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POINT ESTIMATION OF THE PROCESS CAPABILITY INDEX C_{pk}

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Abstract

In this paper point estimation of the process capability index C_{pk} is considered. In order to improve on the natural estimator, a simple alternative estimator is proposed and its mean square error is evaluated numerically.

1 Introduction

Let $X_1, X_2, \cdots, X_n$ be independent random variables from a normal distribution with unknown mean ξ and unknown variance σ^2. Then, the process capability index (PCI) C_{pk} is defined as

$$\begin{aligned} C_{pk} &= min\left\{\frac{USL-\xi}{3\sigma}, \frac{\xi-LSL}{3\sigma}\right\} \\ &= \frac{d-|\xi-m|}{3\sigma}, \end{aligned} \tag{1}$$

where $d=(USL-LSL)/2$, $m=(LSL+USL)/2$, and LSL and USL are lower and upper specification limits, respectively. The values of LSL and USL are known. This PCI is frequently used in the activities of the quality control in order to evaluate whether the processes are satisfactory or not.

Althouth other types of definitions for PCI's have been proposed and a lot of studies on these PCI's have been made (see Kotz and Johnson(1993)), in this paper we will deal with C_{pk} alone. Because C_{pk} is one of the most familiar PCI's for the practitioners, especially for the engineers of Japanese industrial companies.

Suppose that we are interested in the point estimation of C_{pk} in terms of *the mean square error (MSE) criterion*. Since ξ and σ are unknown parameters, they must be estimated from data. Defining $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $\hat{\sigma} = \sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)}$, a natural estimator of C_{pk} is

$$\hat{C}_{pk} = \frac{d - |\bar{X} - m|}{3\hat{\sigma}}. \tag{2}$$

Chou and Owen(1989) derived the *MSE* of $\hat{C}_{pk}$. Kotz and Johnson(1993) evaluated the mean and the standard deviation of $\hat{C}_{pk}$ numerically.

In this paper we consider a simple alternative estimator for C_{pk} in order to improve on the natural estimator $\hat{C}_{pk}$ and evaluate their *MSE*'s numerically.

2 Mean square error of the natural estimator

In this section we introduce several notations and summarize known results.

Define $z = \sqrt{n}(\bar{X} - m)/\sigma$ and let χ_f be a random variable of chi distribution with degrees of freedom f. Since $\hat{\sigma} = \sigma\chi_{n-1}/\sqrt{n-1}$, the estimator (2) is expressed as

$$\hat{C}_{pk} = \frac{d^* - |z|/\sqrt{n}}{3\chi_f/\sqrt{f}}, \tag{3}$$

where $d^* = d/\sigma$ and $f = n - 1$. In this expression, note that z and χ_f are mutually independent. Define $\delta = (\xi - m)/\sigma$. Since z is distributed as $N(\sqrt{n}\delta, 1)$, the statistic $|z|$ has a folded normal distribution. Elandt(1961) showed that

$$E(|z|) = 2\phi(\sqrt{n}\delta) - \sqrt{n}\delta(1 - 2\Phi(\sqrt{n}\delta)) \equiv \theta, \tag{4}$$

$$E(|z|^2) = 1 + n\delta^2, \tag{5}$$

where ϕ and Φ are the probability density function and the cumulative distribution function of $N(0,1)$, respectively. It can be easily shown that

$$E(\chi_f^{-r}) = \frac{\Gamma((f-r)/2)}{2^{r/2}\Gamma(f/2)}. \tag{6}$$

To simplify the notations let us denote $\tau = \sqrt{n}|\delta|$ and $\zeta = \sqrt{n}d^*$. Then, the *MSE* of the natural estimator $\hat{C}_{pk}$ is

$$\begin{aligned} R(\hat{C}_{pk}, C_{pk}) &= E\{(\hat{C}_{pk} - C_{pk})^2\} \\ &= K\{\zeta^2 - 2\theta\zeta + (1+\tau^2) \\ &\quad -2A^*(\zeta-\tau)(\zeta-\theta) + (f-2)(\zeta-\tau)^2/f\}, \end{aligned} \tag{7}$$

where $K = f/\{9n(f-2)\}$ and $A^* = (f-2)E(\chi_f^{-1})/\sqrt{f}$.

The *MSE* depends on n, δ and d^*. Since θ is an even function of δ, the *MSE* is also an even function of δ with fixed n and d^*. The numerical results for this *MSE* will be given in Section 4.

