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PURITY AND GORENSTEIN FILTERED RINGS

Hiroki Miyahara

Abstract. In this paper, we discuss on the existence of filtrations of modules having good properties. In particular, we focus on filtered homomorphisms called strict, and show that there exists a filtration which makes a filtered homomorphism a strict filtered homomorphism. Moreover, by using this result, we study purity for filtered modules over a Gorenstein filtered ring.

1. Introduction

Filtered ring theory was born in the 1950s as the notion of I-adic filtrations in Zariski rings in commutative ring theory, which forms a foundation of singularity theory. It has been used mainly in the study of differential opperator rings, especially Weyl algebras and of the universal enveloping algebras of Lie algebras in noncommutative stages. These rings have excellent filtrations whose associated graded rings are commutative.

On the other hand, homological methods were established in ring theory in the 1950s. M. Auslander and Bridger [1] gave various characterizations of rings satisfying nice condition, which is now called the Auslander condition. In the case of commutative rings, Bass [2] proved the Auslander condition holds if and only if the ring is Gorenstein, i.e., its self-injective dimension is finite. But in the case of non-commutative rings, R. M. Fossum, A. Griffith and I. Reiten [5] gave an example of a ring with a finite self-injective dimension not satisfying the Auslander condition. The rings satisfying the Auslander condition play an important role in non-commutative algebraic geometry appeared in the late 1980s.

Filtered ring theory was also developed by L. Huishi, F. Van. Oystaeyen, J-.E Björk, E. K. Ekström and others in the case its associated graded ring is Auslander Gorenstein, i.e., a ring satisfying the Auslander condition with a finite self-injective dimension, and was applied to non-commutative algebraic geometry. To generalize the above results, we need to study on the existence of filtrations with good properties like J-.E. Björk and E. K. Ekström [4] showed for a pure module as follows:

Theorem 1.1. ([4, Proposition 5.23]) *Let Λ be a Zariskian filtered ring such that its associated graded ring is Auslander-Gorenstein and M a finitely*

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generated Λ-module. If M is pure, then there exists a good filtration such that the associated graded module of M is also pure.

In this paper, for a finitely generated filtered module M over a filtered ring, we define an invariant $\mathrm{pl}(M)$ which we call *pure length* of M. It holds that $\mathrm{pl}(M) = 1$ if and only if M is pure. The following is a main theorem which generalizes Theorem 1.1.

Theorem 5.4. *Let Λ be a Zariskian filtered ring such that its associated graded ring is Auslander-Gorenstein and M a finitely generated Λ-module. Then there exists a good filtration $\mathcal{F}M$ of M such that* $\mathrm{pl}(M) = \mathrm{pl}(\mathrm{gr}_{\mathcal{F}}M)$ *where* $\mathrm{gr}_{\mathcal{F}}M$ *is the associated graded module of M.*

2. Preliminaries

In this section, we recall some definitions and properties of filtered rings. In the last of this section, we give a natural question for the existence of a good filtration. Throughout this paper, we assume that all rings are associative with identity and that modules over rings are unital left or right modules. A ring Λ is said to be a *filtered ring* if it has a family $\mathcal{F}\Lambda = \{\mathcal{F}_p\Lambda \mid p \in \mathbb{Z}\}$ of additive subgroups of Λ, satisfying $1 \in \mathcal{F}_0\Lambda$, $\bigcup_{p\in\mathbb{Z}} \mathcal{F}_p\Lambda = \Lambda$, $\mathcal{F}_p\Lambda \subset \mathcal{F}_{p+1}\Lambda$ and $(\mathcal{F}_p\Lambda)(\mathcal{F}_q\Lambda) \subset \mathcal{F}_{p+q}\Lambda$ for all $p, q \in \mathbb{Z}$. For a filtered ring Λ with a filtration $\mathcal{F}\Lambda$, $\bigoplus_{p\in\mathbb{Z}} \mathcal{F}_p\Lambda/\mathcal{F}_{p-1}\Lambda$ is a graded ring with multiplication $\sigma_p(x) \cdot \sigma_q(y) = \sigma_{p+q}(xy)$, where $\sigma_p : \mathcal{F}_p\Lambda \longrightarrow \mathcal{F}_p\Lambda/\mathcal{F}_{p-1}\Lambda$ is the canonical map. This graded ring is called the *associated graded ring* of Λ and denoted by $\mathrm{gr}_{\mathcal{F}}\Lambda$ or simply $\mathrm{gr}\Lambda$. An abelian group $\widetilde{\Lambda} = \bigoplus_{p\in\mathbb{Z}} \mathcal{F}_p\Lambda$ is naturally a graded ring which is called the *Rees ring* of Λ.

