Math. J. Okayama Univ. **53** (2011), 185–216

TRIANGLE CENTERS DEFINED BY QUADRATIC POLYNOMIALS

Yoshio Agaoka

Abstract. We consider a family of triangle centers whose barycentric coordinates are given by quadratic polynomials, and determine the lines that contain an infinite number of such triangle centers. We show that for a given quadratic triangle center, there exist in general four principal lines through this center. These four principal lines possess an intimate connection with the Nagel line.

Introduction

In our previous paper [1] we introduced a family of triangle centers $\mathcal{P}_C$ based on two conjugates: the Ceva conjugate and the isotomic conjugate. $\mathcal{P}_C$ is a natural family of triangle centers, whose barycentric coordinates are given by polynomials of edge lengths a, b, c. Surprisingly, in spite of its simple construction, $\mathcal{P}_C$ contains many famous centers such as the centroid, the incenter, the circumcenter, the orthocenter, the nine-point center, the Lemoine point, the Gergonne point, the Nagel point, etc.

By plotting the points of $\mathcal{P}_C$, we know that they are not independently distributed in $\mathbf{R}^2$, or rather, they possess a strong mutual relationship such as $2:1$ point configuration on the Euler line (see Figures 2, 4). In [1] we generalized this $2:1$ configuration to the form "generalized Euler lines", which contain an infinite number of centers in $\mathcal{P}_C$. Furthermore, the centers in $\mathcal{P}_C$ seem to have some additional structures, and there frequently appear special lines that contain an infinite number of centers in $\mathcal{P}_C$. In this paper we call such lines "principal lines" of $\mathcal{P}_C$. The most famous principal lines are the Euler line mentioned above, and the Nagel line, which passes through the centroid, the incenter, the Nagel point, the Spieker center, etc. Also in the Book by Kimberling [7], there already listed up many principal lines such as $L(1,6)$, $L(1,7)$, $L(4,9)$, $L(7,8)$, $L(8,9)$, etc. These principal lines of $\mathcal{P}_C$ give a curious network of triangle centers.

In this paper, we study and classify principal lines of $\mathcal{P}_C$ which contain centers of lower degree. As for the linear case, we conjecture that linear triangle centers in $\mathcal{P}_C$ are exhausted by the centers on the Nagel line, and if this conjecture is true, the Nagel line is the unique linear principal line.

Mathematics Subject Classification. Primary 51M04; Secondary 51A20, 05E05.
Key words and phrases. triangle center, generalized Euler line, Nagel line, principal line, Ceva conjugate, isotomic conjugate, symmetric polynomial.

The situation for the quadratic case is much more complicated. We first introduce the set of triangle centers $\mathcal{Q}$, consisting of quadratic triangle centers in $\mathcal{P}_C$, and determine the set of all principal lines of $\mathcal{Q}$. The result is summarized in Theorem 7.

Roughly speaking, for each center in $\mathcal{Q}$, there exist four principal lines passing through this point. Among them, the generalized Euler line introduced in [1] is the most standard one, keeping the $2:1$ point configuration. Remaining three principal lines possess some intimate connection with the Nagel line, which gives a network of linear and quadratic triangle centers in $\mathcal{P}_C$ (for details, see §3, in particular, Figure 3). As a result, the set of centers in $\mathcal{Q}$ forms a pattern like the Milky Way around the Nagel line (Figure 2). Undoubtedly, a similar phenomenon occurs for centers with higher degree, and it is an important and interesting problem to study the structure of such networks of triangle centers.

Now, we briefly explain the contents of this paper. In §1 we first give a decomposition formula of a polynomial space, satisfying certain conditions. In expressing triangle centers in terms of barycentric coordinate, polynomials of this type necessarily appear, and this decomposition gives a fundamental tool throughout this paper. In particular five polynomials s_1, s_2, s_3, t_1, t_2 play a fundamental role. In §2 we review the definition of the set $\mathcal{P}_C$, and prove some property on $\mathcal{P}_C$ (Proposition 4). From Proposition 4 we know that the Feuerbach point is excluded from the set $\mathcal{P}_C$, which indicates that from our standpoint the Feuerbach point takes a singular position in the set of triangle centers. In §3 we introduce a subset $\mathcal{Q}$ of $\mathcal{P}_C$, consisting of quadratic polynomials, and state the main theorem of this paper (Theorem 7). Unfortunately, the explicit form of $\mathcal{P}_C$ is not yet determined, but at present we conjecture that the set of all quadratic centers in $\mathcal{P}_C$ is exhausted by $\mathcal{Q}$. The elements of $\mathcal{Q}$ possess a peculiar form if we express them in terms of the above five fundamental polynomials s_i, t_i, and we may say that the coefficients of this expression give the fundamental invariants of triangle centers in $\mathcal{Q}$. We further explain several mutual relations between principal lines of $\mathcal{Q}$ in connection with the Nagel line. In §4 we prove Theorem 7, and in §5 we give some examples of principal lines. Finally in §6 we summarize open problems concerning the set $\mathcal{P}_C$. We consider that each problem is quite important in triangle geometry, and hope that these problems give some insight for further investigations in this field.

§ 1. **Decomposition of a polynomial space**

Let ΔABC be a triangle in $\mathbf{R}^2$, and let a, b, c be its edge lengths.

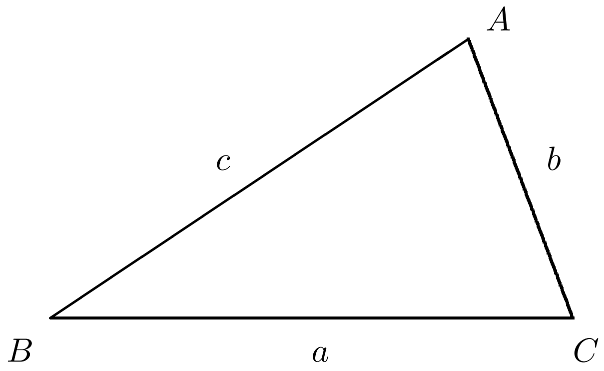

Figure 1

Triangle centers are the points in the plane that are expressed in the form

$$P = \frac{f(a,b,c)A + f(b,c,a)B + f(c,a,b)C}{f(a,b,c) + f(b,c,a) + f(c,a,b)},$$

where $f(a,b,c)$ is a function of a, b, c satisfying

$$\begin{aligned} &f(a,b,c) = f(a,c,b), \\ &f(a,b,c) + f(b,c,a) + f(c,a,b) \neq 0, \\ &f(ka,kb,kc) = k^p f(a,b,c) \quad {}^{\exists}p,\ {}^{\forall}k > 0. \end{aligned}$$

We call the triplet $(f(a,b,c), f(b,c,a), f(c,a,b))$ the barycentric coordinate of the point P. It is essentially determined by the first component $f(a,b,c)$, since the remaining coefficients of B and C are obtained from $f(a,b,c)$ by cyclic permutations of a, b, c.

Remind that the point P is unchanged even if we multiply a symmetric function to $f(a,b,c)$. So we introduce the equivalence relation as follows: $f(a,b,c) \sim f'(a,b,c)$ if and only if $f(a,b,c) = f'(a,b,c)\psi(a,b,c)$ for some symmetric function $\psi(a,b,c)$. We may say that the equivalence class of $f(a,b,c)$ determines the triangle center P. In this paper we treat the case where $f(a,b,c)$ is a homogeneous polynomial of a, b, c.

We denote by $\mathcal{T}$ the space of homogeneous polynomials $f(a,b,c)$ satisfying $f(a,b,c) = f(a,c,b)$. We have a direct sum decomposition $\mathcal{T} = \sum_{n\geq 0} T_n$, where T_n is a subspace of $\mathcal{T}$ with homogeneous degree n. In the following we give a decomposition formula of the space T_n. To explain the result, we put

$$\begin{aligned}
s_1 &= a+b+c, \\
s_2 &= a^2+b^2+c^2-ab-bc-ca, \\
s_3 &= (2a-b-c)(2b-c-a)(2c-a-b), \\
t_1 &= 2a-b-c, \\
t_2 &= s_2+(2b-c-a)(2c-a-b) \\
&= 2a^2-2a(b+c)-b^2+4bc-c^2.
\end{aligned}$$

Note that three polynomials s_1, s_2, s_3 give a free basis of the algebra of symmetric polynomials of a, b, c, and two polynomials t_1 and t_2 satisfy

$$\mathfrak{S}_{a,b,c}\, t_1 = \mathfrak{S}_{a,b,c}\, t_2 = 0,$$

where $\mathfrak{S}_{a,b,c}$ means the cyclic sum with respect to a, b, c. Also note that these polynomials are not independent as functions of a, b, c. In fact, we have relations such as

$$t_2 = {t_1}^2 - 2s_2, \qquad s_3 = t_1({t_1}^2 - 3s_2).$$

We denote by S_n the space of homogeneous symmetric polynomials of a, b, c with degree n. Then we have the following proposition, which plays a crucial role in several places of this paper.

Proposition 1. *We have a direct sum decomposition*:

$$T_n = S_n \oplus S_{n-1}t_1 \oplus S_{n-2}t_2.$$

Proof. We put

$$C_n = \{f(a,b,c) \in T_n \mid \mathfrak{S}_{a,b,c}\, f(a,b,c) = 0\}.$$

Then it is easy to see that we have a direct sum decomposition

$$T_n = S_n \oplus C_n.$$

The explicit decomposition of $f(a,b,c) \in T_n$ is given by

$$f(a,b,c) = \tfrac{1}{3}(f_a+f_b+f_c) + \tfrac{1}{3}(2f_a-f_b-f_c),$$

where $f_a = f(a,b,c)$, $f_b = f(b,c,a)$, $f_c = f(c,a,b)$. Hence to prove the proposition, we have only to show the decomposition

$$C_n = S_{n-1}\, t_1 \oplus S_{n-2}\, t_2.$$

We first show the uniqueness of the decomposition of an element of C_n. Namely we show that if $f_{(1)}t_1 + f_{(2)}t_2 = 0$ ($f_{(i)} \in S_{n-i}$, $i = 1, 2$), then we have $f_{(1)} = f_{(2)} = 0$. Assume $f_{(1)}t_1 + f_{(2)}t_2 = 0$ with $f_{(2)} \neq 0$. Then we have $f_{(1)} \neq 0$. If $f_{(1)}$ and $f_{(2)}$ possess a common symmetric factor, we divide $f_{(1)}$,

$f_{(2)}$ by its common factor in advance. So we may assume $f_{(1)}$ and $f_{(2)}$ have no common symmetric factor. Since $f_{(1)}t_1 = -f_{(2)}t_2$ and

$$t_2 = \{(\sqrt{3}+1)a - (2+\sqrt{3})b + c\}\{(\sqrt{3}-1)a + (2-\sqrt{3})b - c\},$$

the polynomial $t_1 = 2a - b - c$ must be a factor of $f_{(2)}$. Then since $f_{(2)}$ is symmetric, it can be divided by the polynomial $s_3 = (2a-b-c)(2b-c-a)(2c-a-b)$. From the equality $f_{(1)}t_1 = -f_{(2)}t_2$ again, $f_{(1)}$ must be also divided by s_3, which contradicts the above assumption. Hence we have $f_{(2)} = 0$, and so $f_{(1)} = 0$.

Next we show that any polynomial $f(a,b,c) \in C_n$ can be expressed as $f_{(1)}t_1 + f_{(2)}t_2$. For this purpose we calculate the dimension of the spaces T_n, S_n and C_n. First, it is easy to see that

$$\dim T_n = [\tfrac{1}{4}(n+2)^2], \qquad \dim S_n = [\tfrac{1}{12}n(n+6)] + 1,$$

where [ ] is the Gauss symbol. Then taking the difference, we have

$$\dim C_n = [\tfrac{1}{6}(n+1)(n+2)].$$

Now we consider a linear map $S_{n-1} \oplus S_{n-2} \longrightarrow C_n$ defined by $(f_{(1)}, f_{(2)}) \mapsto f_{(1)}t_1 + f_{(2)}t_2$. As seen above, this map is injective, and since $\dim S_{n-1} + \dim S_{n-2} = \dim C_n$, we know that this map is also surjective. Hence we complete the proof of the proposition. q.e.d.

