Journal of the Faculty of Environmental Science and Technology, Okayama University
Vol 5, No.1. pp.57-69, February 2000

The Resolution Modules of A Space and Its Universal Covering Space

Ryousuke FUJITA *

(Received January 12 , 2000)

Let G be a finite group, Y a finite connected G-CW-complex, and let $\Pi(Y)$ denote the G-poset (in the sense of Oliver-Petrie) associated to Y. They defined the abelian group $\Omega(G,\Pi(Y))$ consisting of all equivalent classes of $\Pi(Y)$-complexes. They also defined the subgroup $\Phi(G,\Pi(Y))$ related to $\Pi(Y)$-resolutions. We call $\Phi(G,\Pi(Y))$ the resolution module of Y. Applying the Oliver-Petrie theory to the universal covering space $\tilde{Y}$, we obtain the group $\Omega(\tilde{G},\Pi(\tilde{Y}))$, where $\tilde{G}$ is a certain extension of G by $\pi_1(Y)$. Then the canonical homomorphism $\nu : \Omega(\tilde{G},\Pi(\tilde{Y})) \to \Omega(G,\Pi(Y))$ induced by the projection $\tilde{Y} \to Y$ is an isomorphism. In this paper, for $G = \mathbb{Z}_p \times \mathbb{Z}_q$ we construct a finite G-CW-complex Y such that $\pi_1(Y) \cong \mathbb{Z}_p$ and $\nu(\Phi(\tilde{G},\Pi(\tilde{Y})) \neq \Phi(G,\Pi(Y))$, where p and q are arbitrary distinct primes.

Keywords: G-CW-complex, G-map, G-poset

1 INTRODUCTION

Throughout this paper let G be a finite group and $S(G)$ denote the set of all subgroups of G. Let $f : X \to Y$ be a G-map between finite G-CW-complexes. When does there exist a G-CW-complex $X' \supsetneqq X$ with ${X'}^G = X^G$ and a quasi-equivalence $f' : X' \to Y$ extending f? Here a *quasi-equivalence* $f' : X' \to Y$ means that f' is a G-map inducing an isomorphism on π_1 and integral homology. R.Oliver and T.Petrie treated this problem in [5]. To solve the problem, they introduced the set

$$\Pi(Y) = \coprod_{H \in S(G)} \pi_0(Y^H) \quad \text{(the disjoint union of } \pi_0(Y^H)\text{'s)}.$$

Here Y^H is the H-fixed point set of Y and $\pi_0(Y^H)$ is the set of all connected components of Y^H. The set $\Pi(Y)$ is called a *G-poset associated to Y*. We regard $S(G)$ as a G-set via the action $(g,H) \mapsto gHg^{-1}(g \in G$ and $H \in S(G))$ and as a partially ordered set via

$$H < K \Longleftrightarrow H \supsetneqq K \quad (H, K \in S(G)).$$

Let $S(Y)$ denote the set of all subcomplexes of Y. We also regard $S(Y)$ as a G-set by left traslation, i.e. $(g, A) \mapsto gA$ $(g \in G$ and $A \in S(Y))$. Suppose that $S(G) \times S(Y)$ has the diagonal action, i.e. $(g,(H,A)) \mapsto (gHg^{-1}, gA)$ $(g \in G, H \in S(G), A \in S(Y))$.

For $\alpha \in \Pi(Y)$, there exists uniquely a subgroup $H \in S(G)$ such that $\alpha \in \pi_0(Y^H)$. Hence we can define a map $\rho : \Pi(Y) \to S(G)$ by $\alpha \mapsto H$. In addition, $\Pi(Y)$ is given the partial order $\leqq$ by

$$\alpha \leqq \beta \text{ if and only if } \rho(\alpha) \supseteqq \rho(\beta) \text{ and } |\alpha| \subseteqq |\beta| \quad (\alpha, \beta \in \Pi(Y))$$

where $|\alpha|$ is the underlying space for $\alpha \in \Pi(Y)$.

*Liberal Arts of General Education, Wakayama National College of Technology, Wakayama, 644-0023 Japan.
Current Address : Department of Environmental and Mathematical Sciences, Faculty of Environmental Science and Technology, Okayama University, Okayama, 700-8530 Japan. Communicated with Prof. Masaharu Morimoto

Definition 1.1. We abbreviate $\Pi(Y)$ to Π. A finite G-CW-complex Z with a basepoint q is called a Π-*complex* if it is equipped with a specified set $\{Z_\alpha \mid \alpha \in \Pi\}$ of subcomplexes Z_α of Z, satisfying the following four conditions:
(i) $q \in Z_\alpha$,
(ii) $gZ_\alpha = Z_{g\alpha}$ for $g \in G$, $\alpha \in \Pi$,
(iii) $Z_\alpha \subseteqq Z_\beta$ if $\alpha \leqq \beta$ in Π, and
(iv) for any $H \in S(G)$,

$$Z^H = \bigvee_{\alpha\in\Pi \text{ with } \rho(\alpha)=H} Z_\alpha.$$

Let $\mathcal{F}$ denote the family of all Π-complexes and define the equivalence relation $\sim$ on $\mathcal{F}$ by

$$Z \sim W \iff \chi(Z_\alpha) = \chi(W_\alpha) \quad \text{for all } \alpha \in \Pi \quad (Z, W \in \mathcal{F})$$

where $\chi(Z_\alpha)$ is the Euler characteristic of Z_α.

The set $\Omega(G,\Pi) = \mathcal{F}/\!\sim$ is an abelian group via

$$[Z] + [W] = [Z \vee W] \quad (Z, W \in \mathcal{F}).$$

Moreover it is finitely generated. We call $\Omega(G,\Pi)$ the *Oliver-Petrie module associated with* Π.

The set

$$\Delta(G,\Pi) = \{[Z] \in \Omega(G,\Pi) \mid Z \text{ is contractible } \}$$

is a submodule of $\Omega(G,\Pi)$. By [5, Proposition 2.6] the submodule $\Phi(G,\Pi)$ given below is useful for computing $\Delta(G,\Pi)$, since

$$\Phi(G,\Pi) \supset \Delta(G,\Pi) \text{ and } [\Phi(G,\Pi) : \Delta(G,\Pi)] < \infty.$$

We define

$$\mathcal{P}(\Pi) = \{\alpha \in \Pi \mid \rho(\alpha) \text{ is a subgroup of } G \text{ of prime power order}\}, \text{ and}$$
$$S(G,\ \alpha) = \{K \in S(G) \mid \rho(\alpha) \triangleleft K \subseteqq G_\alpha \text{ and } K/\rho(\alpha) \text{ is cyclic}\}$$

where G_α is the isotropy subgroup at α. We set $\bar{\chi}(Z) = \chi(Z) - 1$ for any space Z. Then the *resolution module* $\Phi(G,\Pi)$ is defined by

$$\Phi(G,\Pi) = \{[Z] \in \Omega(G,\Pi) \mid \bar{\chi}((Z_\alpha)^K) = 0, \text{ for all } \alpha \in \mathcal{P}(\Pi) \text{ and } K \in S(G,\ \alpha)\}.$$

It is easy to check that $\Phi(G,\Pi)$ is a subgroup of $\Omega(G,\Pi)$. This $\Phi(G,\Pi)$ can be defined in the term of Π-resolutions, which will be explained in 2.3. Applying the Oliver-Petrie theory to a covering space, M.Morimoto and K.Iizuka [4] gave a necessary and sufficient condition to extend a G-map $f : X \to Y$ to a pseudo-equivalence $f'' : X'' \to Y$ such that $X''^G = X^G$ when $\pi_1(Y)$ is finite. Here a pseudo-equivalence f'' means a G-map which is a (non-equivariant) homotopy equivalence.

Let G and $\widetilde{G}$ be finite groups, $\sigma : \widetilde{G} \to G$ an epimorphism, Y a finite connected G-CW-complex, $\widetilde{Y}$ a finite connected $\widetilde{G}$-CW-complex, and $(\widetilde{Y},\ p,\ Y)$ a σ-equivariant covering space (i.e. $p(gb) = \sigma(g)p(b)$ for $g \in \widetilde{G}$, $b \in \widetilde{Y}$). Put $\pi = \ker\sigma$. Furthermore assume that π acts freely and transitively on each fiber. Under the conditions, the canonical map $\nu : \Omega(\widetilde{G},\Pi(\widetilde{Y})) \to \Omega(G,\Pi(Y))$ is defined by $[\widetilde{X}] \mapsto [G \times_\sigma \widetilde{X}]$ and it is an isomorphism. As for the resolution submodules, we have $\nu(\Delta(\widetilde{G},\Pi(\widetilde{Y}))) \subseteqq \Delta(G,\Pi(Y))$ and $\nu(\Phi(\widetilde{G},\Pi(\widetilde{Y}))) \subseteqq \Phi(G,\Pi(Y))$ [4, Proposition 3.6]. In the present paper, we study the next problem :

Problem Do there exist G-CW-complexes Y such that

$$\nu(\Phi(\widetilde{G},\Pi(\widetilde{Y})) \neq \Phi(G,\Pi(Y)) \ ?$$

Our result is:

Theorem 1.2. *Let p, q be distinct primes, $G = \mathbb{Z}_p \times \mathbb{Z}_q$ and $\widetilde{G} = \pi \times (\mathbb{Z}_p \times \mathbb{Z}_q)$, where π is a copy of $\mathbb{Z}_p$. Then there exists a finite connected and simply connected $\widetilde{G}$-CW-complex $\widetilde{Y}$ such that the G-CW-complex $Y = \widetilde{Y}/\pi$ satisfies $\pi_1(Y) \cong \pi$ and $\nu(\Phi(\widetilde{G},\Pi(\widetilde{Y})) \neq \Phi(G,\Pi(Y))$.*

This paper is organized as follows. In Section 2, we review basic properties of the Oliver-Petrie module and the resolution module. In Section 3, we study relations between the posets of a base space and its covering space. Finally, in Section 4, we prove Theorem 1.2.

2 BASIC PROPERTIES OF THE OLIVER-PETRIE MODULES

In this section, we recall basic properties needed later from R.Oliver-T.Petrie [5] and M.Morimoto-K.Iizuka [4].

2.1 For a finite G-CW-complex Y, the map $\rho \times |\ | : \Pi(Y) \to S(G) \times S(Y)$ given by $\alpha \mapsto (\rho(\alpha), |\alpha|)$ is injective. We regard $\Pi(Y)$ as a subset of $S(G) \times S(Y)$. Then $\Pi = \Pi(Y)$ has a G-action given by $(g,\ \alpha) \mapsto g(\rho \times |\ |)(\alpha)$. Furthermore Π satisfies the following three conditions:
(i) $\rho(\alpha) \subseteqq G_\alpha$ for $\alpha \in \Pi$,
(ii) if $\alpha \leqq \beta$ then $g\alpha \leqq g\beta$ for $g \in G$, and
(iii) for $\alpha \in \Pi$ and $H \subseteqq \rho(\alpha)$, there exists uniquely $\gamma \in \Pi$ such that $\gamma \geqq \alpha$ and $\rho(\gamma) = H$.