A left or right Λ-module over a filtered ring with a filtration $\mathcal{F}\Lambda$ is said to be a *filtered Λ-module* if it has a family $\mathcal{F}M = \{\mathcal{F}_pM \mid p \in \mathbb{Z}\}$ of additive subgroups of M such that $\bigcup_{p\in\mathbb{Z}} \mathcal{F}_pM = M$, $\mathcal{F}_pM \subset \mathcal{F}_{p+1}M$ and $(\mathcal{F}_p\Lambda)(\mathcal{F}_qM) \subset \mathcal{F}_{p+q}M$ for all $p, q \in \mathbb{Z}$. If a (Λ, Λ')-bimodule M over filtered rings Λ and Λ' has a family $\mathcal{F}M = \{\mathcal{F}_pM \mid p \in \mathbb{Z}\}$ of additive subgroups of M such that both $({}_\Lambda M, \mathcal{F}M)$ and $(M_{\Lambda'}, \mathcal{F}M)$ are filtered modules, we say that M is a *filtered (Λ, Λ')-bimodule*. For a filtered module with a filtration $\mathcal{F}M$, the graded $\mathrm{gr}\Lambda$-module $\mathrm{gr}_{\mathcal{F}}M = \bigoplus_{p\in\mathbb{Z}} \mathcal{F}_pM/\mathcal{F}_{p-1}M$ is called the *associated graded module* of M and the graded $\widetilde{\Lambda}$-module $\widetilde{M} = \bigoplus_{p\in\mathbb{Z}} \mathcal{F}_pM$ is called the *Rees module* of M. We say that a filtration $\mathcal{F}M$ of M is *discrete* if $\mathcal{F}_pM$ is equal to 0 for some $p \in \mathbb{Z}$. We assume that all the filtrations are discrete in this paper. A filtration $\mathcal{F}M$ of M is called *good* if $\mathrm{gr}_{\mathcal{F}}M$ is a finitely generated $\mathrm{gr}\Lambda$-module. For filtered modules $(M, \mathcal{F}M)$ and $(N, \mathcal{F}N)$ over a filtered ring Λ, a Λ-homomorphism $f \in \mathrm{Hom}_\Lambda(M, N)$ is said to be a filtered morphism of degree p if $f(\mathcal{F}_qM) \subset \mathcal{F}_{p+q}N$ for every $q \in \mathbb{Z}$. In particular, a filtered morphism of degree 0 is called a *filtered homomorphism*.

We denote the set of all filtered morphisms of degree p by $\mathcal{F}_p\mathrm{Hom}_\Lambda(M,N)$. Note that if a filtration $\mathcal{F}M$ of M is good, then any Λ-homomorphism $f : M \longrightarrow N$ has a finite degree (see [6, Proposition 6.6]). Hence, if $\mathcal{F}M$ is good and N is a (Λ,Λ')-bimodule, $\mathcal{F}\mathrm{Hom}_\Lambda(M,N) = \{\, \mathcal{F}_p\mathrm{Hom}_\Lambda(M,N) \mid p \in \mathbb{Z}\,\}$ is a filtration of a Λ'-module $\mathrm{Hom}_\Lambda(M,N)$. We recall a strict homomorphism.

Definition 2.1. Let Λ be a filtered ring. Suppose that $(M,\mathcal{F}M)$ and $(N,\mathcal{F}N)$ are filtered Λ-modules and $f : M \longrightarrow N$ is a filtered homomorphism. f is said to be *strict* if $f(\mathcal{F}_pM) = \mathrm{Im}f \cap \mathcal{F}_pN$ for all $p \in \mathbb{Z}$.

Remark 2.2. (1) The composition of two filtered homomorphisms is filtered homomorphism, but it needs not to be strict even if each of them is strict.

(2) Let $f : M \longrightarrow N$ be a filtered homomorphism. Then f induces canonical additive maps $f_p : \mathcal{F}_pM/\mathcal{F}_{p-1}M \longrightarrow \mathcal{F}_pN/\mathcal{F}_{p-1}N$ given by $x + \mathcal{F}_{p-1}M \longmapsto f(x) + \mathcal{F}_{p-1}N$. It is clear that $\mathrm{gr}f = \oplus_{p\in\mathbb{Z}} f_p$ defines a graded homomorphism from $\mathrm{gr}M$ to $\mathrm{gr}N$. Note that $(\mathrm{gr}g)(\mathrm{gr}f) = \mathrm{gr}(gf)$ for any filtered homomorphisms $f : M \longrightarrow N$ and $g : N \longrightarrow L$. Similarly, $\widetilde{f} = \oplus_{p\in\mathbb{Z}} f|_{\mathcal{F}_pM}$ defines a graded homomorphism from $\widetilde{M}$ to $\widetilde{N}$ and $\tilde{g}\widetilde{f} = \widetilde{gf}$ holds.

Definition 2.3. Let $(\Lambda,\mathcal{F}\Lambda)$ be a filtered ring. By (1) of Remark 2.2, we may define a category $\mathrm{Filt}\Lambda$ whose the objects are the filtered left Λ-modules and the morphisms are the filtered homomorphisms. A category $\mathrm{filt}\Lambda$ is the full subcategory of $\mathrm{Filt}\Lambda$ consisting of all finitely generated filtered Λ-modules with good filtration. By (2) of Remark 2.2, we can see that $\mathrm{gr}(-)$ defines a functor from $\mathrm{Filt}\Lambda$ (resp. $\mathrm{filt}\Lambda$) to the category of all graded $\mathrm{gr}\Lambda$-modules (resp. finitely generated graded $\mathrm{gr}\Lambda$-modules) and $\widetilde{(-)}$ a functor from $\mathrm{Filt}\Lambda$ to the category of all graded $\tilde{\Lambda}$-modules.

