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ON MONO-INJECTIVE MODULES AND MONO-OJECTIVE MODULES

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Abstract. In [5] and [6], we have introduced a couple of relative generalized epi-projectivities and given several properties of these projectivities. In this paper, we consider relative generalized injectivities that are dual to these relative projectivities and apply them to the study of direct sums of extending modules. Firstly we prove that for an extending module N, a module M is N-injective if and only if M is mono-N-injective and essentially N-injective. Then we define a mono-ojectivity that plays an important role in the study of direct sums of extending modules. The structure of (mono-)ojectivity is complicated and hence it is difficult to determine whether these injectivities are inherited by finite direct sums and direct summands even in the case where each module is quasi-continuous. Finally we give several characterizations of these injectivities and find necessary and sufficient conditions for the direct sums of extending modules to be extending.

1. Preliminaries

Throughout this paper R will be a ring with identity and all modules considered will be unitary right R-modules. A module M is called *extending* if every submodule of M is essential in a direct summand of M. We use the notation $A \subseteq_e M$ and $B \leq_\oplus M$ to indicate that A is an essential submodule of M and B is a direct summand of M. For a direct sum $M = X \oplus Y$, $p_X : M = X \oplus Y \to X$ denotes the projection of $M = X \oplus Y$ to X.

Let M and N be two modules. M is said to be *essentially N-injective* if every homomorphism with essential kernel from a submodule of N into M extends to M. M is said to be *mono-N-injective* if every monomorphism from a submodule of N into M extends to M. In Section 2, we prove that for an extending module N, a module M is N-injective if and only if M is mono-N-injective and essentially N-injective (Theorem 2.3).

Let M and N be modules. M is said to be *generalized (mono-)N-injective* or *(mono-)N-ojective* if, for any submodule X of N and any homomorphism (monomorphism) $f : X \to M$, there exist the decompositions $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$, a homomorphism (monomorphism) $g_1 : N_1 \to M_1$ and a monomorphism $g_2 : M_2 \to N_2$ satisfying the following property :

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($*$) For any $x \in X$, we express x, $f(x)$ in $N = N_1 \oplus N_2$, $M = M_1 \oplus M_2$ as $x = n_1 + n_2$, $f(x) = m_1 + m_2$. Then $g_1(n_1) = m_1$ and $g_2(m_2) = n_2$.

Let A and B be modules. B is said to be *almost A-injective* if, for any submodule X of A and any homomorphism $f : X \to B$, there exists a homomorphism $g : A \to B$ with $g|_X = f$, or there exist a non-zero direct summand A' of A and a homomorphism $h : B \to A'$ with $h \circ f = p_{A'}|_X$, where $p_{A'}$ is the projection of A onto A'. The almost injectivities are useful for the study of direct sums of uniform modules (cf. [1]). In the case that A is indecomposable, we note that B is A-ojective if and only if B is almost A-injective.

A module M is said to be *weakly generalized mono-N-injective* or *weakly mono-N-ojective* if, for any submodule X of N and any monomorphism $f : X \to M$, there exist an essential submodule Y of X, decompositions $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$ and monomorphisms $g_1 : N_1 \to M_1$, $g_2 : M_2 \to N_2$ satisfying the condition ($*$) for Y, that is,

($*$) For any $y \in Y$, we express y, $f(y)$ in $N = N_1 \oplus N_2$, $M = M_1 \oplus M_2$ as $y = n_1 + n_2$, $f(y) = m_1 + m_2$. Then $g_1(n_1) = m_1$ and $g_2(m_2) = n_2$.

Note that M is (weakly) mono-N-ojective iff N is (weakly) mono-M-ojective. In Section 3, we study several properties of relative mono-ojectivity and find necessary and sufficient conditions for the direct sum of extending modules to be extending in terms of the (weakly) mono-ojectivity (see, Theorem 3.3).

Let $M = M_1 \oplus M_2$ and let $\varphi : M_1 \to M_2$ be a homomorphism. Put $\langle M_1 \overset{\varphi}{\to} M_2 \rangle = \{m_1 - \varphi(m_1) \mid m_1 \in M_1\}$. Then this is a submodule of M which is called the *graph* with respect to $M_1 \to M_2$. Note that $M = M_1 \oplus M_2 = \langle M_1 \overset{\varphi}{\to} M_2 \rangle \oplus M_2$.

For undefined terminologies, the reader is referred to [2], [9] and [12].

The following give several properties of relative essentially injective modules and relative ojective modules.

Proposition 1.1. (cf.[3, pp.16-17])

(1) Let A and B be modules. If A is essentially B-injective, then A is essentially C-injective for any submodule C of B.

(2) Let A be a module and let $\{B_i \mid i \in I\}$ be a family of modules. Then A is essentially $\oplus_I B_i$-injective if and only if A is essentially B_i-injective for all $i \in I$.

(3) Let $A_1, \cdots, A_n, B$ be modules. Then $A_1 \oplus \cdots \oplus A_n$ is essentially B-injective if and only if A_i is essentially B-injective for all $i \in \{1, \cdots, n\}$.

Proposition 1.2. (cf.[8, Proposition 1.4], [10, Proposition 3.8]) *Let A and B be modules. Then*

(1) If A is B-ojective, then A is essentially B-injective.

(2) If A is B-ojective, then A' is B'-ojective for any $A' <_\oplus A$ and $B' <_\oplus B$.

In general, an essentially B-injective module need not be B-ojective. For example, let B be an injective module with exactly one nonzero proper submodule S and let A be an indecomposable non-extending module that contains a simple submodule not isomorphic to S. Then A is essentially B-injective, but not B-ojective ([7, Example 2.3]).

2. Mono-injective modules

We recall the definition of relative mono-injectivity. Let M and N be modules. M is said to be *mono-N-injective* if, for any submodule X of N and any monomorphism $f : X \to M$, there exists a homomorphism $g : N \to M$ with $g|_X = f$.

