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A Note on Duality in Inequality Systems

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1. Introduction

Four decades ago, Tucker (1956) presented an elegant theorem on systems of linear inequalities. That is,

Theorem. (i) If $Ax \geqq 0$ has no solution $x^* \in R^n_+ - \{0\}$, then $pA < 0$ has a solution $p^* \in R^m_+$.

(ii) If $Ax > 0$ has no solution $x^* \in R^n_+$, then $pA \leqq 0$ has a solution $p^* \in R^m_+ - \{0\}$.

In this theorem, A is a given m by n real matrix, x a column n–vector, and p is a row m–vector, while R^n_+ and R^m_+ stand for the nonnegative orthant of the n– and m–dimensional Euclidean spaces respectively.

Though the above theorem can be derived from a more general result in Tucker (1956, Corollary 3 A), it has a simple economic interpretation : (i) if the linear economic system is not reproducible, there exists a price vector under which every process makes losses. Fujimoto (1980) extended the part (i) of the theorem to the case in which the mapping concerned is not necessarily linear but pseudoconcave or homogeneous of a positive degree. Fujimoto–Ranade (1995) considers a proposition due to Okishio (1961) concerning the effect of cost–reducing technical changes on the uniform rate of profit in a model with joint production, externalities and variable returns, and obtain a nonlinear generalization of the part (ii). The proof, however, is based on the assumption that a given map is *concave,* thus lacking in symmetry.

In this note, we prove a nonlinear generalization of the part (ii) assuming the pseudoconcavity. The method is the same to that in Fujimoto (1980).

2. Proposition

We use the same symbol both for a column vector and for its transpose. Thus, xy or $x \cdot y$ for $x,y \in R^n$ means the inner product of x and y. For vector comparison, $x^1 < x^2$ means the strict inequality in each componentwise comparison. Let $F(x)$ be a vector–valued function from R^n_+ to R^m_+ : $F(x) \equiv (F_1(x), \ldots, F_m(x))$. A function $g(x)$ is a function from R^n_+ to R. The domain D is defined as $R^n_+ - \{0\}$. We make the following assumptions.

(A 1) Each $F_i(x)$ is continuously differentiable, and $g(x)$ is differentiable on the domain D.

The $m \times n$ Jacobian of F is denoted by $\nabla F(x)$. Let $\nabla F_i(x)$ be the i–th row of $\nabla F(x)$. The gradient of g is written as $\nabla g(x)$.

(A 2) For each i, if $F_i(x) < 0$ at some $x \in D$, then $\nabla F_i(x) \cdot x < 0$ at the same point x.

(A 3) $g(0) = 0$. $\nabla g(x) > 0$ for any $x \in D$. The set $S \equiv \{x | x \in R^n_+, g(x) = 1\}$ is compact.

(A 4) For any i, if $F_i(x^\circ) = 0$ for $x^\circ \in S$, and $\nabla F_i(x_k) \cdot x_k > 0$ for any vector sequence $\{x_k\} \rightarrow x^\circ$, then $\lim_{x_k \rightarrow x^\circ} (\nabla F_i(x_k) \cdot x_k / F_i(x_k))$ is uniformly finite on S.

We consider a vector–valued function $H(x)$ whose i–th element is

$$H_i(x) \equiv F_i(x) - \varepsilon \cdot Q(x),$$

where ε is an arbitrary positive scalar, and $Q(x) = \sum_{j=1}^{n} x_j^r$ with r being a positive integer. Note that

$$\nabla H_i(x)\cdot x = \nabla F_i(x)\cdot x - r\cdot\varepsilon\cdot Q(x).$$

As in Fujimoto (1980), we define

$$M(x) \equiv \sum_{j=1}^{m} G_i(x)^2,$$

where $G_i(x) \equiv \min\{H_i(x), 0\}$. We also define

$$G(x) \equiv (G_l(x), \cdots, G_m(x)).$$

Proposition. Given the assumptions (A 1)–(A 3), if the system of inequalities $F(x) > 0$ has no solution on S, then there exist a vector $p \in R^m_+ - \{0\}$ and an $x^* \in S$ such that $p\nabla F(x^*) \leqq 0$.

3. Proof

We can proceed as in the proof of Theorem 3.1 in Fujimoto (1980) because the system $H(x) \geqq 0$ has no solution on S. Thus, we consider the :

(Problem P) minimize $M(x)$ subject to $x \in S$.

At an optimal point x^*, we have the Lagrangian multiplier λ^* such that

$$2G(x^*)\cdot\nabla H(x^*) - \lambda^*\nabla g(x^*) \geqq 0 \tag{1}$$

$$2G(x^*)\cdot\nabla H(x^*)\cdot x^* - \lambda^*\nabla g(x^*)\cdot x^* = 0. \tag{2}$$

The point is to prove the positivity of the Lagrangian multiplier λ^* when $H(x)$ may not satisfy (A 2).

First let us take up the case in which we have $F_i(x^*) \leqq 0$ when $H(x^*) < 0$. Exactly the same proof applies, and the limit procedure $\varepsilon \to 0$ will yield the desired result.

Next, suppose the index set $E(x^*) \equiv \{i | \nabla F_i(x^*)\cdot x^* > 0\}$ is not empty. In a more general context, we define a positive integer r^* as

$$r^* \equiv [\max_{x\in S} \max_{i\in E(x)}\{\nabla F_i(x)\cdot x / F_i(x)\}],$$

where $[z]$ is the Gaussian notation meaning the minimum integer not less than z.This r is well defined because of (A 4). Now if r in the definition of $Q(x)$ is designated as the value greater than r^* here obtained, $\nabla H_i(x) \cdot x < 0$ when $H_i(x) < 0$. This is because

$$\nabla H_i(x) \cdot x = \nabla F_i(x) \cdot x - r \cdot \varepsilon \cdot Q(x) < r \cdot F(x) - r \cdot \varepsilon \cdot Q(x) = rH_i(x).$$

Therefore the same proof as the one in Fujimoto (1980) again applies even in this case.

4. Remarks

The problem dealt with in this note is small enough, and yet has been on my mind since 1980. When I first succeeded in extending the part (i) of Tucker's theorem to the case of pseudoconcavity or homogeneity, it seemed not very difficult to generalize the part (ii). Back and forth, occasionally I returned to this problem, and always failed. So, in Fujimoto–Ranade (1995), we made a compromise, assuming concavity to establish a nonlinear version of the part (ii). Then still unsatisfied, I continued to work on this problem, and reached the proposition of this note.

I am not satisfied yet with the assumption (A 4). Some trick on the functional form of $Q(x)$ may allow us to drop (A 4), or the very property of pseudoconcavity may make (A 4) redundant.

The use of separation theorem may dispense with the differentiability assumption altogether, and quasiconcavity will suffice.

References

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The purpose of this note is to present a proposition on a sort of duality relation concerning systems of nonlinear inequalities. A useful theorem due to A.W. Tucker(1956) on duality relations valid for systems of linear inequalities was partially generalized to systems of nonlinear inequalities in Fujimoto(1980), in which a given vector–valued mapping is assumed to be either pseudoconcave or homogeneous of a positive degree.

Another partial extension of Tucker's result is made in Fujimoto–Ranade(1995), where the concavity of a given mapping is required. The assumption of concavity seems too strong in a model with externalities and variable returns, and so in this note a weaker assumption of pseudoconcavity is restored with an additional requirement.