The following is well known (see [6, Theorem 4 and Proposition 8]):

Lemma 2.4. *Let* $(*)\ \ L \xrightarrow{f} M \xrightarrow{g} N$ *be a sequence of filtered homomorphisms among filtered modules such that* $gf = 0$. *It holds that:*

(1) *The sequence*

$$\mathrm{gr}(*)\quad \mathrm{gr}L \xrightarrow{\mathrm{gr}f} \mathrm{gr}M \xrightarrow{\mathrm{gr}g} \mathrm{gr}N$$

is exact if and only if $(*)$ *is exact and* f, g *are strict.*

(2) *The sequence*

$$\widetilde{(*)}\quad \widetilde{L} \xrightarrow{\widetilde{f}} \widetilde{M} \xrightarrow{\tilde{g}} \widetilde{N}$$

is exact if and only if $(*)$ *is exact and* f *is strict.*

Definition 2.5. Let $(N, \mathcal{F}N)$ be a filtered Λ-module, M a Λ-module and $f : M \longrightarrow N$ an additive monomorphism. Then we define the filtration $\mathcal{F}M$ of M as follows:

$$\mathcal{F}_pM = f^{-1}(\mathcal{F}_pN) \text{ for each } p \in \mathbb{Z},$$

then it is clear that f is a strict filtered homomorphism. The filtration $\mathcal{F}M$ of M obtained in such a way is called the *induced filtration* by N. Similarly, for a filtered Λ-module $(M', \mathcal{F}M')$, a Λ-module N' and an additive epimorphism $g : M' \longrightarrow N'$, $\mathcal{F}_pN' = g(\mathcal{F}_pM')$ defines a filtrarion of N' such that g is a strict filtered homomorphism. This filtration $\mathcal{F}N'$ of N' also is called the *induced filtration* by M'.

Remark 2.6. It follows from the definition that $\ker f$ and $\mathrm{cok} f$ exists in $\mathrm{Filt}\Lambda$ for every morphism f. Moreover, we can easily check if $\mathcal{F}M'$ is good then so is $\mathcal{F}N'$. Hence, $\mathrm{cok} f$ exists in $\mathrm{filt}\Lambda$ for every morphism f in $\mathrm{filt}\Lambda$. But $\mathcal{F}M$ needs not to be good even if $\mathcal{F}N$ is good. Thus the category $\mathrm{filt}\Lambda$ is not closed under taking kernels.

Definition 2.7. A filtration $\mathcal{F}\Lambda$ of a ring Λ is said to be *Zariskian* if $\widetilde{\Lambda}$ is left and right Noetherian and $\mathcal{F}_{-1}\Lambda$ is contained in the Jacobson radical $J(\mathcal{F}_0\Lambda)$ of the ring $\mathcal{F}_0\Lambda$.

Note that if $\mathcal{F}\Lambda$ is Zariskian, then $\mathrm{gr}\Lambda$ is left and right Noetherian and a good filtration in $\mathrm{filt}\Lambda$ induces a good filtration on Λ-submodules (see [6], p.83). This fact shows that if $\mathcal{F}\Lambda$ is a Zariskian filtraion then $\ker f$ exists in $\mathrm{filt}\Lambda$ for every morphism f in $\mathrm{filt}\Lambda$.

In the rest of this paper, we will assume that Λ is a filtered ring with a discrete Zariskian filtration $\mathcal{F}\Lambda$.

Finally, we give a natural question for existence of a filtration.

Question 1. Let $M \in \mathrm{filt}\Lambda$, N be a Λ-modules and $f : M \longrightarrow N$ a Λ-homomorphism. Does there exist a good filtration $\mathcal{F}N$ of N such that f is a strict filtered homomorphism ?

3. STRICT HOMOMORPHISMS

The aim of this section is to give the positive answer to Question 1. According to Remark 2.2 (1), the composition of two strict filtered homomorphisms need not be strict. It is natural to ask when it is strict. We begin with the following lemma.

Lemma 3.1. *Let $L, M, N \in \mathrm{Filt}\Lambda$ and $f : L \longrightarrow M$, $g : M \longrightarrow N$ be strict filtered homomorphisms.*

(1) *If g is an injection, then gf is strict.*
(2) *If f is a surjection, then gf is strict.*

Proof. (1) Take $(gf)(x) \in \mathrm{Im}(gf) \cap \mathcal{F}_pN$ where $x \in L$, $p \in \mathbb{Z}$. Since $g(f(x)) \in \mathrm{Im}g \cap \mathcal{F}_pN$ and g is strict, there exists $y \in \mathcal{F}_pM$ such that $g(y) = g(f(x))$.

If g is an injection, $f(x) = y$ and there exists $x' \in \mathcal{F}_pL$ such that $f(x') = f(x)$ because f is strict. Therefore $(gf)(x) = (gf)(x') \in (gf)(\mathcal{F}_pL)$ and $(gf)(\mathcal{F}_pL) \supset \mathrm{Im}(gf) \cap \mathcal{F}_pN$. So the proof of (1) is completed.

(2) Take $(gf)(x) \in \mathrm{Im}(gf) \cap \mathcal{F}_pN$ where $x \in L$, $p \in \mathbb{Z}$. Since g is strict, there exists $y \in \mathcal{F}_pM$ such that $g(y) = (gf)(x)$. Hence there exisits $z \in \mathcal{F}_pL$ such that $f(z) = y$ because f is surjective and strict. Thus, $(gf)(x) = (gf)(z) \in (fg)(\mathcal{F}_pL)$ and gf is strict. □

The following is the key lemma to prove the main theorem in this section.

