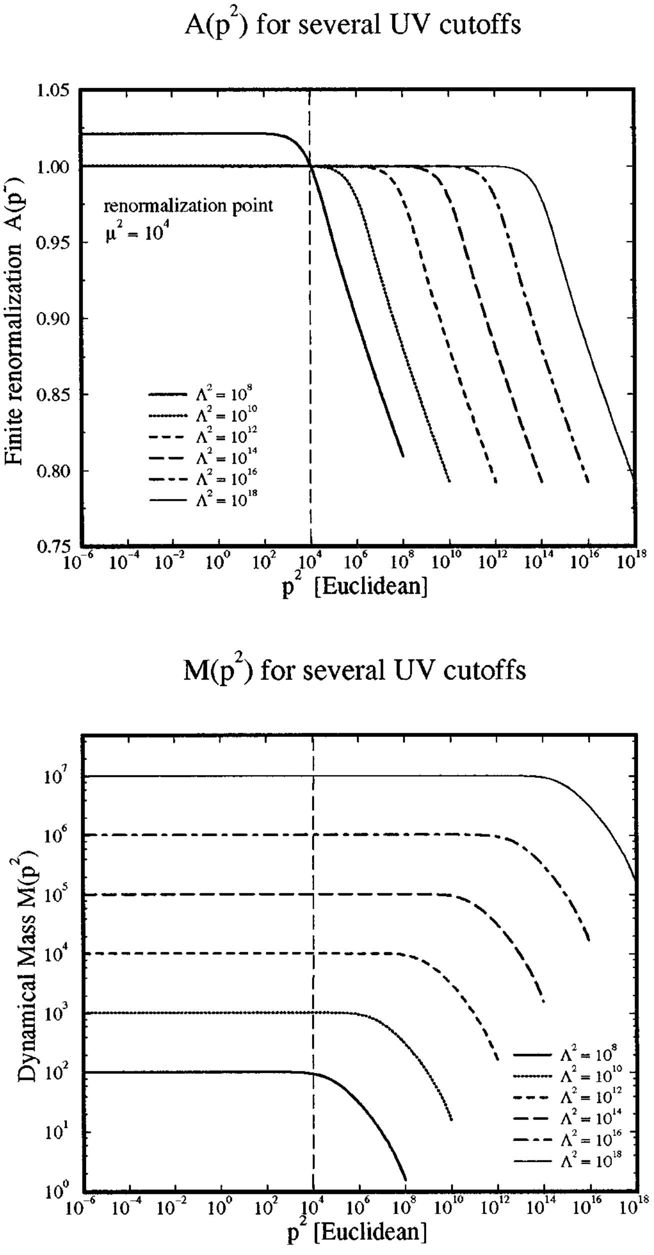

FIG. 1. The behavior of the finite renormalization $A(p^2)$ and the mass function $M(p^2)$ as a function of the ultraviolet cutoff Λ for $m_0(\Lambda)=0$. These solutions were for renormalization point $\mu^2=10^4$, coupling $\alpha=1.15$, and gauge parameter $\xi=0.25$. Clearly as $\Lambda\to\infty$ we find $A(p^2)\to 1$ for all p^2 and $M(p^2)$ diverges with Λ.

III. RESULTS

To motivate our general arguments concerning the chiral limit, let us consider Fig. 1, which shows the nonoscillating solutions $A(p^2)$ and $M(p^2)$ for supercritical coupling and with $m_0(\Lambda)=0$, for a wide range of values of Λ. For any given set of parameters α, $m_0(\Lambda)$, μ, ξ, and Λ it is always possible to find such a nonoscillating solution. It is clear from these numerical solutions that in the continuum limit (i.e., $\Lambda\to\infty$) and for supercritical coupling, we find $A(p^2)\to 1$ for all p^2 and a mass function $M(p^2)$ which diverges proportionally to Λ. This divergent behavior of the mass was verified to the numerical accuracy of our solutions (1 in 10^4). Conversely, for subcritical coupling (i.e., $\alpha<\alpha_c$), the chiral limit does exist and we find simply that $A(\mu;p^2)=(p^2/\mu^2)^{-\alpha\xi/4\pi}$. It seems clear from these numerical studies that above critical coupling there is no finite chiral limit in the continuum quenched theory in any covariant gauge, even in the presence of the renormalization procedure. In the absence of any renormalization program, this

