Math. J. Okayama Univ. **53** (2011), 83–100

FP-GR-INJECTIVE MODULES

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Abstract. In this paper, we give some characterizations of FP-gr-injective R-modules and graded right R-modules of FP-gr-injective dimension at most n. We study the existence of FP-gr-injective envelopes and FP-gr-injective covers. We also prove that (1) $({}^{\perp}\text{gr-}\mathcal{FI}, \text{gr-}\mathcal{FI})$ is a hereditary cotorsion theory if and only if R is a left gr-coherent ring, (2) If R is right gr-coherent with FP-gr-id$(R_R) \leq n$, then $(\text{gr-}\mathcal{FI}_n, \text{gr-}\mathcal{FI}_n^{\perp})$ is a perfect cotorsion theory, (3) $({}^{\perp}\text{gr-}\mathcal{FI}_n, \text{gr-}\mathcal{FI}_n)$ is a cotorsion theory, where gr-$\mathcal{FI}$ denotes the class of all FP-gr-injective left R-modules, gr-$\mathcal{FI}_n$ is the class of all graded right R-modules of FP-gr-injective dimension at most n. Some applications are given.

1. Introduction.

All rings considered are associative with identity element and the R-modules are unital. By R-Mod we will denote the Grothendieck category of all left R-modules. Let G be a multiplicative group with neutral element e. A graded ring R is a ring with identity 1 together with a direct decomposition $R = \bigoplus_{\sigma \in G} R_\sigma$ (as additive subgroups) such that $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for all $\sigma, \tau \in G$. Thus R_e is a subring of R, $1 \in R_e$ and R_σ is an R_e-bimodule for every $\sigma \in G$. A graded left R-module is a left R-module M endowed with an internal direct sum decomposition $M = \bigoplus_{\sigma \in G} M_\sigma$, where each M_σ is a subgroup of the additive group of M satisfying $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ for all $\sigma, \tau \in G$. For graded left R-modules M and N, we put

$$\text{Hom}_{R\text{-gr}}(M, N) = \{f : M \to N | f \text{ is } R\text{-linear and } f(M_\sigma) \subseteq N_\sigma \quad \forall \sigma \in G\}$$

is the group of all morphisms from M to N in the category R-gr of all graded left R-modules. It is well known that R-gr is a Grothendieck category. An R-linear map $f : M \to N$ is said to be a graded morphism of degree τ, $\tau \in G$ if $f(M_\sigma) \subseteq M_{\sigma\tau}$ for all $\sigma \in G$. Graded morphisms of degree σ build an additive subgroup $\text{HOM}_R(M, N)_\sigma$ of $\text{Hom}_R(M, N)$. Then $\text{HOM}_R(M, N) = \bigoplus_{\sigma \in G} \text{HOM}_R(M, N)_\sigma$ is a graded abelian group of type G. We will denote $\text{Ext}^i_{R\text{-gr}}$ and EXT^i_R as the right derived functors of $\text{Hom}_{R\text{-gr}}$ and HOM_R.

Let M be a graded right R-module and N a graded left R-module. The abelian group $M \otimes_R N$ may be graded by putting $(M \otimes_R N)_\sigma$, $\sigma \in G$, equal

Mathematics Subject Classification. Primary 16W50; Secondary 18A40, 18G15.

Key words and phrases. FP-gr-injective module, graded flat module, envelope and cover, cotorsion theory.

This research was partially supported by National Natural Science Foundation of China, TRAPOYT and the Cultivation Fund of the Key Scientific and Technical Innovation Project, Ministry of Education of China.

to the additive subgroup generated by elements $x \otimes y$ with $x \in M_\alpha$, $y \in N_\beta$ such that $\alpha\beta = \sigma$. The object of $\mathbb{Z}$-gr thus defined will be called the graded tensor product of M and N.

If $M = \bigoplus_{\sigma \in G} M_\sigma$ is a graded left R-module and $\sigma \in G$, then $M(\sigma)$ is the graded left R-module obtained by putting $M(\sigma)_\tau = M_{\tau\sigma}$ for all $\tau \in G$; the graded module $M(\sigma)$ is called the σ-suspension of M. We can see the σ-suspension as an isomorphism of categories $T_\sigma : R\text{-gr} \to R\text{-gr}$, given on objects as $T_\sigma(M) = M(\sigma)$ for $M \in R$-gr.

For any element $m = \sum_{\sigma \in G} m_\sigma$ of R, $\mathrm{Supp}(m) = \{\sigma \in G | m_\sigma \neq 0\}$. Consider $\{M_i | i \in I\}$ a set of graded left R-modules and let $\{\prod_{i \in I} M_i, \pi_i\}$ be the direct product in R-Mod of the underlying left R-modules M_i, where $\pi_j : \prod_{i \in I} M_i \to M_j$ denotes the j-th canonical projection for each $j \in I$. Given $m \in \prod_{i \in I} M_i$, define $\mathrm{SUPP}(m) = \bigcup_{i \in I} \mathrm{Supp}(\pi_i(m)) \subset G$. We can define $\prod_{i \in I}^{R\text{-gr}} M_i = \{m \in \prod_{i \in I} M_i | \mathrm{SUPP}(m) \text{ is finite}\}$. Then $\{\prod_{i \in I}^{R\text{-gr}} M_i, \pi_i\}$ is the direct product of the graded left R-modules $\{M_i | i \in I\}$. It is a graded left R-module, where $(\prod_{i \in I}^{R\text{-gr}} M_i)_\sigma = \{m \in \prod_{i \in I}^{R\text{-gr}} M_i | \mathrm{SUPP}(m) \subset \{\sigma\}\}$. Observe that, as R_e-modules $(\prod_{i \in I}^{R\text{-gr}} M_i)_\sigma \cong \prod_{i \in I} (M_i)_\sigma$ for any $\sigma \in G$.