Lemma 3.2.

(1) *Let $M_1, M_2, N \in \mathrm{filt}\Lambda$ and let $f_1 : M_1 \longrightarrow N$, $f_2 : M_2 \longrightarrow N$ be filtered homomorphisms. We make the following pullback diagram from f_1 and f_2:*

$$\begin{array}{ccccc} \ker(\pi_1\nu) & \longrightarrow & P & \xrightarrow{\pi_1\nu} & M_1 \\ {\scriptstyle h}\downarrow & & {\scriptstyle \pi_2\nu}\downarrow & & \downarrow{\scriptstyle f_1} \\ \ker f_2 & \longrightarrow & M_2 & \xrightarrow[f_2]{} & N \end{array}$$

where $\pi_i : M_1 \oplus M_2 \longrightarrow M_i$ $(i = 1, 2)$ are the canonical projections, $P = \ker(f_1\pi_1 - f_2\pi_2)$, $\nu : P \longrightarrow M_1 \oplus M_2$ is the canonical injection and $h = (\pi_2\nu)|_{\ker(\pi_1\nu)}$. Let a filtration $\mathcal{F}P$ of P be induced one by filtered direct sum $M_1 \oplus M_2$ (filtered direct sum means $\mathcal{F}_p(M_1 \oplus M_2) = \mathcal{F}_pM_1 \oplus \mathcal{F}_pM_2$) and filtration $\mathcal{F}(\ker(\pi_1\nu))$ (resp. $\mathcal{F}(\ker f_2)$) be the filtration induced by P (resp. M_2). Then it hold that:

(a) *The isomorphism h is strict.*

(b) *If f_2 is strict, then $\pi_1\nu$ is strict.*

(2) *Let $M, N_1, N_2 \in \mathrm{filt}\Lambda$ and let $g_1 : M \longrightarrow N_1$, $g_2 : M \longrightarrow N_2$ be filtered homomorphisms. We make the following pushout diagram from g_1 and g_2:*

$$\begin{array}{ccccc} M & \xrightarrow{g_1} & N_1 & \longrightarrow & \mathrm{cok} g_1 \\ {\scriptstyle g_2}\downarrow & & {\scriptstyle \pi\nu_1}\downarrow & & \downarrow{\scriptstyle h'} \\ N_2 & \xrightarrow[\pi\nu_2]{} & P' & \longrightarrow & \mathrm{cok}(\pi\nu_2) \end{array}$$

where $\nu_i : N_i \longrightarrow N_1 \oplus N_2$ $(i = 1, 2)$ are the canonical injections, $P' = (N_1 \oplus N_2)/\mathrm{Im}(\nu_1 g_1 - \nu_2 g_2)$, $\pi : N_1 \oplus N_2 \longrightarrow P'$ is the canoniccal epimorphism and h' is induced from $\pi\nu_1$. Let $\mathcal{F}P'$ be the filtration *of P' indeuced by filtered direct sum $N_1 \oplus N_2$ and filtration $\mathcal{F}(\mathrm{cok} g_1)$ (resp. $\mathcal{F}(\mathrm{cok}(\pi\nu_2))$ be the filtration induced by N_1 (resp. P'). Then it hold that:*

(a) *The isomorphism h' is strict.*

(b) *If g_1 is strict, then $\pi\nu_2$ is strict.*

Proof. (1) (a) Take $h(x, y) \in \mathrm{Im} h \cap \mathcal{F}_p(\ker f_2)$ where $(x, y) \in \ker(\pi_1\nu)$ and $p \in \mathbb{Z}$. Then $h(x, y) = y \in \mathcal{F}_p M_2$, $(\pi_1\nu)(x, y) = x = 0$. Of course $0 \in \mathcal{F}_p M_1$, we have $(x, y) \in (\mathcal{F}_p M_1 \oplus \mathcal{F}_p M_2) \cap P = \mathcal{F}_p P$. Therefore $(x, y) \in \mathcal{F}_p(\ker(\pi_1\nu))$ and $h(\mathcal{F}_p \ker(\pi_1\nu)) \supset (\mathrm{Im} h \cap \mathcal{F}_p(\ker f_2)$. So h is strict.

(1) (b) Since ν and π_1 are strict filtered homomorphisms, $\pi_1\nu$ is a filtered homomorphism. Thus $(\pi_1\nu)(\mathcal{F}_p P) \subset \mathrm{Im}(\pi_1\nu) \cap \mathcal{F}_p M_1$ for each $p \in \mathbb{Z}$. Conversely take $(\pi_1\nu)(x, y) \in \mathrm{Im}(\pi_1\nu) \cap \mathcal{F}_p M_1$ where $(x, y) \in \mathcal{F}_q P = P \cap (\mathcal{F}_q M_1 \oplus \mathcal{F}_q M_2)$ for some $q \in \mathbb{Z}$. Then $x = (\pi_1\nu)(x, y) \in \mathcal{F}_p M_1$ and $f_1(x) = f_2(y)$. So $f_2(y) = f_1(x) \in \mathcal{F}_p N$. From the assumption, there exists $y' \in \mathcal{F}_p M_2$ such that $f_2(y) = f_2(y')$. Hence $(x, y') \in \mathcal{F}_p P$ and $(\pi_1\nu)(x, y) = (\pi_1\nu)(x, y')$. Therefore $(\pi_1\nu)(\mathcal{F}_p P) \supset \mathrm{Im}(\pi_1\nu) \cap \mathcal{F}_p M_1$ for each $p \in \mathbb{Z}$ and the proof of (1) is completed.