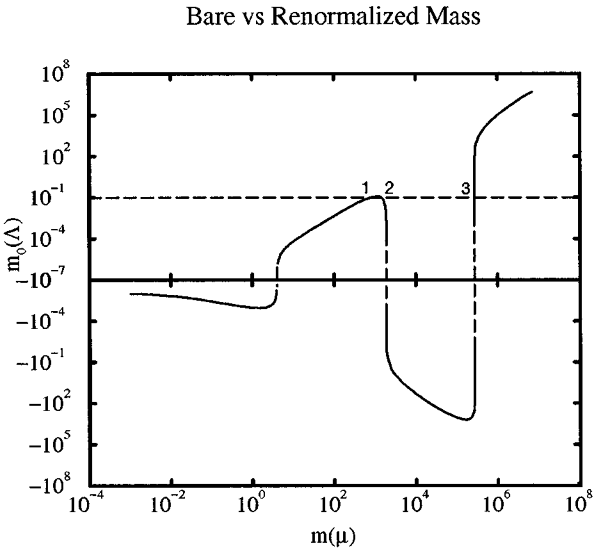

FIG. 2. The relationship between the bare mass $[m_0(\Lambda)]$ and the renormalized mass $[m(\mu)]$ for the renormalized finite solutions. These result from solving for a given $m(\mu)$ and extracting the corresponding $m_0(\Lambda)$. The other parameters for these solutions were renormalization point $\mu^2=10^4$, coupling $\alpha=1.25$, gauge parameter $\xi=0.25$, and $\Lambda^2=10^{14}$. The dashed horizontal line shows, e.g., that for $m_0(\Lambda)=0.1$ there are three solutions. (The dashed vertical lines connecting the upper and lower curves are merely to guide the eye on this back-to-back logarithmic scale.)

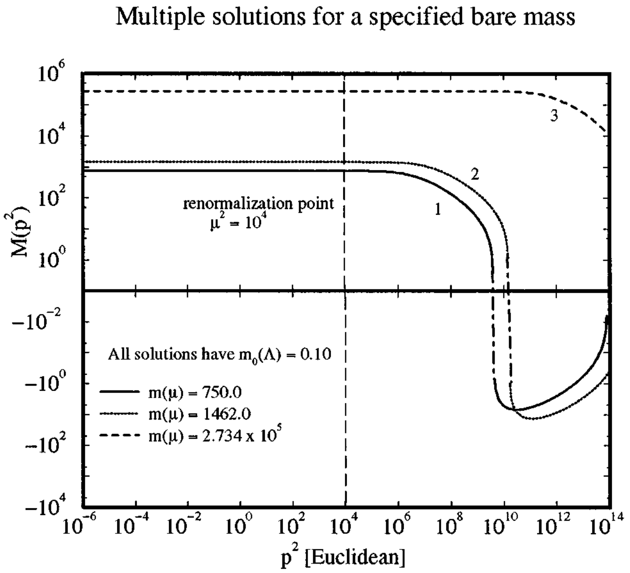

FIG. 3. The three renormalized finite solutions corresponding to the same bare mass, i.e., $m_0(\Lambda)=0.1$, as indicated in Fig. 2. The other parameters for these solutions were renormalization point $\mu^2=10^4$, coupling $\alpha=1.25$, gauge parameter $\xi=0.25$, and $\Lambda^2=10^{14}$. (The short-long dashed vertical lines connecting the upper and lower curves are merely to guide the eye on this back-to-back logarithmic scale.)

divergent behavior of the mass is well known (see, e.g., Refs. [10,11]) and is inevitable since in the absence of ECSB in the quenched theory there is only one external scale (i.e., Λ).