By this proposition each element $f \in T_n$ has a unique expression

$$f = f_{(0)} + f_{(1)}t_1 + f_{(2)}t_2, \qquad f_{(i)} \in S_{n-i}.$$

Explicitly, the components $f_{(i)}$ are given by

$$f_{(0)} = \tfrac{1}{3}(f_a + f_b + f_c),$$
$$f_{(1)} = \frac{(b-c)(2a-b-c)f_a + (c-a)(2b-c-a)f_b + (a-b)(2c-a-b)f_c}{-9(a-b)(b-c)(c-a)},$$
$$f_{(2)} = \frac{(b-c)f_a + (c-a)f_b + (a-b)f_c}{-9(a-b)(b-c)(c-a)}.$$

For example the decomposition of $f(a,b,c) = bc$ is given by

$$bc = \tfrac{1}{9}\{({s_1}^2 - s_2) - s_1t_1 + t_2\}.$$

The center corresponding to $f(a,b,c) = bc$ is X_{75}, by Kimberling's numbering in [7], [9]. We remark that the condition $f(a,b,c)+f(b,c,a)+f(c,a,b) \neq 0$ is equivalent to $f_{(0)} \neq 0$.

In terms of this decomposition we can state a collinearity condition of three triangle centers as follows.

Proposition 2. *Let ΔABC be a scalene triangle. Then three centers of ΔABC represented by*

$$f_{(0)} + f_{(1)}t_1 + f_{(2)}t_2, \quad g_{(0)} + g_{(1)}t_1 + g_{(2)}t_2, \quad h_{(0)} + h_{(1)}t_1 + h_{(2)}t_2$$

are collinear if and only if

$$\begin{vmatrix} f_{(0)} & f_{(1)} & f_{(2)} \\ g_{(0)} & g_{(1)} & g_{(2)} \\ h_{(0)} & h_{(1)} & h_{(2)} \end{vmatrix} = 0.$$

Proof. It is known that three centers defined by $f(a,b,c)$, $g(a,b,c)$, $h(a,b,c)$ are collinear if and only if

$$\begin{vmatrix} f(a,b,c) & f(b,c,a) & f(c,a,b) \\ g(a,b,c) & g(b,c,a) & g(c,a,b) \\ h(a,b,c) & h(b,c,a) & h(c,a,b) \end{vmatrix} = 0.$$

Then substituting the equalities

$$\begin{aligned} f(b,c,a) &= f_{(0)} + f_{(1)}(2b-c-a) + f_{(2)}\{2b^2 - 2b(c+a) - c^2 + 4ca - a^2\}, \\ f(c,a,b) &= f_{(0)} + f_{(1)}(2c-a-b) + f_{(2)}\{2c^2 - 2c(a+b) - a^2 + 4ab - b^2\} \end{aligned}$$

etc., we can easily see that the above determinant is equal to

$$27(a-b)(b-c)(c-a)\begin{vmatrix} f_{(0)} & f_{(1)} & f_{(2)} \\ g_{(0)} & g_{(1)} & g_{(2)} \\ h_{(0)} & h_{(1)} & h_{(2)} \end{vmatrix}.$$

q.e.d.

This proposition implies that the collinearity of triangle centers is essentially reduced to some identity on symmetric polynomials, which is another interesting problem in linear algebra. We remark that the equality in this proposition does not depend on the choice of representatives of f, g, h.

Example. We consider three collinear centers X_4, X_9, X_{10} (we use the numbering in [7], [9]). These are the orthocenter, Mittenpunkt and Spieker center, respectively, and represented by

$$\begin{aligned} f(a,b,c) &= (a^2 - b^2 + c^2)(a^2 + b^2 - c^2), \\ g(a,b,c) &= a(-a+b+c), \\ h(a,b,c) &= b + c. \end{aligned}$$

For these polynomials we have the decompositions

$$\begin{aligned} f &= 3(a+b+c)(-a+b+c)(a-b+c)(a+b-c) \\ &\qquad\qquad\qquad + 4(a^3+b^3+c^3)t_1 + 2(a+b+c)^2 t_2, \\ g &= 3(a^2+b^2+c^2-2ab-2bc-2ca) + (a+b+c)t_1 + 2t_2, \\ h &= 2(a+b+c) - t_1, \end{aligned}$$

up to non-zero constants. Then the collinearity of these three points follows from the identity

$$\begin{vmatrix} 3(a+b+c)(-a+b+c)(a-b+c)(a+b-c) & 4(a^3+b^3+c^3) & 2(a+b+c)^2 \\ 3(a^2+b^2+c^2-2ab-2bc-2ca) & a+b+c & 2 \\ 2(a+b+c) & -1 & 0 \end{vmatrix} = 0.$$

But this is a curious identity on symmetric polynomials, and the collinearity of the above three centers is still mysterious (for further explanations, see §3).

§ 2. A property of $\mathcal{P}_C$

Based on two kinds of conjugates (the isotomic and the Ceva conjugate), we introduced in [1] the set of triangle centers $\mathcal{P}_C$ that plays an important role in triangle geometry. It is defined as follows: We denote by $\mathcal{P}$ the set of equivalence classes of homogeneous polynomials $f(a,b,c)$ satisfying $f(a,b,c) = f(a,c,b)$ and $f(a,b,c) + f(b,c,a) + f(c,a,b) \neq 0$, i.e.,

$$\mathcal{P} = \{f(a,b,c) = f_{(0)} + f_{(1)}t_1 + f_{(2)}t_2 \in \mathcal{T} \mid f_{(0)} \neq 0\}/\sim .$$

Remind that $f(a,b,c) \sim f'(a,b,c)$ if and only if $f(a,b,c) = f'(a,b,c)\psi$ for some symmetric function $\psi = \psi(a,b,c)$.

In the following we express the equivalence class containing a polynomial $f(a,b,c)$ by the same letter, and often consider an element of $\mathcal{P}$ as the polynomial itself that represents the equivalence class. If $f(a,b,c)$ does not possess a symmetric factor, then it is uniquely determined up to a non-zero constant, and in this situation we denote by $\deg f$ the polynomial degree of $f(a,b,c) \in \mathcal{P}$.

Definition 3 [1; §6]. $\mathcal{P}_C$ is a minimum subset of $\mathcal{P}$ satisfying the following conditions:

(i) $1,\, a \in \mathcal{P}_C$,
(ii) $f,\, f' \in \mathcal{P}_C \Longrightarrow f_b {f'}_c + {f'}_b f_c \in \mathcal{P}_C$,
(iii) $f,\, f' \in \mathcal{P}_C \Longrightarrow {f'}_a(-f_a {f'}_a + f_b {f'}_b + f_c {f'}_c) \in \mathcal{P}_C$.

Here, we used the notation $f_a = f(a, b, c)$, $f_b = f(b, c, a)$, $f_c = f(c, a, b)$, etc., as before. Note that the conditions (ii), (iii) do not depend on the choice of representatives of f and f'. Clearly, the initial condition (i) implies that $\mathcal{P}_C$ contains the centroid and the incenter.

The set $\mathcal{P}_C$ contains many famous triangle centers such as the orthocenter, the circumcenter, the nine-point center, the Lemoine point etc. (For details, see [1].) But unfortunately, the explicit form of $\mathcal{P}_C$ is not yet determined (see §6 Problem A). As one typical property of $\mathcal{P}_C$, we here show the following proposition.

Proposition 4. *Let $f(a, b, c)$ be an element of $\mathcal{P}_C$ with* $\deg f = p$, *possessing no symmetric factor. Express f in the form $f = f_{(0)} + f_{(1)}t_1 + f_{(2)}t_2$ as before. Then the coefficient of ${s_1}^p$ in $f_{(0)}$ is not zero.*

Corollary 5. *The Feuerbach point X_{11} is not contained in the set $\mathcal{P}_C$.*

Proof of Corollary 5. Since the barycentric coordinate of the Feuerbach point X_{11} is given by

$$\begin{aligned} f(a, b, c) &= (b - c)^2(-a + b + c) \\ &= \frac{1}{9}(2s_1s_2 + 2s_3 - 2s_2t_1 - s_1t_2), \end{aligned}$$

the coefficient of ${s_1}^3$ is zero. By Proposition 4 this implies $X_{11} \notin \mathcal{P}_C$. q.e.d.

Corollary 5 geometrically implies that the Feuerbach point cannot be obtained by applying the isotomic and the Ceva conjugates successively to the centroid and the incenter.

We remark that under the same assumption and the notation as in Proposition 4, the coefficient of ${s_1}^p$ in $f_{(0)}$ is zero if and only if $f(1, 1, 1) = 0$. This fact follows immediately from the equalities

$$s_1(1, 1, 1) = 3, \qquad s_2(1, 1, 1) = s_3(1, 1, 1) = 0.$$

The condition $f(1, 1, 1) = 0$ means that the center corresponding to $f(a, b, c)$ does not exist for equilateral triangles. As for the Feuerbach point, the incircle and the nine-point circle coincide when ΔABC is equilateral, and so the Feuerbach point disappear in this situation.

Proof of Proposition 4. We prove that the property

$$\text{“the coefficient of } {s_1}^p \text{ in } f_{(0)}\text{”} \neq 0$$

is kept under two operations (ii) and (iii) in Definition 3. We express two elements f, $f' \in \mathcal{P}_C$ as

$$f = f_{(0)} + f_{(1)}t_1 + f_{(2)}t_2, \qquad f' = {f'}_{(0)} + {f'}_{(1)}t_1 + {f'}_{(2)}t_2.$$

We assume that the coefficients of the power of s_1 in $f_{(0)}$ and $f'_{(0)}$ are not zero, and show that $h(a,b,c) = f_b f'_c + f'_b f_c$ also possesses the same property. By definition, we have

$$\begin{aligned} h &= (f_{(0)} + f_{(1)}(t_1)_b + f_{(2)}(t_2)_b)(f'_{(0)} + f'_{(1)}(t_1)_c + f'_{(2)}(t_2)_c) \\ &\qquad + (f'_{(0)} + f'_{(1)}(t_1)_b + f'_{(2)}(t_2)_b)(f_{(0)} + f_{(1)}(t_1)_c + f_{(2)}(t_2)_c) \\ &= h_{(0)} + h_{(1)}t_1 + h_{(2)}t_2, \end{aligned}$$

where

$$\begin{aligned} (t_1)_b &= 2b - c - a, & (t_2)_b &= 2b^2 - 2b(c+a) - c^2 + 4ca - a^2, \\ (t_1)_c &= 2c - a - b, & (t_2)_c &= 2c^2 - 2c(a+b) - a^2 + 4ab - b^2. \end{aligned}$$

Then by simple calculations, we have

$$\begin{aligned} h_{(0)} &= 2f_{(0)}f'_{(0)} - 2f_{(1)}f'_{(1)}s_2 - (f_{(1)}f'_{(2)} + f'_{(1)}f_{(2)})s_3 - 2f_{(2)}f'_{(2)}{s_2}^2, \\ h_{(1)} &= -f_{(0)}f'_{(1)} - f'_{(0)}f_{(1)} + 2(f_{(1)}f'_{(2)} + f'_{(1)}f_{(2)})s_2 + 2f_{(2)}f'_{(2)}s_3, \\ h_{(2)} &= -f_{(0)}f'_{(2)} - f'_{(0)}f_{(2)} + 2f_{(1)}f'_{(1)} - 2f_{(2)}f'_{(2)}s_2. \end{aligned}$$

Here we used the identities

$$\begin{aligned} (t_1)_b(t_1)_c &= -s_2 + t_2, \\ (t_2)_b(t_2)_c &= -{s_2}^2 + s_3t_1 - s_2t_2, \\ (t_1)_b(t_2)_c + (t_2)_b(t_1)_c &= -s_3 + 2s_2t_1. \end{aligned}$$

In the above expression of $h_{(0)}$, the power of s_1 can be appeared only in the first term $2f_{(0)}f'_{(0)}$, whose coefficient is not zero from the assumption. Even if the polynomial $h(a,b,c)$ may be divided by a symmetric polynomial, we easily see that the coefficient of the power of s_1 in the factored polynomial is also not zero.