(2) is dual of (1). □

Now we prove the main theorem in this section. This result is very useful and impotant.

Theorem 3.3. *Suppose that L, N are filtered Λ-module with good filtration $\mathcal{F}L, \mathcal{F}N$ respectively. Let*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be a short exact sequence of Λ-homomorphisms among Λ-modules. Then there exists a good filtration $\mathcal{F}M$ of M and an integer k such that the sequence

$$0 \longrightarrow (L, \mathcal{F}(k)L) \xrightarrow{f} (M, \mathcal{F}M) \xrightarrow{g} (N, \mathcal{F}N) \longrightarrow 0$$

is exact strict with filtered homomorphisms f and g where $\mathcal{F}(k)L$ is the k-shifted filtration of $\mathcal{F}L$ (i.e. $\mathcal{F}(k)_p L = \mathcal{F}_{p+k} L$).

Proof. (1) Since M is finitely generated, M has a good filtration $\mathcal{F}'M$ (see [6, Remark 5.2]). Let $\mathcal{F}'L$ be the filtration of L induced by $(M, \mathcal{F}'M)$. Since Λ is a Zariskian filtered ring, $\mathcal{F}'L$ is good and the identity map $1_L : (L, \mathcal{F}'L) \longrightarrow (L, \mathcal{F}L)$ has the finite degree k. Thus, $1_L : (L, \mathcal{F}'(k)L) \longrightarrow (L, \mathcal{F}L)$ is a filtered homomorphism and $f : (L, \mathcal{F}'(k)L) \longrightarrow (M, \mathcal{F}'(k)M)$

is strict. We make the following pushout diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (L, \mathcal{F}'(k)L) & \xrightarrow{f} & (M, \mathcal{F}'(k)M) & \xrightarrow{g} & (N, \mathcal{F}N) & \longrightarrow & 0 \\
& & {\scriptstyle 1_L}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & (L, \mathcal{F}L) & \xrightarrow[\beta]{} & P & \xrightarrow[\pi]{} & \mathrm{cok}\beta & \longrightarrow & 0.
\end{array}$$

Applying Lemma 3.2 (2) (b), P has a good filtration $\mathcal{F}P$ such that β is strict. Let $\mathcal{F}(\mathrm{cok}\beta)$ be the filtration of $\mathrm{cok}\beta$ induced by P, then h is strict by Lemma 3.2 (2) (a). Since α is an isomorphism, α^{-1} induces another filtration of M such that α^{-1} is strict. Denote this filtration of M by$\mathcal{F}''M$, then we have the following sequence:

$$(L, \mathcal{F}L) \xrightarrow{\beta} (P, \mathcal{F}P) \xrightarrow{\alpha^{-1}} (M, \mathcal{F}''M)$$

Since β is strict and α^{-1} is monomorphism, $\alpha^{-1}\beta = f$ is strict by the lemma 3.1 (1). Similarly, by Lemma 3.1 (1) and (2), $g = h^{-1}\pi\alpha$ is strict because α, h^{-1} are bijections. □

Corollary 3.4. *Let $L, N \in \mathrm{filt}\Lambda$ and*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be a short exact sequence of Λ-homomorphisms among Λ-modules. Then there exists a good filtration of $\mathcal{F}M$ such that

$$0 \longrightarrow \mathrm{gr}_{\mathcal{F}}L \xrightarrow{\mathrm{gr}_{\mathcal{F}}f} \mathrm{gr}_{\mathcal{F}}M \xrightarrow{\mathrm{gr}_{\mathcal{F}}g} \mathrm{gr}_{\mathcal{F}}N \longrightarrow 0$$

is an exact sequence of $\mathrm{gr}_{\mathcal{F}}\Lambda$-homomorphisms among $\mathrm{gr}_{\mathcal{F}}\Lambda$-modules.

Proof. Since $\mathrm{gr}_{\mathcal{F}}L \cong \mathrm{gr}_{\mathcal{F}(k)}L$ as a $\mathrm{gr}\Lambda$-module. So it is due to Lemma 2.4. □

From Theorem 3.3, we get the following result.

Corollary 3.5.

(1) *Let M be a filtered Λ-module with a good filtration $\mathcal{F}M$, N a finitely generated Λ-module and $f : M \longrightarrow N$ an additive monomorphism. Then there exists a good filtration $\mathcal{F}N$ of N such that f is a strict filtered homomorphism.*

(2) *Let N be a filtered Λ-module with a good filtration $\mathcal{F}N$, M a finitely generated Λ-module and $f : M \longrightarrow N$ an additive epimorphism. Then there exists a good filtration $\mathcal{F}M$ of M such that f is a strict filtered homomorphism.*

By Corollary 3.5, we can give the positive answer to Question 1.

