岡山大学経済学会雑誌29(4),1998, 125～132

Factor Price Equalization : Geometrical Conditions

Takao Fujimoto and Ravindra R. Ranade†

1. Introduction

The univalence problem concerning a system of nonlinear functions on a certain domain was first seriously studied by Gale and Nikaido (1965), employing the concept of P–matrix for Jacobian matrix. Their work was to investigate the conjecture raised by Samuelson in the context of 'factor price equalization' theme. (See Samuelson (1948, 1953).) There have been many research papers on this topic, most of which centre on the local characterisation (using the Jacobian matrix at each point) to establish global univalence. Pearce (1967) claimed that the regularity of the Jacobian determinant would be sufficient for univalence. McKenzie (1967 a, 1967 b) gave counter examples to Pearce's assertion showing that the shape of the boundary of a given domain should be too restrictive to be applicable in the context of cost functions. In Kuga (1972) it is proved that the regularity of the Jacobian is sufficient if smooth substitution and strict concavity of production functions are satisfied on the whole non–negative orthant. Mas–Colell (1979 a, 1979 b, 1985) employed index

† Department of Economics, University of Kagawa

theorems to sharpen the result by Gale and Nikaido under an additional assumption of homogeneity. In Mas–Colell (1979 b) he presents a proposition that the positivity of the Jacobian at interior points together with additional positive principal minor determinants on the boundary is sufficient to obtain univalence. (See also Kehoe (1980).) It should be noted that Mas–Colell (1979 a, 1979 b) requires the homogeneity only, and does not assume concavity.

Recently Bandyopadhyay and Biswas (1994) gave another univalence theorem assuming a 'strong' monotonicity of each cost function. Their theorem now allows for singularity of the Jacobian determinant at some points.

Almost in oblivion are two conditions stated in McKenzie (1955) and Kuhn (1959) : the former is geometrical, and the latter is WARP–like. (See Samuelson (1967, p.292).) McKenzie's geometrical condition is explained in Nikaido (1968, pp.360–361) as a condition of 'nearly constant' Jacobian matrix : the cones spanned by the rows of the Jacobian matrices have a common interior point. McKenzie (1955, p.249) showed a local characterisation that if the Jacobian matrix at each point has a dominant diagonal, then his geometrical condition is satisfied. McKenzie's condition is so simple and elegant that none dared to investigate further generalisations.

In this note we present a natural extension of McKenzie's geometrical condition, which is too simple and straightforward, thus almost tautological. The functions concerned are not necessarily concave, and required only to be quasi–concave. Besides, we discuss the relationship of our condition with Kuhn's.

2. Theorems

We adopt the Euclidean space of dimension n, R^n, with its positive cone R^n_+. A given transformation $F \equiv (F_1, F_2, \cdots, F_n)'$ maps from R^n_+ into R^n_+. (The prime indicates transposition.) We assume each F_i is quasi–concave and differentiable with the Jacobian matrix of F at a point x denoted as $J(x)$. The symbol J_i means the i–th row of J. In addition we make

Assumption A1. For each pair of points x_1 and x_2 in R^n_+, the following two inequalities cannot hold simultaneously :

$$J(x_1)(x_2 - x_1) \geq 0, \text{ and} \tag{1}$$

$$J(x_2)(x_2 - x_1) \leq 0. \tag{2}$$

This assumption requires that the two cones spanned respectively by the rows of two Jacobian matrices at x_1 and x_2 cannot be separated by the hyperplane $P \equiv \{x \mid x \in R^n_+, (x_2 - x_1)' \cdot x = 0\}$. It is evident that the geometrical condition in McKenzie (1955) is a special case of the above A1.

It is easy to prove

Theorem 1. Given the assumption A1, the mapping F is univalent.

