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Computational analysis of contamination in Kojima Lake using upwind-type finite element method

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Abstract

We have computed the phase of spreading contaminations in Kojima Lake by using the upwind-type finite element method. We have treated the two cases: the pollutant flows from the Sasagase river and from the Kurashiki River. We see that the upwind-type finite element method is effective in both cases, when the diffusion constant is quite small.

Keywords: Upwind-type, Finite element method, Kojima Lake

1. Introduction

In the previous paper [SIKW], we have made water analysis of Kojima Lake based on the amount of waterfall in the basin of the Sasagase River and that of the Kurashiki River. In that context, we divided Kojima Lake into several large rectangles, and we had each rectangle covered with meshes. Then we considered a stationary convection diffusion equation, with respect to those meshes. We applied the finite element method to these situations. Since the shape of Kojima Lake is complicated, the finite element method seems suitable for analyzing them in contrast to the difference method.

In this paper, we carry out an evolutional model of water analysis in Kojima Lake by using a convection diffusion equation. In [SIKW], our solving it numerically, some oscillations occurred when the diffusion constant was chosen to fit the real circumstances. In order to settle these difficulties, we adopt the upwind-type finite element method. We have gotten satisfactory results in comparison with the 'ordinary' finite element method.

We have used the computers of "The Risk Analysis system" in the Faculty of Environmental Sciences and Technologies, Okayama University.

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2. The upwind-type finite element method

When a diffusion constant is quite small and when a convection phenomenon is surpass, it is known that the upwind-type difference method is much effective measure. On the other hand, since the finite element method is indirectly formulated by weak form, it needs some ideas to realize the upwind-type finite type method [TF].

We denote by $\varphi(x,y,t)$ the density of the pollutant of contamination at (x,y,t). We denote the total area by Ω. The convection diffusion equation for φ is given as follows:

$$\frac{\partial\varphi}{\partial t} - \kappa\Delta\varphi + \mathbf{v}\cdot\operatorname{grad}\varphi = 0 \qquad (1)$$

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where κ is the diffusion constant of contamination, and $\mathbf{v}$ is the velocity vector of flow in Ω.

We multiply the both sides of (1) by a test function ψ vanishing at the estuary, where we impose Dilichlet boundary condition, and integrate it over Ω.

$$\int_\Omega \frac{\partial\varphi}{\partial t}\psi\, dxdy + \kappa \int_\Omega \mathrm{grad}\,\varphi \cdot \mathrm{grad}\,\psi\, dxdy + \int_\Omega (\mathbf{v}\cdot \mathrm{grad}\,\varphi)\psi\, dxdy = 0 \qquad (2)$$

We cover Ω with small meshes, and enumerate all their nodes. We denote by O_l the l-th node. We denote by φ_l the function on Ω which is equal to 1 at O_l, zero at other nodes and which is linear in every small area. We assume that φ is written by the linear combination such as : $\sum_l c_l(t)\varphi_l$, where $c_l(t)$ is a function which depends only on t. Put $\psi = \varphi_k$ in (2). Then, for all k, we have

$$\sum_l \left(\frac{dc_l}{dt} \int_\Omega \varphi_l\varphi_k\, dxdy + \kappa c_l(t) \int_\Omega \mathrm{grad}\,\varphi_l \cdot \mathrm{grad}\,\varphi_k\, dxdy + c_l(t) \int_\Omega (\mathbf{v}\cdot \mathrm{grad}\,\varphi_l)\varphi_k\, dxdy \right) = 0 \qquad (3)$$

Hereafter, we only consider the Kojima Lake situation. Firstly, we compute the stationary flow of Kojima Lake by using the rainfall data in [SIKW]. For each node O, we can define an upwind area S_O of O by the data of the stationary flow.

To use the upwind-type finite element method, in the third term in (3), we change

$$\int_\Omega (\mathbf{v}\cdot \mathrm{grad}\,\varphi_l)\varphi_k\, dxdy$$

to

$$(\mathbf{v}(O_l)\cdot \mathrm{grad}\,\varphi_l(S_k))\,\varphi_k(O_k)\,\mathrm{mesh}(D_{O_k})$$

where $\mathrm{grad}\,\varphi_l(S_k)$ is a gradient of the linear function φ_l on S_k. We denote by $D_{O_{P_k}}$ the barycentric area around O_k and by $\mathrm{mesh}(D_{O_k})$ the square measure of D_{O_k}.

