Math. J. Okayama Univ. **53** (2011), 55–74

PROJECTIVE STRUCTURES AND AUTOMORPHIC PSEUDODIFFERENTIAL OPERATORS

Min Ho Lee

Abstract. Automorphic pseudodifferential operators are pseudodifferential operators that are invariant under an action of a discrete subgroup Γ of $SL(2,\mathbb{R})$, and they are closely linked to modular forms. In particular, there is a lifting map from modular forms to automorphic pseudodifferential operators, which can be interpreted as a lifting morphism of sheaves over the Riemann surface X associated to the given discrete subgroup Γ. One of the questions raised in a paper by Cohen, Manin, and Zagier is whether the difference in the images of a local section of a sheaf under such lifting morphisms corresponding to two projective structures on X can be expressed in terms of certain Schwarzian derivatives. The purpose of this paper is to provide a positive answer to this question for some special cases.

1. Introduction

Pseudodifferential operators are formal Laurent series in the formal inverse ∂^{-1} of the derivative operator ∂ whose coefficients are complex-valued functions, and they have been studied extensively over the years in connection with various topics in pure and applied mathematics. For example, they play a critical role in the theory of nonlinear integrable partial differential equations, also known as soliton equations (see e.g. [3]). In this paper we consider pseudodifferential operators that are invariant under a naturally defined action of a discrete subgroup of $SL(2,\mathbb{R})$.

Given a discrete subgroup Γ of $SL(2,\mathbb{R})$, assuming that the coefficients of a pseudodifferential operator ψ are holomorphic functions on the Poincaré upper half plane $\mathcal{H}$, the usual linear fractional action of Γ on $\mathcal{H}$ induces an operation of the same group on ψ. Pseudodifferential operators that are invariant under such an operation may be called automorphic pseudodifferential operators for Γ, and they are closely linked to classical modular forms. Indeed, given an automorphic pseudodifferential operator ψ for Γ, a certain linear combination of derivatives of coefficients of ψ determines a modular form for Γ, and conversely, each coefficient of ψ can be expressed as a linear combination of derivatives of modular forms for Γ of various weights. These relations can be used to establish a one-to-one correspondence between automorphic pseudodifferential operators and certain sequences of modular

Mathematics Subject Classification. 11F11, 30F30.

Key words and phrases. Automorphic pseudodifferential operators, modular forms, Schwarzian derivatives.

forms. One of the application of this correspondence is the construction of a lifting map from modular forms to automorphic pseudodifferential operators. Various aspects of automorphic pseudodifferential operators were studied systematically by Cohen, Manin, and Zagier in [2] (see also [8]). Similar results involving Hilbert modular forms can also be obtained by considering pseudodifferential operators of several variables (cf. [6]).

It is well-known that modular forms for Γ can be interpreted geometrically as sections of a sheaf or a line bundle over the Riemann surface $X = \Gamma\backslash\mathcal{H}$ corresponding to Γ. Similarly, automorphic pseudodifferential operators for Γ can be identified with sections of a sheaf or a vector bundle over X (see [7]). In particular, we can consider a morphism of sheaves over X corresponding to the lifting from modular forms to automorphic pseudodifferential operators mentioned above. This lifting morphism depends on the structure of the Riemann surface X. A projective structure on a Riemann surface is a maximal atlas of charts in which the transition functions are linear fractional transformations, and it is an important object in the theory of Riemann surfaces. Since each automorphism of the Poincaré upper half plane $\mathcal{H}$ can be given by a linear fractional transformation, the above Riemann surface X has a natural projective structure $\mathfrak{P}$. If $\widetilde{\mathfrak{P}}$ is another projective structure on X, then we are interested in the difference between the two lifting morphisms of sheaves corresponding to the projective structures $\mathfrak{P}$ and $\widetilde{\mathfrak{P}}$.

One of the questions asked by Cohen, Manin, and Zagier in [2] is whether the difference in the images of a local section of a sheaf under those two lifting sheaf morphisms described above can be expressed in terms of certain Schwarzian derivatives. The purpose of this paper is to provide a positive answer to this question for some special cases.

I would like to thank the referee for various highly helpful comments and suggestions.

2. Pseudodifferential operators

In this section we review some basic properties of automorphic pseudodifferential operators including their connections with modular forms studied by Cohen, Manin, and Zagier. More details and some other aspects of automorphic pseudodifferential operators can be found in [2] and [8] (see also [1]).

Let R denote the ring of holomorphic functions on the Poincaré upper half plane $\mathcal{H}$, and let ∂ be the derivative operator d/dz acting on the ring R. A pseudodifferential operator over R is a formal Laurent series in the formal inverse ∂^{-1} of ∂ with coefficients in R, that is, an element of the form

$$\sum_{k=-\infty}^{k_0} \xi_k(z)\partial^k$$

for some $k_0 \in \mathbb{Z}$ with $\xi_k \in R$ for each integer $k \leq k_0$. We denote by ΨDO the set of all pseudodifferential operators over R. Then it has the structure of an algebra over $\mathbb{C}$ whose multiplication is given by the Leibniz rule, that is,

$$\Big(\sum_{k=-\infty}^{k_0} \xi_k(z)\partial^k\Big)\Big(\sum_{\ell=-\infty}^{\ell_0} \eta_\ell(z)\partial^\ell\Big) = \sum_{k=-\infty}^{k_0}\sum_{\ell=-\infty}^{\ell_0}\sum_{r=0}^{\infty}\binom{k}{r}\xi_k(z)\eta_\ell^{(r)}(z)\partial^{k+\ell-r},$$

where $\eta_\ell^{(r)}$ denotes the derivative of η_ℓ of order r with respect to z, and

$$\binom{k}{0} = 1, \quad \binom{k}{r} = \frac{k(k-1)\cdots(k-r+1)}{r!}$$

for $r > 0$. For each integer m, we denote by $\Psi\mathrm{DO}_m$ the subspace of ΨDO given by

$$\Psi\mathrm{DO}_m = \Big\{\sum_{k=0}^{\infty} \xi_k(z)\partial^{m-k} \;\Big|\; \xi_k \in R\Big\}.$$

If $m \geq 0$, then $\Psi\mathrm{DO}_m$ contains the subspace

$$\mathrm{DO}_m = \Big\{\sum_{k=0}^{m} \xi_k(z)\partial^{m-k} \;\Big|\; \xi_k \in R\Big\}$$

consisting of differential operators of order at most m. We define the symbol map $\Xi_m : \Psi\mathrm{DO}_m \to R$ by

$$\Xi_m\Big(\sum_{k=0}^{\infty} \xi_k(z)\partial^{m-k}\Big) = \xi_0(z). \tag{2.1}$$

Since Ξ_m is a complex linear map whose kernel is $\Psi\mathrm{DO}_{m-1}$, we see easily that there is a short exact sequence

$$0 \to \Psi\mathrm{DO}_{m-1} \to \Psi\mathrm{DO}_m \xrightarrow{\Xi_m} R \to 0 \tag{2.2}$$

of complex vector spaces.

We now describe the action of $SL(2,\mathbb{R})$ on pseudodifferential operators induced by its linear fractional action on $\mathcal{H}$. If $\widetilde{\partial}$ denotes the differentiation operator with respect to the transformed coordinate γz of z by an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$, then we see that

$$\widetilde{\partial} = \Big(\frac{d(\gamma z)}{dz}\Big)^{-1}\partial = (cz+d)^2\partial.$$

More generally, we have

$$\widetilde{\partial}^k = ((cz+d)^2\partial)^k = \sum_{\ell=0}^{\infty} \ell! \binom{k}{\ell} \binom{k-1}{\ell} c^\ell (cz+d)^{2k-\ell} \partial^{k-\ell} \tag{2.3}$$

for all $k \in \mathbb{Z}$ and $z \in \mathcal{H}$ (see (1.7) in [2]). Thus the map

$$\partial \mapsto \partial \circ \gamma = \widetilde{\partial}$$

induces a right action $\psi \mapsto \psi \circ \gamma$ of $SL(2, \mathbb{R})$ on $\Psi\mathrm{DO}_m$ for each $m \in \mathbb{Z}$; hence, in particular, $SL(2, \mathbb{R})$ acts on $\Psi\mathrm{DO}$ on the right.