Given a graded left R-module M, we can define the graded character module of M as $M^+ = \mathrm{HOM}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. We note then that it can be seen as $M^+ = \bigoplus_{\sigma \in G} \mathrm{Hom}_{\mathbb{Z}}(M_{\sigma^{-1}}, \mathbb{Q}/\mathbb{Z})$.

The injective objects of R-gr will be called gr-injective modules. Projective (resp. flat) objects of R-gr will be called projective (resp. flat) graded modules because M is gr-projective (resp. gr-flat) if and only if it is a projective (resp. flat) graded module. We will denote the gr-injective dimension of a graded module M by gr-id$M$ and fd$M$ will denote the flat dimension of M. We will denote the gr-injective envelope of M by $E^g(M)$. We will call FP-gr-injective module to those graded R-module M such that $\mathrm{EXT}_R^1(N, M) = 0$ for any finitely presented graded R-module N. It can be proved that if R is gr-noetherian, M is gr-injective if and only if M is FP-gr-injective and that in the case that R is gr-coherent, i.e. a graded ring R such that given a family of graded flat R-modules $\{F_i\}_{i \in I}$, then the graded R-module $\prod_{i \in I}^{R\text{-gr}} F_i$ is flat, M is FP-gr-injective if and only if M^+ is flat. The FP-gr-injective dimension of a graded R-module M will be the least integer n such that $\mathrm{EXT}_R^{n+1}(N, M) = 0$ for any finitely presented graded R-module N.

The forgetful functor $U : R\text{-gr} \to R\text{-Mod}$ associates to M the underlying ungraded R-module. This functor has a right adjoint F which associated to $M \in R$-Mod the graded R-module $F(M) = \bigoplus_{\sigma \in G} ({}^\sigma M)$, where each ${}^\sigma M$ is a copy of M written $\{{}^\sigma x : x \in M\}$ with R-module structure defined by $r * {}^\tau x = {}^{\sigma\tau}(rx)$ for each $r \in R_\sigma$. If $f : M \to N$ is R-linear, then $F(f) : F(M) \to F(N)$ is a graded morphism given by $F(f)({}^\sigma x) = {}^\sigma f(x)$.

Let $\mathcal{F}$ be a class of graded R-modules for a graded ring R. If $\varphi : C \to M$ is a graded morphism, where $C \in \mathcal{F}$ and $M \in R$-gr, then $\varphi : C \to M$ is called an $\mathcal{F}$-precover of M if $\mathrm{Hom}_{R\text{-gr}}(C', C) \to \mathrm{Hom}_{R\text{-gr}}(C', M) \to 0$ is exact for all $C' \in \mathcal{F}$. Moreover, if whenever a graded morphism $f : C \to C$ such that $\varphi \circ f = \varphi$ is an automorphism of C, then $\varphi : C \to M$ is called an $\mathcal{F}$-cover of M. $\mathcal{F}$-envelope and $\mathcal{F}$-preenvelope are defined dually. Let $\varphi : C \to M$ be an $\mathcal{F}$-cover of M. If for any graded morphism $f : C' \to M$ with $C' \in \mathcal{F}$, there is a unique graded morphism $g : C' \to C$ such that $\varphi g = f$, then we say that φ has the unique mapping property. Dually we have the definition of an $\mathcal{F}$-envelope has the unique mapping property.

2. FP-gr-injective envelopes of graded modules.

In this section, we give some characterizations of FP-gr-injective modules and prove that $({}^{\perp}\text{gr-}\mathcal{FI}, \text{gr-}\mathcal{FI})$ is a hereditary cotorsion theory if and only if R is a left gr-coherent ring, where gr-$\mathcal{FI}$ denotes the class of all FP-gr-injective left R-modules.

An exact sequence $0 \to M' \to M \to M'' \to 0$ in R-gr is said to be gr-pure if for any $N \in$ gr-R, the sequence $0 \to N \otimes_R M' \to N \otimes_R M \to N \otimes_R M'' \to 0$ is exact in $\mathbb{Z}$-gr.

Proposition 2.1. *Let R be a ring graded by a group G. Then the following are equivalent for a graded left R-module M:*

(1) M is FP-gr-injective;

(2) The functor $HOM_R(-, M)$ is exact with respect to every exact sequence $0 \to A \to B \to C \to 0$ in R-gr with C finitely presented;

(3) $M(\sigma)$ is FP-gr-injective for all $\sigma \in G$;

(4) $M(\sigma)$ is gr-injective with respect to every exact sequence $0 \to A \to B \to C \to 0$ in R-gr with C finitely presented for all $\sigma \in G$;

(5) M is gr-pure in every graded left R-module that contains it;

(6) M is gr-pure in every gr-injective left R-module that contains it;

(7) M is gr-pure in $E^g(M)$.

Proof. (1) $\Leftrightarrow$ (2) is clear by definition. (3) $\Rightarrow$ (1) and (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are obvious.

