PUBLISHED VERSION

Tuck, E. O.; Kouzoubov, A..
A laminar roughness boundary condition, *Journal of Fluid Mechanics*, 1995; 300:59-70.

Copyright © 1995 Cambridge University Press

PERMISSIONS

http://journals.cambridge.org/action/stream?pageId=4088&level=2#4408

The right to post the definitive version of the contribution as published at Cambridge Journals Online (in PDF or HTML form) in the Institutional Repository of the institution in which they worked at the time the paper was first submitted, or (for appropriate journals) in PubMed Central or UK PubMed Central, no sooner than one year after first publication of the paper in the journal, subject to file availability and provided the posting includes a prominent statement of the full bibliographical details, a copyright notice in the name of the copyright holder (Cambridge University Press or the sponsoring Society, as appropriate), and a link to the online edition of the journal at Cambridge Journals Online. Inclusion of this definitive version after one year in Institutional Repositories outside of the institution in which the contributor worked at the time the paper was first submitted will be subject to the additional permission of Cambridge University Press (not to be unreasonably withheld).

10th December 2010

http://hdl.handle.net/2440/511

J. Fluid Mech. (1995), vol. 300, pp. 59–70
Copyright © 1995 Cambridge University Press

A laminar roughness boundary condition

By E. O. TUCK AND A. KOUZOUBOV†
Applied Mathematics Department, The University of Adelaide, Australia 5005

(Received 23 August 1994 and in revised form 28 April 1995)

A modified slip boundary condition is obtained to represent the effects of small roughness-like perturbations to an otherwise-plane fixed wall which is acting as a boundary to steady laminar flow of a viscous fluid. In its simplest form, for low local Reynolds number and small roughness slope, this boundary condition involves a constant apparent backflow at the mean surface or, equivalently, represents a shift of the apparent plane boundary toward the flow domain. Extensions of the theory are also made to include finite local Reynolds number and finite roughness slope.

1. Introduction

When a viscous fluid is in steady motion with x-wise velocity u over a stationary rough wall whose mean surface is the plane $y = 0$, the no-slip boundary condition $u = 0$ is properly applied not on $y = 0$ but rather on the actual rough surface. Nevertheless, for small roughness, the fluid will appear to be at rest at $y = 0$, and near $y = 0$ the flow will be a shear flow of the form $u = \alpha y$, for some apparent wall shear $u_y = \alpha$ determined by the macroscopic flow far from the wall. If we seek to model the leading-order effect of (microscopic) roughness on such a macroscopic flow by applying a boundary condition on $y = 0$, we must expect that this boundary condition involves a non-zero apparent slip velocity.

Indeed, it is intuitively clear that when $\alpha > 0$, the slip velocity must represent a backflow, with $u < 0$ at the mean wall surface $y = 0$; the ability of the x-directed forcing represented by the wall shear α to move fluid in that direction is inhibited by the roughness. It is also intuitively clear that if ϵ is a measure of the amplitude of height of the roughness, this slip velocity is of second order in ϵ; first-order corrections vary only on the (horizontal) roughness lengthscale $1/k$, and average to zero.

The first result obtained here is that, under certain rather restrictive conditions, the appropriate boundary condition is precisely

$$u = -\epsilon^2 k\alpha. \tag{1.1}$$

The quantities ϵ, k can be defined in terms of means in a spectrum of random roughness, but can be thought of temporarily as the actual amplitude and wavenumber respectively in a pure sinusoidal wavy disturbance to the mean plane $y = 0$. The result (1.1) holds in two-dimensional flow when $k\epsilon$ and $\alpha/(k^2\nu)$ are both small, where ν is the kinematic viscosity. The first restriction is that of small-slope roughness (roughness height $\ll$ roughness length). The second assumption is essentially that of low-Reynolds-number flow, on the roughness lengthscale $1/k$, with typical velocity α/k. We shall seek to relax both of these assumptions.

† Present address: Centre for Advanced Numerical Computation in Engineering and Science, The University of NSW, Australia 2052.

We first give a small-ϵ expansion for the flow that will be used as a fundamental inner solution on the roughness lengthscale, before returning to a more general discussion, and extensions to finite-slope roughness and non-small Reynolds number.

There is of course a vast literature on roughness in fluid mechanics. Much of this literature concerns turbulent flow, or is of an experimental nature, and does not address the apparent-slip boundary-condition concern of the present paper. Nevertheless, the general theme is often how microscopic roughness parameters influence macroscopic flows. Textbooks such as Schlichting (1979) cover the classical literature, including (p. 624) the famous Moody diagram giving the influence of roughness on resistance to flow in pipes. Other useful reviews are in Lachmann (1961). Samples of very recent roughness papers with good bibliographies are Benhalilou, Anselmet & Fulachier (1994), Choudhari (1994), Mazouz, Labraga & Tournier (1994).