Clearly mono-injectivities are inherited by direct summands as follows:

Proposition 2.1. *Let M and N be modules. If M is mono-N-injective, then M' is mono-N'-injective for any direct summands $M' <_\oplus M$ and $N' <_\oplus N$.*

Proof. Straightforward. □

Lemma 2.2. (cf.[11, Lemma 2.2]) *Let $M = M_1 \oplus M_2$ and let X be a submodule of M. If $X_1 \subseteq_e M_1$ for $X_1 \subseteq X$, then $X \supseteq_e X_1 \oplus (M_2 \cap X)$.*

The following result deals with the connection between injectivity, mono-injectivity and essentially injectivity.

Theorem 2.3. *Let M be a module and let N be an extending module. Then M is N-injective if and only if M is mono-N-injective and essentially N-injective.*

Proof. It is enough to prove "if" part. Let X be a submodule of N, let $f : X \to M$ be a homomorphism. Since N is extending, there exists a decomposition $N = N_1 \oplus N_2$ such that $\ker f \subseteq_e N_1$. By Lemma 2.2, we see

$$X \supseteq_e \ker f \oplus (N_2 \cap X) \quad \cdots \ (a).$$

Since M is mono-N_2-injective, there exists a homomorphism $g : N_2 \to M$ such that $g|_{(N_2 \cap X)} = f|_{(N_2 \cap X)}$. Define $g^* : N = N_1 \oplus N_2 \to M$ by $g^*(n_1 + n_2) = g(n_2)$ and put $\varphi = (g^*|_X) - f$. Let $0 \neq x \in X$. By (a), there exists $r \in R$ such that $0 \neq xr \in \ker f \oplus (N_2 \cap X)$, xr can be expressed as $xr = k + y$ with $k \in \ker f$, $y \in N_2 \cap X$. Then $\varphi(xr) = g^*(xr) - f(xr) = g(y) - f(k+y) = f(y) - f(y) = 0$ and so $\ker \varphi \subseteq_e X$.

Since M is essentially N-injective, there exists a homomorphism $\psi : N \to M$ such that $\psi|_X = \varphi$. Put $h = g^* - \psi$. Then, for any $x \in X$, $h(x) = g^*(x) - \psi(x) = g^*(x) - \varphi(x) = g^*(x) - (g^*(x) - f(x)) = f(x)$.

Therefore M is N-injective. □

Proposition 2.4. *Let M be an extending module. If M is mono-M-injective, then M is essentially M-injective.*

Proof. Let X be a submodule of M and let $f : X \to M$ be a homomorphism with $\ker f \subseteq_e X$. As M is extending, there exists a decompositon $M = M_1 \oplus M_2$ such that X is essential in M_1. Define $g : X \oplus M_2 \to M$ by $g(x + m_2) = f(x)$ and put $\varphi = 1_{X \oplus M_2} - g$. Since $\ker g = \ker f \oplus M_2 \subseteq_e M$, φ is a monomorphism. So there exists an endomorphism $\varphi^* : M \to M$ with $\varphi^*|_{X \oplus M_2} = \varphi$. Put $\psi = 1_M - \varphi^*$. Then, for any $x \in X$,

$$\psi(x) = x - \varphi^*(x) = x - \varphi(x) = x - (x - g(x)) = g(x) = f(x).$$

Thus M is essentially M-injective. □

We note that, in general, a mono-N-injective module is not essentially N-injective.

Example 2.5. (cf.[5, Example 2.7]) Let S and S' be simple modules with $S \not\simeq S'$. Let M and K be uniserial modules with the following conditions:
(i) $M \cap K = S$,
(ii) $M \supset S \supset 0$, $K \supset K_1 \supset K_2 \supset S \supset 0$,
(iii) $M/S \simeq S$, $K/K_1 \simeq S'$, $K_1/K_2 \simeq S$, $K_2/S \simeq S'$.

Put $N = M + K$. (Using path algebra, we can see that there exist such modules M, N.)

(I) First we show "M is mono-$N/S(= M/S \oplus K/S)$-injective." Let X be a submodule of N/S and let $f : X \to M$ be a monomorphism. In the case of $f(X) = M$, X is an uniserial module with length 2. So we see $X = K_1/S$. Then the tops of M and K_1/S are not isomorphic, a contradiction. Thus we see $f(X) = S$. Since the socle of N/S is $M/S \oplus K_2/S$ and $X \simeq S$, we see $X = M/S$. Thus there exists a homomorphism $g : N/S = M/S \oplus K/S \to M$ such that $g|_X = f$. Thus M is mono-N/S-injective.

(II) Next we show "M is not essentially N/S-injective." Let $f' : K_1/S \to K_1/K_2$ be the canonical epimorphism and let $\epsilon : K_1/K_2 \to S$ be an isomorphism. Then $f = \epsilon \circ f' : K_1/S \to S$ is a homomorphism with $\ker f = K_2/S \subseteq_e K_1/S$. Assume that f is extended to $g : K/S \to M$. If $\mathrm{Im}\ g = S$ and $\ker g = K_1/S$, then $S = \mathrm{Im}\ g \simeq K/K_1 \simeq S'$, a contradiction. If $\mathrm{Im}\ g = M$, then the map $\varphi = \pi \circ g : K/S \to M/S$ is an epimorphism with $\ker \varphi = K_1/S$, where $\pi : M \to M/S$ is the canonical epimorphism. Then $S \simeq M/S = \mathrm{Im}\ \varphi \simeq K/K_1 \simeq S'$, a contradiction. Thus M is not essentially K/S-injective.

Therefore M is mono-N/S-injective, but not essentially N/S-injective.

As an immediate consequence of Theorem 2.3 and Propositon 2.4, we obtain the following:

Theorem 2.6. (cf. [4]) *A module M is quasi-injective if and only if M is extending and mono-M-injective.*

3. Mono-ojective modules

We recall the definition of relative mono-ojectivity.