In the case where $Y = \{*\}$ (a singleton),

$$\Pi(Y) = \coprod_{H \in S(G)} \pi_0(\{*\}^H) \overset{\rho \times |\ |}{=} \coprod_{H \in S(G)} \{(H,\ \{*\})\} \overset{\text{proj}}{=} S(G).$$

Let Z be a Π-complex. For each cell c in $Z \setminus \{*\}$, there exists a unique element $\alpha(c) \in \Pi$ such that $\rho(\alpha(c)) = G_x,\ \ x \in c,$ and $c \subset Z_{\alpha(c)}$. We say that c *of type* $\alpha(c)$.

2.2 For each $\alpha \in \Pi(Y)$, the G-space $(\alpha)^+ = G/\rho(\alpha) \amalg \{*\}$ is equipped with $\Pi(Y)$-complex structure such that

$$(\alpha)^+_\beta = \{g\rho(\alpha) \mid g \in G,\ \ g\alpha \leqq \beta\} \amalg \{*\} \quad \text{for } \beta \in \Pi(Y).$$

Let $\{\alpha_i \mid 1 \leqq i \leqq s\}$ be the complete representative system of $\Pi(Y)/G$. Then the set $\Omega(G, \Pi(Y))$ is a free abelian group with a basis $\{[(\alpha_i)^+] \mid 1 \leqq i \leqq s\}$ i.e.

$$\Omega(G, \Pi(Y)) = \langle [(\alpha_i)^+] \mid 1 \leqq i \leqq s \rangle_{\mathbf{Z}}.$$

Suppose hereafter that Y is a finite connected G-CW-complex. Then $\pi_0(Y^{\{1\}})$ consists of a unique element which will be denoted by $\mathfrak{m}$. The element $\mathfrak{m}$ is the maximal element in $\Pi(Y)$.

2.3 A finite k-dimensional $\Pi(Y)$-complex Z is called a $\Pi(Y)$-*resolution* if Z satisfies the following three conditions:
(i) Z is connected and simply-connected,
(ii) Z is $(k-1)$-connected, and
(iii) $\widetilde{H}_k(Z\ ; \mathbb{Z})$ is $\mathbb{Z}[G]$-projective.

If Z is a k-dimensional $\Pi(Y)$-resolution, set

$$\gamma_G(Z) = (-1)^k[\widetilde{H}_k(Z\ ; \mathbb{Z})] \in \widetilde{K}_0(\mathbb{Z}[G]),$$

where $\widetilde{K}_0(\mathbb{Z}[G])$ is the Grothendieck group of finitely generated projective $\mathbb{Z}[G]$-modules modulo free modules.

For a $\Pi(Y)$-resolution Z, we get a $\Pi(Y)$-complex Z^* with $\bar{\chi}(Z^*) = 0$ by attaching some free cells $G \times D^i$ to Z. Clearly $\bar{\chi}(Z^*_\alpha) = \bar{\chi}(Z_\alpha)$ for any $\alpha \in \Pi(Y) \setminus \{\mathfrak{m}\}$. Moreover for a k-dimensional $\Pi(Y)$-resolution Z with $k \geqq 1$, there exists a $\Pi(Y)$-resolution W satisfying the following conditions:
(i) $\dim W = k+1$,
(ii) $\gamma_G(Z) = \gamma_G(W)$, and
(iii) $[Z^*] = [W^*]$ in $\Omega(G, \Pi(Y))$.

By [5, Proposition 2.6], $\Phi(G, \Pi(Y))$ defined in Section 1 coincides with

$$\{[Z^*] \in \Omega(G, \Pi(Y)) |\ Z \text{ is a } \Pi(Y)\text{-resolution}\}.$$

Example 2.4. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $Y = \{*\}$ (a singleton). There are three subgroups isomorphic to $\mathbb{Z}_2$. We denote them by $\mathbb{Z}_2^1,\ \mathbb{Z}_2^2,\ \mathbb{Z}_2^3$. By 2.1,

$$\Pi(\{*\}) = S(G) = \{\{1\},\ \mathbb{Z}_2^1,\ \mathbb{Z}_2^2,\ \mathbb{Z}_2^3,\ \mathbb{Z}_2 \times \mathbb{Z}_2\}.$$

The partially ordered set $\Pi(\{*\})$ is illustrated by the diagram below. We arrange the elements of $\Pi(\{*\})$ such that if $a > b$ $(a,\ b \in \Pi(\{*\}))$, then a is situated above b. Furthermore we connect a and b by a segment if and only if $a > b$.

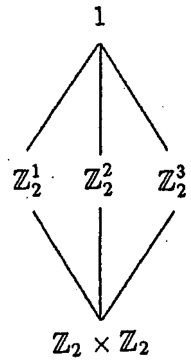

Fig.1

Since G is of prime power order, $\mathcal{P}(\Pi(\{*\}))$ coincides with $\Pi(\{*\})$. As G is abelian, the G-action on $\Pi(\{*\}) = S(G)$ is trivial, which amounts to

$$\Pi(\{*\})/G = S(G)/G = S(G).$$

By 2.2, the free abelian group $\Omega(G, \Pi(Y))$ has the basis

$$\{\ [(\{1\})^+],\ [(\mathbb{Z}_2^1)^+],\ [(\mathbb{Z}_2^2)^+],\ [(\mathbb{Z}_2^3)^+],\ [(\mathbb{Z}_2 \times \mathbb{Z}_2)^+]\ \}.$$

In the following, we show that $\Phi(G,\ \Pi(\{*\}))$ is the trivial group. Each $[Z] \in \Phi(G,\ \Pi(\{*\}))$ is uniquely written in the form:

$$[Z] = n_{\mathbb{Z}_2\times\mathbb{Z}_2}[(\mathbb{Z}_2 \times \mathbb{Z}_2)^+] + n_{\mathbb{Z}_2^1}[(\mathbb{Z}_2^1)^+] + n_{\mathbb{Z}_2^2}[(\mathbb{Z}_2^2)^+] + n_{\mathbb{Z}_2^3}[(\mathbb{Z}_2^3)^+] + n_{\{1\}}[(\{1\})^+],$$

where each coefficient is some integer and satisfies the condition

$$\begin{aligned} \bar{\chi}(Z_\alpha^K) &= n_{\mathbb{Z}_2\times\mathbb{Z}_2}\bar{\chi}((\mathbb{Z}_2 \times \mathbb{Z}_2)_\alpha^{+K}) + n_{\mathbb{Z}_2^1}\bar{\chi}((\mathbb{Z}_2^1)_\alpha^{+K}) + n_{\mathbb{Z}_2^2}\bar{\chi}((\mathbb{Z}_2^2)_\alpha^{+K}) \\ &\quad + n_{\mathbb{Z}_2^3}\bar{\chi}((\mathbb{Z}_2^3)_\alpha^{+K}) + n_{\{1\}}\bar{\chi}((\{1\})_\alpha^{+K}) \\ &= 0 \end{aligned} \tag{2.4.1}$$

for each $\alpha \in \mathcal{P}(\Pi(\{*\}))$ and $K \in S(G,\ \alpha)$. Using (2.4.1), we shall verify that all coefficients vanish.

First, consider the case of $\alpha = \mathbb{Z}_2^1$. Then we have $S(G,\ \alpha) = \{\mathbb{Z}_2^1,\ \mathbb{Z}_2^1 \times \mathbb{Z}_2\}$. For $\alpha = \mathbb{Z}_2^1$ and $K = \mathbb{Z}_2^1$, since

$$\begin{aligned} \bar{\chi}((\mathbb{Z}_2 \times \mathbb{Z}_2)_{\mathbb{Z}_2^1}^{+\mathbb{Z}_2^1}) &= \bar{\chi}(\{\mathbb{Z}_2 \times \mathbb{Z}_2\} \amalg \{*\}) = 1, \\ \bar{\chi}((\mathbb{Z}_2^1)_{\mathbb{Z}_2^1}^{+\mathbb{Z}_2^1}) &= \bar{\chi}(G/\mathbb{Z}_2^1 \amalg \{*\}) = 2, \quad \text{and} \\ \bar{\chi}((\mathbb{Z}_2^2)_{\mathbb{Z}_2^1}^{+\mathbb{Z}_2^1}) &= \bar{\chi}((\mathbb{Z}_2^3)_{\mathbb{Z}_2^1}^{+\mathbb{Z}_2^1}) = \bar{\chi}((\{1\})_{\mathbb{Z}_2^1}^{+\mathbb{Z}_2^1}) = \bar{\chi}(\emptyset \amalg \{*\}) = 0, \end{aligned}$$

the equation (2.4.1) implies

$$n_{\mathbb{Z}_2\times\mathbb{Z}_2} + 2n_{\mathbb{Z}_2^1} = 0. \tag{2.4.2}$$

Next for $\alpha = \mathbb{Z}_2^1$ and $K = \mathbb{Z}_2^1 \times \mathbb{Z}_2$, since

$$\begin{aligned} \bar{\chi}((\mathbb{Z}_2 \times \mathbb{Z}_2)_{\mathbb{Z}_2^1}^{+\mathbb{Z}_2^1\times\mathbb{Z}_2}) &= \bar{\chi}(\{\mathbb{Z}_2 \times \mathbb{Z}_2\} \amalg \{*\}) = 1, \quad \text{and} \\ \bar{\chi}((\mathbb{Z}_2^1)_{\mathbb{Z}_2^1}^{+\mathbb{Z}_2^1\times\mathbb{Z}_2}) &= \bar{\chi}((\mathbb{Z}_2^2)_{\mathbb{Z}_2^1}^{+\mathbb{Z}_2^1\times\mathbb{Z}_2}) = \bar{\chi}((\mathbb{Z}_2^3)_{\mathbb{Z}_2^1}^{+\mathbb{Z}_2^1\times\mathbb{Z}_2}) \\ &= \bar{\chi}((\{1\})_{\mathbb{Z}_2^1}^{+\mathbb{Z}_2^1\times\mathbb{Z}_2}) = \bar{\chi}(\emptyset \amalg \{*\}) = 0, \end{aligned}$$

we obtain

$$n_{\mathbb{Z}_2\times\mathbb{Z}_2} = 0. \tag{2.4.3}$$

We get $n_{\mathbb{Z}_2^1} = 0$, $n_{\mathbb{Z}_2\times\mathbb{Z}_2} = 0$ by (2.4.2) and (2.4.3). Similarly for $\alpha = \mathbb{Z}_2^2$ and $\mathbb{Z}_2^3$, we have $n_{\mathbb{Z}_2^2} = 0$ and $n_{\mathbb{Z}_2^3} = 0$. Moreover the case where $\alpha = \{1\}$, we have

$$S(G,\ \alpha) = \{\{1\},\ \mathbb{Z}_2^1,\ \mathbb{Z}_2^2,\ \mathbb{Z}_2^3\}.$$

Particularly, in the case where $\alpha = \{1\}$, $K = \{1\}$, we have

$$\begin{aligned} 0 &= n_{\{1\}}\bar{\chi}((\{1\})^{+\{1\}}_{\{1\}}) \\ &= n_{\{1\}}\chi(G) \\ &= 4n_{\{1\}}. \end{aligned}$$

Hence $n_{\{1\}} = 0$. Putting all together,

$$n_{\mathbb{Z}_2\times\mathbb{Z}_2} = n_{\mathbb{Z}_2^1} = n_{\mathbb{Z}_2^2} = n_{\mathbb{Z}_2^3} = n_{\{1\}} = 0.$$

This concludes $[Z] = 0$.