While the above conclusions were based on a numerical study with a specific choice of vertex, it is relatively straightforward to construct a general argument which applies irrespective of this choice: Consider any vertex Γ which satisfies the WTI and which leads to multiplicative renormalizability [7]. It automatically follows that $M(p^2)$ and $A(p^2)/Z_2(\mu,\Lambda)$ are renormalization point independent as discussed previously. We can then define dimensionless quantities by appropriately scaling with Λ, i.e., $\hat{\mu}\equiv\mu/\Lambda$, $\hat{p}^2\equiv p^2/\Lambda^2$, $\hat{M}(\hat{p}^2)\equiv M(p^2)/\Lambda$, $\hat{A}(\hat{p}^2)\equiv A(\mu;p^2)/Z_2(\mu,\Lambda)$, and $\hat{m}_0\equiv m_0(\Lambda)/\Lambda$. Note that the renormalization condition $A(\mu;\mu^2)=1$ automatically determines $Z_2(\mu,\Lambda)$ for a given solution for fixed Λ [see Eq. (23)]. The dimensionless functions $\hat{M}(\hat{p}^2)$ and $\hat{A}(\hat{p}^2)$ of the dimensionless variable $\hat{p}^2$ can only depend on dimensionless parameters, i.e., α, ξ, $\hat{\mu}$, and $\hat{m}_0$. Furthermore, since for any fixed Λ they are independent of μ (recall that we are working only in the quenched approximation), then it follows that they must in turn be independent of $\hat{\mu}$. Now for any finite Λ and the choice $m_0(\Lambda)=0$ we have $\hat{m}_0=0$. Hence solving for any μ and Λ with $m_0(\Lambda)=0$ allows us to form $\hat{M}(p^2)$ and $\hat{A}(p^2)$, from which we can read off the solutions for $A(p^2)$ and $M(p^2)$ for any other μ and Λ with vanishing bare mass using the above rules. We see then that the resulting $M(p^2)$ must diverge with Λ as was found numerically. To summarize, we see that above critical coupling any vertex which satisfies the WTI and leads to multiplicative renormalizability will lead to a diverging mass function in the continuum limit for quenched QED in four dimensions.

It is interesting to contrast this result with the well-known behavior [19] found in QCD, which leads to a well-defined and finite chiral limit in the continuum. The asymptotic behavior of QCD has been well established in the ultraviolet regime due to asymptotic freedom in the form of a decreasing running coupling constant $\alpha_s(\mu)$. This can be used in a renormalization-group-improved treatment of the ultraviolet region in the DSE study of DCSB in the quark propagator [1–3,20].

Given that we know that the chiral limit leads to a divergent mass function above critical coupling, what are we then to make of the well-behaved finite solutions? Our conventional thinking about the chiral limit would imply that since ECSB should only increase the mass function above that found in its absence, then any solution above critical coupling which also has ECSB in the conventional sense must also correspond to a divergent mass in the continuum limit. Thus the finite solutions do not correspond to the chiral limit or to any conventional concept of ECSB.

It has by now probably become clear that a given set of the parameters α, $m_0(\Lambda)$, μ, ξ, and Λ actually admits more than one finite solution. If the renormalization point μ is chosen to lie far enough in the infrared that no oscillations in the mass can occur for $p^2<\mu^2$, then specifying the renormalized mass $m(\mu)$ rather than the bare mass $m_0(\Lambda)$ leads to a unique solution. This corresponds to the procedure used in obtaining the numerical solutions in Refs. [6,7]. We explicitly show this behavior in Figs. 2 and 3, where for a given set of parameters α, $m_0(\Lambda)$, μ, ξ, and Λ we see that there are distinct solutions. As $\Lambda\rightarrow\infty$ the number of simultaneous solutions becomes infinite. This can be understood in the following way. The oscillations have a period in terms of $\ln(p^2)$ which is independent of Λ [7]. Thus the higher Λ is, the more oscillations can be squeezed in between the IR and UV cutoffs. As we increase $m(\mu)$ with everything else fixed, we push oscillations along the p^2 axis. Each time that a full oscillation is pushed past the UV cutoff (Λ^2) there will be two solutions which give the same value for $m_0(\Lambda)$.

It seems reasonable to expect that we can also induce these multiple solutions in QCD by lowering $m(\mu)$ below that which corresponds to the chiral limit. This would correspond to a ''negative'' ECSB and would then again lead to multiple solutions. A numerical test of this expectation is currently being pursued by implementing the renormalization program in a QCD-based study of the quark DSE [21]. While the DSE for the quark propagator is not yet well understood in the IR, multiple solutions should manifest themselves in the UV and will not depend on the detailed *Ansätze* used for the QCD propagators and vertex in the IR.

IV. SUMMARY AND CONCLUSIONS

We have studied renormalized quenched strong-coupling QED in four dimensions in arbitrary covariant gauge using a momentum cutoff regularization. We saw that there is no (nonzero) finite chiral limit of the renormalized theory on general grounds above the critical coupling. In addition, we showed that above critical coupling for the gauge-covariance-improved treatment of the Curtis-Pennington proper fermion-photon vertex, there are an infinite number of renormalized finite solutions corresponding to the same bare mass in the continuum limit. All of the (nonzero) finite solutions have oscillations and differ in how far out on the momentum scale the first oscillation occurs. The *only* solutions found above critical coupling were (i) the trivial $M(p^2)=0$ solution with vanishing bare mass, and (ii) the multiple, oscillating solutions.