For the second operation $(f, f') \mapsto f'_a(-f_af'_a + f_bf'_b + f_cf'_c)$ we can similarly show the same property after some calculations. In this case the symmetric part of $f'_a(-f_af'_a + f_bf'_b + f_cf'_c)$ takes a little complicated form

$$\begin{aligned} & f_{(0)}{f'_{(0)}}^2 - 2f'_{(1)}(2f_{(0)}f'_{(1)} + f_{(1)}f'_{(0)})s_2 \\ & - (2f_{(1)}{f'_{(1)}}^2 + 4f_{(0)}f'_{(1)}f'_{(2)} + f_{(2)}f'_{(0)}f'_{(1)} + f_{(1)}f'_{(0)}f'_{(2)})s_3 \\ & - 2(2f_{(2)}{f'_{(1)}}^2 + 4f_{(1)}f'_{(1)}f'_{(2)} + 2f_{(0)}{f'_{(2)}}^2 + f_{(2)}f'_{(0)}f'_{(2)}){s_2}^2 \\ & - 2f'_{(2)}(2f_{(2)}f'_{(1)} + f_{(1)}f'_{(2)})s_2s_3 + 4f_{(2)}{f'_{(2)}}^2{s_2}^3 - 2f_{(2)}{f'_{(2)}}^2{s_3}^2, \end{aligned}$$

though only the first term $f_{(0)}{f'_{(0)}}^2$ is again essential. q.e.d.

§ 3. Quadratic centers

In the previous paper [1] we showed that the following quadratic polynomials are contained in the set $\mathcal{P}_C$ for any l, m, $n \in \mathbf{Z}$.

$$Q_1: \ \ {s_1}^2 - (-\tfrac{1}{2})^{m+n-1}s_2 + \{(-\tfrac{1}{2})^{l+m} + (-\tfrac{1}{2})^{l+n}\}s_1t_1 + (-\tfrac{1}{2})^{l+m+n}t_2,$$

$$Q_2: \ \ {s_1}^2 - (-\tfrac{1}{2})^{m+n}s_2 + \{(-\tfrac{1}{2})^{l+m+1} + (-\tfrac{1}{2})^{l+n+1}\}s_1t_1 + (-\tfrac{1}{2})^{l+m+n}t_2.$$

We remind its construction. Starting from the polynomial $f(a,b,c) \in \mathcal{P}_C$, we construct a family of centers $f_i \in \mathcal{P}_C$ $(i \in \mathbf{Z})$ by

$$f_i = f_a + f_b + f_c + (-\tfrac{1}{2})^i(2f_a - f_b - f_c),$$

where $f_b = f(b,c,a)$, $f_c = f(c,a,b)$ as before (cf. [1; Proposition 5]). Note that these infinite centers all lie on the generalized Euler line (cf. [1; §3]). In case $f = a$ we have $f_i = s_1 + (-\frac{1}{2})^i t_1$. Then we have $f_m f_n \in \mathcal{P}_C$, and further $(f_m f_n)_l \in \mathcal{P}_C$, which is just equal to the above Q_1, up to a non-zero constant. (As far as we consider the equivalence class, we may multiply f by a non-zero constant.) Similarly, we can check that the polynomial

$$\{(f_m)_b(f_n)_c + (f_n)_b(f_m)_c\}_l \in \mathcal{P}_C$$

is equal to Q_2. (See also [1; Proposition 5].) Here, to obtain the final expressions, we used the identities:

$${t_1}^2 = 2s_2 + t_2, \quad (t_1)_b + (t_1)_c = -t_1, \quad (t_1)_b(t_1)_c = -s_2 + t_2.$$

Note that in spite of their different constructions, the final expressions of Q_1 and Q_2 have the following common form:

$$Q(p,q,r,s) = {s_1}^2 - (-\tfrac{1}{2})^p s_2 + \{(-\tfrac{1}{2})^q + (-\tfrac{1}{2})^r\}s_1t_1 + (-\tfrac{1}{2})^s t_2,$$

where p, q, r, $s \in \mathbf{Z}$ satisfy

$$p + q + r = \begin{cases} 2s - 1 & \text{for } Q_1, \\ 2s + 2 & \text{for } Q_2. \end{cases}$$

Clearly we have $Q(p,q,r,s) = Q(p,r,q,s)$.

In terms of these symbols the degree $d(Q(p,q,r,s))$ is equal to $(-1/2)^s$, which is essentially equal to the coefficient of t_2 in the above expression of $Q(p,q,r,s)$. Also we have $q(Q(p,q,r,s)) = (-1/2)^{p+1}$. (For the definition of two invariants $d(f)$, $q(f)$, see [1].) We conjecture that quadratic polynomials in $\mathcal{P}_C$ are exhausted by the union $\mathcal{Q} = \{Q_1\} \cup \{Q_2\}$.

Note that the intersection $\{Q_1\} \cap \{Q_2\}$ is non-empty, and consists of polynomials of the form

$${s_1}^2 - (-\tfrac{1}{2})^{2p}s_2 - (-\tfrac{1}{2})^q s_1t_1 + (-\tfrac{1}{2})^{p+q}t_2,$$

where p, $q \in \mathbf{Z}$. This fact can be easily verified by using the following lemma.

Lemma 6. *If* $(-\frac{1}{2})^{q_1} + (-\frac{1}{2})^{r_1} = (-\frac{1}{2})^{q_2} + (-\frac{1}{2})^{r_2}$ $(q_i,\ r_i \in \mathbf{Z})$, *then up to the order of exponents, we have* $(q_1, r_1) = (q_2, r_2)$ *or* $(r_1, q_2, r_2) =$

$(q_1, q_1 - 1, q_1 - 2)$. *In other words, non-trivial equality in the set* $\{(-\frac{1}{2})^m + (-\frac{1}{2})^n\}_{m,n\in\mathbf{Z}}$ *is essentially exhausted by*

$$(-\tfrac{1}{2})^{q+1} + (-\tfrac{1}{2})^{q+1} = (-\tfrac{1}{2})^{q} + (-\tfrac{1}{2})^{q-1}.$$

This non-trivial relation makes the state of quadratic centers a little complicated. Namely any center of $\mathcal{Q}$ can be expressed in terms of four parameters (p, q, r, s), but not uniquely on account of the above identity. By this lemma it immediately follows that the intersection $\{Q_1\} \cap \{Q_2\}$ is exhausted by $Q(2p, q+1, q+1, p+q) = Q(2p, q, q-1, p+q)$.

We here give a picture of the centers in $\mathcal{Q}$ (Figure 2). As we explain later, the center of the Milky Way in Figure 2 is nothing but the Nagel line. We can also find several $2:1$ point configurations in this figure.

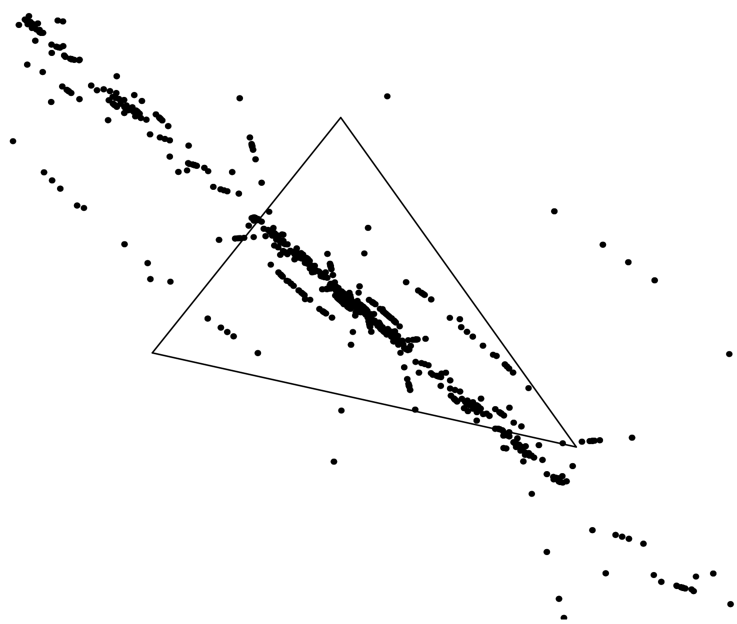

Figure 2

Now we consider principal lines associated with the set $\mathcal{Q}$. We say that the line in $\mathbf{R}^2$ is a *principal line* of $\mathcal{Q}$ if it contains an infinite number of centers in $\mathcal{Q}$. The following is the main result of this paper.

Theorem 7. *We fix an element* $Q(p, q, r, s)$ $(p+q+r = 2s-1$ *or* $2s+2)$ *of* $\mathcal{Q}$. *Then*:

(L_1) *The centers* $Q(p, q+i, r-i, s)$ $(i \in \mathbf{Z})$ *are collinear.*
(L_2) *The centers* $Q(p, q+i, r+i, s+i)$ $(i \in \mathbf{Z})$ *are collinear.*
(L_3) *The centers* $Q(p+i, q, r+i, s+i)$ $(i \in \mathbf{Z})$ *are collinear.*
(L_4) *The centers* $Q(p+i, q+i, r, s+i)$ $(i \in \mathbf{Z})$ *are collinear.*

Conversely, let $\{P_i\}_{i\in\mathbf{Z}}$ *be an infinite subset of centers in* $\mathcal{Q}$ *lying on one line. Then there exists an infinite subset* $\{P_{i_k}\}_{k\in\mathbf{Z}}$ *of* $\{P_i\}_{i\in\mathbf{Z}}$ *which is contained in one of the above* $(L_1)\sim(L_4)$ *for some* p, q, r, s.

Roughly speaking, principal lines in $\mathcal{Q}$ are exhausted by four lines $(L_1)\sim(L_4)$ with parameters (p,q,r,s). Namely, for each center $Q(p,q,r,s)\in\mathcal{Q}$ there pass through in general four principal lines. But there are some exceptional cases. As one case, five lines pass through the given center in $\{Q_1\}\cap\{Q_2\}$. In fact, on the point $Q(2p,q,q-1,p+q)$ there pass through four lines as stated above. Since $Q(2p,q,q-1,p+q)=Q(2p,q+1,q+1,p+q)$, the line (L_3) for $Q(2p,q+1,q+1,p+q)$ also pass through this point. This is the fifth line through the point $Q(2p,q,q-1,p+q)$. (Note that in this case the line (L_4) coincides with (L_3), and it is easy to see that the lines (L_1) and (L_2) are identical to (L_1) and (L_2) for $Q(2p,q,q-1,p+q)$, respectively. cf. Lemma 8 in §4.) As another exceptional case, only three principal lines pass through the center $Q(2p-1,q,q,p+q)$, since $(L_3)=(L_4)$ in this case.

It may happen that different families $\{(L_n)\}$ for distinct p, q, r, s lie in one line (see §5 for explicit examples). Thus, in the converse part of Theorem 7, we must take a suitable infinite subset $\{P_{i_k}\}_{k\in\mathbf{Z}}$ of $\{P_i\}_{i\in\mathbf{Z}}$ to show that it is contained in some (L_n). Of course, if the complement of $\{P_{i_k}\}_{k\in\mathbf{Z}}$ still contains an infinite number of centers, we may repeatedly apply the converse part of Theorem 7 to this complement.

The proof of Theorem 7 will be given in §4. We here give some comments on this theorem. We first consider the direction of these four lines. In general, the direction of the line determined by two centers $f(a,b,c)$ and $g(a,b,c)$ is obtained in the following way: The centers corresponding to these two polynomials are

$$\frac{f(a,b,c)A+f(b,c,a)B+f(c,a,b)C}{f(a,b,c)+f(b,c,a)+f(c,a,b)},$$
$$\frac{g(a,b,c)A+g(b,c,a)B+g(c,a,b)C}{g(a,b,c)+g(b,c,a)+g(c,a,b)}.$$

We decompose two polynomials f, g as before: $f=f_{(0)}+f_{(1)}t_1+f_{(2)}t_2$, $g=g_{(0)}+g_{(1)}t_1+g_{(2)}t_2$. Then it is easy to see that the difference of the above centers is given by

$$\frac{1}{3}\,\mathfrak{S}\left(\frac{f_{(1)}t_1+f_{(2)}t_2}{f_{(0)}}-\frac{g_{(1)}t_1+g_{(2)}t_2}{g_{(0)}}\right)A,$$

where $\mathfrak{S}$ means the cyclic sum with respect to a, b, c and A, B, C. Hence the difference of the ratio $(f_{(1)}t_1+f_{(2)}t_2)/f_{(0)}$ for two centers (up to a symmetric factor) determines the direction of the line connecting these two centers.

