PUBLISHED VERSION

Alexandrou, C.; Rosenfelder, R.; Schreiber, Andreas Wolfgang
Worldline path integral for the massive Dirac propagator: A four-dimensional approach
Physical Review A, 1999; 59(3):1762-1776

© 1999 American Physical Society

http://link.aps.org/doi/10.1103/PhysRevA.59.1762

PERMISSIONS

http://publish.aps.org/authors/transfer-of-copyright-agreement

"The author(s), and in the case of a Work Made For Hire, as defined in the U.S. Copyright Act, 17 U.S.C.

§101, the employer named [below], shall have the following rights (the "Author Rights"):

[...]

3. The right to use all or part of the Article, including the APS-prepared version without revision or modification, on the author(s)' web home page or employer's website and to make copies of all or part of the Article, including the APS-prepared version without revision or modification, for the author(s)' and/or the employer's use for educational or research purposes."

14th March 2013

http://hdl.handle.net/2440/10871

Worldline path integral for the massive Dirac propagator: A four-dimensional approach

C. Alexandrou,[1,2] R. Rosenfelder,[2] and A. W. Schreiber[2,3]
[1]Department of Natural Sciences, University of Cyprus, CY-1678 Nicosia, Cyprus
[2]Paul Scherrer Institute, CH-5232 Villigen PSI, Switzerland
[3]Department of Physics and Mathematical Physics and Research Centre for the Subatomic Structure of Matter, University of Adelaide, Adelaide, S. A. 5005, Australia

(Received 14 September 1998)

We simplify and generalize an approach proposed by Di Vecchia and Ravndal to describe a massive Dirac particle in external vector and scalar fields. Two different path integral representations for the propagator are derived systematically without the usual five-dimensional extension and shown to be equivalent due to the supersymmetry of the action. They correspond to a projection on the mass of the particle either continuously or at the end of the time evolution. It is shown that the supersymmetry transformations are generated by shifting and scaling the supertimes and the invariant difference of two supertimes is given for the general case. A nonrelativistic reduction of the relativistic propagator leads to a three-dimensional path integral with the usual Pauli Hamiltonian. By integrating out the photons we obtain the effective action for quenched QED and use it to derive the gauge-transformation properties of the general Green function of the theory.
[S1050-2947(99)04302-4]

PACS number(s): 03.65.Pm, 12.20.−m, 12.38.Lg, 03.70.+k

I. INTRODUCTION

The problem of how to describe spin in a path integral has a long and twisted history. This is mostly due to the fact that a path integral is determined by the classical Lagrangian (or Hamiltonian) and a classical analog for the internal spin of a particle is not readily available. Martin [1] apparently first suggested to use anticommuting Grassmann variables for this purpose. This can be made plausible when one recalls that the spin operator $\mathbf{s}\equiv\hbar\boldsymbol{\sigma}/2$ of an electron fulfills

$$\{s_i,s_j\}\equiv s_is_j+s_js_i=\frac{\hbar^2}{2}\delta_{ij}\to 0,\quad i,j=1,2,3 \tag{1}$$

in the classical limit. Consequently one can describe a spinning particle by its bosonic part, the usual trajectory $x(t)$, and a fermionic degree of freedom given by a Grassmann valued function $\zeta(t)$.[1] Brink *et al.* [3] noted an important supersymmetry between the bosonic and fermionic parts of a relativistic massless Dirac particle and Berezin and Marinov [4] showed that *massive* particles can be described by adding a fifth component $\zeta_5(t)$ to the spin variable. The reason for this peculiar addition is that in the rest frame of the particle the spin is intrinsically three-dimensional [see Eq. (1)] and a covariant four-dimensional description therefore has superfluous degrees of freedom which have to be cancelled by the fifth spin variable [5].

There is now a vast amount of literature about spin in path integrals (a partial list of references is [6–10]) which discusses various aspects of this approach. In particular, Fradkin and Gitman [11] have given a straightforward way of constructing the corresponding relativistic propagator. In addition to the dependence on bosonic and fermionic trajectories mentioned above, their formulation has the special feature that as well as the usual Schwinger proper time a Grassmannian partner to it is required. Representing Dirac particles in a first quantized form in the ''world line formalism'' has become popular for perturbative calculations in QED and QCD [12–14]. These one-loop calculations of the effective action are simplified by the fact that only Green functions on a circle (with simpler boundary conditions) are needed. More recently, the method has also been used in order to derive derivative expansions of the one-loop effective action in (2+1)- and (3+1)-dimensional QED [15].

Although sufficient for many purposes the Berezin-Marinov introduction of the fifth spin variable is an awkward one: there is no clear physical picture associated with it and the corresponding multiplication of the propagator with the Dirac matrix γ_5 [11] is very unnatural in a parity conserving theory. A four-dimensional approach, which has not received very much attention up to now, is that proposed by Di Vecchia and Ravndal [16,17] in which the unwanted spin degrees of freedom are simply projected out.[2]

It is the purpose of the present paper to develop this latter approach further and to show that it has attractive features. In particular, in Sec. II, we will calculate the propagator for a Dirac particle in an external vector field and demonstrate that the projection mentioned above can be done in two different ways: either at each time step during the evolution of the system or at the end. We will refer to the former as the ''local''projection method and to the latter as the ''global'' projection method. In Sec. III both procedures are shown to be equivalent due to the supersymmetry between bosonic

[1]It should be noted that there are other approaches, e.g., using coherent state path integrals [2], which we will not consider here.

[2]During the course of this work there appeared a publication [18] in which the fifth spin variable is also eliminated but by a different nonlinear technique.

 ©1999 The American Physical Society

and fermionic variables. However, the global projection leads, in general, to simpler expressions without a Grassmann proper time. In addition, an inherent coupling between orbital and spin parts which is already present for a free particle is removed by the global projection method. Section IV contains the nonrelativistic reduction where we start directly from the path integral representation of the Dirac propagator and show that this reduces to the three-dimensional spin path integral of the nonrelativistic theory. This is to be contrasted with Ref. [9] where the nonrelativistic propagator was derived starting with the nonrelativistic Hamiltonian and introducing three-dimensional Grassmann variables instead of obtaining it from the relativistic path integral for the Dirac propagator. In Sec. V we show that one can also describe a Yukawa interaction of the fermion (i.e., the particle in an external scalar field) in such a four-dimensional framework. As an application we derive the effective action in quantum electrodynamics in Sec. VI and finally we summarize our results.

Since we aim in making this paper self-contained we include an Appendix with a derivation of the spin path integral which is somewhat different, more explicit and simpler than the one given by Fradkin and Gitman. Our conventions follow Bjorken and Drell [19] and in general we use $\alpha - \iota$ and $\xi - \omega$ to denote Grassmann variables, with some exceptions to comply with the standard notation found in the literature.

II. DIRAC PROPAGATOR IN AN EXTERNAL VECTOR POTENTIAL

We are looking for the path integral representation for the propagator of a Dirac particle

$$G(x,y) = \left\langle x \left| \frac{1}{\hat{\not p} - g\not A(\hat{x}) - M + i0} \right| y \right\rangle \tag{2}$$

in an external field $A_\mu(x)$ where throughout this paper quantum-mechanical operators are denoted by hats over the corresponding symbols. In the spinless (bosonic) case this can be achieved by using Schwinger's proper time representation for the quantum-mechanical resolvent

$$\frac{1}{E - \hat{H} + i0} = -i \int_0^\infty dT \exp[i(E - \hat{H} + i0)T]. \tag{3}$$

However, in the fermionic case we have to make sure that the operator $\hat{H}$ which plays the role of a Hamiltonian for the quantum-mechanical system contains an *even* number of Dirac matrices. This is because in the classical limit (and also in the path integral) only an even, commuting object can represent a physical quantity.

Fradkin and Gitman [11] achieved this by multiplying numerator and denominator in Eq. (2) by γ_5 and extending the Dirac algebra to five dimensions. However, it is much simpler to use the representation of Di Vecchia and Ravndal [16,17] where we write

$$\frac{1}{\hat{\not\Pi} - M + i0} = (\hat{\not\Pi} + M) \frac{1}{\hat{\not\Pi}^2 - M^2 + i0}. \tag{4}$$

It is now possible either to exponentiate only the denominator which gives

$$\frac{1}{\hat{\not\Pi} - M + i0} = -\frac{i}{2\kappa_0} \int_0^\infty dT (\hat{\not\Pi} + M) \times \exp\left(-\frac{iM^2 T}{2\kappa_0}\right) \exp\left(\frac{i}{2\kappa_0} \hat{\not\Pi}^2 T\right) \tag{5}$$

or *both* numerator and denominator leading to

$$\frac{1}{\hat{\not\Pi} - M + i0} = \int_0^\infty dT \int d\chi \exp\left[-\frac{i}{2\kappa_0}(M^2 T + M\chi)\right] \times \exp\left[\frac{i}{2\kappa_0}(\hat{\not\Pi}^2 T + \hat{\not\Pi}\chi)\right]. \tag{6}$$

The latter only holds if $\hat{\Pi}$ commutes with $\hat{\Pi}^2$ which is proved in Ref. [17]. Here

$$\hat{\Pi}^\mu = \hat{p}^\mu - gA^\mu(\hat{x}), \tag{7}$$

and the Berezin integrals over Grassmann variables are defined as [20]

$$\int d\chi = 0, \qquad \int d\chi \chi = 1. \tag{8}$$

κ_0 is a parameter which reparametrizes the proper times $T \to \kappa_0 T, \chi \to \kappa_0 \chi$ without changing the physics and is a remnant of the local reparametrization invariance of the action. It is thus related to the "einbein" [3]. χ is either called a (one-dimensional) "gravitino" field or, more appropriately in the present context, as the supersymmetric partner of the proper time, the "supertime."

The Di Vecchia–Ravndal representation has several advantages compared to the standard Berezin-Marinov form [4] for the description of a *massive* spinning particle: no five-dimensional extension and multiplication with γ_5 are necessary and, as we will see in Sec. III, the supersymmetric transformations are much simpler and more transparent. It can be considered as the result obtained by integrating out the fifth spin variable. A certain disadvantage is that not all exponents in Eq. (6) are Grassmann even. The odd term

$$\exp\left(-\frac{iM}{2\kappa_0}\chi\right) \tag{9}$$

is to be considered as part of an operator which projects out $\hat{\not\Pi} = M$ [17] and not as part of the evolution operator. In Eq. (5) this projection is done at the end ("global") whereas in Eq. (6) it is done at each time step during the evolution ("local").