(2) $\Rightarrow$ (3) Let $0 \to A \to B \to C \to 0$ be exact in R-gr with C finitely presented. Then

$$0 \longrightarrow \mathrm{HOM}_R(C, M)_\sigma \longrightarrow \mathrm{HOM}_R(B, M)_\sigma \longrightarrow \mathrm{HOM}_R(A, M)_\sigma \longrightarrow 0$$

is exact for all $\sigma \in G$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{HOM}_R(C,M)_{\sigma\tau} & \longrightarrow & \mathrm{HOM}_R(B,M)_{\sigma\tau} & \longrightarrow & \mathrm{HOM}_R(A,M)_{\sigma\tau} & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(C,M(\sigma\tau)) & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(B,M(\sigma\tau)) & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(A,M(\sigma\tau)) & & \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \mathrm{HOM}_R(C,M(\sigma))_{\tau} & \longrightarrow & \mathrm{HOM}_R(B,M(\sigma))_{\tau} & \longrightarrow & \mathrm{HOM}_R(A,M(\sigma))_{\tau} & &
\end{array}$$

with the upper row exact for every $\tau \in G$. So

$$0 \longrightarrow \mathrm{HOM}_R(C,M(\sigma)) \longrightarrow \mathrm{HOM}_R(B,M(\sigma)) \longrightarrow \mathrm{HOM}_R(A,M(\sigma)) \longrightarrow 0$$

is exact, which means that $M(\sigma)$ is FP-gr-injective for all $\sigma \in G$.

(2) $\Leftrightarrow$ (4) By $\mathrm{HOM}_R(-,M)_\sigma = \mathrm{Hom}_{R\text{-gr}}(-,M(\sigma))$ for every $\sigma \in G$.

(1) $\Rightarrow$ (5) Let $0 \to M \to L \to L/M \to 0$ be exact, N a finitely presented graded left R-module. Then

$$\begin{aligned} 0 \longrightarrow & \mathrm{HOM}_R(N,M) \longrightarrow \mathrm{HOM}_R(N,L) \\ & \longrightarrow \mathrm{HOM}_R(N,L/M) \longrightarrow \mathrm{EXT}^1_R(N,M) = 0 \end{aligned}$$

is exact. So M is gr-pure in L by [9, Proposition 3.1].

(7) $\Rightarrow$ (1) Let N be any finitely presented graded left R-module. Then

$$\begin{aligned} 0 \longrightarrow & \mathrm{HOM}_R(N,M) \longrightarrow \mathrm{HOM}_R(N,E^g(M)) \\ & \longrightarrow \mathrm{HOM}_R(N,E^g(M)/M) \longrightarrow 0 \end{aligned}$$

is exact, and so $\mathrm{EXT}^1_R(N,M) = 0$, which implies that M is FP-gr-injective. $\square$

Remark 2.2. *By the definition and Proposition 2.1, we see that the class of all FP-gr-injective R-modules is closed under graded direct summands, graded direct products and graded pure submodules.*

Lemma 2.3. *Let R be a ring graded by a group G. If M is an FP-injective left R-module, then $F(M)$ is FP-gr-injective.*

Proof. Let $0 \to A \xrightarrow{f} B \to C \to 0$ be exact in R-gr with C finitely presented, $g : A \to F(M)$ a graded morphism. Since F is a right adjoint functor of the forgetful functor, we have the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & \swarrow & & & & & \\
 & & M & & & & & &
\end{array}$$

Now, again by the adjoint situation between the forgetful functor and F we have a graded morphism $B \to F(M)$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & \swarrow & & & & & \\ & & F(M) & & & & & & \end{array}$$

which shows that $F(M)$ is gr-injective with respect to the exact sequence $0 \to A \xrightarrow{f} B \to C \to 0$. Let $\sigma \in G$ and $g : A \to F(M)(\sigma)$ be a graded morphism. Since $0 \to A(\sigma^{-1}) \xrightarrow{T_{\sigma^{-1}}(f)} B(\sigma^{-1}) \to C(\sigma^{-1}) \to 0$ is exact and $C(\sigma^{-1})$ is finitely presented, there exists a graded morphism $h : B(\sigma^{-1}) \to F(M)$ such that $hT_{\sigma^{-1}}(f) = T_{\sigma^{-1}}(g)$, and so $T_\sigma(h)f = g$ for $T_\sigma(h) : B \to F(M)(\sigma)$, which gives that $F(M)(\sigma)$ is gr-injective with respect to the exact sequence $0 \to A \to B \to C \to 0$ for all $\sigma \in G$. Therefore $F(M)$ is FP-gr-injective by Proposition 2.1. □

Corollary 2.4. *Let R be a ring graded by a finite group G and $M \in R$-gr. Then M is FP-gr-injective if and only if M is an FP-injective left R-module.*

Proof. "$\Leftarrow$" By Lemma 2.3, $F(M)$ is FP-gr-injective, and so M is FP-gr-injective since M is a direct summand of $F(M)$.