As for the line (L_1), we have

$$\begin{aligned} f_{(0)} &= {s_1}^2 - (-\tfrac{1}{2})^p s_2, \\ f_{(1)}t_1 &= \{(-\tfrac{1}{2})^q + (-\tfrac{1}{2})^r\} s_1 t_1, \\ f_{(2)}t_2 &= (-\tfrac{1}{2})^s t_2, \end{aligned}$$

where p, $q+r$, s are constant. Hence up to a symmetric factor, the difference is equal to t_1, which implies that (L_1) is parallel to the Nagel line. Since $Q(p, q+(r-q-i), r-(r-q-i), s) = Q(p, r-i, q+i, s) = Q(p, q+i, r-i, s)$, the centers on (L_1) turn back at some i, and actually lie on a half line.

As for the line (L_3) we can similarly calculate its direction. Up to a symmetric factor, it is equal to

$$\{{s_1}^2 + (-\tfrac{1}{2})^{p+q-r} s_2\} t_1 + (-\tfrac{1}{2})^{s-r} s_1 t_2.$$

This value essentially depends only on $s-r$, because $p+q-r = 2(s-r)-1$ or $2(s-r)+2$, according as $p+q+r$ is odd or even. The direction of the line (L_4) can be similarly obtained, and is equal to

$$\{{s_1}^2 + (-\tfrac{1}{2})^{p-q+r} s_2\} t_1 + (-\tfrac{1}{2})^{s-q} s_1 t_2.$$

The line (L_2) is nothing but the generalized Euler line, which we introduced in the paper [1]. In fact, centers on the generalized Euler line can be expressed in the form

$$f_{(0)} + (-\tfrac{1}{2})^i (f_{(1)}t_1 + f_{(2)}t_2)$$

for fixed $f_{(i)} \in S_{n-i}$ ($i = 0,\ 1,\ 2$). For the center $Q(p, q+i, r+i, s+i)$, we have the expression

$${s_1}^2 - (-\tfrac{1}{2})^p s_2 + (-\tfrac{1}{2})^i \left[\{(-\tfrac{1}{2})^q + (-\tfrac{1}{2})^r\} s_1 t_1 + (-\tfrac{1}{2})^s t_2\right],$$

which just fits to the above form. The direction of this line is given by

$$\{(-\tfrac{1}{2})^{q-s} + (-\tfrac{1}{2})^{r-s}\} s_1 t_1 + t_2.$$

Next we consider the limit points of (L_3). By putting $i \to \infty$ in the expression

$$Q(p+i, q, r+i, s+i) = {s_1}^2 - (-\tfrac{1}{2})^{p+i} s_2 + \{(-\tfrac{1}{2})^q + (-\tfrac{1}{2})^{r+i}\} s_1 t_1 + (-\tfrac{1}{2})^{s+i} t_2,$$

this point converges to the center

$${s_1}^2 + (-\tfrac{1}{2})^q s_1 t_1,$$

which is nothing but the center $s_1 + (-\frac{1}{2})^q t_1$ on the Nagel line. Conversely if we put $i \to -\infty$, this point converges to

$$-s_2 + (-\tfrac{1}{2})^{r-p} s_1 t_1 + (-\tfrac{1}{2})^{s-p} t_2.$$

Since $Q(p+i,q,r+i,s+i)$ converges for both cases $i\to\pm\infty$, the centers on (L_3) actually lie on a line segment. A similar phenomenon occurs for the case (L_4). In this case $Q(p+i,q+i,r,s+i)$ converges to

$$s_1+(-\tfrac{1}{2})^r t_1 \quad \text{and} \quad -s_2+(-\tfrac{1}{2})^{q-p}s_1t_1+(-\tfrac{1}{2})^{s-p}t_2.$$

Two limit points of (L_3) and (L_4) for $i\to-\infty$

$$-s_2+(-\tfrac{1}{2})^{r-p}s_1t_1+(-\tfrac{1}{2})^{s-p}t_2, \qquad -s_2+(-\tfrac{1}{2})^{q-p}s_1t_1+(-\tfrac{1}{2})^{s-p}t_2$$

are not contained in $\mathcal{Q}$ (or more strongly, not in $\mathcal{P}_C$. cf. Proposition 4). But the line connecting these two limit points is also parallel to the Nagel line in case $q\neq r$.

We remark that the line in the example at the end of §1, passing through X_4, X_9, X_{10}, is nothing but the line $(L_3)=(L_4)$ for the center $X_9=Q(-2,1,1,-1)$. The limit point of this line for $i\to\infty$ is the Spieker center X_{10}, corresponding to the linear polynomial $s_1+(-\frac{1}{2})t_1\sim b+c$. As we examined before, this line also pass through the orthocenter X_4, defined by quartic polynomial $(a^2-b^2+c^2)(a^2+b^2-c^2)$.

We here give an illustrative picture around $Q(p,q,r,s)$ to explain the above complicated relationship among the centers in $\mathcal{Q}$ (Figure 3).

By the above arguments, we know that the points $s_1+(-\frac{1}{2})^n t_1$ on the Nagel line are all accumulation points of the set $\mathcal{Q}$. This gives one explanation on the existence of the Milky Way around the Nagel line (see Figure 2).

§ 4. **Proof of the main theorem**

Before the proof of Theorem 7, we prepare several lemmas.

Lemma 8. *Three centers $Q(p_1,q_1,r_1,s_1)$, $Q(p_2,q_2,r_2,s_2)$, $Q(p_3,q_3,r_3,s_3)$ in $\mathcal{Q}$ are collinear if and only if*

$$\begin{vmatrix} 1 & (-\frac{1}{2})^{q_1}+(-\frac{1}{2})^{r_1} & (-\frac{1}{2})^{s_1} \\ 1 & (-\frac{1}{2})^{q_2}+(-\frac{1}{2})^{r_2} & (-\frac{1}{2})^{s_2} \\ 1 & (-\frac{1}{2})^{q_3}+(-\frac{1}{2})^{r_3} & (-\frac{1}{2})^{s_3} \end{vmatrix}=0$$

and

$$\begin{vmatrix} (-\frac{1}{2})^{p_1} & (-\frac{1}{2})^{q_1}+(-\frac{1}{2})^{r_1} & (-\frac{1}{2})^{s_1} \\ (-\frac{1}{2})^{p_2} & (-\frac{1}{2})^{q_2}+(-\frac{1}{2})^{r_2} & (-\frac{1}{2})^{s_2} \\ (-\frac{1}{2})^{p_3} & (-\frac{1}{2})^{q_3}+(-\frac{1}{2})^{r_3} & (-\frac{1}{2})^{s_3} \end{vmatrix}=0.$$

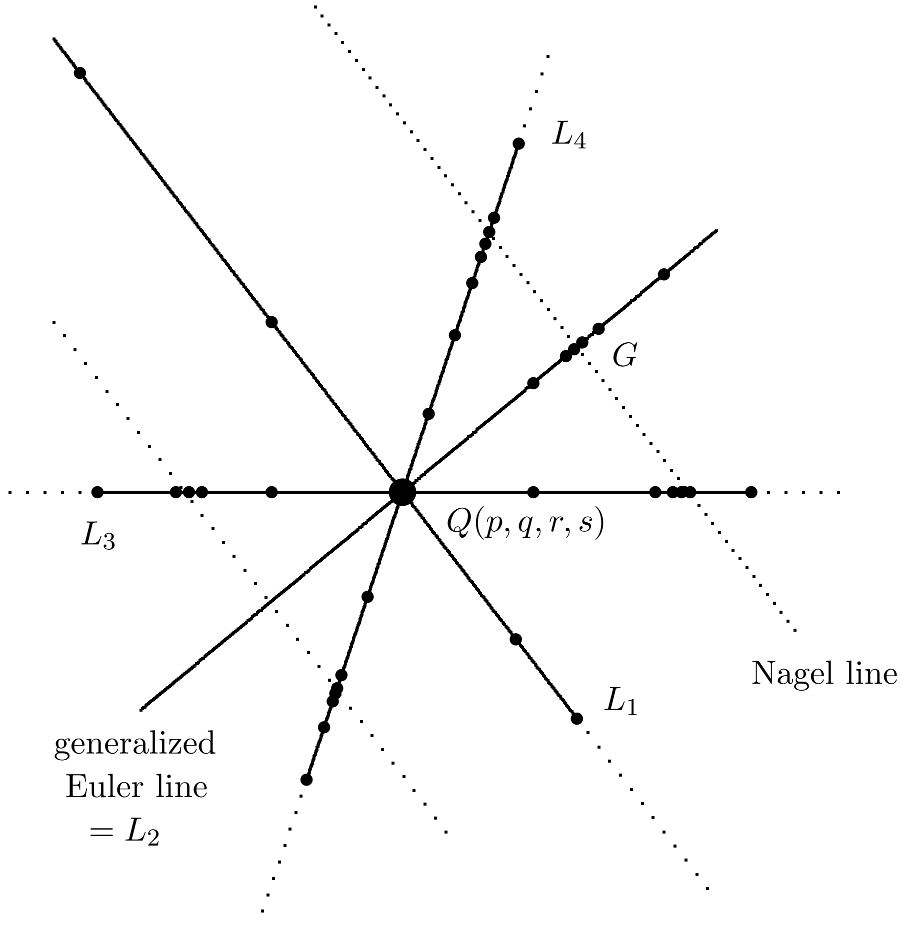

Figure 3

Proof. This lemma is an immediate consequence of Proposition 2, since we have

$$f_{(0)} = {s_1}^2 - (-\tfrac{1}{2})^{p_i} s_2, \quad f_{(1)} = \{(-\tfrac{1}{2})^{q_i} + (-\tfrac{1}{2})^{r_i}\} s_1, \quad f_{(2)} = (-\tfrac{1}{2})^{s_i}$$

in the decomposition $f = f_{(0)} + f_{(1)} t_1 + f_{(2)} t_2$. q.e.d.

Hereafter, do not confuse the first terms of the sequence of integers $\{s_i\}$ and the polynomials $s_1 = a + b + c$, etc., which we introduced in §1.

Lemma 9. *Let $\{x_i\}$, $\{y_i\}$ $(i \in \mathbf{N})$ be two sequences of real numbers such that $x_i \neq x_j$ for $i \neq j$. If they satisfy the equalities*

$$\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = 0$$

for all i, j, $k \in \mathbf{N}$, then there exist real numbers k, l such that $y_i = k + lx_i$ for all $i \in \mathbf{N}$.

Proof. Assume $i \geq 3$. Then from the equality

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_i & y_i \end{vmatrix} = 0,$$

we have

$$y_i = \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2} + \frac{y_1 - y_2}{x_1 - x_2} x_i.$$

Since this equality also holds in the cases $i = 1, 2$, we may put

$$k = \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}, \qquad l = \frac{y_1 - y_2}{x_1 - x_2}.$$

q.e.d.

Lemma 10. *Let $\{x_i\}$, $\{y_i\}$ $(i \in \mathbf{N})$ be two sequences of integers such that*

$$(-\tfrac{1}{2})^{x_i} + (-\tfrac{1}{2})^{y_i} \neq a \quad \textit{and} \quad (-\tfrac{1}{2})^{x_i} + (-\tfrac{1}{2})^{y_i} \to a$$

as $i \to \infty$. Then there exists a subsequence $\{x_{i_k}\}$ of $\{x_i\}$ or $\{y_{i_k}\}$ of $\{y_i\}$ such that $(-\frac{1}{2})^{x_{i_k}} \to 0$ or $(-\frac{1}{2})^{y_{i_k}} \to 0$ as $k \to \infty$.