It is essential that in both procedures the "Hamiltonian" which governs the proper-time evolution is even. In the global projection method it is given by

$$\mathcal{H}(\hat{\Pi}, \hat{x}, \gamma) = -\frac{1}{2\kappa_0} \hat{\not\Pi}^2 = -\frac{\hat{\Pi}^2}{2\kappa_0} + \frac{i}{4\kappa_0} g F_{\mu\nu}(\hat{x}) \gamma^\mu \gamma^\nu$$

$$(F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu), \tag{10}$$

whereas for the local projection method it reads

$$\mathcal{H}'(\hat{\Pi},\hat{x},\gamma)=\mathcal{H}(\hat{\Pi},\hat{x},\gamma)-\frac{1}{2\kappa_0 T}\hat{\Pi}_\mu\gamma^\mu\chi. \tag{11}$$

In both cases the parameter κ_0 can be interpreted as the ''mass'' of the quantum-mechanical particle.

A. Global projection

We will first consider the projection after the end of the evolution, i.e.,

$$\begin{aligned}G(x,y)&=-\frac{i}{2\kappa_0}\int d^4z\langle x|\hat{\not p}-gA(\hat{x})+M|z\rangle\int_0^\infty dT\\&\quad\times\exp\left(-\frac{iM^2T}{2\kappa_0}\right)\langle z|\exp(-i\hat{\mathcal{H}}T)|y\rangle\\&=-\frac{i}{2\kappa_0}(i\not\partial_x-g\not A(x)+M)\int_0^\infty dT\exp\left(-\frac{iM^2T}{2\kappa_0}\right)\\&\quad\times\langle x|\exp(-i\hat{\mathcal{H}}T)|y\rangle.\end{aligned} \tag{12}$$

The remaining proper-time evolution operator can be written in path integral form following Fradkin and Gitman [11] but staying within a four-dimensional framework. In the Appendix we show that

$$\begin{aligned}&\langle x|\exp(-i\hat{\mathcal{H}}T)|y\rangle\\&\quad=\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_{x(0)=y}^{x(T)=x}\mathcal{D}x\mathcal{D}p\mathcal{D}\xi> N^{\rm spin}\\&\qquad\times\exp\left\{i\int_0^T dt[i\xi\cdot\dot{\xi}-p\cdot\dot{x}-\mathcal{H}(\Pi,x,2\xi+\Gamma)]\right\}_{\Gamma=0},\end{aligned} \tag{13}$$

where $N^{\rm spin}$ is a normalization factor for the four-dimensional spin path integral as given in Eq. (A19) and we use antiperiodic boundary conditions for the spin variable $\xi(t)$

$$\xi_\mu(0)+\xi_\mu(T)=0. \tag{14}$$

Equation (13) can be further simplified by shifting to the new spin variables

$$\zeta_\mu(t)=\frac{1}{2}\Gamma_\mu+\xi_\mu(t) \tag{15}$$

so that the boundary condition becomes

$$\zeta_\mu(0)+\zeta_\mu(T)=\Gamma_\mu. \tag{16}$$

This introduces an additional boundary term $-\frac{1}{2}\Gamma\cdot[\zeta(T)-\zeta(0)]=\zeta(T)\cdot\zeta(0)$. After shifting to the momentum (7) as integration variable we obtain

$$\begin{aligned}G(x,y)&=-\frac{i}{2\kappa_0}(i\not\partial_x-g\not A(x)+M)\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty dT\\&\quad\times\exp\left(-\frac{i}{2\kappa_0}M^2T\right)N^{\rm spin}\int\mathcal{D}x\mathcal{D}\Pi\mathcal{D}\zeta\\&\quad\times\exp\left\{\zeta(0)\cdot\zeta(T)+i\int_0^T dt[i\zeta\cdot\dot{\zeta}-(\Pi+gA(x))\cdot\dot{x}\right.\\&\quad\left.-\mathcal{H}(\Pi,x,2\zeta)]\right\}_{\Gamma=0}.\end{aligned} \tag{17}$$

As usual we can perform the functional Π-integration since the Hamiltonian is at most quadratic in the kinematical momentum. We then obtain the final expression

$$\begin{aligned}G(x,y)&=-\frac{i}{2\kappa_0}(i\not\partial_x-g\not A(x)+M)\\&\quad\times\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty dT N(T)\\&\quad\times\exp\left(-\frac{i}{2\kappa_0}M^2T\right)\cdot\int\mathcal{D}x\mathcal{D}\zeta\exp\{iS[x,\zeta]\}_{\Gamma=0},\end{aligned} \tag{18}$$

where

$$\begin{aligned}N(T)&=\left[\int\mathcal{D}\zeta\exp\left(\zeta(0)\cdot\zeta(T)\right.\right.\\&\quad\left.\left.-\int_0^T dt\zeta\cdot\dot{\zeta}\right)\right]^{-1}\cdot\int\mathcal{D}\Pi\exp\left(i\int_0^T dt\frac{\Pi^2}{2\kappa_0}\right)\end{aligned} \tag{19}$$

provides the proper normalization and

$$S[x,\zeta]\equiv\int_0^T dt L(x,\dot{x},\zeta,\dot{\zeta})-i\zeta(0)\cdot\zeta(T),$$

$$L(x,\dot{x},\zeta,\dot{\zeta})=-\frac{\kappa_0}{2}\dot{x}^2+i\zeta\cdot\dot{\zeta}-g\dot{x}\cdot A(x)-\frac{ig}{\kappa_0}F_{\mu\nu}(x)\zeta^\mu\zeta^\nu \tag{20}$$

are the action and the Lagrangian, respectively. The first two terms in Eq. (20) correspond, respectively, to contributions from the orbital and spin degrees of freedom to the kinetic energy, while the last two terms are the contributions of the photon field coupling to both the electron's convection current and its spin current. The canonically conjugate momenta are given by

$$p_\mu=\frac{\partial L}{\partial\dot{x}^\mu}=-\kappa_0\dot{x}_\mu-gA_\mu(x) \tag{21}$$

and

$$\eta_\mu=\frac{\partial L}{\partial\dot{\zeta}^\mu}=-i\zeta_\mu \tag{22}$$

so that the canonical Hamiltonian becomes

$$H=\sum_{q_i=x,\zeta}\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}-L=-\frac{\kappa_0}{2}\dot{x}^2+\frac{ig}{\kappa_0}F_{\mu\nu}\zeta^\mu\zeta^\nu \tag{23}$$

when expressed in terms of (generalized) coordinates and velocities. In terms of coordinates and momenta we have the relation $H=\mathcal{H}|_{p\to -p}$. This is a consequence of our metric which gives $\exp(-ip\cdot x)$ as plane wave and therefore leads to the form $\int dt[-p\cdot\dot{x}-\mathcal{H}]$ for the action in the phase space path integral (13).

The free Dirac propagator in momentum space is readily obtained from Eq. (18) since orbital and spin variables decouple. The ζ-path integral cancels against the normalization factor N^{spin} and the x-path integral gives just the usual free bosonic evolution kernel. Thus

$$\begin{aligned}G^{(0)}(p)&=\int d^4x e^{ip\cdot x}\left(-\frac{1}{2\kappa_0}\right)(i\partial\!\!\!/_x+M)\int_0^\infty dT\\&\quad\times\exp\left(-\frac{i}{2\kappa_0}M^2T\right)\\&\quad\times\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\exp\left(\frac{i}{2\kappa_0}k^2T\right)\\&=\frac{p\!\!\!/+M}{p^2-M^2+i0}=\frac{1}{p\!\!\!/-M+i0},\end{aligned} \tag{24}$$

independent of the reparametrization parameter κ_0.

B. Local projection

The local projection method follows along the same lines with two differences: first we have an additional integration over the supertime χ and second there is an extra term in the action due to the additional term in Eq. (11). Thus

$$\begin{aligned}G'(x,y)&=\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty dT N(T)\\&\quad\times\int d\chi\exp\left[-\frac{i}{2\kappa_0}(M^2T+M\chi)\right]\\&\quad\times\int\mathcal{D}x\mathcal{D}\zeta\exp\{iS'[x,\zeta]\}_{\Gamma=0}\end{aligned} \tag{25}$$

with

$$S'[x,\zeta]\equiv\int_0^T dt L'(x,\dot{x},\zeta,\dot{\zeta};\chi)-i\zeta(0)\cdot\zeta(T)$$

$$\begin{aligned}L'(x,\dot{x},\zeta,\dot{\zeta};\chi)&=L(x,\dot{x},\zeta,\dot{\zeta})+\frac{1}{T}\dot{x}^\mu\zeta_\mu\chi\\&=-\frac{\kappa_0}{2}\dot{x}^2+i\zeta\cdot\dot{\zeta}+\frac{1}{T}\dot{x}\cdot\zeta\chi-g\dot{x}\cdot A(x)\\&\quad-\frac{ig}{\kappa_0}F_{\mu\nu}(x)\zeta^\mu\zeta^\nu.\end{aligned} \tag{26}$$

Note that there is now a coupling between orbital movement and spin, even for the free particle. This is the same mechanism which at high energy aligns the spin of a Dirac particle along (or opposite) to the momentum whereas a nonrelativistic particle with spin is unaffected.

Since the spin degrees of freedom appear at most quadratically it is also possible to integrate them out completely and reduce the path integral to a bosonic one modified by a ‘‘spin factor’’ [21,22]. The price to be paid is that this spin factor is highly nonlinear in the external fields. This prevents an analytic integration over the boson fields to obtain an effective interaction for the fermion only, as is done in Sec. VI.

III. BOSONIC AND FERMIONIC TRANSFORMATIONS

We next discuss the transformation properties of the Lagrange function in the local formulation [3,23]. The corresponding ones for the global formulation can be obtained by setting $\chi=0$. There are two kinds of transformations which leave the Lagrange function L' in Eq. (26) invariant (up to a total derivative).

