REFERENCE FALLIBLE ENDGAME PLAY

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ABSTRACT

A reference model of fallible endgame play is defined in terms of a spectrum of endgame players whose play ranges in competence from random to optimal choice of move. They may be used as suitable practice partners, to differentiate between otherwise equi-optimal moves, to promote or expedite a result, to assess an opponent, to run Monte Carlo simulations, and to identify the difficulty of a position or a whole endgame.

1.   INTRODUCTION

When a player is armed with perfect information from an endgame table (EGT), questions still arise as to how to play a fallible opponent. In a drawn position, how best can the attacker pressure the defender or the defender resist the attacker (Levy 1987, 1991; Nunn, 2002; Schaeffer 1991, 1997)? How can the opponent be given the best opportunity to cede depth in a situation where a draw claim is likely (Haworth, 2000, 2001)? How can a player identify and adapt to apparent opponent fallibility? More generally, one might ask how well a fallible player is likely to perform in an endgame, and how 'difficult' that endgame or a specific position is.

A model of fallible endgame play is defined here, providing a way to address these questions. It defines a spectrum of reference endgame players, REPs, with defined levels of competence ranging from random to optimal play. A 'better' move (in terms of value and depth) is at least as likely to be chosen as a 'worse' move.

Section 2 introduces the key concepts while section 3 states the first results. Section 4 demonstrates some basic uses of fallibility and section 5 demonstrates how to assess the competence of the opponent by revisiting the historic Browne-BELLE KQKR games. Section 6 surveys ways of exploiting these reference fallible players and a fallible opponent.

2.   KEY CONCEPTS

White here will, as is customary, be the winning or attacking player: Black is losing or defending. In scenario *BFL*, Black is a fallible player F_c in a lost btm position P of known depth, while White is infallible[2]. Here, Depth to Conversion (DTC)[3] is used (Tamplin, 2002) but DTM(ate), DTZ[4] or DTR[5] are equally valid. Below are listed some basic concepts and notation:

P a btm (wtm) position with n_a successors of depth $a > 0$ ($\leq d$-1), conversions to loss (win) being ignored

$d(P)$ the depth of a position, taken to be max(depth)+ω for a drawn position, say $\omega = 1$

F_c a fallible player with competence level c: specific White and Black fallible players are FW_w and FB_b.

$p_c(a)$ the probability of loser F_c moving to a specific position of depth $a \equiv \theta.a^c$, θ ensuring Σ $p_c(a) = 1$

$q_c(a)$ the probability of winner F_c moving to a specific position of depth $a \equiv \theta.a^{-c}$

$d_c(P)$ E[depth after move by loser F_c from P] = $\Sigma_a n_a.p_c(a).a = (\Sigma_a n_a.a^c.a)/(\Sigma_a n_a.a^c) = (\Sigma_a n_a.a^{c+1})/(\Sigma_a n_a.a^c)$

$e_c(P)$ E[depth after move by winner F_c from P] = $\Sigma_a n_a.q_c(a).a = (\Sigma_a n_a.a^{-c}.a)/(\Sigma_a n_a.a^{-c})$

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[2] Scenario *WFL* has fallible White in a won wtm position against an infallible Black
[3] *Conversion* is effected by mate or by change of force, capture and/or promotion, on the board.
[4] DTZ ≡ Depth to move-count zeroing move, i.e. to mate, force-change or Pawn advance.
[5] DTR ≡ Depth by The Rule .. which zeroes the move-count after a capture or advance of a Pawn (Haworth 2000, 2001).

3.   FIRST RESULTS

The proofs involve only elementary algebra and analysis and are given in Appendix A.

Theorem 1. Scenario *BFL*: $d_c(P) \le d_{c+1}(P)$, i.e. increased competence increases expected depth.
In this sense, FB_{c+1} is better than FB_c. Equality occurs only if all successor positions have the same depth.

Theorem 2. Scenario *BFL*: $d_c(P) \to d(P)$, i.e. $\forall\, r$, $\exists\, c(r)$ such that $c > c(r) \Rightarrow d_c(P) > d(P) - r$.
e.g., for a losing defender with competence $c \ge c(1)$, $d_c(P) > d(P) - 1$.

Theorem 3. Scenario *WFL*: $e_c(P) \ge e_{c+1}(P)$, i.e. increased competence lowers expected depth.

Theorem 4. Scenario *WFL*: $e_c(P) \to d(P) - 1$, i.e. $\forall\, r$, $\exists$ a $c(r)$ such that $c > c(r) \Rightarrow d_c(P) < d(P) - 1 + r$.
Therefore, when facing a winning attacker with competence $\ge c(1)$, speculative defending can be excluded.