Proof. Assume that any subsequence of $\{(-\frac{1}{2})^{x_i}\}$, $\{(-\frac{1}{2})^{y_i}\}$ does not converge to 0. Then there exists an integer N_0 such that the inequalities

$$|(-\tfrac{1}{2})^{x_i}| \geq 2^{N_0}, \quad |(-\tfrac{1}{2})^{y_i}| \geq 2^{N_0}$$

hold for all $i \in \mathbf{N}$. We put $A = \{2^{N_0} \cdot n \,|\, n \in \mathbf{Z}\}$. Then A is a discrete subset of $\mathbf{R}$, which is closed under addition. Since $(-\frac{1}{2})^{x_i}$, $(-\frac{1}{2})^{y_i} \in A$, we have $(-\frac{1}{2})^{x_i} + (-\frac{1}{2})^{y_i} \in A$ for all $i \in \mathbf{N}$. Since $(-\frac{1}{2})^{x_i} + (-\frac{1}{2})^{y_i}$ converges to a and A is discrete, there exists an integer N_1 such that $(-\frac{1}{2})^{x_i} + (-\frac{1}{2})^{y_i} = a$ for $i > N_1$. But this contradicts the assumption $(-\frac{1}{2})^{x_i} + (-\frac{1}{2})^{y_i} \neq a$. q.e.d.

Lemma 11. *Let $\{x_i\}$, $\{y_i\}$ $(i \in \mathbf{N})$ be two sequences of integers such that*

$$(-\tfrac{1}{2})^{x_i} + (-\tfrac{1}{2})^{y_i} = a$$

for all $i \in \mathbf{N}$. Then for an infinite numbers of $i \in \mathbf{N}$ we have $x_i =$ constant, $y_i =$ constant.

Proof. Assume $i \geq 2$. From Lemma 6 we know that the pair of integers (q, r) satisfying

$$(-\tfrac{1}{2})^q + (-\tfrac{1}{2})^r = (-\tfrac{1}{2})^{x_1} + (-\tfrac{1}{2})^{y_1}$$

is finite in number. Hence for an infinite number of $i \in \mathbf{N}$ x_i and y_i are constants. q.e.d.

Proof of Theorem 7. We can easily show that centers in $(L_1) \sim (L_4)$ are collinear, by using Lemma 8 and elementary properties of determinants. We show the converse part of the theorem.

Assume that the centers $\{P_i\}_{i\in\mathbf{N}}$ $(P_i \in \mathcal{Q})$ lie on one line. Then one of $\{Q_1\}$, $\{Q_2\}$ contains an infinite subset of $\{P_i\}$. Now assume $\{Q_1\}$ contains infinite elements. We express them as $Q(p_i, q_i, r_i, s_i)$. The integers p_i, q_i, r_i, s_i satisfy the relations $p_i + q_i + r_i = 2s_i - 1$, and

$$(D_1) \qquad \begin{vmatrix} 1 & (-\frac{1}{2})^{q_i} + (-\frac{1}{2})^{r_i} & (-\frac{1}{2})^{s_i} \\ 1 & (-\frac{1}{2})^{q_j} + (-\frac{1}{2})^{r_j} & (-\frac{1}{2})^{s_j} \\ 1 & (-\frac{1}{2})^{q_k} + (-\frac{1}{2})^{r_k} & (-\frac{1}{2})^{s_k} \end{vmatrix} = 0,$$

$$(D_2) \qquad \begin{vmatrix} (-\frac{1}{2})^{p_i} & (-\frac{1}{2})^{q_i} + (-\frac{1}{2})^{r_i} & (-\frac{1}{2})^{s_i} \\ (-\frac{1}{2})^{p_j} & (-\frac{1}{2})^{q_j} + (-\frac{1}{2})^{r_j} & (-\frac{1}{2})^{s_j} \\ (-\frac{1}{2})^{p_k} & (-\frac{1}{2})^{q_k} + (-\frac{1}{2})^{r_k} & (-\frac{1}{2})^{s_k} \end{vmatrix} = 0.$$

We divide the proof into several cases.

[A] The case $\{p_i\}_{i\in\mathbf{N}}$ takes the same value for an infinite number of $i \in \mathbf{N}$.

We take out a subsequence $\{Q(p_{i_k}, q_{i_k}, r_{i_k}, s_{i_k})\}_{k\in\mathbf{N}}$ such that all p_{i_k} take the same value, and express this new infinite set as $\{Q(p_i, q_i, r_i, s_i)\}_{i\in\mathbf{N}}$ again ($p_i = p =$ constant). In the following we usually express a subsequence by the same letter as the original one. In the case [A] two conditions (D_1) and (D_2) are equivalent.

[A-1] The case $\{s_i\}_{i\in\mathbf{N}}$ takes the same value for an infinite number of $i \in \mathbf{N}$.

As above, we take out a subsequence such that s_{i_k} take the same value, and express it as $\{Q(p, q_i, r_i, s)\}_{i\in\mathbf{N}}$. Then from the condition $p + q_i + r_i = 2s - 1$, the sum $q_i + r_i$ is constant. This is the case (L_1).

[A-2] The case $\{s_i\}_{i\in\mathbf{N}}$ contains infinitely many different values.

In this case we can take out a suitable infinite subset such that $s_i \neq s_j$ for $i \neq j$. Then from the condition (D_1) and Lemma 9 there exist real numbers k, l such that

$$(R) \qquad (-\tfrac{1}{2})^{q_i} + (-\tfrac{1}{2})^{r_i} = k + l(-\tfrac{1}{2})^{s_i}$$

for all $i \in \mathbf{N}$. In this situation we have $l \neq 0$. In fact if $l = 0$, then we have

$$(-\tfrac{1}{2})^{q_i} + (-\tfrac{1}{2})^{r_i} = k,$$

which implies that $q_i =$ constant, $r_i =$ constant for an infinite number of i (Lemma 11). But this contradicts the assumption $p + q_i + r_i = 2s_i - 1$ and $s_i \neq s_j$ for $i \neq j$.

Since $s_i \in \mathbf{Z}$ are all different, we may assume that $s_i \to \infty$ or $-\infty$ in case $i \to \infty$, by taking a suitable infinite subset of $\{s_i\}$.

[A-2-i] The case $s_i \to \infty$.

From the relation (R) and $l \neq 0$, we have

$$(-\tfrac{1}{2})^{q_i} + (-\tfrac{1}{2})^{r_i} \neq k \quad \text{and} \quad (-\tfrac{1}{2})^{q_i} + (-\tfrac{1}{2})^{r_i} \to k$$

as $i \to \infty$. Then by Lemma 10 a suitable subsequence of $\{(-\frac{1}{2})^{q_i}\}$ or $\{(-\frac{1}{2})^{r_i}\}$ converges to 0. We express this subsequence by the same letter. Assume $(-\frac{1}{2})^{r_i} \to 0$. Then we have $(-\frac{1}{2})^{q_i} \to k$. If $k \neq 0$, then for sufficiently large i, the value q_i is constant, and $(-\frac{1}{2})^{q_i} = k$ for such i. We take out this part as a subsequence. Then we have $q_i = q$, $(-\frac{1}{2})^q = k$, and so $(-\frac{1}{2})^{r_i} = l(-\frac{1}{2})^{s_i}$ from (R). The last equality implies $r_i = s_i + m$ for some m. Then substituting into the equality $p + q + r_i = 2s_i - 1$, we have $s_i =$ constant, which is a contradiction. Hence we have $k = 0$. In the case $(-\frac{1}{2})^{q_i} \to 0$ we have similarly $k = 0$.

Thus we have

$$(-\tfrac{1}{2})^{q_i} + (-\tfrac{1}{2})^{r_i} = l(-\tfrac{1}{2})^{s_i},$$

which implies

$$(-\tfrac{1}{2})^{q_i - s_i} + (-\tfrac{1}{2})^{r_i - s_i} = l.$$

Hence by Lemma 11, for an infinite number of i, we have $q_i - s_i$, $r_i - s_i =$ constant. This is the case (L_2).

[A-2-ii] The case $s_i \to -\infty$.

From the relation (R) we have

$$(-\tfrac{1}{2})^{q_i - s_i} + (-\tfrac{1}{2})^{r_i - s_i} = k(-\tfrac{1}{2})^{-s_i} + l.$$

Assume $k \neq 0$. Then we have

$$(-\tfrac{1}{2})^{q_i - s_i} + (-\tfrac{1}{2})^{r_i - s_i} \neq l, \qquad (-\tfrac{1}{2})^{q_i - s_i} + (-\tfrac{1}{2})^{r_i - s_i} \to l.$$

Then from Lemma 10 a subsequence of $\{(-\frac{1}{2})^{q_i - s_i}\}$ or $\{(-\frac{1}{2})^{r_i - s_i}\}$ converges to 0. Taking a subsequence, we assume $(-\frac{1}{2})^{r_i - s_i} \to 0$. Then we have $(-\frac{1}{2})^{q_i - s_i} \to l$, and since $l \neq 0$, $q_i - s_i$ is constant for sufficiently large i. We take this part as a subsequence. Then we have $q_i - s_i = m$ for some constant m. Substituting into $p + q_i + r_i = 2s_i - 1$, we have $r_i - s_i =$ constant. This contradicts the assumption $(-\frac{1}{2})^{r_i - s_i} \to 0$. Hence we have $k = 0$. The case $(-\frac{1}{2})^{q_i - s_i} \to 0$ can be treated in the same way. Thus we have

$$(-\tfrac{1}{2})^{q_i - s_i} + (-\tfrac{1}{2})^{r_i - s_i} = l,$$

and by the same reason as in the case of [A-2-i], these centers are contained in (L_2).

[B] The case $\{p_i\}_{i\in\mathbf{N}}$ contains infinitely many different values.

In this case, by taking a suitable subsequence, we assume that $p_i \neq p_j$ for $i \neq j$.

[B-1] The case $\{s_i\}_{i\in\mathbf{N}}$ takes the same value for an infinite number of $i \in \mathbf{N}$.

We assume $s_i = s$ for all $i \in \mathbf{N}$. Then we have $p_i + q_i + r_i = 2s - 1 =$ constant. The condition (D_1) is automatically satisfied in this case. From the condition (D_2) we have

$$(-\tfrac{1}{2})^{q_i} + (-\tfrac{1}{2})^{r_i} = k + l(-\tfrac{1}{2})^{p_i}$$

for some real numbers k, l (Lemma 9). By the same reason as in the case of [A-2] we can show $l \neq 0$.

Since $p_i \in \mathbf{Z}$ are all different in this case, we assume that $p_i \to \infty$ or $-\infty$, by taking a suitable infinite subset of $\{p_i\}$.

[B-1-i] The case $p_i \to \infty$.

Since $l \neq 0$, we have

$$(-\tfrac{1}{2})^{q_i} + (-\tfrac{1}{2})^{r_i} \neq k, \qquad (-\tfrac{1}{2})^{q_i} + (-\tfrac{1}{2})^{r_i} \to k.$$

By the same method as in [A-2-i], we can show $k = 0$. Thus we have

$$(-\tfrac{1}{2})^{q_i - p_i} + (-\tfrac{1}{2})^{r_i - p_i} = l.$$

Then for an infinite number of i we have $q_i - p_i = m$, $r_i - p_i = n$ for some m, n. Then from the equality $p_i + q_i + r_i =$ constant, we have $p_i =$ constant, which is a contradiction. Hence this case does not occur.

[B-1-ii] The case $p_i \to -\infty$.

In this case we have

$$(-\tfrac{1}{2})^{q_i - p_i} + (-\tfrac{1}{2})^{r_i - p_i} = k(-\tfrac{1}{2})^{-p_i} + l.$$

Assume $k \neq 0$. Then we have

$$(-\tfrac{1}{2})^{q_i - p_i} + (-\tfrac{1}{2})^{r_i - p_i} \neq l, \qquad (-\tfrac{1}{2})^{q_i - p_i} + (-\tfrac{1}{2})^{r_i - p_i} \to l.$$

We follow a similar argument as in [A-2-ii]. Assume $(-\frac{1}{2})^{r_i - p_i} \to 0$. Then since $l \neq 0$, we have $q_i - p_i =$ constant and $(-\frac{1}{2})^{q_i - p_i} = l$ for an infinite number of i. For such i we have

$$(-\tfrac{1}{2})^{r_i - p_i} = k(-\tfrac{1}{2})^{-p_i},$$

which implies that $r_i =$ constant. Substituting these equalities into $p_i + q_i + r_i =$ constant, we have a contradiction. Hence we have $k = 0$. Then from

Lemma 11, we have $q_i - p_i = m$, $r_i - p_i = n$ for an infinite number of i. From the equality $p_i + q_i + r_i =$ constant, we have $p_i =$ constant, which is a contradiction. Hence this case also does not occur.