Since above critical coupling a finite continuum chiral limit does not exist, these finite solutions in the supercritical region may in some sense correspond to explicit chiral-symmetry-breaking modes built on the unstable chiral-symmetric solution. It seems likely that this behavior for the finite solutions is independent of the detailed vertex choice and furthermore that the same behavior can be induced in QCD by forcing the renormalized mass into an unphysical regime [by choosing $m(\mu)\equiv M(\mu^2)$ below the value corresponding to the chiral limit]. It also seems likely that unquenching the theory will not remove this rather undesirable behavior in QED$_4$, since the running coupling increases with scale rather than decreasing as in QCD. These latter conjectures are the subject of current investigation [21]. This study supports the assertion that in four dimensions strong-coupling QED does not have a continuum limit in the conventional sense.

ACKNOWLEDGMENTS

We thank C. Roberts for a critical reading of the manuscript and for some helpful discussions. This work was partially supported by the Australian Research Council, by the U.S. Department of Energy through Contract No. DE-FG05-86ER40273, and by the Florida State University Supercomputer Computations Research Institute, which is partially funded by the Department of Energy through Contract No. DE-FC05-85ER250000. This research was also partly supported by grants of supercomputer time from the U.S. National Energy Research Supercomputer Center and the Australian National University Supercomputer Facility.

[1] C. D. Roberts and A. G. Williams, *Dyson-Schwinger Equations and their Application to Hadronic Physics*, in *Progress in Particle and Nuclear Physics* (Pergamon Press, Oxford, 1994), Vol. 33, p. 477.

[2] V. A. Miranskii, *Dynamical Symmetry Breaking in Quantum Field Theories* (World Scientific, Singapore, 1993).

[3] P. I. Fomin, V. P. Gusynin, V. A. Miransky, and Yu. A. Sitenko, Riv. Nuovo Cimento **6**, 1 (1983).

[4] H. J. Rothe, *Lattice Gauge Theories: An Introduction* (World Scientific, Singapore, 1992).

[5] A. H. Blin, B. Hiller, and M. Schaden, Z. Phys. A **331**, 75 (1988); E. Quack and S. P. Klevansky, Phys. Rev. C **49**, 3283 (1994); S. P. Klevansky, Rev. Mod. Phys. **64**, 649 (1992).

[6] F. T. Hawes and A. G. Williams, Phys. Rev. D **51**, 3081 (1995).

[7] F. T. Hawes, A. G. Williams, and C. D. Roberts, Phys. Rev. D **54**, 5361 (1996).

[8] D. C. Curtis and M. R. Pennington, Phys. Rev. D **42**, 4165 (1990).

[9] D. C. Curtis and M. R. Pennington, Phys. Rev. D **44**, 536 (1991).

[10] D. C. Curtis and M. R. Pennington, Phys. Rev. D **46**, 2663 (1992).

[11] D. C. Curtis and M. R. Pennington, Phys. Rev. D **48**, 4933 (1993).

[12] C. D. Roberts, in *QCD Vacuum Structure*, edited by H. M. Fried and B. Müller (World Scientific, Singapore, 1993).

[13] J. S. Ball and T. W. Chiu, Phys. Rev. D **22**, 2542 (1980); **22**, 2550 (1980).

[14] A. Kizilersü, M. Reenders, and M. R. Pennington, Phys. Rev. D **52**, 1242 (1995).

[15] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[16] L. von Smekal, P. A. Amundsen, and R. Alkofer, Nucl. Phys. **A529**, 633 (1991); M. Becker, Ph.D. thesis, W. W. U. Münster, 1995.

[17] Z. Dong, H. Munczek, and C. D. Roberts, Phys. Lett. B **333**, 536 (1994).

[18] A. Bashir and M. R. Pennington, Phys. Rev. D **50**, 7679 (1994); Phys. Rev. D **53**, 4694 (1996).

[19] H. D. Politzer, Nucl. Phys. **B117**, 397 (1976).

[20] A. G. Williams, G. Krein, and C. D. Roberts, Ann. Phys. (N.Y.) **210**, 464 (1991).

[21] F. T. Hawes, T. Sizer, and A. G. Williams (in progress).