3 RELATIONS BETWEEN THE POSETS OF A BASE SPACE AND ITS COVERING SPACE

In this section let G and $\widetilde{G}$ be finite groups, $\sigma : \widetilde{G} \to G$ an epimorphism, Y a finite connected G-CW-complex, $\widetilde{Y}$ a finite connected $\widetilde{G}$-CW-complex, and $p : \widetilde{Y} \to Y$ a σ-equivariant covering space. We put $\pi = \ker\sigma$. Moreover we assume that π acts freely and transitively on each fiber. Remark that the $\widetilde{G}$-action on $\widetilde{Y}$ gives a $\widetilde{G}$-poset $\widetilde{\Pi} = \Pi(\widetilde{Y})$ and a $\widetilde{G}$-map $\widetilde{\rho} : \widetilde{\Pi} \to S(\widetilde{G})$.

Let $\widetilde{\alpha}$ be an element of $\Pi(\widetilde{Y})$. Then $|\widetilde{\alpha}|$ is a connected component of $\widetilde{Y}^{\widetilde{\rho}(\widetilde{\alpha})}$. Hence $p(|\widetilde{\alpha}|)$ is connected. Moreover we have $p(|\widetilde{\alpha}|) \subseteq Y^{\sigma(\widetilde{\rho}(\widetilde{\alpha}))}$. Thus there exists a unique connected component $\alpha \in \Pi(Y)$ such that $\rho(\alpha) = \sigma(\widetilde{\rho}(\widetilde{\alpha}))$ and $|\alpha| \supseteq p(|\widetilde{\alpha}|)$. Now we define the map $\mu : \Pi(\widetilde{Y}) \to \Pi(Y)$ by $\widetilde{\alpha} \mapsto \alpha$.

Lemma 3.1. *In the above situation, $\rho(\mu(\widetilde{\alpha})) = \sigma(\widetilde{\rho}(\widetilde{\alpha}))$ and $|\mu(\widetilde{\alpha})| = p(|\widetilde{\alpha}|)$ hold for any $\widetilde{\alpha} \in \Pi(\widetilde{Y})$.*

Proof. We have already showed $\rho(\mu(\widetilde{\alpha})) = \sigma(\widetilde{\rho}(\widetilde{\alpha}))$. It suffices to show that $|\alpha| \subseteq p(|\widetilde{\alpha}|)$, where $\alpha = \mu(\widetilde{\alpha})$. First we take $\widetilde{y}_0 \in |\widetilde{\alpha}|$, and set $y_0 = p(\widetilde{y}_0)$. Take $y_1 \in |\alpha|$ arbitrarily. Remark that $y_0 \in |\alpha|$ and $y_1 \in |\alpha|$. Then there exists a path $y(t) : I \to |\alpha|$ such that $y(0) = y_0$ and $y(1) = y_1$, where $I = [0,\ 1]$. Then we have a lift $\widetilde{y}(t) : I \to \widetilde{Y}$ of $y(t)$ with $\widetilde{y}(0) = \widetilde{y}_0$. On the other hand, for any $\widetilde{g} \in \widetilde{\rho}(\widetilde{\alpha})$, a path $\widetilde{g}\widetilde{y}(t) : I \to \widetilde{Y}$ is also a lift of $y(t)$ with $\widetilde{g}\widetilde{y}(0) = \widetilde{y}_0$. Hence we have $\widetilde{g}\widetilde{y}(t) = \widetilde{y}(t)$ for any $\widetilde{g} \in \widetilde{\rho}(\widetilde{\alpha})$. It follows at once that $\widetilde{y}(1) \in \widetilde{Y}^{\widetilde{\rho}(\widetilde{\alpha})}$. Since $\widetilde{y}_0 \in |\widetilde{\alpha}| \subseteq \widetilde{Y}^{\widetilde{\rho}(\widetilde{\alpha})}$, we have $\widetilde{y}(1) \in |\widetilde{\alpha}|$. Thus $y_1 = p(\widetilde{y}(1)) \in p(|\widetilde{\alpha}|)$. This means that $|\alpha| \subseteq p(|\widetilde{\alpha}|)$. □

By Lemma 3.1, the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\Pi} = \Pi(\widetilde{Y}) & \xrightarrow{\widetilde{\rho}\times|\ |} & S(\widetilde{G}) \times S(\widetilde{Y}) \\ \mu\downarrow & & \downarrow\sigma\times p \\ \Pi = \Pi(Y) & \xrightarrow[\rho\times|\ |]{} & S(G) \times S(Y). \end{array}$$

Proposition 3.2. *For any $\alpha \in \Pi(Y)$, $\mu^{-1}(\alpha)$ is non-empty. Moreover π acts transitively on $\mu^{-1}(\alpha)$.*

Proof. We first show that for any $\alpha \in \Pi(Y)$, $\mu^{-1}(\alpha)$ is non-empty. Arbitrarily choose and fix $y \in |\alpha|$. Since $p : \widetilde{Y} \to Y$ is surjective, there exists $\widetilde{y} \in p^{-1}(y)$. Now, remark that $\sigma|\ \widetilde{G}_{\widetilde{y}} : \widetilde{G}_{\widetilde{y}} \to G_y$ is an isomorphism. Since $y \in |\alpha| \subseteq Y^{\rho(\alpha)}$, we have $\rho(\alpha) \subseteq G_y$. Put $\widetilde{H} = (\sigma|\ \widetilde{G}_{\widetilde{y}})^{-1}(\rho(\alpha))$. Since $\widetilde{H} \subseteq \widetilde{G}_{\widetilde{y}}$, $\widetilde{y}$ lies in $\widetilde{Y}^{\widetilde{H}}$. Hence there exists $\widetilde{\alpha} \in \pi_0(\widetilde{Y}^{\widetilde{H}})$ with $\widetilde{y} \in |\widetilde{\alpha}|$, which implies $\widetilde{\rho}(\widetilde{\alpha}) = \widetilde{H}$. Thus we obtain $\rho(\mu(\widetilde{\alpha})) = \sigma(\widetilde{\rho}(\widetilde{\alpha})) = \sigma(\widetilde{H}) = \rho(\alpha)$, $y = p(\widetilde{y}) \in p(|\widetilde{\alpha}|) = |\mu(\widetilde{\alpha})|$, and $y \in |\mu(\widetilde{\alpha})| \cap |\alpha| \neq \emptyset$. It follows at once that $\mu(\widetilde{\alpha}) = \alpha$. Namely, $\mu^{-1}(\alpha)$ is non-empty.

Next we shall prove that π $(=\ker\sigma)$ acts transitively on $\mu^{-1}(\alpha)$. Let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be elements of $\mu^{-1}(\alpha)$. It suffices to show that $\widetilde{h}\widetilde{\alpha}=\widetilde{\beta}$ for some $\widetilde{h}\in\pi$. By the definition of μ, we have $\sigma(\widetilde{\rho}(\widetilde{\alpha}))=\rho(\alpha)=\sigma(\widetilde{\rho}(\widetilde{\beta}))$ and $p(|\widetilde{\alpha}|)=|\alpha|=p(|\widetilde{\beta}|)$. Let $\widetilde{a}$ and $\widetilde{b}$ be the points on $|\widetilde{\alpha}|$ and $|\widetilde{\beta}|$ respectively such that $p(\widetilde{a})=y=p(\widetilde{b})$. Then there exists $\widetilde{h}\in\pi$ such that $\widetilde{h}\widetilde{a}=\widetilde{b}$ because π acts transitively on each fiber. Now, it should be noted that $\widetilde{\rho}(\widetilde{\alpha})\subseteqq\widetilde{G}_{\widetilde{a}}$ and $\widetilde{\rho}(\widetilde{\beta})\subseteqq\widetilde{G}_{\widetilde{b}}$. Observe that $\widetilde{G}_{\widetilde{b}}=\widetilde{G}_{\widetilde{h}\widetilde{a}}=\widetilde{h}\widetilde{G}_{\widetilde{a}}\widetilde{h}^{-1}$. Remark that $\sigma|\ \widetilde{G}_{\widetilde{b}}$ is an isomorphism from $\widetilde{G}_{\widetilde{b}}$ to G_y. Now, since $\widetilde{\rho}(\widetilde{\alpha})\subseteqq\widetilde{G}_{\widetilde{a}}$, we have $\widetilde{h}\widetilde{\rho}(\widetilde{\alpha})\widetilde{h}^{-1}\subseteqq\widetilde{h}\widetilde{G}_{\widetilde{a}}\widetilde{h}^{-1}=\widetilde{G}_{\widetilde{b}}$. Moreover since $\widetilde{\rho}(\widetilde{\beta})\subseteqq\widetilde{G}_{\widetilde{b}}$, we have $\sigma(\widetilde{h}\widetilde{\rho}(\widetilde{\alpha})\widetilde{h}^{-1})=\sigma(\widetilde{h})\sigma(\widetilde{\rho}(\widetilde{\alpha}))\sigma(\widetilde{h}^{-1})=\rho(\alpha)$. Recalling that $\sigma(\widetilde{\rho}(\widetilde{\beta}))=\rho(\alpha)$, we get $\widetilde{h}\widetilde{\rho}(\widetilde{\alpha})\widetilde{h}^{-1}=\widetilde{\rho}(\widetilde{\beta})$, that is, $\widetilde{\rho}(\widetilde{h}\widetilde{\alpha})=\widetilde{\rho}(\widetilde{\beta})$. Therefore we have $\widetilde{h}\widetilde{\alpha},\ \widetilde{\beta}\in\pi_0(\widetilde{Y}^{\widetilde{\rho}(\widetilde{\beta})})$. Remark that $\widetilde{b}=\widetilde{h}\widetilde{a}\in\widetilde{h}|\widetilde{\alpha}|=|\widetilde{h}\widetilde{\alpha}|$. It follows at once that $\widetilde{b}\in|\widetilde{h}\widetilde{\alpha}|\cap|\widetilde{\beta}|\neq\emptyset$. Thus $\widetilde{h}\widetilde{\alpha}=\widetilde{\beta}$. □

