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A CAUCHY-KOWALEVSKI THEOREM FOR INFRAMONOGENIC FUNCTIONS

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Abstract. In this paper we prove a Cauchy-Kowalevski theorem for the functions satisfying the system $\partial_x f \partial_x = 0$ (called inframonogenic functions).

1. Introduction

Let $\mathbb{R}_{0,m}$ be the 2^m-dimensional real Clifford algebra constructed over the orthonormal basis $(e_1, \dots, e_m)$ of the Euclidean space $\mathbb{R}^m$ (see [3]). The multiplication in $\mathbb{R}_{0,m}$ is determined by the relations $e_j e_k + e_k e_j = -2\delta_{jk}$, $j, k = 1, \dots, m$, where δ_{jk} is the Kronecker delta. A general element of $\mathbb{R}_{0,m}$ is of the form

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R},$$

where for $A = \{j_1, \dots, j_k\} \subset \{1, \dots, m\}$, $j_1 < \dots < j_k$, $e_A = e_{j_1} \dots e_{j_k}$. For the empty set $\emptyset$, we put $e_\emptyset = 1$, the latter being the identity element.

Notice that any $a \in \mathbb{R}_{0,m}$ may also be written as $a = \sum_{k=0}^m [a]_k$ where $[a]_k$ is the projection of a on $\mathbb{R}_{0,m}^{(k)}$. Here $\mathbb{R}_{0,m}^{(k)}$ denotes the subspace of k-vectors defined by

$$\mathbb{R}_{0,m}^{(k)} = \left\{ a \in \mathbb{R}_{0,m} : \ a = \sum_{|A|=k} a_A e_A, \quad a_A \in \mathbb{R} \right\}.$$

Observe that $\mathbb{R}^{m+1}$ may be naturally identified with $\mathbb{R}_{0,m}^{(0)} \oplus \mathbb{R}_{0,m}^{(1)}$ by associating to any element $(x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1}$ the "paravector" $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^m x_j e_j$.

Conjugation in $\mathbb{R}_{0,m}$ is given by

$$\overline{a} = \sum_A a_A \overline{e}_A,$$

where $\overline{e}_A = \overline{e}_{j_k} \dots \overline{e}_{j_1}$, $\overline{e}_j = -e_j$, $j = 1, \dots, m$. One easily checks that $\overline{ab} = \overline{b}\,\overline{a}$ for any $a, b \in \mathbb{R}_{0,m}$. Moreover, by means of the conjugation a norm

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$|a|$ may be defined for each $a \in \mathbb{R}_{0,m}$ by putting

$$|a|^2 = [a\overline{a}]_0 = \sum_A a_A^2.$$

Let us denote by $\partial_x = \partial_{x_0} + \partial_{\underline{x}} = \partial_{x_0} + \sum_{j=1}^m e_j \partial_{x_j}$ the generalized Cauchy-Riemann operator and let Ω be an open set of $\mathbb{R}^{m+1}$. According to [11], an $\mathbb{R}_{0,m}$-valued function $f \in C^2(\Omega)$ is called an inframonogenic function in Ω if and only if it fulfills in Ω the "sandwich" equation $\partial_x f \partial_x = 0$.

It is obvious that monogenic functions (i.e. null-solutions of ∂_x) are inframonogenic. At this point it is worth mentioning that the monogenic functions are the central object of study in Clifford analysis (see [2, 4, 5, 7, 8, 9, 10, 14]). Furthermore, the concept of monogenicity of a function may be seen as the higher dimensional counterpart of holomorphy in the complex plane.

Moreover, as

$$\Delta_x = \sum_{j=0}^m \partial_{x_j}^2 = \partial_x \overline{\partial}_x = \overline{\partial}_x \partial_x,$$

every inframonogenic function $f \in C^4(\Omega)$ satisfies in Ω the biharmonic equation $\Delta_x^2 f = 0$ (see e.g. [1, 6, 12, 15]).