Before closing this section we give several values of A^* in Table 1. (Kotz and Johnson(1993, p.50) reported the values of $b_f = 1/\{\sqrt{f}\ E(\chi_f^{-1})\}$ for f =4(5)59.)

Table 1: Values of A^*

f	4	9	14	19	24	29	34	39	44	49
A^*	.627	.851	.907	.932	.947	.956	.963	.967	.971	.974

We can observe from Table 1 that A^* increases as f increases and is less than one for at least moderate values of f.

3 Components of C_{pk} and their estimation

The PCI C_{pk} is rewritten as

$$C_{pk} = C_p - \frac{|\delta|}{3}, \tag{8}$$

where $C_p = d^*/3$. C_p is also one of the traditional and familiar PCI's. It should be noted, however, that as C_p does not include the information about population mean ξ it is useful only when ξ coincides with m.

Let us call C_p and $|\delta|/3$ the components of C_{pk}. In this section we consider the estimation of C_p and $|\delta|/3$.

The natural estimators of C_p and $|\delta|/3$ are

$$\hat{C}_p = \frac{\hat{d}^*}{3} = \frac{d}{3\hat{\sigma}}, \tag{9}$$

$$\frac{|\hat{\delta}|}{3} = \frac{|\bar{X} - m|}{3\hat{\sigma}}. \tag{10}$$

In order to improve on the natural estimator $\hat{C}_p$ in terms of the *MSE* criterion, let us consider a constant multiple of $\hat{C}_p$:

$$\phi(A) = A\hat{C}_p = \frac{1}{3}A\sqrt{f}d^*\chi_f^{-1}. \tag{11}$$

Since the *MSE* of $\phi(A)$ is

$$\begin{aligned} R(\phi(A), C_p) &= E\{(\phi(A) - C_p)^2\} \\ &= \frac{fd^{*2}}{9(f-2)}\left\{A^2 - 2A^*A + \frac{f-2}{f}\right\}, \end{aligned} \tag{12}$$

the *MSE* is minimized at $A = A^*$.

Thus, the natural estimator $\phi(1) = \hat{C}_p$ can be improved by $\phi(A^*)$ uniformly. Note that the *MSE* $R(\phi(A), C_p)$ does not depend on δ. Let us define the relative improvement as

$$RI = \frac{R(\hat{C}_p, C_p) - R(\phi(A^*), C_p)}{R(\hat{C}_p, C_p)} \times 100\ (\%). \tag{13}$$

Note that RI depends on f alone.

We give the the values of $R(\hat{C}_p, C_p)$, $R(\phi(A^*), C_p)$ and RI for $f =$ 9(10)49 in Table 2. We can see from Table 2 that RI is significantly large for small f and that RI decreases as f increases.

Next, let us consider the improvement on the natural estimator $|\hat{\delta}|/3$. Consider a constant multiple of $|\hat{\delta}|/3$:

$$\psi(B) = B\frac{|\hat{\delta}|}{3} = \frac{1}{3}B\sqrt{\frac{f}{n}}|z|\chi_f^{-1}. \tag{14}$$

Table 2: Values of $R(\hat{C}_p, C_p)$, $R(\phi(A^*), C_p)$ and RI

f	d^*	$R(\hat{C}_p, C_p)$	$R(\phi(A^*), C_p)$	RI (%)
9	2	.0432	.0305	29.33
	3	.0972	.0687	29.33
	4	.1729	.1222	29.33
	5	.2701	.1909	29.33
	6	.3889	.2749	29.33
19	2	.0152	.0129	15.10
	3	.0341	.0290	15.10
	4	.0607	.0515	15.10
	5	.0948	.0805	15.10
	6	.1365	.1159	15.10
29	2	.0091	.0082	10.18
	3	.0204	.0183	10.18
	4	.0363	.0326	10.18
	5	.0567	.0510	10.18
	6	.0817	.0734	10.18
39	2	.0065	.0060	7.68
	3	.0145	.0134	7.68
	4	.0258	.0239	7.68
	5	.0404	.0373	7.68
	6	.0582	.0537	7.68
49	2	.0050	.0047	6.16
	3	.0113	.0106	6.16
	4	.0201	.0188	6.16
	5	.0313	.0294	6.16
	6	.0451	.0423	6.16