Proof. Suppose that there exist two points x_1 and x_2 such that $x_1 \neq x_2$ and $F(x_1) = F(x_2)$. Since every F_i is quasi–concave, and $F(x_1) = F(x_2)$, we have

$$J(x_1)(x_2 - x_1) \geq 0, \text{ and}$$

$$J(x_2)(x_2 - x_1) \leq 0.$$

These are impossible at the same time by the assumption A1. Thus, F is univalent. □

Remark 1. In the example of McKenzie (1967 a), the inequalities (1) and (2) do hold simultaneously for the two points given therein.

Remark 2. Strangely most propositions on univalence so far put forward cannot cover simple linear cases such as the one in which a given matrix has a negative determinant. See for example Gale and Nikaido (1965) and Mas–Colell (1979 b). Kuga (1972) assumes strict concavity of the production functions except on the rays through the origin. Note that our Theorem 1 is able to deal with the linear case in which a given matrix is regular.

Now suppose that the inequalities (1) and (2) are both valid, and take two arbitrary non–negative vectors c_1 and c_2. Premultiply (1) and (2) by these vectors respectively and sum them up, and we have

$$(c_1' \cdot J(x_1) - c_2' \cdot J(x_2)) \cdot (x_1 - x_2) \leq 0. \qquad (3)$$

Theorem 2. Suppose that for any two vectors x_1 and x_2 in R_+^n, there exist two non–negative vectors c_1 and c_2, both of which can depend upon x_1 and x_2, such that

$$(c_1' \cdot J(x_1) - c_2' \cdot J(x_2)) \cdot (x_1 - x_2) > 0. \qquad (4)$$

then F is univalent.

Proof. The inequality (4) is the negation of (3), and by contraposition the assumption A1 is to be satisfied. Thus by Theorem 1, we have the desired result. □

Corollary 1. If $x' \cdot J(x)$ is strictly monotone function on R_+^n, then F is univalent.

Proof. In Theorem 2, c_1 and c_2 can be chosen as x_1 and x_2 respectively. □

Remark 3. When every F_i is assumed to be concave, this Corollary 1 may be vacuous because each row of $J(x)$ satisfies

$$((J_i(x_i) - J_i(x_2)) \cdot (x_1 - x_2) \leq 0.$$

For monotonicity of vector functions, see Ortega and Rheinboldt (1970).

3. Kuhn's WARP-like Condition

We continue to assume that each F_i is differentiable. In Kuhn's framework (1959), each F_i is assumed to be homogeneous of degree one, and his condition is :

Assumption A2. $(x_1 - x_2)' \cdot (F(x_1) - F(x_2)) > 0$ for x_1 and x_2 in R^n_+ such that $e \cdot x_1 = 1$ and $e \cdot x_2 = 1$, where $e = (1, 1, \cdots, 1)$.

Proposition 1. Given the assumption A2, the following two inequalities cannot hold simultaneously for non-zero vectors x_1 and x_2 in R^n_+ such that $e \cdot x_1 = 1$ and $e \cdot x_2 = 1$:

$$(x_2 - x_1)' \cdot J(x_1) \geq 0 \text{ and} \tag{5}$$

$$(x_2 - x_1)' \cdot J(x_2) \leq 0. \tag{6}$$

Proof. Suppose to the contrary that the inequalities (5) and (6) are both valid. Postmultiplying (5) and (6) respectively by $-x_1$ and x_2, and adding the resulting two inequalities give $(x_1 - x_2)' \cdot (F(x_1) - F(x_2)) \leq 0$ by using Euler's identities for homogeneous functions of degree one $J(x_1) \cdot x_1 = F(x_1)$ and $J(x_2) \cdot x_2 = F(x_2)$. The contraposition establishes the proposition. □

The inequalities (5) and (6) means that two cones spanned by the columns of two Jacobian matrices cannot be separated by the hyperplane $P \equiv \{x \mid x \in R^n_+, (x_2 - x_1)' \cdot x = 0\}$. Unfortunately the logical flow is reverse to that for the case of the assumption A1, i. e., from Kuhn's condition to the inseparability.

