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# Mixed expansion formula for the rectangular Schur functions and the affine Lie algebra A(1)((1))

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# MIXED EXPANSION FORMULA FOR THE RECTANGULAR SCHUR FUNCTIONS AND THE AFFINE LIE ALGEBRA $A_1^{(1)}$

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Abstract. Formulas are obtained that express the Schur $S$-functions indexed by Young diagrams of rectangular shape as linear combinations of "mixed" products of Schur's $S$- and $Q$-functions. The proof is achieved by using representations of the affine Lie algebra of type $A_1^{(1)}$. A realization of the basic representation that is of "$D_2^{(2)}$"-type plays the central role.

## 1. Introduction

We derive formulas of combinatorial nature that express the Schur $S$-functions indexed by Young diagrams of rectangular shape, the rectangular $S$-functions for short, as linear combinations of "mixed" products of $S$- and $Q$-functions.

The rectangular $S$-functions are studied in [4, 7] from a viewpoint of representations of the affine Lie algebra of type $A_1^{(1)}$ and $A_2^{(2)}$. These functions appear as certain distinguished weight vectors in the so called *homogeneous* realization of the basic representation $L(\Lambda_0)$ of $A_1^{(1)}$ (see [5]). On the other hand, the Schur $Q$-functions arise naturally in the representation of $D_{\ell+1}^{(2)}$-type Lie algebras ([8]). In the subsequent pursuit of various realizations of $L(\Lambda_0)$, our formula has come out as an application of the isomorphism $D_2^{(2)} \cong A_1^{(1)}$. Roughly speaking, we can realize the space $L(\Lambda_0)$ as a tensor product of the spaces of the Schur $S$- and $Q$-functions. We call such a "mixed" realization as the homogeneous realization of type $D_2^{(2)}$.

Let us describe our main result in more detail. Let $\mu$ be a partition and $S_\mu(t)$ be the corresponding Schur $S$-function, where $t = (t_1, t_2, t_3, \ldots)$, and each $t_j$ ($j = 1, 2, \ldots$) is the $j$-th *power sum* $p_j$ divided by $j$. Let $Q_\lambda(t)$ denote the Schur $Q$-function indexed by a *strict* partition $\lambda$, where $t = (t_1, t_3, t_5, \ldots)$. Let $\square(m, n)$ denote the Young diagram of the rectangular shape $(n^m)$. Set also $S_\mu(t^{(2)}) = S_\mu(t_2, t_4, t_6, \ldots)$. Note that the set

$$\{Q_\lambda(t) S_\mu(t^{(2)}); \lambda \text{ is a strict partition and } \mu \text{ is a partition}\}$$

forms a basis of the space of the symmetric functions.

Let $m, n$ be non-negative integers. Our formula (Theorem 3.1), called "mixed expansion formula", reads:

$$\sum_{\mu} \delta(\mu) Q_{\mu[0]}(t) S_{\mu[1]}(t^{(2)}) = S_{\Box(m,n)}(t), \tag{1}$$

where the summation runs over a certain finite set of strict partitions determined by $m$ and $n$. For each strict partition $\mu$, one associates a strict partition $\mu[0]$, a partition $\mu[1]$ and a sign $\delta(\mu) = \pm 1$ in a combinatorial way. We prove the formula (1) by comparing two realizations of $L(\Lambda_0)$ mentioned above. The left hand side stems from combinatorial descriptions of actions of Chevalley generators in the homogeneous realization of type $D_2^{(2)}$, whereas the right hand side is obtained via "vertex operator calculus" (as employed in [4]) in the homogeneous realization of type $A_1^{(1)}$.

Here we explain the background of our study of rectangular Schur functions. As written in the above, our formula arose from a study of the homogeneous realization of the basic $A_1^{(1)}$-module. We have two pictures of the principal realization of the basic $A_1^{(1)}$-module; one is described in terms of the 2-reduced Schur functions and is relevant to the KdV hierarchy; the other is the twisted version, which is best described by the $Q$-functions and is relevant to the 4-reduced BKP hierarchy. On the other hand, the homogeneous realization of that module is connected with the nonlinear Schrödinger (NLS) hierarchy. Using an intertwining operator between the (non twisted) principal and the homogeneous realizations, one can derive an expression of the rectangular Schur functions and certain $\tau$-functions of the NLS hierarchy ([4]).

The paper is organized as follows. In Section 2 we recall some combinatorial materials related to partitions. In Section 3 we state our main theorem on rectangular Schur functions. In Section 4 we recall the spin representation of $A_1^{(1)}$ and describe the action of $A_1^{(1)}$ in terms of Young diagrams. In Section 5 through the boson-fermion correspondence, we obtain weight vectors as a sum of products of $S$- and $Q$- functions. In Section 6 we consider $f_i$-action ($i = 0, 1$) and obtain the rectangular Schur functions appearing in the right hand side of our formula through a vertex operator calculus. Section 7 is devoted to the proof of the main theorem.

## 2. Combinatorics of Partitions

2.1. **Partition.** A *partition* is any non-increasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ containing only finitely many non-zero terms. We regard two partitions as the same that differ only by a string of zeros at the end. The non-zero $\lambda_i$ are called the *parts* of $\lambda$. The number of parts is the *length* of $\lambda$, denoted by $\ell(\lambda)$. A partition is *strict* if all parts are distinct. Denote by $\mathcal{P}$ (resp. $\mathcal{SP}$) the set of all partitions (resp. strict partitions).

2.2. ***i*-addable node.** Let $\lambda \in \mathcal{SP}$. To each node $x \in \lambda$ in the $j$-th column, we assign a color $c(x)$ by the following rule:

$$c(x) = \begin{cases} 0 & (j \equiv 0, 1 \mod 4) \\ 1 & (j \equiv 2, 3 \mod 4) \end{cases}.$$

For example, the nodes of $\lambda = (5, 4, 2, 1)$ is colored as

$$\begin{array}{ccccc} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \\ 0 & 1 & & & \\ 0 & & & & \end{array}.$$

We say that a node $x$ is $i$-addable to $\lambda$, if $\lambda \cup \{x\}$ is a strict partition and $c(x) = i$. The following nodes indicated by dots are the 1-addable nodes:

$$\begin{array}{cccccc} 0 & 1 & 1 & 0 & 0 & \bullet \\ 0 & 1 & 1 & 0 & & \\ 0 & 1 & \bullet & & & \\ 0 & & & & & \end{array}.$$

Set

$$I_i^\ell(\lambda) = \{\mu \in \mathcal{SP} \mid \mu \supset \lambda,\ |\mu| = |\lambda| + \ell,\ \forall x \in \mu - \lambda,\ c(x) = i\}.$$

It is the set of strict partitions obtained from $\lambda$ by adding $i$-nodes $\ell$ times in succession. Put

$$\begin{cases} c_m = (4m - 3, \dots, 5, 1) & (m > 0) \\ c_m = \emptyset & (m = 0) \\ c_m = (-4m - 1, \dots, 7, 3) & (m < 0). \end{cases}$$

If $m > 0$, we have $I_1^\ell(c_m) = \emptyset$ for $\ell > 2m$ and $I_0^\ell(c_{-m}) = \emptyset$ for $\ell > 2m + 1$.

The strict partitions $c_m$ ($m \in \mathbb{Z}$) are called 4-*bar cores*, introduced in [1, 8].

**Example 2.1.** *For $m = -2$ and $i = 0$ we have*

$$I_0^1(c_{-2}) = \{(8, 3), (7, 4), (7, 3, 1)\},$$

$$I_0^2(c_{-2}) = \{(9, 3), (8, 4), (8, 3, 1), (7, 4, 1), (7, 5)\}$$

*and*

$$I_0^3(c_{-2}) = \{(9, 4), (8, 5), (9, 3, 1), (8, 4, 1), (7, 5, 1)\}.$$

2.3. **4-bar quotient.** Let us introduce the notion of 4-bar quotient. We shall give a bijection

$$\mathcal{SP} \to \mathbb{Z} \times \mathcal{SP} \times \mathcal{P}, \quad \lambda \mapsto (m, \lambda[0], \lambda[1]).$$

For $\lambda \in \mathcal{SP}$, the pair $(\lambda[0], \lambda[1])$ is called the 4-bar quotient of $\lambda$.