Definition 1. An element $\psi \in \Psi\mathrm{DO}$ is an *automorphic pseudodifferential operator for* Γ if it is invariant under the action of Γ, that is,

$$\psi \circ \gamma = \psi$$

for all $\gamma \in \Gamma$. We denote by $\Psi\mathrm{DO}^\Gamma$ the space of all automorphic pseudodifferential operators for Γ.

Definition 2. Given an integer m, a *modular form of weight* m *for* Γ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying

$$(f|_m\gamma)(z) := (cz+d)^{-m} f(\gamma z) = f(z) \tag{2.4}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. We denote by $\mathcal{M}_m(\Gamma)$ the space of all modular forms of weight m for Γ.

Remark. The usual definition of modular forms also contains the finiteness condition at the cusps. However, we have suppressed this condition in order to allow modular forms of negative weight.

If Ξ_{-w} is the symbol map given by (2.1) for $m = -w \in \mathbb{Z}$, we see from (2.3) that the image $\Xi_{-w}(\psi)$ of an automorphic pseudodifferential operator $\psi \in \Psi\mathrm{DO}^\Gamma_{-w}$ under Ξ_{-w} is a modular form belonging to $\mathcal{M}_{2w}(\Gamma)$. Thus the sequence (2.2) induces the short exact sequence

$$0 \to \Psi\mathrm{DO}^\Gamma_{-w-1} \to \Psi\mathrm{DO}^\Gamma_{-w} \xrightarrow{\Xi_{-w}} \mathcal{M}_{2w}(\Gamma) \to 0 \tag{2.5}$$

for each $w \in \mathbb{Z}$. In fact, this sequence splits as can be seen below.

Given a positive integer p and an element $f \in R$, we set

$$\mathcal{L}_p(f) = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{(\ell+p)!(\ell+p-1)!}{\ell!(\ell+2p-1)!} f^{(\ell)} \partial^{-p-\ell},$$

$$\mathcal{L}_{-p}(f) = \sum_{\ell=0}^{p-1} \frac{(2p-\ell)!}{\ell!(p-\ell)!(p-\ell-1)!} f^{(\ell)} \partial^{p-\ell}.$$

Then the maps $f \mapsto \mathcal{L}_p(f)$ and $f \mapsto \mathcal{L}_{-p}(f)$ determine the linear maps

$$\mathcal{L}_p : R \to \Psi\mathrm{DO}_{-p}, \quad \mathcal{L}_{-p} : R \to \mathrm{DO}_p$$

for each $p \geq 1$. Thus, if we denote the identity map on R by $\mathcal{L}_0$, we obtain a complex linear map

$$\mathcal{L}_w : R \to \Psi\mathrm{DO}_{-w} \tag{2.6}$$

for each integer w.

Proposition 2.1. Each linear map $\mathcal{L}_w : R \to \Psi\mathrm{DO}_{-w}$ in (2.6) satisfies the condition

$$\mathcal{L}_w(f \mid_{2w} \gamma) = \mathcal{L}_w(f) \circ \gamma$$

for all $f \in R$ and $\gamma \in SL(2, \mathbb{C})$.

Proof. See Proposition 1 in [2]. □

If $f \in \mathcal{M}_{2w}(\Gamma)$ with $w \in \mathbb{Z}$ for some discrete subgroup Γ of $SL(2, \mathbb{R})$, then it follows from Proposition 2.1 that $\mathcal{L}_w(f)$ is an automorphic pseudodifferential operator belonging to $\Psi\mathrm{DO}^{\Gamma}_{-w}$; hence the map $\mathcal{L}_w$ in (2.6) induces the linear map

$$\mathcal{L}_w : \mathcal{M}_{2w}(\Gamma) \to \Psi\mathrm{DO}^{\Gamma}_{-w}. \tag{2.7}$$

Since we see easily that $(\Xi_{-w} \circ \mathcal{L}_w)f = f$, the short exact sequence (2.5) splits and the linear map $\mathcal{L}_w$ in (2.7) may be regarded as a lifting from modular forms of weight $2w$ to automorphic pseudodifferential operators for Γ.

3. Projective structures

There are two important structures on each Riemann surface that are finer than the complex structure, namely, the affine and projective structures. The projective structure is closely linked to the differential operator known as the Schwarzian derivative. In this section we review some basic properties of the projective structure and its relation with the Schwarzian derivative (see, e.g., [4, Section 9] for more details). We also describe one of the questions asked by Cohen, Manin, and Zagier in [2] and state our main theorem in this paper which provide a partial answer to that question for some special cases.

Definition 3. Let U and V be open subdomains of $\mathbb{C}$, and let $f : U \to V$ be a complex analytic local homeomorphism. The *Schwarzian derivative* of f is the function $\mathcal{S}f$ defined by

$$\mathcal{S}f = \Big(\frac{f''}{f'}\Big)' - \frac{1}{2}\Big(\frac{f''}{f'}\Big)^2 = \frac{2f'f''' - 3(f'')^2}{2(f')^2}. \tag{3.1}$$

Lemma 3.1. *Let $f : U \to V$ be a complex analytic local homeomorphism of open subdomains U and V of $\mathbb{C}$. Then $\mathcal{S}f = 0$ if and only if f is a linear fractional transformation.*

Proof. If $f : U \to V$ is a linear fractional transformation given by

$$f(z) = \frac{az+b}{cz+d}$$

with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{C})$, then we have

$$\frac{f''(z)}{f'(z)} = -2c(cz+d)^{-1};$$

hence we see that

$$(\mathcal{S}f)(z) = 2c^2(cz+d)^{-2} - \frac{1}{2}(-2c(cz+d)^{-1})^2 = 0$$

for all $z \in U$. To verify the converse, we now assume that f satisfies $\mathcal{S}f = 0$. Then, using the fact that $\mathcal{S}f(z)$ can be written in the form

$$(\mathcal{S}f)(z) = -2f'(z)^{1/2}\frac{d^2}{dz^2}\Big(f'(z)^{-1/2}\Big),$$

we obtain

$$\frac{d^2}{dz^2}f'(z)^{-1/2} = 0.$$

This shows that

$$f'(z)^{-1/2} = cz+d, \quad f'(z) = (cz+d)^{-2}, \quad f(z) = \frac{az+b}{cz+d}$$

for some $a, b, c, d \in \mathbb{C}$. Since f is a local homeomorphism, we have $f'(z) \neq 0$, which implies that $ad - bc \neq 0$; hence f is a linear fractional transformation. □

If $f = h \circ g$ is the composite of complex analytic local homeomorphisms $g : U \to V$ and $h : V \to W$ for some open subdomains U, V and W of $\mathbb{C}$, then it can be easily shown that

$$(\mathcal{S}f)(z) = (\mathcal{S}h)(g(z)) \cdot g'(z)^2 + (\mathcal{S}g)(z) \tag{3.2}$$

for all $z \in U$. In particular, if g is a linear fractional transformation of the form

$$g(z) = \gamma z = \frac{az+b}{cz+d}$$

for some $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{C})$, then we have

$$(\mathcal{S}f)(z) = (\mathcal{S}h)(g(z)) \cdot g'(z)^2 = (\mathcal{S}h)(\gamma z)(cz+d)^{-4}. \tag{3.3}$$

Since $d(\gamma z) = (cz+d)^{-2}dz$, we see from (3.3) that

$$(\mathcal{S}h)(\gamma z)d(\gamma z)^2 = (\mathcal{S}h)(z)dz^2; \tag{3.4}$$

hence the quadratic differential $(\mathcal{S}h)(z)dz^2$ is invariant under linear fractional transformations.

Example 3.2. Let Γ be a discrete subgroup of $SL(2,\mathbb{R})$, which acts on $\mathcal{H}$ by linear fractional transformations. If f is a holomorphic function on $\mathcal{H}$, then by (3.4) the associated quadratic differential $\mathcal{S}f(z)dz^2$ on $\mathcal{H}$ is Γ-invariant; hence it induces a quadratic differential form on the Riemann surface $X = \Gamma\backslash\mathcal{H}$. On the other hand, given a holomorphic quadratic differential hdz^2 on X, by solving locally the differential equation $\mathcal{S}f(z) = h$ in f we can recover the charts of a projective structure on X. Here we need to use the fact that any two local solutions of $\mathcal{S}f(z) = h$ differ by composition with a linear fractional transformation (see [5, Proposition 1]).