[B-2] The case $\{s_i\}_{i\in\mathbf{N}}$ contains infinitely many different values.

Taking a suitable subsequence, we assume $s_i \neq s_j$ for $i \neq j$. From the condition (D_1) we have

$$(-\tfrac{1}{2})^{q_i} + (-\tfrac{1}{2})^{r_i} = k + l(-\tfrac{1}{2})^{s_i}$$

for some k, l. If $k = 0$, then we have

$$(-\tfrac{1}{2})^{q_i - s_i} + (-\tfrac{1}{2})^{r_i - s_i} = l,$$

and we have $q_i - s_i = m$, $r_i - s_i = n$ for an infinite number of i. By substituting these equalities into $p_i + q_i + r_i = 2s_i - 1$, it follows that $p_i =$ constant, which is a contradiction. Hence we have $k \neq 0$.

Next, substituting the equality

$$(-\tfrac{1}{2})^{q_i} + (-\tfrac{1}{2})^{r_i} = k + l(-\tfrac{1}{2})^{s_i}$$

into (D_2), we have

$$\begin{vmatrix} 1 & (-\frac{1}{2})^{p_i} & (-\frac{1}{2})^{s_i} \\ 1 & (-\frac{1}{2})^{p_j} & (-\frac{1}{2})^{s_j} \\ 1 & (-\frac{1}{2})^{p_k} & (-\frac{1}{2})^{s_k} \end{vmatrix} = 0.$$

Hence by Lemma 9 there are real numbers t, u such that

$$(-\tfrac{1}{2})^{p_i} = t + u(-\tfrac{1}{2})^{s_i}.$$

Since p_i are all different, we have $u \neq 0$.

If $\{s_i\}$ contains a subsequence such that $s_i \to \infty$, then $(-\frac{1}{2})^{p_i} \to t$. Since p_i are different for all i, we have $t = 0$. Hence we have $(-\frac{1}{2})^{p_i} = u(-\frac{1}{2})^{s_i}$, which implies $p_i = s_i + m$ for an infinite number of i. If $\{s_i\}$ contains a subsequence such that $s_i \to -\infty$, then $(-\frac{1}{2})^{p_i - s_i} = t(-\frac{1}{2})^{-s_i} + u \to u\,(\neq 0)$. Thus we have also $p_i = s_i + m$ for an infinite number of i and $t = 0$.

Now assume $l = 0$. Then we have

$$(-\tfrac{1}{2})^{q_i} + (-\tfrac{1}{2})^{r_i} = k,$$

which implies that $q_i = q$, $r_i = r$ for an infinite number of i. Substituting into the equality $p_i + q_i + r_i = 2s_i - 1$, we have $s_i =$ constant. This is a contradiction, and hence we have $l \neq 0$. We assume $s_i \to \infty$ or $-\infty$, by taking a suitable subsequence of $\{s_i\}$.

[B-2-i] The case $s_i \to \infty$.

In this case we have

$$(-\tfrac{1}{2})^{q_i} + (-\tfrac{1}{2})^{r_i} \neq k, \qquad (-\tfrac{1}{2})^{q_i} + (-\tfrac{1}{2})^{r_i} \to k.$$

We assume $(-\frac{1}{2})^{r_i} \to 0$. Then we have $(-\frac{1}{2})^{q_i} \to k\,(\neq 0)$. Hence by taking a suitable subsequence, we have $q_i = q$. Then from the equality $p_i+q_i+r_i = 2s_i-1$, we have $r_i = s_i-q-m-1$. Thus we have $(p_i, q_i, r_i, s_i) = (s_i+m, q, s_i-q-m-1, s_i)$, and this just coincides with (L_3). If we assume $(-\frac{1}{2})^{q_i} \to 0$ instead, then the final subsequence coincides with (L_4).

[B-2-ii] The case $s_i \to -\infty$.

In this case we have

$$(-\tfrac{1}{2})^{q_i-s_i} + (-\tfrac{1}{2})^{r_i-s_i} \neq l, \qquad (-\tfrac{1}{2})^{q_i-s_i} + (-\tfrac{1}{2})^{r_i-s_i} \to l.$$

Then by the same argument as above, we have $(p_i, q_i, r_i, s_i) = (s_i+m, q, s_i+n, s_i)$ or (s_i+m, s_i+n, r, s_i) for some m, n, i.e., this subsequence coincides with (L_3) or (L_4). This completes the proof of the converse part of Theorem 7 in case $\{Q_1\}$ contains infinite elements of the initial set $\{P_i\}_{i\in\mathbf{N}}$.

In case $\{Q_2\}$ contains infinite elements, we can prove the converse part completely in the same way. Thus we complete the proof of Theorem 7.

q.e.d.

§ 5. **Examples**

In this section we show several examples of principal and also exceptional lines.

5.1. The lines (L_1):

For fixed p the centers $Q(p-1, i, p-i, p)$ $(i \in \mathbf{Z})$ on the line (L_1) have a special form, i.e., it can be expressed as a product of linear forms:

$$Q(p-1, i, p-i, p) = f_i f_{p-i},$$

where $f_i = s_1 + (-\frac{1}{2})^i t_1$. As examples, we have lines connecting X_6 and X_{2321} ($p = 0$, $i = 0, 1$), X_{346} and X_{1743} ($p = -2$, $i = -1, 0$), X_{594} and X_{1100} ($p = 2$, $i = 1, 0$). We may call these lines *quadratic Nagel lines* since the family $\{f_i\}$ itself defines the Nagel line. (For the data on X_k, see Appendix.)

In some special cases two different families of centers lie on the same line (L_1). For example, we can easily see that for each fixed p, s, two families

$$Q(p, s-p-i, s-1+i, s) \quad \text{and} \quad Q(p, s-p-j, s+2+j, s) \quad (i, j \in \mathbf{Z})$$

lie on the same line.

5.2. Generalized Euler lines (L_2):

Consider the line connecting the Lemoine point X_6 and the centroid X_2. In Kimberling's list [7; p.128], [9] the following quadratic centers also lie on this line.

$$X_{69}, \quad X_{86}, \quad X_{141}, \quad X_{193}, \quad X_{1213}, \quad X_{1654}.$$

But they are divided into two different families from our algebraic viewpoint: $Q(-1, i, i, i)$ (X_6, X_{69}, X_{141}, X_{193}) and $Q(2, i, i, i)$ (X_{86}, X_{1213}, X_{1654}), though they lie geometrically on the same line. Note that the third family $Q(5, i-1, i-2, i)$ also lies on this line.

5.3. The lines (L_3), (L_4):
One example of this type is the line through the centers

$$X_1, \quad X_6, \quad X_9, \quad X_{37}.$$

They are all of the form $Q(i-1, 0, i, i)$ except X_1. The incenter X_1, lying on the Nagel line, is one limit point of this line, and another limit point is X_{44}, corresponding to the polynomial $a(-2a+b+c)$.

Another example of such line is given by the centers

$$X_7, \quad X_8, \quad X_{69}, \quad X_{75}, \quad X_{319}.$$

They are of the form $Q(i, -1, i, i)$ except X_8. The Nagel point X_8 and the center X_{320}, corresponding to $a^2 - b^2 + bc - c^2$, are two limit points of this line.

Implicitly, many lines of type (L_3), (L_4) are already listed up in [7].

5.4. Exceptional lines:
We consider three centers

$$Q(2p+1, q, q, p+q+1), \quad Q(2p, q, q, p+q-1), \quad Q(2p-1, q, q, p+q).$$

Note that the second center coincides with $Q(2p, q-1, q-2, p+q-1)$. By using Lemma 8, we can easily show that these three centers are collinear. And if we put $p = q = -1$, these centers correspond to X_{69}, X_{144}, X_{346}, respectively, where their collinearlity is already shown in [7; p.132].

It is clear that this line is not a principal line for any (p, q) because it is not listed up in Theorem 7. This means that the number of centers in $\mathcal{Q}$ lying on this line is finite. But we can show the following stronger result on this line.

Proposition 12. *Let (p, q) be any pair of integers. Then centers of $\mathcal{Q}$ lying on the above line are exhausted by the above three centers.*

Proof. Assume three centers $Q(2p+1,q,q,p+q+1)$, $Q(2p,q,q,p+q-1)$, $Q(p',q',r',s')$ are collinear. Then from Lemma 8 we have

$$\begin{vmatrix} 1 & 2(-\frac{1}{2})^q & (-\frac{1}{2})^{p+q+1} \\ 1 & 2(-\frac{1}{2})^q & (-\frac{1}{2})^{p+q-1} \\ 1 & (-\frac{1}{2})^{q'}+(-\frac{1}{2})^{r'} & (-\frac{1}{2})^{s'} \end{vmatrix} = 0,$$

$$\begin{vmatrix} (-\frac{1}{2})^{2p+1} & 2(-\frac{1}{2})^q & (-\frac{1}{2})^{p+q+1} \\ (-\frac{1}{2})^{2p} & 2(-\frac{1}{2})^q & (-\frac{1}{2})^{p+q-1} \\ (-\frac{1}{2})^{p'} & (-\frac{1}{2})^{q'}+(-\frac{1}{2})^{r'} & (-\frac{1}{2})^{s'} \end{vmatrix} = 0.$$

From these equalities we have

$$(-\tfrac{1}{2})^{q'}+(-\tfrac{1}{2})^{r'} = 2(-\tfrac{1}{2})^q,$$
$$(-\tfrac{1}{2})^{p'+q}+(-\tfrac{1}{2})^{p+s'} = 2(-\tfrac{1}{2})^{2p+q+1}.$$

Then by using Lemma 6, we know that (p',q',r',s') must be one of the following:

$$(2p+1,q,q,p+q+1),\quad (2p,q,q,p+q-1),\quad (2p-1,q,q,p+q),$$
$$(2p+1,q-1,q-2,p+q+1),\quad (2p,q-1,q-2,p+q-1),$$
$$(2p-1,q-1,q-2,p+q).$$

The fourth and the last one satisfy the equality of type $p+q+r=2s-4$, and we must omit them. The fifth one coincides with the second, as we stated above. Thus we complete the proof of the proposition. q.e.d.

Remark. In several examinations like the above, the tuple (p,q,r,s) with $p+q+r=2s-4$ often appears. So the restriction $p+q+r=2s-1$ or $2s+2$, which we imposed on the parameters may be too restrictive. It may occur that the set of centers $Q(p,q,r,s)$ with $p+q+r \equiv 2s-1 \pmod 3$ actually coincides with the quadratic centers in $\mathcal{P}_C$, though we cannot at present even show $Q(p,q,r,s) \in \mathcal{P}_C$ for the case $p+q+r=2s-4$. If this is the case, the statement in Theorem 7 must be enlarged into a form which is fitted to a wider class of quadratic centers.

There is another exceptional line, consisting of polynomials with different degrees.

Proposition 13. *Let (p,q) be any pair of integers. Then two quadratic centers $Q(2p-1,q,q,p+q)$, $Q(2p+1,q+1,q+1,p+q+2)$ in $\mathcal{Q}$ and the linear center defined by $s_1+(-\frac{1}{2})^{q-1}t_1$ lying on the Nagel line are collinear.*

In addition, linear or quadratic centers lying on this line are exhausted by these three centers.

We can prove this proposition completely in the same way as Proposition 12. If we put $p = q = -1$, we obtain a line through X_6, X_{145} and X_{346}, which is already listed up in [7; p.129]. Note that if we allow the centers of type $p+q+r = 2s-4$ or $2s-7$, additional centers appear on this line. The latter also satisfies the condition of type $p+q+r \equiv 2s-1 \pmod 3$.

