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The determination of small dimensions of the Hilbert Modular Type Cusp forms of Weight Two over Real Quadratic Fields

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The author has proved the dimension formula of the space of the Hilbert modular type cusp forms of weight two. We present further refinements of the dimension formula for a real quadratic number field $Q(\sqrt{D})$, and calculate it effectively. We tabulate the dimensions for square-free natural numbers D below 10^4 with the computer assistance. We also determine all the spaces of their dimension below 10.

1. INTRODUCTION

For a square-free natural number D, let k be a real quadratic number field generated by $\sqrt{D}$ over Q. $\mathfrak{g}$ denotes the ring of integers in k, and E the group of units in $\mathfrak{g}$. Let H^2 be 2-fold product of the complex upper half plane. Let Γ stand for the Hilbert modular group $SL_2(\mathfrak{g})$ embedded in $G=SL_2(R)^2$. We consider that G acts on H^2 by the linear fractional transformation.

In this paper, we tabulate the dimensions of the space of the cusp forms of weight two over $Q(\sqrt{D})$, where D is below 10^4, with H^2 computer's aid, and determine all D whose space have a dimension below 10. This result was partially announced in [4].

The trace formula of Hecke operators was investigated by the author in [3], which was based on more general situations. Since the first Betti number of the Hilbert modular surface H^2/Γ vanishes, its arithmetic genus minus one coincides with the dimension of the cusp forms of weight two. The geometric aspects were studied by Hirzebruch in [2].

We present further refinements of the dimension formula. The formula is made up of the contributions from the identity, the elliptic elements, the parabolic ones, and the Eisenstein series. The contribution from the identity refers to the special value of the Dedekind zeta function, and that from elliptic

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elements to the ideal class numbers of CM-fields. When k has a unit of negative norm, there is no contribution from the parabolic elements. On the other case, its contribution refers to the special value of L-function, which was introduced in [8] and was investigated in [1].

2. The Dimension Formula

By a cusp form of weight two belonging to Γ, we understand a function $f(z)$ on H^2, satisfying the following conditions:

i) $f(z)$ is holomorphic on H^2

ii) $f(\gamma z)=j(\gamma, z)^{-1}f(z)$ for $\gamma \in \Gamma$

where $j(\gamma, z)=\Pi(c_i z_i+d_i)^{-2}$ $\qquad (\gamma = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix})$

iii) $f(z)$ vanishes at every parabolic point of Γ.

The linear space of all $f(z)$ is denoted by $S_2(D)$.

Theorem 1. ([3, Theorem 1])

(1) $\dim S_2(D)= t_0+t_1+t_2+t_3$

(2) $$t_0 = \frac{1}{8\pi^2} d_k^{3/2} \zeta_k(2)$$

$$t_1 = \sum_{n\geq 1} \frac{1}{n} \sum_{[g]} e(g)$$

$$t_2 = \begin{cases} 0 & \text{if } k \text{ has a unit of negative norm} \\ -\pi^{-2} d_k^{1/2} \sum_{i=1}^{h_k} \overline{\chi}\left(A_i^2\right) L\left(1,\chi,A_i^2\right) & \text{otherwise.} \end{cases}$$

$$t_3 = -1$$

Here, t_0, t_1, t_2 and t_3 are the contributions from the identity, elliptic elements, parabolic elements, and the Eisenstein series.

The notations are as follows.

d_k, h_k, ζ_k, χ, or $L(s,\chi,A)$ stands for the discriminant of k, the class number of k, the Dedekind zeta function of k, a character of norm signature type, or L-series taken over the ideals in A. $[g]$ runs over all Γ-inequivalent elliptic elements of order $2n$ in Γ, and A_i runs over all ideal classes in $\mathfrak{o}$. $e(g)$ is decided by an eigenvalue of g.

3. The Contribution From The Identity

By the functional equation of the Dedekind zeta function, t_0 is equal to $(1/2)\zeta_k(-1)$. The next lemma is due to Siegel [9].