Corollary 3.6. *Let M and N be Λ-modules and $f : M \longrightarrow N$ be a Λ-homomorphism.*

(1) *If M is a filtered Λ-module with a good filtration $\mathcal{F}M$, then there exists a good filtration $\mathcal{F}N$ of N such that f is a strict filtered homomorphism.*
(2) *If N is a filtered Λ-module with a good filtration $\mathcal{F}N$, then there exists a good filtration $\mathcal{F}M$ of M which such that f is a strict filtered homomorphism.*

Proof. (1) Consider the following decomposition of f:

$$M \xrightarrow{f} \mathrm{Im} f \xrightarrow{\nu} N$$

where ν is the inclusion map. If M is a filtered Λ-module with a good filtration $\mathcal{F}M$, f induces a good filtration $\mathcal{F}(\mathrm{Im} f)$ of $\mathrm{Im} f$. So there exists a good filtration $\mathcal{F}N$ of N such that ν is strict by Theorem 3.5 (1). Since $f : (M, \mathcal{F}M) \longrightarrow (\mathrm{Im} f, \mathcal{F}(\mathrm{Im} f))$, $\nu : (\mathrm{Im} f, \mathcal{F}(\mathrm{Im} f)) \longrightarrow (N, \mathcal{F}N)$ are strict and ν is a monomorphism, $f : (M, \mathcal{F}M) \longrightarrow (N, \mathcal{F}N)$ is strict from the lemma 3.1.

(2) If N is a filtered Λ-module with good filtration $\mathcal{F}N$, ν induces a good filtration $\mathcal{F}(\mathrm{Im} f)$ of $\mathrm{Im} f$ and there exists a good filtration of $\mathcal{F}M$ of M by Theorem 3.5 (2). Then $f : (M, \mathcal{F}M) \longrightarrow (N, \mathcal{F}N)$ is strict because f, ν are strict and ν is a monomorphism. □

4. Purity of modules over Auslander-Gorenstein Rings

In this section, we will define an invariant for purity of modules. First of all, we recall the definition of Auslander-Gorenstein ring.

Definition 4.1. Let R be a left and right Noetherian ring and M be a finitely generated R-module. We denote the grade of M by $j(M)$, that is, minimum number of i such that $\mathrm{Ext}^i_R(M, R) \neq 0$.

We say M satisfies the *Auslander condition* if $j(U) \geq k$ for any non-negative integer k and any submodule U of $\mathrm{Ext}^k_R(M, R)$. Moreover R is called an *Auslander-Gorenstein ring* if R has finite left and right injective dimension and every finitely generated R-module satisfies the Auslander condition.

An Auslander-Gorenstein ring has many good properties for the grade. For example, the followings are well known (see [4]):

Lemma 4.2. *Let R be an Auslander-Gorenstein ring and M a finitely generated R-module. Then*

(1) *$j(X) \geq j(M)$ for every subfactor X of M.*
(2) *$j(M) = \min\{\, j(N), j(M/N) \,\}$ for every submodule N of M.*

We recall the definition of a pure module.

Definition 4.3. Let R be an Auslander-Gorenstein ring and M a finitely generated R-module. M is called *pure* if $j(N) = j(M)$ for every nonzero submodule N of M.

J-.E. Björk and E. K. Ekström [4] give the following characterization of pure modules:

Theorem 4.4. ([4] Theorem 2.12) *Let R be an Auslander-Gorenstein ring and M a finitely generated R-module. Then M is pure if and only if $\mathrm{Ext}^i_R(\mathrm{Ext}^i_R(M, \Lambda), \Lambda) = 0$ for any $i \neq j(M) \geq 0$.*

In the rest of this paper, we assume that a ring R is an Auslander-Gorenstein ring. We will give some preliminary lemmas.

Lemma 4.5. *Let M be a finitely generated R-module and N, L submodules of M such that $M \supsetneq N \supsetneq L$ and $j(M) < j(L)$. Then $j(M/N) = j(N/L)$ if and only if $j(N) = j(M)$.*

Proof. Since $j(M) < j(L)$, $j(M) = j(M/L)$. If $j(M/N) = j(N/L)$ then it follows from $j(M/L) = \min\{\, j(M/N),\ j(N/L)\,\} = j(N/L)$. Hence

$$j(M) = j(M/L) = j(N/L) \geq j(N).$$

Since $j(M) \leq j(N)$ in general, we have $j(M) = j(N)$.

Conversely, if $j(N) = j(M)$, then $j(N) = \min\{\, j(L),\ j(N/L)\,\} = j(N/L)$. Thus we have

$$j(M/L) = j(M) = j(N) = j(N/L). \quad \square$$

Corollary 4.6. *Let M be a finitely generated R-module and*

$$(*)\ M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{n-1} \supsetneq M_n = \{\,0\,\}$$

a chain of submodules of M. Then the followings are equivalent:

(1) *$j(M_i) < j(M_{i+1})$ for any $1 \leq i \leq n-1$.*

(2) *$j(M_i/M_{i+1}) < j(M_{i+1}/M_{i+2})$ for any $1 \leq i \leq n-2$.*

Lemma 4.7. *Let M be a finitely generated R-module and N a submodule of M such that $j(M) < j(N)$. Then the followings are equivalent:*

(1) *M/N is pure.*

(2) *N is a maximal element of the set $\{\, L \subset M \mid j(M) < j(L)\,\}$.*

Moreover, if N and N' are both maximal elements in (2), then $N = N'$.

Proof. Let L be a submodule of M such that $L \supsetneq N$. Since $j(M) < j(N)$, it follows from the lemma 4.5 that $j(M/N) = j(L/N)$ if and only if $j(L) = j(M)$. Thus (1) and (2) are equivalent.