Theorem 5. The numbers of positions $n_a \ge 0$ may be modified by weights $w \ge 0$.
Weighting can reflect presumed search-depths or strategies, preferences and/or competencies on the part of the opponent (Jansen, 1992b, 1993), e.g. "plays only value preserving moves", "will always identify a win or loss within 10 plies", "tends to play moves with high tactical threat value" and so on.

4.   BASIC USES OF FALLIBLE PLAYERS

The assumption of a fallible opponent enables a player to distinguish between metric-optimal[6] moves. in the Browne-BELLE position **BB2-22b**, q.v. Table 1 which lists all positions used, the infallible defender Black has a choice between DTC-optimal Rf6 and Rf7. Figures 1-4 have White's competence c as the x-axis and the expected depth, given c, after White's next move as the y-axis. Figure 1 shows that the expected depth after White's reply to Rf7 is greater than after Rf6 – regardless of White's actual competence c.

Position		Position	
BB1-01w	wKa8Qa5/bKf6Re8	**BB2-12b**	wKc5Qc3/bKe2Ra4
BB2-01w	wKc8Qd8/bKc3Rc4	**BB2-22b**	wKe4Qe3/bKg4Rf5
BB1-09b	wKd6Qd3/bKf4Re4	**BB2-24b**	wKe4Qg3/bKh5Rf7
BB2-10b	wKc5Qg6/bKe3Rf4	**J1-1b**	wKe3Qc5/bKf5Ra6

Table 1. KQKR positions.

At position **BB2-10b**, Black has a similar decision between DTC-optimals Ra4 and Re4. As Figure 2 shows, no move dominates across the whole range of c: Ra4 is preferable against poor players but Re4 is significantly better for $c > 6.3$. A probability distribution for c is therefore required: section 5 below addresses this issue.

The ranking of Black's options for $c = 0$ is determined by the average depth of White's choices; as $c \to \infty$ it is determined by the profile of White's best responses to Black's moves. Thus, it is not surprising that the two scenarios shown in Figures 1 and 2 can arise.

For position **BB2-12b**, Black has a choice between DTC-optimals Re4, Rf4, Rg4 or Rh4. The fallibility model shows that Rh4 is never the best option, while Rf4, Re4 and Rg4 are in turn best, respectively, for $c \in [0, 19.7)$, $c \in [19.7, 36]$ and $c > 36$.

For position **BB2-24b**, Black has a choice between DTC-optimals Rb7, Rf6 and Rf8, q.v. Figure 3. The move Rf8 is just dominated throughout by Rf6 which, however, is best only for the range $c \in (4.25, 28.5)$. Rb7 is best against both the zero-skill random mover and the near-infallible player.

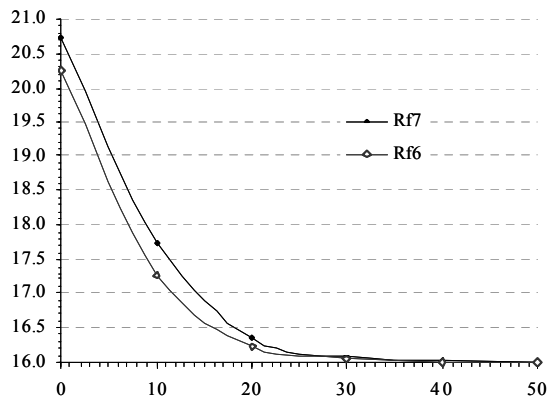[6] ... the metric used here being DTC.

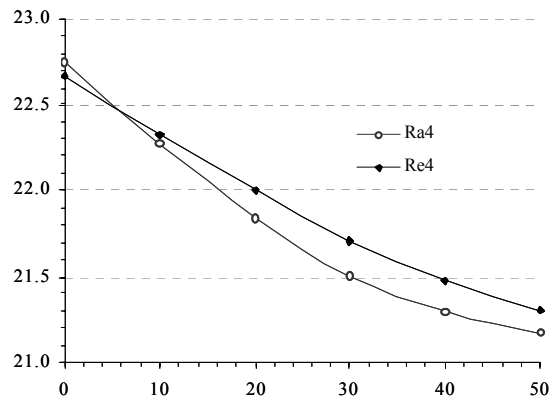**Figure 1.** BB2-22b: Rf7 dominates Rf6.

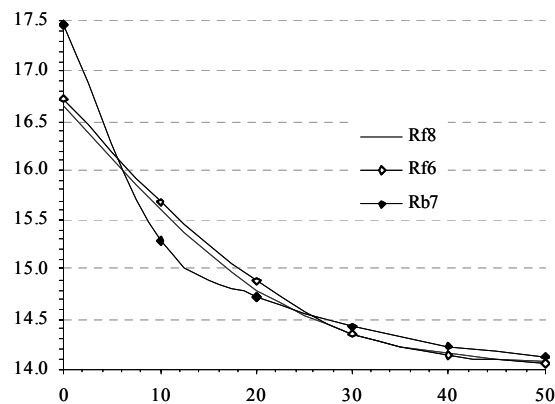**Figure 2.** BB2-10b: Ra4 best ... then Re4.