We impose Dilichlet condition on each estuary that the density is constant and Neumann type condition that $\dfrac{\partial\varphi}{\partial\mathbf{n}} = 0$ on the lock. Concerning about time component t, we have employed the implicit method where we have put $\theta = 0.55$ [K].

When we treat the upwind-type finite element method, approximating solutions do not necessarily converge to the real solution if there exist obtuse-angle triangles. In fact, the oscillation occurred when we executed the upwind-type finite element approximation with the meshes in [SIKW].

On the other hand, in the case that all triangles have no obtuse angle, it is known that approximating solutions converge to the real solution when the diameter of all triangles tend towards zero [T]. To get acute-angle triangles or right-angle triangles, we divide the region by lines crossing at right angle. In our calculations, the following meshes in Kojima Lake are used (Figure 1):

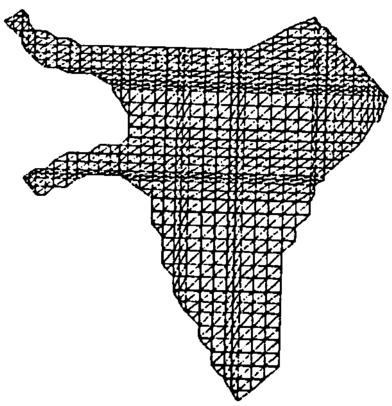

Figure 1: Mesh of Kojima Lake

This method has the advantage that all triangulate meshes have right-angle and stand in line in any cases. If we limit Ω to the original lake area, the occurrence of the oscillation increases. We make Ω contain the estuary of rivers.

The stationary flow computed by the above meshes is shown in Figure 2.

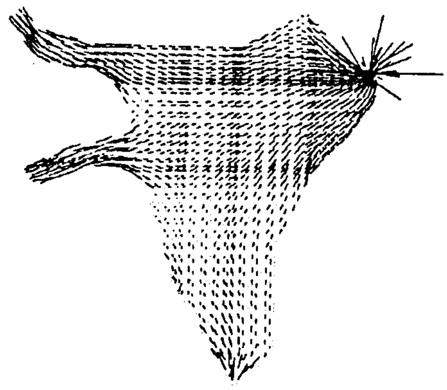

Figure 2: Stationary Flow in Kojima Lake

3. Simulations and Discussion

In this section, we show several results of our simulations. We consider the two cases:

- Case 1. Water from the Sasagase River contains the pollutant.
- Case 2. Water from the Kurashiki River contains the pollutant.

We note that the flow of water is the same in these cases as we mentioned in Section 2.

Here, we display the results by figures. Gray-scale color of each box shows the concentration of the pollutant at its point. Black means high concentration and white means low.

In Case 1, we adopt $\kappa = 10^{-3}$ and $\Delta t = 10^{-2}$ as the diffusion constant and the step size of time. We carry out the simulations in this case.

An overflow occurs at early time step when we compute the numerical solution obtained by the 'ordinary' finite element method. Here 'ordinary' means the finite element method without the upwind-type modification.

On the other hand, we can obtain good results by applying the upwind-type finite element method (Figure 3). The pollutant moves downstream from the Sasagase river to the lock and the solution becomes stationary at about 6000th time-step.

