Journal of the Faculty of Environmental Science and Technology, Okayama University
Vol.1, No.1, pp.15-20, March 1996

Smash product extensions of separable algebras

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(Received November 17 , 1995)

Abstract

Let A be a bialgebra and let S be a right A-comodule algebra which has an A-comodule subalgebra T with common identity. We show that if S is a separable extension of T, then for a left A-module algebra K, $K\sharp S$ is a separable extension of $K\sharp T$. Similar result holds for left A-module algebras and right A-comodule algebras.

KEYWORDS: Separable extension, smash product, module algebra and comodule algebra.

1991 MATHEMATICS SUBJECT CLASSIFICATION: 16A16, 16A24.

1. Introduction

Let R be a commutative ring with identity and let $R \subset T \subset S$ be ring extensions with common identity. It is known that some properties of a ring extension S/T are preserved by taking a tensor product. For example, if S/T is a separable extension, then for any R-algebra K, $K \otimes_R S$ is a separable extension of $K \otimes_R T$.

In this note we will consider a similar things at the standpoint of the smash product. Throughout the follwoing, R is a commutative ring with identity and all algebras, modules, maps and $\otimes$ are considered over R unless otherwise stated. We freely use the terminologies and notations in the new fundamental book [M] for Hopf algebras and their actions.

2. Definitions and examples.

A *bialgebra* is both an algbera and a coalgebra such that the coalgebra structure maps Δ and ε are algebra maps. Let A be a bialgebra and X an algebra. X is called a left *A-module algebra* if X is a left A-module and

$$a(xy) = \sum_{(a)} a_1(x)a_2(y) \quad and \quad a(1) = \varepsilon(a)1 \quad (a \in A, x, y \in X,)$$

where $\Delta(a) = \sum_{(a)} a_1 \otimes a_2 \in A \otimes A$ is the Sweedler's sigma notation. We omit the summation index (a) in case it is clear. X is called a right *A-comodule algebra* if X is a right A-comodule with the structure map $\rho : X \to X \otimes A$ such that ρ is an algebra map. For a left A-module algebra K and a right A-comodule algebra S, we define the product by

$$(k \otimes s)(\ell \otimes t) = \sum k(s_1\ell) \otimes s_0 t$$

where $\rho(s) = \sum s_1 \otimes s_0 \in A \otimes S$ Then $K \otimes S$ is an algebra. We denote this algebra by $K\sharp S$ which is called the *smash product* of K and S, and we also denote an element in $K\sharp S$ by $k\sharp u$. $K\sharp S$ has an identity $1\sharp 1$ and the maps

$$1_K : K \ni k \mapsto k\sharp 1 \in K\sharp S \quad and \quad 1_S : S \ni s \mapsto 1\sharp s \in K\sharp S$$

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are algebra maps. By these maps $K\sharp S$ is a left K- and a right S-module with respect to the product in $K\sharp S$. If A is commutative and cocommutative, then by

$$a(k\sharp s) = ak\sharp s \quad and \quad k\sharp s \mapsto \sum k\sharp s_0 \otimes s_1,$$

$K\sharp S$ is a left A-module algebra and a right A-comodule algebra, respectively.

Example 2.1. (1) Let K be an algebra and G a subgroup of the algebra automorphism group of K. It is known that the crossed product algebra of K by G is the smash product of a left RG-module algebra K and a right RG-comodule algebra RG.

(2) Let σ is an algebra automorphism of K and let D be a σ-derivation of K, that is, D is a module emdomorphism of K such that $D(k\ell) = \sigma(k)D(\ell) + D(k)\ell$ $(k, \ell \in K)$. We set

$$K^{\sigma} = \{k \in K | \sigma(k) = k\}, \qquad K^{D} = \{k \in K | D(k) = 0\}.$$

Since these maps are R-linear, R is contained in $K^{\sigma} \cap K^{D}$. Let $A = R[\sigma, D]$ be the non-commutative free algebra on variables σ and D which has the following coalgebra structure

$$\Delta(\sigma) = \sigma \otimes \sigma, \quad \varepsilon(\sigma) = 1, \quad \Delta(D) = \sigma \otimes D + D \otimes 1, \quad \varepsilon(D) = 0.$$