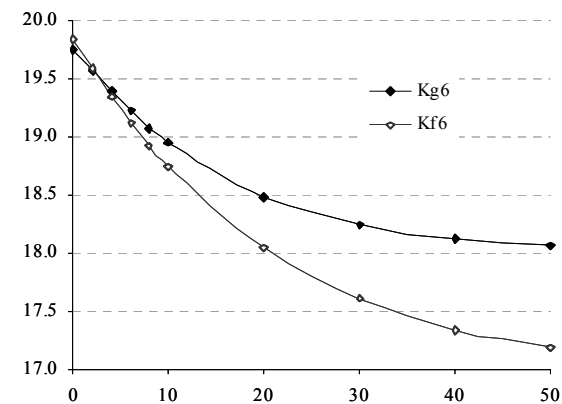**Figure 3.** BB2-24b: Rb7, Rf6 ... and Rb7 best.

Figure 4. J1-1b: Speculative Kf6 only for $c < 2.5$.

Finally, we recall Jansen's (1993) suggestion that DTC-sub-optimal moves should also be considered: he illustrated this with another KQKR position, **J1-1b**. Certainly the move Kf6 which sacrifices one move in depth is at some stage better but, as Figure 4 shows, this is only for F_c with $c < 2.5$. Theorem 2 above helps limit the set of sub-optimal moves that need to be examined. For example, if it is known that $c > 22.5$, Kf6 must be preferred to any move ceding one or more moves in depth. Speculative play has greater potential in the harder endings when fallible opponents are likely to have less competence and lower apparent competence factors c.

5. IDENTIFYING COMPETENCE LEVELS

Perhaps the earliest demonstration of infallible player was in two KQKR games between the sporting volunteer GM Walter Browne and Ken Thompson's BELLE (Fenner, 1979; Jansen, 1992a; Levy and Newborn, 1991), c.f. Appendix B. BELLE, defending, had Thompson's newly calculated KQKR DTC EGT and apparently chose at random between equi-optimal moves.

The two initial positions were maxDTC positions with DTC = 31. BELLE claimed a 50-move draw in the first game but Browne, after a further bout of exhaustive preparation, won exactly on move 50 in the second game. It is intriguing to speculate as to whether BELLE could have squeezed out a second draw. Could it have set bigger problems for Browne by calculating which equi-optimal move to choose, or even by playing suboptimally (Jansen, 1993)? We address the first half of this question and speculate about the second.

Here we propose a way in which BELLE might have developed a probability distribution for Browne's competence c, and the same technique can be used by an infallible observer when both players are fallible.

Let us suppose that BELLE had made a conventional, neutral, 'know nothing' initial assumption about Browne's competence level *c*, for example[7] that *c* was equally likely to be 0, 10, 20, 30, 40 or 50. The initial $1/6^{th}$ probabilities could then be adjusted by the following rule of Bayesian inference:

Posterior_Prob[*c*] $\propto$ Prior_ Prob[*c*] * Prob[observed move | *c*]

and then Σ_cProb(*c*).(Expected depth | *c*) could be calculated for each move option.