The simplest result (1.1) of the present work actually has much more in common with a classical study of G. I. Taylor (1951) on swimming motions of a sinusoidally waving plate. The common feature is that when one carries a perturbation expansion to second order in ϵ, there appears to be a uniform stream far from the wall, which in Taylor's case is interpreted as the swimming speed, but which in the present case is an apparent backflow. The extension to finite local Reynolds number has a similar analogy to inclusion of inertia in the swimming problem, as was treated by Reynolds (1965) and Tuck (1968). The extension to finite roughness slope is also similar to work on Stokes flow over periodic boundaries done in some other contexts, a recent example being Wang (1994). The actual boundary condition (1.1) or its generalizations, is equivalent to the slipping boundary conditions which were in vogue in the early years of viscous fluid theory, and associated with the names of Navier and Helmholtz, as reviewed by Bateman in Dryden, Murnaghan & Bateman (1956, e.g. p. 159).

Some related work has also been done by Jansons (1988), Richardson (1973), and Nye (1969, 1970), using the opposite assumption to that in the present paper, namely that the true (microscopic) boundary condition is one of zero shear rather than exact non-slip, and showing that this leads to a macroscopic boundary condition which to leading order is of a non-slip type, but as a correction allows for a small apparent slip velocity. Richardson (1973) also solves a finite-slope problem with non-zero wall shear as in the present §5, finding an apparent slip velocity proportional to the shear, as in (1.1).

Hocking (1976) carries this Richardson approach further, and finds numerical values for the apparent slip velocity for a sinusoidal wavy wall, as a function of the slope $k\epsilon$ of this sinusoid. By curve-fitting to the small-slope behaviour of these computations, Hocking conjectures an empirical formula whose term of second-order in the slope is in agreement with (1.1). A very recent paper by Miksis & Davis (1994) also obtains an apparent slip velocity for general roughness geometry, but this slip velocity is of first-order, rather than second-order, in the roughness amplitude ϵ. This is consistent with the present results and those of Hocking (1976) in those cases where the origin of coordinates is located at a plane such that the mean roughness is zero, since then the first-order apparent slip predicted by Miksis & Davis would vanish.

2. Shear flow over a wavy wall

The canonical (inner) problem to be solved is as sketched in figure 1. We seek a solution of the Navier–Stokes equations such that $(u, v) \to (\alpha y, 0)$ as $y \to +\infty$, subject to boundary conditions $(u, v) = (0, 0)$ on $y = \epsilon \cos kx$. We use a stream function $\psi(x, y; \epsilon)$ and expand for small ϵ, retaining terms up to order ϵ^2.

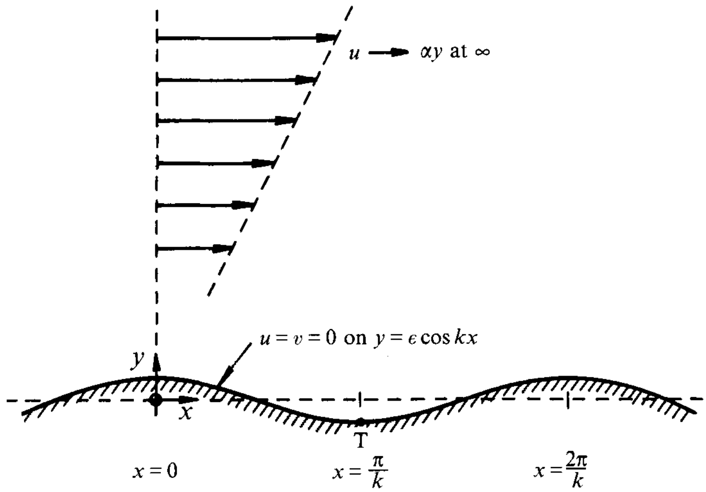

FIGURE 1. Sketch of shear flow over a wavy wall.

That is, we write

$$\psi(x, y; \epsilon) = \tfrac{1}{2}\alpha y^2 + \epsilon\psi_1(x, y) + \epsilon^2\psi_2(x, y) + \dots. \tag{2.1}$$

The result of substitution into the Navier–Stokes equation

$$\frac{\partial(\nabla^2\psi, \psi)}{\partial(x, y)} = \nu\nabla^4\psi \tag{2.2}$$

is

$$\nu\nabla^4\psi_1 - \alpha y\nabla^2\psi_{1x} = 0 \tag{2.3}$$

and

$$\nu\nabla^4\psi_2 - \alpha y\nabla^2\psi_{2x} = J, \tag{2.4}$$

where

$$J = \frac{\partial(\nabla^2\psi_1, \psi_1)}{\partial(x, y)}. \tag{2.5}$$

Similarly, substituting the expansion (2.1) into the exact boundary conditions on $y = \epsilon \cos kx$ yields conditions on $y = 0$ for the separate terms in the expansion, namely

$$\psi_{1y} = -\alpha \cos kx, \quad \psi_{2y} = -\psi_{1yy} \cos kx \tag{2.6}$$

and

$$\psi_{1x} = 0, \quad \psi_{2x} = -\psi_{1xy} \cos kx = -k\alpha \sin kx \cos kx. \tag{2.7}$$

We now assume that the x-dependence is sinusoidal, in the sense that

$$\psi_1(x, y) = \mathrm{Re}[\Psi_1(y)\, \mathrm{e}^{\mathrm{i}kx}], \tag{2.8}$$

and

$$\psi_2(x, y) = \mathrm{Re}[\Psi_2(y)\, \mathrm{e}^{2\mathrm{i}kx}] + \Psi(y) \tag{2.9}$$

for some to-be-determined functions $\Psi_1(y)$, $\Psi_2(y)$, $\Psi(y)$; however, we shall have no interest in the second-harmonic term in Ψ_2.