Then A is a bialgebra and K is a left A-module algebra. Let $R[X]$ be the polynomial ring over R with indeterminate X. Define a linear map

$$\rho : R[X] \ni X \mapsto X \otimes \sigma + 1 \otimes D \in R[X] \otimes A.$$

Then $R[X]$ is a right A-comodule algebra. Let $K[X; \sigma, D]$ be the skew polynomial ring in which the multiplication is given by

$$Xk = \sigma(k)X + D(k) \quad (k \in K).$$

Then it is easy to see that the map

$$\varphi : K\sharp R[X] \ni \sum k_i \sharp X^i \mapsto \sum k_i X^i \in K[X; \sigma, D]$$

is an algebra isomorphism. Thus the skew polynomial ring $K[X; \sigma, D]$ is a special case of a smash product of a left A-module algebra K and a right A-comodule algebra $R[X]$ (cf. [N]). Of course it is generalized to n-variables.

Let A be a bialgebra, K a left A-module algebra and S a right A-comodule algebra. A subalgebra L of K is called a *A-module subalgebra* if L is a left A-submodule of K. Similarly, a subalgebra T of S is called an *A- comodule subalgebra* if T is a right A-subcomodule of S. Then there exist the canonical algebra maps

$$L\sharp S \to K\sharp S \quad and \quad K\sharp T \to K\sharp S.$$

Thus $K\sharp S$ is an $L\sharp S$-bimodule and a $K\sharp T$-bimodule with respect to the multiplication in $K\sharp S$. Moreover $(K\sharp S) \otimes_{L\sharp S} (K\sharp S)$ and $(K\sharp S) \otimes_{K\sharp T} (K\sharp S)$ are also a $K\sharp S$-bimodule canonically.

In the following, for the sake of simplicity, we assume A is a commutative and cocommutative bialgebra. K is a left A-module algebra with an A-module subalgebra L and S is a right A-comodule algebra with an A-comodule subalgebra T. We fix these notations.

3. Extensions by module algebras.

In this section we define some module structures on $K \otimes S \otimes_T S$ and proves that the separability of extensions of comodule algebras is preserved by taking the smash product $K\sharp(-)$.

Lemma 3.1. *$K \otimes S \otimes_T S$ has a $K\sharp S$- bimodule structure.*

Proof. For any $k, \ell \in K$ and $s, u, v \in S$, we define

$$\begin{aligned}(k\sharp s)(\ell \otimes u \otimes v) &= \sum k(s_1\ell) \otimes s_0 u \otimes v,\\ (\ell \otimes u \otimes v)(k\sharp s) &= \sum \ell(u_1 v_1 k) \otimes u_0 \otimes v_0 s.\end{aligned}$$

Then $K \otimes S \otimes_T S$ is a $K\sharp S$-bimodule.

Lemma 3.2. *The map*

$$\varphi : (K\sharp S) \otimes_{K\sharp T} (K\sharp S) \ni (\ell\sharp u) \otimes (m\sharp v) \mapsto \sum \ell(u_1 m) \otimes u_0 \otimes v \in K \otimes S \otimes_T S$$

is a $K\sharp S$-bimodule isomorphism.

Proof. For any $k, \ell, m \in K$ and $s, u, v \in S$, we have

$$\begin{aligned}(k\sharp s)\varphi((\ell\sharp u) \otimes (m\sharp v)) &= (k\sharp s)\sum \ell(u_1 m) \otimes u_0 \otimes v\\ &= \sum k(s_1(\ell(u_1 m))) \otimes s_0 u_0 \otimes v\\ &= \sum k(s_2\ell)(s_1 u_1 m) \otimes s_0 u_0 \otimes v,\\ \varphi((k\sharp s)(\ell\sharp u) \otimes (m\sharp v)) &= \varphi(\sum (k(s_1\ell)\sharp s_0 u) \otimes (m\sharp v))\\ &= \sum k(s_1\ell)((s_0 u_1)_1 m) \otimes (s_0 u_1)_0) \otimes v\\ &= \sum k(s_2\ell)(s_1 u_1 m) \otimes s_0 u_0 \otimes v.\end{aligned}$$