"$\Rightarrow$" Let $0 \to A \to B \to C \to 0$ be exact in R-Mod with C finitely presented. Then $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact in R-gr and $F(C)$ is finitely presented since G is finite. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(F(C), M) & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(F(B), M) & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(F(A), M) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(C, M) & \longrightarrow & \mathrm{Hom}_R(B, M) & \longrightarrow & \mathrm{Hom}_R(A, M) & & \end{array}$$

with the upper row exact. Therefore M is an FP-injective left R-module. □

Theorem 2.5. *Let R be a ring graded by a group G. Then every graded R-module has an FP-gr-injective preenvelope.*

Proof. Let M be a graded R-module. We take $\mathcal{N}_\beta$ an infinite cardinal number such that $\mathrm{Card}(M)\mathrm{Card}(R)\mathrm{Card}(G) \leq \mathcal{N}_\beta$. Set

$$Y = \{A | A \text{ is an FP-gr-injective } R\text{-module and } \mathrm{Card}(A) \leq \mathcal{N}_\beta\}.$$

Let $\{A_i\}_{i \in I}$ be a family of representatives of this class with the index set I. Let $H_i = \mathrm{Hom}_{R\text{-gr}}(M, A_i)$ for every $i \in I$ and let $B = \prod_{i \in I}^{R\text{-gr}} (\prod_{j \in H_i}^{R\text{-gr}} (A_i)_j)$, where $(A_i)_j = A_i$ for each $j \in H_i$. Then B is FP-gr-injective. Define $\varphi : M \to B$ so that the composition of φ with the projective map $B \to \prod_{j\in H_i}^{R\text{-gr}}(A_i)_j$ maps $x \in B_\sigma$ to $(h(x))_{h\in H_i}$ for any $\sigma \in G$. Then φ is a graded morphism. We claim that $\varphi : M \to B$ is an FP-gr-injective preenvelope. Let $\varphi' : M \to B'$ with B' an FP-gr-injective R-module. By [9, Lemma 2.3], the graded submodule $\varphi'(M)$ can be enlarged to a graded pure submodule $\varphi'(M)^* \subseteq B'$ with $\mathrm{Card}(\varphi'(M)^*) \leq \mathcal{N}_\beta$ and $\varphi'(M)^*$ is FP-gr-injective by Remark 2.2. Thus $\varphi'(M)^*$ is isomorphic to one of the A_i. By the construction of the map φ, it is easy to see that φ' can be factored through φ. □

Definition 2.6. *([9]) A pair $(\mathcal{F},\mathcal{C})$ of classes of graded R-modules is a cotorsion theory in R-gr if the following properties are satisfied:*
$Ext^1_{R\text{-}gr}(F,C) = 0$ for every $F \in \mathcal{F}$, $C \in \mathcal{C}$.
$Ext^1_{R\text{-}gr}(F,C) = 0$ for every $F \in \mathcal{F}$, implies $C \in \mathcal{C}$.
$Ext^1_{R\text{-}gr}(F,C) = 0$ for every $C \in \mathcal{C}$, implies $F \in \mathcal{F}$.
A cotorsion theory $(\mathcal{F},\mathcal{C})$ is called hereditary if whenever $0 \to F' \to F \to F'' \to 0$ is exact in R-gr with F, $F'' \in \mathcal{F}$, then F' is also in $\mathcal{F}$. A cotorsion theory $(\mathcal{F},\mathcal{C})$ is said to be perfect if every graded R-module has an $\mathcal{F}$-cover and an $\mathcal{C}$-envelope.

Let $\mathcal{FI}$ denote the class of all FP-injective left R-modules. It is well known that $({}^\perp\mathcal{FI},\mathcal{FI})$ is a hereditary cotorsion theory if and only if R is a left coherent ring. Here we have a graded version.

Theorem 2.7. *Let gr-$\mathcal{FI}$ denote the class of all FP-gr-injective left R-modules. Then $({}^\perp\text{gr-}\mathcal{FI}, \text{gr-}\mathcal{FI})$ is a hereditary cotorsion theory if and only if R is a left gr-coherent ring.*

Proof. "⇒" Let I be a finitely generated graded left ideal of R, N an FP-injective left R-module and let $0 \to N \to E \to C \to 0$ be exact in R-Mod with E injective. Then $0 \to F(N) \to F(E) \to F(C) \to 0$ is exact in R-gr with $F(E)$ gr-injective, and so $F(C)$ is FP-gr-injective by Lemma 2.3 and hypothesis. Hence

$$\mathrm{Ext}^1_{R\text{-gr}}(I,F(N)) \cong \mathrm{Ext}^2_{R\text{-gr}}(R/I,F(N)) \cong \mathrm{Ext}^1_{R\text{-gr}}(R/I,F(C)) = 0.$$

Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(I,F(N)) & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(I,F(E)) & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(I,F(C)) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(I,N) & \longrightarrow & \mathrm{Hom}_R(I,E) & \longrightarrow & \mathrm{Hom}_R(I,C) & & \end{array}$$

with the upper row exact. Thus $\mathrm{Ext}_R^1(I,N)=0$, which means that I is finitely presented.

"$\Leftarrow$" Let $X \in {}^{\perp}\text{gr-}\mathcal{FI}$. Then $X(\sigma) \in {}^{\perp}\text{gr-}\mathcal{FI}$ for all $\sigma \in G$ by a proof dual to that of Lemma 2.3. Let $M \in ({}^{\perp}\text{gr-}\mathcal{FI})^{\perp}$ and N be a finitely presented graded left R-module. Then $N \in {}^{\perp}\text{gr-}\mathcal{FI}$ and $M(\sigma) \in ({}^{\perp}\text{gr-}\mathcal{FI})^{\perp}$ for all $\sigma \in G$ by analogy with the proof of Lemma 2.3. Thus $\mathrm{EXT}_R^1(N,M)_\sigma = \mathrm{Ext}_{R\text{-gr}}^1(N, M(\sigma)) = 0$, and so $\mathrm{EXT}_R^1(N,M)=0$, which implies that $M \in \text{gr-}\mathcal{FI}$. Let $0 \to A \to B \to C \to 0$ be exact in R-gr with A and B FP-gr-injective. Then $0 \to C^+ \to B^+ \to A^+ \to 0$ is exact and A^+, B^+ are flat, and so C^+ is flat. Hence C is FP-gr-injective. It follows that $({}^{\perp}\text{gr-}\mathcal{FI}, \text{gr-}\mathcal{FI})$ is a hereditary cotorsion theory. □

3. FP-gr-injective covers of graded modules.

In this section, we give some characterizations of gr-coherent rings and prove that if R is left gr-coherent, then every graded left R-module has an FP-gr-injective cover. Some applications are given.