This paper is intended to study the following Cauchy-type problem for the inframonogenic functions. Given the functions $A_0(\underline{x})$ and $A_1(\underline{x})$ analytic in an open and connected set $\underline{\Omega} \subset \mathbb{R}^m$, find a function $F(x)$ inframonogenic in some open neighbourhood $\widetilde{\Omega}$ of $\underline{\Omega}$ in $\mathbb{R}^{m+1}$ which satisfies

$$F(x)|_{x_0=0} = A_0(\underline{x}), \tag{1.1}$$

$$\partial_{x_0} F(x)|_{x_0=0} = A_1(\underline{x}). \tag{1.2}$$

2. Cauchy-type problem for inframonogenic functions

Consider the formal series

$$F(x) = \sum_{n=0}^{\infty} x_0^n A_n(\underline{x}). \tag{2.1}$$

It is clear that F satisfies conditions (1.1) and (1.2). We also see at once that

$$\partial_x \left(x_0^n A_n\right) \partial_x = n(n-1)x_0^{n-2} A_n + n x_0^{n-1} \big(\partial_{\underline{x}} A_n + A_n \partial_{\underline{x}}\big) + x_0^n \partial_{\underline{x}} A_n \partial_{\underline{x}}.$$

We thus get

$$\partial_x F \partial_x = \sum_{n=0}^{\infty} x_0^n \Big((n+2)(n+1) A_{n+2} + (n+1)\big(\partial_{\underline{x}} A_{n+1} + A_{n+1}\partial_{\underline{x}}\big) + \partial_{\underline{x}} A_n \partial_{\underline{x}}\Big).$$

From the above it follows that F is inframonogenic if and only if the functions A_n satisfy the recurrence relation

$$A_{n+2} = -\frac{1}{(n+2)(n+1)}\Big((n+1)\big(\partial_{\underline{x}} A_{n+1} + A_{n+1}\partial_{\underline{x}}\big) + \partial_{\underline{x}} A_n \partial_{\underline{x}}\Big), \quad n \geq 0.$$

It may be easily proved by induction that

$$A_n = \frac{(-1)^{n+1}}{n!}\left(\sum_{j=0}^{n-2} \partial_{\underline{x}}^{n-j-1} A_0 \partial_{\underline{x}}^{j+1} + \sum_{j=0}^{n-1} \partial_{\underline{x}}^{n-j-1} A_1 \partial_{\underline{x}}^{j}\right), \quad n \geq 2. \tag{2.2}$$

We now proceed to examine the convergence of the series (2.1) with the functions A_n $(n \geq 2)$ given by (2.2). Let $\underline{y}$ be an arbitrary point in $\underline{\Omega}$. Then there exist a ball $B\big(\underline{y}, R(\underline{y})\big)$ of radius $R(\underline{y})$ centered at $\underline{y}$ and a positive constant $M(\underline{y})$, such that

$$\big|\partial_{\underline{x}}^{n-j} A_s(\underline{x}) \partial_{\underline{x}}^{j}\big| \leq M(\underline{y})\frac{n!}{R^n(\underline{y})}, \;\; \underline{x} \in B\big(\underline{y}, R(\underline{y})\big), \;\; j = 0, \ldots, n, \;\; s = 0, 1.$$

It follows that

$$|A_n(\underline{x})| \leq M(\underline{y})\frac{n + R(\underline{y}) - 1}{R^n(\underline{y})}, \quad \underline{x} \in B\big(\underline{y}, R(\underline{y})\big),$$

and therefore the series (2.1) converges normally in

$$\widetilde{\Omega} = \bigcup_{\underline{y}\in\underline{\Omega}} \big(-R(\underline{y}), R(\underline{y})\big) \times B\big(\underline{y}, R(\underline{y})\big).$$

Note that $\widetilde{\Omega}$ is a x_0-normal open neighbourhood of $\underline{\Omega}$ in $\mathbb{R}^{m+1}$, i.e. for each $x \in \widetilde{\Omega}$ the line segment $\{x + t : t \in \mathbb{R}\} \cap \widetilde{\Omega}$ is connected and contains one point in $\underline{\Omega}$.

We thus have proved the following.

Theorem 2.1. *The function* $\mathsf{CK}[A_0, A_1]$ *given by*

$$\begin{aligned} \mathsf{CK}[A_0, A_1](x) &= A_0(\underline{x}) + x_0 A_1(\underline{x}) \\ &\quad - \sum_{n=2}^{\infty} \frac{(-x_0)^n}{n!}\left(\sum_{j=0}^{n-2} \partial_{\underline{x}}^{n-j-1} A_0(\underline{x}) \partial_{\underline{x}}^{j+1} + \sum_{j=0}^{n-1} \partial_{\underline{x}}^{n-j-1} A_1(\underline{x}) \partial_{\underline{x}}^{j}\right) \end{aligned} \tag{2.3}$$

is inframonogenic in a x_0*-normal open neighbourhood of* $\underline{\Omega}$ *in* $\mathbb{R}^{m+1}$ *and satisfies conditions (1.1)-(1.2).*

It is worth noting that if in particular $A_1(\underline{x}) = -\partial_{\underline{x}} A_0(\underline{x})$, then

$$\mathsf{CK}[A_0, -\partial_{\underline{x}} A_0](x) = \sum_{n=0}^{\infty} \frac{(-x_0)^n}{n!}\, \partial_{\underline{x}}^n A_0(\underline{x}),$$

which is nothing else but the left monogenic extension (or CK-extension) of $A_0(\underline{x})$. Similarly, it is easy to see that $\mathsf{CK}[A_0, -A_0\partial_{\underline{x}}](x)$ yields the right monogenic extension of $A_0(\underline{x})$ (see [2, 5, 13, 16, 17]).