It is shown that the MSE of $\psi(B)$ is

$$R(\psi(B), \frac{|\delta|}{3}) = E\left\{\left(\psi(B) - \frac{|\delta|}{3}\right)^2\right\}$$

Table 3: Values of $R(|\hat{\delta}|/3, |\delta|/3)$, $R(\psi(A^*), |\delta|/3)$ and RI

f	$\|\delta\|$	$R(\|\hat{\delta}\|/3, \|\delta\|/3)$	$R(\psi(A^*), \|\delta\|/3)$	RI (%)
9	0	.0143	.0104	27.57
	0.5	.0151	.0107	29.45
	1.0	.0251	.0180	28.34
	1.5	.0386	.0275	28.68
	2.0	.0575	.0409	28.89
19	0	.0062	.0054	13.12
	0.5	.0069	.0060	13.60
	1.0	.0100	.0086	13.87
	1.5	.0147	.0126	14.27
	2.0	.0214	.0183	14.52
29	0	.0040	.0036	8.60
	0.5	.0045	.0041	8.84
	1.0	.0063	.0057	9.18
	1.5	.0091	.0082	9.49
	2.0	.0131	.0118	9.70
39	0	.0029	.0027	6.40
	0.5	.0033	.0031	6.56
	1.0	.0045	.0042	6.86
	1.5	.0066	.0061	7.11
	2.0	.0094	.0087	7.28
49	0	.0023	.0022	5.10
	0.5	.0026	.0025	5.23
	1.0	.0036	.0034	5.47
	1.5	.0051	.0048	5.68
	2.0	.0073	.0069	5.83

$$= \frac{f}{9n(f-2)}\left\{(1+\tau^2)B^2 - 2\tau\theta A^* B + \frac{(f-2)}{f}\tau^2\right\}. \qquad (15)$$

This MSE is minimized at $B = \tau\theta A^*/(1+\tau^2)(\equiv B^*)$. As B^* depends on the unknown parameter δ, the estimator $\psi(B^*)$ is not available. Noting, however, that $\theta \leq \sqrt{1+\tau^2}$, it is shown that $B^* < A^*$ uniformly. Since $A^* < 1$ from Table 1, it follows that

$$R\left(\psi(B^*), \frac{|\delta|}{3}\right) < R\left(\psi(A^*), \frac{|\delta|}{3}\right) < R\left(\frac{|\hat{\delta}|}{3}, \frac{|\delta|}{3}\right), \tag{16}$$

uniformly. As A^* does not depend on the unknown parameter, the estimator $\psi(A^*)$ is available.

Thus, the natural estimator $\psi(1) = |\hat{\delta}|/3$ can be improved by $\psi(A^*)$ uniformly. Note that the *MSE* $R(\psi(B), |\delta|/3)$ does not depend on d^*. Let us define the relative improvement RI as

$$RI = \frac{R(|\hat{\delta}|/3, |\delta|/3) - R(\psi(A^*), |\delta|/3)}{R(|\hat{\delta}|/3, |\delta|/3)} \times 100\ (\%). \tag{17}$$

We give the values of $R(|\hat{\delta}|/3, |\delta|/3)$, $R(\psi(A^*), |\delta|/3)$ and RI for $f =$ 9(10)49 in Table 3. We can see from Table 3 that RI is significantly large for small f and that RI decreases as f increases.

In this section we have shown that both of the natural estimators for the components of C_{pk} can be improved by some multiples of the natural estimators, respectively and that the multipliers can be set as the same value.

4 An alternative estimator for C_{pk} and its mean square error

In this section we consider an alternative estimator for C_{pk} in order to improve on the natural estimator $\hat{C}_{pk}$.

Let us consider the estimator

$$\rho(A, B) = \phi(A) - \psi(B), \tag{18}$$

where $\phi(A)$ and $\psi(B)$ were defined in Section 3. Note that $\rho(1,1) = \hat{C}_{pk}$. The *MSE* of $\rho(A, B)$ is derived as

$$\begin{aligned} R(\rho(A,B), C_{pk}) &= E\{(\rho(A,B) - C_{pk})^2\} \\ &= K\{\zeta^2 A^2 - 2\theta\zeta AB + (1+\tau^2)B^2 \\ &\quad -2A^*(\zeta - \tau)(\zeta A - \theta B) + (f-2)(\zeta - \tau)^2/f\}, \end{aligned} \tag{19}$$

Table 4: Values of $R(\hat{C}_{pk}, C_{pk})$, $R(A^*\hat{C}_{pk}, C_{pk})$ and RI for f=9.