Assume contrary, if $N \neq N'$, then $N + N' \supsetneqq N$. Since $(N + N')/N' \cong N/(N \cap N')$, we have

$$j(M) < j(N) \leq j(N/(N \cap N')) = j((N + N')/N').$$

Hence $j(N + N') = \min\{\, j(N'), j((N + N')/N')\,\} > j(M)$. But this contradicts to the maximality of N. □

By using the spectral sequence determined by Ischebeck complex, T. Levasseur [7] showed that for any finitely generated R-module M, there exsits a filtration

$$M = \Delta_0(M) \supseteq \Delta_1(M) \supseteq \cdots \supseteq \Delta_{d-1}(M) \supseteq \Delta_d(M) = \{\,0\,\}$$

of submodules of M which satisfies the followings for each $0 \leq i \leq d$:

(1) $\Delta_i(M)/\Delta_{i+1}(M)$ is 0 or a pure module with the grade i,
(2) $\Delta_i(M)/\Delta_{i+1}(M) = 0$ if and only if $\mathrm{Ext}^i_\Lambda(\mathrm{Ext}^i_\Lambda(M, \Lambda), \Lambda) = 0$.

This filtration is called a *dimension filtration* of M. It follows from Lemma 4.6 that $j(\Delta_i(M)) < j(\Delta_{i+1}(M))$ if $\Delta_i(M) \neq \Delta_{i+1}(M)$. Thus we have the following result:

Lemma 4.8. *Let M be a finitely generated R-module. Then there exists a unique chain*

$$(*)\ M = M_0 \supsetneqq M_1 \supsetneqq \cdots \supsetneqq M_{n-1} \supsetneqq M_n = \{\,0\,\}$$

of submodules of M such that M_i/M_{i+1} is pure for each $0 \leq i \leq n-1$.

Proof. Let

$$(*)\ M = M_0 \supsetneqq M_1 \supsetneqq \cdots \supsetneqq M_{n-1} \supsetneqq M_n = \{\,0\,\}$$

be a properly maximal descending subchain of the dimension filtration. Then the chain $(*)$ satisfies the assertion from Lemma 4.7. □

Definition 4.9. Since the length of $(*)$ in Lemma 4.8 depends only on M, we denote this length n by $\mathrm{pl}(M)$.

We give some remarks for $\mathrm{pl}(M)$.

Remark 4.10. Let M be a finitely generated R-module and

$$(*)\quad M = M_0 \supsetneqq M_1 \supsetneqq \cdots \supsetneqq M_{n-1} \supsetneqq M_n = \{\,0\,\}$$

the chain in Lemma 4.8.

(1) It follows from the definition that $\mathrm{pl}(M) = 1$ if and only if M is pure.
(2) The chain $(*)$ is different from critical composition series with respect to canonical dimension in the sense of Levasseur [8]. For a simple R-module S, $S \oplus S$ is pure, but the critical composition length of $S \oplus S$ is equal to 2.

(3) Since the chain $(*)$ is the properly maximal descending subchain of the dimension filtration, we have

$$\mathrm{pl}(M) = \#\{\, i \mid \mathrm{Ext}^i_\Lambda(\mathrm{Ext}^i_\Lambda(M,\Lambda),\Lambda) \neq 0 \,\}.$$

Moreover, set $\mathrm{id}R = d$ and $M_{i,j} = M_i$ for $j \in \mathbb{N}$, then the following chain

$$M = M_{0,0} = \cdots = M_{0,j(M_0)} \supsetneq M_{1,j(M_0)+1} = \cdots = M_{1.j(M_1)} \supsetneq \cdots$$
$$= M_{n-1,j(M_{n-1})} \supsetneq M_{n,j(M_{n-1})+1} = \cdots = M_{n,d} = \{\,0\,\}$$

is the dimension filtration of M.

(4) Let $(**)$ $M_0 \supsetneq M'_1 \supsetneq \cdots \supsetneq M'_{n-1} \supsetneq M'_n = \{\,0\,\}$ be a chain of submodules of M such that $j(M'_i) < j(M'_{i+1})$ for each $0 \le i \le n-1$. Then $n \le \mathrm{pl}(M)$ by Lemma 4.7.

5. Purity of filtered modules over a Gorenstein filtered Ring

The aim of this section is to show the existence of a good filtration $\mathcal{F}M$ of M such that $\mathrm{pl}(M) = \mathrm{pl}(\mathrm{gr}_{\mathcal{F}}M)$. We begin with recalling the definition of Gorenstein filtered ring in the sense of J-.E. Björk and E. K. Ekström [4].

Definition 5.1. Let Λ be a Zariskian filtered ring. We say that Λ is a *Gorenstein filtered ring* if $\mathrm{gr}\Lambda$ is Auslander-Gorenstein.

It follow from the [3, Theorem 3.9] that a Gorenstein filtered ring is Auslander-Gorenstein. In what follows, we assume that a filtered ring Λ is a Gorenstein filtered ring. We show a valuabile inequality for the pure length of M and $\mathrm{gr}_{\mathcal{F}}M$.

Proposition 5.2. *Let* $(M, \mathcal{F}M) \in \mathrm{filt}\Lambda$. *Then* $\mathrm{pl}(M) \le \mathrm{pl}(\mathrm{gr}_{\mathcal{F}}M)$.