Henceforth let $\{\widetilde{\alpha}_1,\ \widetilde{\alpha}_2,\ \cdots,\ \widetilde{\alpha}_s\}$ be a complete representative system of $\Pi(\widetilde{Y})/\widetilde{G}$, that is,

$$\Pi(\widetilde{Y})=\coprod_{i=1}^{s}\widetilde{G}\widetilde{\alpha}_i\ \text{(disjoint union)}.$$

Lemma 3.3. *For $i\neq j$, one has $\mu(\widetilde{G}\widetilde{\alpha}_i)\cap\mu(\widetilde{G}\widetilde{\alpha}_j)=\emptyset$.*

Proof. Suppose that $\mu(\widetilde{G}\widetilde{\alpha}_i)\cap\mu(\widetilde{G}\widetilde{\alpha}_j)\ni\alpha$. Then α is written in two ways: $\alpha=\mu(\widetilde{g}_1\widetilde{\alpha}_i)=\mu(\widetilde{g}_2\widetilde{\alpha}_j)$ for $\widetilde{g}_1,\ \widetilde{g}_2\in\widetilde{G}$. Since $\mu^{-1}(\alpha)\ni\widetilde{g}_1\widetilde{\alpha}_i,\ \widetilde{g}_2\widetilde{\alpha}_j$, by Proposition 3.2 there exists $\widetilde{h}\in\pi$ such that $\widetilde{g}_1\widetilde{\alpha}_i=\widetilde{h}(\widetilde{g}_2\widetilde{\alpha}_j)$. This means $\widetilde{g}_1\widetilde{\alpha}_i\in\widetilde{G}\widetilde{\alpha}_i\cap\widetilde{G}\widetilde{\alpha}_j$, so we get a contradiction. □

Next we shall show that μ is a σ-equivariant map.

Lemma 3.4. *For $\widetilde{g}\in\widetilde{G}$, $\widetilde{\alpha}\in\Pi(\widetilde{Y})$, one has $\mu(\widetilde{g}\widetilde{\alpha})=\sigma(\widetilde{g})\mu(\widetilde{\alpha})$.*

Proof. It suffices to show that $(\rho\times|\ \ |)(\mu(\widetilde{g}\widetilde{\alpha}))=(\rho\times|\ \ |)(\sigma(\widetilde{g})\mu(\widetilde{\alpha}))$. The following hold:

$$\begin{aligned}\rho(\mu(\widetilde{g}\widetilde{\alpha}))&=\sigma(\widetilde{\rho}(\widetilde{g}\widetilde{\alpha}))\\&=\sigma(\widetilde{g}\widetilde{\rho}(\widetilde{\alpha})\widetilde{g}^{-1})\\&=\sigma(\widetilde{g})\sigma(\widetilde{\rho}(\widetilde{\alpha}))\sigma(\widetilde{g})^{-1}\\&=\sigma(\widetilde{g})\rho(\mu(\widetilde{\alpha}))\sigma(\widetilde{g})^{-1}\\&=\rho(\sigma(\widetilde{g})\mu(\widetilde{\alpha})),\quad\text{and}\\|\mu(\widetilde{g}\widetilde{\alpha})|&=p(|\widetilde{g}\widetilde{\alpha}|)\\&=p(\widetilde{g}|\widetilde{\alpha}|)\\&=\sigma(\widetilde{g})p(|\widetilde{\alpha}|)\\&=\sigma(\widetilde{g})|\mu(\widetilde{\alpha})|\\&=|\sigma(\widetilde{g})\mu(\widetilde{\alpha})|.\end{aligned}$$

Hence we have

$$(\rho\times|\ \ |)(\mu(\widetilde{g}\widetilde{\alpha}))=(\rho\times|\ \ |)(\sigma(\widetilde{g})\mu(\widetilde{\alpha})).\quad\square$$

Using Lemmas 3.3 and 3.4, we show that $\Omega(\widetilde{G},\Pi(\widetilde{Y}))$ and $\Omega(G,\Pi(Y))$ are abstractly isomorphic.

Proposition 3.5. *Both $\Omega(\widetilde{G},\Pi(\widetilde{Y}))$ and $\Omega(G,\Pi(Y))$ have the same rank.*

Proof. Note that μ is surjective by Proposition 3.2. We have the following:

$$\begin{aligned}\Pi(Y)&=\mu(\Pi(\widetilde{Y}))\\&=\mu(\coprod_{i=1}^{s}\widetilde{G}\widetilde{\alpha}_i)\\&=\coprod_{i=1}^{s}\mu(\widetilde{G}\widetilde{\alpha}_i)\\&=\coprod_{i=1}^{s}\sigma(\widetilde{G})\mu(\widetilde{\alpha}_i)\\&=\coprod_{i=1}^{s}G\mu(\widetilde{\alpha}_i).\end{aligned}$$

Thus $\{\mu(\widetilde{\alpha}_1),\ \mu(\widetilde{\alpha}_2),\ \cdots,\ \mu(\widetilde{\alpha}_s)\}$ is a complete representatie system of $\Pi(Y)/G$. By 2.2, rank $(\Omega(\widetilde{G},\Pi(\widetilde{Y})))$ coincides with rank $(\Omega(G,\Pi(Y)))$. □

In the remainder of this section, we shall show that the canonical map $\nu : \Omega(\widetilde{G},\Pi(\widetilde{Y})) \to \Omega(G,\Pi(Y))$ is an isomorphism.

Definition 3.6. Given a $\widetilde{G}$-space $\widetilde{X}$, let $(g,\widetilde{x})$, $(g',\widetilde{x}') \in G \times \widetilde{X}$. Then we write $(g,\widetilde{x}) \sim (g',\widetilde{x}')$ to mean that there exists $\widetilde{g} \in \widetilde{G}$ such that $g' = g\sigma(\widetilde{g})^{-1}$, $\widetilde{x}' = \widetilde{g}\widetilde{x}$. This relation $\sim$ can be easily verified to be an equivalence relation. The quotient space $(G \times \widetilde{X})/\sim$ is denoted by $G \times_\sigma \widetilde{X}$.

Remark that G-action on $G \times_\sigma \widetilde{X}$ is naturally defined by $(g',[g,\widetilde{x}]) \mapsto [g'g,\widetilde{x}]$ for g', $g \in G$, and $\widetilde{x} \in \widetilde{X}$. We regard $G \times_\sigma \widetilde{X}$ as a G-space with respect to this action.

Suppose that $\widetilde{X}$ has a $\Pi(\widetilde{Y})$-complex structure $(\widetilde{X},\ \{\widetilde{X}_{\widetilde{\alpha}} \mid \widetilde{\alpha} \in \Pi(\widetilde{Y})\})$. Setting $X = G \times_\sigma \widetilde{X}$, we define the map $p_{\widetilde{X}} : \widetilde{X} \to X$ by $\widetilde{x} \mapsto [1,\widetilde{x}]$. Take the point of X to which $p_{\widetilde{X}}$ maps the basepoint of $\widetilde{X}$. For $\alpha \in \Pi(Y)$, we define

$$X_\alpha = \bigcup_{\widetilde{\alpha}\in\mu^{-1}(\alpha)} p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\alpha}}).$$

Let $\widetilde{\alpha}$ be an element of $\mu^{-1}(\alpha)$. Then $X_\alpha = p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\alpha}})$ holds. Indeed, it is easy to see that $p_{\widetilde{X}}$ is σ-equivariant. For $\widetilde{\beta} \in \mu^{-1}(\alpha)$, by Proposition 3.2 there exists $\widetilde{h} \in \pi$ such that $\widetilde{h}\widetilde{\alpha} = \widetilde{\beta}$. Thus we have

$$p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\beta}}) = p_{\widetilde{X}}(\widetilde{X}_{\widetilde{h}\widetilde{\alpha}}) = p_{\widetilde{X}}(\widetilde{h}\widetilde{X}_{\widetilde{\alpha}}) = \sigma(\widetilde{h})p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\alpha}}) = p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\alpha}}).$$

We need the next lemma to prove Lemma 3.8, and Proposition 3.10 will follow from Lemmas 3.8 and 3.9.

Lemma 3.7. *For $\widetilde{\alpha}$, $\widetilde{\beta} \in \Pi(\widetilde{Y})$ such that $|\widetilde{\alpha}| \cap |\widetilde{\beta}| = \emptyset$, one has $\widetilde{X}_{\widetilde{\alpha}} \cap \widetilde{X}_{\widetilde{\beta}} = \{*\}$.*

Proof. Suppose that $\widetilde{X}_{\widetilde{\alpha}} \cap \widetilde{X}_{\widetilde{\beta}} \neq \{*\}$. Then we can take a cell $\widetilde{e} \subseteqq (\widetilde{X}_{\widetilde{\alpha}} \cap \widetilde{X}_{\widetilde{\beta}})\backslash\{*\}$ and a point $\widetilde{x} \in \widetilde{e}$. Let $\widetilde{\gamma} \in \Pi(\widetilde{Y})$ be the type of $\widetilde{e}$. By 2.1, $\widetilde{\rho}(\widetilde{\gamma}) = \widetilde{G}_{\widetilde{x}}$ and $\widetilde{X}_{\widetilde{\gamma}} \supset \widetilde{e}$ hold. On the other hand, $\widetilde{x} \in \widetilde{X}_{\widetilde{\alpha}}\backslash\{*\} \subseteqq \widetilde{X}^{\widetilde{\rho}(\widetilde{\alpha})}$. Hence we have $\widetilde{\rho}(\widetilde{\alpha}) \subseteqq \widetilde{G}_{\widetilde{x}} = \widetilde{\rho}(\widetilde{\gamma})$, and $\widetilde{Y}^{\widetilde{\rho}(\widetilde{\alpha})} \supseteqq \widetilde{Y}^{\widetilde{\rho}(\widetilde{\gamma})}$. For each $\widetilde{\gamma}' \in \pi_0(\widetilde{Y}^{\widetilde{\rho}(\widetilde{\gamma})})$, there exists a unique $\widetilde{\alpha}' \in \pi_0(\widetilde{Y}^{\widetilde{\rho}(\widetilde{\alpha})})$ such that $\widetilde{\gamma}' \leqq \widetilde{\alpha}'$. Thus we obtain the map $f : \pi_0(\widetilde{Y}^{\widetilde{\rho}(\widetilde{\gamma})}) \to \pi_0(\widetilde{Y}^{\widetilde{\rho}(\widetilde{\alpha})})$ such that $\widetilde{\gamma}' \leqq f(\widetilde{\gamma}')$ for any $\widetilde{\gamma}' \in \pi_0(\widetilde{Y}^{\widetilde{\rho}(\widetilde{\gamma})})$. If $f(\widetilde{\gamma}) \neq \widetilde{\alpha}$, then by Definition 1.1(iv),

$$\widetilde{X}_{f(\widetilde{\gamma})} \cap \widetilde{X}_{\widetilde{\alpha}} = \{*\}.$$