This computation, also reasonably assuming Browne would never choose a drawing or losing move, shows that his competence profiles in the two games are remarkably similar. The first fourteen moves elevate E[*c*] = ΣProb(*c*).c towards the maximum possible in the model before a few sub-optimal moves bring it sharply down.
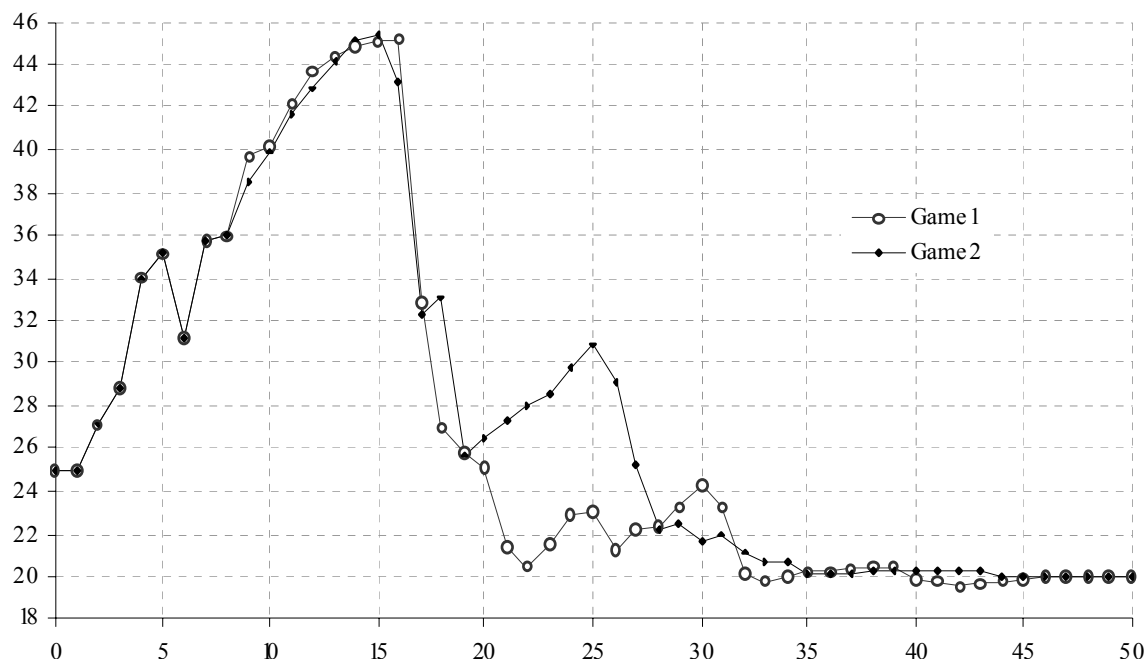

Figure 5. Competency measure *c* of Walter Browne's play during the two KQKR games.

However, in moves 20-22 and 32^+ of the second game, Browne progressed where he had stalled in game 1. Even so, the final E[*c*] values for each game were similar, q.v. Figure 5. Browne ceded DTC depth as follows:

game 1: moves 6w (+1/1), 17w (+3/4), 18w (+2/6), 19w (+1/7), 20w (+1/8), 21w (+2/10), 22w (+1/11),
  26w (+2/13), 31w (+1/14), 32w (+4/18), 33w (+2/20) ... 40w (+3/23), 41w (+2/25), 42w (+2/27)
game 2: moves 6w (+1/1), 16w (+1/2), 17w (+3/5), 19w (+3/8), 26w (+1/9), 27w (+2/11), 28w (+2/13),
  30w (+1/14), 32w (+1/15), 33w (+1/16), 35w (+2/18), 44w (+1/19)

BELLE had equi-optimal choices in game 2 at moves 10b, 12b, 22b and 24b, hence their use in section 4 above. It picked the correct option only on move 22b. Assuming that Browne was roughly equivalent to player F_{20}, the opponent model shows a 0.3 move advantage for the best choices, so perhaps BELLE might have got a second draw. It is unlikely that speculative play would ever have been worthwhile.

In general, a set of fallible players, of which the set $\{F_c\}$ is one example, may be incorporated into a 'PrOM' Probabilistic Opponent-Modelling Strategy (Donkers, Uiterwijk and van den Herik, 2001).

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[7] Relatively course-grain for convenience of calculation: a finer-grain initial *c*-profile is *c* = 0(1)100, probability 1/101.

6. EXPLOITING FALLIBLE PLAYERS

6.1 Endgame Practice Partners

The current 7-hour schedule of play is (40/2º, 20/1º, All/30'), involving fast third-phase rates of play to a finish. Good endgame technique is therefore increasingly important. The fallible players F_c would make ideal practice partners, being available, tireless, house-trained, uncritical and non-deterministic in behaviour. They can comment on moves played, be set manually or automatically to a level of difficulty c, and even help you win if c is set negative. These benefits recommend them as an adjunct to existing chess engines.

6.2 Changing the game result

The most significant use of the F_c is perhaps to change the game score by differentiating between drawn positions. Attacker can pressure fallible defender and equally, defender can resist fallible attacker (Levy 1987, 1991; Nunn, 2002; Schaeffer 1991, 1997). Note that, as usual, the estimated values of positions may be backed up, in this case factored by the calculated probabilities of them occurring.

The suggestion that theoretically drawn positions should be given a depth of max(depth)+ω was inevitably arbitrary on two counts. First, although ω must be positive to make draws less likely for an attacker than wins, it is not otherwise defined. Secondly, not all draws are the same. Blunder moves leading to capture and shallow 'draws in n' are less likely than play which merely retains the draw in the endgame.

6.3 Avoiding draw claims

Haworth (2000, 2001) establishes the DTR, Depth by The Rule, metric as necessary and sufficient for best avoiding k-move draw claims, whether $k = 50$ or not. If the opponent is identified as playing to the DTM or DTC metrics, it may be possible to induce them to sacrifice depth in DTR terms. However, in general, if the attacker assesses the opponent's apparent competence as in section 6.2 above, this will inform their choice of equi-optimal or even sub-optimal moves.