Let us identify the strict partition $\lambda$ with the subset $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_s\}$ of $\mathbb{N}$. For $a = 0, 1, 2, 3$, we set $\boldsymbol{\lambda}^{(a)} = \{\lambda_j \in \boldsymbol{\lambda} \mid \lambda_j \equiv a \mod 4\}$. Namely

$$\boldsymbol{\lambda}^{(a)} = \boldsymbol{\lambda} \cap (4\mathbb{N} + a) \quad (a = 0, 1, 2, 3)$$

and we have $\boldsymbol{\lambda} = \sqcup_{a=0}^{3} \boldsymbol{\lambda}^{(a)}$. The even part $\boldsymbol{\lambda}^{(0)} \cup \boldsymbol{\lambda}^{(2)} \subset 2\mathbb{N}$ of $\boldsymbol{\lambda}$ gives a strict partition $\lambda[0]$ via the inclusion

$$\boldsymbol{\lambda}^{(0)} \cup \boldsymbol{\lambda}^{(2)} \subset 2\mathbb{N} \longrightarrow \mathbb{N}, \quad 2k \mapsto k.$$

From the odd parts $\boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(3)}$, we define a partition $\lambda[1]$ in the following way: First consider two bijections

$$\iota : 4\mathbb{N} + 1 \longrightarrow \mathbb{Z}_{\geq 0} \quad (4k + 1 \mapsto k), \quad \iota^* : 4\mathbb{N} + 3 \longrightarrow \mathbb{Z}_{<0} \quad (4k + 3 \mapsto -k - 1).$$

Then define a subset

$$\mathcal{M}(\lambda) = \iota(\boldsymbol{\lambda}^{(1)}) \cup (\mathbb{Z}_{<0} - \iota^*(\boldsymbol{\lambda}^{(3)}))$$

of $\mathbb{Z}$. This is a "Maya diagram" in the sense that, if we express $\mathcal{M}(\lambda)$ as an descending sequence $i_1 > i_2 > i_3 > \cdots$, then the integer $m = \sharp\boldsymbol{\lambda}^{(1)} - \sharp\boldsymbol{\lambda}^{(3)}$ satisfies $i_k = -k + m$ for $k \ll 0$. Then we can define

$$\lambda[1] = (i_1 + 1 - m, i_2 + 2 - m, i_3 + 3 - m, \ldots) \in \mathcal{P}.$$

The integer $m$ is called the *charge* of $\mathcal{M}(\lambda)$.

**Lemma 2.2.** *(cf.* [1]*) The map*

$$\mathcal{SP} \to \mathbb{Z} \times \mathcal{SP} \times \mathcal{P}, \quad \lambda \mapsto (m, \lambda[0], \lambda[1])$$

*is a bijection.*

We can illustrate the above construction. Let us look at a particular example, $\lambda = (11, 9, 6, 2, 1, 0)$. We draw a "4-bar abacus":

$$\begin{array}{ccc}
0 & 1 & 3 \\
2 & & \\
4 & 5 & 7 \\
6 & & \\
8 & 9 & 11 \\
10 & & \\
12 & 13 & 15
\end{array}$$

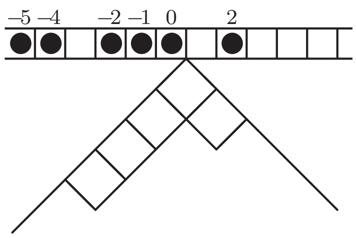
Here we do not put a bead on 0. We can read $\boldsymbol{\lambda}^{(0)} \bigcup \boldsymbol{\lambda}^{(1)}$ from the first column. Then we have $\lambda[0] = (3, 1)$. From the second and the third columns, we can read

$$\iota(\boldsymbol{\lambda}^{(1)}) = (2, 0)$$
$$\iota^*(\boldsymbol{\lambda}^{(3)}) = (-3).$$

Then we obtain

$$M = (2, 0, -1, -2, -4, -5, \cdots)$$

and draw a Maya diagram;

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Finally we have $\lambda[1] = (2, 1, 1, 1)$ and $m = 1$.

2.4. **Sign.** Each strict partition $\mu$ in $I_1^\ell(c_m)$ or $I_0^\ell(c_m)$ has its own sign determined by bead configuration.

**Definition 2.3.** *Put a bead on* 0 *of the 4-bar abacus of* $\lambda \in I_i^\ell(c_m)$ $(i = 0, 1)$, *if and only if* $m < 0$ *and* $\lambda_{-m+1} = 0$. *Let* $g(\lambda)$ *be the number of pair of beads on the central runner at the positions bigger than that of each bead on the leftmost runner. For a strict partition* $\lambda \in I_i^\ell(c_m)$, *we define the sign by*

$$\delta(\lambda) = (-1)^{g(\lambda)}.$$

**Example 2.4.** *We consider the case of* $i = 1, m = 3, \ell = 3$ *and* $\lambda = (11, 5, 2)$.

$$\begin{array}{ccc}
0 & 1 & 3 \\
2 & & \\
4 & 5 & 7 \\
6 & & \\
8 & 9 & 11
\end{array}$$

*We have* $\delta(\lambda) = (-1)^1 = -1$. *In the case of* $i = 0, m = -4, \ell = 5$ *and* $\lambda = (15, 13, 8, 5)$, *we have to put a bead on 0.*

$$\begin{array}{ccc} \textcircled{0} & 1 & 3 \\ 2 & & \\ 4 & \textcircled{5} & 7 \\ 6 & & \\ \textcircled{8} & 9 & 11 \\ 10 & & \\ 12 & \textcircled{13} & \textcircled{15} \end{array}$$

*We have* $\delta(\lambda) = (-1)^{2+1} = -1$.

## 3. Main Result

Define $h_n(t)$ by $\exp\left(\sum_{n=1}^{\infty} t_n z^n\right) = \sum_{n=0}^{\infty} h_n(t) z^n$. Let $\lambda$ be a partition. The Schur $S$-function with shape $\lambda$ is defined as

$$S_\lambda(t) = \det(h_{\lambda_i+j-i}(t)).$$

Define $q_n(t)$ by $\exp\left(\sum_{n=1}^{\infty} t_{2n-1} z^{2n-1}\right) = \sum_{n=0}^{\infty} q_n(t) z^n$. For $m > n \geq 0$, we put

$$Q_{m,n}(t) = q_m(t) q_n(t) + 2 \sum_{i=1}^{n} (-1)^i q_{m+i}(t) q_{n-i}(t).$$

If $m \leq n$ we define $Q_{m,n}(t) = -Q_{n,m}(t)$. Let $\lambda = (\lambda_1, \ldots, \lambda_{2n})$ be a strict partition, where $\lambda_1 > \cdots > \lambda_{2n} \geq 0$. Then the $2n \times 2n$ matrix $M_\lambda = (Q_{\lambda_i,\lambda_j})$ is skew-symmetric. The $Q$-function $Q_\lambda$ is defined as

$$Q_\lambda(t) = \mathrm{Pf}(M_\lambda).$$

We can now state our main result which we call the *mixed expansion formulas.*

**Theorem 3.1.** *For non-negative integers* $m$ *and* $n$, *we have*

$$\sum_{\mu \in I_1^n(c_m)} \delta(\mu) Q_{\mu[0]}(t) S_{\mu[1]}(t^{(2)}) = S_{\square(2m-n,n)}(t),$$

$$\sum_{\mu \in I_0^n(c_{-m})} \delta(\mu) Q_{\mu[0]}(t) S_{\mu[1]}(t^{(2)}) = S_{\square(n,2m+1-n)}(t),$$

*where* $t = (t_1, t_2, t_3, \cdots)$ *and* $S_\nu(t^{(2)}) = S_\nu(u)|_{u_j \mapsto t_{2j}}$. *If* $I_i^n(c_{\pm m}) = \emptyset$, *we agree that both sides are equal to 0.*

**Example 3.2.** *For $m = 3$ and $n = 2$, we have*

$$S_{1^4}(t^{(2)}) + S_{21^2}(t^{(2)}) + S_{2^2}(t^{(2)}) + Q_{53}(t) - Q_{51}(t)S_1(t^{(2)}) + Q_{31}(t)S_2(t^{(2)}) = S_{\square(4,2)}(t).$$

*For $m = -2$ and $n = 2$, we have*

$$-S_3(t^{(2)}) - S_{21}(t^{(2)}) + S_{1^2}(t^{(2)})Q_2(t) + S_1(t^{(2)})Q_4(t) + Q_{42}(t) = S_{\square(2,3)}(t).$$

## 4. The spin representation of $A_1^{(1)}$

We consider the associative $\mathbb{C}$-algebra $\mathbb{B}$ defined by the generators $\beta_n$ ($n \in \mathbb{Z}$) and the anti-commutation relations:

$$[\beta_m, \beta_n]_+ = \beta_m\beta_n + \beta_n\beta_m = (-1)^m\delta_{m+n,0}.$$

These generators are often called the neutral free fermions. Note that $\beta_0^2 = 1/2$. Let $\mathcal{F}$ be the Fock module which is a left $\mathbb{B}$-module generated by the vacuum $|\emptyset\rangle$ with

$$\beta_n|\emptyset\rangle = 0 \quad (n < 0).$$

Similarly we consider the right $\mathbb{B}$-module $\mathcal{F}^\dagger$ which is generated by the vacuum $\langle\emptyset|$ with

$$\langle\emptyset|\beta_n = 0 \quad (n > 0).$$

Elements of $\mathcal{F}$ and $\mathcal{F}^\dagger$ are sometimes called "states". We have a bilinear pairing

$$\mathcal{F}^\dagger \otimes_{\mathbb{B}} \mathcal{F} \to \mathbb{C}, \quad \langle\emptyset|u \otimes_{\mathbb{B}} v|\emptyset\rangle \mapsto \langle\emptyset|uv|\emptyset\rangle.$$

This pairing is called the *vacuum expectation value.* The vacuum expectation value is uniquely determined by putting $\langle\emptyset|\emptyset\rangle = 1$ and $\langle\emptyset|\beta_0|\emptyset\rangle = 0$.