Lemma 1. *For a real quadratic field k with a discriminant d_k, we have*

(3) $\zeta_k(-1) = (1/60)\Sigma\,\sigma((d_k - b^2)/4)$

where the summation runs over all b such that $|b| < \sqrt{d_k}$, $b^2 \equiv d_k$ mod 4, and $\sigma(x)$ is the sum of divisors of x.

4. The Elliptic Contribution

By [8, No.26], we can rewrite t_1 as follows.

Lemma 2.

(4) $$t_1 = \frac{1}{h_k} \sum_{s,\boldsymbol{f}} \frac{h(\mathfrak{o}(s,\boldsymbol{f}))}{w(\mathfrak{o}(s,\boldsymbol{f}))}\, e(\mathfrak{o}(s,\boldsymbol{f}))$$

(5) $$e(\mathfrak{o}(s,\boldsymbol{f})) = \begin{cases} (1/2)(w(\mathfrak{o}(s,\boldsymbol{f}))-1) & \text{if the norm map induced by } N_{K/k} \text{ from the ideal class group} \\ & \text{of } \mathfrak{o}(s,\boldsymbol{f}) \text{ to the narrow ideal class group of } \mathfrak{g} \text{ is surjective} \\ \sum_{g}\left\{\prod_{i=1,2}\frac{\zeta(g_i)}{1-\zeta(g_i)^2} + \prod_{i=1,2}\frac{\bar{\zeta}(g_i)}{1-\bar{\zeta}(g_i)^2}\right\} & \text{otherwise} \end{cases}$$

Here, s, $\boldsymbol{f}$ run over integers in k, integral ideals in k such that $4-s^2$ is totally positive and $(s^2-4)\boldsymbol{f}^{-2}$ is integral (taken up to $\left(s \pm \sqrt{s^2-4}\right)/2$ mod(E)). $\mathfrak{o}=\mathfrak{o}(s,\boldsymbol{f})$, $h(\mathfrak{o})$, and $w(\mathfrak{o})$ are the order in K $= k\left(\sqrt{s^2-4}\right)$ with discriminant $(s^2-4)\boldsymbol{f}^{-2}$, the class number of $\mathfrak{o}$, and the index of E in the group of units in $\mathfrak{o}$. g runs over all elements in $\Gamma(\mathfrak{o}(s,\boldsymbol{f}))-\{\pm 1\}$ modulo $\{\pm 1\}$.

$\zeta(g_i)$, $\bar{\zeta}(g_i)$ denote eigenvalues of g_i satisfying $g_i z_i = \zeta(g_i)^2 z_i$ ($z=(z_1, z_2)$ being the fixed point of g in H^2).

s is only 0 or ± 1 except $D=5$, when s satisfies the condition above. For $D=5$, $s=\pm\left(1 \pm \sqrt{5}\right)/2$ also satisfies it. The possible $\boldsymbol{f}$ other than $\boldsymbol{f}=\mathfrak{g}$ are $\mathfrak{p}$, $\mathfrak{p}^2$ for $s=0, D\equiv 3$ mod(4), $\mathfrak{p}$ for $s=0, D\equiv 2$ mod(4), and $\mathfrak{q}$ for $s=\pm 1, D\equiv 0$ mod(3), where $\mathfrak{p}^2=2\mathfrak{g}$, $\mathfrak{q}^2=3\mathfrak{g}$.

LEMMA 3. *When* (s, f) *satisfies the condition in Lemma 2, we get*

(6) $$w(\mathfrak{o}(0, f)) = \begin{cases} 4 & if D=2, f=\mathfrak{p} \\ 2 & otherwise \end{cases}, \quad w(\mathfrak{o}(\pm 1, f)) = \begin{cases} 6 & if D=3, f=\mathfrak{q} \\ 3 & otherwise \end{cases}$$

$$w(\mathfrak{o}(\pm(1 \pm \sqrt{5})/2), \mathfrak{g})) = 5 \quad if D=5$$

Therefore there are elliptic points of order 2, 3, 4, 5, 6 in Γ. A point of order 4, 5, or 6 appears only when $D=2$, 5, or 3, respectively.