Definition 4. A *projective structure* on a Riemann surface X is a maximal atlas of charts on X such that each transition function is a linear fractional transformation.

Remark. Let X be a Riemann surface equipped with a projective structure. Then by Lemma 3.1 the Schwarzian derivative of each transition function is equal to zero. Furthermore, by (3.2) the Schwarzian derivative of the composite of two transition functions is also zero. In fact, a projective structure can also be defined to be a maximal atlas of charts in which the Schwarzian derivative of each transition function is equal to zero.

Let $\mathcal{O}_X$ be the sheaf of germs of holomorphic functions on a Riemann surface X. If an open subset U of X has a local coordinate z and if m is an integer, we set

$$\mathcal{E}_X^m(U) = \Big\{ \sum_{k=0}^{\infty} h_k(z)\partial_z^{m-k} \;\Big|\; h_k \in \mathcal{O}_X(U) \text{ for each } k \geq 1 \Big\}, \tag{3.5}$$

where $\partial_z = \partial/\partial z$ and $\mathcal{O}_X(U)$ is the ring of holomorphic functions on U. We denote by $\mathcal{E}_X^m$ the sheaf on X associated to the presheaf $U \mapsto \mathcal{E}_X^m(U)$. Let ω_X be the sheaf of holomorphic differentials on X. Then it is an invertible sheaf, and we can consider its tensor power $\omega_X^{\otimes m}$ for each integer m. If (U, z) is an open chart and if $\omega_X^{\otimes m}(U)$ denotes the space of sections of $\omega_X^{\otimes m}$, then an element of $\omega_X^{\otimes m}(U)$ can be written in the form $f(dz)^m$ for some $f \in \mathcal{O}_X(U)$.

We now consider the case where $X = \Gamma\backslash\mathcal{H}$ is the quotient of $\mathcal{H}$ by a discrete subgroup Γ of $SL(2,\mathbb{R})$, where the quotient is taken with respect to the linear fractional action of Γ on $\mathcal{H}$. If U_α is an open subset of X, then there is an isomorphism $z_\alpha : U_\alpha \to \widetilde{U}_\alpha$ of U_α onto an open subset $\widetilde{U}_\alpha$ of $\mathcal{H}$ such that (U_α, z_α) is a chart. Using this and the fact that automorphisms of $\mathcal{H}$ are given by linear fractional transformations, we can consider a natural maximal atlas $(U_\alpha, z_\alpha)_{\alpha\in I}$ whose transition functions are linear fractional. We shall call this the *natural projective structure* on X and denote it by $\mathfrak{P}$. Sections of the sheaf $\omega_X^{\otimes m}$ are Γ-invariant forms of the form $f(dz)^m$ with $f \in \mathcal{O}_X(\mathcal{H})$. Since $d(\gamma z) = (cz + d)^{-2}dz$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we see that sections of $\omega_X^{\otimes m}$ on X can be identified with modular forms belonging to $\mathcal{M}_{2m}(\Gamma)$. On the other hand, by (3.5) sections of the sheaf $\mathcal{E}_X^m$ may be regarded as automorphic pseudodifferential operators belonging to $\Psi\mathrm{DO}_m^\Gamma$. Thus the sequence (2.5) induces the short exact sequence of the type

$$0 \to \mathcal{E}_X^{-m-1} \to \mathcal{E}_X^{-m} \to \omega_X^{\otimes m} \to 0. \tag{3.6}$$

In particular, there is a canonical isomorphism

$$\mathcal{E}_X^{-m}/\mathcal{E}_X^{-m-1} \cong \omega_X^{\otimes m}$$

of sheaves on X for each $m \geq 0$.

If (U_α, z_α) is a chart on X belonging to the projective structure $\mathfrak{P}$, we set

$$\Lambda_0^{\mathfrak{P}}(f) = f \tag{3.7}$$

$$\Lambda_k^{\mathfrak{P}}(f(dz_\alpha)^k) = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{(\ell+k)!(\ell+k-1)!}{\ell!(\ell+2k-1)!} f^{(\ell)} \partial_{z_\alpha}^{-k-\ell}, \tag{3.8}$$

$$\Lambda_{-k}^{\mathfrak{P}}(f(dz_\alpha)^{-k}) = \sum_{\ell=0}^{k-1} \frac{(2k-\ell)!}{\ell!(k-\ell)!(k-\ell-1)!} f^{(\ell)} \partial_{z_\alpha}^{k-\ell} \tag{3.9}$$

for each positive integer k and an element $f \in \mathcal{O}_X(U_\alpha)$, where $\partial_{z_\alpha} = \partial/\partial_{z_\alpha}$. The next lemma shows that the short exact sequence (3.6) splits.

Lemma 3.3. *Let $\mathfrak{P}$ be the natural projective structure on X described above. Then the formulas* (3.7), (3.8) *and* (3.9) *determine a morphism*

$$\Lambda_m^{\mathfrak{P}} : \omega_X^{\otimes m} \to \mathcal{E}_X^{-m} \tag{3.10}$$

of sheaves on X for each $m \in \mathbb{Z}$.

Proof. First, to see that $\Lambda_m^{\mathfrak{P}}$ is well-defined we consider two charts (U_α, z_α) and (U_β, z_β). Since transition functions are linear fractional transformations, $z_\alpha = \gamma_{\alpha\beta} z_\beta$ for some $\gamma_{\alpha\beta} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$, so that $dz_\alpha = (cz_\beta + d)^{-2} z_\beta$. Hence, if $f \in \mathcal{O}_X(U_\alpha)$, we have

$$f(dz_\alpha)^m = (f \mid_{2m} \gamma_{\alpha\beta})(dz_\beta)^m \tag{3.11}$$

on $U_\alpha \cap U_\beta$. We note that

$$\Lambda_m^{\mathfrak{P}}(f(dz_\alpha)^m) = \mathcal{L}_m(f), \tag{3.12}$$

where $\mathcal{L}_m$ is as in (2.6). Thus the fact that $\Lambda_m^{\mathfrak{P}}$ is well-defined follows from (3.11), (3.12) and Proposition 2.1. By (3.7) the map $\Lambda_0^{\mathfrak{P}}$ is simply the inclusion morphism of $\mathcal{O}_X = \omega_X^0$ into $\mathcal{E}_X^0$. On the other hand, given a positive integer k and a chart (U_α, z_α) in the projective structure $\mathfrak{P}$ on X, the sections of the sheaves $\omega_X^{\otimes k}$ and $\omega_X^{\otimes(-k)}$ over U_α are generated by elements of the form $f(dz_\alpha)^k$ and $h(dz_\alpha)^{-k}$, respectively, for some $f, h \in \mathcal{O}_X(U_\alpha)$. Hence the lemma follows from this and (3.5). □

We now consider another projective structure $\widetilde{\mathfrak{P}}$ corresponding to a local coordinate $\widetilde{z}$ on X, and set

$$J = \frac{d\widetilde{z}}{dz}. \tag{3.13}$$

By (3.1) the Schwarzian derivative of the coordinate transformation map $z \mapsto \widetilde{z}$ can be written as

$$\mathcal{S}_z^{\widetilde{z}} = \frac{2JJ'' - 3(J')^2}{2J^2} = J^{-1}J'' - \frac{3}{2}J^{-2}(J')^2. \tag{3.14}$$

One of the questions asked by Cohen, Manin, and Zagier in [2, Section 8] may be paraphrased as follows.

Question 3.4. Let (U, z) and $(\widetilde{U}, \widetilde{z})$ be charts on X with $U \cap \widetilde{U} \neq \emptyset$ belonging to the projective structures $\widetilde{\mathfrak{P}}$ and $\mathfrak{P}$, respectively, with $U \cap \widetilde{U} \neq \emptyset$, and let $f \in \mathcal{O}_X(U \cap \widetilde{U})$. Is it true that

$$(\Lambda_m^{\widetilde{\mathfrak{P}}} - \Lambda_m^{\mathfrak{P}})(f(dz)^m)$$

can be expressed in terms of f and the Schwarzian derivative $\mathcal{S}_z^{\widetilde{z}}$ for each $m \in \mathbb{Z}$?