**Definition 4.1.** *Let $\lambda$ be a strict partition, which we may write in the form $\lambda = (\lambda_1, \dots, \lambda_{2k})$ where $\lambda_1 > \cdots > \lambda_{2k} \geq 0$. Here*

$$2k = \begin{cases} \ell(\lambda) & \ell(\lambda) \equiv 0 \pmod 2, \\ \ell(\lambda) + 1 & \ell(\lambda) \equiv 1 \pmod 2. \end{cases}$$

*Let $|\lambda\rangle$ denote the state*

$$|\lambda\rangle = \beta_{\lambda_1} \cdots \beta_{\lambda_{2k}}|\emptyset\rangle \in \mathcal{F}.$$

*For $\lambda = \emptyset$, we define $|\emptyset\rangle = |\emptyset\rangle$.*

Set $f_i^\infty = (-1)^i\beta_{i+1}\beta_{-i}$ ($i \geq 0$). They have the following combinatorial property whose proof is left to the reader.

**Proposition 4.2.** *Let $\lambda$ be a strict partition. If $i > 0$, then we have*

$$f_i^{\infty}|\lambda\rangle = \begin{cases} |\mu\rangle & \text{if } i \text{ is a part of } \lambda \text{ and } i+1 \text{ is not,} \\ 0 & \text{otherwise,} \end{cases}$$

*where $\mu$ is obtained from $\lambda$ by replacing its part $i$ by $i+1$. If $1$ is not a part of $\lambda$, then we have*

$$f_0^{\infty}|\lambda\rangle = \begin{cases} 2^{-1}|\mu\rangle & 1 \text{ is not a part of } \lambda \text{ and } \ell(\lambda) \equiv 1 \pmod 2, \\ |\mu\rangle & 1 \text{ is not a part of } \lambda \text{ and } \ell(\lambda) \equiv 0 \pmod 2, \\ 0 & \text{otherwise,} \end{cases}$$

*where $\mu$ is obtained from $\lambda$ by adding a part $1$.*

We shall use standard notation of the affine Lie algebra $A_1^{(1)}$ ([5]). Let $e_i, f_i, h_i (i = 0,1)$ be the Chevalley generators, $\alpha_0, \alpha_1$ are the simple roots, $\delta = \alpha_0 + \alpha_1$ is the fundamental imaginary root, $\Lambda_i (i = 0,1)$ are the fundamental weights. The affine Lie algebra $A_1^{(1)}$ acts on $\mathcal{F}$ by

$$f_0 = \sqrt{2}\sum_{n\in\mathbb{Z}} \beta_{-4n+1}\beta_{4n}, \quad f_1 = \sqrt{2}\sum_{n\in\mathbb{Z}} \beta_{-4n-1}\beta_{4n+2},$$
$$e_0 = \sqrt{2}\sum_{n\in\mathbb{Z}} \beta_{4n}\beta_{-4n-1}, \quad e_1 = \sqrt{2}\sum_{n\in\mathbb{Z}} \beta_{4n-2}\beta_{-4n+1},$$
$$h_1 = -h_0 + 1 = 2\sum_{n\in\mathbb{Z}} : \beta_{4n-1}\beta_{-4n+1} :,$$

where we define the normal ordering for the quadratic elements by

$$: \beta_n\beta_m := \beta_n\beta_m - \langle\emptyset|\beta_n\beta_m|\emptyset\rangle.$$

Let $\mathcal{F}_0$ be the $A_1^{(1)}$-submodule of $\mathcal{F}$ generated by $|\emptyset\rangle$ . $\mathcal{F}_0$ is isomorphic to the irreducible highest weight module $L(\Lambda_0)$.

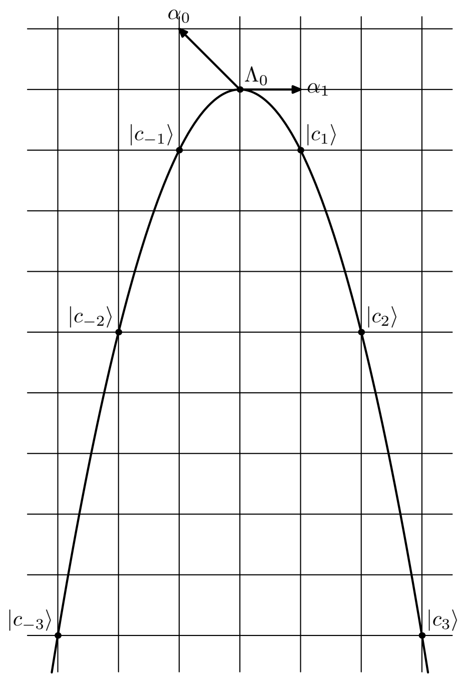
Note the following expressions:

$$f_0 = \sqrt{2}\sum_{j\geq 0} f_{4j}^{\infty} + \sqrt{2}\sum_{j\geq 0} f_{4j+3}^{\infty}, \quad f_1 = \sqrt{2}\sum_{j\geq 0} f_{4j+1}^{\infty} + \sqrt{2}\sum_{j\geq 0} f_{4j+2}^{\infty}.$$

We need the following combinatorial lemmas:

**Lemma 4.3.** [8] *A weight vector of the weight $\Lambda_0 - m^2\delta + m\alpha_1$ is given by $|c_m\rangle$ in $\mathcal{F}_0$.*

The weight diagram of $L(\Lambda_0)$ looks as follows. Maximal weights correspond to the lattice points on the parabola, and other weights are on the lattice points under this parabola.




**Lemma 4.4.**

$$\frac{f_i^\ell}{\ell!}|c_m\rangle = \sqrt{2}^{-\varepsilon_m} \sum_{\lambda \in I_i^\ell(c_m)} \sqrt{2}^{a(\lambda)}|\lambda\rangle.$$

*where* $a(\lambda) = \sharp\{j \mid \lambda_j \equiv 0 \mod 2\}$ *for* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{\ell(\lambda)+\varepsilon_{\ell(\lambda)}})$ *with* $\varepsilon_m = 1$ *if* $m$ *is odd, and* $\varepsilon_m = 0$ *if* $m$ *is even.*

*Proof.* Firstly we consider the case of $i = 1$. For $\lambda \in I_1^\ell(c_m)$ $(m > 0)$, put

$$\lambda - c_m = (r_1, r_2, \cdots, r_m).$$

We compute

$$\begin{aligned} r_1!r_2!\cdots r_m! &= 2^{\sharp\{j; r_j=2\}} \\ &= 2^{(\ell - \sharp\{j; r_j=1\})/2} \\ &= 2^{(\ell - a(\lambda)+\varepsilon_m)/2}, \end{aligned}$$

where we note that $a(\lambda)$ counts a 0 if $m \equiv 1 \pmod 2$. Then the coefficient of $|\lambda\rangle$ is

$$\frac{\sqrt{2}^\ell}{\ell!}\frac{\ell!}{r_1!r_2!\cdots r_m!} = \sqrt{2}^{a(\lambda)-\varepsilon_m}.$$

Secondary we consider the case of $i = 0$. In this case we have to take $\beta_1\beta_0$-part into account. Let

$$\lambda - c_{-m} = (r_1, r_2, \cdots, r_m, r_{m+1})$$

for $\lambda \in I_0^\ell(c_{-m})$ $(m \geq 0)$. Then the coefficient of $|\lambda\rangle$ is

$$\frac{1}{2^{r_{m+1}\varepsilon_m}}\frac{\sqrt{2}^\ell}{\ell!}\frac{\ell!}{r_1!r_2!\cdots r_m!}. \tag{2}$$

A computation similar to the case $i = 1$ above

$$r_1!r_2!\cdots r_m! = 2^{\sharp\{j;r_j=2\}} = 2^{(\ell-\sharp\{j;r_j=1,j\leq m\}-r_{m+1})/2}.$$

Here we divide our argument into two cases. First we assume that $m$ is even. We have

$$\sharp\{j;r_j = 1, j \leq m\} = \begin{cases} a(\lambda) & (\lambda_{m+1} = 0) \\ a(\lambda) - 1 & (\lambda_{m+1} = 1) \end{cases}$$

Secondly we assume that $m$ is odd. We have

$$\sharp\{j;r_j = 1, j \leq m\} = \begin{cases} a(\lambda) - 1 & (\lambda_{m+1} = 0) \\ a(\lambda) & (\lambda_{m+1} = 1) \end{cases}$$

By substituting these four results into (2) we obtain

$$\frac{1}{2^{r_{m+1}\varepsilon_m}}\frac{\sqrt{2}^\ell}{\ell!}\frac{\ell!}{r_1!r_2!\cdots r_m!} = \sqrt{2}^{a(\lambda)-\varepsilon_m}.$$

□

**Example 4.5.** $f_0|c_{-2}\rangle = \sqrt{2}\,\beta_8\beta_3|\emptyset\rangle + \sqrt{2}\,\beta_7\beta_4|\emptyset\rangle + \sqrt{2}\,\beta_7\beta_3\beta_1\beta_0|\emptyset\rangle$.