6.4 Monte Carlo Simulation

From a position P, games F_c-F_∞ may be played out to demonstrate how well F_c can progress the win: similarly, games F_∞-F_c examine defensive capability. *Engine*-F_c games can benchmark a chess engine's endgame capability. Matches F_c-F_c may be set up to determine whether, in general, an endgame is harder to win that to defend.

6.5 Assessing Endgame difficulty

Here, we attempt to derive some aggregate characteristics of an endgame as a whole rather than examining play from specific positions.

An endgame is modelled here as a Markov system: depth i corresponds to state i, and endgame play is the Markov process. Averaged across the endgame, a White F_c will move the endgame from depth i to depth j with probability $p_{i,j,c}$: then $\underline{u}_j = [p_{i,j,c} \mid \text{fixed } i]$ is the depth probability vector after White's move. Thus $MW_c = [p_{i,j,c}]$ is a transition matrix for White's moves: MB_c is the corresponding matrix $[q_{i,j,c}]$ for Black's moves. For the infallible players, MW_∞ has $p_{i,i-1,\infty} = 1$ and otherwise 0: MB_∞ is the identity matrix. If $\underline{d}_n$ is a depth probability vector after n moves, $\underline{d}_n.(MW_w.MB_b)^m$ is the same vector after a further m moves by White F_w and Black F_b.

The computation of the matrices MW_c and MB_c are feasible for 3-5-man endgames, given DTC depths ≤ 114 and the the EGT sizes involved. With them, we may address questions such as:

- what is the expected depth $ED_{w,n}$ after n moves of F_w-F_∞ play from a given position?
  Jansen (1992a) compared random KQKR play with optimal play, q.v. his Figures 9 and 10.
- what is the expected number of moves V_w required to win F_w-F_∞ from a given position?

Two theorems, proved in Appendix A, and two likely conjectures follow.

Theorem 6. Let $E[\underline{d}] \equiv$ expected depth associated with probability vector $\underline{d}$. Then $E[\underline{d}.MW_c] \geq E[\underline{d}.MW_{c+1}]$. Symmetrically, $E[\underline{d}.MB_c] \leq E[\underline{d}.MB_{c+1}]$. See Appendix A for the proof.

Theorem 7. If $\underline{u_i} \equiv$ Prob[depth=i]=1, then $E[\underline{u_i}] = i$ and $\exists$ a c^* s.t. $c > c^* \Rightarrow E[\underline{u_i}.MW_c] < i < E[\underline{u_{i+1}}.MW_c] \; \forall \; i$.

Conjecture 1. There is a w^* s.t. $w2 > w1 > w^* \Rightarrow ED_{w1,n} \geq ED_{w2,n}$ and $V_{w1,n} \geq V_{w2,n}$.

Conjecture 2. $\exists \; w = w_d^*$(endgame) s.t. $w > w_d^* \Leftrightarrow F_w$ can secure all wins of depth d, i.e. $ED_{w,n} \to 0$.

Certainly, the graphs $ED_{w,n}$ from the aggregate maxDTC positions should indicate where the difficulties lie in depth terms. The parameters $\{ w_d^* \}$ are characteristics of the endgame and may usefully indicate its relative difficulty.

7.   SUMMARY

A number of questions arise as to how best to attack or defend against a fallible opponent. How best can a result be created or expedited? How competent is the opponent? This note has defined a reference spectrum of fallible endgame players whose use enables these and other questions to be addressed. Some reasonably tractable computations have been defined which could assess the aggregate difficulty of at least 3-5 man endgames.

Thanks goes to John Tamplin for many 3-5-man pawnless endgame DTC EGTs, including KQKR used here.

8.   REFERENCES

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APPENDIX A: PROOFS OF THEOREMS

Theorem 1. Scenario *BFL*: $d_c(P) \le d_{c+1}(P)$, i.e. expected depth after move increases with competence.

Proof. $d_{c+1}(P) - d_c(P) = (\Sigma_a n_a.a^{c+2})/(\Sigma_a n_a.a^{c+1}) - (\Sigma_b n_b.b^{c+1})/(\Sigma_b n_b.b^c)$
$= \{(\Sigma_a n_a.a^{c+2}).(\Sigma_b n_b.b^c) - (\Sigma_a n_a.a^{c+1}).(\Sigma_b n_b.b^{c+1})\}/\{(\Sigma_a n_a.a^{c+1}).(\Sigma_b n_b.b^c)\}$
$= \kappa.\{\Sigma_{a=b}(n_a^2.a^{2c+1} - n_a^2.a^{2c+1}) + \Sigma_{a\neq b} n_a.n_b.a^c.b^c(a^2 + b^2 - 2a.b)\}, \; \kappa > 0$
$= \kappa.\{\Sigma_{a\neq b} n_a.n_b.a^c.b^c(a-b)^2\} \ge 0$, and $= 0$ iff all successors are the same depth
Note that c is not constrained to be either integer or non-negative: F_c with negative skill are conceivable.
Note also, in anticipation of Theorem 5, that the n_a are constrained only to be non-negative.