(i) Bosonic transformations (reparametrizations)

$$\begin{aligned}\delta x^\mu&=b(t)\dot{x}^\mu,\\\delta\zeta^\mu&=b(t)\dot{\zeta}^\mu,\end{aligned} \tag{27}$$

$$\delta\kappa_0=-\kappa_0^2\frac{d}{dt}\left(\frac{b(t)}{\kappa_0}\right)\Rightarrow\delta L'=\frac{d}{dt}[b(t)L'], \tag{28}$$

where $b(t)$ is the infinitesimal parameter of the transformation which, in principle, could have an arbitrary time-dependence (local transformations). However, for quantization the reparametrization ‘‘gauge’’ has to be fixed [4,7] which in our case, by construction, was taken to be $\kappa_0=\mathrm{const}$. This means that we only can allow $\delta\kappa_0=\mathrm{const}$ or

$$b(t)=b_0+b_1t. \tag{29}$$

Note that $b_1=0$ corresponds to proper time *translations*, e.g., $\delta x^\mu=x^\mu(t+b_0)-x^\mu(t)=b_0\dot{x}^\mu+\cdots$, and $b_0=0$ to proper time *scalings*, e.g., $\delta x^\mu=x^\mu(t+b_1t)-x^\mu(t)=b_1t\dot{x}^\mu+\cdots$.

(ii) Fermionic (supersymmetric) transformations

$$\begin{aligned}\delta x^\mu&=i\alpha(t)\zeta^\mu,\\\delta\zeta^\mu&=\frac{\kappa_0}{2}\alpha(t)\left[\dot{x}^\mu-\frac{1}{\kappa_0T}\zeta^\mu\chi\right]\\\delta\kappa_0&=\frac{\kappa_0}{T}\alpha(t)\chi,\\\delta\chi&=-i\kappa_0T\dot{\alpha}(t)\end{aligned} \tag{30}$$

$$\Rightarrow\delta L'=i\frac{d}{dt}\left[\alpha(t)\left(-\frac{\kappa_0}{2}\dot{x}\cdot\zeta+gA\cdot\zeta\right)\right], \tag{31}$$

where $\alpha(t)$ is the infinitesimal Grassmannian parameter of the transformation. Again, since κ_0 and χ are by construction time independent, one can only allow transformations with $\alpha(t)=\text{const}$. For $\chi=0$ Ravndal [17] has shown that similar to the bosonic case it is also possible to generate the fermionic transformations by a shift in the proper time if a Grassmannian partner of the proper time t is added. This allows for a concise supersymmetric formulation of the action. In this section we will show the generalization of Ravndal's transformations to the case with spin-orbit coupling, which includes a special scaling of the ''supertime'' in addition to a shift.

Since the change of the full Lagrange function is a total derivative, Noether's theorem [24] allows us to define quantities which are conserved classically. For the bosonic transformation with $b_1=0$ (i.e., proper time translations) we have $\delta\kappa_0=0$ and Eq. (28) therefore leads to the conservation of the canonical Hamiltonian (23). It can be shown that proper time scalings $(b_0=0)$ lead to the same result.[3] For the fermionic transformations we find from Eq. (31) that the projection of the spin variable on the kinematical momentum $-\kappa_0\dot{x}$

$$Q=-\kappa_0\dot{x}_\mu\zeta^\mu \tag{32}$$

is conserved classically without the spin-orbit term [17].

Quantum mechanically the Noether charges either become conserved operators or, in the functional formalism, their conservation implies that certain averages, i.e., Green functions with the Noether charges as insertions, stay time independent. For quantum-mechanical averages we will use the following notation:

$$\langle\mathcal{O}\rangle_S\equiv\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int\mathcal{D}x\mathcal{D}\zeta\mathcal{O}(x,\zeta)e^{iS[x,\zeta]}\big|_{\Gamma=0}. \tag{33}$$

To be specific, we consider the fermionic transformations with $\chi=0$ because their Noether charge (32) does not dependent (explicitly) on the interaction and we make a *local*, time-dependent transformation [25]

$$x(t)=x'(t)+i\alpha(t)\zeta(t),\qquad \zeta(t)=\zeta'(t)+\frac{\kappa_0}{2}\alpha(t)\dot{x}(t) \tag{34}$$

in the path integral. We assume that $\alpha(0)=\alpha(T)=0$ so that we do not have to consider boundary contributions. The Jacobian for this transformation is $1+\mathcal{O}(\alpha^2)$. Since the path integral does not change its value we obtain (omitting the primes)

$$0=\left\langle i\int_0^T dt\,\delta L\right\rangle_S, \tag{35}$$

where

$$\delta L=i\alpha(t)\frac{d}{dt}\left(-\frac{\kappa_0}{2}\dot{x}\cdot\zeta-gA\cdot\zeta\right)+i\dot{\alpha}(t)\left(-\frac{3\kappa_0}{2}\dot{x}\cdot\zeta-gA\cdot\zeta\right). \tag{36}$$

The first term is what we obtain for a global, time-independent transformation in Eq. (31). Performing an integration by parts in the second term (no boundary terms) the result is then

$$0=\left\langle\int_0^T dt\,\alpha(t)\left[-\kappa_0\frac{d}{dt}(\dot{x}\cdot\zeta)\right]\right\rangle_S \tag{37}$$

or since $\alpha(t)$ is arbitrary

$$\frac{d}{dt}\langle-\kappa_0\dot{x}(t)\cdot\zeta(t)\rangle_S=0 \tag{38}$$

for all times.

A. Supersymmetric formulation

It is convenient to write the Lagrange function for a relativistic spinning particle in explicit supersymmetric form by combining orbital and spin degrees of freedom into a ''superfield'' [3] or ''superposition'' [17]

$$X^\mu(t,\theta)=x^\mu(t)+a\theta\zeta^\mu(t). \tag{39}$$

Here θ is an additional time-independent Grassmann variable which acts as a superpartner of the proper time t and a a suitably chosen constant. If, in addition, a ''superderivative'' is defined as

$$D=\frac{\partial}{\partial\theta}-\theta\frac{\partial}{\partial t} \tag{40}$$

then

$$L_0=\int d\theta\left(-\frac{\kappa_0}{2}\right)DX_\mu D^2X^\mu=-\frac{\kappa_0}{2}\dot{x}^2+i\zeta\cdot\dot{\zeta} \tag{41}$$

generates all terms in the free Lagrangian of the spinning particle provided the constant a is chosen as

$$a=i\sqrt{\frac{2i}{\kappa_0}}. \tag{42}$$

Note that in this compact form only first-order derivatives appear since $D^2=-\partial/\partial t$. In the local projection approach one needs an additional χ-dependent factor [3,13]

$$e(\theta\chi)=1+\frac{a}{iT}\theta\chi \tag{43}$$

in the integrand of Eq. (41) to account for the explicit spin-orbit coupling. Thus the corresponding free action is

[3]In this case the reparametrization parameter κ_0, which is also changed, is not a dynamical variable for which the equations of motion can be used. Consequently the change of the corresponding Noether charge $Q=tH$ with time is proportional to $\kappa_0\partial L/\partial\kappa_0=H$, which gives no new information.

$$S_0'[X]=\int_0^T dt\int d\theta e(\theta\chi)\left(-\frac{\kappa_0}{2}\right)DX\cdot D^2X$$
$$=\int_0^T dt\left[-\frac{\kappa_0}{2}\dot{x}^2+i\zeta\cdot\dot{\zeta}+\frac{1}{T}\zeta\cdot\dot{x}\chi\right]. \qquad (44)$$

The interaction of the Dirac particle with an electromagnetic field takes the equally simple form

$$L_{\mathrm{e.m.}}=g\int d\theta DX_\mu A^\mu(X) \qquad (45)$$

which is easily proved by expanding the ''superposition'' X and performing the Berezin integration. Equation (45) thus contains *both* the convection current and the spin current interaction.

For $\chi=0$ Ravndal and Di Vecchia [16,17] have given a simple way of generating both the bosonic (with $b_1=0$) as well as the fermionic transformations by a shift in the proper times t and θ:

$$t\to t+b_0+\epsilon\theta,$$
$$\theta\to\theta+\epsilon, \qquad (46)$$

where ϵ and b_0 are constants which may be zero. Indeed, the superfield changes into

$$X(t,\theta)\to X'(t,\theta)=x(t+b_0+\epsilon\theta)+a(\theta+\epsilon)\zeta(t+b_0+\epsilon\theta)$$
$$=x+b_0\dot{x}+a\epsilon\zeta+a\theta\left[\zeta+b_0\dot{\zeta}-\frac{\epsilon}{a}\dot{x}\right]+\cdots, \qquad (47)$$

and if we set

$$\epsilon=\frac{i}{a}\alpha \qquad (48)$$

we obtain both transformations (27) and (30) for the individual components of the superfield in the special case $\chi=0$. This is not only more transparent but also treats bosonic and fermionic transformations on an equal footing. The equations of motion and the conserved quantities can also be formulated compactly in this formalism.

We can generalize the transformations (46) to the case $\chi\neq0$ by observing that any change in t,θ leaves κ_0,χ unchanged, since these quantities are by construction time-independent. This means that necessarily

$$\delta\kappa_0=0, \qquad (49)$$
$$\delta\chi=0. \qquad (50)$$

While the latter condition is fulfilled by a constant parameter α in the fermionic transformation [see Eq. (30)] the former one requires that the bosonic scaling parameter b_1 is not arbitrary but given by

$$b_1=\frac{1}{T}\alpha\chi. \qquad (51)$$

Using Eq. (48) we then find that

$$t'=\left(1+\frac{a}{iT}\epsilon\chi\right)t+b_0+\epsilon\theta,$$
$$\theta'=\left(1+\frac{a}{2iT}\epsilon\chi\right)\theta+\epsilon, \qquad (52)$$

generate the χ-dependent supersymmetric transformations with $\delta\kappa_0=\delta\chi=0$. Although this constitutes a scaling of the bosonic time t by a factor

$$\ell=\left(1+\frac{a}{iT}\epsilon\chi\right) \qquad (53)$$

the fermionic time θ is only scaled by $\sqrt{\ell}$. Consequently D scales by $1/\sqrt{\ell}$. Since the spin-orbit factor (43) scales again with ℓ and the Berezin integral over θ transforms inversely compared to a bosonic one the free action is easily found to be invariant under scaling.

We also note that for two times $t_1,t_2,\theta_1,\theta_2$ the combination

$$T_{12}\equiv\frac{t_1-t_2}{\sqrt{e(\theta_1\chi)e(\theta_2\chi)}}+\theta_1\theta_2 \qquad (54)$$

is invariant under the shift and scaling (52) of proper times. This is the generalization of a result which is well known for $\chi=0$ [21] and is important for extensions of the polaron variational approach to QED [26].