## 5. Bosonization

In this section we will establish the bozon-fermion correspondence and see the states as the polynomials. In the course, products of Schur's $S$- and $Q$-functions arise naturally.

We introduce the operators $\phi_n, \psi_n, \psi_n^*$ $(n \in \mathbb{Z})$ by

$$\beta_{4n} = \phi_{2n}, \quad \beta_{4n+1} = \sqrt{-1}\,\psi_n, \quad \beta_{4n+2} = \sqrt{-1}\,\phi_{2n+1}, \quad \beta_{4n+3} = \sqrt{-1}\,\psi^*_{-n-1}, \tag{3}$$

which satisfy the anti-commutation relations:

$$\begin{aligned} &[\psi_m, \psi_n^*]_+ = \delta_{m,n}, \quad [\psi_m^*, \psi_n^*]_+ = [\psi_m, \psi_n]_+ = 0, \\ &[\phi_m, \phi_n]_+ = (-1)^m\delta_{m+n,0}, \\ &[\psi_m^*, \phi_n]_+ = [\psi_m, \phi_n]_+ = 0. \end{aligned}$$

Let us introduce the bosonic current operators

$$H_{2m}=\sum_{k\in\mathbb{Z}}:\psi_k\psi^*_{k+m}:,\quad H_{2m+1}=\frac{1}{2}\sum_{k\in\mathbb{Z}}(-1)^{k+1}\phi_k\phi_{-k-(2m+1)}.$$

These operators generate an infinite-dimensional Heisenberg algebra

$$\mathfrak{H}=\oplus_{n\neq 0}\mathbb{C}H_n\oplus\mathbb{C}c,$$

where $c$ denotes the central element of $\mathfrak{H}$. One has

$$[H_m,H_n]=\frac{m}{2}\,\delta_{m+n,0}\,c.$$

We have a canonical $\mathfrak{H}$-module $S[\mathfrak{H}_-]$, where $\mathfrak{H}_-=\oplus_{n<0}\mathbb{C}H_n$, and $S$ stands for the symmetric algebra. Let $t_n=\frac{2}{n}H_{-n}$ $(n>0)$. Then we can identify $S[\mathfrak{H}_-]$ with the ring $\mathbb{C}[\,t\,]=\mathbb{C}[\,t_1,t_2,t_3,\dots]$ of polynomials in infinitely many variables $t_n$. The representation of $\mathfrak{H}$ on $\mathbb{C}[\,t\,]$ is described as follows:

$$H_nP(t)=\frac{\partial}{\partial t_n}P(t),\quad H_{-n}P(t)=\frac{n}{2}\,t_nP(t)\quad(n>0,\ P(t)\in\mathbb{C}[\,t\,]),$$

and $c$ acts as identity.

If we introduce the space of highest weight vectors with respect to $\mathfrak{H}$ by

$$\Omega=\{|v\rangle\in\mathcal{F}\ ;\ H_m|v\rangle=0\ (\forall m>0)\},$$

then $\Omega$ has a basis $\{|\sigma,m\rangle;\ m\in\mathbb{Z},\ \sigma=0,1\}$, where

$$|0,m\rangle=\begin{cases}\psi_{m-1}\cdots\psi_0|\emptyset\rangle & (m>0)\\ |\emptyset\rangle & (m=0)\\ \psi^*_m\cdots\psi^*_{-1}|\emptyset\rangle & (m<0)\end{cases},\quad |1,m\rangle=\begin{cases}\sqrt{2}\phi_0\psi_{m-1}\cdots\psi_0|\emptyset\rangle & (m>0)\\ \sqrt{2}\phi_0|\emptyset\rangle & (m=0)\\ \sqrt{2}\phi_0\psi^*_m\cdots\psi^*_{-1}|\emptyset\rangle & (m<0)\end{cases}.$$

Note that

$$\phi_n|\sigma,m\rangle=0\quad(n<0),\quad \psi_n|\sigma,m\rangle=0\quad(n<m),\quad \psi^*_n|\sigma,m\rangle=0\quad(n\geq m).$$

**Lemma 5.1.**

$$|c_m\rangle=(\sqrt{-1})^{-|m|}\,\sqrt{2}^{-\varepsilon_m}|\,\varepsilon_m,m\rangle.$$

*Proof.* We can easily obtain the equation by direct calculation. □

We introduce formal symbols $\theta$ and $e^{m\alpha}$ which satisfies $\theta^2=1$ and define

$$\mathbf{\Omega}=\bigoplus_{m\in\mathbb{Z},\,\sigma=0,1}\mathbb{C}\,\theta^\sigma e^{m\alpha}.$$

Then $H_n$ act on $\mathbb{C}[\,t\,]\otimes\mathbf{\Omega}$ by $H_n\otimes id$.

**Proposition 5.2.** [2, 3] *There exists a canonical isomorphism of $\mathfrak{H}$-modules*

$$\Phi : \mathcal{F} \longrightarrow \mathbb{C}[\,t\,] \otimes_{\mathbb{C}} \mathbf{\Omega}$$

*such that* $\Phi(|\sigma, m\rangle) = \theta^{\sigma} e^{m\alpha} (m \in \mathbb{Z}, \sigma = 0, 1)$.

We will see $\mathbb{C}[\,t\,] \otimes \mathbf{\Omega}$ as an $A_1^{(1)}$-module via $\Phi$ (cf. Proposition 6.1).

When we write $\Phi(|v\rangle) = \sum_{m,\sigma} P_{m,\sigma}(t)\, \theta^{\sigma} e^{m\alpha}$ for $|v\rangle \in \mathcal{F}$, the coefficient $P_{m,\sigma}(t) \in \mathbb{C}[t]$ can be expressed in terms of the vacuum expectation value on $\mathbb{B}$ as follows:

$$P_{m,\sigma}(t) = \langle m, \sigma | e^{H(t)} | v \rangle, \quad H(t) = \sum_{n=1}^{\infty} t_n H_n.$$

Introduce the states $\langle m, \sigma | \in \mathcal{F}^{\dagger}$ $(m \in \mathbb{Z}, \sigma = 0, 1)$ which are characterized by $\langle m, \sigma | \sigma', n \rangle = \delta_{m,n} \delta_{\sigma,\sigma'}$ $(m, n \in \mathbb{Z}, \sigma, \sigma' = 0, 1)$ and

$$\langle m, \sigma | \phi_n = 0 \quad (n > 0), \quad \langle m, \sigma | \psi_n = 0 \quad (n \geq m), \quad \langle m, \sigma | \psi_n^* = 0 \quad (n < m).$$

We denote by $\mathbb{W}_{\phi}$ the linear subspace of $\mathbb{B}$ spanned by $\phi_n$ $(n \in \mathbb{Z})$.

**Lemma 5.3.** *If* $\langle u | \in \mathcal{F}^{\dagger}, |v\rangle \in \mathcal{F}$ *be such that* $\langle u | \phi_n = 0$ $(n > 0)$, $\phi_n | v \rangle = 0$ $(n < 0)$, *then for* $w_i \in \mathbb{W}_{\phi}$ $(i = 1, \ldots, 2k)$ *we have*

$$\langle u | w_1 \cdots w_{2k} | v \rangle = \langle u | v \rangle \, \mathrm{Pf}(\langle \emptyset | w_i w_j | \emptyset \rangle)$$

*Proof.* A bilinear form on $\mathbb{W}_{\phi}$ is defined by $(a, b) \mapsto \langle u | ab | v \rangle$, which has all the properties of vacuum expectation value on $\mathbb{W}_{\phi}$ except for the normalization condition. Obviously, the normalization factor is given by $\langle u | v \rangle$. Hence the lemma follows. $\square$

**Lemma 5.4.** [2, 3] *We have*

$$\Phi(\psi_{i_1} \cdots \psi_{i_s} | 0, m \rangle) = S_{(i_1 - m, i_2 - m, \ldots, i_s - m) - \delta_s}(t^{(2)})\, e^{(m+s)\alpha} \quad (i_1 > \cdots > i_s > m),$$
$$\Phi(\phi_{j_1} \cdots \phi_{j_a} | \emptyset \rangle) = \sqrt{2}^{-a} Q_{j_1, \ldots, j_a}(t)\, \theta^a \quad (j_1 > \cdots > j_a \geq 0),$$

*where* $\delta_s = (s-1, s-2, \ldots, 1, 0)$ *and* $t^{(2)} = (t_2, t_4, \ldots)$.