Definition. Let M and N be modules. M is said to be *generalized (mono-)N-injective* or *(mono-)N-ojective* if, for any submodule X of N and any homomorphism (monomorphism) $f : X \to M$, there exist decompositions $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$, a homomorphism (monomorphism) $g_1 : N_1 \to M_1$ and a monomorphism $g_2 : M_2 \to N_2$ satisfying the following property :

($*$) For any $x \in X$, we express x, $f(x)$ in $N = N_1 \oplus N_2$, $M = M_1 \oplus M_2$ as $x = n_1 + n_2$, $f(x) = m_1 + m_2$. Then $g_1(n_1) = m_1$ and $g_2(m_2) = n_2$.

Note that "mono-ojectivity" is named "symmetrically injective" in [7].

Proposition 3.1. *Let M be a module and let N be an extending module. If M is N-ojective, then M is mono-N-ojective.*

Proof. Let X be any submodule of N and $g : X \longrightarrow M$ a monomorphism. Since N is extending, there exists the decomposition $N = N_1 \oplus N_2$ such that $X \subseteq_e N_1$. By Proposition 1.2, M is N_1-ojective and hence there exist the decompositions $N_1 = \overline{N_1} \oplus \overline{\overline{N_1}}$ and $M = \overline{M} \oplus \overline{\overline{M}}$ together with the homomorphism $h_1 : \overline{N_1} \longrightarrow \overline{\overline{M}}$ and monomorphism $h_2 : \overline{M} \longrightarrow \overline{\overline{N_1}}$ such that for $x = \overline{n_1} + \overline{\overline{n_1}}$ and $g(x) = \overline{m} + \overline{\overline{m}}$ one has $h_1(\overline{n_1}) = \overline{\overline{m}}$ and $h_2(\overline{m}) = \overline{\overline{n_1}}$. Since X is essential in N_1 and g is a monomorphism, we see that h_1 is a monomorphism. Now we have the decompositions $N = (\overline{\overline{N_1}} \oplus N_2) \oplus \overline{N_1}$ and $M = \overline{M} \oplus \overline{\overline{M}}$ together with the monomorphisms $h_1 : \overline{N_1} \longrightarrow \overline{\overline{M}}$ and $h_2 : \overline{M} \longrightarrow \overline{\overline{N_1}} \oplus N_2$ such that for $x = \overline{n_1} + (\overline{\overline{n_1}} + n_2)$ and $g(x) = \overline{\overline{m}} + \overline{m}$ one has $h_1(\overline{n_1}) = \overline{\overline{m}}$ and $h_2(\overline{m}) = \overline{\overline{n_1}} + n_2$. Thus M is mono-N-ojective. □

We recall the definition of relative weakly mono-ojectivity. A module M is said to be *weakly generalized mono-N-injective* or *weakly mono-N-ojective* if, for any submodule X of N and any monomorphism $f : X \to M$, there exist an essential submodule Y of X, decompositions $M = M_1 \oplus M_2$, $N = N_1 \oplus N_2$ and monomorphisms $g_1 : N_1 \to M_1$, $g_2 : M_2 \to N_2$ satisfying the following condition ($*$):

($*$) For any $y \in Y$, we express y, $f(y)$ in $N = N_1 \oplus N_2$, $M = M_1 \oplus M_2$ as $y = n_1 + n_2$, $f(y) = m_1 + m_2$. Then $g_1(n_1) = m_1$ and $g_2(m_2) = n_2$.

In this case, the decompositons $N = N_1 \oplus N_2$, $M = M_1 \oplus M_2$ and monomorphisms $g_1 : N_1 \to M_1$, $g_2 : M_2 \to N_2$ are said to be satisfy the condition $(*)$ for Y.

Since any mono-N-ojective module is weakly mono-N-ojective, for any extending module N, we see the following:

N-injective $\Rightarrow$ N-ojective $\Rightarrow$ mono-N-ojective $\Rightarrow$ weakly mono-N-ojective.

We note that, in general, a mono-N-ojective module is not N-ojective. For example, $\mathbb{Z}/2\mathbb{Z}$ is mono-$\mathbb{Z}/8\mathbb{Z}$-ojective but is not essentially $\mathbb{Z}/8\mathbb{Z}$-injective (not $\mathbb{Z}/8\mathbb{Z}$-ojective).

Theorem 3.2. *Let N be an extending module. Then the module M is N-ojective if and only if M is essentially N-injective and mono-N'-ojective for any direct summand N' of N.*

Proof. "Only if" part is clear by Proposition 1.2.

"If" part: Let X be a submodule of N and let $f : X \to M$ be a homomorphism. By Zorn's lemma, $\Gamma = \{(X_i, f_i) \mid X \subseteq X_i \subseteq N,\ f_i|_X = f\}$ has a maximal element (X^*, f^*). Since N is extending, there exists a decomposition $N = N^* \oplus N^{**}$ such that $\ker f^* \subseteq_e N^*$. Since M is essentially N^*-injective and $\ker f^* \subseteq_e X^* \cap N^*$, there exists a homomorphism $g^* : N^* \to M$ with $g^*|_{X^* \cap N^*} = f^*|_{X^* \cap N^*}$. Let $x^* + n^* \in X^* + N^*$. If $x^* + n^* = 0$, then $n^* = -x^* \in N^* \cap X^*$ and so $g^*(n^*) = -f^*(x^*)$. Thus we can define $\varphi : X^* + N^* \to M$ by $\varphi(x^* + n^*) = f^*(x^*) + g^*(n^*)$. By the maximality of $(X^*,\ f^*) \in \Gamma$, $X^* + N^* = X^*$ and so $N^* \subseteq X^*$. Hence we see $X^* = N^* \oplus (X^* \cap N^{**})$.