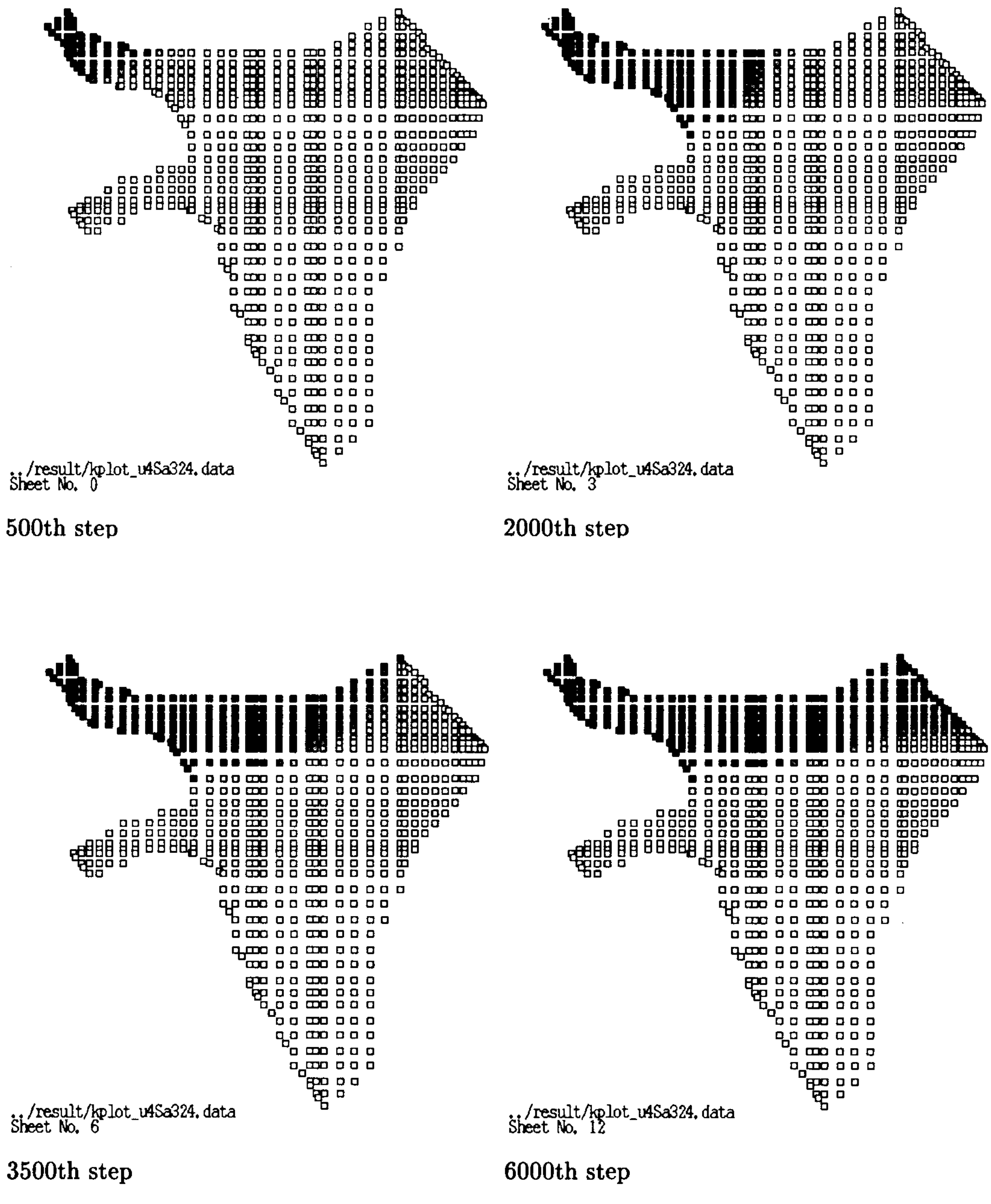

Figure 3: Upwind-type FEM; The pollutant from the Sasagase River

In Case 2, we adopt κ as the same as in Case 1 and $\Delta t = 10^{-2}, 10^{-3}$. For $\Delta t = 10^{-2}$, the result is unstable. Hence we change the step size of time smaller: $\Delta t = 10^{-3}$.

By using the upwind-type finite element method, we obtain the result as shown in Figure 4. The pollutant moves downstream from the Kurashiki River to the lock, while there are some points in the east part of the region where the solution becomes unstable. The solution becomes stationary at about 70000 time-step.