Lemma 5.3 and 5.4 give us

**Lemma 5.5.** *Let* $j_1 > \cdots > j_a \geq 0$, $i_1 > \cdots > i_s > m$. *We have*

$$\Phi(\phi_{j_1} \cdots \phi_{j_a} \psi_{i_1} \cdots \psi_{i_s} | 0, m \rangle) = \sqrt{2}^{-a}\, Q_{j_1, \ldots, j_a}(t)\, S_{(i_1 - m, \ldots, i_s - m) - \delta_s}(t^{(2)}) \theta^a e^{(m+s)\alpha}.$$

Consequently we obtain the following proposition.

**Proposition 5.6.** *Let* $\lambda \in I_i^{\ell}(c_m)$. *There exists a* 4*-th root of unity* $\zeta_{m,\ell,i}(\lambda)$ *such that*

$$\Phi(\sqrt{2}^{a(\lambda)} | \lambda \rangle) = \zeta_{m,\ell,i}(\lambda)\ \ Q_{\lambda[0]}(t) S_{\lambda[1]}(t^{(2)})\, \theta^{m+\ell}\, e^{(m+(-1)^i \ell)\alpha}.$$

## 6. Vertex operators

In this section we realize $f_i$ on $\mathbb{B}$ in terms of vertex operators. We introduce the formal generating functions

$$\phi(z)=\sum_{n\in\mathbb{Z}}\phi_n z^n,\quad \psi(z)=\sum_{n\in\mathbb{Z}}\psi_n z^{2n},\quad \psi^*(z)=\sum_{n\in\mathbb{Z}}\psi^*_{-n}z^{2n-2}.$$

For $t=(t_1,t_2,t_3,\ldots)$, set

$$\xi(t,z)=\sum_{n=1}^{\infty}t_n z^n,\quad \xi_0(t,z)=\sum_{n=1}^{\infty}t_{2n}z^{2n},\quad \xi_1(t,z)=\sum_{n=1}^{\infty}t_{2n-1}z^{2n-1}.$$

On the space $\mathbf{\Omega}$, we define the operators $\theta, e^{\pm\alpha}$ and $z^{H_0}$by

$$\theta.(\theta\, e^{m\alpha})=e^{m\alpha},\qquad\qquad \theta.\, e^{m\alpha}=\theta\, e^{m\alpha},$$
$$e^{\pm\alpha}.(\theta\, e^{m\alpha})=-\theta\, e^{(m\pm1)\alpha},\qquad\qquad e^{\pm\alpha}.e^{m\alpha}=e^{(m\pm1)\alpha},$$

and

$$z^{H_0}.(\theta\, e^{m\alpha})=z^m(\theta\, e^{m\alpha})\quad(\sigma=0,1).$$

**Proposition 6.1.** [2, 3] *One has*

$$\Phi\ \phi(z)\Phi^{-1}=\sqrt{2}^{-1}e^{\xi_1(t,z)}e^{-2\xi_1(\widetilde{\partial}_t,z^{-1})}\theta,$$
$$\Phi\ \psi(z)\Phi^{-1}=e^{\xi_0(t,z)}e^{-2\xi_0(\widetilde{\partial}_t,z^{-1})}e^{\alpha}z^{2H_0},$$
$$\Phi\ \psi^*(z)\Phi^{-1}=e^{-\xi_0(t,z)}e^{2\xi_0(\widetilde{\partial}_t,z^{-1})}e^{-\alpha}z^{-2H_0},$$

*where* $\tilde{\partial}_t=(\frac{\partial}{\partial t_1},\frac{1}{2}\frac{\partial}{\partial t_2},\frac{1}{3}\frac{\partial}{\partial t_3},\cdots)$.

**Lemma 6.2.** [2, 3] *Let* $V_1(z)=\sqrt{2}\ \Phi\,\phi(-z)\psi^*(z)\,\Phi^{-1}$, $V_0(z)=\sqrt{2}\ \Phi\,\phi(z)\psi(z)\,\Phi^{-1}$. *Then we have*

$$V_1(z)=e^{-\xi(t,z)}e^{2\,\xi(\widetilde{\partial}_t,z^{-1})}\theta e^{-\alpha}z^{-2H_0},\quad V_0(z)=e^{\xi(t,z)}e^{-2\,\xi(\widetilde{\partial}_t,z^{-1})}\theta e^{\alpha}z^{2H_0}.$$

Due to Lemma 6.2, we can write the actions of $f_i$ on $\mathbb{C}[t]\otimes_{\mathbb{C}}\mathbf{\Omega}$ in terms of formal contour integrals

$$f_0=\sqrt{-1}^{-1}\oint z^{-1}V_0(z)dz,\quad f_1=-\oint V_1(z)dz,$$

where we set $\oint A(z)dz=A_{-1}$ for $A(z)=\sum_n A_n z^n$.

**Lemma 6.3.**

$$V_i(z_\ell)\cdots V_i(z_2)V_i(z_1)=(-1)^{\frac{\ell(\ell-1)}{2}}\Delta(z)^2 e^{(-1)^i\sum_j\xi(t,z_j)}e^{2(-1)^{i+1}\sum_j\xi(\widetilde{\partial}_t,z_j^{-1})}\theta^\ell e^{(-1)^i\ell\alpha}(z_1\cdots z_\ell)^{2(-1)^iH_0}.$$

*Here* $\Delta(z)=\det(z_i^{j-1})_{1\le i,j\le\ell}$.

*Proof.* By $V_i^0(z)$, we denote the "zero mode" $\theta e^{(-1)^i\alpha}z^{2(-1)^iH_0}$ of $V_i(z)$. Then by using the relations $\theta e^{\pm\alpha} = -e^{\pm\alpha}\theta$, and $z_j^{\pm 2H_0}e^{\pm\alpha} = z_j^2 \cdot e^{\pm\alpha}z_j^{\pm 2H_0}$, we have

$$V_i^0(z_\ell)\cdots V_i^0(z_2)V_i^0(z_1) = (-1)^{\frac{\ell(\ell-1)}{2}}\left(\prod_{j=1}^{\ell} z_j^{2j-2}\right)\theta^\ell e^{(-1)^i\ell\alpha}(z_1\cdots z_\ell)^{2(-1)^iH_0}.$$

On the other hand, by the standard calculus of vertex operators, we have

$$V_i^+(z_2)V_i^-(z_1) = \left(1-\frac{z_1}{z_2}\right)^2 V_i^-(z_1)V_i^+(z_2),$$

where we set $V_i^-(z) = e^{(-1)^i\xi(t,z)}$, $V_i^+(z) = e^{(-1)^{i+1}2\xi(\widetilde{\partial}_t,z^{-1})}$. Then the lemma follows immediately. $\square$

For $\lambda \in \mathcal{P}$, we denote by

$$\mathrm{S}^\lambda(z) = \det(z_i^{\lambda_j+j-1})/\det(z_i^{j-1})$$

the Schur function with respect to $z = (z_1,\ldots,z_\ell)$. We use the well-known orthogonality relation

$$\frac{1}{(2\pi\sqrt{-1})^\ell}\int_{T^\ell}\mathrm{S}^\lambda(z)\overline{\mathrm{S}^\mu(z)}\,|\Delta(z)|^2\frac{dz_1}{z_1}\cdots\frac{dz_\ell}{z_\ell} = \ell!\,\delta_{\lambda,\mu},$$

where we denote by $T^\ell = \{z = (z_j) \in \mathbb{C}^\ell\ ;\ |z_j| = 1\}$, the $\ell$-dimensional torus. Since $\overline{\mathrm{S}^\mu(z)} = \mathrm{S}^\mu(z^{-1}) = \mathrm{S}^\mu(z_1^{-1},\ldots,z_\ell^{-1})$ for $z \in T^\ell$, we can rewrite this relation as

$$\oint\cdots\oint \mathrm{S}^\lambda(z)\mathrm{S}^\mu(z^{-1})(-1)^{\frac{\ell(\ell-1)}{2}}\Delta(z)^2(z_1\cdots z_\ell)^{-\ell}\,dz_1\cdots dz_\ell = \ell!\,\delta_{\lambda,\mu}. \tag{4}$$

We also utilize the following form of the Cauchy identity:

$$e^{\sum_{j=1}^{\ell}\xi(t,z_j)} = \sum_{\ell(\lambda)\le\ell}\mathrm{S}^\lambda(z)S_\lambda(t). \tag{5}$$

We remark

$$e^{-\sum_{j=1}^{\ell}\xi(t,z_j)} = \sum_{\ell(\lambda)\le\ell}(-1)^{|\lambda|}\mathrm{S}^\lambda(z)S_{\lambda'}(t).$$

Here $\lambda'$ is the conjugate of $\lambda$.