The first row: values of $R(\hat{C}_{pk}, C_{pk})$,
the second row: values of $R(A^*\hat{C}_{pk}, C_{pk})$,
the third row: values of RI (%).

d^*	$\lvert\delta\rvert$				
	0	0.5	1.0	1.5	2.0
2	.0360	.0354	.0250	.0170	.0143
	.0409	.0259	.0180	.0123	.0104
	-13.49	26.77	28.31	27.85	27.57
3	.0793	.0780	.0575	.0386	.0251
	.0791	.0565	.0409	.0275	.0180
	.31	27.57	28.88	28.68	28.33
4	.1442	.1421	.1115	.0818	.0575
	.1325	.1023	.0790	.0581	.0409
	8.11	28.04	29.09	29.02	28.89
5	.2307	.2279	.1871	.1466	.1115
	.2012	.1634	.1325	.1039	.0791
	12.77	28.32	29.18	29.16	29.10
6	.3388	.3353	.2843	.2331	.1871
	.2852	.2397	.2012	.1650	.1325
	15.82	28.51	29.23	29.22	29.19

where K was defined in Section 2.

As it was shown in Section 3 that $\phi(1)$ and $\psi(1)$ are improved by $\phi(A^*)$ and $\psi(A^*)$, respectively, let us deal with the estimator

$$\rho(A^*, A^*) = \phi(A^*) - \psi(A^*) = A^*\hat{C}_{pk}. \tag{20}$$

This estimator (20) can also be derived in the following way. Differentiating $R(\rho(A, B), C_{pk})$ with respect to A and B, we obtain the minimizers

Table 5: Values of $R(\hat{C}_{pk}, C_{pk})$, $R(A^*\hat{C}_{pk}, C_{pk})$ and RI for f=19.
The first row: values of $R(\hat{C}_{pk}, C_{pk})$,
the second row: values of $R(A^*\hat{C}_{pk}, C_{pk})$,
the third row: values of RI (%).

d^*	$\|\delta\|$				
	0	0.5	1.0	1.5	2.0
2	.0154	.0144	.0100	.0072	.0062
	.0183	.0124	.0086	.0062	.0054
	-18.97	13.99	13.87	13.38	13.12
3	.0313	.0296	.0214	.0147	.0100
	.0344	.0253	.0183	.0126	.0086
	-9.77	14.46	14.52	14.27	13.87
4	.0548	.0523	.0403	.0299	.0214
	.0569	.0446	.0344	.0255	.0183
	-3.76	14.69	14.80	14.69	14.52
5	.0859	.0826	.0669	.0527	.0403
	.0859	.0704	.0569	.0448	.0344
	.10	14.80	14.92	14.87	14.80
6	.1246	.1205	.1010	.0830	.0669
	.1213	.1025	.0859	.0706	.0569
	2.71	14.87	14.98	14.95	14.92

$A = A^*(1 - \tau/\zeta) = A^*(1 - |\delta|/d^*)(\equiv A^\dagger)$ and $B = 0$. Since $|\delta|/d^* = |\xi - m|/d$ involves an unknown parameter ξ, let us estimate it by $|\bar{X} - m|/d = |z|/(\sqrt{n}d^*)$. Then, we obtain

$$\rho(\hat{A}^\dagger, 0) = \phi(\hat{A}^\dagger) = \frac{1}{3}A^*\sqrt{f}d^*\chi_f^{-1} - \frac{1}{3}A^*\sqrt{\frac{f}{n}}|z|\chi_f^{-1}, \tag{21}$$

which is equivalent to the estimator (20).

