Journal of the Faculty of Environmental Science and Technology, Okayama University
Vol.3, No.1, pp.5-9, January 1998

# *The Fourier coefficients of certain Maass wave form for $\Gamma_0(2)$*

Hirofumi ISHIKAWA *

(Received November 5, 1997)

We investigate the Maass wave form for $\Gamma_0(2)$ whose eigenvalue of Laplacian $\Delta$ is $1/4 - \pi^2/\log^2(\sqrt{2} - 1)$. In this note, we study the methods of calculation of its Fourier coefficients and carry out the numerical calculations.

## 1. INTRODUCTION

Let $\Gamma_0(2)$ denote the congruence subgroup of $SL(2,\mathbf{Z})$ which consists of all elements $g$:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, c \equiv 0 \bmod 2$$

$\Gamma_0(2)$ has two generators such as:

$$\{U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, V = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\}$$

We denote by $H$ the complex upper half plane, and let the group $GL(2, \mathbf{R})$ act on $H$ by the linearly fractional transformation. The standard fundamental domain $F$ of $\Gamma_0(2)$ in $H$ is shown in Figure 1. There are two cusps $\{\infty, 0\}$, and one elliptic point $\rho = -0.5+0.5i$ in $F$ ( 0, -1 are equivalent under the action of $\Gamma_0(2)$).

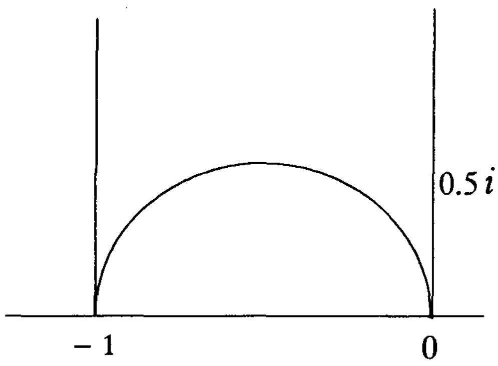


Fig 1. The Fundamental domain of $\Gamma_0(2)$

Maass [6] studied an automorphic function on the modular group which is an eigen-function of Laplacian; it is not holomorphic, but is real analytic. We call a function $f(z)$ on $H$ satisfying the following conditions a wave form for $\Gamma_0(2)$ :

* Department of Environmental and Mathematical Sciences, Faculty of Environmental Science and Technology, Okayama University, Tsushima, Okayama, 700 Japan

(1) $f(gz) = f(z)$, for all $g \in \Gamma_0(2)$

(2) $\Delta f(z) = \lambda f(z)$, where $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is the Laplace - Beltami operator on $H$

(3) $f(z)$ is bounded

(4) $f(z)$ is cuspidal

Note that $\lambda$ is non-negative as $\Delta$ is an self-adjoint operator.

For $SL(2,\mathbb{Z})$, Cartier [1] calculated 30 small spectra of $\Delta$, but some of them were not actual eigen values. The investigation of its spectra on a full scale was carried out by Hejhal [3]. Stark [7] calculated the Fourier coefficients of primes below 1,000 for the first even wave form.

For the congruence subgroup $\Gamma_0(2)$, it follows from Jacquet-Langrands theory that Gelbart [2, § 7, Theorem 2.11] has proved the principal series representation $\pi_s$ with $s = \frac{2\pi i}{\log\left(\sqrt{2}-1\right)}$ occurs in $L^2(\Gamma_0(2)\backslash SL(2,\mathbb{R}))$. Then in this note, we treat the wave form with the eigenvalue $\lambda$ of $\Delta$:

$$\lambda = \frac{1-s^2}{4} = \frac{1}{4} - \frac{\pi^2}{\log^2\left(\sqrt{2}-1\right)} = 12.955146626242823584$$

## 2. THE FOURIER COEFFICIENTS

As $\Gamma_0(2)$ contains the element $U$, a wave form $f(z)$ has the Fourier expansion at $\infty$ of the shape:

$$f(z) = \sum_{n=1}^{\infty} c_n y^{1/2} K_{ir}\left(2\pi ny\right)\cos(2\pi nx)$$

where $K_{ir}$ denotes $K$-Bessel function of imaginary order $ir$. $K_{ir}(x)$ has the distinctive features. The more violently $K_{ir}(x)$ oscillates, the more close $x$ tends towards zero. Figure 2 shows the graph of $K_{ir}(x)$ $(0.00001<x<0.0001)$. On the other hand, $K_{ir}(x)$ is decreasing rapidly, when x is going beyond 20.0 (Figure 3). Therefore, it is difficult to calculate $f(z)$ for $z$ in $F$ belonging to the neighborhood of the cusp at -1. On the other hand, $f(z)$ nearly vanishes for $z$ in $F$ with the imaginary part greater than 3.2.

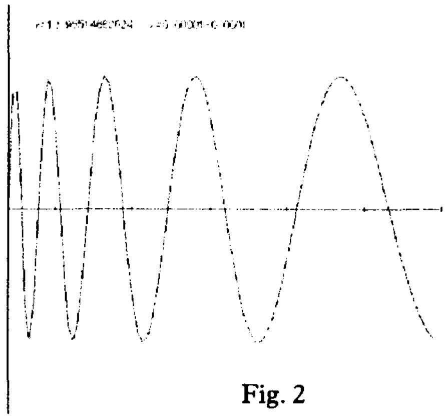

Fig. 2

$K_{ir}(x)$, $10^{-5}<x<10^{-4}$

We assume that $f(z)$ is normalized, *i.e.*, $c_1=1$. Firstly, we determine the coefficient $c_2$. Since $\Gamma_0(2)$ is generated by $U$ and $V$, the definition (2) means that $f(z)$ satisfies the following formula ($f(z+1)=f(z)$ being automatically fulfilled):

(2') $f(Vz) = f(z)$ for $z \in H$

For $z \in F, n \geq 1$, put

$$I(z,n) = \sqrt{y^*}\, K_{ir}(2\pi n y^*) \cos(2\pi n x^*) - \sqrt{y}\, K_{ir}(2\pi n y) \cos(2\pi n x)$$

where $Vz = z^* = x^* + iy^*$.