Let $\mathfrak{o}_0$ be the principal order of K. Considering the numbers of residue classes for the conductor f of $\mathfrak{o}$ in $\mathfrak{o}_0$, we have

LEMMA 4. *When* f *is the conductor of an order* $\mathfrak{o}$ *in* K, *then*

(7) $$\frac{h(\mathfrak{o})}{w(\mathfrak{o})} = \frac{h(\mathfrak{o}_0)}{w(\mathfrak{o}_0)} N(f) \prod_{\mathfrak{p} \mid f} (1-(\frac{K}{\mathfrak{p}})N(\mathfrak{p})^{-1})$$

where $(K/\mathfrak{p})$ *stands for the Artin symbol, and* N *for the norm.* $\mathfrak{p}$ *runs over all prime ideals dividing* f.

For any cases except $s=\pm\left(1 \pm \sqrt{5}\right)/2$, K is a composite field of k and an imaginary quadratic field of $Q(\sqrt{-1})$ or $Q(\sqrt{-3})$, so $h(\mathfrak{o}_0)$ can be expressed by the class numbers of the quadratic fields. The next lemma is given by the consideration of the Dedekind zeta function of K.

LEMMA 5. *For* $K=Q(\sqrt{d_1}, \sqrt{-d_2})$ $(d_1 \geqq 2, d_2 \geqq 1)$, *the ideal class number of* K *is given by*

(8) $$h(\mathfrak{o}_0) = \delta_2 (1/2) h(d_1) h(-d_2) h(-d_1 d_2)$$

where $h(d)$ *denotes the class number of* $Q(\sqrt{d})$. $\delta_2 = 2$ *or* 1, *according to* $(d_1, d_2) = (2, 1), (2, 2)$ *or not.*

On the other hand, by a direct calculation we get

(9) $$h(\mathfrak{o}(\pm(1 \pm \sqrt{5})/2, \mathfrak{g})) = 1$$

It is easy to see that $e(\mathfrak{o}(0, f)) = (1/2)(w(\mathfrak{o}(0, f)) - 1)$ for any f. As to $s=\pm 1$, $e(\mathfrak{o}(\pm 1, \mathfrak{g}))=1$ for $(3, D)=1$, $e(\mathfrak{o}(\pm 1, f))=2/3$ for $D \equiv 6 \bmod 9$ ($f=\mathfrak{g}$, $\mathfrak{q}$), and $e(\mathfrak{o}(\pm 1, \mathfrak{q}))=4/3$, $e(\mathfrak{o}(\pm 1, \mathfrak{g}))=3/2$ for $D \equiv 3 \bmod 9$ $(D \neq 3)$. For $D=3$, the elliptic contribution of order 3 and 6 becomes to 17/24 by a direct calculation.

Summing up above things, we obtain

THEOREM 2.

(10) $$t_1 = a(D)h(-D) + b(D)h(-3D) + c(D)$$

$a(D)$, $b(D)$ and $c(D)$ *are given in the following table.*

TABLE 1

D	$D\equiv 1$ mod(4)	$D\equiv 2$ mod(4) $D\neq 2$	$D\equiv 3$ mod(8) $D\neq 3$	$D\equiv 7$ mod(8)	$D=2$	$D=3$
$8\,a(D)$	1	3	10	4	5	3

D	$D\equiv 1,2$	$D\equiv 3$ mod(9)	$D\equiv 6$ mod(9)	$D=3$
$24\,b(D)$	4	16	8	17

D	$D=5$	$D\neq 5$
$5\,c(D)$	2	0

By using the computation of a class number of an imaginary quadratic field, we get t_1 effectively.

5. THE CUSP CONTRIBUTION

When k has a unit of negative norm, t_2 vanishes. Thus we assume k does not have such a unit. Moreover, if D has no prime factors that are 3 modulo 4, the narrow ideal class represented by the principal ideal with negative norm is a square element. Then t_2 vanishes. From now on, we also assume that D is divisible by at least one prime factor $p\equiv 3$ (mod 4). Next lemma is due to [1, Theorem 1.2].

LEMMA 6.

$$\sum_i \bar{\chi}(A_i)L\left(1,\chi,A_i^2\right) = \sum_j L(s,\chi_j) \tag{11}$$

where χ_j runs over all real characters of norm signature type.