Theorem 2. Scenario *BFL*: $d_c(P) \to d(P)$, i.e. for each r, there is a $c(r)$ such that $d_c(P) > d(P) - r$.

Proof. Let $d(P) = d$: $d_c(P) = (\Sigma_a n_a.a^{c+1})/(\Sigma_a n_a.a^c)$ for $a \le d$.
Dividing numerator and denominator by d^c ...
$d_c(P) = (\Sigma_a n_a.a^{c+1}/d^c)/(\Sigma_a n_a.a^c/d^c)$
$= (n_d.d + \Sigma_{a<d} n_a.a^{c+1}/d^c)/(n_d + \Sigma_{a<d} n_a.a^c/d^c)$
$\to d = d(P)$ as $c \to \infty$

Theorem 3. Scenario *WFL*: $e_c(P) \ge e_{c+1}(P)$, i.e. expected depth after move decreases with competence.

Proof. $e_c(P) - e_{c+1}(P) = (\Sigma_a n_a.a^{1-c})/(\Sigma_a n_a.a^{-c}) - (\Sigma_b n_b.b^{-c})/(\Sigma_b n_b.b^{-c-1})$
$= \{(\Sigma_a n_a.a^{1-c}).(\Sigma_b n_b.b^{-c-1}) - (\Sigma_a n_a.a^{-c}).(\Sigma_b n_b.b^{-c})\}/\{(\Sigma_a n_a.a^{-c}).(\Sigma_b n_b.b^{-c-1})\}$
$= \kappa.\{\Sigma_{a=b}(n_a^2.a^{1-c}.a^{-c-1} - n_a^2.a^{-c}.a^{-c}) + \Sigma_{a\neq b} n_a.n_b.a^{-c-1}.b^{-c-1}(a^2 + b^2 - 2a.b)\}, \; \kappa > 0$
$= \kappa.\{\Sigma_{a\neq b} n_a.n_b.a^{-c-1}.b^{-c-1}(a-b)^2\} \ge 0$, and $= 0$ iff all successors are the same depth

Theorem 4. Scenario *WFL*: $e_c(P) \to d(P) - 1$, i.e. for each r, there is a $c(r)$ such that $e_c(P) < d(P) + r - 1$.

Proof. The '-1' derives from the convention that depth is counted in winner's move, i.e. White's moves here.
Let $e = d(P) - 1$. $e_c(P) = (\Sigma_a n_a.a^{1-c})/(\Sigma_a n_a.a^{-c})$ for $a \ge e$.
Dividing numerator and denominator by e^{-c} ...
$e_c(P) = (\Sigma_a n_a.a^{1-c}/e^{-c})/(\Sigma_a n_a.a^{-c}/e^{-c})$
$= (n_e.e + \Sigma_{a>e} n_a.a^{1-c}/e^{-c})/(n_e + \Sigma_{a>e} n_a.a^{-c}/e^{-c}) = (n_e.e + \Sigma_{a>e} n_a.e.(e/a)^{c-1})/(n_e + \Sigma_{a>e} n_a.(e/a)^c)$
$\to e = d(P) - 1$ as $c \to \infty$

Theorem 5. The numbers of positions $n_a \ge 0$ may be modified by weights $w \ge 0$.

Proof. The proofs of Theorems 1 & 3 require only that $\forall \, a, b, n_a.n_b \ge 0$. Thus, $n_a \to w.n_a$ is allowable.

Theorem 6. Let $\underline{d}$ be a depth probability vector and let $E[\underline{d}] = E[\Sigma_i d_i.\underline{u}_i]$ = expected depth associated with $\underline{d}$. Then $E[\underline{d}.MW_c] \ge E[\underline{d}.MW_{c+1}]$. Symmetrically, $E[\underline{d}.MB_c] \le E[\underline{d}.MB_{c+1}]$.