As M is mono-N^{**}-ojective, there exist decompositions $N^{**} = \overline{N^{**}} \oplus \overline{\overline{N^{**}}}$, $M = \overline{M} \oplus \overline{\overline{M}}$ and monomorphisms $g_1 : \overline{N^{**}} \to \overline{M}$, $g_2 : \overline{\overline{M}} \to \overline{\overline{N^{**}}}$ such that $g_1(\overline{n^{**}}) = \overline{m}$ and $g_2(\overline{\overline{m}}) = \overline{\overline{n^{**}}}$ for any $y = \overline{n^{**}} + \overline{\overline{n^{**}}} \in X^* \cap N^{**}$ and $f^*(y) = \overline{m} + \overline{\overline{m}}$.

Define $\alpha : N^* \to \overline{\overline{N^{**}}}$ by $\alpha(n^*) = g_2\ p_{\overline{\overline{M}}}\ g^*(n^*)$, where $p_{\overline{\overline{M}}} : M = \overline{M} \oplus \overline{\overline{M}} \to \overline{\overline{M}}$ is the projection. Put $\rho = p_{\overline{M}} \circ g^* \circ \beta : \langle N^* \overset{\alpha}{\to} \overline{\overline{N^{**}}} \rangle \to \overline{M}$, where $\beta : \langle N^* \overset{\alpha}{\to} \overline{\overline{N^{**}}} \rangle \to N^*$ is the canonical isomorphism and $p_{\overline{M}} : M = \overline{M} \oplus \overline{\overline{M}} \to \overline{M}$ is the projection.

Put $M_1 = \overline{M}$, $M_2 = \overline{\overline{M}}$, $N_1 = \langle N^* \overset{\alpha}{\to} \overline{\overline{N^{**}}} \rangle \oplus \overline{N^{**}}$, $N_2 = \overline{\overline{N^{**}}}$, $\varphi_2 = g_2$ and define $\varphi_1 = \rho + g_1 : N_1 \to M_1$ by $\varphi_1((n^* - \alpha(n^*)) + \overline{n^{**}}) = p_{\overline{M}}(g^*(n^*)) + g_1(\overline{n^{**}})$.

For any $x^* \in X^*$, we express x^* in $N = N_1 \oplus N_2$ as $x^* = n_1 + n_2 = (n^* - \alpha(n^*) + \overline{n^{**}}) + \overline{\overline{n^{**}}}$. By $n^* \in X^*$, $f^*(x^*) = f^*(n^*) + f^*(-\alpha(n^*) + \overline{n^{**}} + \overline{\overline{n^{**}}}) =$

$g^*(n^*) + g_1(\overline{n^{**}}) + g_2^{-1}(-\alpha(n^*)) + g_2^{-1}(\overline{\overline{n^{**}}}) = p_{\overline{M}}(g^*(n^*)) + p_{\overline{\overline{M}}}(g^*(n^*)) + g_1(\overline{n^{**}}) + g_2^{-1}(-g_2 p_{\overline{\overline{M}}} g^*(n^*)) + g_2^{-1}(\overline{\overline{n^{**}}}) = p_{\overline{M}}(g^*(n^*)) + g_1(\overline{n^{**}}) + g_2^{-1}(\overline{\overline{n^{**}}})$.

Put $m_1 = p_{\overline{M}}(g^*(n^*)) + g_1(\overline{n^{**}}) \in M_1$ and $m_2 = g_2^{-1}(\overline{\overline{n^{**}}}) \in M_2$. Then

$$\varphi_1(n_1) = \varphi_1(n^* - \alpha(n^*) + \overline{n^{**}}) = p_{\overline{M}}(g^*(n^*)) + g_1(\overline{n^{**}}) = m_1$$

and

$$\varphi_2(m_2) = g_2(g_2^{-1}(\overline{\overline{n^{**}}})) = \overline{\overline{n^{**}}} = n_2.$$

Thus M is N-ojective. □

Let $\{M_i \mid i \in I\}$ be a family of modules. The decomposition $M = \oplus_{i\in I} M_i$ is said to be *exchangeable* if, for any direct summand X of M, there exists $\overline{M_i} \subseteq M_i$ $(i \in I)$ such that $M = X \oplus (\oplus_{i\in I} \overline{M_i})$. A module M is said to have the (finite) internal exchange property if, any (finite) direct sum decomposition $\oplus_{i\in I} M_i$ of M is exchangeable.

The following proof is essentially due to [8, Theorem 2.1].

Theorem 3.3. *Let M_1 and M_2 be extending modules with the finite internal exchange property and put $M = M_1 \oplus M_2$. Then the following conditions are equivalent:*

(1) M is extending with the finite internal exchange property,

(2) M is extending and the decomposition $M = M_1 \oplus M_2$ is exchangeable,

(3) M_1 and M_2 are mutually relative ojective,

(4) M_1 is M_2-ojective and M_2 is essentially M_1-injective,

(5) M_1 and M_2 are mutually relative essentially injective, N_1 is mono-N_2-ojective for all direct summands N_1 of M_1 and N_2 of M_2,

(6) M_1 and M_2 are mutually relative essentially injective, N_1 is weakly mono-N_2-ojective for all direct summands N_1 of M_1 and N_2 of M_2.

Proof. (1)⇔(2)⇔(3) : By [8, Theorem 2.15].

(3)⇒(4)⇒(5) : By Proposition 1.2 and Proposition 3.1.

(5)⇒(6) is clear.