There is a one-to-one correspondence between a set of real characters of norm signature type and a set of pairs of discriminants of imaginary quadratic fields (d_1, d_2) satisfying $d_k = d_1 \cdot d_2$.

LEMMA 7. ([9]) *For a character χ of norm signature type corresponding with (d_1, d_2), we have*

$$L(1,\chi)= 4\pi^2 d_k^{-1/2}\, h(-d_1)\, h(-d_2)/w(-d_1)\, w(-d_2) \tag{12}$$

where $h(-d)$ denotes a class number of $Q(\sqrt{-d}\,)$, and $w(-d)$ an order of the unit group of $Q(\sqrt{-d}\,)$.

Summing up above things, we get

THEOREM 3.

$$(13) \qquad t_2 = \begin{cases} 0 & \textit{if } k \textit{ has a unit of negative norm or } d_k \textit{ has no prime factors } 3 \pmod 4 \\ -4\, \Sigma\, h(-d_1)h(-d_2)/w(-d_1)w(-d_2) & \textit{otherwise} \end{cases}$$

where (d_1, d_2) *runs over all discriminants of imaginary quadratic fields satisfying* $d_k = d_1 \cdot d_2$.

To decide whether k has a unit of negative norm or not, we calculate a period of an infinite continued fraction of ω where $[1,\omega]$ is a basis of $\mathfrak{g}$. When t_2 does not vanish, we have to get a set of pairs (d_1, d_2) satisfying the condition in Theorem 3, and calculate the class numbers of the imaginary quadratic fields $Q(\sqrt{-d_1}), Q(\sqrt{-d_2})$.

6. A TABLE OF DIMENSIONS

A table of dim $S_2(D)$ appears as appendix in Supplement section for $1<D<6000$.

7. THE ESTIMATION OF THE LOWER BOUND

Now we estimate the lower bound of dim $S_2(D)$.

LEMMA 8. ([6]) *For a discriminant* $-d$ *of an imaginary quadratic field, we have*

$$(14) \qquad \frac{h(-d)}{w(-d)} \le \frac{\sqrt{d}}{4\pi}(2+\log d)$$

Also we use the estimation:

$$(15) \qquad \zeta_k(2) > \zeta_Q(4) = \pi^4/90$$

Therefore dim $S_2(D)$ is greater than

$$(16) \qquad (1/720)d_k^{3/2} - (1/16\pi^2)\, d_k^{1/2}\{16 + 8\log d_k + (\log d_k)^2\}\,\mathrm{div}(d_k) - 1$$

where $\mathrm{div}(d_k)$ is the number of pairs (d_1, d_2) satisfying the condition in Theorem 3. We estimate roughly

$$(17) \qquad \mathrm{div}(d_k) \leqq d_k^{1/2}$$

Put

$$(18) \qquad f(x) = (1/720)\, x^{3/2} - (1/16\pi^2)\, x\{16 + 8\log x + (\log x)^2\} - 1$$

We can check easily that $f(x)>10$ when $x>x_0=68134$. For a small discriminant d_k, we list the upper bound of $\mathrm{div}(d_k)$ in Table 2.

TABLE 2

$d_k \equiv 1 \mod(4)$	21~	105~	4389~	21945~285285
$\mathrm{div}(d_k)$	1	2	4	8

$4 \parallel d_k$	12~	60~	84~	420~	60060~384540
$\mathrm{div}(d_k)$	1	2	4	8	16

$8 \parallel d_k$	24~	120~	1560~	9240~120120
$\mathrm{div}(d_k)$	1	2	4	8

As a consequence of the table above and of (16), we find

(19) $\dim S_2(D)>10, \quad \text{for } D>5983$

Therefore, a complete table of small dimensions is given by the table of dimensions in this supplementary section.