It was pointed out by Cohen, Manin, and Zagier in [2, Section 8] that

$$\Lambda_m^{\widetilde{\mathfrak{P}}} - \Lambda_m^{\mathfrak{P}} = 0$$

for $m = 1, 0, -1, -2$. To see the cases of $m = 0, -1, -2$, note that

$$\partial_{\widetilde{z}} = \frac{d}{d\widetilde{z}} = \frac{1}{d\widetilde{z}/dz}\frac{d}{dz} = J^{-1}\partial, \tag{3.15}$$
$$\partial_{\widetilde{z}}^2 = J^{-1}\partial(J^{-1}\partial) = J^{-1}(-J^{-2}J'\partial + J^{-1}\partial^2) = J^{-2}\partial^2 - J'J^{-3}\partial.$$

On the other hand, since $d\widetilde{z} = J dz$, we have

$$f(dz)^m = f(J^{-1}d\widetilde{z})^m = J^{-m}f(d\widetilde{z})^m, \tag{3.16}$$

so that

$$(\Lambda_m^{\widetilde{\mathfrak{P}}} - \Lambda_m^{\mathfrak{P}})(f(dz)^m) = \Lambda_m^{\widetilde{\mathfrak{P}}}(J^{-m}f(d\widetilde{z})^m) - \Lambda_m^{\mathfrak{P}}(f(dz)^m) \tag{3.17}$$

for each $m \in \mathbb{Z}$. For $m = 0$, we have

$$\Lambda_0^{\widetilde{\mathfrak{P}}}(f(d\widetilde{z})^0) = \Lambda_0^{\mathfrak{P}}(f(dz)^0) = f;$$

hence we see that

$$(\Lambda_0^{\widetilde{\mathfrak{P}}} - \Lambda_0^{\mathfrak{P}})(f) = 0.$$

For $m = -1$, from (3.9) we obtain

$$\Lambda^{\mathfrak{P}}_{-1}(f(dz)^{-1}) = \sum_{n=0}^{0} \frac{(2-n)!}{n!(1-n)!(-n)!} f^{(n)} \partial^{1-n} = 2f\partial.$$

Thus we have

$$(\Lambda^{\widetilde{\mathfrak{P}}}_{-1} - \Lambda^{\mathfrak{P}}_{-1})(f(dz)^{-1}) = 2(Jf(z)\partial_{\widetilde{z}} - f\partial) = 2(Jf(z)J^{-1}\partial - f(z)) = 0$$

For $m = -2$, we see from (3.9) that

$$\Lambda^{\mathfrak{P}}_{-2}(f(dz)^{-2}) = \sum_{n=0}^{1} \frac{(4-n)!}{n!(2-n)!(1-n)!} f^{(n)}(z)\partial^{2-n} = 12f(z)\partial^2 + 6f'(z)\partial,$$

and therefore we have

$$(\Lambda^{\widetilde{\mathfrak{P}}}_{-2} - \Lambda^{\mathfrak{P}}_{-2})(f(dz)^{-2}) = 12(J^2 f\partial^2_{\widetilde{z}} - f\partial^2) + 6\left(\frac{d}{d\widetilde{z}}(J^2 f)\partial_{\widetilde{z}} - f'(z)\partial\right),$$

where

$$\frac{d}{d\widetilde{z}}(J^2 f) = J^{-1}\frac{d}{dz}(J^2 f) = J^{-1}(2JJ'f + J^2 f') = 2J'f + Jf'.$$

From this and (3.15) it follows that

$$\begin{aligned}(\Lambda^{\widetilde{\mathfrak{P}}}_{-2} - \Lambda^{\mathfrak{P}}_{-2})(f(dz)^{-2}) &= 12(J^2 f(J^{-2}\partial^2 - J'J^{-3}) - f\partial^2) \\ &\quad + 6((J^{-1}(2J'f + Jf' - f'\partial)J^{-1}\partial - f'\partial) = 0.\end{aligned}$$

Cohen, Manin, and Zagier also gave a formula for $\Lambda^{\widetilde{\mathfrak{P}}}_m - \Lambda^{\mathfrak{P}}_m$ in the case of $m = -3$, which we show in the next lemma using our notation.

Lemma 3.5. *Given charts (U, z) and $(\widetilde{U}, \widetilde{z})$ belonging to the projective structures $\widetilde{\mathfrak{P}}$ and $\mathfrak{P}$, respectively, on X with $U \cap \widetilde{U} \neq \emptyset$, we have*

$$(\Lambda^{\widetilde{\mathfrak{P}}}_{-3} - \Lambda^{\mathfrak{P}}_{-3})(f(dz)^{-3}) = -24\mathcal{S}^{\widetilde{z}}_{z} f\partial \tag{3.18}$$

for $f \in \mathcal{O}_X(U \cap \widetilde{U})$.

Proof. Using (3.9), we obtain

$$\begin{aligned}\Lambda^{\mathfrak{P}}_{-3}(f(dz)^{-3}) &= \sum_{n=0}^{2} \frac{(6-n)!}{n!(3-n)!(2-n)!} f^{(n)}\partial^{3-n} \\ &= 60f\partial^3 + 60f'\partial^2 + 12f''\partial;\end{aligned}$$

hence from (3.17) we see that

$$(\Lambda^{\widetilde{\mathfrak{P}}}_{-3} - \Lambda^{\mathfrak{P}}_{-3})(f(dz)^{-3}) = 60(J^3 f\partial^3_{\widetilde{z}} - f\partial^3) \tag{3.19}$$

$$+ 60\left(\frac{d}{d\widetilde{z}}(J^3 f)\partial_{\widetilde{z}}^2 - f'\partial^2)\right)$$
$$+ 12\left(\frac{d^2}{d\widetilde{z}^2}(J^3 f)\partial_{\widetilde{z}} - f''(z)\partial\right).$$

Here $\partial_{\widetilde{z}}$ and $\partial_{\widetilde{z}}^2$ satisfy (3.15), and

$$\begin{aligned}\partial_{\widetilde{z}}^3 &= J^{-1}\partial(J^{-2}\partial^2 - J'J^{-3}\partial)\\ &= J^{-1}(-2J^{-3}J'\partial^2 + J^{-2}\partial^3\\ &\qquad - (J''J^{-3} - 3(J')^2J^{-4})\partial - J'J^{-3}\partial^2)\\ &= J^{-3}\partial^3 - 3J'J^{-3}\partial^2 + (3(J')^2 - JJ'')J^{-5}\partial.\end{aligned} \tag{3.20}$$

On the other hand, we have

$$\begin{aligned}\frac{d}{d\widetilde{z}}(J^3 f) &= J^{-1}(J^3 f)'\\ &= J^{-1}(3J^2J'f + J^3 f') = 3JJ'f + J^2 f',\\ \frac{d^2}{d\widetilde{z}^2}(J^3 f) &= J^{-1}(3JJ'f + J^2 f')'\\ &= J^{-1}(3(J')^2 + 3JJ'')f + 3JJ'f' + 2JJ'f' + J^2 f'')\\ &= (3(J')^2J^{-1} + 3J'')f + 5J'f' + Jf''.\end{aligned}$$

Thus we obtain

$$\begin{aligned}\frac{1}{60}\Lambda^{\widetilde{\mathfrak{P}}}_{-3}(f(dz)^{-3}) &= J^3 f\partial_{\widetilde{z}}^3 + \frac{d}{d\widetilde{z}}(J^3 f)\partial_{\widetilde{z}}^2 + \frac{1}{5}\frac{d^2}{d\widetilde{z}^2}(J^3 f)\partial_{\widetilde{z}}\\ &= J^3 f(J^{-3}\partial^3 - 3J'J^{-4}\partial^2 + (3(J')^2 - JJ'')J^{-5}\partial)\\ &\qquad + (3JJ'f + J^2 f')(J^{-2}\partial^2 - J'J^{-3}\partial)\\ &\qquad + \frac{1}{5}((3(J')^2J^{-1} + 3J'')f + 5J'f' + Jf'')J^{-1}\partial\\ &= f\partial^3 - 3J'J^{-1}f\partial^2 + (3(J')^2J^{-2} - J''J^{-1}f)\partial\\ &\qquad + (3J'J^{-1}f + f')\partial^2 - (3(J')^2J^{-2}f + J'J^{-1}f')\partial\\ &\qquad + \frac{1}{5}(3(J')^2J^{-2}f + 3J''J^{-1}f + 5J'J^{-1}f' + f'')\partial\\ &= (f\partial^3 + f'\partial^2 + f''\partial) + \frac{1}{5}(3(J')^2J^{-2} - 2J''J^{-1})f\partial,\end{aligned}$$

which implies that

$$\frac{1}{60}(\Lambda^{\widetilde{\mathfrak{P}}}_{-3} - \Lambda^{\mathfrak{P}}_{-3})(f(dz)^{-3}) = \frac{1}{5}(3(J')^2J^{-2} - 2J''J^{-1})f\partial.$$