On the other hand, since $\widetilde{\gamma} \leqq f(\widetilde{\gamma})$, we have $\widetilde{X}_{\widetilde{\gamma}} \subseteqq \widetilde{X}_{f(\widetilde{\gamma})}$, and hence

$$\widetilde{X}_{f(\widetilde{\gamma})} \cap \widetilde{X}_{\widetilde{\alpha}} \supseteqq \widetilde{X}_{\widetilde{\gamma}} \cap \widetilde{X}_{\widetilde{\alpha}} \supseteqq \widetilde{e}.$$

This is a contradiction, which concludes $f(\widetilde{\gamma}) = \widetilde{\alpha}$. This implies $\widetilde{\gamma} \leqq \widetilde{\alpha}$. By an argument similar to the above, we have $\widetilde{\gamma} \leqq \widetilde{\beta}$. Then since $|\widetilde{\gamma}| \subseteqq |\widetilde{\alpha}|$ and $|\widetilde{\gamma}| \subseteqq |\widetilde{\beta}|$, $|\widetilde{\alpha}| \cap |\widetilde{\beta}|$ contains $|\widetilde{\gamma}|$, which is not empty. This contradicts the assumption that $|\widetilde{\alpha}| \cap |\widetilde{\beta}| = \emptyset$. □

Lemma 3.8. *For α, $\beta \in \pi_0(Y^H)$ such that $\alpha \neq \beta$, one has $X_\alpha \cap X_\beta = \{*\}$.*

Proof. Let $\widetilde{\gamma}$ be an element of $\mu^{-1}(\gamma)$ for each $\gamma \in \pi_0(Y^H)$. As noted previously, $X_\alpha = p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\alpha}})$ and $X_\beta = p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\beta}})$. Suppose that $X_\alpha \cap X_\beta \neq \{*\}$. We take $x \in (X_\alpha \cap X_\beta)\backslash\{*\}$. Then x is written in two ways: $x = p_{\widetilde{X}}(\widetilde{a}) = p_{\widetilde{X}}(\widetilde{b})$, where $\widetilde{a} \in \widetilde{X}_{\widetilde{\alpha}}\backslash\{*\}$ and $\widetilde{b} \in \widetilde{X}_{\widetilde{\beta}}\backslash\{*\}$. Now, by the defintion of $p_{\widetilde{X}}$, there exists $\widetilde{h} \in \pi$ with $\widetilde{h}\widetilde{a} = \widetilde{b}$. Since $\widetilde{a} \in \widetilde{X}_{\widetilde{\alpha}}$, we have $\widetilde{b} = \widetilde{h}\widetilde{a} \in \widetilde{h}\widetilde{X}_{\widetilde{\alpha}}\backslash\{*\} = \widetilde{X}_{\widetilde{h}\widetilde{\alpha}}\backslash\{*\}$, hence $\widetilde{b} \in (\widetilde{X}_{\widetilde{h}\widetilde{\alpha}} \cap \widetilde{X}_{\widetilde{\beta}})\backslash\{*\}$. Moreover by Lemma 3.7, since $|\widetilde{h}\widetilde{\alpha}| \cap |\widetilde{\beta}| \neq \emptyset$, we have $|\alpha| \cap |\beta| = p(|\widetilde{h}\widetilde{\alpha}|) \cap p(|\widetilde{\beta}|) \supseteqq p(|\widetilde{h}\widetilde{\alpha}| \cap |\widetilde{\beta}|) \neq \emptyset$. Both α and β are connected components of Y^H, and so we obtain $|\alpha| = |\beta|$, hence $\alpha = \beta$. This is a contradiction, which implies $X_\alpha \cap X_\beta = \{*\}$. □

Lemma 3.9. *For any subgroup H of G,*

$$X^H = \bigcup_{\widetilde{\alpha}\in\Pi(\widetilde{Y})\ \text{s.t.}\ \rho(\mu(\widetilde{\alpha}))=H} p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\alpha}}).$$

Proof. For each $\widetilde{\alpha} \in \Pi(\widetilde{Y})$ with $\rho(\mu(\widetilde{\alpha})) = H$, we have $\sigma(\widetilde{\rho}(\widetilde{\alpha})) = \rho(\mu(\widetilde{\alpha})) = H$ by definition. Since $p_{\widetilde{X}}(\widetilde{X}^{\widetilde{\rho}(\widetilde{\alpha})}) \subseteqq X^{\sigma(\widetilde{\rho}(\widetilde{\alpha}))}$ and $\widetilde{X}$ is a $\Pi(\widetilde{Y})$-complex, we obtain $p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\alpha}}) \subseteqq p_{\widetilde{X}}(\widetilde{X}^{\widetilde{\rho}(\widetilde{\alpha})}) \subseteqq X^{\sigma(\widetilde{\rho}(\widetilde{\alpha}))} = X^H$.

Conversely, take $x \in X^H \backslash \{*\}$ arbitrarily. Since $p_{\widetilde{X}}$ is surjective, there exists $\widetilde{x} \in p_{\widetilde{X}}^{-1}(x)$, and then we have $\sigma(\widetilde{G}_{\widetilde{x}}) = G_x$. Indeed, noting that $p_{\widetilde{X}}$ is σ-equivarent and π acts transitively on each fibre of $p_{\widetilde{X}}$, one can easily verify that $\sigma(\widetilde{G}_{\widetilde{x}}) = G_x$. Take a cell $\widetilde{e} \subset \widetilde{X}$ such that $\widetilde{e} \ni \widetilde{x}$. Let $\widetilde{\gamma} \in \Pi(\widetilde{Y})$ be the type of $\widetilde{e}$. By 2.1, $\widetilde{\rho}(\widetilde{\gamma}) = \widetilde{G}_{\widetilde{x}}$ and $\widetilde{e} \subseteqq \widetilde{X}_{\widetilde{\gamma}}$. Take $\widetilde{y} \in |\widetilde{\gamma}|$, and we have $p(\widetilde{y}) \in p|\widetilde{\gamma}| = |\mu(\widetilde{\gamma})|$. Set $y = p(\widetilde{y}), \gamma = \mu(\widetilde{\gamma})$, and $\widetilde{H} = (\sigma|\widetilde{G}_{\widetilde{y}})^{-1}(H)$ respectively. Putting all together, we get the following:

$$\begin{array}{c} \widetilde{H} \subseteqq \widetilde{G}_{\widetilde{x}} = \widetilde{\rho}(\widetilde{\gamma}) \subseteqq \widetilde{G}_{\widetilde{y}} \\ \big\downarrow \sigma|\widetilde{G}_{\widetilde{y}} \quad \text{iso} \\ H \subseteqq G_x = \rho(\gamma) \subseteqq G_y \end{array}$$

where each of the upper sets corresponds to each of the lower sets via the isomorphism $\sigma|\widetilde{G}_{\widetilde{y}} : \widetilde{G}_{\widetilde{y}} \longrightarrow G_y$. By the above diagram, $\widetilde{x} \in \widetilde{X}^{\widetilde{H}}$ holds. Since $\widetilde{X}$ is the $\Pi(\widetilde{Y})$-complex, we get $\widetilde{x} \in \bigcup_{\widetilde{\alpha}} \widetilde{X}_{\widetilde{\alpha}}$, where $\widetilde{\alpha} \in \Pi(\widetilde{Y})$ with $\rho(\widetilde{\alpha}) = \widetilde{H}$. Mapping two sides by $p_{\widetilde{X}}$, we have $x = p_{\widetilde{X}}(\widetilde{x}) \in \bigcup_{\widetilde{\alpha}} p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\alpha}})$. On the other hand, $\rho(\mu(\widetilde{\alpha})) = \sigma(\widetilde{\rho}(\widetilde{\alpha})) = \sigma(\widetilde{H}) = H$, as was to be shown. □

Proposition 3.10. *The above space X is a $\Pi(Y)$-complex.*

Proof. We must verify that X satisfies Definition 1.1(i)–(iv). Condition (i) is clearly fulfilled. We shall verify (ii)–(iv). First let $\widetilde{\alpha} \in \mu^{-1}(\alpha)$ and $\widetilde{g} \in \sigma^{-1}(g)$. Then $\mu(\widetilde{g}\widetilde{\alpha}) = \sigma(\widetilde{g})\mu(\widetilde{\alpha}) = g\alpha$. This means $\widetilde{g}\widetilde{\alpha} \in \mu^{-1}(g\alpha)$. Hence we have $X_{g\alpha} = p_{\widetilde{X}}(\widetilde{X}_{\widetilde{g}\widetilde{\alpha}}) = p_{\widetilde{X}}(\widetilde{g}\widetilde{X}_{\widetilde{\alpha}}) = \sigma(\widetilde{g})p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\alpha}}) = gX_\alpha$, which verifies (ii). Second, let $\alpha \leqq \beta \in \Pi(Y)$. Let $\widetilde{\alpha}$ be the fixed element of $\mu^{-1}(\alpha)$. Take $\widetilde{y} \in |\widetilde{\alpha}|$ and set $y = p(\widetilde{y})$ $(\in p(|\widetilde{\alpha}|) = |\alpha| \subseteqq Y^{\rho(\alpha)})$. By assumption, $Y^{\rho(\alpha)} \subseteqq Y^{\rho(\beta)}$. Hence we get $y \in Y^{\rho(\beta)}$. Then we have $\rho(\beta) \subseteqq G_y$. Recall $\sigma|\widetilde{G}_{\widetilde{y}} : \widetilde{G}_{\widetilde{y}} \to G_y$ is an isomorphism. Setting $\widetilde{H} = (\sigma|\widetilde{G}_{\widetilde{y}})^{-1}(\rho(\beta))$, we obtain an element $\widetilde{\beta} \in \pi_0(\widetilde{Y}^{\widetilde{H}})$ with $|\widetilde{\beta}| \supseteqq |\widetilde{\alpha}|$. Since $\widetilde{\rho}(\widetilde{\beta}) = \widetilde{H} \subseteqq \widetilde{\rho}(\widetilde{\alpha})$, we have $\widetilde{\alpha} \leqq \widetilde{\beta}$. We get at once $\sigma(\widetilde{\rho}(\widetilde{\beta})) = \sigma(\widetilde{H}) = \rho(\beta)$. The space $p(|\widetilde{\beta}|)$ $(\supseteqq |\alpha|)$ is a connected component of $Y^{\sigma(\widetilde{\rho}(\widetilde{\beta}))} = Y^{\rho(\beta)}$, and $|\beta|$ $(\supseteqq |\alpha|)$ is also a connected component of $Y^{\rho(\beta)}$. This means $|\beta| \supseteqq p(|\widetilde{\beta}|)$. By the definition of μ, we have $\mu(\widetilde{\beta}) = \beta$, that is, $\widetilde{\beta} \in \mu^{-1}(\beta)$. Therefore $X_\alpha = p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\alpha}}) \subseteqq p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\beta}}) = X_\beta$, which finishes the verification of (iii). Finally Lemmas 3.8 and 3.9 guarantee (iv). □

The next lemma will be used to prove Theorem 3.12.