The Navier–Stokes equations (2.2), (2.3) then give fourth-order ordinary differential equations for these functions of y alone. The equation for Ψ_1 is best expressed in terms of a vorticity function

$$\begin{aligned} W(y) &= \Psi_1''(y) - k^2\Psi_1 \\ &= \mathrm{D}^2\Psi_1(y), \end{aligned} \tag{2.10}$$

where

$$\mathrm{D}^2 = \frac{\mathrm{d}^2}{\mathrm{d}y^2} - k^2. \tag{2.11}$$

Then (2.3) shows that W satisfies the Airy equation

$$\frac{d^2W}{dy^2} - \left(k^2 + \frac{ik\alpha y}{\nu}\right) W = 0. \tag{2.12}$$

Once $W(y)$ is determined by solving the homogeneous second-order ODE (2.12), $\Psi_1(y)$ follows by solving the inhomogeneous second-order ODE (2.10).

Similarly, (2.4) can be used to give equations for $\Psi_2(y)$ and $\Psi(y)$. We are only interested in the latter, and, if an overbar denotes an average with respect to x, the required equation is

$$\nu \frac{d^4\Psi}{dy^4} = \bar{J}(y), \tag{2.13}$$

where J is given by (2.5). After some manipulation, it can be seen that

$$\bar{J}(y) = \frac{d^2}{dy^2}\frac{1}{2}k \operatorname{Re} i\, \Psi_1'(y)\, \Psi_1^*(y), \tag{2.14}$$

where a star denotes a complex conjugate.

A similar x-representation of the boundary conditions gives

$$\Psi_1(0) = 0, \quad \Psi_1'(0) = -\alpha \tag{2.15}$$

and

$$\Psi(0) = 0, \quad \Psi'(0) = -\tfrac{1}{2} \operatorname{Re} \Psi_1''(0), \tag{2.16}$$

noting that the right-hand side of (2.7) has zero x-average.

Our task is to solve the fourth-order ODEs (2.12), (2.13). We need four boundary conditions; we already have two on the mean wall surface $y = 0$. The other two conditions express the fact that we must return to the shear flow $u = \alpha y$ as $y \to \infty$. That is, each term in the stream function expansion (2.1) must be of smaller magnitude with respect to y than the leading $O(y^2)$ term as $y \to \infty$. Hence we may take as our two extra boundary conditions a simple requirement that the second and third derivatives of ψ with respect to y tend to zero at infinity.

In practice, this means for the first-order term Ψ_1 that this function and all of its derivatives must tend to zero, since the only alternative is an exponential growth. On the other hand, no such limitation applies to the mean second-order function $\Psi(y)$. Thus although we do demand that the second and third (and hence all higher) derivatives of $\Psi(y)$ tend to zero at infinity, this still allows $\Psi(y)$ to behave as a linear function of y. Indeed, the non-zero limiting value as $y \to \infty$ of the derivative $\Psi'(y)$ is the prime output of the present study.

Before attempting to solve the above ODE problems, we first discuss the use we shall make of their solution, and in particular of the value of $\Psi'(\infty)$, in the context of an apparent wall boundary condition.

3. Matching conditions

Let us for the time being retain the assumption that the wall is a pure sinusoidal wavy perturbation to the plane $y = 0$, with wavelength $\lambda = 2\pi/k$. In the present section, it is not necessary to assume that this wave has a small slope $k\epsilon$ or ϵ/λ, so long as the roughness has only a small effect on the flow far from the wall. This can be so if both ϵ and λ are small relative to some outer lengthscale L, even if their ratio is not small.

For example, L might be the width of a channel within which occurs a plane

Poiseuille flow. If this flow has (negative) pressure gradient p_x, and is symmetric relative to the mid-plane $y = L/2$ of the channel, then it has velocity

$$u = u_0 + \frac{p_x}{2\rho\nu}(y^2 - Ly), \tag{3.1}$$

where u_0 is the wall slip velocity at $y = 0$. The latter is usually taken to be zero, but we now allow for a more general boundary condition with non-zero slip. The behaviour of this flow near the wall $y = 0$ is thus of the form

$$u = u_0 + \alpha y, \tag{3.2}$$

with

$$\alpha = -L\frac{p_x}{2\rho\nu}. \tag{3.3}$$

Equation (3.2) may be considered as an 'inner expansion of the outer expansion' in the sense of matched asymptotic expansions with respect to a small parameter λ/L. The outer expansion (on the lengthscale L) is one where no actual roughness wave is seen, but where there is an apparent slip velocity u_0 at $y = 0$. The magnitude of this apparent slip velocity can only be determined by matching with an inner expansion, on the lengthscale $\lambda \ll L$. This matching principle (Van Dyke 1975) asserts that (3.2) is also the 'outer expansion of the inner expansion': that is, an apparent outer boundary condition as $y \rightarrow \infty$ (relative to λ), to be applied to the solution of the problem discussed in the previous section.