THEOREM 4. *A complete set of D that satisfies the condition* $\dim S_2(D)\leq 10$ *is given in Table 3.*

TABLE 3

$\dim S_2(D)$	D
0	2 3 5 6 7 13 15 17 21 33
1	10 11 14 29 37 41 57 69 105
2	19 22 23 30 53 61 65 73 77 93
3	31 35 42 85 89 97 141 165
4	26 39 101 109 113 133 161
5	34 38 46 47 129 137 213
6	55 145 149 157 177
7	43 51 173 181 285

8	62 78 185 197 217 237 253
9	59 70 193 201 205 209 221 229 273 357
10	58 66 71

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Supplement to
The determination of small dimensions of the Hilbert Modular Type Cusp forms of Weight Two over Real Quadratic Fields

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TABLE

$\dim S_2(D)$ for a square-free number D, $1<D<6000$

In this table, the number D is given by

(1) $D=i+100j$ (i=row number, j=column number).

When the mark "-" appears after a figure, $Q(\sqrt{D})$ has a unit of negative norm. The mark "**" means D is not square-free.

i\j	0	1	2	3	4	5	6	7	8	9
1	**	4-	9	12	24-	21	50-	39-	**	59-
2	0-	17	55-	81	130	200	215	**	435	391
3	0	20	41	86	151	166	**	336	339	429
5	0-	1	9	16	**	38	**	45	43	74
6	0	25-	49	**	165	189	264	375	365	471
7	0	21	**	100	126	**	271	271	388	523
9	**	4-	9	12	28-	23-	39	43-	67-	**
10	1-	16	39	101	139	175	304-	277	**	491
11	1	20	66	88	147	231	243	**	449	451
13	0-	4-	5	18-	16	**	35-	40	42	83
14	1	22	64	96-	**	233	253	315	452	470-
15	0	19	46	**	162	179	233	343	356	427
17	0-	**	8	13-	25	22	41-	31	71	44
18	**	24	56-	84	150	169	243	355	358-	**
19	2	17	61	110	144	212	305	307	**	570
21	0	**	9	18	21-	35-	**	60	46-	85
22	2	24-	48	97	135	**	263	**	377	520-
23	2	23	61	89	**	221	219	325	454	397
26	4-	**	73-	105	147	237	254-	**	461	468
27	**	29	54	94	161	171	257	362	359	**
29	1-	5	9-	14	13	**	31-	**	52-	78-
30	2	32-	53	89	158	196-	**	391-	366	433
31	3	28	53	125	145	**	324	321	414	**
33	0	4	11-	**	29-	25-	45	42-	**	49
34	5	31	**	115	135	227	332-	310	436	553
35	3	**	65	92	135	224	233	**	455	431
37	1-	5-	8	21-	16	34	**	53	**	90-
38	5	23	59	**	153	228-	246	**	427	415
39	4	35	61	113	180	**	**	410	378	531
41	1-	3	13-	12	**	30-	48-	35	**	53-
42	3	29	**	**	181-	184	260	369	385-	458
43	7	24	**	**	151	216	298	302	385	491
45	**	6-	**	13	23-	36	27	66	**	**
46	5	34	73	130-	154	207	333	345-	**	588
47	5	**	74	105	146	246	244	**	**	446
49	**	6-	12	16-	28-	**	53	36	77	65-
51	7	38	69	**	185	196	283	423	407	510
53	2-	**	8	18-	18	36	33-	58	51-	79-
54	**	35	68	120	193	221-	290	408-	403	**
55	6	29	55	117	135	211	318	305	**	533
57	1	6-	12-	9	32-	27-	**	47-	65-	45
58	10-	30	66	127	153-	**	287	324	393	522
59	9	34	85	112	**	258	286	337	505	467
61	2-	4	**	**	21-	39	37-	59-	45	**
62	8	**	75	107-	141	262	268	327	468	444-
63	**	42	68	**	191	202	253	393	385	**
65	2-	3	14-	17-	23	29-	37	**	84-	55-
66	10	43	67	122	208	218	**	408	411	523
67	12	32	73	131	156	**	313	318	**	564
69	1	**	11-	**	24	41-	34	71-	44	87
70	9	41-	**	138-	155	201	308	305	431	593-
71	10	**	87	113	186	283	266	393	498	491
73	2-	7-	9	19-	27	23	57-	38-	**	60
74	15-	38	98-	119	175	261	295	**	495	503
77	2	6	13-	20	**	43-	34-	53	56-	80-
78	8	48	70	**	190	**	279	411-	371	494
79	15	44	**	153	163	245	349	359	460	593
81	**	7-	15-	16	36-	26	54	44	74-	**
82	16-	35	73	132	165	233	313	299	**	576
83	14	42	91	109	151	260	275	**	508	442
85	3-	8-	7	19	23-	**	39-	60-	43	100-
86	16	43	84	135	**	289-	**	396	532	495-
87	12	45	67	**	204	213	300	396	382	479
89	3-	**	**	18-	33	32	52	41	84	58
90	**	44	84-	111	**	218	293	413	421	**
91	17	45	93	141	187	244	360	343	**	639
93	2	9-	12-	22	25-	38-	**	72	46	85
94	16	51	**	165-	178	**	370	364-	464	601
95	14	35	96	113	**	255	280	363	490	485
97	3-	8-	**	18-	24	28	59-	41-	63	65-
98	**	**	103-	113	186	245	282-	339	515	483
99	**	56	81	135	230	244	325	435	411	**