Thus (3.18) follows from this and (3.14). □

Remark. The formula given Cohen, Manin and Zagier in [2] has the right hand side of (3.18) in the form

$$\frac{1}{5}\left(2\frac{J''}{J} - 3\frac{(J')^2}{J^2}\right) f\partial = \frac{1}{5}\mathcal{S}_z^{\widetilde{z}} f\partial$$

in our notation. The right hand side of (3.18) would coincide with this if the formula (3.19) is normalized by dividing its right hand side by the the coefficient of $f\partial^3$.

We now state our main theorem in this paper, which provides a positive answer to Question 3.4 for some special cases. The proof of this theorem will be given in the next two sections.

Theorem 3.6. *Let (U, z) and $(\widetilde{U}, \widetilde{z})$ be charts belonging to the projective structures $\widetilde{\mathfrak{P}}$ and $\mathfrak{P}$, respectively, on X with $U \cap \widetilde{U} \neq \emptyset$, and let $f \in \mathcal{O}_X(U \cap \widetilde{U})$. Then we have*

$$(\Lambda_{-4}^{\widetilde{\mathfrak{P}}} - \Lambda_{-4}^{\mathfrak{P}})(f(dz)^{-4}) = -800\mathcal{S}_z^{\widetilde{z}} f\partial^2 - 400(\mathcal{S}_z^{\widetilde{z}} f)'\partial, \tag{3.21}$$

$$\begin{aligned}(\Lambda_{-5}^{\widetilde{\mathfrak{P}}} - \Lambda_{-5}^{\mathfrak{P}})(f(dz)^{-5}) = &-210\mathcal{S}_z^{\widetilde{z}} f\partial^3 - 210(\mathcal{S}_z^{\widetilde{z}} f)'\partial^2 \\ &- 30\big(37(\mathcal{S}_z^{\widetilde{z}})'' f + 16(\mathcal{S}_z^{\widetilde{z}})^2 f \\ &+ 65(\mathcal{S}_z^{\widetilde{z}})' f' + 30\mathcal{S}_z^{\widetilde{z}} f''\big)\partial\end{aligned} \tag{3.22}$$

for $f \in \mathcal{O}_X(U \cap \widetilde{U})$, where $\Lambda_m^{\mathfrak{P}}$ and $\Lambda_m^{\widetilde{\mathfrak{P}}}$ for $m = -4, -5$ are morphisms of sheaves on X in (3.10) *and $\mathcal{S}_z^{\widetilde{z}}$ is the Schwarzian derivative of the coordinate transformation map $z \mapsto \widetilde{z}$ given by* (3.14).

4. Degree four case

In this section we prove the relation (3.21) in Theorem 3.6. Let $f \in \mathcal{O}_X(U\cap\widetilde{U})$, where (U, z) and $(\widetilde{U}, \widetilde{z})$ with $U\cap\widetilde{U} \neq \emptyset$ are charts on X belonging to the projective structures $\widetilde{\mathfrak{P}}$ and $\mathfrak{P}$, respectively. Then, using (3.9), we have

$$\begin{aligned}\Lambda_{-4}^{\mathfrak{P}}(f(dz)^{-4}) &= \sum_{n=0}^{3} \frac{(8-n)!}{n!(4-n)!(3-n)!} f^{(n)}\partial^{4-n} \\ &= \frac{8!}{4!3!} f\partial^4 + \frac{7!}{3!2!} f'\partial^3 + \frac{6!}{2!2!} f''\partial^2 + \frac{5!}{3!} f'''\partial \\ &= 280 f\partial^4 + 420 f'\partial^3 + 180 f''\partial^2 + 20 f'''\partial.\end{aligned} \tag{4.1}$$

From this and (3.16) we obtain

$$\Lambda_{-4}^{\widetilde{\mathfrak{P}}}(f(dz)^{-4}) = \Lambda_{-4}^{\widetilde{\mathfrak{P}}}(J^4 f(d\widetilde{z})^{-4}) \tag{4.2}$$

$$= 280J^4 f\partial_{\widetilde{z}}^4 + 420\frac{d}{d\widetilde{z}}(J^4 f)\partial_{\widetilde{z}}^3$$
$$+ 180\frac{d^2}{d\widetilde{z}^2}(J^4 f)\partial_{\widetilde{z}}^2 + 20\frac{d^3}{d\widetilde{z}^3}(J^4 f)\partial_{\widetilde{z}},$$

where J is as in (3.13). Since $\partial_{\widetilde{z}} = J^{-1}\partial$, using (3.20), we see that

$$\begin{aligned}
(4.3)\quad \partial_{\widetilde{z}}^4 &= J^{-1}\partial(J^{-3}\partial^3 - 3J'J^{-4}\partial^2 + (3(J')^2J^{-5} - J''J^{-4})\partial) \\
&= J^{-1}\big[-3J^{-4}J'\partial^3 + J^{-3}\partial^4 - 3(J''J^{-4} - 4(J')^2J^{-5})\partial^2 \\
&\qquad - 3J'J^{-4}\partial^3 + (6J''J'J^{-5} - 15(J')^3J^{-6} - J'''J^{-4} \\
&\qquad\qquad + 4J''J'J^{-5})\partial + (3(J')^2J^{-5} - J''J^{-4})\partial^2\big] \\
&= -3J^{-5}J'\partial^3 + J^{-4}\partial^4 - 3(J''J^{-5} - 4(J')^2J^{-6})\partial^2 - 3J'J^{-5}\partial^3 \\
&\qquad + (10J''J'J^{-6} - 15(J')^3J^{-7} - J'''J^{-5})\partial \\
&\qquad\qquad + (3(J')^2J^{-6} - J''J^{-5})\partial^2 \\
&= J^{-4}\partial^4 - 6J'J^{-5}\partial^3 + \big[15(J')^2J^{-6} - 4J''J^{-5}\big]\partial^2 \\
&\qquad + \big[10J''J'J^{-6} - 15(J')^3J^{-7} - J'''J^{-5}\big]\partial.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\frac{d}{d\widetilde{z}}(J^4 f) &= J^{-1}\frac{d}{dz}(J^4 f) \\
&= J^{-1}(4J^3J'f + J^4 f') = 4J^2J'f + J^3 f', \\
\frac{d^2}{d\widetilde{z}^2}(J^4 f) &= J^{-1}(4J^2J'f + J^3 f')' \\
&= J^{-1}\big[(8J(J')^2 + 4J^2J'')f + 4J^2J'f' + 3J^2J'f' + J^3 f''\big] \\
&= (8(J')^2 + 4J''J)f + 7J'Jf' + J^2 f'', \\
\frac{d^3}{d\widetilde{z}^3}(J^4 f) &= J^{-1}\big[(8(J')^2 + 4J''J)f + 7J'Jf' + J^2 f''\big]' \\
&= J^{-1}\big[(16J'J'' + 4J'''J + 4J''J')f + (8(J')^2 + 4J''J)f' \\
&\qquad + 7(J''J + (J')^2)f' + 7J'Jf'' + 2J'Jf'' + J^2 f'''\big] \\
&= (20J''J'J^{-1} + 4J''')f + (8(J')^2J^{-1} + 4J'')f' \\
&\qquad + 7(J'' + (J')^2J^{-1})f' + 7J'f'' + 2J'f'' + Jf''' \\
&= (20J''J'J^{-1} + 4J''')f \\
&\qquad + (15(J')^2J^{-1} + 11J'')f' + 9J'f'' + Jf'''.
\end{aligned}$$