We have in fact used (3.2) already as our boundary condition as $y \rightarrow \infty$, except that u_0 plays no input role, since it is dominated by the shear term αy. On the other hand, u_0 appears as an output (of second order with respect to ϵ), and takes the value

$$u_0 = \epsilon^2 \Psi'(\infty). \tag{3.4}$$

That is, once we have solved the inner problem for the x-averaged second-order stream function $\Psi(y)$, this immediately yields the apparent slip velocity u_0 for use in the outer problem.

We now consider the effect of a general distribution of small-amplitude roughness. At least for gentle roughness with $k\epsilon$ small, this can be considered as a stationary random process, obtained by summing infinitely many sinusoidal waves of random phase, each with amplitude ϵ proportional to the square root of a power spectrum which is a general function of wavenumber k. The nature of the x-averaging process at second order in ϵ is such that the final contribution to the slip velocity is just proportional to this power spectrum. That is, we have to average (3.4) over all members of the ensemble of sinusoidal waves making up the roughness spectrum.

4. Low Reynolds number limit for small-slope roughness

The formulation of the inner problem for gentle roughness with $k\epsilon$ small in §2 can be rendered non-dimensional using the lengthscale $1/k$ and velocity scale α/k. Although there is no need to do such an explicit scaling, the resulting equations would involve the Reynolds number

$$R = \frac{\alpha}{k^2\nu} \tag{4.1}$$

and the inner problem is consistent when R is order 1. In the present section we discuss the further simplification that results when R is small.

In effect, this is the limit as $\nu \rightarrow \infty$. The Airy ODE (2.12) reduces to a constant-coefficient equation, and the final solution for Ψ_1 satisfying all boundary conditions is simply

$$\Psi_1 = -\alpha y \,\mathrm{e}^{-ky}. \tag{4.2}$$

Turning to the solution for the second-order stream function $\Psi(y)$, we see first that the Jacobian $\bar{J}$ is zero (since Ψ_1 is real), so Ψ must have zero fourth derivative, i.e. must be a cubic expression in y. The y^2 and y^3 terms must vanish by the boundary condition at infinity, and the constant term must be zero by the boundary condition on the wall. Hence Ψ is just a multiple of y, namely

$$\begin{aligned}\Psi(y) &= -\tfrac{1}{2}\mathrm{Re}\,\Psi_1''(0)\,y \\ &= -k\alpha y.\end{aligned} \tag{4.3}$$

Thus we have shown that $\Psi'(\infty) = -k\alpha$, and hence using (3.4) have found the apparent wall slip velocity

$$u_0 = -\epsilon^2 k\alpha, \tag{4.4}$$

verifying (1.1) since (3.2) has $u = u_0$ on $y = 0$. In this low-R limit, the whole $O(\epsilon^2)$ mean flow is nothing more than a uniform stream of magnitude u_0. A similar conclusion was obtained by Taylor (1951) in his study of swimming.

Equation (4.4) holds in the first place for a pure sinusoidal wave of amplitude ϵ and wavenumber k. It extends immediately to a general periodic roughness of zero mean, represented by a Fourier series

$$y = \epsilon \sum_{n=1}^{\infty} a_n \cos nkx + b_n \sin nkx \tag{4.5}$$

for some coefficients a_n, b_n. The pure sine wave has $a_1 = 1$ and all other Fourier coefficients zero. The apparent slip velocity corresponding to the general periodic roughness (4.5) is

$$u_0 = -\epsilon^2 k\alpha \sum_{n=1}^{\infty} n(a_n^2 + b_n^2), \tag{4.6}$$

which reduces to (4.4) for the pure sine wave. The discussion at the end of the previous section can now be formalized to generalize (4.6) further to the case of a non-periodic random roughness, where the coefficients a_n, b_n are suitably random. In a sense, the general result (4.6), whether for periodic or random roughness, may be considered to be still of the simple form (4.4), provided the constant k appearing in it is re-defined as a suitable weighted mean wavenumber of the roughness, and ϵ is the RMS roughness amplitude.