Table 6: Values of $R(\hat{C}_{pk}, C_{pk})$, $R(A^*\hat{C}_{pk}, C_{pk})$ and RI for f=29.
The first row: values of $R(\hat{C}_{pk}, C_{pk})$,
the second row: values of $R(A^*\hat{C}_{pk}, C_{pk})$,
the third row: values of RI (%).

d^*	$\|\delta\|$				
	0	0.5	1.0	1.5	2.0
2	.0100	.0090	.0063	.0046	.0040
	.0118	.0082	.0057	.0042	.0036
	-17.94	9.44	9.18	8.80	8.60
3	.0198	.0181	.0131	.0091	.0063
	.0220	.0163	.0118	.0082	.0057
	-10.94	9.79	9.70	9.49	9.18
4	.0342	.0317	.0244	.0182	.0131
	.0363	.0286	.0220	.0164	.0118
	-6.09	9.95	9.92	9.83	9.70
5	.0531	.0499	.0403	.0318	.0244
	.0546	.0449	.0363	.0286	.0220
	-2.89	10.02	10.02	9.98	9.92
6	.0765	.0726	.0607	.0499	.0403
	.0770	.0653	.0546	.0449	.0363
	-.68	10.07	10.07	10.05	10.02

Let us define the relative improvement as

$$RI = \frac{R(\hat{C}_{pk}, C_{pk}) - R(A^*\hat{C}_{pk}, C_{pk})}{R(\hat{C}_{pk}, C_{pk})} \times 100\ (\%). \qquad (22)$$

We give the values of $R(\hat{C}_{pk}, C_{pk})$, $R(A^*\hat{C}_{pk}, C_{pk})$ and RI for f = 9(10)49 in Tables 4, 5, 6, 7 and 8, respectively. We selected the same values of parameters d^* and δ as in Table 2.4 of Kotz and Johnson(1993).

Table 7: Values of $R(\hat{C}_{pk}, C_{pk})$, $R(A^*\hat{C}_{pk}, C_{pk})$ and RI for f=39.

The first row: values of $R(\hat{C}_{pk}, C_{pk})$,
the second row: values of $R(A^*\hat{C}_{pk}, C_{pk})$,
the third row: values of RI (%).

d^*	$\|\delta\|$				
	0	0.5	1.0	1.5	2.0
2	.0075	.0066	.0045	.0033	.0029
	.0087	.0061	.0042	.0031	.0027
	-16.60	7.10	6.86	6.56	6.40
3	.0146	.0130	.0094	.0066	.0045
	.0162	.0121	.0087	.0061	.0042
	-10.84	7.38	7.28	7.11	6.86
4	.0249	.0227	.0175	.0130	.0094
	.0266	.0210	.0162	.0121	.0087
	-6.72	7.51	7.46	7.39	7.28
5	.0385	.0356	.0288	.0227	.0175
	.0400	.0329	.0266	.0210	.0162
	-3.95	7.57	7.55	7.51	7.46
6	.0553	.0518	.0433	.0356	.0288
	.0564	.0478	.0400	.0329	.0266
	-2.02	7.60	7.59	7.57	7.55

From Tables 4–8 we observe the following points:(a)RI is positive unless $|\delta| = 0$; (b)RI is significantly large when f is small and $|\delta| \neq 0$, and RI decreases as f increases; (c)the extent of improvement by $A^*\hat{C}_{pk}$ is comparable to those in Tables 2 and 3 when $|\delta| \neq 0$.

From above observation, we may conclude that $A^*\hat{C}_{pk}$ is useful when we have some information that $|\delta|$ is not close to zero.

Finally, we give a remark that any member of the class $\{A\hat{C}_{pk}$; *A is*

Table 8: Values of $R(\hat{C}_{pk}, C_{pk})$, $R(A^*\hat{C}_{pk}, C_{pk})$ and RI for f=49.
The first row: values of $R(\hat{C}_{pk}, C_{pk})$,
the second row: values of $R(A^*\hat{C}_{pk}, C_{pk})$,
the third row: values of RI (%).

d^*	$\|\delta\|$				
	0	0.5	1.0	1.5	2.0
2	.0060	.0051	.0036	.0026	.0023
	.0069	.0048	.0034	.0025	.0022
	-15.44	5.68	5.47	5.22	5.10
3	.0116	.0102	.0073	.0051	.0036
	.0128	.0096	.0069	.0048	.0034
	-10.48	5.92	5.83	5.68	5.47
4	.0197	.0177	.0136	.0102	.0073
	.0210	.0166	.0128	.0096	.0069
	-6.85	6.02	5.98	5.92	5.83
5	.0303	.0277	.0224	.0177	.0136
	.0316	.0260	.0210	.0166	.0128
	-4.38	6.07	6.05	6.02	5.98
6	.0434	.0402	.0336	.0277	.0224
	.0445	.0378	.0316	.0260	.0210
	-2.66	6.10	6.09	6.07	6.05