Proof. Let

$$(*)\ M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{n-1} \supsetneq M_n = \{\,0\,\}$$

be the chain in Lemma 4.8. For each $1 \le i \le n-1$, let $\mathcal{F}M_i$ be the filtration of M_i induced by $(M, \mathcal{F}M)$. Since $\mathrm{gr}_{\mathcal{F}}M_i$ is isomorphic to a submodule of $\mathrm{gr}_{\mathcal{F}}M$ and $j(M_i) = j(\mathrm{gr}_{\mathcal{F}}M_i)$ for each $i \ge 0$ (see [4, Corollary 5.8] or [9, Theorem 2.8]), $\mathrm{gr}_{\mathcal{F}}M$ has a chain

$$\mathrm{gr}_{\mathcal{F}}M = N_0 \supsetneq N_1 \supsetneq \cdots \supsetneq N_{n-1} \supsetneq N_n = \{\,0\,\}$$

such that $j(N_0) < j(N_1) < \cdots < j(N_n)$. By Remark 4.10 (4), we have $\mathrm{pl}(M) = n \le \mathrm{pl}(\mathrm{gr}_{\mathcal{F}}M)$. □

It is natural to ask when an equality holds in Proposition 5.2. The following example shows that this is not true in general.

Example 5.3. Assume that Λ is the first Weyl algebra $A_1(\mathbb{C})$ and $\mathcal{B}$ is the Bernstein filtration of Λ. we set $\mathcal{F} = \{\, \mathcal{F}_p\Lambda \mid p \in \mathbb{Z}\,\}$ the filtration of ${}_\Lambda\Lambda$ given by

$$\mathcal{F}_p\Lambda = \mathcal{B}_p\Lambda + (\mathcal{B}_p\Lambda)\partial$$

for each $p \in \mathbb{Z}$. Since

$$\partial \in \mathcal{B}_1\Lambda,\ 1 \in \mathcal{F}_0\Lambda,\ \partial \cdot 1 \in \mathcal{F}_0\Lambda,$$

$\overline{\partial} \cdot \overline{1} = \overline{0}$ in $\mathrm{gr}_{\mathcal{F}}\Lambda$. It is well-known $\mathrm{gr}_{\mathcal{B}}\Lambda$ is isomorphic to a polynomial ring with two variables. Since $\mathrm{gr}_{\mathcal{F}}\Lambda \cong k[x,y]/(y) \oplus k[x,y]$ as $k[x,y]$-module and $j(k[x,y]/(y)) > 0$, we see that $\mathrm{gr}_{\mathcal{F}}\Lambda$ is not pure. On the other hand, we can show easily that ${}_\Lambda\Lambda$ is pure.

Now we give a proof of the main theorem.

Theorem 5.4. *Let Λ be a Zariskian filtered ring such that its associated graded ring is Auslander-Gorenstein and M a finitely generated Λ-module. Then there exists a good filtration $\mathcal{F}M$ of M such that* $\mathrm{pl}(M) = \mathrm{pl}(\mathrm{gr}_{\mathcal{F}}M)$.

Proof. We prove by induction on $\mathrm{pl}(M)$. In the case of $\mathrm{pl}(M) = 1$, i.e., M is pure, it is proved by J-.E. Björk and E. K. Ekström [4]. Assume that $\mathrm{pl}(M) = n$. Take a submodule N of M such that $j(M) < j(N)$ and M/N is pure. Then $\mathrm{pl}(N) = n - 1$. By induction hypthesis, there exists a filtration $\mathcal{F}N$ of N such that $\mathrm{pl}(\mathrm{gr}_{\mathcal{F}}N) = n - 1$. Since M/N is pure, there exists a filtration $\mathcal{F}(M/N)$ of M/N such that $\mathrm{gr}_{\mathcal{F}}(M/N)$ is pure. Thus it follows from Collorary 3.4 that there exists a filtration $\mathcal{F}M$ of M such that the sequence

$$0 \longrightarrow \mathrm{gr}_{\mathcal{F}}N \longrightarrow \mathrm{gr}_{\mathcal{F}}M \longrightarrow \mathrm{gr}_{\mathcal{F}}(M/N) \longrightarrow 0$$

is exact. Then $\mathrm{pl}(\mathrm{gr}_{\mathcal{F}}M) = n$. Hence the proof is completed. □

We can see the following from Remark 4.10(3) and Theorem 5.4.

Corollary 5.5. *Let M be a finitely genertated Λ-module. Then there exists a good filtration $\mathcal{F}M$ of M such that* $\mathrm{Ext}^i_\Lambda(\mathrm{Ext}^i_\Lambda(M,\Lambda),\Lambda) = 0$ *if and only if* $\mathrm{Ext}^i_{\mathrm{gr}_{\mathcal{F}}\Lambda}(\mathrm{Ext}^i_{\mathrm{gr}_{\mathcal{F}}\Lambda}(\mathrm{gr}_{\mathcal{F}}M, \mathrm{gr}_{\mathcal{F}}\Lambda), \mathrm{gr}_{\mathcal{F}}\Lambda) = 0$.

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Faculty of Engineering
University of Yamanashi,
4-3-11, Takeda, Kofu,
Yamanashi 400-8511, JAPAN
e-mail address: hmiyahara@yamanashi.ac.jp

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