Since $\alpha = u_y$, equation (4.4) can also be written as

$$u + \delta u_y = 0 \tag{4.7}$$

on $y = 0$, where the constant δ is given by

$$\delta = \epsilon^2 k. \tag{4.8}$$

This type of slip boundary condition (4.7) is of some antiquity, being sometimes referred to as Navier's or Helmholtz's boundary condition (Dryden *et al.* 1956, p. 163). In one sense it is trivial, since it is also the condition that one would obtain simply by transferring the no-slip condition from $y = 0$ to $y = \delta$, as discussed by Butcher (1876). Thus if there was a no-slip plane wall at $y = \delta$, we would have $u(x, \delta) = 0$, which is approximated for small δ by the truncated Taylor series $u(x, 0) + \delta u_y(x, 0) = 0$, and this is identical to (4.7). Hence δ may be interpreted as a displacement thickness, and the Navier boundary condition (4.7) simply reflects the role of the roughness in creating a planar displacement of the apparent boundary.

Yet another interpretation of this result is that, for a given non-sinusoidal (especially random) roughness distribution, it may not be obvious in advance what to choose as a fluid-dynamically significant 'mean' plane. What we have shown is that if $y = 0$ defines the geometric mean surface (i.e. a plane with as much volume of roughness peaks above it as roughness troughs below it), the flow behaves as if there were a non-slip plane boundary at $y = \delta$ rather than at $y = 0$. Intuitively, this apparent plane boundary is nearer to the peaks of the roughness than it is to the troughs, since (at least in the low-Reynolds-number limit) fluid trapped in the microscopic roughness pores appears to the macroscopic flow as if it were almost solid, so effectively removing some of the deeper troughs.

In this context, it is interesting to note that Hocking (1976) locates his origin of coordinates for a pure-sinusoidal wavy wall not at the mean surface, but rather at the crests of the sine wave. He then finds an apparent slip coefficient which contains a (positive) term of the first order in the wave amplitude, but this first-order term simply shifts the apparent origin of coordinates back to the geometric mean surface. The remainder of Hocking's apparent slip coefficient is negative, and to leading (second) order in wave slope agrees with the present results. The correct location for a fluid-dynamically significant apparent mean surface is neither at the geometric mean plane nor at the roughness crests, but somewhere in between.

5. Low Reynolds number solution for finite-slope periodic roughness

To illustrate the effect of the small-slope assumption, let us relax it while retaining the low-Reynolds-number assumption. Suppose that the boundary is a wall $y = f(x)$ of a general periodic nature, the function $f(x)$ being specified on a half-wavelength $0 < x < \pi/k$. Our main example (for which figure 1 remains relevant) is the same sinusoidal wave $f(x) = \epsilon \cos kx$ as was used earlier, but without the assumption that $k\epsilon$ is small.

We shall attempt to solve the biharmonic equation

$$\nabla^4 \psi = 0 \tag{5.1}$$

for the stream function $\psi(x, y)$, assuming symmetry with respect to x at both planes $x = 0$ and $x = \pi/k$, and zero velocity $\psi_x = \psi_y = 0$ on the wall $y = f(x)$. The boundary condition at infinity is that we recover the shear flow (3.2).

There are a number of techniques for solving the biharmonic equation (5.1) numerically, subject to these boundary conditions. In particular, finite difference methods are suitable and of good general applicability, but with the defect that the grid must in some way be extended toward $y = \infty$. An alternative (cf. Wang 1994; Tuck & Schwartz 1994) is to use a truncated eigenfunction expansion, as follows. Richardson (1973) used a special inverse solution for a similar purpose.

A representation as a sum of elementary biharmonic functions with the appropriate symmetries is

$$\psi(x, y) = \tfrac{1}{2}\alpha y^2 + u_0 y + \sum_{j=1}^{\infty} [a_j + y b_j]\, \mathrm{e}^{-jky} \cos(jkx), \tag{5.2}$$

where u_0, a_j, b_j are all constants to be determined. Since the series part of (5.2) tends to zero exponentially fast as $y \to \infty$, (3.2) holds in that limit, upon y-differentiation.

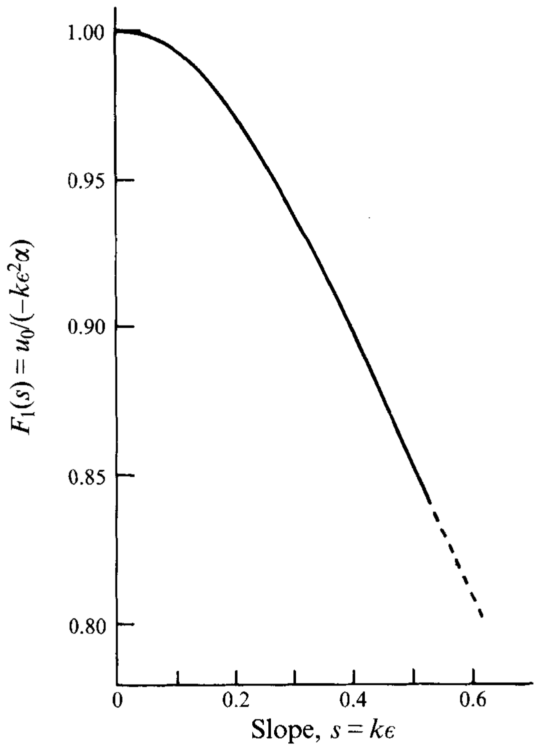

FIGURE 2. Reduction factor on apparent slip velocity, due to finite roughness slope.