Proof. This follows directly from Theorem 3 and, exchanging White for Black, from Theorem 1.
$E[\underline{d}] = E[\Sigma_i d_i.\underline{u}_i]$ is a linear function on the vector space $\{\underline{d}\}$.
Therefore we need only prove the theorem for a typical component of $\underline{d}$.
Let $\underline{u}_i = [0, ..., d_i = 1, 0, ..., 0]$, representing the fact that play is at any position of depth i.
From theorem 3, we have $e_c(P) \ge e_{c+1}(P)$.
Summing over all x positions of depth n:
$\Sigma e_c(P) \ge \Sigma e_{c+1}(P) \Rightarrow (\Sigma e_c(P))/x \ge (\Sigma e_{c+1}(P))/x \Rightarrow E[\underline{u}_i.MW_c] \ge E[\underline{u}_i.MW_{c+1}]$.
Therefore $E[\underline{d}.MW_c] \ge E[\underline{d}.MW_{c+1}]$ for an arbitrary initial state vector $\underline{d}$.

Theorem 7. If $\underline{u}_i \equiv \text{Prob}[\text{depth}=i]=1$, then $E[\underline{u}_i] = i$ and $\exists$ a c^* s.t. $c > c^* \Rightarrow E[\underline{u}_i.MW_c] < i < E[\underline{u}_{i+1}.MW_c] \; \forall \, i$.

Proof. Let P be a position of depth i. By theorem 4, there is a $c'(P)$ s.t. $c > c' \Rightarrow e_c(P) < d(P) = i$.
Therefore, there is a $c''(i)$ s.t. $c > c'' \Rightarrow E[\underline{u}_i.MW_c] < i < E[\underline{u}_{i+1}.MW_c]$.
Therefore, there is a c^* s.t. $c > c^* \Rightarrow E[\underline{u}_i.MW_c] < i < E[\underline{u}_{i+1}.MW_c] \; \forall \, i$.

It is thought that c^* is quite small.
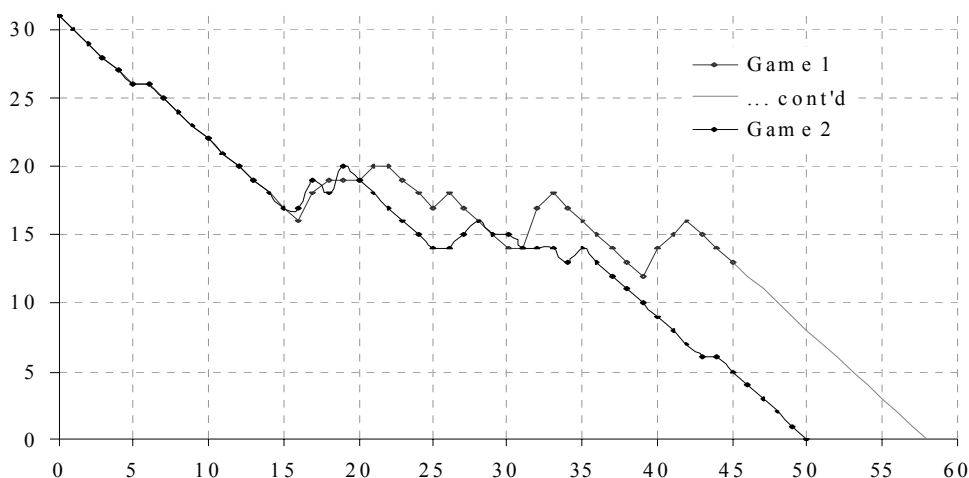
APPENDIX B: THE BROWNE-BELLE GAMES

Figure 6. The two Browne-BELLE games: depth after move *m*.

These games have been annotated with respect to the DTC metric as follows:

> " ≡ only value-preserving move, ' ≡ only optimal move, º ≡ only legal move,
> {+*i*/+*j*: ...} ≡ move ceding *i* moves in depth, making a total loss of *j* moves in depth in the game
> [Re4, Rg4, Rh4] ≡ DTC-optimal moves for Black
> ☺ ≡ DTC equi-optimal move preferred by the model of opponent fallibility