So φ is a left $K\sharp S$-module map. Moreover by

$$\begin{aligned}\varphi((\ell\sharp u) \otimes (m\sharp v))(k\sharp s) &= \sum \ell(u_1 m) \otimes u_0 \otimes v)(k\sharp s)\\ &= \sum \ell(u_2 m)(u_1 v_1 k) \otimes u_0 \otimes v_0 s,\\ \varphi(((\ell\sharp u) \otimes (m\sharp v))(k\sharp s)) &= \varphi((\ell\sharp u) \otimes \sum m(v_1 k)\sharp v_0 s)\\ &= \sum \ell u_1(m(v_1 k)) \otimes u_0 \otimes v_0 s\\ &= \sum \ell(u_2 m)(u_1 v_1 k) \otimes u_0 \otimes v_0 s,\end{aligned}$$

φ is a right $K\sharp S$-module map. Define a map

$$\psi : K \otimes S \otimes_T S \ni \ell \otimes u \otimes v \mapsto (\ell\sharp u) \otimes (1\sharp v) \in (K\sharp S) \otimes_{K\sharp T} (K\sharp S).$$

Then as is easily seen, ψ is a left $K\sharp S$-module map such that $\varphi\psi = 1$. Moreover using the following equality, we see

$$\begin{aligned}(\ell\sharp u) \otimes (m\sharp v) &= (\ell\sharp u) \otimes (m\sharp 1)(1\sharp v)\\ &= (\ell\sharp u)(m\sharp 1) \otimes (1\sharp v)\\ &= \sum (\ell(u_1 m)\sharp u_0) \otimes (1\sharp v).\end{aligned}$$

Thus ψ is a right $K\sharp S$-module map and $\psi\varphi = 1$.

Lemma 3.3. *Let $\mu : S \otimes_T S \ni u \otimes v \mapsto uv \in S$. Then the map $1 \otimes \mu : K \otimes S \otimes_T S \to K \otimes S$ is a $K\sharp S$-bimodule map, where $K \otimes S$ is the canonical $K\sharp S$- bimodule by the multiplication and the $K\sharp S$-bimodule structure of $K \otimes S \otimes_T S$ is defined in Lemma 3.1.*

Proof. By

$$\begin{aligned}(k\sharp s)(1\otimes\mu)(\ell\otimes u\otimes v) &= (k\sharp s)(\ell\otimes uv)\\ &= \sum k(s_1\ell)\otimes s_0uv,\\ (1\otimes\mu)((k\sharp s)(\ell\otimes u\otimes v)) &= (1\otimes\mu)(\sum k(s_1\ell)\otimes s_0u\otimes v)\\ &= \sum k(s_1\ell)\otimes s_0uv,\end{aligned}$$

$1\otimes\mu$ is a left $K\sharp S$-module map. Similarly $1\otimes\mu$ is a right $K\sharp S$-module map.

Now, since S is a right A-comodule algebra, $S\otimes_T S$ is a right A-comodule by

$$\rho: S\otimes_T S\ni u\otimes v\mapsto \sum u_0\otimes v_0\otimes u_1v_1\in S\otimes_T S\otimes A,$$

and $\mu: S\otimes_T S\to S$ is a right A-comodule map. We call that a right A-comodule algebra S is a *separable extension* of T in the category of right A-comodule algebras if there exists a right A-comodule and an S-bimodule map $\tau: S\to S\otimes_T S$ such that $\mu\tau = 1$.

Example 3.4. Let G be a group and $A = RG$ the group algebra. If S is a right A-comodule algebra, then S is a G-graded algebra and T is a G-graded subalgebra of S. That S is a separable extension of R in the category of right RG-comodule algebras means a G-graded separable extension in the sense of [CGO]. In case of $T = R$ and A is a commutative and cocommutative Hopf algebra, it was discussed in [L].