All we have left to do is to force (5.2) to satisfy the zero-velocity conditions on the wall $y = f(x)$. Hocking (1976) did this semi-analytically for the special sinusoidal case, expanding in a double series and collecting Fourier terms. We do it here in the somewhat cruder way, collocating at N distinct points inside the interval $0 < x < \pi/k$, and truncating the series (5.2) after the Nth term. There are then a total of $2N+1$ unknowns, namely u_0 and (a_j, b_j), $j = 1, 2, \ldots, N$. We get $2N$ equations by this interior collocation, and add one more by demanding that $u = 0$ on the wall at the right-hand end of the interval $x = \pi/k$. Note that $v = 0$ already at both ends, by symmetry; the choice of making $u = 0$ at the right-hand end is arbitrary, and consistent numerical results were also obtained using the left-hand end $x = 0$.

The method was tested on the sinusoidal wave with amplitude ϵ. The results for $u_0/(-k\epsilon^2\alpha)$ are given as a function of the slope $k\epsilon$ in figure 2. The small-slope theory indicates that this ratio should be 1, and this is indeed the limit as $k\epsilon \rightarrow 0$. The present method gives good results for moderate slopes (in agreement with Hocking 1976), but fails suddenly when the nonlinearity becomes too severe, specifically when $k\epsilon > 0.5$, because the exponentials in late terms of the series (5.2) overflow at the troughs where $y < 0$. Meanwhile, however, it indicates that nonlinearity reduces the apparent backflow u_0 by up to about 15% at just below the failure point.

Failure of the method occurs just at the point where (according to small-slope theory) actual inner-region backflow would begin in the troughs of the roughness wave. To illustrate this, figure 3 shows a plot of the computed wall shear $u_y = \psi_{yy}(\pi/k, -\epsilon)$ at the extreme point of the trough in the roughness wave (point T of figure 1), as a function of the wave slope. It is not hard to use the small-slope theory (carrying the analysis a step further to give the second-harmonic term $\Psi_2(y)$ in (2.9))

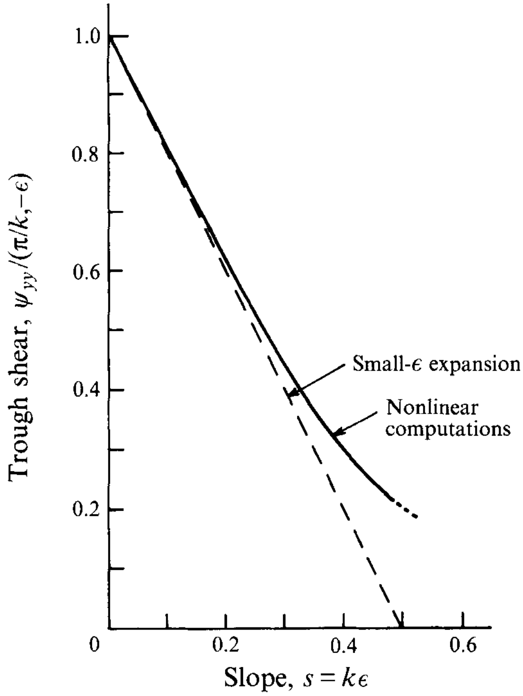

FIGURE 3. Actual wall shear at the extreme trough point, as a function of roughness slope.

to show that, according to that theory, the wall shear at the trough decreases linearly with wall slope, becoming negative for $k\epsilon > 0.5$. This is shown by the dashed line on figure 3. If that conclusion was sustained by the nonlinear theory, local backflow would occur, and there would be a standing eddy in the trough whenever $k\epsilon > 0.5$. However, the actual nonlinear computations, the solid curve in figure 3, suggest that the wall shear remains positive beyond $k\epsilon = 0.5$, up to significantly greater slopes, perhaps for ever.

This question could only be settled properly by use of a numerical method such as finite differences which handled extreme nonlinearity better than does the series expansion (5.2). However, it is notable that Wang (1994) does obtain standing eddies for walls with a periodic array of perpendicular plate projections, using a series truncation method. We have carried out some preliminary finite-difference computations for a stepped (square-wave) wall, giving apparent-slip results similar to those obtained by the series method, and indicating reversed flow in the troughs at sufficiently high steps. However, further work is needed to confirm this trend; for such a stepped roughness, the series in (4.6) diverges, so that there is no finite small-ϵ limit to the ratio u_0/ϵ^2.

In effect, the present results generalize (1.1) to

$$u = -\epsilon^2 k\alpha F_1(k\epsilon) \tag{5.3}$$

for some function $F_1(s)$ such that $F_1(0) = 1$. Our results as in figure 2 suggest that $F_1(s) < 1$ for slopes $s > 0$, at least for sinusoidal perturbations. The program has also been tested on some other periodic perturbations, e.g. a saw-tooth wall, with a similar qualitative conclusion of reduced backflow due to finite-slope effects. The nonlinear dependence of the slip velocity on the wavenumber k makes it difficult to extend the present results to random (non-periodic) roughness.