B. Equivalence of local and global projection

We are now able to prove the equivalence between the local projection method and the global one. We give here a somewhat different and more explicit derivation than the one sketched in Ref. [27]. We start from the local formulation and perform the χ integration. This gives

$$G'(x,y)=\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty dTN(T)\exp\left(-\frac{i}{2\kappa_0}M^2T\right)$$
$$\times\int_{x(0)=y}^{x(T)=x}\mathcal{D}x\int_{\zeta(0)+\zeta(T)=\Gamma}\mathcal{D}\zeta e^{iS}$$
$$\times\left[-\frac{i}{2\kappa_0}M-\frac{i}{T}\int_0^T dt\dot{x}\cdot\zeta\right]_{\Gamma=0}. \qquad (55)$$

As we have seen in Eq. (38) the supersymmetry of the action S leads to the result that the expectation value of $\dot{x}\cdot\zeta$ is time independent and thus can be evaluated at any time t, in particular at $t=T$. We then can perform the t integral and obtain

$$G'(x,y)=\int_0^\infty dTN(T)\exp\left(-\frac{i}{2\kappa_0}M^2T\right)$$
$$\times\left\langle-\frac{i}{2\kappa_0}M-i\dot{x}(T)\cdot\zeta(T)\right\rangle_S, \qquad (56)$$

where the average with respect to the action S is defined in Eq. (33). For the calculation of the last average we use the well-known fact (see, e.g., Ref. [28]) that the expectation value of time-ordered products of Heisenberg operators $\hat{\mathcal{O}}_H(t)=\exp(i\hat{\mathcal{H}}t)\hat{\mathcal{O}}\exp(-i\hat{\mathcal{H}}t)$ is given by the insertion of $\mathcal{O}(t)$ in the corresponding path integral. Thus

$$\left\langle x,T\left|\mathcal{T}\left[\hat{x}_H(t_1)\cdot\frac{\hat{\gamma}_H(t_2)}{2}\right]\right|y,0\right\rangle \equiv\left\langle x\left|e^{-i\hat{\mathcal{H}}T}\mathcal{T}\left[\hat{x}_H(t_1)\cdot\frac{\hat{\gamma}_H(t_2)}{2}\right]\right|y\right\rangle=\langle x(t_1)\cdot\zeta(t_2)\rangle_S \tag{57}$$

since Eq. (17) tells us that the (Weyl ordered) γ matrices are to be replaced by 2ζ. Differentiating with respect to t_1 (the equal time contribution vanishes) and putting $t_1=t_2=T$ we obtain

$$\langle\dot{x}(T)\cdot\zeta(T)\rangle_S=\langle x|i[\hat{\mathcal{H}},\hat{x}_\mu]\frac{\gamma^\mu}{2}e^{-i\hat{\mathcal{H}}T}|y\rangle. \tag{58}$$

Evaluating the commutator with the help of Eq. (10) and the canonical commutation relations we find

$$\langle\dot{x}(T)\cdot\zeta(T)\rangle_S=-\frac{1}{2\kappa_0}\langle x|\hat{\not\Pi}e^{-i\hat{\mathcal{H}}T}|y\rangle =\frac{i}{2\kappa_0}[\not\partial_x+ig\not A(x)]\langle x|e^{-i\hat{\mathcal{H}}T}|y\rangle \tag{59}$$

which, inserted into Eq. (56), gives exactly the same result for the propagator as the global projection method, i.e.,

$$G'(x,y)=G(x,y). \tag{60}$$

IV. NONRELATIVISTIC LIMIT

If the mass M of the fermion becomes large the integral over the proper time T is dominated by the stationary points of its integrand which approximately occur at $\Pi_0^2=M^2$. Therefore we make the ansatz

$$\Pi_0=sM+E,\quad s=\pm1. \tag{61}$$

For the nonrelativistic limit it is very convenient and natural to take

$$\kappa_0=M \tag{62}$$

and to assume

$$E=\mathcal{O}\left(\frac{1}{M}\right). \tag{63}$$

In this section we will write $\mathcal{D}^dx,\mathcal{D}^dp$, and $N_d^{\rm spin}$ with $d=3,4$ to stress the different dimensionality of relativistic and nonrelativistic path integrals. In the global projection method we then obtain from Eq. (17)

$$G(x,y)\simeq-\frac{i}{2M}(i\not\partial_x-g\not A+M)\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\times\sum_{s=\pm1}\int_0^\infty dT\exp\left(-\frac{i}{2}MT\right)N_4^{\rm spin}\times\int\mathcal{D}^4x\mathcal{D}^3\Pi\mathcal{D}^4\zeta\int\mathcal{D}E\exp\{iS_s[x,\mathbf{\Pi},E,\zeta]\}_{\Gamma=0} \tag{64}$$

with

$$S_s[x,\mathbf{\Pi},E,\zeta]=-i\zeta(0)\cdot\zeta(T)+\int_0^T dt\left[\frac{M}{2}-Ms\dot{x}_0+\mathbf{\Pi}\cdot\dot{\mathbf{x}}-gA_0\dot{x}_0+g\mathbf{A}\cdot\dot{\mathbf{x}}-\frac{\mathbf{\Pi}^2}{2M}+i\zeta\cdot\dot{\zeta}-\frac{ig}{M}F_{\mu\nu}\zeta^\mu\zeta^\nu+sE-E\dot{x}_0+\frac{E^2}{2M}\right]. \tag{65}$$

According to our assumption the last term in the square bracket is $\mathcal{O}(1/M^3)$ which we neglect. The path integral over E then gives a functional δ function

$$\delta[\dot{x}_0-s]=\lim_{N\to\infty}\prod_{k=1}^N\delta\left(\frac{x_{0,k}-x_{0,k-1}}{\Delta t}-s\right),\quad \Delta t=\frac{T}{N}. \tag{66}$$

The functional integration over x_0 can now be performed trivially, with the result that the time coordinate has the proper time dependence

$$x_0(t)=y_0+st. \tag{67}$$

However, one δ-function remains because there are only $(N-1)$ integrations in the discretized path integral for the coordinates

$$G(x,y)\simeq-\frac{i}{2M}(i\not\partial_x-g\not A+M)\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\times\sum_{s=\pm1}\int_0^\infty dT\delta(x_0-y_0-sT)\times\exp(-iMT)N_4^{\rm spin}\int\mathcal{D}^3x\mathcal{D}^3\Pi\mathcal{D}^4\zeta\times\exp\left\{\zeta(0)\cdot\zeta(T)+i\int_0^T dt\left[i\zeta\cdot\dot{\zeta}+\mathbf{\Pi}\cdot\dot{\mathbf{x}}-\frac{\mathbf{\Pi}^2}{2M}-gsA_0+g\mathbf{A}\cdot\dot{\mathbf{x}}-\frac{ig}{M}F_{\mu\nu}\zeta^\mu\zeta^\nu\right]\right\}_{\Gamma=0}. \tag{68}$$

The remaining δ function enforces the boundary condition $x_0(T)=x_0$ and can be used to perform the integration over the proper time T yielding

$$T=s(x_0-y_0). \tag{69}$$

In other words, in the nonrelativistic limit the proper time becomes the ordinary time (difference), as expected. Since the proper time is positive, the $(s=+1)$ term describes forward propagation of the particle whereas the $(s=-1)$ term describes backward propagation of the antiparticle, which is also contained in the Feynman propagator but decouples in the heavy mass limit. Furthermore, the global projection operator in front of the propagator (68) can be replaced by

$$-\frac{i}{2M}(i\partial\!\!\!/_x-gA\!\!\!/+M)\rightarrow -i\frac{1}{2}(1+s\gamma_0)+\mathcal{O}\left(\frac{1}{M}\right) \quad (70)$$

as the x_0 derivative acting on the phase factor $\exp[-iMs(x_0-y_0)]$ gives the leading contribution. Since

$$\gamma_0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad (71)$$

the (anti)particle propagator acts only on the (lower) upper components of Dirac spinors if the remaining path integral is diagonal in 2×2 Dirac space (which will turn out to be the case). Shifting back to integration over $\mathbf{p}$ we therefore obtain

$$\begin{aligned}G(x,y)\simeq &-i\sum_{s=\pm 1}\Theta(s(x_0-y_0))\frac{1}{2}(1+s\gamma_0)e^{-iMs(x_0-y_0)}\\ &\times\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)N_4^{\text{spin}}\int\mathcal{D}^3x\mathcal{D}^3p\mathcal{D}^4\zeta\\ &\times\exp\left\{\zeta(0)\cdot\zeta(T)+i\int_0^{T=s(x_0-y_0)}dt\right.\\ &\times\left[i\zeta\cdot\dot{\zeta}+\mathbf{p}\cdot\dot{\mathbf{x}}-\frac{1}{2M}(\mathbf{p}-g\mathbf{A})^2-gsA_0\right.\\ &\left.\left.-\frac{ig}{M}F_{\mu\nu}\zeta^\mu\zeta^\nu\right]\right\}_{\Gamma=0}. \end{aligned}\quad (72)$$

The time dependence of the electromagnetic potentials and fields is fixed by Eq. (67). Substituting

$$\begin{aligned}t'=y_0+st,\quad t'\in[y_0,x_0],\\ \mathbf{x}(t)=\mathbf{x}'(t'),\mathbf{p}(t)=\mathbf{p}'(t'),\quad \zeta(t)=\zeta'(t')\end{aligned}\quad (73)$$

the boundary conditions for the coordinate space path integral become the usual ones for a nonrelativistic path integral [29]

$$\mathbf{x}'(y_0)=\mathbf{y},\quad \mathbf{x}'(x_0)=\mathbf{x}. \quad (74)$$

Omitting the primes, the action in the phase space path integral now reads

$$\begin{aligned}S_s[\mathbf{x},\mathbf{p},\zeta]=&-i\zeta_\mu(y_0)\zeta^\mu(x_0)+\int_{y_0}^{x_0}dt[i\zeta^\mu\dot{\zeta}_\mu+\mathbf{p}\cdot\dot{\mathbf{x}}\\ &-\mathcal{H}_s(\mathbf{x},\mathbf{p},2\zeta)],\end{aligned}\quad (75)$$

$$\mathcal{H}_s(\mathbf{x},\mathbf{p},2\zeta)=s\left[M+\frac{(\mathbf{p}-g\mathbf{A})^2}{2M}\right]+gA_0+\frac{igs}{M}F_{\mu\nu}\zeta^\mu\zeta^\nu. \quad (76)$$

Here we have absorbed the phase factor $\exp[-iMs(x_0-y_0)]$ into the Hamiltonian $\mathcal{H}_s$.