There are quite many exceptional lines other than the above, containing at least three but only a finite number of centers in $\mathcal{Q}$.

§ 6. **Open problems**

Finally, in this section we present several open problems concerning the set $\mathcal{P}_C$.

Problem A : Determine the set $\mathcal{P}_C$ explicitly.

As for the cubic centers in $\mathcal{P}_C$ we have the following partial result. First we put

$$\begin{aligned} &Q = {s_1}^2 - (-\tfrac{1}{2})^i s_2 + \{(-\tfrac{1}{2})^j + (-\tfrac{1}{2})^k\} s_1 t_1 + (-\tfrac{1}{2})^l t_2, \\ &f = s_1 + (-\tfrac{1}{2})^m t_1. \end{aligned}$$

Then the cubic polynomial $C = (Q_b f_c + f_b Q_c)_n$ is contained in $\mathcal{P}_C$ if $i+j+k = 2l-1$ or $2l+2$ (cf. [1; Proposition 5]). Then after some calculations, we know that it is equal to

$$\begin{aligned} &{s_1}^3 - \{(-\tfrac{1}{2})^i + (-\tfrac{1}{2})^{j+m} + (-\tfrac{1}{2})^{k+m}\} s_1 s_2 + (-\tfrac{1}{2})^{l+m+1} s_3 \\ &\quad + \{(-\tfrac{1}{2})^{j+n+1} + (-\tfrac{1}{2})^{k+n+1} + (-\tfrac{1}{2})^{m+n+1}\} {s_1}^2 t_1 \\ &\qquad + \{(-\tfrac{1}{2})^{l+m+n} - (-\tfrac{1}{2})^{i+m+n+1}\} s_2 t_1 \\ &\qquad\quad + \{(-\tfrac{1}{2})^{j+m+n} + (-\tfrac{1}{2})^{k+m+n} + (-\tfrac{1}{2})^{l+n+1}\} s_1 t_2, \end{aligned}$$

up to a non-zero constant. Here, we used the identities

$$(t_1)_b (t_2)_c + (t_2)_b (t_1)_c = -s_3 + 2 s_2 t_1, \quad (t_2)_b + (t_2)_c = -t_2.$$

We plot these centers in Figure 4. From this figure we see that there also exist many principal lines in the set of cubic centers.

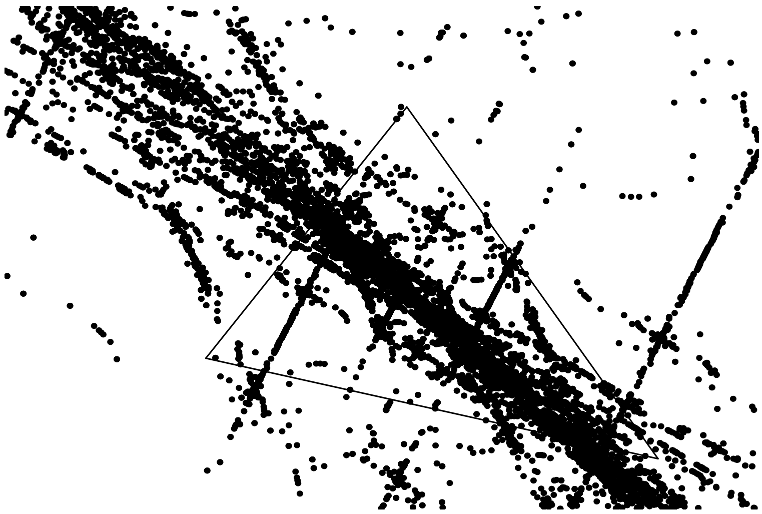

Figure 4

Undoubtedly the above polynomials do not exhaust the set of cubic centers in $\mathcal{P}_C$. But we conjecture that general cubic centers in $\mathcal{P}_C$ are expressed in a similar form as above, i.e., the coefficients of s_1s_2, s_3, ${s_1}^2t_1$, s_2t_1 and s_1t_2 are sums of $\pm(-\frac{1}{2})^*$, where exponents depend on five parameters i, j, k, m, n. Note that the degree $d(C) = (-\frac{1}{2})^{l+m+1}$ is equal to the coefficient of s_3 in this case.

In a similar way we can successively construct a polynomial in $\mathcal{P}_C$ with higher degree $(C_bf_c+f_bC_c)_u$ etc, by using a linear form $f = s_1+(-\frac{1}{2})^v t_1$ with two new parameters u, $v \in \mathbf{Z}$. Finally, we can see that the polynomial with degree p constructed in this successive method is expressed in the following form in terms of $2p-1$ parameters $l_1 \sim l_{2p-1}$

$$\sum_{i+2j+3k=p}\left(\sum_{\alpha=1}^{A}(-\tfrac{1}{2})^{a^\alpha_{ijk}} - \sum_{\alpha=1}^{B}(-\tfrac{1}{2})^{b^\alpha_{ijk}}\right){s_1}^i{s_2}^j{s_3}^k$$
$$+\sum_{i+2j+3k+1=p}\left(\sum_{\alpha=1}^{C}(-\tfrac{1}{2})^{c^\alpha_{ijk}} - \sum_{\alpha=1}^{D}(-\tfrac{1}{2})^{d^\alpha_{ijk}}\right){s_1}^i{s_2}^j{s_3}^k t_1$$
$$+\sum_{i+2j+3k+2=p}\left(\sum_{\alpha=1}^{E}(-\tfrac{1}{2})^{e^\alpha_{ijk}} - \sum_{\alpha=1}^{F}(-\tfrac{1}{2})^{f^\alpha_{ijk}}\right){s_1}^i{s_2}^j{s_3}^k t_2,$$

if it is not divided by a symmetric polynomial during the successive step. Here, $a^\alpha_{ijk} \sim f^\alpha_{ijk}$ are linear polynomials of parameters $l_1 \sim l_{2p-1}$, possibly containing non-zero constant terms. We can see that the summation numbers $A \sim F$ are given by

$$\begin{cases} A = B = 2^{j-2}\begin{pmatrix} i+2j+3k \\ i \end{pmatrix}\left\{\begin{pmatrix} j+k \\ j \end{pmatrix} + \begin{pmatrix} j+k-1 \\ j \end{pmatrix}\right\} & j \geq 2, \\ A = k\begin{pmatrix} i+3k+2 \\ i \end{pmatrix}, \qquad B = (k+1)\begin{pmatrix} i+3k+2 \\ i \end{pmatrix} & j = 1, \\ A = \begin{pmatrix} i+3k \\ i \end{pmatrix}, \qquad B = 0 & j = 0, \end{cases}$$

$$\begin{cases} C = D = 2^{j-1}\begin{pmatrix} i+2j+3k+1 \\ i \end{pmatrix}\begin{pmatrix} j+k \\ j \end{pmatrix} & j \geq 1, \\ C = \begin{pmatrix} i+3k+1 \\ i \end{pmatrix}, \qquad D = 0 & j = 0, \end{cases}$$

$$\begin{cases} E = F = 2^{j-1}\begin{pmatrix} i+2j+3k+2 \\ i \end{pmatrix}\begin{pmatrix} j+k \\ j \end{pmatrix} & j \geq 1, \\ E = \begin{pmatrix} i+3k+2 \\ i \end{pmatrix}, \qquad F = 0 & j = 0. \end{cases}$$

But in $\mathcal{P}_C$ there appear other polynomials constructed by a different method. So the final expression of $\mathcal{P}_C$ may take a more complicated form, though we conjecture that it is not so far from the above expression. We here note that the number of plus part and the minus part in the above expression are not uniquely determined since we have the identity such as

$$(-\tfrac{1}{2})^n = -(-\tfrac{1}{2})^{n+1} - (-\tfrac{1}{2})^{n+1}.$$

The degree $d(f)$ of the above polynomial is equal to

$$\begin{cases} (-\frac{1}{2})^{a^1_{00k}} = \text{the coefficient of } {s_3}^k & p = 3k, \\ (-\frac{1}{2})^{c^1_{00k}} = \text{the coefficient of } {s_3}^k t_1 & p = 3k+1, \\ (-\frac{1}{2})^{e^1_{00k}} = \text{the coefficient of } {s_3}^k t_2 & p = 3k+2, \end{cases}$$

if the above polynomial is not divided by a symmetric polynomial and if we set $a^1_{p00} = 0$ by multiplying a suitable constant. Note that the equality $a^1_{p00} = 0$ holds if and only if the coefficient of ${s_1}^p$ is 1. In the above calculation of $d(f)$, we used the equalities:

$$s_1(1,1,1) = 3,\ s_2(1,1,1) = s_3(1,1,1) = t_1(1,1,1) = t_2(1,1,1) = 0,$$

$$s_1(1,\omega,\omega^2) = s_2(1,\omega,\omega^2) = 0,\ s_3(1,\omega,\omega^2) = 27,$$

$$t_1(1,\omega,\omega^2) = 3,\ t_2(1,\omega,\omega^2) = 9.$$

The above expression of $d(f)$ supports our Conjecture 2 in [1].

Problem B : Determine the set of all principal lines in $\mathcal{P}_C$.
Problem C : Determine the set of all exceptional lines in $\mathcal{P}_C$.

These are the next fundamental problems in our setting. Here an *exceptional line* means a line of $\mathbf{R}^2$ containing at least three, but only a finite number of centers of $\mathcal{P}_C$. Of course the generalized Euler lines which we introduced in [1] are principal lines.

Problem D : Determine all triangles $\Delta A'B'C'$ possessing the same set of centers $\mathcal{P}_C$ as ΔABC. In other words, to what extent does the set $\mathcal{P}_C$ determine the triangle.

It happens that different triangles define the same set of centers $\mathcal{P}_C$. For example, the successive sequence of medial and anticomplementary triangles of ΔABC satisfy this condition. Are there any other triangles possessing the same property?

Concerning this problem, we consider the triangle centers of the tangential triangle $\Delta A'B'C'$ of ΔABC. We denote by $\mathcal{P}_C(\Delta A'B'C')$ the set of triangle centers of $\Delta A'B'C'$, constructed in the same way as in the case of ΔABC. Then we conjecture $\mathcal{P}_C(\Delta A'B'C') \subsetneq \mathcal{P}_C$. For example the incenter, the centroid, the Nagel point and the Gergonne point of $\Delta A'B'C'$ coincide with the circumcenter, X_{154}, X_{1498} and the Lemoine point of ΔABC, respectively. We can see X_{154}, $X_{1498} \in \mathcal{P}_C$ since X_{154} is the X_3-Ceva conjugate of X_6, X_{1498} is the X_{20}-Ceva conjugate of X_3, and X_3, X_6, $X_{20} \in \mathcal{P}_C$. On the contrary, we can easily show the property $\mathcal{P}_C(\Delta A'B'C') \neq \mathcal{P}_C$ because the incenter of ΔABC is not contained in $\mathcal{P}_C(\Delta A'B'C')$. In fact the incenter of ΔABC coincides with the center of $\Delta A'B'C'$ corresponding to the function

$$f(a', b', c') = \sqrt{\frac{a'b'(a' - b' + c')}{-a' + b' + c'}} + \sqrt{\frac{a'c'(a' + b' - c')}{-a' + b' + c'}},$$

where a', b', c' are edge lengths of $\Delta A'B'C'$. Since this function is not a polynomial of a', b', c', this center does not contained in $\mathcal{P}_C(\Delta A'B'C')$.

Similar phenomenon occurs for the orthic triangle of ΔABC.

Problem E : Consider several variants of the set $\mathcal{P}_C$.

In defining the set $\mathcal{P}_C$ (Definition 3 (i)), if we replace the incenter with the Feuerbach point, we obtain another set of triangle centers. This set contains the center X_{100}, which possesses some interesting properties (see [7]). Further, we may consider the set of triangle centers, generated by the centroid, the incenter and the Feuerbach point, or other different type centers. It is also a natural problem to characterize and study such variant sets of triangle centers.

Problem F : Determine the closure of the set of points in $\mathcal{P}_C$ for general scalene triangles. Is it equal to the whole space $\mathbf{R}^2$? As a similar problem, determine the closure of the set of points in $\mathcal{Q}$.