4. Other Remarks

The assumption A1 is simple and geometrically clear, meaning that the cones spanned by the rows of Jacobian matrix do not move around wildly so that the two cones corresponding to the two end points are not separated by the hyperplane orthogonal to the vector of changes in 'factor prices'. However, the local characterisation in terms of Jacobian matrix is desirable. The task may become somewhat clumsy depending on individual cases. It should be noted again that we assume quasi–concavity in place of concavity.

If we assume concavity and homogeneity of the first degree for each function together with differentiability, it can be proved that the strict monotonicity of $x' \cdot J(x)$ implies Kuhn's condition. As is remarked in the above, however, this seems void.

We will in the next work deal with systems of quasi–convex and/or homogeneous of degree zero functions.

References

T. Bandyopadhyay and T. Biswas, Global univalence when mappings are not necessarily continuous, *Journal of Mathematical Economics* 23 (1994) 435–450.

D. Gale and H. Nikaido, The Jacobian matrix and the global univalence of mappings, *Mathematischen Annalen* 159 (1965) 81–93.

T. Kehoe, An index theorem for general equilibrium models with production, *Econometrica* 48 (1980) 1211–1232.

K. Kuga, The factor price equalization theorem, Econometrica 40 (1972) 723–736.

H. W. Kuhn, Factor Endowments and Factor Prices : Mathematical Appendix [to A. H. Land's article], *Economica* 26 (1959) 142–144.

A. Mas–Colell, Homeomorphisms of compact, convex sets and the Jacobian matrix, *SIAM Journal on Mathematical Analysis* 10 (1979 a) 1105–1109.

A. Mas–Colell, Two propositions on the global univalence of systems of cost functions, in : J. R. Green and J. A. Scheinkman, eds., *General Equilibrium, Growth, and Trade* (Academic Press, New York, 1979 b).

A. Mas–Colell, *The Theory of General Equilibrium* (Cambridge University Press, Cambridge, 1985).

L. W. McKenzie, Equality of factor prices in world trade, *Econometrica* 23 (1955) 239–257.

L. W. McKenzie, The inversion of cost functions : a counter–example, *International Economic Review* 8 (1967 a) 271–278.

L. W. McKenzie, Theorem and counter–example, *International Economic Review* 8 (1967 b) 279–285.

H. Nikaido, *Convex Structures and Economic Theory* (Academic Press, New York, 1968).

J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables* (Academic Press, New York, 1970).

I. F. Pearce, More about factor price equalization, *International Economic Review* 8 (1967) 255–270.

P. A. Samuelson, International Trade and the equalization of factor prices, *Economic Journal* 58 (1948) 163–184.

P. A. Samuelson, Prices of factors and goods in general equilibrium, *Review of Economic Studies* 21 (1953) 1–20.

P. A. Samuelson, Summary of factor–price equalization, *International Economic Review* 8 (1967) 286–295.

Factor Price Equalization : Geometrical Conditions

Takao Fujimoto and Ravindra R. Ranade

Abstract

This paper presents a geometrical approach to the univalence problem for a system of cost functions. We present a natural (almost tautological) extension of a geometrical theorem due to McKenzie : our sufficient condition is related to the non–separability of two cones formed by convex combinations of the rows of the Jacobian matrix. This means that the cones spanned by the rows of Jacobian matrix (i.e., production coefficients) do not move wildly so that the two cones corresponding to the two end points (i.e., factor price vectors) cannot be separated by the hyperplane orthogonal to the vector of changes in factor prices. Unlike most of the previous propositions, our condition can naturally include as a special case such linear systems as having a non–singular matrix. We also give an alternative condition employing the concept of monotone functions.

Dual to the above result is one more condition, which is shown to be closely connected with Kuhn's WARP–like requirement when the given functions are concave as well as homogeneous of degree one.