Lemma 3.11. *Let $\widetilde{\alpha}$ be an element of $\Pi(\widetilde{Y})$ and set $\alpha = \mu(\widetilde{\alpha})$. Then $G \times_\sigma (\widetilde{\alpha})^+$ is isomorphic to $(\alpha)^+$ as $\Pi(Y)$-complexes.*

Proof. We start with two definitions:

$$\begin{aligned} &(\widetilde{\alpha})^+ = \widetilde{G}/\widetilde{\rho}(\widetilde{\alpha}) \amalg \{*\}, \text{ and} \\ &(\widetilde{\alpha})^{\pm}_{\widetilde{\beta}} = \{\widetilde{g}\widetilde{\rho}(\widetilde{\alpha}) \mid \widetilde{g} \in \widetilde{G},\ \widetilde{g}\widetilde{\alpha} \leqq \widetilde{\beta}\} \amalg \{*\} \quad \text{for } \widetilde{\beta} \in \Pi(\widetilde{Y}). \end{aligned}$$

Set $\widetilde{X} = (\widetilde{\alpha})^+$ and $X = G \times_\sigma (\widetilde{\alpha})^+ = G \times_\sigma \widetilde{X}$. First we investigate the cardinality of $\widetilde{X}$ and X respectively. It is obvious that $|\widetilde{X}| = |\widetilde{G}/\widetilde{\rho}(\widetilde{\alpha})| + 1$, where $|\widetilde{X}|$ is the the cardinality of $\widetilde{X}$. Notice that

$$\begin{aligned} |X| &= |\widetilde{G}/\pi\widetilde{\rho}(\widetilde{\alpha})| + 1 \\ &= |G/\sigma(\widetilde{\rho}(\widetilde{\alpha}))| + 1 \\ &= |G/\rho(\alpha)| + 1 \\ &= |(\alpha)^+|. \end{aligned}$$

Next we shall define a map $f : X \to (\alpha)^+$ given by $[1, \widetilde{g}\widetilde{\rho}(\widetilde{\alpha})] \mapsto \sigma(\widetilde{g})\rho(\alpha)$, where the basepoint is mapped to the basepoint. This map is well-defined, σ being surjective, with the result that f is surjective. Since

$|X|$ equals $|(\alpha)^+|$, f is also injective. In the following we shall verify that f is a G-map. Choose $\widetilde{a} \in \sigma^{-1}(a)$ for any $a \in G$. Then

$$\begin{aligned} f(a[1, \widetilde{g}\widetilde{\rho}(\widetilde{\alpha})]) &= f([a, \widetilde{g}\widetilde{\rho}(\widetilde{\alpha})]) \\ &= f([\sigma(\widetilde{a}), \widetilde{g}\widetilde{\rho}(\widetilde{\alpha})]) \\ &= f([1, \widetilde{a}\widetilde{g}\widetilde{\rho}(\widetilde{\alpha})]) \\ &= \sigma(\widetilde{a}\widetilde{g})\rho(\alpha) \\ &= \sigma(\widetilde{a})\sigma(\widetilde{g})\rho(\alpha) \\ &= af([1, \widetilde{g}\widetilde{\rho}(\widetilde{\alpha})]). \end{aligned}$$

Thus f is a G-CW-complex isomorphism. It remains to prove that f is a $\Pi(Y)$-map. Remark that the basepoint of X is mapped to the basepoint of $\widetilde{X}$ by f. For $x \in X_\beta \backslash \{*\}$, it suffices to verify that $f(x) \in (\alpha)^+_\beta$ for any $\beta \in \Pi(Y)$. Let $\widetilde{\beta}$ be an element of $\mu^{-1}(\beta)$. Since $p_{\widetilde{X}} : \widetilde{X} \to X$ is surjective and $X_\beta = p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\beta}})$, there exists $\widetilde{x} \in \widetilde{X}_{\widetilde{\beta}}$ such that $x = p_{\widetilde{X}}(\widetilde{x}) = [1, \widetilde{x}]$. By the definition of $\widetilde{X}_{\widetilde{\beta}} = (\widetilde{\alpha})^+_{\widetilde{\beta}}$, the point $\widetilde{x}$ is written in the form: $\widetilde{x} = \widetilde{g}_0\widetilde{\rho}(\widetilde{\alpha})$ with $\widetilde{g}_0\widetilde{\alpha} \leqq \widetilde{\beta}$, where $\widetilde{g}_0$ is a certain element of $\widetilde{G}$. The following holds:

$$\begin{aligned} f(x) &= f([1, \widetilde{x}]) \\ &= f([1, \widetilde{g}_0\widetilde{\rho}(\widetilde{\alpha})] \\ &= \sigma(\widetilde{g}_0)\rho(\alpha) \quad \text{with} \quad \sigma(\widetilde{g}_0)\mu(\widetilde{\alpha}) \leqq \mu(\widetilde{\beta}). \end{aligned}$$

Hence we have $f(x)$ lies in

$$(\alpha)^+_\beta = \{g\rho(\alpha) \mid g \in G, \ g\alpha \leqq \beta\} \amalg \{*\},$$

which asserts f is a $\Pi(Y)$-map. It follows at once that f is an isomorphism between $\Pi(Y)$-complexes. □

For each $\alpha \in \Pi(Y)$, take $\widetilde{\alpha} \in \mu^{-1}(\alpha)$. Suppose that $[\widetilde{X}] = [\widetilde{Z}]$. Then $\chi(\widetilde{X_{\widetilde{\gamma}}}) = \chi(\widetilde{Z_{\widetilde{\gamma}}})$ for all $\widetilde{\gamma} \in \Pi(\widetilde{Y})$. We have already seen

$$\begin{aligned} (G \times_\sigma \widetilde{X})_\alpha &= p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\alpha}}), \quad \text{and} \\ (G \times_\sigma \widetilde{Z})_\alpha &= p_{\widetilde{X}}(\widetilde{Z}_{\widetilde{\alpha}}). \end{aligned}$$

Now,

$$\chi(p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\alpha}})) = \chi(\widetilde{X}_{\widetilde{\alpha}})/|\pi| = \chi(\widetilde{Z}_{\widetilde{\alpha}})/|\pi| = \chi(p_{\widetilde{X}}(\widetilde{Z}_{\widetilde{\alpha}})).$$

Hence we have $\chi((G \times_\sigma \widetilde{X})_\alpha) = \chi((G \times_\sigma \widetilde{Z})_\alpha)$ for all $\alpha \in \Pi(Y)$, which means $[G \times_\sigma \widetilde{X}] = [G \times_\sigma \widetilde{Z}]$. Thus the canonical correspondence $[\widetilde{X}] \mapsto [G \times_\sigma \widetilde{X}]$ gives a well-defined map $\Omega(\widetilde{G}, \Pi(\widetilde{Y})) \to \Omega(G, \Pi(Y))$ and it has been denoted by ν.

Theorem 3.12. ([4, Proposition 3.5]) *The map ν is an isomorphism.*

Proof. For two elements $[\widetilde{X}_1], [\widetilde{X}_2] \in \Omega(\widetilde{G}, \Pi(\widetilde{Y}))$, it is easily verified that

$$p_{\widetilde{X}}(\widetilde{X}_{1\widetilde{\alpha}} \bigvee \widetilde{X}_{2\widetilde{\alpha}}) = p_{\widetilde{X}_1}(\widetilde{X}_{1\widetilde{\alpha}}) \bigvee p_{\widetilde{X}_2}(\widetilde{X}_{2\widetilde{\alpha}}).$$

Then we have the following:

$$\begin{aligned} \nu([\widetilde{X}_1] + [\widetilde{X}_2]) &= \nu([\widetilde{X}_1 \bigvee \widetilde{X}_2]) \\ &= [G \times_\sigma (\widetilde{X}_1 \bigvee \widetilde{X}_2)] \\ &= [G \times_\sigma \widetilde{X}_1] + [G \times_\sigma \widetilde{X}_2] \\ &= \nu([\widetilde{X}_1]) + \nu([\widetilde{X}_2]). \end{aligned}$$

Thus ν is a homomorphism. By 2.2,

$$\Omega(G, \Pi(Y)) = \bigoplus_\alpha \langle [(\alpha)^+] \rangle_{\mathbb{Z}}$$

where $[\alpha]$ runs over $\Pi(Y)/G$, hence by Proposition 3.2 and Lemma 3.11, ν is surjective. We can write

$$[\widetilde{X}_1] = \sum_{\widetilde{\alpha}\in\Pi(\widetilde{Y})/\widetilde{G}} n_{\widetilde{\alpha}}^{\widetilde{X}_1}[(\widetilde{\alpha})^+], \quad \text{and}$$
$$[\widetilde{X}_2] = \sum_{\widetilde{\alpha}\in\Pi(\widetilde{Y})/\widetilde{G}} n_{\widetilde{\alpha}}^{\widetilde{X}_2}[(\widetilde{\alpha})^+],$$

where $n_{\widetilde{\alpha}}^{\widetilde{X}_1}$, $n_{\widetilde{\alpha}}^{\widetilde{X}_2} \in \mathbb{Z}$. By Lemma 3.11, it holds that

$$\nu([\widetilde{X}_1]) = \sum_{\widetilde{\alpha}\in\Pi(\widetilde{Y})/\widetilde{G}} n_{\widetilde{\alpha}}^{\widetilde{X}_1}[G\times_\sigma(\widetilde{\alpha})^+] = \sum_{\widetilde{\alpha}\in\Pi(\widetilde{Y})/\widetilde{G}} n_{\widetilde{\alpha}}^{\widetilde{X}_1}[(\mu(\widetilde{\alpha}))^+], \quad \text{and}$$
$$\nu([\widetilde{X}_2]) = \sum_{\widetilde{\alpha}\in\Pi(\widetilde{Y})/\widetilde{G}} n_{\widetilde{\alpha}}^{\widetilde{X}_2}[G\times_\sigma(\widetilde{\alpha})^+] = \sum_{\widetilde{\alpha}\in\Pi(\widetilde{Y})/\widetilde{G}} n_{\widetilde{\alpha}}^{\widetilde{X}_2}[(\mu(\widetilde{\alpha}))^+].$$

Note that $\{[\mu((\widetilde{\alpha})^+)] \mid \widetilde{\alpha} \in \Pi(\widetilde{Y})/\widetilde{G}\}$ is a basis of $\Omega(G,\Pi(Y))$ by Proposition 3.5. Thus $\nu([\widetilde{X}_1]) = \nu([\widetilde{X}_2])$ implies that each of the coeffients is equal, hence only if $[\widetilde{X}_1] = [\widetilde{X}_2]$. This shows that ν is injective, and therefore an isomorphism. □