(6)⇒(2) : Let X be a submodule of M and put $X_i = M_i \cap X$ $(i = 1, 2)$. Since M_i is extending, there exists the decomposition $M_i = M_i' \oplus M_i''$ with $X_i \subseteq_e M_i''$ $(i = 1, 2)$. Put $M' = M_1' \oplus M_2'$ and $X' = M' \cap X$. Then $X \supseteq_e X_1 \oplus X_2 \oplus X'$ by Lemma 2.2. Define $f : p_{M_1'}(X') \to p_{M_2'}(X')$ by $f(p_{M_1'}(x')) = p_{M_2'}(x')$ and then f is an isomorphism, where $p_{M_i'} : M' = M_1' \oplus M_2' \to M_i'$ $(i = 1, 2)$ is the projection. As M_i' is extending, there exists a decomposition $M_i' = N_i \oplus T_i$ with $p_{M_i'}(X') \subseteq_e T_i$ $(i = 1, 2)$. Since T_2 is weakly mono-T_1-ojective, there exist an essential submodule Y of $p_{M_1'}(X')$, the decompositons $T_i = T_i' \oplus T_i''$ $(i = 1, 2)$ and monomorphisms $g_1 : T_1' \to T_2''$,

$g_2 : T_2' \to T_1''$ with the condition $(*)$ for Y. Then we see

$$\langle Y \stackrel{f}{\to} f(Y)\rangle \subseteq_e \langle T_1' \stackrel{g_1}{\to} T_2''\rangle \oplus \langle T_2' \stackrel{g_2}{\to} T_1''\rangle$$

and

$$X_1 \oplus X_2 \oplus \langle Y \stackrel{f}{\to} f(Y)\rangle \subseteq_e X_1 \oplus X_2 \oplus X' \subseteq_e X \ \cdots (i).$$

Put $Z = M_1'' \oplus M_2'' \oplus \langle T_1' \stackrel{g_1}{\to} T_2''\rangle \oplus \langle T_2' \stackrel{g_2}{\to} T_1''\rangle$ and $Q_i = T_i'' \oplus N_i$ $(i = 1, 2)$. Then $M = Z \oplus Q_1 \oplus Q_2$. For any $x \in X$, we express x in $M = Z \oplus Q_1 \oplus Q_2$ as $x = z + q$, where $z \in Z$ and $q \in Q_1 \oplus Q_2$. By $X' \cap (Q_1 \oplus Q_2) = 0$ and (i), we define a homomorphism $\gamma : p_Z(X) \to p_{Q_1 \oplus Q_2}(X)$ by $\gamma(z) = q$ and then $\ker \gamma \subseteq_e p_Z(X)$. By (1) $\Rightarrow$ (3) and Proposition 1.1, $Q_1 \oplus Q_2$ is essentially Z-injective and hence there exists a homomorphism $\gamma^* : Z \to Q_1 \oplus Q_2$ with $\gamma^*|_{p_Z(X)} = \gamma$.

Thus we see

$$X = \langle p_Z(X) \stackrel{\gamma}{\to} p_{Q_1 \oplus Q_2}(X)\rangle \subseteq_e \langle Z \stackrel{\gamma^*}{\to} Q_1 \oplus Q_2\rangle$$

and

$$M = \langle Z \stackrel{\gamma^*}{\to} Q_1 \oplus Q_2\rangle \oplus Q_1 \oplus Q_2.$$

Therefore M is extending and the decompositon $M = M_1 \oplus M_2$ is exchangeable. $\square$

Corollary 3.4. *Let A be a semisimple module and let B be an extending module with the finite internal exchange property. If A is essentially B-injective, then $M = A \oplus B$ is extending with the finite internal exchange property.*

Now we consider whether weakly mono-ojectivities are inherited by direct summands, finite direct sums in the case that each module is quasi-continuous.

Proposition 3.5. *Let M be quasi-continuous and let N be extending with the finite internal exchange property. If M is weakly mono-N-ojective, then M is weakly mono-N'-ojective for any direct summand $N' <_{\oplus} N$.*

Proof. Let X be a submodule of N' and let $f : X \to M$ be a monomorphism. As N' is extending, we can assume that X is essential in N'. Since M is weakly mono-N-ojective, there exist an essential submodule Y of X, decompositions $N = N_1 \oplus N_2$, $M = M_1 \oplus M_2$ and monomorphisms $g_1 : N_1 \to M_1$, $g_2 : M_2 \to N_2$ with the condition $(*)$ for Y. As N satisfies the finite internal exchange property, there exists a direct summand $\overline{\overline{N_i}}$ of N_i $(i = 1, 2)$ such that $N = N' \oplus \overline{\overline{N_1}} \oplus \overline{\overline{N_2}}$. Let $N_i = \overline{N_i} \oplus \overline{\overline{N_i}}$. Define $\alpha : \overline{N_1} \oplus \overline{N_2} = p_{\overline{N_1} \oplus \overline{N_2}}(N') \to p_{\overline{\overline{N_1}} \oplus \overline{\overline{N_2}}}(N')$ by $\alpha(p_{\overline{N_1} \oplus \overline{N_2}}(n')) = p_{\overline{\overline{N_1}} \oplus \overline{\overline{N_2}}}(n')$ for any $n' \in N'$. Put $\alpha_i = \alpha|_{\overline{N_i}}$ and $Q_i = \langle \overline{N_i} \stackrel{\alpha_i}{\to} \overline{\overline{N_1}} \oplus \overline{\overline{N_2}}\rangle$. Now define

$\alpha_i^* : \overline{N_i} \to \overline{\overline{N_i}}$ by $\alpha_i^*(\overline{n_i}) = p_{\overline{\overline{N_i}}}(\alpha_i(\overline{n_i}))$ and define $\beta_i^* : \langle \overline{N_i} \overset{\alpha_i^*}{\to} \overline{\overline{N_i}} \rangle \to \overline{\overline{N_j}}$ by $\beta_i^*(\overline{n_i} - \alpha_i^*(\overline{n_i})) = p_{\overline{\overline{N_j}}}\alpha_i(\overline{n_i})$ for $i \neq j$. Then we see

$$Q_i = \langle \langle \overline{N_i} \overset{\alpha_i^*}{\to} \overline{\overline{N_i}} \rangle \overset{\beta_i^*}{\to} \overline{\overline{N_j}} \rangle \qquad (i, j = 1, 2, \quad i \neq j).$$