Finally we simplify the spin degrees of freedom by using

$$F_{\mu\nu}\zeta^\mu\zeta^\nu=2\zeta_0\mathbf{E}\cdot\boldsymbol{\zeta}-\mathbf{B}\cdot(\boldsymbol{\zeta}\times\boldsymbol{\zeta}) \quad (77)$$

and observing that the first term in Eq. (77) is linear in ζ_0. After shifting $\zeta_0=\Gamma_0/2+\xi_0$, the ξ_0 integration can be performed and leads to a term in the remaining action which is of $\mathcal{O}(1/M^2)$: the Fourier transform of a Gaussian is again a Gaussian. In that process part of the spin normalization factor is cancelled. The same argument can be applied with respect to the $\boldsymbol{\zeta}$ integration so that in leading order only $igs\Gamma_0\mathbf{E}\cdot\boldsymbol{\Gamma}/M$ survives from the first term in Eq. (77). Performing the required differentiations with respect to Γ_0 we see that we obtain a contribution to Eq. (70) of the same order which was already neglected. Notice that $2\zeta_0\mathbf{E}\cdot\boldsymbol{\zeta}$ is "odd" in the sense of connecting large and small components in the Dirac equation; from the standard Foldy-Wouthuysen transformation of the Dirac Hamiltonian it also follows that the odd parts are suppressed by a factor $1/M$ compared to the "even" ones. In addition, since

$$\gamma_i\gamma_j=\begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}\begin{pmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{pmatrix}=-\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}\begin{pmatrix}\sigma_j & 0\\ 0 & \sigma_j\end{pmatrix} \quad (78)$$

one can set

$$\exp(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}_\Gamma)\rightarrow\exp(i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}_\Gamma) \quad (79)$$

for the remaining even part of the action. After changing the signs of the spin terms by the substitution $\boldsymbol{\zeta}\rightarrow i\boldsymbol{\zeta}$ the nonrelativistic limit for the Dirac propagator finally becomes

$$\begin{aligned}G(\mathbf{x},&x_0;\mathbf{y},y_0)\\ \simeq&-i\sum_{s=\pm 1}\Theta[s(x_0-y_0)]\frac{1}{2}(1+s\gamma_0)e^{\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}_\Gamma}\overline{N}_3^{\text{spin}}\\ &\times\int_{\mathbf{x}(y_0)=\mathbf{y}}^{\mathbf{x}(x_0)=\mathbf{x}}\mathcal{D}^3x\mathcal{D}^3p\int_{\boldsymbol{\zeta}(y_0)+\boldsymbol{\zeta}(x_0)=\boldsymbol{\Gamma}}\mathcal{D}^3\zeta\\ &\times\exp\left\{\boldsymbol{\zeta}(y_0)\cdot\boldsymbol{\zeta}(x_0)+i\int_{y_0}^{x_0}dt[i\boldsymbol{\zeta}\cdot\dot{\boldsymbol{\zeta}}+\mathbf{p}\cdot\dot{\mathbf{x}}\right.\\ &\left.-\mathcal{H}_s(\mathbf{x},\mathbf{p},2\boldsymbol{\zeta})]\right\}_{\boldsymbol{\Gamma}=0}\end{aligned}\quad (80)$$

with

$$\begin{aligned}\overline{N}_3^{\text{spin}}=&\left\{\int_{\boldsymbol{\zeta}(y_0)+\boldsymbol{\zeta}(x_0)=\boldsymbol{\Gamma}}\mathcal{D}^3\zeta\exp\left[\boldsymbol{\zeta}(y_0)\cdot\boldsymbol{\zeta}(x_0)\right.\right.\\ &\left.\left.-\int_{y_0}^{x_0}dt\boldsymbol{\zeta}\cdot\dot{\boldsymbol{\zeta}}\right]\right\}^{-1}.\end{aligned}\quad (81)$$

The Hamiltonian

$$\mathcal{H}_s(\mathbf{x},\mathbf{p},2\boldsymbol{\zeta})=s\left[M+\frac{(\mathbf{p}-g\mathbf{A})^2}{2M}\right]+gA_0+\frac{igs}{M}\mathbf{B}\cdot(\boldsymbol{\zeta}\times\boldsymbol{\zeta}) \quad (82)$$

coincides exactly with the standard Foldy-Wouthuysen Hamiltonian for particles and antiparticles [see, e.g., Ref. [19], Eq. (4.5)]

$$H_{\rm FW}(\mathbf{x},\mathbf{p},\boldsymbol{\sigma}) = \gamma_0\left[M+\frac{(\mathbf{p}-g\mathbf{A})^2}{2M}\right]+gA_0-\frac{g}{2M}\gamma_0\boldsymbol{\sigma}\cdot\mathbf{B} + \mathcal{O}\left(\frac{1}{M^2}\right) \tag{83}$$

if the last term (the so-called Pauli term) is rewritten using

$$\boldsymbol{\sigma}\times\boldsymbol{\sigma} = 2i\boldsymbol{\sigma}. \tag{84}$$

This ensures that the time evolution is governed by a Grassmann-even Hamiltonian. It is, of course, straightforward to start from the nonrelativistic spin-dependent Hamiltonian (83) and by using the identity (84) to derive the three-dimensional path integral (80) for the propagator as has been done in Ref. [9], Sec. 5. Here we proceeded in the reverse order showing how the nonrelativistic limit can be taken within the path integral representation of the Dirac propagator. It should also be possible to evaluate higher-order terms in the nonrelativistic reduction in this way or to obtain the semiclassical limit of the propagator [30].

V. DIRAC PROPAGATOR IN AN EXTERNAL SCALAR POTENTIAL

For some applications one needs the propagator of a fermion which moves in an external scalar field $S(x)$ as well. For example, in the Walecka model [31] the exchange of a scalar meson generates attraction between nucleons whereas massive vector mesons are responsible for repulsion at shorter distances. In such cases we need to evaluate the following Green function

$$G(x,y) = \left\langle x\left|\frac{1}{\Pi\!\!\!/-M^\star(x)+i0}\right|y\right\rangle, \tag{85}$$

where

$$M^\star(x) = M+S(x) \tag{86}$$

is the effective, position-dependent mass of the fermion.

The previous method of multiplying numerator and denominator in Eq. (85) by $\Pi\!\!\!/+M^\star$ obviously does not work anymore since

$$(\Pi\!\!\!/-M^\star)(\Pi\!\!\!/+M^\star) = \Pi\!\!\!/^2-M^{\star 2}+[\Pi\!\!\!/,M^\star] \tag{87}$$

is not Grassmann even. Consequently there are statements in the literature [27] that in this case a five-dimensional formalism is the only possible approach. However, this is *not* the case: the problem to rationalize the denominator of the Green function is analogous to the problem of inverting complex matrices by using only real arithmetic. This is easily achieved by writing

$$\frac{1}{A+iB} = (1-iA^{-1}B)\frac{1}{A+BA^{-1}B}. \tag{88}$$

Therefore we have

$$\frac{1}{\Pi\!\!\!/-M^\star} = \left(M+\frac{M}{M^\star}\Pi\!\!\!/\right)\frac{1}{\Pi\!\!\!/(M/M^\star)\Pi\!\!\!/-MM^\star+i0} \tag{89}$$

as the Di Vecchia-Ravndal representation for the present case. Again we have the choice to project on $\Pi\!\!\!/=M^\star$ either during the evolution or at the end. If we adopt the latter approach the quantum-mechanical Hamiltonian which governs the proper time evolution is now

$$\hat{\mathcal{H}} = -\frac{1}{2\kappa_0}\hat{\Pi\!\!\!/}\frac{M}{M^\star(\hat{x})}\hat{\Pi\!\!\!/}. \tag{90}$$

The phase-space path integral representation of the propagator is now determined by evaluating the Wigner transform of Eq. (90) (see the Appendix). With the abbreviation $U(x) = M/M^\star(x)$ one obtains

$$\mathcal{H}(p,x,\gamma) = -\frac{1}{2\kappa_0}\Pi^2U(x)+\frac{ig}{4\kappa_0}\gamma_\mu\gamma_\nu F^{\mu\nu}(x)U(x) - \frac{ig}{4\kappa_0}\gamma_\mu\gamma_\nu[\partial^\mu U(x)\Pi^\nu-\Pi^\mu\partial^\nu U(x)] - \frac{1}{8\kappa_0}\partial^2U(x). \tag{91}$$

Since the Hamiltonian is quadratic in $\Pi = p-gA$ the momentum path integral can still be performed so that the Lagrangian path integral representation of the propagator reads

$$G(x,y) = -\frac{i}{2\kappa_0}[U(x)[i\partial\!\!\!/-gA\!\!\!/(x)]+M]\int_0^\infty dT N(T) \times\exp\left(-\frac{iMM^\star(x)T}{2\kappa_0}\right) \times\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_{x(0)=y}^{x(T)=x}\mathcal{D}x\int_{\zeta(0)+\zeta(T)=\Gamma}\mathcal{D}\zeta \times\exp\left\{\zeta(0)\cdot\zeta(T)+i\int_0^T dt L(x,\dot{x},\zeta,\dot{\zeta})\right\}\Bigg|_{\Gamma=0} \tag{92}$$

with

$$L(x,\dot{x},\zeta,\dot{\zeta}) = i\zeta\cdot\dot{\zeta}-\frac{\kappa_0}{2U(x)}\dot{x}^2-gA(x)\cdot\dot{x} - \frac{ig}{\kappa_0}U(x)F^{\mu\nu}(x)\zeta_\mu\zeta_\nu-2i\dot{x}\cdot\zeta\frac{1}{U(x)}\zeta\cdot\partial U(x) + \frac{1}{8\kappa_0}\partial^2U(x)+2i\delta(0)\ln U(x). \tag{93}$$

The last term arises from the quadratic fluctuations

$$\prod_k \frac{1}{U^2(x_k)} = \exp\left(-2\sum_k \log U(x_k)\right) = \exp\left(i2i\frac{1}{\Delta t}\Delta t\sum_k \ln U(x_k)\right) \tag{94}$$

in the discretized momentum path integral which are now position dependent due to the effective mass $M^\star(x)$. The awkward $\delta(0)$ appears as the formal limit of $1/\Delta t$ when the time slicing Δt is made infinitesimal (see Ref. [32], Chap. 19) and cancels consistently against other divergencies [33].