From these, (3.15), (3.20), (4.2) and (4.3) we obtain

$$\frac{1}{20}\Lambda_{-4}^{\widetilde{\mathfrak{P}}}(f(dz)^{-4})$$

$$
\begin{aligned}
&= 14f(\widetilde{z})\partial_{\widetilde{z}}^4 + 21\frac{d}{d\widetilde{z}}f(\widetilde{z})\partial_{\widetilde{z}}^3 + 9\frac{d^2}{d\widetilde{z}^2}f(\widetilde{z})\partial_{\widetilde{z}}^2 + \frac{d^3}{d\widetilde{z}^3}f(\widetilde{z})\partial_{\widetilde{z}} \\
&= 14J^4 f\big(J^{-4}\partial^4 - 6J'J^{-5}\partial^3 + \big[15(J')^2J^{-6} - 4J''J^{-5}\big]\partial^2 \\
&\qquad\qquad + \big[10J''J'J^{-6} - 15(J')^3J^{-7} - J'''J^{-5}\big]\partial\big) \\
&\quad + 21(4J^2J'f + J^3f')(J^{-3}\partial^3 - 3J'J^{-4}\partial^2 + (3(J')^2 - JJ'')J^{-5}\partial) \\
&\qquad + 9((8(J')^2 + 4J''J)f + 7J'Jf' + J^2f'')(J^{-2}\partial^2 - J'J^{-3}\partial) \\
&\qquad + \big((20J''J'J^{-1} + 4J''')f \\
&\qquad\qquad + (15(J')^2J^{-1} + 11J'')f' + 9J'f'' + Jf'''\big)(J^{-1}\partial) \\
&= 14f\partial^4 - 84J'J^{-1}f\partial^3 + \big[210(J')^2J^{-2}f - 56J''J^{-1}f\big]\partial^2 \\
&\quad + \big[140J''J'J^{-2}f - 210(J')^3J^{-3}f - 14J'''J^{-1}f\big]\partial \\
&\quad + \big[84J'J^{-1}f + 21f'\big]\partial^3 - \big[252(J')^2J^{-2}f + 63J'J^{-1}f'\big]\partial^2 \\
&\quad + \big[252(J')^3J^{-3}f - 84J''J'J^{-2}f + 63(J')^2J^{-2}f' - 21J''J^{-1}f'\big]\partial \\
&\quad + \big[72(J')^2J^{-2}f + 36J''J^{-1}f + 63J'J^{-1}f' + 9f''\big]\partial^2 \\
&\quad - \big[72(J')^3J^{-3}f + 36J''J'J^{-2}f + 63(J')^2J^{-2}f' + 9J'J^{-1}f''\big]\partial \\
&\quad + \big[20J''J'J^{-2}f + 4J'''J^{-1}f \\
&\qquad\qquad + 15(J')^2J^{-2}f' + 11J''J^{-1}f' + 9J'J^{-1}f'' + f'''\big]\partial.
\end{aligned}
$$

Using this and (4.1), we have

$$
\begin{aligned}
&\frac{1}{20}(\Lambda^{\widetilde{\mathfrak{P}}}_{-4} - \Lambda^{\mathfrak{P}}_{-4})(f(dz)^{-4}) \\
&\qquad = 10\big[3(J')^2J^{-2} - 2J''J^{-1}\big]f\partial^2 + 5\big[3(J')^2J^{-2} - 2J''J^{-1}\big]f'\partial \\
&\qquad\quad + 10\big[4J''J'J^{-2} - 3(J')^3J^{-3} - J'''J^{-1}\big]f\partial.
\end{aligned}
$$

If $\mathcal{S}^{\widetilde{z}}_z$ is as in (3.14), its derivative with respect to z is given by

$$
\begin{aligned}
(\mathcal{S}^{\widetilde{z}}_z)' &= J'''J^{-1} - J''J'J^{-2} - \frac{3}{2}(2J''J'J^{-2} - 2(J')^3J^{-3}) \\
&= J'''J^{-1} + 3(J')^3J^{-3} - 4J''J'J^{-2}.
\end{aligned}
\tag{4.4}
$$

Thus we have

$$
\begin{aligned}
-\frac{1}{400}(\Lambda^{\widetilde{\mathfrak{P}}}_{-4} - \Lambda^{\mathfrak{P}}_{-4})(f(dz)^{-4}) &= 2\mathcal{S}^{\widetilde{z}}_z f\partial^2 + (\mathcal{S}^{\widetilde{z}}_z f' + (\mathcal{S}^{\widetilde{z}}_z)'f)\partial \\
&= 2\mathcal{S}^{\widetilde{z}}_z f\partial^2 + (\mathcal{S}^{\widetilde{z}}_z f)'\partial,
\end{aligned}
$$

which verifies the relation (3.21) in Theorem 3.6.

5. Degree five case

In this section we prove the relation (3.22) in Theorem 3.6. If $f \in \mathcal{O}_X(U \cap \widetilde{U})$ is as in Section 4, using (3.9) and (3.16), we have

(5.1)

$$\begin{aligned}\Lambda^{\mathfrak{P}}_{-5}(f(dz)^{-5}) &= \sum_{n=0}^{4} \frac{(10-n)!}{n!(5-n)!(4-n)!} f^{(n)}\partial^{5-n} \\ &= \frac{10!}{5!4!} f\partial^5 + \frac{9!}{4!3!} f'\partial^4 + \frac{8!}{2!3!2!} f''\partial^3 + \frac{7!}{3!2!} f'''\partial^2 + \frac{6!}{4!} f^{(4)}\partial \\ &= 30\Big(42f\partial^5 + 84f'\partial^4 + 56f''\partial^3 + 14f'''\partial^2 + f^{(4)}\partial\Big).\end{aligned}$$

(5.2)

$$\begin{aligned}\Lambda^{\widetilde{\mathfrak{P}}}_{-5}(f(dz)^{-5}) &= \Lambda^{\widetilde{\mathfrak{P}}}_{-5}(J^5 f(d\widetilde{z})^{-5}) \\ &= 30\Bigg(42J^5 f\partial^5_{\widetilde{z}} + 84\frac{d}{d\widetilde{z}}(J^5 f)\partial^4_{\widetilde{z}} + 56\frac{d^2}{d\widetilde{z}^2}(J^5 f)\partial^3_{\widetilde{z}} \\ &\qquad + 14\frac{d^3}{d\widetilde{z}^3}(J^5 f)\partial^2_{\widetilde{z}} + \frac{d^4}{d\widetilde{z}^4}(J^5 f)\partial_{\widetilde{z}}\Bigg).\end{aligned}$$

Since $\partial^5_{\widetilde{z}} = J^{-1}\partial\partial^4_{\widetilde{z}}$, by using (4.3) we see that

$$\begin{aligned}J\partial^5_{\widetilde{z}} &= \partial\big[J^{-4}\partial^4 - 6J'J^{-5}\partial^3 + (15(J')^2J^{-6} - 4J''J^{-5})\partial^2 \\ &\qquad + (10J''J'J^{-6} - 15(J')^3J^{-7} - J'''J^{-5})\partial\big] \\ &= -4J^{-5}J'\partial^4 + J^{-4}\partial^5 - 6(J''J^{-5}\partial^3 - 5(J')^2J^{-6}\partial^3 + J'J^{-5}\partial^4) \\ &\quad + 15(2J''J'J^{-6}\partial^2 - 6(J')^3J^{-7}\partial^2 + (J')^2J^{-6}\partial^3) \\ &\quad - 4(J'''J^{-5}\partial^2 - 5J''J'J^{-6}\partial^2 + J''J^{-5}\partial^3) \\ &\quad + 10(J'''J'J^{-6}\partial + (J'')^2J^{-6}\partial - 6J''(J')^2J^{-7}\partial + J''J'J^{-6}\partial^2) \\ &\quad - 15(3J''(J')^2J^{-7}\partial - 7(J')^4J^{-8}\partial + (J')^3J^{-7}\partial^2) \\ &\quad - (J^{(4)}J^{-5}\partial - 5J'''J'J^{-6}\partial + J'''J^{-5}\partial^2);\end{aligned}$$