Let $\mathsf{P}(k)$ ($k \in \mathbb{N}_0$ fixed) denote the set of all $\mathbb{R}_{0,m}$-valued homogeneous polynomials of degree k in $\mathbb{R}^m$. Let us now take $A_0(\underline{x}) = P_k(\underline{x}) \in \mathsf{P}(k)$ and $A_1(\underline{x}) = P_{k-1}(\underline{x}) \in \mathsf{P}(k-1)$. Clearly,

$$\begin{aligned}\mathsf{CK}[P_k, P_{k-1}](x) &= P_k(\underline{x}) + x_0 P_{k-1}(\underline{x}) \\ &\quad - \sum_{n=2}^{k} \frac{(-x_0)^n}{n!} \left(\sum_{j=0}^{n-2} \partial_{\underline{x}}^{n-j-1} P_k(\underline{x}) \partial_{\underline{x}}^{j+1} + \sum_{j=0}^{n-1} \partial_{\underline{x}}^{n-j-1} P_{k-1}(\underline{x}) \partial_{\underline{x}}^{j} \right),\end{aligned}$$

since the other terms in the series (2.3) vanish. Moreover, we can also claim that $\mathsf{CK}[P_k, P_{k-1}](x)$ is a homogeneous inframonogenic polynomial of degree k in $\mathbb{R}^{m+1}$.

Conversely, if $P_k(x)$ is a homogeneous inframonogenic polynomial of degree k in $\mathbb{R}^{m+1}$, then $P_k(x)|_{x_0=0} \in \mathsf{P}(k)$, $\partial_{x_0} P_k(x)|_{x_0=0} \in \mathsf{P}(k-1)$ and obviously $\mathsf{CK}[P_k|_{x_0=0}, \partial_{x_0} P_k|_{x_0=0}](x) = P_k(x)$.

Call $\mathsf{I}(k)$ the set of all homogeneous inframonogenic polynomials of degree k in $\mathbb{R}^{m+1}$. Then $\mathsf{CK}[.,.]$ establishes a bijection between $\mathsf{P}(k) \times \mathsf{P}(k-1)$ and $\mathsf{I}(k)$.

It is easy to check that

$$P_k(\underline{x}) = P_k(\partial_{\underline{u}}) \frac{\langle \underline{x}, \underline{u} \rangle^k}{k!}, \quad P_k(\underline{x}) \in \mathsf{P}(k),$$

where $P_k(\partial_{\underline{u}})$ is the differential operator obtained by replacing in $P_k(\underline{u})$ each variable u_j by ∂_{u_j}. Therefore, in order to characterize $\mathsf{I}(k)$, it suffices to calculate $\mathsf{CK}\big[\langle \underline{x}, \underline{u} \rangle^k e_A, 0\big]$ and $\mathsf{CK}\big[0, \langle \underline{x}, \underline{u} \rangle^{k-1} e_A\big]$ with $\underline{u} \in \mathbb{R}^m$.

A simple computation shows that

$$\begin{aligned}\mathsf{CK}\big[\langle \underline{x}, \underline{u} \rangle^k e_A, 0\big](x) &= \langle \underline{x}, \underline{u} \rangle^k e_A \\ &\quad - \sum_{n=2}^{k} \binom{k}{n} (-x_0)^n \langle \underline{x}, \underline{u} \rangle^{k-n} \left(\sum_{j=0}^{n-2} \underline{u}^{n-j-1} e_A \underline{u}^{j+1} \right),\end{aligned}$$

$$\begin{aligned}\mathsf{CK}\big[0, \langle \underline{x}, \underline{u} \rangle^{k-1} e_A\big](x) &= x_0 \langle \underline{x}, \underline{u} \rangle^{k-1} e_A \\ &\quad - \frac{1}{k} \sum_{n=2}^{k} \binom{k}{n} (-x_0)^n \langle \underline{x}, \underline{u} \rangle^{k-n} \left(\sum_{j=0}^{n-1} \underline{u}^{n-j-1} e_A \underline{u}^{j} \right).\end{aligned}$$

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