VI. EFFECTIVE ACTION FOR QUENCHED QED

In order to reduce the number of degrees of freedom it is advantageous for some applications to integrate out the bosons which mediate the interactions. The price to be paid is, of course, a more complicated two-time effective interaction. We will outline this procedure by considering quantum electrodynamics (QED) (or the Walecka model without scalar mesons[4])

$$\mathcal{L} = \mathcal{L}_0(A) + \bar{\psi}(i\partial\!\!\!/ - gA\!\!\!/ - M_0)\psi, \tag{95}$$

where

$$\mathcal{L}_0(A) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A^2 - \frac{1}{2}\lambda(\partial\cdot A)^2 \tag{96}$$

is the Stückelberg Lagrangian with a gauge parameter λ. We have given the photons a mass m in order to regularize infrared divergencies.

The generating functional for the two-point function with an arbitrary number of photons is

$$Z'[j,x] = \int \mathcal{D}A\langle x|\frac{1}{i\partial\!\!\!/ - gA\!\!\!/ - M_0}|0\rangle \exp\{i\mathcal{A}_0[A] + (j,A)\}. \tag{97}$$

Here the free vector meson action is denoted by $\mathcal{A}_0[A] = \int d^4x \mathcal{L}_0(A)$ and we have neglected closed fermion loops (quenched approximation) [34] in order to have a single world line for the fermion. For integrating out the vector field A_μ we use the path integral representation of the Dirac propagator in an external vector field in its supersymmetric form and the Gaussian integration formula

$$\int \mathcal{D}A_\mu \exp\left[\frac{i}{2}[A_\mu,(G^{-1})^{\mu\nu}A_\nu] + (A_\mu,h^\mu)\right] \propto \exp\left[\frac{i}{2}(h_\mu,G^{\mu\nu}h_\nu)\right]. \tag{98}$$

Here

$$G^{\mu\nu}(k) = -\left[\frac{g^{\mu\nu}-k^\mu k^\nu/m^2}{k^2-m^2} + \frac{k^\mu k^\nu/m^2}{k^2-m^2/\lambda}\right] \tag{99}$$

is the standard propagator for massive vector particles (Ref. [24]). From the linear terms in A^μ we read off

$$h_\mu(y) = j_\mu(y) + ig\int_0^T dt\int d\theta DX_\mu(t,\theta)\,\delta^4(y - X(t,\theta)). \tag{100}$$

For the present purposes the local projection method is preferable because the whole dependence on the photon field resides in the free photon action and the electron-photon interaction. After integration over the photon field we then obtain for the generating functional (97)

$$Z'[j,x] = \text{const}\ \exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty dTN(T) \times\exp\left(-\frac{i}{2\kappa_0}M^2T\right)\cdot\int d\chi \times\exp\left(-\frac{i}{2\kappa_0}M\chi\right)\mathcal{D}x\mathcal{D}\zeta e^{iS_{\text{eff}}[X,j]}|_{\Gamma=0}, \tag{101}$$

where the effective action is given by

$$S_{\text{eff}}[X,j] = S_0'[X] + \frac{1}{2}(h_\mu, G^{\mu\nu}h_\nu). \tag{102}$$

As in Ref. [34], it is advantageous to split it up into terms involving zero, one or two external sources $j(y)$. The latter one leads to disconnected diagrams and can be discarded. We then have

$$S_{\text{eff}}[X,j] = S_0'[X] + S_1[X] + S_2[X,j], \tag{103}$$

where the free action is given in Eq. (44) and the interaction part by

$$S_1[X] = \frac{g^2}{2}\int_0^T dt_1\int d\theta_1\int_0^T dt_2\int d\theta_2\int\frac{d^4k}{(2\pi)^4} \times G^{\mu\nu}(k)DX_\mu(t_1,\theta_1)DX_\nu(t_2,\theta_2) \times\exp\{-ik\cdot[X(t_1,\theta_1) - X(t_2,\theta_2)]\}. \tag{104}$$

Note that the ''current''

$$J_\mu(X) \equiv DX_\mu(t,\theta) = -\theta\dot{x}_\mu(t) + a\zeta_\mu(t) \tag{105}$$

looks similar to scalar QED but is Grassmann odd and does not depend on the integration variable k. Therefore the k integration can be performed easily giving the photon propagator in configuration space with argument $X(t_1,\theta_1) - X(t_2,\theta_2)$ [14]. Written in components the interaction term

[4]If one also wants to integrate out the scalar mesons, the five-dimensional Berezin-Marinov description has to be used because only then is a Gaussian path integral for the scalar mesons obtained; our four-dimensional form (93) is highly nonlinear in $S(x)$.

$$S_1[x,\zeta]=-\frac{g^2}{2}\int_0^T dt_1 dt_2\int\frac{d^4k}{(2\pi)^4}\times G^{\mu\nu}(k)\left[\dot{x}_\mu(t_1)+\frac{2}{\kappa_0}\zeta_\mu(t_1)k\cdot\zeta(t_1)\right]\times\left[\dot{x}_\nu(t_2)-\frac{2}{\kappa_0}\zeta_\nu(t_2)k\cdot\zeta(t_2)\right]e^{-ik\cdot[x(t_1)-x(t_2)]}\tag{106}$$

is seen to contain up to *quartic* terms in the spin variable ζ. This means that, unlike the case of external fields, the Grassmann variables cannot be integrated out anymore to give a ''spin factor.'' Vice versa, it is impossible to eliminate the photon field starting from the spin factor formulation for the propagator.

The source term becomes

$$S_2[X,j]=ig\int d^4y j_\mu(y)\int_0^T dt\int d\theta\int\frac{d^4k}{(2\pi)^4}\times G^{\mu\nu}(k)DX_\nu(t,\theta)\exp\{-ik\cdot[X(t,\theta)-y]\}.\tag{107}$$

It is also possible to use the global projection method which does not have a spin-orbit coupling. However, there is an additional dependence on the photon field in the covariant derivative acting on the path integral in Eq. (18) which makes it less suitable for deriving an effective action.

To conclude this section, we note that the effective action in Eq. (103) allows a particularly concise derivation of the transformation properties of Green functions[5] under a change of the gauge parameter λ: We see from the photon propagator $G_{\mu\nu}(k)$ in Eq. (99) that a change in λ only effects the term proportional to $k_\mu k_\nu$. For this term the integrals over the proper times t_i and θ_i occurring in the effective action may be performed exactly as the integrand is a total derivative, i.e.,

$$\int_0^T dt\int d\theta k\cdot DX(t,\theta)e^{-ik\cdot X(t,\theta)}=i\int_0^T dt\int d\theta De^{-ik\cdot X(t,\theta)}=i(1-e^{-ik\cdot x}).\tag{108}$$

The change in S_1 [Eq. (104)] and S_2 [Eq. (107)] induced by a change in λ from λ_1 to λ_2, say, is therefore only dependent on the end points of the path $x(t)$ and not on the path itself. If we define $\Delta(x^2)$ to be the Fourier transform of the change of the coefficient $[\equiv\tilde{\Delta}(k^2)]$ of $k_\mu k_\nu$ in the photon propagator, i.e.,

$$\Delta(x^2)=-\frac{1}{m^2}\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2-m^2/\lambda_2}-\frac{1}{k^2-m^2/\lambda_1}\right)e^{-ik\cdot x},\tag{109}$$

then the corresponding change in S_1 is given by

$$\delta S_1=S_1^{\lambda=\lambda_2}[X]-S_1^{\lambda=\lambda_1}[X]=\frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\tilde{\Delta}(k^2)i^2(1-e^{-ik\cdot x})(1-e^{ik\cdot x})=g^2[\Delta(x^2)-\Delta(0)]\tag{110}$$

while the change in S_2 is

$$\delta S_2=-ig\int d^4y j(y)\cdot\partial_y[\Delta([y-x]^2)-\Delta(y^2)].\tag{111}$$

Note that not only is the change in $S_{1,2}$ independent of the path, so that it may be pulled out of the path integral in Eq. (101), it also does not involve the Grassmann valued Γ nor is it dependent on the proper time T. Hence the generating function for the Green functions with gauge parameter λ_2 is related to that with gauge parameter λ_1 in a very simple way, namely,

$$Z'_{\lambda_2}[j,x]=e^{i(\delta S_1+\delta S_2)}Z'_{\lambda_1}[j,x]=\exp\left\{ig^2[\Delta(x^2)-\Delta(0)]+g\int d^4y j(y)\cdot\partial_y[\Delta([y-x]^2)-\Delta(y^2)]\right\}\times Z'_{\lambda_1}[j,x].\tag{112}$$

As special cases we can derive the transformation laws for the propagator and the electron-photon vertex from this expression.[6] Setting $j=0$ we obtain

$$G^{\lambda_2}(x,0)=e^{ig^2[\Delta(x^2)-\Delta(0)]}G^{\lambda_1}(x,0),\tag{113}$$

while by differentiating once with respect to the current and then setting $j=0$ we obtain the (untruncated) vertex function

$$G_{2,1}^{\lambda_2\mu}(y;x,0)=g\{\partial_y^\mu[\Delta([y-x]^2)-\Delta(y^2)]\}G^{\lambda_2}(x,0)+e^{ig^2[\Delta(x^2)-\Delta(0)]}G_{2,1}^{\lambda_1\mu}(y;x,0).\tag{114}$$

It should be noted that these relations are valid even if the photon mass (which violates gauge invariance) is kept nonzero in the photon propagator.

---

[5]These transformations were first derived for the electron propagator and the electron-photon vertex by Landau and Khalatnikov [35] and extended to general Green functions by Fradkin and Zumino [36].

[6]See Ref. [35]. Note that in that paper the photon propagator is defined with a minus sign with respect to ours. Hence our function $\Delta(y^2)$ is $-\Delta_F(y)$ of Ref. [35] and our untruncated vertex function is the negative of the function B_μ defined by Landau *et al.*

VII. SUMMARY AND CONCLUSIONS

The main purpose of this work is to explore a four-dimensional path integral representation for the Dirac propagator general enough to describe the particle's motion in both vector and scalar fields. Although the four-dimensional approach, for an external vector field, was proposed by Di Vecchia and Ravndal almost twenty years ago, it had received limited attention up to now. Instead it is standard to use the Berezin-Marinov approach where one introduces a fifth component to eliminate the extra spin degree of freedom. However, the fifth component has no clear physical meaning and the necessity of introducing a γ_5 for the evaluation of the propagator seems rather unnatural. The four-dimensional representation avoids these difficulties; in addition the supersymmetry transformations become easier and more natural to generate.