Lemma 3.1. *Let R be a graded ring, A a finitely generated graded left R-module. Then A is finitely presented if and only if $Hom_{R\text{-}gr}(A, \varinjlim M_i) \cong \varinjlim Hom_{R\text{-}gr}(A, M_i)$, where $\{M_i\}_{i\in I}$ is a family of gr-injective left R-modules.*

Proof. "$\Rightarrow$" By [15, Chap.V, Proposition 3.4].

"$\Leftarrow$" Let E be a gr-injective cogenerator of R-gr. Define $H : R\text{-gr} \to R\text{-gr}$ as follows. Let $H(N) = \prod_{i\in I_N}^{R\text{-gr}} E_i$, where $E_i = E$ and $I_N = \mathrm{Hom}_{R\text{-gr}}(N,E)$. If $\alpha \in \mathrm{Hom}_{R\text{-gr}}(N_1,N_2)$, let $\alpha^* : \mathrm{Hom}_{R\text{-gr}}(N_2,E) \to \mathrm{Hom}_{R\text{-gr}}(N_1,E)$ be canonical. Then $H(\alpha) : H(N_1) \to H(N_2)$ via $\beta \mapsto \beta \cdot \alpha^*$. Note that $H(N)$ is gr-injective. The evaluation map $h_N : N \to H(N)$ yields a natural transformation.

Let (X_i, φ_{ji}) be a direct system of graded R-modules. Then $(H(X_i), H(\varphi_{ji}))$ is a direct system and

$$0 \to \varinjlim X_i \to \varinjlim H(X_i) \to \varinjlim H(X_i)/X_i \to 0$$

is exact. So we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(A, \varinjlim X_i) & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(A, \varinjlim H(X_i)) & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(A, \varinjlim H(X_i)/X_i) \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
0 & \longrightarrow & \varinjlim \mathrm{Hom}_{R\text{-gr}}(A, X_i) & \longrightarrow & \varinjlim \mathrm{Hom}_{R\text{-gr}}(A, H(X_i)) & \longrightarrow & \varinjlim \mathrm{Hom}_{R\text{-gr}}(A, H(X_i)/X_i)
\end{array}$$

Since β is an isomorphism, α is monic. Similarly, we have γ is monic. So α is an isomorphism, which implies that A is finitely presented by [15, Chap.V, Proposition 3.4]. □

Theorem 3.2. *The following are equivalent for a ring R graded by a group G:*

(1) R is left gr-coherent;

(2) Any direct limit of FP-gr-injective left R-modules is FP-gr-injective;

(3) $EXT^1_R(N, \varinjlim M_i) \to \varinjlim EXT^1_R(N, M_i)$ is an isomorphism for any finitely presented graded left R-module N and direct system $\{M_i\}_{i\in\Lambda}$ of graded left R-modules;

(4) $EXT^2_R(N, M) = 0$ for any finitely presented graded left R-module N and FP-gr-injective left R-module M.

Proof. (1) ⇒ (4) and (3) ⇒ (2) are obvious.

(1) ⇒ (3) Let N be any finitely presented graded left R-module. Then there exists an exact sequence $0 \to K \to P \to N \to 0$ in R-gr with P finitely generated projective and K finitely generated . Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{HOM}_R(P, \varinjlim M_i) & \longrightarrow & \mathrm{HOM}_R(K, \varinjlim M_i) & \longrightarrow & \mathrm{EXT}^1_R(N, \varinjlim M_i) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
\varinjlim \mathrm{HOM}_R(P, M_i) & \longrightarrow & \varinjlim \mathrm{HOM}_R(K, M_i) & \longrightarrow & \varinjlim \mathrm{EXT}^1_R(N, M_i) & \longrightarrow & 0
\end{array}$$

So $\mathrm{EXT}^1_R(N, \varinjlim M_i) \to \varinjlim \mathrm{EXT}^1_R(N, M_i)$ is an isomorphism.

(2) ⇒ (1) Let I be a finitely generated graded left ideal of R and $\{M_i\}_{i\in\Lambda}$ be a family of gr-injective left R-modules. Then $\varinjlim M_i$ is FP-gr-injective, and so $\mathrm{EXT}^1_R(R/I, \varinjlim M_i) = 0$. Thus we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(R/I, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(R, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(I, \varinjlim M_i) & \longrightarrow & 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \longrightarrow & \varinjlim \mathrm{Hom}_{R\text{-gr}}(R/I, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}_{R\text{-gr}}(R, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}_{R\text{-gr}}(I, M_i) & \longrightarrow & 0
\end{array}$$

Since α, β are isomorphisms, then γ is an isomorphism, and so I is finitely presented by Lemma 3.1.