Proposition 3.13. *The set $\nu(\Phi(\widetilde{G},\Pi(\widetilde{Y})))$ is contained in $\Phi(G,\Pi(Y))$.*

Proof. Let $x \in \Phi(\widetilde{G},\Pi(\widetilde{Y}))$. Then x is represented by $\widetilde{X^*}$ for some $\Pi(\widetilde{Y})$-resolution $\widetilde{X}$. Then $\nu([\widetilde{X^*}]) = [G\times_\sigma \widetilde{X^*}]$. Since $\chi(\widetilde{X^*}) = 0$,

$$\chi(G\times_\sigma \widetilde{X^*}) = \chi(\widetilde{X^*})/|\pi| = 0.$$

For $\alpha \in \Pi(Y)$ with $\alpha \neq \mathfrak{m}$ (where $\mathfrak{m}$ is a unique maximal element of $\Pi(Y)$),

$$\begin{aligned}\chi((G\times_\sigma \widetilde{X}^*)_\alpha) &= \chi(p_{\widetilde{X}^*}(\widetilde{X}^*_{\widetilde{\beta}})) \qquad \text{(for an arbitrarily chosen } \widetilde{\beta}\in\mu^{-1}(\alpha))\\ &= \chi(p_{\widetilde{X}}(\widetilde{X}_{\widetilde{\beta}}))\\ &= \chi((G\times_\sigma \widetilde{X})_\alpha).\end{aligned}$$

Since $G\times_\sigma \widetilde{X}$ is a $\Pi(Y)$-resolution, we have $\nu(x) = \nu([\widetilde{X^*}]) \in \Phi(G,\Pi(Y))$. □

4 PROOF OF THEOREM 1.2

In the following, we shall first define groups π, G and $\widetilde{G}$, second define a finite $\widetilde{G}$-CW-complex $\widetilde{Y}$ using the join operator $*$, and finally check that $\widetilde{Y}$ is connected and simply connected, and that the G-CW-complex $Y = \widetilde{Y}/\pi$ satisfies $\pi_1(Y) \cong \pi$ and $\nu(\Phi(\widetilde{G},\Pi(\widetilde{Y})) \neq \Phi(G,\Pi(Y))$. Define

$$\pi = \mathbb{Z}_p, \quad G = \mathbb{Z}_p\times\mathbb{Z}_q, \quad \text{and} \quad \widetilde{G} = \pi\times G.$$

Let $\mathbb{Z}'_p$ be a subgroup of $\pi\times\mathbb{Z}_p$ of order p such that $\mathbb{Z}'_p \neq \pi\times\{1\}$ nor $\{1\}\times\mathbb{Z}_p$. Next define

$$\begin{aligned}B(\mathbb{Z}'_p\times\mathbb{Z}_q,+_1) &= (\widetilde{G}/(\mathbb{Z}'_p\times\mathbb{Z}_q) * \widetilde{G}/(\mathbb{Z}'_p\times\mathbb{Z}_q))\times\{1\},\\ B(\mathbb{Z}'_p\times\mathbb{Z}_q,+_2) &= (\widetilde{G}/(\mathbb{Z}'_p\times\mathbb{Z}_q) * \widetilde{G}/(\mathbb{Z}'_p\times\mathbb{Z}_q))\times\{2\},\\ B(\mathbb{Z}_p\times\mathbb{Z}_q,-_1) &= (\widetilde{G}/(\mathbb{Z}_p\times\mathbb{Z}_q) * \widetilde{G}/(\mathbb{Z}_p\times\mathbb{Z}_q))\times\{1\},\\ B(\mathbb{Z}_p\times\mathbb{Z}_q,-_2) &= (\widetilde{G}/(\mathbb{Z}_p\times\mathbb{Z}_q) * \widetilde{G}/(\mathbb{Z}_p\times\mathbb{Z}_q))\times\{2\},\end{aligned}$$

and

$$\begin{aligned}B(\mathbb{Z}'_p,+) &= B(\mathbb{Z}'_p\times\mathbb{Z}_q,+_1) * B(\mathbb{Z}'_p\times\mathbb{Z}_q,+_2),\\ B(\mathbb{Z}_p,-) &= B(\mathbb{Z}_p\times\mathbb{Z}_q,-_1) * B(\mathbb{Z}_p\times\mathbb{Z}_q,-_2),\\ B(\mathbb{Z}_q,1) &= B(\mathbb{Z}'_p\times\mathbb{Z}_q,+_1) * B(\mathbb{Z}_p\times\mathbb{Z}_q,-_1),\\ B(\mathbb{Z}_q,2) &= B(\mathbb{Z}'_p\times\mathbb{Z}_q,+_2) * B(\mathbb{Z}_p\times\mathbb{Z}_q,-_2).\end{aligned}$$

Further set

$$\widetilde{Y} = (B(\mathbb{Z}'_p,+) \amalg B(\mathbb{Z}_p,-) \amalg B(\mathbb{Z}_q,1) \amalg B(\mathbb{Z}_q,2)) * \widetilde{G}.$$

Then clearly $\widetilde{Y}$ is a finite $\widetilde{G}$-CW-complex, moreover connected and simply connected. Define $Y = \widetilde{Y}/\pi$. Since π acts freely on $\widetilde{Y}$, $\pi_1(Y)$ is isomorphic to π.

In the remainder of this section, we shall prove that $\Phi(\widetilde{G}, \widetilde{\Pi}) = 0$ and $\Phi(G, \Pi) \neq 0$, where $\widetilde{\Pi} = \Pi(\widetilde{Y})$ and $\Pi = \Pi(Y)$, which concludes the proof of Theorem 1.2.

Proposition 4.1. *The module* $\Phi(\widetilde{G}, \widetilde{\Pi})$ *is a trivial group.*

Proof. It is easy to see that $\widetilde{\Pi}$ consists of 9 elements, that is,

$$\begin{aligned}\widetilde{\Pi} = \{&\beta(\mathbb{Z}_p' \times \mathbb{Z}_q, +_1),\ \beta(\mathbb{Z}_p' \times \mathbb{Z}_q, +_2),\ \beta(\mathbb{Z}_p \times \mathbb{Z}_q, -_1),\ \beta(\mathbb{Z}_p \times \mathbb{Z}_q, -_2),\ \beta(\mathbb{Z}_p', +),\\ &\beta(\mathbb{Z}_p, -),\ \beta(\mathbb{Z}_q, 1),\ \beta(\mathbb{Z}_q, 2),\ \widetilde{m}\}\end{aligned}$$

such that

$$\begin{aligned}|\beta(\mathbb{Z}_p' \times \mathbb{Z}_q, +_1)| &= B(\mathbb{Z}_p' \times \mathbb{Z}_q, +_1), & \rho(\beta(\mathbb{Z}_p' \times \mathbb{Z}_q, +_1)) &= \mathbb{Z}_p' \times \mathbb{Z}_q,\\ |\beta(\mathbb{Z}_p' \times \mathbb{Z}_q, +_2)| &= B(\mathbb{Z}_p' \times \mathbb{Z}_q, +_2), & \rho(\beta(\mathbb{Z}_p' \times \mathbb{Z}_q, +_2)) &= \mathbb{Z}_p' \times \mathbb{Z}_q,\\ |\beta(\mathbb{Z}_p \times \mathbb{Z}_q, -_1)| &= B(\mathbb{Z}_p \times \mathbb{Z}_q, -_1), & \rho(\beta(\mathbb{Z}_p \times \mathbb{Z}_q, -_1)) &= \mathbb{Z}_p \times \mathbb{Z}_q,\\ |\beta(\mathbb{Z}_p \times \mathbb{Z}_q, -_2)| &= B(\mathbb{Z}_p \times \mathbb{Z}_q, -_2), & \rho(\beta(\mathbb{Z}_p \times \mathbb{Z}_q, -_2)) &= \mathbb{Z}_p \times \mathbb{Z}_q,\end{aligned}$$

and

$$\begin{aligned}|\beta(\mathbb{Z}_p', +)| &= B(\mathbb{Z}_p', +), & \rho(\beta(\mathbb{Z}_p', +)) &= \mathbb{Z}_p',\\ |\beta(\mathbb{Z}_p, -)| &= B(\mathbb{Z}_p, -), & \rho(\beta(\mathbb{Z}_p, -)) &= \mathbb{Z}_p,\\ |\beta(\mathbb{Z}_q, 1)| &= B(\mathbb{Z}_q, 1), & \rho(\beta(\mathbb{Z}_q, 1)) &= \mathbb{Z}_q,\\ |\beta(\mathbb{Z}_q, 2)| &= B(\mathbb{Z}_q, 2), & \rho(\beta(\mathbb{Z}_q, 2)) &= \mathbb{Z}_q,\\ |\widetilde{m}| &= \widetilde{Y}, & \rho(\widetilde{m}) &= \{1\}.\end{aligned}$$

The $\widetilde{G}$-poset $\widetilde{\Pi}$ is illustrated in Figure 2.

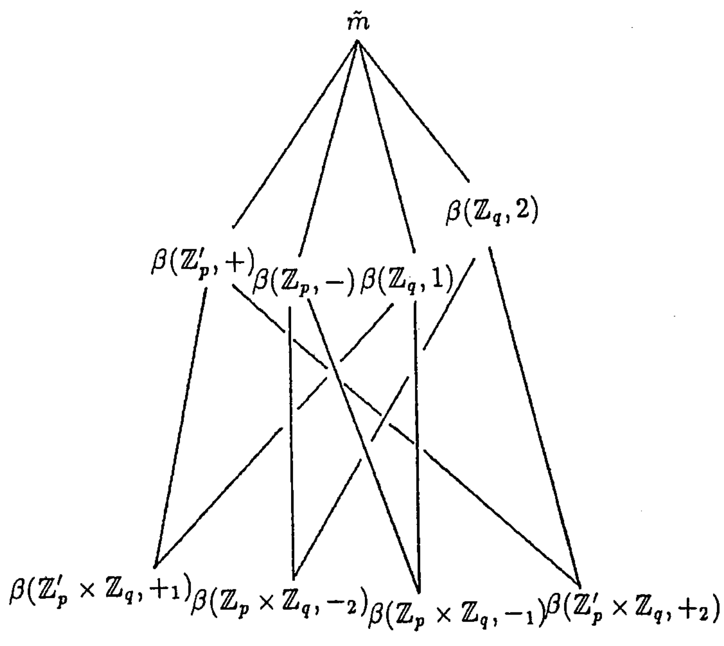

Fig.2

We recall

$$\begin{aligned}\mathcal{P}(\widetilde{\Pi}) &= \{\alpha \in \widetilde{\Pi} \mid \rho(\alpha) \ \text{ is a subgroup of } \widetilde{G} \text{ of prime power order}\}, \quad \text{and}\\ S(\widetilde{G},\ \alpha) &= \{K \in S(\widetilde{G}) \mid \rho(\alpha) \triangleleft K \subseteqq \widetilde{G}_\alpha \ \text{ and } \ K/\rho(\alpha) \ \text{ is cyclic}\}.\end{aligned}$$

We set $\widetilde{\mathcal{K}} = \{(\alpha, K) \mid \alpha \in \mathcal{P}(\widetilde{\Pi}), \quad K \in S(\widetilde{G}, \alpha)\}$. Then, define the homomorphism

$$\bar{\chi}_{(\alpha, K)} : \Omega(\widetilde{G}, \widetilde{\Pi}) \to \mathbb{Z}$$

by $\bar{\chi}_{(\alpha, K)}([Z]) = \bar{\chi}(Z_\alpha^K)$ for $[Z] \in \Omega(\widetilde{G}, \widetilde{\Pi})$ and $(\alpha, K) \in \widetilde{\mathcal{K}}$, and the homomorphism

$$\bar{\chi}_\alpha : \Omega(\widetilde{G}, \widetilde{\Pi}) \to \mathbb{Z}$$

by $\bar{\chi}_\alpha([Z]) = \bar{\chi}(Z_\alpha)$ for $[Z] \in \Omega(\widetilde{G}, \widetilde{\Pi})$ and $\alpha \in \widetilde{\Pi}$.