Working within this four-dimensional formalism we have presented two alternative methods to project out the unwanted spin degree of freedom. The first method projects onto the final state after the time evolution and is hence termed global, whereas in the second method the projection is done at each step in the time evolution and it is therefore referred to as local. Extending previous work by Reuter, Schmidt, and Schubert we have shown that due to the supersymmetry the two methods are completely equivalent and may be used according to convenience. The main difference between the two approaches is that the path integral representation using the local projection has an explicit spin-orbit coupling term. It was therefore crucial for the proof of equivalence to generalize the results of Ravndal and Di Vecchia regarding the supersymmetry transformations to apply also in the case where the spin-orbit term appears. In Refs. [16,17] it was pointed out that, in the case where no spin-orbit term was present, a simple way of generating both bosonic and fermionic transformations is to shift the times t and θ. We show in this paper that in the presence of a spin-orbit term in addition to a shift an appropriate scaling of the times t and θ is needed in order to generate the correct supersymmetry transformations. This scaling is such that the parameter κ_0 and the supertime χ remain unchanged.

For the case of a Dirac particle in an external scalar potential it was generally believed that a five-dimensional approach was unavoidable. We have here shown that this is *not* the case and we used the four-dimensional description to obtain the Dirac propagator in an external scalar field.

Despite the attention given to spin in the path integrals, a nonrelativistic reduction starting directly from the Dirac propagator was still missing. By expanding the relativistic expression in powers of $1/M$ we were able to reduce the path integrals to three-dimensional form and to obtain the leading nonrelativistic result described by the Foldy-Wouthuysen Hamiltonian.

Finally we applied the four-dimensional approach to quenched QED in order to obtain a supersymmetric formulation for the generating functional of Green functions with one electron line and an arbitrary number of external photon lines. It was possible to do this as in the quenched approximation the photons can be integrated out, yielding a path integral only in the electron degrees of freedom, albeit with a complicated nonlocal interaction. In this form one can apply methods along the same lines as those used in the study of the polaron problem as described in Ref. [26]. Furthermore, we showed that it is a rather simple matter to derive the Landau-Khalatnikov transformations for the propagator, vertex function, and indeed any higher-point function from this formalism.

From a field-theoretic point of view, the world line technique is particularly appropriate whenever one deals with a situation where internal fermion loops may either be neglected or taken into account perturbatively. As this situation arises quite naturally in the nonrelativistic regime, the technique would appear to be particularly appropriate in that setting. We think that the reason it has not received a great deal of attention by physicists working in that area is partly due to the fact that the commonly used five-dimensional representation appears artificial within this context. In this paper we have tried to convey the message that for most problems the five-dimensional formulation is not only unnecessary but in fact less transparent than the four-dimensional one. It is our hope, therefore, that this paper makes world line techniques more accessible to a wider audience than they have been up to now.

Note added in proof. Recently it was pointed out to us that the elimination of the fifth spin variable was also considered by T. Allen using Hamiltonian methods [T. Allen, Phys. Lett. B **214**, 87 (1988); see also T. Allen, Ph.D. thesis, California Institute of Technology (1988)]. Also, J. W. van Holten has advocated the use of a commuting rather than anticommuting fifth spin variable and a different four-dimensional approach [see the first reference in [10] as well as a more concise discussion of the problem in Nucl. Phys. B (Proc. Suppl.) **49**, 319 (1996)]. Finally, another important contribution to the literature on spin in path integrals missing from Ref. [6] is the paper by M. Halpern, A. Jevicki, and P. Senjanovic [Phys. Rev. D **16**, 2476 (1977)]. We are grateful to Professor T. Allen, Professor M. Halpern, and Professor J. W. van Holten for correspondence regarding these references.

ACKNOWLEDGMENTS

We would like to thank Michael Marinov for helpful discussions. One of us (A.W.S.) was supported by the Australian Research Council.

APPENDIX: SPIN PATH INTEGRAL FOR THE TIME EVOLUTION OPERATOR

Here we consider the matrix element of the time evolution operator

$$U(x,y)=\langle x|\exp(-i\hat{\mathcal{H}}T)|y\rangle, \tag{A1}$$

where

$$\hat{\mathcal{H}}=\mathcal{H}(\hat{p},\hat{x},\gamma) \tag{A2}$$

is a Weyl-ordered Hamiltonian. Breaking up the evolution operator in N time steps we obtain in the usual way

$$U(x,y)=\lim_{N\to\infty}\int d^4x_1\cdots d^4x_{N-1}\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\times\exp\left[-i\sum_{i=1}^N p_i\cdot(x_i-x_{i-1})\right]\times\exp[-i\mathcal{H}_W(p_N,x_N,\gamma_N)\Delta t]\cdots\times\exp[-i\mathcal{H}_W(p_1,x_1,\gamma_1)\Delta t] \tag{A3}$$

with $x_0=y$ and $x_N=x$. Here

$$\mathcal{H}_W(p,x,\gamma)=\int d^4y\left\langle x-\frac{y}{2}\middle|\hat{\mathcal{H}}\middle|x+\frac{y}{2}\right\rangle e^{-ip\cdot y} \tag{A4}$$

is the Wigner transform (or Weyl symbol) of the Hamiltonian which is the closest classical analog to the (Weyl-ordered) quantum operator [37]. We will suppress the subscript W in the following.

There are two essential steps to derive a path integral with spin.

(i) Because the Dirac matrices do not commute, the ordering of the factors is essential and the exponentials cannot be combined with impunity. As is well known this also happens in ordinary quantum mechanics for time dependent Hamiltonians. We therefore have assigned an artificial time-dependence to the Dirac matrices and can write now the time evolution operator as a time-ordered path integral [29]

$$\begin{aligned}U(x,y)&=\int \mathcal{D}x\mathcal{D}p\mathcal{T}\times\exp\left\{-i\int_0^T dt\{p\cdot\dot{x}+\mathcal{H}[p(t),x(t),\gamma(t)]\}\right\}\\&=\int\mathcal{D}x\mathcal{D}p\exp\left\{-i\int_0^T dt\left[p\cdot\dot{x}+\mathcal{H}\left(p(t),x(t),\frac{\delta}{\delta\rho(t)}\right)\right]\right\}\mathcal{T}\times\exp\left[\int_0^T dt\rho^\mu(t)\gamma_\mu(t)\right]_{\rho^\mu=0}.\end{aligned} \tag{A5}$$

Here $\rho^\mu(t)$ are Grassmann sources which are assumed to anticommute with the Dirac matrices. The boundary conditions for the x-space path integral are

$$x_\mu(0)=y_\mu,\quad x_\mu(T)=x_\mu. \tag{A6}$$

The time-ordering symbol $\mathcal{T}$ would be disastrous for further manipulation of the path integral. However, in the special case it can be eliminated by the relation

$$\begin{aligned}V(T)&\equiv\mathcal{T}\exp\left\{\int_0^T dt\rho^\mu(t)\gamma_\mu(t)\right\}\\&=\exp\left\{-\int_0^T dt_1\int_0^{t_1}dt_2\rho^\mu(t_1)\rho_\mu(t_2)\right\}\times\exp\left\{\int_0^T dt\rho^\mu(t)\gamma_\mu\right\}.\end{aligned} \tag{A7}$$

This can be proved by solving the corresponding evolution equation

$$\frac{\partial V(T)}{\partial T}=\rho^\mu(T)\gamma_\mu V(T),\quad V(0)=1 \tag{A8}$$

by using the Magnus expansion [38]

$$V(T)=\exp\left\{\int_0^T dt\rho^\mu(t)\gamma_\mu+\frac{1}{2}\int_0^T dt_1\int_0^{t_1}dt_2[\rho_\mu(t_1)\gamma^\mu,\rho_\nu(t_2)\gamma^\nu]+\cdots\right\}. \tag{A9}$$

The commutator yields $-2\rho_\mu(t_1)\rho^\mu(t_2)$ which is a commuting c number so that all higher terms in the expansion which involve multiple commutators vanish. On the right-hand side of Eq. (A7) we can now drop the artificial time dependence of the Dirac matrices.

(ii) The differentiations with respect to $\rho^\mu(t)$ which are required in Eq. (A5) can only be performed easily if they appear linearly in the exponent. This can be achieved by ''undoing the square,'' which is a standard procedure [29]. However, because $\rho^\mu(t)$ is anticommuting and one needs an even object in the exponent as evolution operator, we have to do it with the help of a Grassmann path integral. We thus use the identity

$$\begin{aligned}\exp&\left\{-\int_0^T dt_1\int_0^{t_1}dt_2\rho^\mu(t_1)\rho_\mu(t_2)\right\}\\&=\int\mathcal{D}\xi\exp\left\{\int_0^T dt[-\xi_\mu(t)\dot{\xi}^\mu(t)+2\rho^\mu(t)\xi_\mu(t)]\right\}\times\left[\int\mathcal{D}\xi\exp\left(-\int_0^T dt\xi_\mu(t)\dot{\xi}^\mu(t)\right)\right]^{-1}\end{aligned} \tag{A10}$$

and the antiperiodic boundary condition $\xi_\mu(0)+\xi_\mu(T)=0$ for the Grassmann path integral. The standard way of proving this identity in the continuum formulation is by solving the (differential) equations of motion which should give the exact result for quadratic actions. However, it is very useful (and reassuring) to have an unambiguous formulation with finite time steps Δt, which we will present now: the discretized form of

$$S=-\int_0^T dt[-\xi_\mu(t)\dot{\xi}^\mu(t)+2\rho^\mu(t)\xi_\mu(t)] \tag{A11}$$

may be written as

$$S=\sum_{i=1}^{N}\left[\xi_{i,\mu}\left(\frac{\xi^{\mu}_{i+1}-\xi^{\mu}_{i-1}}{2}\right)-\frac{\Delta t}{2}\rho_{i,\mu}(\xi^{\mu}_{i+1}+2\xi^{\mu}_{i}+\xi^{\mu}_{i-1})\right], \tag{A12}$$

where $\Delta t=T/N$ and N needs to be even for the path integral to be an even quantity. In discretized form the path integral over ξ in Eq. (A10) can now be done by the stationary phase method. The (difference) equation of motion

$$\xi^{\mu}_{k+1}-\xi^{\mu}_{k-1}=-\frac{\Delta t}{2}(\rho^{\mu}_{k+1}+2\rho^{\mu}_{k}+\rho^{\mu}_{k-1}) \tag{A13}$$

can be solved using antiperiodic boundary conditions for ξ, i.e., $\xi^{\mu}_{N}=-\xi^{\mu}_{0}$, $\quad \xi^{\mu}_{N+1}=-\xi^{\mu}_{1}$. It is convenient (but not necessary) to impose the equivalent boundary conditions for ρ. Note that the particular discretization of $\rho(t)\cdot\xi(t)$ in Eq. (A12) is chosen so that the equations of motion (A13) for the odd and even sites are coupled. This avoids the infamous ‘‘fermion doubling’’ problem. The solution to the equation of motion is

$$\xi^{\mu}_{\mathrm{cl}j}=\frac{\Delta t}{2}\rho^{\mu}_{j}-\Delta t\sum_{k=1}^{j}\rho^{\mu}_{k}+\frac{\Delta t}{2}\sum_{k=1}^{N}\rho^{\mu}_{k}. \tag{A14}$$