(4) ⇒ (1) By analogy with the proof of Theorem 2.7. □

Theorem 3.3. *Let R be left gr-coherent. Then every graded left R-module has an FP-gr-injective cover.*

Proof. Let M be any graded left R-module and $A \to M$ be any graded morphism with A FP-gr-injective. We want to show that $A \to M$ can be factored through an FP-gr-injective left R-module B with $\mathrm{Card}(B) \leq \mathcal{N}_\beta$ for some cardinal number $\mathcal{N}_\beta$. If $\mathrm{Card}(A) \leq \mathcal{N}_\beta$, set $A = B$. So suppose that

$\mathrm{Card}(A) > \mathcal{N}_\beta$. Consider a graded submodule $S \subseteq A$ maximal with respect to the two properties that S is gr-pure in A and that $S \subseteq \mathrm{Ker}(A \to M)$. Let $B = A/S$. Then B is FP-gr-injective by Remark 2.2 and Theorem 2.7. We wish to argue that $\mathrm{Card}(B) \le \mathcal{N}_\beta$. Consider a submodule $S' \subseteq A$ maximal with respect to the two properties that S' is pure in A and that $S' \subseteq \mathrm{Ker}(A \to M)$. Then $S' \subseteq S$ and $\mathrm{Card}(A/S') \le \mathcal{N}_\beta$ by the proof of [14, Lemma 2.5]. Since $0 \to S/S' \to A/S' \to A/S \to 0$ is exact, we have $\mathrm{Card}(B) \le \mathcal{N}_\beta$.

Set $Y = \{B | B$ is an FP-gr-injective left R-module and $\mathrm{Card}(B) \le \mathcal{N}_\beta\}$. Let $\{B_i\}_{i\in I}$ be a family of representatives of this class with the index set I. Then $\bigoplus_{i\in I} B_i^{(\mathrm{Hom}_{R\text{-gr}}(B_i,M))} \to M$ is an FP-gr-injective precover by analogy with the proof of [14, Lemma 2.4], which implies that every graded left R-module has an FP-gr-injective cover by Theorem 3.2 and [1, Theorem 2.10]. □

Lemma 3.4. *Let R be a ring graded by a group G. Then $0 \to A \to B \to C \to 0$ is a gr-pure exact sequence in R-gr if and only if $0 \to A(\sigma) \to B(\sigma) \to C(\sigma) \to 0$ is gr-pure exact for all $\sigma \in G$.*

Proof. "$\Rightarrow$" Let M be a graded right R-module and $\sigma \in G$. We have to prove the exactness of

$$0 \longrightarrow M \otimes_R A(\sigma) \longrightarrow M \otimes_R B(\sigma) \longrightarrow M \otimes_R C(\sigma) \longrightarrow 0,$$

which is equivalent to proving the exactness of each of the homogeneous components

$$0 \longrightarrow (M \otimes_R A(\sigma))_\tau \longrightarrow (M \otimes_R B(\sigma))_\tau \longrightarrow (M \otimes_R C(\sigma))_\tau \longrightarrow 0,$$

i.e., the exactness of

$$0 \longrightarrow M_\alpha \otimes_{R_e} A(\sigma)_\beta \longrightarrow M_\alpha \otimes_{R_e} B(\sigma)_\beta \longrightarrow M_\alpha \otimes_{R_e} C(\sigma)_\beta \longrightarrow 0$$

with $\alpha\beta = \tau$. Since $0 \to A \to B \to C \to 0$ is gr-pure exact, we have

$$0 \longrightarrow M_\alpha \otimes_{R_e} A_{\beta\sigma} \longrightarrow M_\alpha \otimes_{R_e} B_{\beta\sigma} \longrightarrow M_\alpha \otimes_{R_e} C_{\beta\sigma} \longrightarrow 0$$

is exact with $\alpha\beta\sigma = \tau\sigma$, which implies that $0 \to A(\sigma) \to B(\sigma) \to C(\sigma) \to 0$ is gr-pure exact.

"$\Leftarrow$" is trivial. □

A graded left R-module Q is called pure gr-injective if for every pure sequence $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ in R-gr and every graded morphism $\varphi : L \to Q$, there exists $\psi : M \to Q$ such that $\psi\alpha = \varphi$.

Lemma 3.5. *Let R be a ring graded by a group G. Then H is a pure gr-injective left R-module if and only if $H(\sigma)$ is pure gr-injective for all $\sigma \in G$.*

Proof. By analogy with the proof of Lemma 2.3. □

Proposition 3.6. *The following are true for any graded ring R of type G:*

(1) A graded left R-module M is FP-gr-injective if and only if for any pure gr-injective left R-module H, every graded morphism $f : M \to H$ factors through a gr-injective left R-module.

(2) If M is a pure gr-injective left R-module and $f : C \to M$ is an FP-gr-injective cover of M, then C is gr-injective.

Proof. (1) "$\Rightarrow$" Consider the exact sequence $0 \to M \to E^g(M) \to C \to 0$. Then the sequence is gr-pure by Proposition 2.1. So there exists a graded morphism $g : E^g(M) \to H$ such that $g|_M = f$, as required.