We can show that any point of $\mathcal{P}_C$ is an accumulation point of the set $\mathcal{P}_C$, i.e., $\mathcal{P}_C$ has no isolated points. This fact can be verified in the following way. For any center $f(a,b,c) \in \mathcal{P}_C$ we construct a sequence of centers $\{g_n(a,b,c)\}_{n\in\mathbf{N}}$ inductively by

$$\begin{aligned} g_1(a,b,c) &= 1, \\ g_{n+1}(a,b,c) &= \{f_b(g_n)_c + f_c(g_n)_b\}^{-1} \\ &\qquad = \{f_c(g_n)_a + f_a(g_n)_c\}\{f_a(g_n)_b + f_b(g_n)_a\}, \end{aligned}$$

where $f_a = f(a,b,c)$, $(g_n)_a = g_n(a,b,c)$, etc. Then by Proposition 5 in [1], we have $g_n(a,b,c) \in \mathcal{P}_C$, and we can show that the point defined by g_n converges to the point defined by f as $n \to \infty$, though the degree of the polynomial g_n does not coincide with the degree of f.

We remark that the above fact does not imply $\overline{\mathcal{P}_C} = \mathbf{R}^2$. Also note that we have $\overline{\mathcal{P}_C} \neq \mathcal{P}_C$, as we already verified. (Remind the limit points of principal lines (L_3), (L_4) for $i \to -\infty$, which we explained at the end of §3.)

Even if $\overline{\mathcal{P}_C} = \mathbf{R}^2$, the triangle centers in $\mathcal{P}_C$ are not uniformly distributed in $\mathbf{R}^2$. (For example, see Figures 2, 4.) So it may be possible to introduce the concept "density" in the set of triangle centers.

Problem G : Determine the invariants of $\mathcal{P}_C$.

The values $d(f)$ and also $q(f)$ for the quadratic case are the invariants of $\mathcal{P}_C$. They coincides with the coefficients of ${s_3}^k$, ${s_3}^k t_1$, ${s_3}^k t_2$, etc. in the decomposition $f = f_{(0)} + f_{(1)}t_1 + f_{(2)}t_2$, when the coefficient of the power of s_1 is normalized to 1. Clearly other coefficients of ${s_1}^i {s_2}^j {s_3}^k$ etc. can be considered as invariants of $\mathcal{P}_C$. But there should be many relations among these invariants because there are only $2p-1$ independent parameters as we conjectured above. What's the fundamental invariant of $\mathcal{P}_C$, and how can we recover $2p-1$ parameters $l_1 \sim l_{2p-1}$ from the fundamental invariants?

Problem H : Find a good family of centers of a tetrahedron in $\mathbf{R}^3$.

In defining a center of a tetrahedron, if we only impose the condition on the invariance by Euclidean transformations including homotheties and also the invariance by the exchange of vertices, then as in the case of triangles, all points in $\mathbf{R}^3$ becomes a center of a tetrahedron. In fact, if there exist four centers not lying on one plane, we can specify any points in $\mathbf{R}^3$ in terms of these four centers. (And we can see that such four centers actually exist.) So to define a good family of centers of a tetrahedron, we must impose some additional conditions.

One candidate may be obtained by using two concepts "isotomic conjugate" and "Ceva conjugate", as in the case of triangles. But the situation is a little different. We cannot define a set similar to $\mathcal{P}_C$ for a tetrahedron since the concept "Ceva conjugate" does not exist for the three dimensional case. Two concepts "cevian tetrahedron" and "anticevian tetrahedron" exist as in the triangle case, but we can see that four lines connecting the corresponding vertices of these tetrahedra do not meet at one point in general. So we cannot define a set similar to $\mathcal{P}_C$, though the concept "isotomic conjugate" exists for a tetrahedron. We must introduce a new viewpoint to consider a good family of centers of a tetrahedron.

Appendix

In Appendix we list up barycentric coordinates of triangle centers $X_k \in \mathcal{Q}$ in Kimberling's list [7], [9], including the centers on the Nagel line and other typical quadratic centers in $\mathcal{Q}$. We express $Q(p,q,r,s)$ in $\{Q_i\}$ as $Q_i(p,q,r,s)$ ($i = 1, 2$). See the references [7], [9] for the exact definition of X_k and their properties. It is surprising that the quadratic polynomials in the following list have a single common expression

$$Q(p,q,r,s) = {s_1}^2 - (-\tfrac{1}{2})^p s_2 + \{(-\tfrac{1}{2})^q + (-\tfrac{1}{2})^r\} s_1 t_1 + (-\tfrac{1}{2})^s t_2,$$

in spite of their various apperances.

X_1 (Incenter) $f = a$.

X_2 (Centroid) $f = 1$.

X_6 (Lemoine point) $f = a^2$, $Q_1(-1,0,0,0)$.

X_7 (Gergonne point) $f = (a-b+c)(a+b-c)$,
$Q_1(-2,-2,-1,-2) = Q_2(-2,0,0,-2)$.

X_8 (Nagel point) $f = -a+b+c$.

X_9 (Mittenpunkt) $f = a(-a+b+c)$,
$Q_1(-2,-1,0,-1) = Q_2(-2,1,1,-1)$.

X_{10} (Spieker center) $f = b+c$.

X_{37} $f = a(b+c)$, $Q_1(0,0,1,1) = Q_2(0,2,2,1)$.

X_{69} $f = -a^2+b^2+c^2$, $Q_1(-1,-1,-1,-1)$.

X_{75} $f = bc$, $Q_1(0,-1,0,0) = Q_2(0,1,1,0)$.

X_{86} $f = (a+b)(a+c)$, $Q_1(2,0,1,2) = Q_2(2,2,2,2)$.

X_{141} $f=b^2+c^2$, $Q_1(-1,1,1,1)$.

X_{142} $f=a(b+c)-(b-c)^2$, $Q_1(-2,0,1,0)=Q_2(-2,2,2,0)$.

X_{144} $f=3a^2-2a(b+c)-(b-c)^2$,
$Q_1(-2,-3,-2,-3)=Q_2(-2,-1,-1,-3)$.

X_{145} $f=-3a+b+c$.

X_{192} $f=a(b+c)-bc$, $Q_1(0,-2,-1,-1)=Q_2(0,0,0,-1)$.

X_{193} $f=-3a^2+b^2+c^2$, $Q_1(-1,-2,-2,-2)$.

X_{319} $f=a^2-b^2-bc-c^2$, $Q_1(1,-1,1,1)$.

X_{346} $f=(-a+b+c)^2$, $Q_1(-3,-1,-1,-2)$.

X_{594} $f=(b+c)^2$, $Q_1(1,1,1,2)$.

X_{894} $f=a^2+bc$, $Q_2(1,0,1,0)$.

X_{1100} $f=a(2a+b+c)$, $Q_1(1,0,2,2)$.

X_{1125} $f=2a+b+c$.

X_{1213} $f=(b+c)(2a+b+c)$, $Q_1(2,1,2,3)=Q_2(2,3,3,3)$.

X_{1278} $f=a(b+c)-3bc$, $Q_1(0,-3,-2,-2)=Q_2(0,-1,-1,-2)$.

X_{1654} $f=a^2-a(b+c)-b^2-bc-c^2$, $Q_1(2,-1,0,1)=Q_2(2,1,1,1)$.

X_{1743} $f=a(-3a+b+c)$, $Q_1(-3,-2,0,-2)$.

X_{2321} $f=(b+c)(-a+b+c)$, $Q_1(-1,-1,1,0)$.

$f=(-a+b+c)(2a+b+c)$, $Q_1(0,-1,2,1)$.

$f=(b+c)(-3a+b+c)$, $Q_1(-2,-2,1,-1)$.

$f=(2a+b+c)(-3a+b+c)$, $Q_1(-1,-2,2,0)$.

$f=a(b+c)+2bc$, $Q_1(0,1,2,2)=Q_2(0,3,3,2)$.

$f=a(b+c)+b^2+c^2$, $Q_2(1,1,2,1)$.

$f = a(b+c) + (b-c)^2$, $Q_2(-1, 0, 1, -1)$.

$f = a(b+c) + 2(b^2 + bc + c^2)$, $Q_1(1, 1, 3, 3)$.

$f = a^2 + (b-c)^2$, $Q_1(-3, 0, 0, -1)$.

$f = a^2 + a(b+c) - bc$, $Q_1(1, 0, 0, 1)$.

$f = a^2 + a(b+c) - 2bc$, $Q_1(-1, -2, 0, -1)$.

$f = a^2 - a(b+c) + 2bc$, $Q_2(-1, -1, 0, -2)$.

$f = a^2 - a(b+c) + b^2 + c^2$, $Q_1(-3, 1, 1, 0)$.

$f = a^2 - a(b+c) - b^2 + bc - c^2$, $Q_2(1, -1, 0, -1)$.

$f = a^2 - a(b+c) - 2(b-c)^2$, $Q_2(-3, -1, 0, -3)$.

$f = a^2 + 2a(b+c) - bc$, $Q_2(2, 0, 2, 1)$.

$f = a^2 + 2a(b+c) + b^2 + bc + c^2$, $Q_2(3, 2, 3, 3)$.

$f = 2a^2 + b^2 + c^2$, $Q_1(-1, 2, 2, 2)$.

$f = 2a^2 - b^2 + bc - c^2$, $Q_2(0, -2, 0, -2)$.

$f = 2a^2 - b^2 - bc - c^2$, $Q_1(0, -3, 0, -1)$.

$f = 2a^2 + a(b+c) + (b+c)^2$, $Q_2(1, 2, 3, 2)$.

$f = 2a^2 + a(b+c) + (b-c)^2$, $Q_1(-1, 0, 2, 1)$.

$f = 2a^2 + a(b+c) - (b-c)^2$, $Q_2(0, 0, 2, 0)$.

$f = 2a^2 + a(b+c) - b^2 - c^2$, $Q_1(0, -2, 1, 0)$.

$f = 2a^2 - a(b+c) + (b+c)^2$, $Q_2(-1, 1, 2, 0)$.

$f = 2a^2 - a(b+c) - b^2 - c^2$, $Q_2(0, -1, 1, -1)$.

$f = 2a^2 - a(b+c) + 2(b^2 - bc + c^2)$, $Q_1(-3, 2, 2, 1)$.

$f = 2a^2 + 2a(b+c) + b^2 + c^2$, $Q_1(1, 2, 2, 3)$.

References

[1] Y. Agaoka, *Degree of triangle centers and a generalization of the Euler line*, Beiträge Algebra Geom., **51** (2010), 63–89. Also available at `http://ir.lib.hiroshima-u.ac.jp/00025256`.

[2] G. Ambrus and A. Bezdek, *On iterative processes generating dense point sets*, Period. Math. Hungarica, **53** (2006), 27–44.

[3] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Math. Assoc. America, Washington, 1967.

[4] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, Math. Assoc. Amer., Washington, 1995.

[5] C. Kimberling, *Central points and central lines in the plane of a triangle*, Math. Magazine, **67** (1994), 163–187.

[6] C. Kimberling, *Constructive systems in triangle geometry*, Nieuw Arch. Wiskunde Ser.4, **15** (1997), 163–173.

[7] C. Kimberling, *Triangle Centers and Central Triangles*, Congressus Numerantium, **129**, Utilitas Math. Publ. Incorp., Winnipeg, 1998.

[8] C. Kimberling, *Bicentric pairs of points and related triangle centers*, Forum Geom., **3** (2003), 35–47.

[9] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`

[10] C. Kimberling and F. van Lamoen, *Central triangles*, Nieuw Arch. Wiskunde Ser.4, **17** (1999), 1–19.

[11] T. Lalesco, *La Géométrie du Triangle*, Éditions Jacques Gabay, 2003.

[12] M. S. Longuet-Higgins, *On the principal centers of a triangle*, Elemente Math., **56** (2001), 122–129.

Department of Mathematics, Graduate School of Science,
Hiroshima University, Higashi-Hiroshima 739-8521, Japan
e-mail address: agaoka@mis.hiroshima-u.ac.jp

(*Received January 10, 2010*)