Since M is extending, there exists a direct summand M_1' of M_1 such that $g_1(\langle \overline{N_1} \overset{\alpha_1^*}{\to} \overline{\overline{N_1}} \rangle) \subseteq_e M_1'$. By $Y \subseteq_e N' = Q_1 \oplus Q_2$, we see $Q_i \cap Y \subseteq_e Q_i$ $(i = 1, 2)$. For $x_1 \in Q_1 \cap Y$, we express x_1 in $\langle \langle \overline{N_1} \overset{\alpha_1^*}{\to} \overline{\overline{N_1}} \rangle \overset{\beta_1^*}{\to} \overline{\overline{N_2}} \rangle$ as $x_1 = n_1 - \beta_1^*(n_1)$, where $n_1 \in \langle \overline{N_1} \overset{\alpha_1^*}{\to} \overline{\overline{N_1}} \rangle$. Then $g_1(n_1) = 0$ imply $g_2^{-1}(\beta_1^*(n_1)) = 0$, since g_1 and g_2 are monomorphisms. Hence the natural map $\gamma_1 : g_1(\langle \overline{N_1} \overset{\alpha_1^*}{\to} \overline{\overline{N_1}} \rangle) \to g_2^{-1}(\beta_1^*(\langle \overline{N_1} \overset{\alpha_1^*}{\to} \overline{\overline{N_1}} \rangle))$ is a homomorphism. Since M is quasi-continuous, M_2 is M_1'-injective. So there exists a homomorphism $\gamma_1^* : M_1' \to M_2$ such that $\gamma_1^*|_{g_1(\langle \overline{N_1} \overset{\alpha_1^*}{\to} \overline{\overline{N_1}} \rangle)} = \gamma_1$.

Now we put $\varphi_1 = \epsilon_2 g_1 \epsilon_1 : Q_1 = \langle \langle \overline{N_1} \overset{\alpha_1^*}{\to} \overline{\overline{N_1}} \rangle \overset{\beta_1^*}{\to} \overline{\overline{N_2}} \rangle \to \langle M_1' \overset{\gamma_1^*}{\to} M_2 \rangle$, where $\epsilon_1 : Q_1 \to \langle \overline{N_1} \overset{\alpha_1^*}{\to} \overline{\overline{N_1}} \rangle$ and $\epsilon_2 : M_1' \to \langle M_1' \overset{\gamma_1^*}{\to} M_2 \rangle$ are canonical isomorphisms.

Then, for any $x_1 = n_1 - \beta_1^*(n_1) \in Q_1 \cap Y$, $\varphi_1(x_1) = \epsilon_2 g_1 \epsilon_1(x_1) = \epsilon_2 g_1(n_1) = g_1(n_1) - \gamma_1^* g_1(n_1) = g_1(n_1) - \gamma_1 g_1(n_1) = g_1(n_1) - g_2^{-1}(\beta_1^*(n_1)) = f(x_1)$.

Thus φ_1 is a monomorphism with $\varphi_1|_{Q_1 \cap Y} = f|_{Q_1 \cap Y}$.

On the other hand, by $p_{N_2}(Q_2 \cap Y) \subseteq g_2(M_2)$, there exists a direct summand M_2' of M_2 with $g_2^{-1}(p_{N_2}(Q_2 \cap Y)) \subseteq_e M_2'$. Let $\pi : N_2 = \langle \overline{N_2} \overset{\alpha_2^*}{\to} \overline{\overline{N_2}} \rangle \oplus \overline{\overline{N_2}} \to \langle \overline{N_2} \overset{\alpha_2^*}{\to} \overline{\overline{N_2}} \rangle$ be the projection and put $\gamma_2^* = g_1 \beta_2^* \pi (g_2|_{M_2'}) : M_2' \to M_1$. For $x_2 \in Q_2 \cap Y$, we express x_2 in $\langle \langle \overline{N_2} \overset{\alpha_2^*}{\to} \overline{\overline{N_2}} \rangle \overset{\beta_2^*}{\to} \overline{\overline{N_1}} \rangle$ as $x_2 = n_2 - \beta_2^*(n_2)$, where $n_2 \in \langle \overline{N_2} \overset{\alpha_2^*}{\to} \overline{\overline{N_2}} \rangle$. Then $f(x_2) = f(n_2 - \beta_2^*(n_2)) = g_2^{-1}(n_2) - g_1(\beta_2^*(n_2))$. By $n_2 \in p_{N_2}(Q_2 \cap Y)$, we see $g_2^{-1}(n_2) \in M_2'$. Hence $\gamma_2^*(g_2^{-1}(n_2)) = g_1 \beta_2^* \pi g_2(g_2^{-1}(n_2)) = g_1 \beta_2^*(n_2)$ and so $f(x_2) = g_2^{-1}(n_2) - \gamma_2^*(g_2^{-1}(n_2))$. Thus $f(Q_2 \cap Y) \subseteq \langle M_2' \overset{\gamma_2^*}{\to} M_1 \rangle$. By $g_2^{-1}(p_{N_2}(Q_2 \cap Y)) \subseteq_e M_2'$, we see $f(Q_2 \cap Y) \subseteq_e \langle M_2' \overset{\gamma_2^*}{\to} M_1 \rangle$. Now we put $\varphi_2 = \epsilon_4 \pi g_2 \epsilon_3 : \langle M_2' \overset{\gamma_2^*}{\to} M_1 \rangle \to Q_2$, where $\epsilon_3 : \langle M_2' \overset{\gamma_2^*}{\to} M_1 \rangle \to M_2'$ and $\epsilon_4 : \langle \overline{N_2} \overset{\alpha_2^*}{\to} \overline{\overline{N_2}} \rangle \to Q_2$ are canonical isomorphisms. Then $\varphi_2|_{f(Q_2 \cap Y)} = f^{-1}|_{f(Q_2 \cap Y)}$.

By $g_2^{-1}(p_{N_2}(Q_2 \cap Y)) \subseteq_e M_2'$, $\pi|_{g_2(M_2')}$ is a monomorphism and so φ_2 is a monomorphism.