Theorem 3.5. *If S is a separable extension of T in the categrory of right A-comodule algebras, then $K\sharp S$ is a separable extension of $K\sharp T$ in the category of right A-comodule algebras.*

Proof. Let $\tau: S\to S\otimes_T S$ be a right A-comodule and an S-bimodule map such that $\mu\tau = 1$. Then $1\otimes\tau: K\otimes S\to K\otimes S_T\otimes S$ is a module map. Since $K\otimes S$ has the canonical $K\sharp S$-bimodule structure and $K\otimes S\otimes_T S$ has also a $K\sharp S$-bimodule defined in Lemma 3.1, then by

$$\begin{aligned}(\ell\sharp u)(1\otimes\tau)(k\otimes s) &= (\ell\sharp u)(k\otimes\tau(s))\\ &= \sum \ell(u_1k)\otimes(u_0\otimes 1)\tau(s)\\ &= \sum \ell(u_1k)\otimes(u_0s)\\ &= (1\otimes\tau)(\sum \ell(u_1k)\otimes u_0s)\\ &= (1\otimes\tau)((\ell\sharp u)(k\sharp s)),\end{aligned}$$

$1\otimes\tau$ is a left $K\sharp S$-module map. Moreover by

$$\begin{aligned}((1\otimes\tau)(k\otimes s))(\ell\sharp u) &= (k\otimes\tau(s))(\ell\sharp u)\\ &= \sum k((\tau(s))_1\ell)\otimes(\tau(s))_0(1\otimes u)\\ &= \sum k(s_1\ell)\otimes\tau(s_0)(1\otimes u)\\ &= \sum k(s_1\ell)\otimes\tau(s_0u)\\ &= (1\otimes\tau)((k\otimes s)(\ell\sharp u)),\end{aligned}$$

$1\otimes\tau$ is a right $K\sharp S$-module map. Now consider the following sequences of maps from $(K\sharp S)\otimes_{K\sharp T}(K\sharp S)$ to $K\sharp S$:

$$\begin{aligned}(1\otimes\mu)\varphi &: (K\sharp S)\otimes_{K\sharp T}(K\sharp S)\to K\otimes S\otimes_T S\to K\otimes S,\\ \theta\mu_{K\sharp S} &: (K\sharp S)\otimes_{K\sharp T}(K\sharp S)\to K\sharp S\to K\otimes S,\end{aligned}$$

where θ is the canonical isomorphism, $\mu_{K\sharp S}$ is the multiplication of $K\sharp S$ and φ is defined in Lemma 3.2. Then we see $(1\otimes\mu)\varphi = \theta\mu_{K\sharp S}$ and by Lemma 3.2, φ is a $K\sharp S$-bimodule isomorphism. Thus $\varphi^{-1}(1\otimes\tau)\theta$ is a $K\sharp S$-bimodule map such that $\varphi^{-1}(1\otimes\tau)\theta\mu_{K\sharp S} = 1$.

It is know that a ring extension S/T is a separable extension if and only if there exist finite elements x_i, $y_i \in S$ such that

$$\sum x_i y_i = 1 \quad and \quad \sum x x_i \otimes_T y_i = \sum x_i \otimes_T y_i x \quad for\ any\ x \in S$$

(cf. [DI, Chap.II, Prop.1.1]). $\sum x_i \otimes_T y_i$ is called a *separability idempotent* and $\{x_i, y_i\}$ is called a *separable coordinate system*. Using this fact, we prove the following theorem.

Theorem 3.6. *Let K be a left A-module algebra such that R is an R-direct summand of K. If $K\sharp S$ is a separable extension of K and if the projection map $p : K \to R$ is a left A-module map, then S is a separable algebra.*

Proof. Let $x_i = \sum_j k_{ij}\sharp s_{ij}$ and $y_i = \sum_j \ell_{ij}\sharp u_{ij}$ be a separble coordinate system of $K\sharp S$ over K. We set

$$a_i = \sum_j p(k_{ij})s_{ij} \quad and \quad b_i = \sum_j p(\ell_{ij})u_{ij}.$$