Since that $\Phi(\widetilde{G}, \widetilde{\Pi}) = \{[Z] \in \Omega(\widetilde{G}, \widetilde{\Pi}) \mid \bar{\chi}(Z_\alpha^K) = 0$, for all $\alpha \in \mathcal{P}(\widetilde{\Pi})$ and $K \in S(\widetilde{G}, \alpha)\}$,

$$\begin{aligned}\Phi(\widetilde{G}, \widetilde{\Pi}) &= \ker \left[\bigoplus_{(\alpha, K) \in \widetilde{\mathcal{K}}} \bar{\chi}_{(\alpha, K)} : \Omega(\widetilde{G}, \widetilde{\Pi}) \to \bigoplus_{(\alpha, K) \in \widetilde{\mathcal{K}}} \mathbb{Z} \right] \\ &\subset \ker \left[\bigoplus_{(\alpha, K) \in \widetilde{\mathcal{K}'}} \bar{\chi}_{(\alpha, K)} : \Omega(\widetilde{G}, \widetilde{\Pi}) \to \bigoplus_{(\alpha, K) \in \widetilde{\mathcal{K}'}} \mathbb{Z} \right]\end{aligned}$$

where $\widetilde{\mathcal{K}'} := \{(\alpha, K) \in \widetilde{\mathcal{K}} \mid \widetilde{Y_\alpha}^K$ is connected$\}$. It suffices to prove that

$$\ker(\oplus_{(\alpha, K) \in \widetilde{\mathcal{K}'}} \bar{\chi}_{(\alpha, K)})$$

is a trivial group. Since $\widetilde{Y_\alpha}^K$ is connected for $(\alpha, K) \in \widetilde{\mathcal{K}'}$, we define $\phi : \widetilde{\mathcal{K}} \to \widetilde{\Pi}$ by $\phi(\alpha, K) =$ the component of $\widetilde{Y_\alpha}^K$. Furthermore $Z_\alpha^K = Z_{\phi(\alpha, K)}$ for $(\alpha, K) \in \widetilde{\mathcal{K}'}$, and so we have $\bar{\chi}_{(\alpha, K)}([Z]) = \bar{\chi}_{\phi(\alpha, K)}([Z])$. Remark that $\phi(\widetilde{\mathcal{K}'}) = \widetilde{\Pi}$. It follows at once that $\ker(\oplus_{(\alpha, K) \in \widetilde{\mathcal{K}'}} \bar{\chi}_{(\alpha, K)})$ is a trivial group. □

Proposition 4.2. *The module $\Phi(G, \Pi)$ is not a trivial group.*

Proof. The G-poset $\Pi = \Pi(Y)$ consists of 9 elements as follows:

$$\begin{aligned}\Pi(Y) &= \coprod_{H \in S(G)} \pi_0(Y^H) \\ &= \coprod_{H \in S(G)} \pi_0((\widetilde{Y}/\mathbb{Z}_p)^H) \\ &= \pi_0((\widetilde{Y}/\mathbb{Z}_p)^{\mathbb{Z}_p \times \mathbb{Z}_q}) \coprod \pi_0((\widetilde{Y}/\mathbb{Z}_p)^{\mathbb{Z}_p}) \coprod \pi_0((\widetilde{Y}/\mathbb{Z}_p)^{\mathbb{Z}_q}) \coprod \pi_0((\widetilde{Y}/\mathbb{Z}_p)^{\{1\}}) \\ &= \{\mu(\beta(\mathbb{Z}_p' \times \mathbb{Z}_q, +_1)),\ \mu(\beta(\mathbb{Z}_p' \times \mathbb{Z}_q, +_2)),\ \mu(\beta(\mathbb{Z}_p \times \mathbb{Z}_q, -_1)), \\ &\quad \mu(\beta(\mathbb{Z}_p \times \mathbb{Z}_q, -_2))\} \coprod \{\mu(\beta(\mathbb{Z}_p', +)),\ \mu(\beta(\mathbb{Z}_p, -))\} \coprod \{\mu(\beta(\mathbb{Z}_q, 1)), \\ &\quad \mu(\beta(\mathbb{Z}_q, 2))\} \coprod \{\mu(\widetilde{m})\}\end{aligned}$$

We write the elements of Π as follows: $\alpha_1 := \mu(\beta(\mathbb{Z}_p' \times \mathbb{Z}_q, +_1))$, $\alpha_2 := \mu(\beta(\mathbb{Z}_p' \times \mathbb{Z}_q, +_2))$, $\alpha_3 := \mu(\beta(\mathbb{Z}_p \times \mathbb{Z}_q, -_1))$, $\alpha_4 := \mu(\beta(\mathbb{Z}_p \times \mathbb{Z}_q, -_2))$, $\alpha_5 := \mu(\beta(\mathbb{Z}_p', +))$, $\alpha_6 := \mu(\beta(\mathbb{Z}_p, -))$, $\alpha_7 := \mu(\beta(\mathbb{Z}_q, 1))$, $\alpha_8 := \mu(\beta(\mathbb{Z}_q, 2))$, $m := \mu(\widetilde{m})$.

It suffices to prove that $\omega = [(\alpha_1)^+] + [(\alpha_4)^+] - [(\alpha_2)^+] - [(\alpha_3)^+]$ lies in $\Omega(G, \Pi)$ and $\omega \neq 0$. However, by 2.5, it is clear that $\omega \neq 0$. Since $G = \mathbb{Z}_p \times \mathbb{Z}_q$, we have that $\mathcal{P}(\Pi) = \{m,\ \alpha_5,\ \alpha_6,\ \alpha_7,\ \alpha_8\}$. We must show that

$$\bar{\chi}(X_\alpha^K) = 0 \quad \text{for all } \alpha \in \mathcal{P}(\Pi) \text{ and } K \in S(G, \alpha),$$

where X is a Π-complex representing ω.

Consider the case of $\alpha = \alpha_5$. Then, $S(G, \alpha) = \{\mathbb{Z}_p, \ \mathbb{Z}_p \times \mathbb{Z}_q\}$. For $K = \mathbb{Z}_p$, the following hold:

$$\begin{aligned}&\bar{\chi}((\alpha_1)_{\alpha_5}^{+\mathbb{Z}_p}) = \chi(G/(\mathbb{Z}_p \times \mathbb{Z}_q)) = 1, \\ &\bar{\chi}((\alpha_4)_{\alpha_5}^{+\mathbb{Z}_p}) = \bar{\chi}(\{*\}) = 0, \\ &\bar{\chi}((\alpha_2)_{\alpha_5}^{+\mathbb{Z}_p}) = \chi(G/(\mathbb{Z}_p \times \mathbb{Z}_q)) = 1, \quad \text{and} \\ &\bar{\chi}((\alpha_3)_{\alpha_5}^{+\mathbb{Z}_p}) = \bar{\chi}(\{*\}) = 0.\end{aligned}$$

For $K = \mathbb{Z}_p \times \mathbb{Z}_q$, the following hold:

$$\begin{aligned}
\bar{\chi}((\alpha_1)_{\alpha_6}^{+\mathbb{Z}_p\times\mathbb{Z}_q}) &= \chi(G/(\mathbb{Z}_p \times \mathbb{Z}_q)) = 1,\\
\bar{\chi}((\alpha_4)_{\alpha_5}^{+\mathbb{Z}_p\times\mathbb{Z}_q}) &= \bar{\chi}(\{*\}) = 0,\\
\bar{\chi}((\alpha_2)_{\alpha_5}^{+\mathbb{Z}_p\times\mathbb{Z}_q}) &= \chi(G/(\mathbb{Z}_p \times \mathbb{Z}_q)) = 1, \quad \text{and}\\
\bar{\chi}((\alpha_3)_{\alpha_5}^{+\mathbb{Z}_p\times\mathbb{Z}_q}) &= \bar{\chi}(\{*\}) = 0.
\end{aligned}$$

Hence we obtain

$$\bar{\chi}(X_\alpha^K) = 0.$$

By arguments similar to the above, we obtain

$$\bar{\chi}(X_\alpha^K) = 0 \quad \text{for all} \quad \alpha = \alpha_6,\ \alpha_7,\ \alpha_8,\ m, \text{ and } K \in S(G,\ \alpha).$$

Therefore ω lies in $\Phi(G, \Pi)$. □

Remark 4.3. *Further computation proves that* $\Phi(G, \Pi) \cong \mathbb{Z}$.

References

[1] Dovermann K.H and Rothenberg M, *The generalized whitehead torsion of a G-fibre homotopy equivalence*, Lecure Notes in Math, Transformation Groups (K.Kawakubo,ed), **1375**, (1989), Springer-Verlag 60–88.

[2] tom Dieck, T, *Transformation Groups and Representation Theory*, Lecture Notes in Math, **766**, Springer-Verlag, 1978.

[3] Kawakubo K, *The Theory of Transformation Groups*, Oxford University Press, London, (1991).

[4] Morimoto M. and Iizuka K, *Extendibility of G-maps to pseudo-equivalences to finite G-CW-complexes whose fundamental groups are finite*, Osaka J. Math. **21** (1984), 59–69.

[5] Oliver R. and Petrie T, *G-CW-surgery and* $K_0(\mathbf{Z}G)$, Math. Z. **179** (1982), 11–42.

[6] Rim D.S, *Modules over finite groups*, Ann. Math. **69** (1958), 700–712.

[7] Rotman J.J, *An introduction to Algebraic Topology*, GTM. 119, Springer-Verlag, 1988.

[8] Swan R.G, *Induced representations and projective modules*, Ann. Math. **71** (1960), 552–578.