constant} can not improve on the natural estimator $\hat{C}_{pk}$ uniformly when f is more than at least 19 (see Appendix for details), although the natural estimators for the components of C_{pk} can be improved uniformly. We have not been able to prove whether there exist some estimators which belong to the class $\{\rho(A, B);\ A \text{ and } B \text{ are constants}\}$ and improve on $\hat{C}_{pk}$ uniformly. We have evaluated the MSE's of $\rho(A, B)$'s for several (A, B)'s in moderately large neighborhood of (A^*, A^*) numerically. We have, however, found no uniformly improving estimator. The author thinks that $A^*\hat{C}_{pk}$ is a simple and good choice as an alternative estimator.

Since the natural estimator $\hat{C}_{pk}$ is not a smooth estimator, it is shown from the well-known decision theory that it is not admissible. In this paper we have tried to look for an improving estimator, but we have not been able to find such an estimator in the class of simple estimators. The author will continue this study in the future work.

Appendix

Let us consider the class of estimators $\{A\hat{C}_{pk};\ A \text{ is constant}\}$. Since it follows from (19) that

$$\begin{aligned} R(A\hat{C}_{pk}, C_{pk}) &= E\{(A\hat{C}_{pk} - C_{pk})^2\} \\ &= K\{((\zeta^2 - 2\theta\zeta + (1+\tau^2))A^2 \\ &\quad -2A^*(\zeta-\tau)(\zeta-\theta)A + (f-2)(\zeta-\tau)^2/f\}, \end{aligned} \tag{23}$$

we obtain the minimizer

$$A = \frac{A^*(\zeta-\tau)(\zeta-\theta)}{\zeta^2 - 2\theta\zeta + 1 + \tau^2} \ (\equiv A^\dagger). \tag{24}$$

We give the values of $A^\dagger$ in Table 9. We observe from Table 9 that when $f \geq 19$ $A^\dagger > 1$ for some parameter values and that $A^\dagger < 1$ for other parameter values.

If we set the constant A which is smaller than one, it follows that for some parameter values which implies $A^\dagger > 1$

$$R(A\hat{C}_{pk}, C_{pk}) > R(\hat{C}_{pk}, C_{pk}) > R(A^\dagger\hat{C}_{pk}, C_{pk}). \tag{25}$$

Similarly, if we set the constant A which is larger than one, it follows that for some parameter values which implies $A^\dagger < 1$

$$R(A\hat{C}_{pk}, C_{pk}) > R(\hat{C}_{pk}, C_{pk}) > R(A^\dagger\hat{C}_{pk}, C_{pk}). \tag{26}$$

Thus, no constant multiple of the natural estimator can not improve on the natural estimator uniformly.

Table 9: Values of $A^{\dagger}$

f	d^*	$\|\delta\|$				
		0	0.5	1.0	1.5	2.0
9	2	.9625	.8282	.7740	.6079	.0000
	3	.9248	.8448	.8304	.8149	.7737
	4	.9060	.8489	.8418	.8377	.8303
	5	.8949	.8504	.8458	.8442	.8417
	6	.8875	.8511	.8477	.8469	.8458
19	2	1.018	.9138	.8877	.7768	.0000
	3	.9888	.9257	.9206	.9118	.8877
	4	.9744	.9290	.9270	.9247	.9206
	5	.9658	.9303	.9292	.9283	.9270
	6	.9602	.9310	.9302	.9298	.9292
29	2	1.027	.9424	.9252	.8435	.0000
	3	1.003	.9511	.9481	.9421	.9252
	4	.9913	.9535	.9525	.9509	.9481
	5	.9842	.9545	.9540	.9534	.9525
	6	.9795	.9550	.9547	.9544	.9540
39	2	1.030	.9569	.9439	.8795	.0000
	3	1.009	.9636	.9615	.9568	.9439
	4	.9984	.9655	.9648	.9636	.9615
	5	.9921	.9663	.9660	.9655	.9648
	6	.9880	.9667	.9665	.9663	.9660
49	2	1.030	.9656	.9551	.9020	.0000
	3	1.011	.9711	.9693	.9656	.9551
	4	1.002	.9726	.9720	.9711	.9693
	5	.9964	.9732	.9730	.9726	.9720
	6	.9926	.9735	.9734	.9732	.9730

References

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