Put

$$f_i^{(\ell)} = \frac{1}{\ell!}f_i^\ell\ (i = 0,1).$$

**Lemma 6.4.** *For $m > 0$, we have*

$$f_0^{(\ell)}\theta^m e^{-m\alpha} = \zeta'_{-m,\ell,1}S_{\square(\ell,2m+1-\ell)}(t)\,\theta^{m+\ell}e^{(\ell-m)\alpha},$$

*where*

$$\zeta'_{-m,\ell,0} = \sqrt{-1}^{(2m-1)\ell}\ (1\le\ell\le 2m-1).$$

*Similarly we have*

$$f_1^{(\ell)}\theta^m e^{m\alpha} = \zeta'_{m,\ell,1} S_{\square(2m-\ell,\ell)}(t)\,\theta^{m+\ell}e^{(m-\ell)\alpha},$$

*where*

$$\zeta'_{m,\ell,1} = (-1)^{m\ell}.$$

*Proof.* In view of the relation $e^{\ell\alpha}\cdot\theta^m = (-1)^{\ell m}\theta^m\cdot e^{\ell\alpha}$, we have by Lemma 6.3

$$V_0(z_\ell)\cdots V_0(z_1)\,\theta^m e^{-m\alpha} = (-1)^{\ell m}(-1)^{\frac{\ell(\ell-1)}{2}}\Delta(z)^2 e^{\sum_j \xi(t,z_j)}\theta^{m+\ell}e^{(-m+\ell)\alpha}(z_1\cdots z_\ell)^{-2m}.$$

Using this, we have

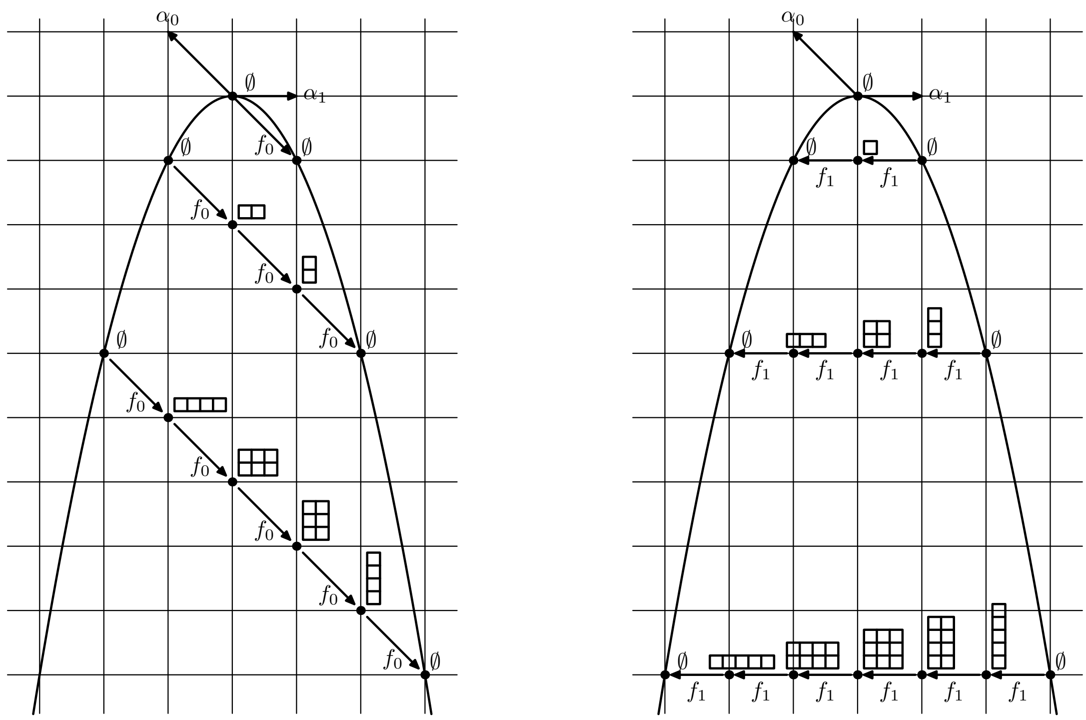
$$\begin{aligned}
&f_0^{(\ell)}\theta^m e^{-m\alpha}\\
&= \frac{\sqrt{-1}^{-\ell}}{\ell!}\oint\cdots\oint(-1)^{\frac{\ell(\ell-1)}{2}}\Delta(z)^2(z_1\cdots z_\ell)^{-2m-1}e^{\sum_j\xi(t,z_j)}dz_1\cdots dz_\ell\cdot(-1)^{\ell m}\theta^{m+\ell}e^{(-m+\ell)\alpha}\\
&= \sqrt{-1}^{-\ell}(-1)^{\ell m}S_{\square(\ell,2m+1-\ell)}(t)\,\theta^{m+\ell}e^{(-m+\ell)\alpha}
\end{aligned}$$

where we carried out the contour integral by using (4) and (5). Here we remark that

$$\mathrm{S}^{\square(\ell,m)}(z) = (z_1z_2\cdots z_\ell)^m.$$

In a similar way, we have $\zeta'_{m,\ell,1} = (-1)^{m\ell}$. We just note that $S_{\square(2m-\ell,\ell)}(-t) = (-1)^{\ell(2m-\ell)}S_{\square(\ell,2m-\ell)}(t)$. Detail of the calculation is left to the reader. $\square$

The following pictures express the $f_0$- and $f_1$-action to each maximal weight.



## 7. Proof of the main theorem

First we have

$$\begin{aligned}\Phi\left(f_i^{(\ell)}|c_m\rangle\right) &= \sqrt{2}^{-\varepsilon_m} \sum_{\lambda\in I_i^\ell(c_m)} \Phi\left(\sqrt{2}^{a(\lambda)}|\lambda\rangle\right)\\ &= \sqrt{2}^{-\varepsilon_m} \sum_{\lambda\in I_i^\ell(c_m)} \zeta_{m,\ell,i}(\lambda)\, Q_{\lambda[0]}(t)S_{\lambda[1]}(t^{(2)})\,\theta^{m+\ell}\, e^{(m+(-1)^i\ell)\alpha}.\end{aligned}$$

Second we have seen in the previous section that

$$f_1^{(\ell)}\Phi(|c_m\rangle) = \sqrt{2}^{-\varepsilon_m}\sqrt{-1}^{-m}\zeta'_{m,\ell,1}S_{\square(2m-\ell,\ell)}(t)\theta^{m+\ell}e^{(m-\ell)\alpha}$$

and

$$f_0^{(\ell)}\Phi(|c_{-m}\rangle) = \sqrt{2}^{-\varepsilon_m}\sqrt{-1}^{-m}\zeta'_{-m,\ell,0}S_{\square(\ell,2m-\ell+1)}(t)\theta^{m+\ell}e^{(\ell-m)\alpha},$$

for $m>0$. Therefore all we have to show is the following:

**Lemma 7.1.** *For $\lambda\in I_i^\ell(c_m)$, we have*

$$\zeta_{m,\ell,i}(\lambda) = \sqrt{-1}^{-|m|}\zeta'_{m,\ell,i}\delta(\lambda).$$

We prove this lemma together with looking at some examples for help. We set

$$|\lambda\rangle = \beta_{\lambda_1}\cdots\beta_{\lambda_{2s}}|\emptyset\rangle,$$

where $\lambda_1>\cdots>\lambda_{2s}\geq 0$. If we ignore the factor $\sqrt{-1}$ in (3), the set $\{\beta_{\lambda_1},\ldots,\beta_{\lambda_{2s}}\}$ is decomposed into the three parts

$$\mathcal{I}=\{\psi_{i_1},\ldots,\psi_{i_N}\},\quad \mathcal{J}=\{\psi^*_{j_1},\ldots,\psi^*_{j_{N^*}}\},\quad \mathcal{K}=\{\phi_{k_1},\ldots,\phi_{k_a}\},$$

where $a$, $N$ and $N^*$ the number of $\phi$'s, $\psi$'s and $\psi^*$'s respectively and $i_1>\cdots>i_N\geq 0>j_1>\cdots>j_{N^*}$, $k_1>\cdots>k_a\geq 0$. Actually, $I=\{i_1,\ldots,i_N\}$ ( resp. $J=\{j_1,\ldots,j_{N^*}\}$) is nothing but $\iota(\boldsymbol{\lambda}^{(1)})$ (resp. $\iota^*(\boldsymbol{\lambda}^{(3)})$), and $K=\{k_1,\ldots,k_a\}$ corresponds to $\boldsymbol{\lambda}^{(0)}\cup\boldsymbol{\lambda}^{(2)}$. According to the following operations, we shall rewrite $|\lambda\rangle$ into its "normal form" such as in Lemma 5.5.

{OP. 0} Rewrite $\beta$'s into $\phi$'s, $\psi$'s and $\psi^*$'s according to (3).

{OP. 1} Rewrite the vacuum $|\emptyset\rangle$ into $\psi_{-1}\psi_{-2}\cdots\psi_{j_{N^*}}|0,j_{N^*}\rangle$, i.e.,

$$|\emptyset\rangle = \psi_{-1}\psi_{-2}\cdots\psi_{j_{N^*}}|0,j_{N^*}\rangle.$$

{OP. 2} Repeat the following operations in order of $m=j_1,j_2,\cdots,j_{N^*}$:

Move $\psi^*_m$ to the left side of $\psi_m$ and remove $\psi^*_m$ by using the relation

$$\psi^*_m\psi_m = 1-\psi_m\psi^*_m.$$

{OP. 3} Move $\phi_j$'s to the left of $\psi$'s in order of $j=k_1,k_2\cdots,k_a$.