i\j	10	11	12	13	14	15	16	17	18	19
1	77	72	141-	84-	160	114	172-	**	258-	149-
2	523	636	612	731	994-	860	**	1297	1125	1308
3	581	559	671	898	757	**	1175	1050	1293	1466
5	55	117-	77	**	107	141	109	213	**	229
6	607	581	**	952-	812	1032	1229	1133-	1345	1702-
7	450	**	713	691	849	1107	976	1123	1444	1230
9	109-	68-	113	91	148-	109	214-	125-	**	160
10	520-	595	**	744	837	1077	943	**	1518-	1233
11	567	704	669	825	1073	928	**	1342	1220	**
13	56-	91	90-	124-	**	185	107-	193	**	213-
14	**	772-	666	**	1049	963-	1150	1394-	1254	1413
15	597	581	**	859	824	923	1185	**	**	1621
17	**	80-	106-	86	164-	99	**	147	186	**
18	596-	532	653	899	811-	897	1224-	1024	**	1495
19	531	644	859	730	909	**	1066	**	1521	1356
21	68-	105	69	158-	**	**	143-	193-	148	277-
22	457	623	781	715-	**	1092-	964	1095	1416	**
23	527	682	624	**	1011	894	1028	1321	1127	1379
26	**	741	721-	793	1029	949	1179	1448	1208	**
27	593	**	711	917	792	992	1161	1024	**	1545
29	**	128-	79-	144	119-	159	**	231	132	257
30	637	591-	685	887	815	**	1192	1162-	1283	1605-
31	540	659	837	**	**	1179	1047	1299	1592	1309
33	106-	58	**	92	137-	101	188	124-	189	172-
34	509	**	884	801	937	1194	1038	**	1547	1411
35	**	740	627	801	987	926	1107	1330	1225	**
37	61-	105	85-	110	91	192	108-	**	142	212-
38	546	732-	634	799	936	901	**	1369	1177	1349
39	663	597	751	989	895	**	1340	1141	1332	1705
41	100	84	123-	**	175	104	197	150-	202	157
42	604-	580	**	862	793	991	1265-	1042	1320	1518
43	501	**	815	703	887	1116	949	1179	1487	1260
45	61	106-	73	162	**	161	123	196-	**	270
46	542	645	853	815	1009	1199-	1071	**	1507	1357
47	554	713	666	799	988	855	**	1374	1146	1437
49	95-	71	147-	82	**	130-	185-	123	**	149-
51	689	603	**	995	899	1077	1321	1167	1385	1697
53	**	121-	70	139	116-	150-	107	224-	125-	**
54	681	635	789	979-	863	1033	1382	1226-	**	1767
55	500	587	856	765	917	1169	1008	**	1485	1295
57	102	71-	126	88	130	**	204-	112	225	168
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21	605-	**	722	1063-	697	1339-	663	1314	918-	1150
22	**	6536-	5501	6407	7465	5894	6977	7632-	6373	**
23	6637	5203	6392	6855	5619	6643	7738	6561	**	8522
26	6869	5997	6753	7513	6482-	**	8315-	6737	7792	9081
27	5373	5854	6819	5735	**	7738	5981	6997	7962	6608
29	680	940	**	**	659-	1191	869-	1153-	759	**
30	5613	**	7398	5785-	6563	7619	6660	7455	8243	7079-
31	**	7397	6007	7007	7794	6623	7232	8367	**	**
33	844	599	1167-	584-	1100	732	1031	**	1411	705-