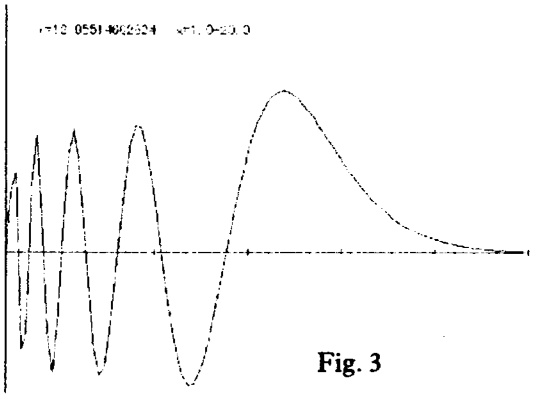

Fig. 3

$K_{ir}(x)$, $1<x<20$

By using (2'), we get the equality:

$$\sum_{n=1}^{\infty} c_n I(z,n) = 0$$

When we choose N points $\{z_j\}$ in $F$ ($N$ being large enough) , the following linear equations will be effected approximately because of K-Bessel function's property of decreasing rapidly.

$$\sum_{n=1}^{N} c_n I(z_j,n) = 0, \qquad (c_1 = 1,\ 1 \leq j \leq N-1)$$

Now we choose 30 points $\{z_j\}$ in $F$ whose imaginary parts are near 0.5. For many cases of the selections of a set of seventeen and / or eighteen points ($N$=18~19) from $\{z_j\}$, we have solved those equations . In all cases, $I(z_j,n)$ s' have been calculated down to thirty decimal places. We get

$c_2 = -1.736$

For $n \geq 1$, we denote by $T(n)$ the Hecke operator with respect to $\Gamma_0(2)$. As $f(z)$ is normalized, the Fourier coefficient $c_n$ satisfies:

$T(n)\, f(z) = c_n f(z)$

The multiplicative properties of the Fourier coefficients follow from those of the Hecke operators, *i.e.*,

(1) $c_m \cdot c_n = c_{mn}$ if $(m,n) = 1$

(2) $c_{p^2} = c_p^2 - 1,\ c_{p^3} = c_p^3 - 2c_p,\ c_{p^4} = c_p^4 - 3c_p^2 + 1$ if $p \neq 2$

(3) $c_{2^k} = c_2^k$

Using the relation $c_n = T(n)\, f(z) \,/\, f(z)$, we calculate the Fourier coefficients $c_p$'s for all primes $p$ below 200 repeatedly. The following table shows the required $n$-th approximate values $c_n$ to get the value $c_p$ for prime $p$ below 50, when one can choose suitable $z$ ($z \approx -0.5 + 0.9\,i$).

TABLE 1

| $p$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 8 | 9 | 15 | 21 | 32 | 38 | 49 | 55 | 66 | 84 | 89 | 106 | 118 | 124 | 135 |

We give the approximate values of the $p$-th Fourier coefficient of the wave form for prime $p$ below 17 in Table 2. For the next prime 19, it is difficult to determine $c_{19}$ because we need the rough estimations of 43-th, 47-th and 53-th Fourier coefficients and the more precise estimations of the small ordinal number ones.

TABLE 2

| $p$ | $c_p$ | $p$ | $c_p$ |
|---|---|---|---|
| 2 | -1.736 | 11 | 0.59 |
| 3 | 0.37 | 13 | 0.18 |
| 5 | 0.11 | 17 | 0.83 |
| 7 | 1.9 | | |

## REFERENCES


1. Cartier, P., Some Numerical Computations Relating to Automorphic Functions, Computers in Number theory, Proceedings of the Science Research Council Atlas Symposium No. 2 held at Oxford, Academic Press, 1971, 37-48.
2. Gelbart, S. S., Automorphic Forms on Adele groups, Annals of Math. studies, Princeton University Press, 1975.
3. Hejhal, D. A., The Selberg trace formula for $PSL(2, \mathbf{Z})$, Lecture Notes in Mathematics, No. 1001, Springer-Verlag, New York, 1983.
4. Hejhal, D. A., Eigenvalues of the Laplacian for $PSL(2, \mathbf{Z})$: some new results and computational techniques, International Symposium in memory of Hua Loo Keng, vol. 1. Science Press and Springer-Verlag, New York, 1991, 59-102.
5. Hejhal, D. A. & Arno, S., On Fourier coefficients of Maass waveforms for PSL(2,Z), Math. Computation vol. 61, 203, (1993), 245-267.
6. Maass, H., Uber eine neue Art von nichtanalytischen automorphen Functionen und die

Bestimmung Dirichletscher Reihen durch Functionalgeichungen, Math. Ann. 121 (1949), 141-183.

7. Stark, H. M., Fourier coefficients of Maass waveforms, Modular forms, Ellis Horwood Ltd., 1984, 263-269.