We divide our argument into four cases.

Case 1: We consider the case of $i = 1$ and $m = 2n > 0$. We have

$$\zeta'_{m,\ell,1} = (-1)^{m\ell} = 1.$$

Our purpose is to rewrite $|\lambda\rangle$ into its normal form and compute a factor $\zeta_{m,\ell,i}(\lambda)$. We employ the following example for our understanding;

$$m = 6, \ell = 6 \text{ and } \mu = (21, 19, 13, 10, 7, 2) \in I_0^6(c_6).$$

In this case, since all elements of $K$ are odd, {OP. 0} gives a factor $\sqrt{-1}^m$. In our example, we have

$$\beta_{21}\beta_{19}\beta_{13}\beta_{10}\beta_7\beta_2|\emptyset\rangle \overset{\{\text{OP.0}\}}{=} \sqrt{-1}^6 \psi_5\psi^*_{-5}\psi_3\phi_3\psi^*_{-2}\phi_1|\emptyset\rangle.$$

We neglect the factor $\sqrt{-1}^m$ for the moment. After rewriting the vacuum according to {OP. 1}, we move $\psi^*_{j_1}$ to the left side of $\psi_{j_1}$. Then $\psi^*_{j_1}$ jumps $-j_1 - 1$ elements of $\mathcal{I} \cup \mathcal{K}$ and $\psi_{-1}, \cdots, \psi_{j_1+1}$. Therefore this operation gives a factor

$$(-1)^{(-j_1-1)+(-j_1-1)} = 1.$$

We apply this operation to $\psi^*_{j_m}$ in oder of $m = 2, 3, \cdots, N^*$. Then we have a factor

$$(-1)^{(-j_m-m)+(-j_m-m)} = 1$$

for each $m$. Therefore {OP. 2} gives a factor

$$(-1)^{\sum_{m=1}^{N^*} 2(-j_m-1)} = 1.$$

In our example, we have

$$\begin{aligned}
\sqrt{-1}^6 \psi_5\psi^*_{-5}\psi_3\phi_3\psi^*_{-2}\phi_1|\emptyset\rangle &\overset{\{\text{OP.1}\}}{=} \sqrt{-1}^6 \psi_5\psi^*_{-5}\psi_3\phi_3\underline{\underline{\psi^*_{-2}}}\phi_1\psi_{-1}\psi_{-2}\psi_{-3}\psi_{-4}\psi_{-5}|0,-5\rangle \\
&\overset{\{\text{OP.2}\}}{=} (-1)^{1+1}\sqrt{-1}^6 \psi_5\underline{\underline{\psi^*_{-5}}}\psi_3\phi_3\phi_1\psi_{-1}\psi_{-3}\psi_{-4}\psi_{-5}|0,-5\rangle \\
&\overset{\{\text{OP.2}\}}{=} (-1)^{1+1}(-1)^{3+3}\sqrt{-1}^6 \psi_5\psi_3\phi_3\phi_1\psi_{-1}\psi_{-3}\psi_{-4}|0,-5\rangle \\
&= \sqrt{-1}^6 \psi_5\psi_3\phi_3\phi_1\psi_{-1}\psi_{-3}\psi_{-4}|0,-5\rangle.
\end{aligned}$$

From the Definition 2.3, the factor occurred by {OP. 3} is $\delta(\lambda)$. We see this fact through our example. We compute

$$\begin{aligned}
\sqrt{-1}^6 \psi_5\psi_3\underline{\underline{\phi_3}}\phi_1\psi_{-1}\psi_{-3}\psi_{-4}|0,-5\rangle &\overset{\{\text{OP.3}\}}{=} \sqrt{-1}^6 (-1)^2 \phi_3\psi_5\psi_3\underline{\underline{\phi_1}}\psi_{-1}\psi_{-3}\psi_{-4}|0,-5\rangle \\
&\overset{\{\text{OP.3}\}}{=} \sqrt{-1}^6 (-1)^{2+2} \phi_3\phi_1\psi_5\psi_3\psi_{-1}\psi_{-3}\psi_{-4}|0,-5\rangle
\end{aligned}$$

and obtain $\delta(\mu) = (-1)^{2+2}$ from 4-bar abacus of $\mu$;

$$\begin{array}{ccc}
0 & 1 & 3 \\
\textcircled{2} & & \\
4 & 5 & \textcircled{7} \\
6 & & \\
8 & 9 & 11 \\
\textcircled{10} & & \\
12 & \textcircled{13} & 15 \\
14 & & \\
18 & 17 & \textcircled{19} \\
20 & & \\
22 & \textcircled{21} & 23
\end{array}.$$

Now we obtain

$$|\lambda\rangle = \delta(\lambda)\sqrt{-1}^{\,m}\phi_{k_1}\cdots\phi_{k_a}\psi_{i_1}\cdots\psi_{i_N}\psi_{-1}\psi_{-2}\cdots\widehat{\psi_{j_1}}\cdots\widehat{\psi_{j_2}}\cdots\widehat{\psi_{j_{N^*}}}|0, j_{N^*}\rangle.$$

and $\zeta_{m,\ell,1}(\lambda) = \delta(\lambda)\sqrt{-1}^{m}$. Since $m$ is even, we have $\zeta_{m,\ell,1}(\lambda) = \delta(\lambda)\sqrt{-1}^{\,-m}\zeta'_{m,\ell,1}$,

Case 2: We consider the case of $i = 1$ and $m = 2n+1 > 0$. Then we have

$$\zeta'_{m,\ell,1} = (-1)^{m\ell} = (-1)^{\ell}.$$

The only difference from the case 1 is the existence of $\phi_0 = \beta_0$ in the right end of $\beta$'s. The element $\phi_0$ causes a factor $(-1)^{N+N^*}$, because $\phi_0$ is jumped by the elements of $\mathcal{J}$ ({OP. 2}) and jump the elements of $\mathcal{I}$ ({OP. 3}). For example, if $\mu = (19, 13, 10, 7, 2) \in I_1^6(c_5)$, then we have

$$\begin{aligned}
|\mu\rangle &= \beta_{19}\beta_{13}\beta_{10}\beta_7\beta_2\beta_0|\emptyset\rangle \\
&\overset{\{\text{OP.}0\}}{=} \sqrt{-1}^{\,5}\psi^*_{-5}\psi_3\phi_3\psi^*_{-2}\phi_1\phi_0|\emptyset\rangle \\
&\overset{\{\text{OP.}1\}}{=} \sqrt{-1}^{\,5}\psi^*_{-5}\psi_3\phi_3\psi^*_{-2}\phi_1\phi_0\psi_{-1}\psi_{-2}\psi_{-3}\psi_{-4}\psi_{-5}|0, -5\rangle.
\end{aligned}$$

In this example $\phi_0$ causes a factor $(-1)^{2+3}$. Therefore we have

$$|\lambda\rangle = \delta(\lambda)\sqrt{-1}^{\,m}(-1)^{N+N^*}\phi_{k_1}\cdots\phi_{k_a}\psi_{i_1}\cdots\psi_{i_N}\psi_{-1}\psi_{-2}\cdots\widehat{\psi_{j_1}}\cdots\widehat{\psi_{j_2}}\cdots\widehat{\psi_{j_{N^*}}}|0, j_{N^*}\rangle.$$

By using the relations

$$\begin{cases} m = N + N^* + a \\ \ell = a + 2N^*, \end{cases}$$

we have $(-1)^{N+N^*}$ is equal to $(-1)^{m-\ell}$. Thus we have $\zeta_{m,\ell,1}(\lambda) = \delta(\lambda)\sqrt{-1}^{\,m}(-1)^{m-\ell}$ and $\zeta_{m,\ell,i}(\lambda) = \delta(\lambda)\sqrt{-1}^{\,-|m|}\zeta'_{m,\ell,i}$, since $m$ is odd.

The following case 3 and case 4 are for $i = 0$. In these cases, we should put a bead on 0 of the 4-bar abacus when $\lambda_{m+1} = 0$ $(m > 0)$. If we take care of this point, we can rewrite $|\lambda\rangle$ into its normal form and determine $\zeta_{m,\ell,0}(\lambda)$ as same as the case 1 and case 2.

Case 3: We consider the case of $i = 0$ and $m = 2n + 1 > 0$. Then we have

$$\zeta'_{-m,\ell,0} = \sqrt{-1}^{(2m-1)\ell} = \sqrt{-1}^{\ell}.$$

Take an element $\lambda \in I_0^\ell(c_{-m})$. Remark that $\lambda_{m+1} = 0$ or 1. We further divide this case into subcases:

(a) $\lambda_{m+1} = 0$, i.e., $|c_{-m}\rangle$ has the end term $\beta_0 = \phi_0$,
(b) $\lambda_{m+1} = 1$, i.e., $|c_{-m}\rangle$ does not have the end term $\beta_0 = \phi_0$.