hence we obtain

$$\begin{aligned}(5.3)\qquad \partial^5_{\widetilde{z}} &= J^{-5}\partial^5 - 10J'J^{-6}\partial^4 + (-10J''J^{-6} + 45(J')^2J^{-7})\partial^3 \\ &\quad + (60J''J'J^{-7} - 105(J')^3J^{-8} - 5J'''J^{-6})\partial^2 \\ &\quad + (-105J''(J')^2J^{-8} + 105(J')^4J^{-9} + 15J'''J'J^{-7} \\ &\qquad + 10(J'')^2J^{-7} - J^{(4)}J^{-6})\partial\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\frac{d}{d\widetilde{z}}(J^5 f) &= J^{-1}(J^5 f)' = J^{-1}(5J^4 J' f + J^5 f') = 5J^3 J' f + J^4 f',\\
\frac{d^2}{d\widetilde{z}^2}(J^5 f) &= J^{-1}(5J^3 J' f + J^4 f')'\\
&= J^{-1}\big[(15J^2(J')^2 + 5J^3 J'')f + 5J^3 J' f' + 4J^3 J' f' + J^4 f''\big]\\
&= (15(J')^2 J + 5J'' J^2) f + 9J' J^2 f' + J^3 f'',\\
\frac{d^3}{d\widetilde{z}^3}(J^5 f) &= J^{-1}\big[(15(J')^2 J + 5J'' J^2) f + 9J' J^2 f' + J^3 f''\big]'\\
&= J^{-1}\big[(30J'' J' J + 15(J')^3 + 5J''' J^2 + 10J'' J' J) f\\
&\qquad + (15(J')^2 J + 5J'' J^2) f' + (9J'' J^2 + 18(J')^2 J) f'\\
&\qquad + 9J' J^2 f'' + 3J' J^2 f'' + J^3 f'''\big]\\
&= (40J'' J' + 15(J')^3 J^{-1} + 5J''' J) f\\
&\qquad + (33(J')^2 + 14J'' J) f' + 12J' J f'' + J^2 f''',\\
\frac{d^4}{d\widetilde{z}^4}(J^5 f) &= J^{-1}\big[(40J'' J' + 15(J')^3 J^{-1} + 5J''' J) f\\
&\qquad + (33(J')^2 + 14J'' J) f' + 12J' J f'' + J^2 f'''\big]'\\
&= J^{-1}\big[(40J''' J' + 40(J'')^2 + 45J''(J')^2 J^{-1}\\
&\qquad - 15(J')^4 J^{-2} + 5J^{(4)} J + 5J''' J') f\\
&\qquad + (40J'' J' + 15(J')^3 J^{-1} + 5J''' J) f'\\
&\qquad + (66J'' J' + 14J''' J + 14J'' J') f' + (33(J')^2 + 14J'' J) f''\\
&\qquad + (12J'' J + 12(J')^2) f'' + 12J' J f''' + 2J' J f''' + J^2 f^{(4)}\big]\\
&= (45J''' J' J^{-1} + 40(J'')^2 J^{-1} + 45J''(J')^2 J^{-2} - 15(J')^4 J^{-3}\\
&\qquad + 5J^{(4)}) f + (120J'' J' J^{-1} + 15(J')^3 J^{-2} + 19J''') f'\\
&\qquad + (45(J')^2 J^{-1} + 26J'') f'' + 14J' f''' + J f^{(4)}.
\end{aligned}$$

Combining these relations with (5.1), (5.2) and (5.3) as well as (3.15), (3.20) and (4.3), we obtain

$$\begin{aligned}
&\frac{1}{30}(\Lambda^{\widetilde{\mathfrak{P}}}_{-5} - \Lambda^{\mathfrak{P}}_{-5})(f(dz)^{-5})\\
&\quad = 42J^5 f(J^{-5}\partial^5 - 10J' J^{-6}\partial^4 + (-10J'' J^{-6} + 45(J')^2 J^{-7})\partial^3\\
&\qquad + (60J'' J' J^{-7} - 105(J')^3 J^{-8} - 5J''' J^{-6})\partial^2\\
&\qquad + (-105J''(J')^2 J^{-8} + 105(J')^4 J^{-9} + 15J''' J' J^{-7}
\end{aligned}$$

$$
\begin{aligned}
&+ 10(J'')^2J^{-7} - J^{(4)}J^{-6})\partial \\
&+ 84(5J^3J'f + J^4f')(J^{-4}\partial^4 - 6J'J^{-5}\partial^3 + \left[15(J')^2J^{-6} - 4J''J^{-5}\right]\partial^2 \\
&\qquad + \left[10J''J'J^{-6} - 15(J')^3J^{-7} - J'''J^{-5}\right]\partial) \\
&+ 56((15(J')^2J + 5J''J^2)f + 9J'J^2f' + J^3f'') \\
&\qquad \times (J^{-3}\partial^3 - 3J'J^{-4}\partial^2 + (3(J')^2 - JJ'')J^{-5}\partial) \\
&+ 14((40J''J' + 15(J')^3J^{-1} + 5J'''J)f \\
&+ (33(J')^2 + 14J''J)f' + 12J'Jf'' + J^2f''')(J^{-2}\partial^2 - J'J^{-3}\partial) \\
&+ ((45J'''J'J^{-1} + 40(J'')^2J^{-1} + 45J''(J')^2J^{-2} - 15(J')^4J^{-3} + 5J^{(4)})f \\
&\qquad + (120J''J'J^{-1} + 15(J')^3J^{-2} + 19J''')f' \\
&\qquad + (45(J')^2J^{-1} + 26J'')f'' + 14J'f''' + Jf^{(4)})(J^{-1}\partial)
\end{aligned}
$$

Thus, if we write

$$
\frac{1}{30}(\Lambda^{\widetilde{\mathfrak{P}}}_{-5} - \Lambda^{\mathfrak{P}}_{-5})(f(dz)^{-5}) = C_5\partial^5 + C_4\partial^4 + C_3\partial^3 + C_2\partial^2 + C_1\partial,
$$

then the coefficients $C_1, \ldots, C_5$ can be written in the form

$$
C_5 = 42f - 42f = 0,
$$

$$
C_4 = -420J'J^{-1}f + 420J' + 84f' - 84f = 0,
$$

$$
\begin{aligned}
C_3 &= -420J''J^{-1}f + 1890(J')^2J^{-2}f - 2520(J')^2J^{-2}f - 504J'J^{-1}f' \\
&\quad + 840(J')^2J^{-2}f + 280J''J^{-1}f + 504J'J^{-1}f' + 56f'' - 56f'' \\
&= 210(J')^2J^{-2}f - 140J''J^{-1}f = 70\mathcal{S}^{\tilde{z}}_z f,
\end{aligned}
$$

$$
\begin{aligned}
C_2 &= 2520J''J'J^{-2}f - 4410(J')^3J^{-3}f - 210J'''J^{-1}f + 6300(J')^3J^{-3}f \\
&\quad + 1260(J')^2J^{-2}f' - 1680J''J'J^{-2}f - 336J''J^{-1}f' - 2520(J')^3J^{-3}f \\
&\quad - 840J''J'J^{-2}f - 1512(J')^2J^{-2}f' - 168J'J^{-1}f'' + 560J''J'J^{-2}f \\
&\quad + 210(J')^3J^{-3}f + 70J'''J^{-1}f + 462(J')^2J^{-2}f' + 196J''J^{-1}f' \\
&\quad + 168J'J^{-2}f'' + 14f''' - 14f''' \\
&= 560J''J'J^{-2}f - 420(J')^3J^{-3}f - 140J'''J^{-1}f \\
&\qquad + 210(J')^2J^{-2}f' - 140J''J^{-1}f' \\
&= 70((\mathcal{S}^{\tilde{z}}_z)'f + \mathcal{S}^{\tilde{z}}_z f') = 70(\mathcal{S}^{\tilde{z}}_z f)',
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= -4410J''(J')^2J^{-3}f + 4410(J')^4J^{-4}f + 630J'''J'J^{-2}f \\
&\quad + 420(J'')^2J^{-2}f - 42J^{(4)}J^{-1}f + 4200J''(J')^2J^{-3}f
\end{aligned}
$$