Since f is a monomorphism, we see

$$\langle M_1' \stackrel{\gamma_1^*}{\to} M_2\rangle \cap \langle M_2' \stackrel{\gamma_2^*}{\to} M_1\rangle = 0$$

by $(Q_1 \cap Y) \cap (Q_2 \cap Y) = 0$ and $f(Q_i \cap Y) \subseteq_e \langle M_i' \stackrel{\gamma_i^*}{\to} M_j\rangle$ $(i, j = 1, 2,\ \ i \neq j)$. As M is quasi-continuous,

$$\langle M_1' \stackrel{\gamma_1^*}{\to} M_2\rangle \oplus \langle M_2' \stackrel{\gamma_2^*}{\to} M_1\rangle <_\oplus M.$$

Thus M is weakly mono-N'-ojective. □

Let M and N be quasi-continuous and let M be weakly mono-N-ojective. Since weakly mono-ojectivity is symmetric, M' is weakly mono-N'-ojective for any direct summands $N' <_\oplus N$ and $M' <_\oplus M$.

Proposition 3.6. *Let M be a quasi-continuous module and let $N = N_1 \oplus \cdots \oplus N_t$ be an extending module with the finite internal exchange property. If M is weakly mono-N_i-ojective for all $i \in \{1, \cdots, t\}$, then M is weakly mono-N-ojective.*

Proof. Let X be a submodule of N, let $f : X \to M$ be a monomorphism and put $F = \{1, \cdots, t\}$. Since N is extending with the finite internal exchange property, there exists a direct summand X^* of N such that $X \subseteq_e X^*$ and $N = X^* \oplus N_1'' \oplus \cdots \oplus N_t''$, where $N_i = N_i' \oplus N_i''$ $(i \in F)$. Hence there exists an isomorphism $\alpha : N_1' \oplus \cdots \oplus N_t' \to X^*$. Put $X_i^* = \alpha(N_i')$ and $X_i = X_i^* \cap X$ for any $i \in F$. Then we see $X_i \subseteq_e X_i^*$.

By Proposition 3.5, M is weakly mono-X_i^*-ojective for any $i \in F$ and so there exist an essential submodule Y_i of X_i, decompositions $X_i^* = \overline{X_i^*} \oplus \overline{\overline{X_i^*}}$, $M = \overline{M_i} \oplus \overline{\overline{M_i}}$ and monomorphisms $g_i : \overline{X_i^*} \to \overline{M_i}$, $h_i : \overline{\overline{M_i}} \to \overline{\overline{X_i^*}}$ with the condition $(*)$ for Y_i.

As $\overline{M_i}$ is extending, there exists a direct summand K_i of $\overline{M_i}$ with $g_i(\overline{X_i^*}) \subseteq_e K_i$. Since g_i and h_i are monomorphisms and $Y_i \subseteq_e X_i^*$, we see $f(Y_i) \subseteq_e K_i \oplus \overline{\overline{M_i}}$. Hence $(K_i \oplus \overline{\overline{M_i}}) \cap (K_j \oplus \overline{\overline{M_j}}) = 0$ for any $i \neq j$. As M is quasi-continuous, there exists a direct summand T of M such that $M = T \oplus K_1 \oplus \cdots \oplus K_t \oplus \overline{\overline{M_1}} \oplus \cdots \oplus \overline{\overline{M_t}}$.

Put $\overline{N} = \overline{X_1^*} \oplus \cdots \oplus \overline{X_t^*}$, $\overline{\overline{N}} = \overline{\overline{X_1^*}} \oplus \cdots \oplus \overline{\overline{X_t^*}} \oplus N_1'' \oplus \cdots \oplus N_t''$, $\overline{M} = K_1 \oplus \cdots \oplus K_t \oplus T$ and $\overline{\overline{M}} = \overline{\overline{M_1}} \oplus \cdots \oplus \overline{\overline{M_t}}$. Then, $g = g_1 + \cdots + g_t : \overline{N} \to \overline{M}$ and $h = h_1 + \cdots + h_t : \overline{\overline{M}} \to \overline{\overline{N}}$ are monomorphisms that satisfy the condition $(*)$ for $X_1 \oplus \cdots \oplus X_t$. Thus M is weakly mono-N-ojective. □

Corollary 3.7. *Let N be quasi-continuous and let $M = M_1 \oplus \cdots \oplus M_t$ be extending with the finite internal exchange property. If M_i is weakly mono-N-ojective for all $i \in \{1, \cdots, t\}$, then M is weakly mono-N-ojective.*

Theorem 3.8. (cf. [8].) *Let $M_1, \cdots, M_n$ be quasi-continuous modules and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent:*

(1) M is extending with the finite internal exchange property,

(2) M is extending and the decomposition $M = M_1 \oplus \cdots \oplus M_n$ is exchangeable,

(3) M_i is mono-M_j-ojective and essentially M_j-injective for $i \neq j$,

(4) M_i is weakly mono-M_j-ojective and essentially M_j-injective for $i \neq j$.

Proof. By [8, Theorem 2.15], (1) $\Leftrightarrow$ (2) holds.

(1) $\Rightarrow$ (3) holds by Theorems 3.2 and 3.3.

(3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (2) : By Theorem 3.3 and Proposition 3.5, the statement holds for $n = 2$.

Assume that the statement holds for $n = k$ ($k \geq 2$), and consider the case $n = k + 1$; $M = M_1 \oplus \cdots \oplus M_k \oplus M_{k+1}$. Let X be a submodule of M and put $M^* = M_1 \oplus \cdots \oplus M_k$, $X^* = M^* \cap X$, $X_{k+1} = M_{k+1} \cap X$. By assumption, there exists a decomposition $M^* = T \oplus M_1' \oplus \cdots \oplus M_k'$ such that $X^* \subseteq_e T$ and $M_i' \subseteq M_i$ ($i = 1, \cdots, k$). As M_{k+1} is extending, there exists a decomposition $M_{k+1} = M_{k+1}' \oplus M_{k+1}''$ with $X_{k+1} \subseteq_e M_{k+1}''$. Put $M' = M_1' \oplus \cdots \oplus M_k' \oplus M_{k+1}'$ and $X' = M' \cap X$. By Lemma 2.2, we see

$$X \supseteq_e X^* \oplus X_{k+1} \oplus X' \qquad \cdots \ (i).$$

Let p_1 and p_2 be the projections : $M' \to M_1' \oplus \cdots \oplus M_k'$, $M' \to M_{k+1}'$, respectively. As $(M_1' \oplus \cdots \oplus M_k') \cap X' = M_{k+1}' \cap X' = 0$, the canonical map $f : p_1(X') \to p_2(X')$ given by $p_1(x') \to p_2(x')$ is an isomorphism.