Let us first consider the subcase (a). We have

$$|\lambda\rangle = \delta(\lambda)\sqrt{-1}^{(m+1)-a}(-1)^{N^*}\phi_{k_1}\cdots\phi_{k_a}\psi_{i_1}\cdots\psi_{i_N}\psi_{-1}\psi_{-2}\cdots\widehat{\psi_{j_1}}\cdots\widehat{\psi_{j_2}}\cdots\widehat{\psi_{j_{N^*}}}|0, j_{N^*}\rangle$$

and

$$\zeta_{-m,\ell,0}(\lambda) = \delta(\lambda)\sqrt{-1}^{m+1-a}(-1)^{N^*}.$$

Here we remark the factor $(-1)^{N^*}$ caused by $\phi_0$ ($\{$OP. 2$\}$) and the factor $(-1)^N$ caused by $\phi_0$ is included by $\delta(\lambda)$ ($\{$OP. 3$\}$). For example, set $\mu = (12, 9, 3, 0) \in I_0^4(c_{-3})$. Then we compute

$$\begin{aligned}
|\mu\rangle &= \beta_{12}\beta_9\beta_3\beta_0|\emptyset\rangle \\
&\overset{\{\text{OP.0}\}}{=} \sqrt{-1}^2\phi_6\psi_2\psi^*_{-1}\phi_0|\emptyset\rangle \\
&\overset{\{\text{OP.1}\}}{=} \sqrt{-1}^2\phi_6\psi_2\psi^*_{-1}\phi_0\psi_{-1}|0,-1\rangle \\
&\overset{\{\text{OP.2}\}}{=} \sqrt{-1}^2(-1)^1\phi_6\psi_2\phi_0|0,-1\rangle \\
&\overset{\{\text{OP.3}\}}{=} \sqrt{-1}^2(-1)^1(-1)^1\phi_6\phi_0\psi_2|0,-1\rangle
\end{aligned}$$

and $\delta(\mu) = (-1)^{1+0}$ from the 4-bar abacus of $\mu$

$$\begin{array}{ccc}
\textcircled{0} & 1 & \textcircled{3} \\
2 & & \\
4 & 5 & 7 \\
6 & & \\
8 & \textcircled{9} & 11 \\
10 & & \\
\textcircled{12} & 13 & 15
\end{array}.$$

Now we have the relations

$$\begin{cases} N - N^* = -m + \ell \\ a + N + N^* = m + 1. \end{cases}$$

We eliminate $N$ from these equations to get $a + 2N^* = 2m - \ell + 1$. Then we have

$$\zeta_{-m,\ell,0}(\lambda) = \delta(\lambda)\sqrt{-1}^{-m+\ell},$$

which is equal to $\delta(\lambda)\sqrt{-1}^{-m}\zeta'_{-m,\ell,0}$. For subcase (b), we have $\psi_0$ instead of the absence of $\phi_0$. So the same factor $(-1)^{N^*}$ occurs when we exchange $\psi_0$ and $\psi^*$'s. Then the formula is the same as (a). The readers can check this fact by using an example $\mu = (12, 9, 3, 1) \in I_0^4(c_{-3})$.

<u>Case 4</u>: We consider the case of $i = 0$ and $m = 2n > 0$. Then we have

$$\zeta'_{-m,\ell,0} = \sqrt{-1}^{(2m-1)\ell} = \sqrt{-1}^{-\ell}.$$

Consider $\lambda \in I_0^\ell(c_{-m})$. We further divide this case into the subcases:

(c) $\lambda_{m+1} = 1$ i.e., $|\lambda\rangle$ ends with $\psi_0\phi_0$

(d) $\lambda_{m+1} = 0$ i.e., $|\lambda\rangle$ does not contain $\phi_0$ nor $\psi_0$.

In the case (c), we remark, by the similar argument of the case 1, {OP. 2} does not cause any sign change and {OP. 3} causes $\delta(\lambda)(-1)^N$. We have

$$|\lambda\rangle = \sqrt{-1}^{(m+2)-a}\delta(\lambda)(-1)^N\phi_{k_1}\cdots\phi_{k_a}\psi_{i_1}\cdots\psi_{i_N}\psi_{-1}\psi_{-2}\cdots\widehat{\psi_{j_1}}\cdots\widehat{\psi_{j_2}}\cdots\widehat{\psi_{j_{N^*}}}|0, j_{N^*}\rangle$$

and

$$\zeta_{-m,\ell,0}(\lambda) = \sqrt{-1}^{m+2-a}\delta(\lambda)(-1)^N.$$

For example, if $\mu = (15, 13, 9, 4, 1) \in I_0^6(c_{-4})$, then we compute

$$\begin{aligned}
|\mu\rangle &= \beta_{15}\beta_{13}\beta_9\beta_4\beta_1\beta_0|\emptyset\rangle \\
&\overset{\{\text{OP.0,1}\}}{=} \sqrt{-1}^4\underline{\underline{\psi^*_{-4}}}\psi_3\psi_2\phi_2\psi_0\phi_0\psi_{-1}\psi_{-2}\psi_{-3}\psi_{-4}|0,-4\rangle \\
&\overset{\{\text{OP.2}\}}{=} \sqrt{-1}^4(-1)^{3+2+3}\psi_3\psi_2\underline{\underline{\phi_2}}\psi_0\phi_0\psi_{-1}\psi_{-2}\psi_{-3}|0,-4\rangle \\
&\overset{\{\text{OP.3}\}}{=} \sqrt{-1}^4(-1)^2\phi_2\psi_3\psi_2\psi_0\underline{\underline{\phi_0}}\psi_{-1}\psi_{-2}\psi_{-3}|0,-4\rangle \\
&\overset{\{\text{OP.3}\}}{=} \sqrt{-1}^4(-1)^2(-1)^3\phi_2\phi_0\psi_3\psi_2\psi_0\psi_{-1}\psi_{-2}\psi_{-3}|0,-4\rangle
\end{aligned}$$

and $\delta(\mu) = (-1)^2$ from the 4-bar abacus:

$$\begin{array}{ccc} 0 & \textcircled{1} & 3 \\ 2 & & \\ \textcircled{4} & 5 & 7 \\ 6 & & \\ 8 & \textcircled{9} & 11 \\ 10 & & \\ 12 & \textcircled{13} & \textcircled{15} \end{array}.$$

Since we have

$$\begin{cases} a + 2N = \ell + 2 \\ m \text{ is odd,} \end{cases}$$

we can see $\zeta_{-m,\ell,0} = \delta(\lambda)\sqrt{-1}^{-m}\zeta'_{-m,\ell,0}$. The subcase (d) is the most cumbersome one. By the definition, we include the sign $(-1)^N$ in $\delta(\lambda)$ arising from the exchanges of the dummy "$\phi_0$" and $\psi$'s. So we have to compensate the same factor to get

$$\zeta_{-m,\ell,0}(\lambda) = \sqrt{-1}^{m-a}\delta(\lambda)(-1)^N.$$

The readers can check this fact by using an example $\mu = (15, 13, 9, 4) \in I_0^5(c_{-4})$. Now using $a + 2N = \ell$, we have $\zeta_{-m,\ell,0}(\lambda) = \delta(\lambda)\sqrt{-1}^{m-\ell}$. Since $m$ is even, we have $\sqrt{-1}^{m-\ell} = \sqrt{-1}^{-m-\ell} = \sqrt{-1}^{-m}\zeta'_{m,\ell,0}$.

## References


[1] C. Bessenrodt and J. B. Olsson, The 2-blocks of the covering groups of the symmetric groups. Adv. Math. 129, no. 2 (1997), 261–300.

[2] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Transformation groups for soliton equations. IV. A new hierarchy of soliton equations of KP-type. Phys. D 4, no. 3 (1981/82), 343–365.

[3] E. Date, M. Kashiwara, M. Jimbo and T. Miwa, Transformation groups for soliton equations, in Nonlinear Integrable Systems—Classical Theory and Quantum Theory, ed. by M. Jimbo and T. Miwa, 39-119, World Sci. Publishing, Singapore, 1983.

[4] T. Ikeda and H.-F. Yamada, Polynomial $\tau$-functions of the NLS-Toda hierarchy and the Virasoro singular vectors, Lett. Math. Phys. 60 (2002), 147-156.

[5] V. G. Kac, *Infinite Dimensional Lie Algebras, 3rd. ed.* , Cambridge, 1990.

[6] I. G. Macdonald, *Symmetric Functions and Hall Polynomials, 2nd. ed.* , Oxford, 1995.

[7] H. Mizukawa and H.-F. Yamada , Rectangular Schur functions and the basic representation of affine Lie algebras, Discrete Math. 298 (2005), 285-300.

[8] T. Nakajima and H.-F. Yamada, Schur's $Q$-functions and twisted affine Lie algebras, Adv. Stud. in Pure Math. 28 (2000), 241-259.

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