34	6343	7111	5960-	6807	7989	6814	**	8785	7051-	7865
35	6791	5747	6175	7417	6061	**	**	6733	7663	8456
37	571	1093	586-	**	740-	**	710	1358-	661-	1274
38	6260	5427	**	7309	5836	6913	7507	6147	7163	8638
39	5581	**	**	6189	**	8199	6820	7417	9065	7018
41	**	589	1097	**	1086-	745	1392-	711-	**	911-
42	5396-	6153	6794-	5854	6292	7429	6103	**	8329	6570
43	**	6500	**	6359	7638	5821	**	7945	6310	7379
45	557-	**	759-	1008	**	1344-	655	1201	835	1213
46	**	7439	5794	**	8029	6637	7738	**	7335	7835
47	**	5579	**	7096	5886	**	7635	6481	7388	8435
49	**	762	1004	712	1310-	646	1221	901-	1225-	**
51	5777	6365	7799	6074	6863	8135	6872	**	8635	7500
53	668	940-	605	1176-	583	**	803-	1035	768	1377-
54	**	6556	7525	6041-	**	8030-	6671	7651	8992	7580-
55	6077	6817	5710	**	7694	6439	6855	8445	6648	7579
57	864	**	1142	608	1135	755-	1071-	683	1355-	673
58	**	6639	5629	6137	7190-	6043	7063	8240	6388-	**
59	6890	5775	6825	7923	6399	7443	8455	6726	**	9150
61	649	1240-	651-	1134	838	1089	**	1428	750-	1365
62	6667	5377-	6148	7119	5930	**	7534	6688	7097	8085
63	5390	6105	6812	5809	**	7441	6223	7259	8335	6745
65	1176-	577-	**	751-	1020-	671	1365	701-	1159	913-
66	5645	**	7722-	6442	6829	**	6647	**	8999	7590
67	**	6827	5471	6184	7215	6332	7073	7839	6871	**
69	611	1069	742	1031	731	1401-	696-	**	882-	1172
70	**	7179	5667	6741	7630	6192-	**	8355-	6804	7581
71	7018	5866	6635	7967	6499	**	8618	6797	7613	9295
73	957	704	959-	**	1212-	648-	1097	798	1080	753
74	7130	6025	**	7546	6311	7553	8697-	6715	8033	8893
77	685-	912	618	1231	608-	**	808	1090-	**	1401
78	5152	5877	6961	5889	6681	7657-	6363	**	8154	**
79	6534	7273	6263	6781	7858	6441	**	8928	7060	8235
81	941-	657	1257-	644-	**	866-	1143	767	1508-	739-
82	**	6900	5704	**	7713-	5949	6888	**	6687-	7706
83	6699	5323	**	7317	6207	6798	7552	6484	7171	8363
85	**	1171-	577	1105	805-	1036-	747	1365-	667	**
86	7376	5933	6839	7679-	6254	**	8584	7139	**	9303
87	5279	5979	7121	5829	6565	7769	**	**	**	6938
89	1176	618-	1159	769-	1117	**	1376-	720	1333	965
90	5572-	6179	**	**	**	7967	6539	7465	8495	6905
91	6571	7548	5995	**	**	6640	7647	8909	7293	7862
93	564	**	726	964-	658	1237	671-	1212	804	1178-
94	**	**	6323	7127	7869	6812-	7261	8509	7217	**
95	6714	5769	6419	7335	6175	7067	8185	6599	**	8395
97	1025	694-	910-	633	1216	653-	**	827	1101-	754
98	6869-	5539	6088	7432	5944-	**	7815	6649	7443	8259
99	5727	6400	7533	6482	**	8104	6991	7692	8929	7339