6. Finite Reynolds number solution for small-slope roughness

The general solution of the ODE (2.12) can be expressed as a linear combination of Airy functions, or equivalently in terms of Bessel functions of order 1/3. Choosing the latter approach, a particular solution is

$$W = W_0(y) = z^{1/3} K_{1/3}(z), \tag{6.1}$$

where K is the modified Bessel function of the third kind (Abramowitz & Stegun, 1964, p. 374), and

$$z = \frac{2}{3}\left(\frac{ik\alpha}{\nu}\right)^{1/2}\left(y - \frac{ik\nu}{\alpha}\right)^{3/2}. \tag{6.2}$$

The general solution of (2.12) is a linear combination of the solution W_0 defined in (6.1) and another linearly independent solution involving the first-kind modified Bessel function $I_{1/3}(z)$. However, we exclude the latter on the basis that it becomes unbounded as $y \to +\infty$ (assuming $\operatorname{Re} z > 0$).

Thus the appropriate solution of (2.12) for our purpose is

$$W(y) = aW_0(y) \tag{6.3}$$

for some constant a to be determined. Then (at fixed a) the solution of the inhomogeneous ODE (2.10) for Ψ_1 is

$$\Psi_1(y) = \frac{a}{k}\int_0^y \sinh k(y-t)\, W_0(t)\, \mathrm{d}t - \frac{\alpha}{k}\sinh ky, \tag{6.4}$$

which satisfies both boundary conditions on the plane $y = 0$. However, it is not yet assured of correct behaviour at infinity, and we must choose the constant a so this is so. Namely, if we let $y \to +\infty$ and cancel the exponentially growing term, we find

$$a = \frac{\alpha}{\int_0^\infty \mathrm{e}^{-kt}\, W_0(t)\, \mathrm{d}t}. \tag{6.5}$$

Equation (6.4) subject to (6.5) is the complete solution for the first-order stream function Ψ_1. Let us for the present assume that this function and hence the Jacobian $J(y)$ is fully known. If (2.13) is integrated twice, we have

$$\nu\frac{\mathrm{d}^2\Psi}{\mathrm{d}y^2} = \tfrac{1}{2}k \operatorname{Re} \mathrm{i}\Psi_1'(y)\, \Psi_1^*(y). \tag{6.6}$$

In general, a linear function of y might be added to (6.6), but this is incompatible with the boundary condition at $y = +\infty$; note that the right-hand side of (6.6) tends to zero exponentially rapidly as $y \to +\infty$.

Integrating (6.6) twice more and applying the boundary conditions (2.16) gives

$$\Psi(y) = -\tfrac{1}{2}\operatorname{Re}\Psi_1''(0)\, y + \frac{k}{2\nu}\operatorname{Re}\mathrm{i}\int_0^y (y-t)\, \Psi_1'(t)\, \Psi_1^*(t)\, \mathrm{d}t. \tag{6.7}$$

Hence on differentiation and letting $y \to +\infty$, we obtain our fundamental output quantity

$$\Psi'(\infty) = -\tfrac{1}{2}\operatorname{Re}\Psi_1''(0) + \frac{k}{2\nu}\operatorname{Re}\mathrm{i}\int_0^\infty \Psi_1'(t)\, \Psi_1^*(t)\, \mathrm{d}t. \tag{6.8}$$

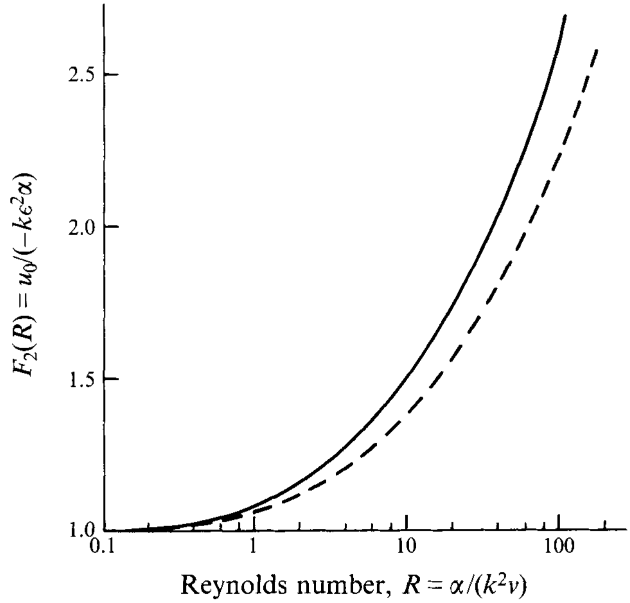

FIGURE 4. Increased apparent slip factor (dashed curve) due to finite Reynolds number effects. Solid curve shows the result of ignoring inertia terms.