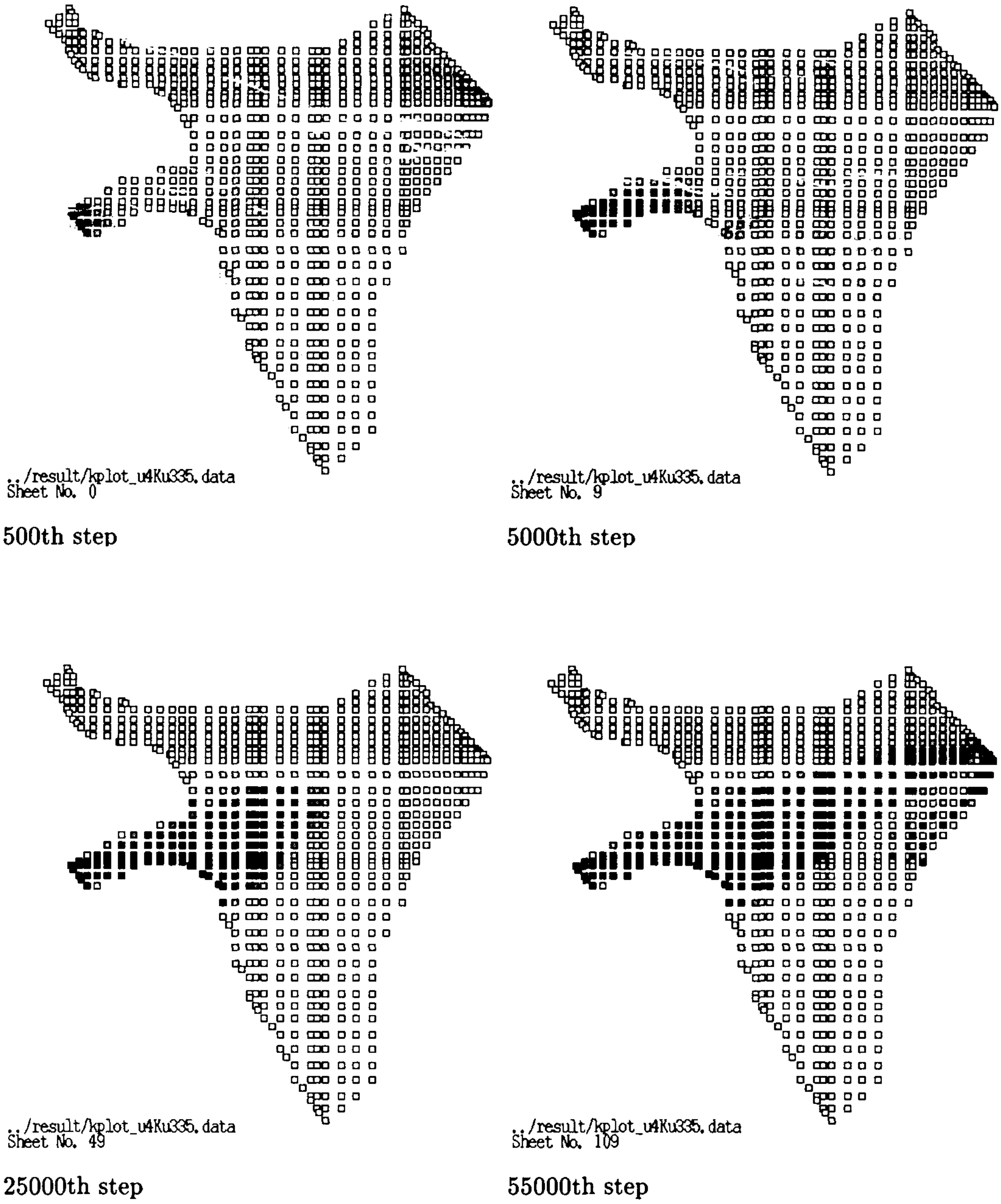

Figure 4: Upwind-type FEM; The pollutant comes from the Kurashiki River

Note that the result of the 'ordinary' finite element method at 1000th time-step is as shown in Figure 5: oscillations occur near the estuary and we see the solution is unstable. Therefore we see that the upwind-type finite element methods is effective for small κ.

10000th time-step

Figure 5: Unstable result by the 'ordinary' FEM in Case 2

In both cases, we obtain the results that the pollutant spreads from the estuary to the lock. Moreover, both numerical solutions become stationary at nearly same time.

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