Since $M_1' \oplus \cdots \oplus M_k'$ and M_{k+1}' are extending with the finite internal exchange property, there exist decompositions $M_1' \oplus \cdots \oplus M_k' = A \oplus A'$, $M_{k+1}' = B \oplus B'$ with $p_1(X') \subseteq_e A$, $p_2(X') \subseteq_e B$, respectively. By Propositions 3.5, 3.6, B is weakly mono-A-ojective. Hence there exist an essential submodule Y of $p_1(X')$, decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$ and monomorphisms $g_1 : A_1 \to B_1$, $g_2 : B_2 \to A_2$ with the condition $(*)$ for Y. Thus we see

$$\langle Y \xrightarrow{f} f(Y) \rangle \subseteq_e \langle A_1 \xrightarrow{g_1} B_1 \rangle \oplus \langle B_2 \xrightarrow{g_2} A_2 \rangle \quad \text{and} \quad \langle Y \xrightarrow{f} f(Y) \rangle \subseteq_e X'.$$

Put $Z = T \oplus M_{k+1}'' \oplus \langle A_1 \xrightarrow{g_1} B_1 \rangle \oplus \langle B_2 \xrightarrow{g_2} A_2 \rangle$, $\overline{A} = A_2 \oplus A'$ and $\overline{M_{k+1}} = B_1 \oplus B'$. Let q_1 and q_2 be the projections : $M = Z \oplus \overline{A} \oplus \overline{M_{k+1}} \to Z$, $M \to \overline{A} \oplus \overline{M_{k+1}}$, respectively. By (i), the natural map

$$\varphi : q_1(X) \to q_2(X) \quad \text{via} \quad \varphi : q_1(x) \mapsto q_2(x)$$

is a homomorphism with $\ker \varphi \subseteq_e q_1(X)$.

By Theorem 3.3 (1) $\Rightarrow$ (5) and Proposition 1.1, $\overline{A} \oplus \overline{M_{k+1}}$ is essentially Z-injective and hence there exists a homomorphism $\varphi^* : Z \to \overline{A} \oplus \overline{M_{k+1}}$

with $\varphi^*|_{q_1(X)} = \varphi$. Thus we obtain

$$X = \langle q_1(X) \overset{\varphi}{\to} q_2(X)\rangle \subseteq_e \langle Z \overset{\varphi^*}{\to} \overline{A} \oplus \overline{M_{k+1}}\rangle$$

and

$$M = \langle Z \overset{\varphi^*}{\to} \overline{A} \oplus \overline{M_{k+1}}\rangle \oplus \overline{A} \oplus \overline{M_{k+1}}.$$

Finally, we show that there exsits a submodule $\overline{M_i}$ of M_i $(i = 1, \cdots, k)$ such that $M = \langle Z \overset{\varphi^*}{\to} \overline{A} \oplus \overline{M_{k+1}}\rangle \oplus \overline{M_1} \oplus \cdots \oplus \overline{M_k} \oplus \overline{M_{k+1}}$. By $\overline{A} \subseteq M^* = M_1 \oplus \cdots \oplus M_k$,

$$M^* = \overline{A} \oplus (\langle Z \overset{\varphi^*}{\to} \overline{A} \oplus \overline{M_{k+1}}\rangle \oplus \overline{M_{k+1}}) \cap M^* \quad \cdots \ (ii).$$

Since the decomposition $M^* = M_1 \oplus \cdots \oplus M_k$ is exchangeable, there exists a submodule $\overline{M_i}$ of M_i $(i = 1, \cdots, k)$ such that

$$M^* = \overline{M_1} \oplus \cdots \oplus \overline{M_k} \oplus (\langle Z \overset{\varphi^*}{\to} \overline{A} \oplus \overline{M_{k+1}}\rangle \oplus \overline{M_{k+1}}) \cap M^* \quad \cdots \ (iii).$$

By (ii) and (iii), $\overline{A} = \langle \overline{M_1} \oplus \cdots \oplus \overline{M_k} \to (\langle Z \overset{\varphi^*}{\to} \overline{A} \oplus \overline{M_{k+1}}\rangle \oplus \overline{M_{k+1}}) \cap M^*\rangle$. Thus we see

$$\begin{aligned} M &= \langle Z \overset{\varphi^*}{\to} \overline{A} \oplus \overline{M_{k+1}}\rangle \oplus \overline{A} \oplus \overline{M_{k+1}} \\ &= \langle Z \overset{\varphi^*}{\to} \overline{A} \oplus \overline{M_{k+1}}\rangle \oplus \langle \overline{M_1} \oplus \cdots \oplus \overline{M_k} \to \langle Z \overset{\varphi^*}{\to} \overline{A} \oplus \overline{M_{k+1}}\rangle \oplus \overline{M_{k+1}}\rangle \\ &\qquad \oplus \overline{M_{k+1}} \\ &= \langle Z \overset{\varphi^*}{\to} \overline{A} \oplus \overline{M_{k+1}}\rangle \oplus \overline{M_1} \oplus \cdots \oplus \overline{M_k} \oplus \overline{M_{k+1}}. \end{aligned}$$

Therefore M is extending and the decomposition $M = M_1 \oplus \cdots \oplus M_k \oplus M_{k+1}$ is exchangeable.

□

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