The apparent slip velocity is given by (3.4), and $\Psi'(\infty)$ is now known, since (6.8) involves only the quantity $\Psi_1(y)$ which is prescribed by (6.4) and (6.5).

It is notable that, in contrast to the low-R theories, the final results from this finite-R theory depend nonlinearly on the wall shear α. Thus (1.1) generalizes (in a similar manner to (5.3)) to

$$u = -\epsilon^2 k\alpha F_2\left(\frac{\alpha}{k^2\nu}\right) \tag{6.9}$$

for some function $F_2(R)$ such that $F_2(0) = 1$. This function can be determined by numerical integration of equations (6.8). We used Simpson's rule on the integrals in (6.4), (6.5) and (6.7), with appropriate series or asymptotic representations for the Bessel function in (6.1).

Our numerical results are given in the dashed curve of figure 4. We find that $F_2(R) > 1$ for $R > 0$: that is, the apparent slip velocity is increased by finite-Reynolds-number effects. This increase is mainly due to the first term of (6.8), which arises from boundary-condition corrections. The second term of (6.8), which arises from the quadratic inertia terms in the Navier–Stokes equations, has the effect of reducing this increase. This conclusion about the negative effect of the inertia terms is similar to that obtained by Tuck (1968) for the Taylor swimming problem. In that problem, the net result was a decreased swimming speed, whereas an earlier study by Reynolds (1965) which omitted the second (inertia) term had reached an opposite conclusion. Figure 4 also shows (solid curve) the result in the present case that would hold if only the first term of (6.8) was retained. The present numerical computations have been checked by a low-R asymptotic expansion, showing that $F_2 \approx 1 + R^2/8$; if only the first term of (6.8) is retained, the result is $F_2 \approx 1 + 3R^2/16$.

The nonlinear dependence of (6.9) on α makes it difficult to use the Butcher interpretation of the boundary condition in the form (4.7) as a shift of the apparent plane boundary. As with the finite-slope extension (5.3), the nonlinear dependence on wavenumber k also makes it difficult to extend these sinusoidal results by summation to the case of random roughness.

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1964 *Handbook of Mathematical Functions*. Dover.

BENHALILOU, M., ANSELMET, F. & FULACHIER, L. 1994 Conditional Reynolds stress on a V-grooved surface. *Phys. Fluids* **6**, 2101–2117.

BUTCHER, J. G. 1876 On viscous fluids in motion. *Proc. Lond. Math. Soc.* (1) **8**, 103–135.

CHOUDHARI, M. 1994 Roughness-induced generation of crossflow vortices in three-dimensional boundary layers. *Theor. Comput. Fluid Dyn.* **6**, 1–30.

DRYDEN, H. L., MURNAGHAN, F. D. & BATEMAN, H. 1956 *Hydrodynamics*. Dover.

HOCKING, L. M. 1976 A moving fluid interface on a rough surface. *J. Fluid Mech.* **76**, 801–817.

JANSONS, K. M. 1988 Determination of the macroscopic (partial) slip boundary condition for a viscous flow over a randomly rough surface with a perfect slip microscopic boundary condition. *Phys. Fluids* **31**, 15–17.

LACHMANN, G. V. 1961 *Boundary Layer and Flow Control*, Vol. 2. Pergamon.

MAZOUZ, A., LABRAGA, C. & TOURNIER, C. 1994 Behaviour of the Reynolds stress on rough walls. *Exp. Fluids* **17**, 39–44.

MIKSIS, M. J. & DAVIS, S. D. 1994 Slip over rough and coated surfaces. *J. Fluid Mech.* **273**, 125–139.

NYE, J. F. 1969 A calculation of the sliding of ice over a wavy surface using a Newtonian viscous approximation. *Proc. R. Soc. Lond.* A **311**, 445–467.

NYE, J. F. 1970 Glacier sliding without cavitation in a linear viscous approximation. *Proc. R. Soc. Lond.* A **315**, 381–403.

REYNOLDS, A. J. 1965 The swimming of minute organisms. *J. Fluid Mech.* **23**, 241–260.

RICHARDSON, S. 1973 On the no-slip boundary condition. *J. Fluid Mech.* **59**, 707–719.

SCHLICHTING, H. 1979 *Boundary Layer Theory*, 7th edn. McGraw-Hill.

TAYLOR, G. I. 1951 Swimming of a waving plate. *Proc. R. Soc. Lond.* A **209**, 447–461.

TUCK, E. O. 1968 A note on a swimming problem. *J. Fluid Mech.* **31**, 305–308.

TUCK, E. O. & SCHWARTZ, L. W. 1994 The slumping liquid bridge. Preprint report, Applied Mathematics Department, University of Adelaide.

VAN DYKE, M. 1975 *Perturbation Methods in Fluid Mechanics*. Parabolic.

WANG, C. Y. 1994 The Stokes drag due to the sliding of a smooth plate over a finned plate. *Phys. Fluids* **6** 2248–2252.