Game 1: {**BB1-1w**: wKa8Qa5/bKf6Re8+w, *dc* = 31} 1. Kb7 Re7+' 2. Kc6 Re6+' 3. Kd7 Re7+' 4. Kd8' Re4' 5. Qc5 Re5' 6. Qd4 {+1/+1: Qc3, Qc4, Qf8+ and Qg1 optimal} Kf5' 7. Kd7 Re4' 8. Qd3 Kf4 ☺ [Ke5] 9. Kd6 Re3 [Re1 ☺] 10. Qd4+ Re4' 11. Qf2+ Kg4' 12. Kd5 Re8 [Re7 ☺] 13. Qf6 Re3 [Re1 ☺] 14. Kd4 Rf3' 15. Qg6+ Kf4' 16. Qg2 Ra3' 17. Qc6 {+3/+4: Qd2+, Qe4+ optimal} Ra1' 18. Qc7+ {+2/+6: Qc2'} Kf5' 19. Qc2+ {+1/+7: Qc8+'} Ke6' 20. Qd2 {+1/+8: Qc4+'} Ra7 ☺ [Ra4+] 21. Qb4 {+2/+10: Qe1+, Qe2+, Qe3+, Qh6+ optimal} Re7' 22. Ke4 {+1/+11: Qa5, Qb3+, Qc3, Qc5, Qe1+ optimal} Kf6+' 23. Kf4' Ke6' 24. Qd4 Rf7+' 25. Ke4' Rf6' 26. Qd5+ {+2/+13: Qd8'} Ke7º 27. Ke5' Rh6 [Rg6 ☺] 28. Qb7+ Kd8' 29. Qf7' Rc6 ☺ [Ra6] 30. Kd5' Rb6' 31. Kc5 {+1/+14: Qf4'} Ra6' 32. Qc4 {+4/+18: Qd5+'} Rf6' 33. Qh4 {+2/+20: Kd5'. *dc* = 18 so Black has the draw} Ke7' 34. Kd5' Kf7' 35. Ke5 Re6+' 36. Kf5' Rd6' 37. Qc4+' Ke7' 38. Ke5' Rh6 ☺ [Rg6] 39. Qc7+' Kf8' 40. Kf5 {+3/+23: Qd7'} Ke8' 41. Qc1 {+2/+25: Qe5+'} Rd6' 42. Qc8+ {+2/+27: Qc7'} Ke7' 43. Qc7+ Rd7' 44. Qc5+' Kd8' 45. Ke6 Rb7' {*dc* = 13: draw agreed}. ½-½.

White ceded 27 moves in depth over moves 6, 17-22, 26, 31-33 and 40-42. Black had DTC-optimal choices on moves 8, 9, 12, 13, 20, 27, 29 and 38, and made the correct choice on only moves 8, 20 and 38.

Game 2 {**BB2-1w**: wKc8Qd8/bKc3Rc4+w, *dc* = 31}1. Kb7 Rb4+' 2. Kc6 Rc4+' 3. Kb5 Rb4+' 4. Ka5' Re4' 5. Qd6 Rd4' 6. Qe5 {+1/+1: Qa3+, Qe6, Qf6 and Qh2 optimal} Kd3' 7. Kb5 Re4' 8. Qf6 Ke3' 9. Kc5 Rf4' 10. Qg6 Ra4 [Re4 ☺] 11. Qg3+ Ke2' 12. Qc3 Rf4 [Re4, Rg4 ☺, Rh4] 13. Kd5' Rh4' 14. Qc2+' Ke3' 15. Qd1' Kf2' 16. Qd2+ {+1/+2: Qd3'} Kf3' 17. Qe1 {+3/+5: Qd3+'} Rg4' 18. Qd1+ Kf4' 19. Qe2 {+3/+8: Qc1+'} Rg5+' 20. Kd4' Rf5' 21. Qe3+ Kg4º 22. Ke4 Rf7 ☺ [Rf6] 23. Qg1+' Kh5' 24. Qg3' Rf8 [Rb7☺, Rf6] 25. Ke5' Rf7' 26. Ke6 {+1/+9: Qd3'} Rf8' 27. Qa3 {+2/+11: Qe5+'} Rf4' 28. Qh3+ {+2/+13: Qg3'} Kg5' 29. Qg3+ Rg4' 30. Qe5+ {+1/14: Qe3+'} Kh4' 31. Qh2+' Kg5º 32. Ke5 {+1/+15: Qd2+'} Kg6' 33. Qh8 {+1/+16: Qc2+ and Qh3 optimal} Rg5+' 34. Ke6' {and now, this pattern repeats at positions 37b, 40b, 43b and 46b) Rg4' 35. Qg8+ {+2/+18: Qe5'} Kh5' 36. Qh7+' Kg5º 37. Ke5' Rg3' 38. Qg7+' Kh4' 39. Qh6+' Kg4º 40. Ke4' Rg2' 41. Qg6+' Kh3' 42. Qh5+' Kg3º 43. Ke3' Rg1' 44. Qg5+ {+1/+19: Qe5', and now White cannot slip again} Kh2' 45. Qh4+' Kg2º 46. Ke2' Ra1' 47. Qe4+ Kh3' 48. Qh7+' Kg3 [Kg2, Kg4] 49. Qg7+' Kh3 50. Qxa1' {Just in time! Bl. resigns} 1-0.

White ceded 19 moves in depth over moves 6, 16-17, 19, 26-28, 30, 32-33, 35 and 44. Black had DTC-optimal choices on moves 10, 12, 22 and 24, and made the correct choice on move 22 only.