$$
\begin{aligned}
&- 6300(J')^4J^{-4}f - 420J'''J'J^{-2}f + 840J''J'J^{-2}f' \\
&- 1260(J')^3J^{-3}f' - 84J'''J^{-1}f' + 2520(J')^4J^{-4}f \\
&+ 840J''(J')^2J^{-3}f + 1512(J')^3J^{-3}f' + 168(J')^2J^{-2}f'' \\
&- 840J''(J')^2J^{-3}f - 280(J'')^2J^{-2}f - 504J''J'J^{-2}f' \\
&- 56J''J^{-1}f'' - 560J''(J')^2J^{-3}f - 210(J')^4J^{-4}f \\
&- 70J'''J'J^{-2}f - 462(J')^3J^{-3}f' - 196J''J'J^{-2}f' \\
&- 168(J')^2J^{-2}f'' - 14J'J^{-1}f''' + 45J'''J'J^{-2}f + 40(J'')^2J^{-2}f \\
&+ 45J''(J')^2J^{-3}f - 15(J')^4J^{-4}f + 5J^{(4)}J^{-1}f + 120J''J'J^{-2}f' \\
&+ 15(J')^3J^{-3}f' + 19J'''J^{-1}f' + 45(J')^2J^{-2}f'' + 26J''J^{-1}f'' \\
&+ 14J'J^{-1}f''' + f^{(4)} - f^{(4)} \\
= &\big[-725J''(J')^2J^{-3} + 405(J')^4J^{-4} + 185J'''J'J^{-2} \\
&\qquad + 180(J'')^2J^{-2} - 37J^{(4)}J^{-1}\big]f \\
&+ \big[260J''J'J^{-2} - 195(J')^3J^{-3} - 65J'''J^{-1}\big]f' \\
&+ \big[45(J')^2J^{-2} - 30J''J^{-1}\big]f'' \\
= &\,37\big[5J'''J'J^{-2} + 4(J'')^2J^{-2} - 17J''(J')^2J^{-3} \\
&\qquad + 9(J')^4J^{-4} - J^{(4)}J^{-1}\big]f \\
&+ 8\big[9(J')^4J^{-4} - 12J''(J')^2J^{-3} + 4(J'')^2J^{-2}\big]f \\
&+ 65\big[4J''J'J^{-2} - 3(J')^3J^{-3} - J'''J^{-1}\big]f' \\
&+ 15\big[3(J')^2J^{-2} - 2J''J^{-1}\big]f''.
\end{aligned}
$$

If $\mathcal{S}_z^{\widetilde{z}}$ is as in (3.14), we have

$$
\begin{aligned}
(\mathcal{S}_z^{\widetilde{z}})^2 &= \frac{1}{4}(3(J')^2J^{-2} - 2J''J^{-1})^2 \\
&= \frac{1}{4}(9(J')^4J^{-4} - 12J''(J')^2J^{-3} + 4(J'')^2J^{-2}).
\end{aligned}
$$

On the other hand, using (4.4), we obtain

$$
\begin{aligned}
(\mathcal{S}_z^{\widetilde{z}})'' &= J^{(4)}J^{-1} - J'''J'J^{-2} + 9J''(J')^2J^{-3} - 9(J')^4J^{-4} \\
&\qquad - 4J'''J'J^{-2} - 4(J'')^2J^{-2} + 8J''(J')^2J^{-3} \\
&= -5J'''J'J^{-2} - 4(J'')^2J^{-2} + 17J''(J')^2J^{-3} - 9(J')^4J^{-4} + J^{(4)}J^{-1}).
\end{aligned}
$$

Thus it follows that

$$
-\frac{1}{30}(\Lambda^{\widetilde{\mathfrak{P}}}_{-5} - \Lambda^{\mathfrak{P}}_{-5})(f(dz)^{-5})
$$

$$= 70\mathcal{S}_z^{\widetilde{z}} f\partial^3 + 70(\mathcal{S}_z^{\widetilde{z}} f)'\partial^2 + (37(\mathcal{S}_z^{\widetilde{z}})'' f$$
$$+ 16(\mathcal{S}_z^{\widetilde{z}})^2 f + 65(\mathcal{S}_z^{\widetilde{z}})' f' + 30\mathcal{S}_z^{\widetilde{z}} f'')\partial.$$

Hence we obtain the formula (3.22), and the proof of Theorem 3.6 is complete.

6. Concluding remarks

In the previous sections we obtained expressions of $(\Lambda_m^{\widetilde{\mathfrak{P}}} - \Lambda_m^{\mathfrak{P}})(f(dz)^m)$ in terms of $f \in \mathcal{O}_X(U \cap \widetilde{U})$ and the Schwarzian derivative of the coordinate transformation map for

$$m = 0, -1, -2, -3, -4, -5.$$

When $m = -6$, from (3.9) we obtain

$$(\Lambda_{-6}^{\widetilde{\mathfrak{P}}} - \Lambda_{-6}^{\mathfrak{P}})(f(dz)^{-6})$$
$$= \sum_{n=0}^{5} \frac{(12-n)!}{n!(6-n)!(5-n)!} \Big[((\partial_{\widetilde{z}}^n (J^6 f)\partial_{\widetilde{z}}^{6-n} - (\partial^n f)\partial^{6-n}\Big]$$

with $\partial_{\widetilde{z}} = J^{-1}\partial$. Thus, in addition to $\partial_{\widetilde{z}}^{\ell}$ for $0 \le \ell \le 5$ used in Section 5, we also need to calculate $\partial_{\widetilde{z}}^6 = J^{-1}\partial\partial_{\widetilde{z}}^5$ as well as the derivatives

$$\partial_{\widetilde{z}}^n (J^6 f) = \Big(J^{-1}\frac{d}{dz}\Big)^n (J^6 f)$$

for $1 \le n \le 6$. The computations would be much more complicated than the ones we had in Section 5. It does not seem likely that the general case can be calculated inductively.

As is mentioned in Question 3.4, Cohen, Manin and Zagier [2] asks if $(\Lambda_m^{\widetilde{\mathfrak{P}}} - \Lambda_m^{\mathfrak{P}})(f(dz)^m)$ could be expressed essentially as multiplication by $\widetilde{\mathfrak{P}} - \mathfrak{P} = \mathcal{S}_z^{\widetilde{z}}(dz)^2$ in a suitable sense for all $m \in \mathbb{Z}$. For $m > 0$, they also mention that $\Lambda_1^{\widetilde{\mathfrak{P}}} - \Lambda_1^{\mathfrak{P}} = 0$, so that their claim holds trivially at least for $m = 1$. Note, however, that our method in this paper cannot apply to the cases of $m > 0$, even to the simplest case of $m = 1$.

References

[1] Y. Choie and M. H. Lee, *Rankin-Cohen brackets on pseudodifferential operators*, J. Math. Anal. Appl. **326** (2007), 882–895.

[2] P. B. Cohen, Y. Manin, and D. Zagier, *Automorphic pseudodifferential operators*, Algebraic Aspects of Nonlinear Systems, Birkhäuser, Boston, 1997, pp. 17–47.

[3] L. Dickey, *Soliton equations and Hamiltonian systems*, World Scientific, Singapore, 1991.

[4] R. Gunning, *Lectures on Riemann surfaces*, Princeton Univ. Press, Princeton, 1966.

[5] I. Kra, *Deformations of fuchsian groups*, Duke Math. J. **36** (1969), 537–546.

[6] M. H. Lee, *Hilbert modular pseudodifferential operators*, Proc. Amer. Math. Soc. **129** (2001), 3151–3160.
[7] M. H. Lee, *Automorphic pseudodifferential operators and vector bundles*, Complex Variables Theory Appl. **48** (2003), 937–945.
[8] D. Zagier, *Modular forms and differential operators*, Proc. Indian Acad. Sci. Math. Sci. **104** (1994), 57–75.

Min Ho Lee
Department of Mathematics
University of Northern Iowa
Cedar Falls, IA 50614, U.S.A.
e-mail address: lee@math.uni.edu

(*Received January 22, 2009*)
(*Revised April 1, 2010*)