"$\Leftarrow$" It is enough to show that the exact sequence $0 \to M \to E^g(M) \to C \to 0$ is gr-pure. Let H be a graded right R-module. Then $H^+(\sigma^{-1})$ is pure gr-injective for all $\sigma \in G$ by Lemma 3.5. For any graded morphism $f : M \to H^+(\sigma^{-1})$, there are a gr-injective left R-module E and graded morphisms $g : M \to E$, $h : E \to H^+(\sigma^{-1})$ such that $f = hg$ by hypothesis. Thus there exists a graded morphism $k : E^g(M) \to E$ such that $k|_M = g$, and so $hk|_M = f$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}_{R\text{-gr}}(E^g(M), H^+(\sigma^{-1})) & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(M, H^+(\sigma^{-1})) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{HOM}_R(E^g(M), H^+)_{\sigma^{-1}} & \longrightarrow & \mathrm{HOM}_R(M, H^+)_{\sigma^{-1}} & & \\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}_{\mathbb{Z}}((H \otimes_R E^g(M))_\sigma, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}((H \otimes_R M)_\sigma, \mathbb{Q}/\mathbb{Z}) & &
\end{array}$$

with the upper row exact. Then $0 \to (H \otimes_R M)_\sigma \to (H \otimes_R E^g(M))_\sigma$ is exact for all $\sigma \in G$. Therefore $0 \to H \otimes_R M \to H \otimes_R E^g(M) \to H \otimes_R C \to 0$ is exact and M is FP-gr-injective.

(2) By (1), there exist a gr-injective left R-module E and graded morphisms $g : C \to E$, $h : E \to M$ such that $f = hg$, and so there is a graded morphism $k : E \to C$ such that $fk = h$ since f is a cover. Thus $fkg = f$ and kg is an isomorphism, which implies that C is isomorphic to a direct summand of E, and hence C is gr-injective. □

Lemma 3.7. *Let R be left gr-coherent and M a graded left R-module. Then FP-gr-id$M \leq n$ if and only if there is an exact sequence $0 \to M \to E^0 \to \cdots \to E^n \to 0$ in R-gr with each E^i FP-gr-injective.*

Proof. Easy. □

Proposition 3.8. *The following are equivalent for a left gr-coherent ring R of type G:*

(1) ${}_RR$ is FP-gr-injective;

(2) Every (finitely presented) graded left R-module has an epic FP-gr-injective cover;

(3) Every (finitely presented) graded right R-module has a monic gr-flat preenvelope;

(4) Every (finitely presented) graded right R-module is a graded submodule of a gr-flat right R-module.

Proof. (2) $\Rightarrow$ (1) and (3) $\Leftrightarrow$ (4) are obvious.

(1) $\Rightarrow$ (2) Let M be a graded left R-module. Then M has an FP-gr-injective cover $f : C \to M$. On the other hand, there is an exact sequence $\bigoplus_{\sigma\in S} R(\sigma) \to M \to 0$ for some $S \subseteq G$. Let N be any finitely presented graded left R-module and $0 \to K \to P \to N \to 0$ be exact in R-gr, where P is finitely generated projective and K is finitely generated. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{HOM}_R(N, \oplus_{\sigma\in S} R(\sigma)) & \to & \mathrm{HOM}_R(P, \oplus_{\sigma\in S} R(\sigma)) & \to & \mathrm{HOM}_R(K, \oplus_{\sigma\in S} R(\sigma)) & & \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
\mathrm{Hom}_R(N, \oplus_{\sigma\in S} R(\sigma)) & \to & \mathrm{Hom}_R(P, \oplus_{\sigma\in S} R(\sigma)) & \to & \mathrm{Hom}_R(K, \oplus_{\sigma\in S} R(\sigma)) & & \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
\oplus_{\sigma\in S}\mathrm{HOM}_R(N, R(\sigma)) & \to & \oplus_{\sigma\in S}\mathrm{HOM}_R(P, R(\sigma)) & \to & \oplus_{\sigma\in S}\mathrm{HOM}_R(K, R(\sigma)) & \longrightarrow & 0
\end{array}$$

with the lower row exact. Then the upper row exact. Hence

$$\mathrm{EXT}^1_R(N, \oplus_{\sigma\in S} R(\sigma)) = 0$$

and $\bigoplus_{\sigma\in S} R(\sigma)$ is FP-gr-injective. So f is epic.

(1) $\Rightarrow$ (3) Let E be any gr-injective right R-module. Then there exists an exact sequence $\bigoplus_{\sigma\in S} R(\sigma) \to E^+ \to 0$ for some $S \subseteq G$, and hence $0 \to E^{++} \to (\bigoplus_{\sigma\in S} R(\sigma))^+$ is exact. Since E^{++} is gr-injective and $(\bigoplus_{\sigma\in S} R(\sigma))^+$ is flat, we have E^{++} is flat, and so E is flat and (3) follows.

(3) $\Rightarrow$ (1) Since $({}_RR)^+$ has a monic gr-flat preenvelope, $({}_RR)^+$ is flat, and so ${}_RR$ is FP-gr-injective. $\square$

We denote l.FP-gr-dim$R = \sup\{$FP-gr-id$M | M$ is a graded left R-module$\}$.

Proposition 3.9. *The following are equivalent for a left gr-coherent ring R of type G:*

(1) l.FP-gr-dim$R \leq 1$;

(2) Every graded left R-module has a monic FP-gr-injective cover;

(3) Every graded right R-module has an epic gr-flat preenvelope;

(4) The kernel of any FP-gr-injective (pre)cover of a graded left R-module is FP-gr-injective;

(5) The cokernel of any FP-gr-injective preenvelope of a graded left R-module is FP-gr-injective;

(6) The cokernel of any gr-flat preenvelope of a graded right R-module is gr-flat;

(7) The kernel of any gr-flat (pre)cover of a graded right R-module is gr-flat.

Proof. (1) $\Rightarrow$ (2) Let M be any graded left R-module. Then M has an FP-gr-injective cover $f : C \to M$. Since $0 \to \mathrm{Ker}f \to C \to \mathrm{Im}f \to 0$ is exact, we have $\mathrm{Im}f$ is FP-gr-injective by Lemma 3.7. So the inclusion $\mathrm{Im}f \to M$ is a monic FP-gr-injective cover.