Since

$$1\sharp 1 = \sum x_i y_i = \sum_{i,j,n,(s_{ij})} k_{ij}(s_{ij})_1\ell_{in}\sharp(s_{ij})_0 u_{in},$$

we have

$$\begin{aligned} 1 = p(1)1 &= \sum_{i,j,n,(s_{ij})} p(k_{ij}(s_{ij})_1\ell_{in})\sharp(s_{ij})_0 u_{in} \\ &= \sum_{i,j,n,(s_{ij})} p(k_{ij})(s_{ij})_1 p(\ell_{in})\sharp(s_{ij})_0 u_{in} \\ &= \sum_{i,j,n} p(k_{ij})s_{ij}p(\ell_{in})\sharp u_{in} \\ &= \sum_i a_i b_i \end{aligned}$$

Moreover by $\sum(1\sharp s)x_i \otimes y_i = \sum x_i \otimes y_i(1\sharp s)$, we get

$$\sum_{i,j,(s)} (s_1(k_{ij})\sharp s_0 s_{ij}) \otimes (\ell_{in}\sharp u_{in}) = \sum_{i,j,n} (k_{ij}\sharp s_{ij}) \otimes (\ell_{in}\sharp u_{in}s).$$

Applying the map $(p\sharp 1)\otimes(p\sharp 1)$ on the both side, we have

$$\sum_{i,j,n} p(k_{ij})ss_{ij} \otimes p(\ell_{in})u_{in} = \sum_{i,j,n} p(k_{ij})s_{ij} \otimes p(\ell_{in})u_{in}s,$$

that is,

$$\sum sa_i \otimes b_i = \sum a_i \otimes b_i s.$$

This shows that $\{a_i, b_i\}$ is a separable coordinate system of S over R.

4. Extension by comodule algebras.

In this section we dualize the results in section 3 by taking the smash product $(-)\sharp S$. Since the proofs are similar to the corresponding results, we omit them.

Lemma 4.1. *For any k, ℓ, $m \in K$ and s, $u \in S$, if we define*

$$(k\sharp s)(\ell \otimes m \otimes u) = \sum k(s_2\ell) \otimes s_1 m \otimes s_0 u,$$
$$(\ell \otimes m \otimes u)(k\sharp s) = \sum \ell \otimes m(u_1 k) \otimes u_0 s,$$

then $K \otimes_L K \otimes S$ is a $K\sharp S$-bimodule structure.

Lemma 4.2. *The map*

$$\varphi : (K\sharp S) \otimes_{L\sharp S} (K\sharp S) \ni (\ell\sharp u) \otimes (m\sharp v) \mapsto \sum \ell \otimes (u_1 m) \otimes u_0 v \in K \otimes_L K \otimes S$$

is a $K\sharp S$-bimodule isomorphism.

Lemma 4.3. *Let $\mu : K \otimes_L K \ni k \otimes \ell \mapsto k\ell \in K$. Then the map $\mu \otimes 1 : K \otimes_L K \otimes S \to K \otimes S$ is a $K\sharp S$-bimodule map, where $K \otimes S$ is the canonical $K\sharp S$-bimodule by the multiplication and $K\sharp S$-bimodule structure of $K \otimes_L K \otimes S$ is defined in Lemma 4.1*

Let K be an A-module algebra. Then $K \otimes_L K$ is a left A-module by $a(k \otimes \ell) = \sum a_1 k \otimes a_0 \ell$ and $\mu : K \otimes_L K \to K$ is a left A-module map. We call that a left A-module algebra K is *separable extension* of L in the category of left A-module if there exists a left A-module and a K-bimodule map $\tau : K \to K \otimes_L K$ such that $\mu\tau = 1$.

Theorem 4.4. *If K is a separable extension of L in the category of left A-module algebras, then $K\sharp S$ is a separable extension of $L\sharp S$ in the category of left A-module algebras.*

Theorem 4.5. *Let S be a right A-comodule algebra such that R is an R-direct summand of S. If $K\sharp S$ is a separable extension of S and if the projection map $p : S \to R$ is a right A-comodule map, then K is a separable algebra.*

For a Galois extension S/T in the category of right A-comodule algebras, or in the category of left A-module algebras, we can show that the corresponding theorems 3.5, 3.6, 4.4 and 4.5 hold for Galois extensions, and the assumption that A is commutative and cocommutative also leave out. Moreover we have an application of our theorems to the skew polynomial ring with respect to separable extensions and Galois extensions. These results will be showed in the forthcomming paper.

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