Substituting the solution ξ_{cl} into Eq. (A12) we find

$$S_{\mathrm{cl}}=(\Delta t)^2\sum_{i=1}^{N}\rho_{\mu,i}\sum_{k=1}^{i-1}\rho^{\mu}_{k}+\frac{(\Delta t)^2}{8}\sum_{i=1}^{N}\rho_{\mu,i}(\rho^{\mu}_{i+1}-\rho^{\mu}_{i-1}). \tag{A15}$$

The path integral over ξ can now be performed yielding

$$\frac{\int\mathcal{D}\xi\exp[-S]}{\int\mathcal{D}\xi\exp\left[-\int_0^T\xi_{\mu}\dot{\xi}^{\mu}\right]}=\exp[-S_{\mathrm{cl}}] \tag{A16}$$

since the determinant from the quantum fluctuations is canceled by the denominator. Taking the continuous limit of S_{cl} only the first term in Eq. (A15) survives and we obtain the required result. Having proven the relation (A10) by writing the functional integrals in a well defined discretized form we can now use it in Eq. (A5) with all manipulations formally done in the continuum.

Using the representation

$$\exp\left\{\int_0^T dt\rho^{\mu}(t)\gamma_{\mu}\right\}=\exp\left\{\gamma_{\mu}\frac{\partial}{\partial\Gamma_{\mu}}\right\}\exp\left\{\int_0^T dt\rho^{\mu}(t)\Gamma_{\mu}\right\}\Bigg|_{\Gamma_{\mu}=0} \tag{A17}$$

we obtain

$$U(x,y)=\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int\mathcal{D}x\mathcal{D}p\mathcal{D}\xi N^{\mathrm{spin}}\exp\left\{-i\int_0^T dt[p\cdot\dot{x}-i\xi\cdot\dot{\xi}+\mathcal{H}(p,x,2\xi+\Gamma)]\right\}\Bigg|_{\Gamma=0}. \tag{A18}$$

Here

$$N^{\mathrm{spin}}=\left[\int\mathcal{D}\xi\exp\left(-\int_0^T dt\xi_{\mu}\dot{\xi}^{\mu}\right)\right]^{-1} \tag{A19}$$

is a normalization factor for the spin integral. Note that the operation in Eq. (A17) is in general *not* just a replacement of the boundary variable Γ by the corresponding Dirac γ matrix but involves an antisymmetrization as well. For example, $\exp\{\gamma\cdot\partial/\partial\Gamma\}\Gamma_{\mu}|_{\Gamma=0}=\gamma_{\mu}$, but

$$\begin{aligned}\exp\left\{\gamma^{\mu}\frac{\partial}{\partial\Gamma^{\mu}}\right\}\Gamma_{\mu}\Gamma_{\nu}\Bigg|_{\Gamma=0}&=\frac{1}{2}\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)^2\Gamma_{\mu}\Gamma_{\nu}\Bigg|_{\Gamma=0}\\&=\frac{1}{2}\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)(\gamma_{\mu}\Gamma_{\nu}+\Gamma_{\mu}\gamma_{\nu})\Bigg|_{\Gamma=0}\\&=\frac{1}{2}(-\gamma_{\nu}\gamma_{\mu}+\gamma_{\mu}\gamma_{\nu}).\end{aligned} \tag{A20}$$

This is the inverse transformation of the Weyl representation for fermionic operators.

[1] J. L. Martin, Proc. R. Soc. London, Ser. A **251**, 536 (1959); **251**, 543 (1959).

[2] E. Fradkin and M. Stone, Phys. Rev. B **38**, 7215 (1988); M. Bergeron, Fortschr. Phys. **40**, 119 (1992).

[3] L. Brink, S. Deser, B. Zumino, P. Di Vecchia, and P. Howe, Phys. Lett. **64B**, 435 (1976).

[4] F. A. Berezin and M. S. Marinov, JETP Lett. **21**, 320 (1975); Ann. Phys. (N.Y.) **104**, 336 (1977).

[5] M. S. Marinov (private communication).

[6] A. Barducci, R. Casalbuoni, and L. Lusanna, Nuovo Cimento A **35**, 377 (1976); Cl. Teitelboim, Phys. Rev. Lett. **38**, 1106 (1976); C. A. P. Galvao and Cl. Teitelboim, J. Math. Phys. **21**, 1863 (1980).

[7] L. Brink, P. Di Vecchia, and P. Howe, Nucl. Phys. B **118**, 76 (1977).

[8] M. Henneaux and Cl. Teitelboim, Ann. Phys. (N.Y.) **143**, 127 (1982).

[9] T. M. Aliev, V. Ya. Fainberg, and N. K. Pak, Nucl. Phys. B **429**, 321 (1994).

[10] J. W. van Holten, Nucl. Phys. B **457**, 375 (1995); E. D'Hoker and D. G. Gagné, *ibid.* **467**, 272 (1996); D. G. Gagné, e-print hep-th/9604149 (unpublished).

[11] E. S. Fradkin and D. M. Gitman, Phys. Rev. D **44**, 3230 (1991); D. M. Gitman, Nucl. Phys. B **488**, 490 (1997).

[12] M. J. Strassler, Nucl. Phys. B **385**, 145 (1992); D. G. C. McKeon and A. Rebhan, Phys. Rev. D **48**, 2891 (1993); M. G. Schmidt and C. Schubert, e-print hep-ph/9412358 (unpublished); M. Mondragón, L. Nellen, M. G. Schmidt, and C. Schubert, Phys. Lett. B **366**, 212 (1996); D. Fliegner, M. Reu-

ter, M. G. Schmidt, and C. Schubert, Theor. Math. Phys. **113**, 1442 (1997).

[13] M. Mondragón, L. Nellen, M. G. Schmidt, and C. Schubert, Phys. Lett. B **351**, 200 (1995).

[14] M. G. Schmidt and C. Schubert, Phys. Rev. D **53**, 2150 (1996).

[15] V. P. Gusynin and I. A. Shovkovy, Can. J. Phys. **74**, 282 (1996); e-print hep-th/9804143 (unpublished).

[16] P. Di Vecchia and F. Ravndal, Phys. Lett. **73A**, 371 (1979).

[17] F. Ravndal, Phys. Rev. D **21**, 2823 (1980).

[18] F. Delduc, E. Ivanov, and E. Sokatchev, Mod. Phys. Lett. A **12**, 937 (1997).

[19] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[20] F. A. Berezin, *The Method of Second Quantization* (Academic, London, 1966).

[21] A. M. Polyakov, *Gauge Fields and Strings* (Harwood, Chur, 1987).

[22] D. M. Gitman and S. I. Zlatev, Phys. Rev. D **55**, 7701 (1997).

[23] V. Ya. Fainberg and A. V. Marshakov, Nucl. Phys. B **306**, 659 (1988).

[24] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[25] M. E. Peskin and D. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, MA, 1995).

[26] R. Rosenfelder, C. Alexandrou and A. W. Schreiber, in *Proceedings of the Workshop on Nonperturbative Methods in Quantum Field Theory*, edited by A. W. Schreiber, A. G. Williams, and A. W. Thomas (World Scientific, Singapore, 1998); A. W. Schreiber, C. Alexandrou, and R. Rosenfelder, Nucl. Phys. A **631**, 635c (1998).

[27] M. Reuter, M. G. Schmidt, and C. Schubert, Ann. Phys. (N.Y.) **259**, 313 (1997).

[28] Ta-Pei Cheng and Ling-Fong Li, *Gauge Theory of Elementary Particle Physics* (Clarendon, Oxford, 1988).

[29] L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).

[30] J. Bolte and S. Keppeler, Phys. Rev. Lett. **81**, 1987 (1998).

[31] J. D. Walecka, Ann. Phys. (N.Y.) **83**, 491 (1973); B. D. Serot and J. D. Walecka, Adv. Nucl. Phys. **16**, 1 (1986); B. D. Serot, Rep. Prog. Phys. **55**, 1855 (1992); B. D. Serot and J. D. Walecka, Int. J. Mod. Phys. E **6**, 515 (1997).

[32] T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood, Chur, 1982).

[33] T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962); E. S. Abers and B. W. Lee, Phys. Rep. **9**, 1 (1973); F. Bastianelli *et al.*, Phys. Rev. D **58**, 044 002 (1998).

[34] R. Rosenfelder and A. W. Schreiber, Phys. Rev. D **53**, 3337 (1996).

[35] L. D. Landau and I. M. Khalatnikov, Sov. Phys. JETP **2**, 69 (1956); L. D. Landau, A. Abrikosov, and I. M. Khalatnikov, Nuovo Cimento Suppl. **3**, 80 (1956).

[36] E. S. Fradkin, Sov. Phys. JETP **2**, 361 (1956); B. Zumino, J. Math. Phys. **1**, 1 (1960).

[37] E. Wigner, Phys. Rev. **40**, 749 (1932); for a review see: M. Hillery *et al.*, Phys. Rep. **106**, 121 (1984).

[38] R. M. Wilcox, J. Math. Phys. **8**, 962 (1967).