(2) $\Rightarrow$ (4) Let $f : L \to M$ be an FP-gr-injective precover of a graded left R-module M and $K = \mathrm{Ker}f$ and let $g : C \to M$ be a monic FP-gr-injective cover. Consider the pullback of f and g:

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & C \\ \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle g} \\ L & \xrightarrow{f} & M \end{array}$$

By the definition of precover, there is a factorization $C \to L \to M$ of the graded morphism $C \to M$. This means that there is a graded morphism $\gamma : C \to P$ such that $\alpha\gamma = 1_C$, and so $P \cong K \oplus C$ since $\mathrm{Ker}\alpha \cong K$. Similarly $P \cong L$. Thus $K \oplus C \cong L$, which gives that K is FP-gr-injective.

(4) $\Rightarrow$ (1) It is enough to show that any quotient of an FP-gr-injective left R-module is FP-gr-injective. Let M be a quotient of an FP-gr-injective left R-module. Note that M has an FP-gr-injective cover $f : C \to M$. Then f is an epimorphism. Since $\mathrm{Ker}f$ is FP-gr-injective, we have $\mathrm{Ker}f^+$ and C^+ are flat, and so M^+ is flat. Thus M is FP-gr-injective since R is left gr-coherent.

(1) $\Rightarrow$ (3) Let M be a graded right R-module. Then M has a gr-flat preenvelope $f : M \to L$. Consider the exact sequence $0 \to \mathrm{Im}f \to L \to L/\mathrm{Im}f \to 0$. Then $0 \to (L/\mathrm{Im}f)^+ \to L^+ \to \mathrm{Im}f^+ \to 0$ is exact in R-gr and L^+ is FP-gr-injective, and hence $\mathrm{Im}f^+$ is FP-gr-injective by Lemma 3.7. Therefore $f : M \to \mathrm{Im}f$ is an epic gr-flat preenvelope.

(3) $\Rightarrow$ (6) The proof is dual to that of (2) $\Rightarrow$ (4).

(6) $\Rightarrow$ (1) By a proof dual to that of (4) $\Rightarrow$ (1), we can show that any graded submodule of a gr-flat right R-module is gr-flat. Let M be any graded left R-module. Then FP-gr-id$M$= fd$M^+ \leq 1$, and hence l.FP-gr-dim$R \leq 1$.

(1) $\Leftrightarrow$ (5) is obvious.

(1) $\Leftrightarrow$ (7) By analogy with the proof of (1) $\Leftrightarrow$ (6). □

Proposition 3.10. *The following are equivalent for a graded ring R of type G:*

(1) R is left gr-coherent and l.FP-gr-dim$R \leq 2$;

(2) Every graded left R-module has an FP-gr-injective cover with the unique mapping property.

Proof. (1) $\Rightarrow$ (2) Let M be any graded left R-module. Then M has an FP-gr-injective cover $f : C \to M$ by (1). It is enough to show that, for any FP-gr-injective left R-module A and any graded morphism $g : A \to C$ such that $fg = 0$, we have $g = 0$. In fact, there is a morphism in R-Mod $\beta : C/\mathrm{Img} \to M$ such that $\beta\eta = f$, where $\eta : C \to C/\mathrm{Im}g$ is the natural map, and so there exists a graded morphism $\beta' : C/\mathrm{Im}g \to M$ such that $\beta'\eta = f$ by [15, Lemma I.2.1]. Since l.FP-gr-dim$R \leq 2$, $C/\mathrm{Im}g$ is FP-gr-injective. Thus there exists a graded morphism $\alpha : C/\mathrm{Im}g \to C$ such that $\beta' = f\alpha$, which gives the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}g & \xrightarrow{i} & A & \xrightarrow{g} & C & \xrightarrow{\eta} & C/\mathrm{Im}g & \longrightarrow & 0 \\ & & & & & \searrow^{0} & \downarrow^{f} & \swarrow_{\beta'} & & & \\ & & & & & & M & & & & \end{array}$$

Thus $f\alpha\eta = f$, and hence $\alpha\eta$ is an isomorphism. It follows that η is monic, and so $g = 0$.

(2) $\Rightarrow$ (1) We first prove that R is left gr-coherent. Let $\{C_i, \varphi_j^i\}$ be a direct system with each C_i FP-gr-injective. Then $\varinjlim C_i$ has an FP-gr-injective cover $\alpha : E \to \varinjlim C_i$ with the unique mapping property. Let $\alpha_i : C_i \to \varinjlim C_i$ satisfy $\alpha_i = \alpha_j\varphi_j^i$ whenever $i \leq j$. Then there is a graded morphism $f_i : C_i \to E$ such that $\alpha_i = \alpha f_i$ for any i. It follows that $\alpha f_i = \alpha f_j \varphi_j^i$, and so $f_i = f_j\varphi_j^i$ whenever $i \leq j$. Therefore, by the definition of direct limits and [15, Lemma I.2.1], there exists a graded morphism $\beta : \varinjlim C_i \to E$ such that the following diagram is commutative:

$$\begin{array}{ccc} C_i & \xrightarrow{\alpha_i} & \varinjlim C_i \\ \downarrow{\scriptstyle \varphi_j^i} & & \downarrow{\scriptstyle \beta} \\ C_j & \xrightarrow{f_j} & E \end{array}$$

Thus $(\alpha\beta)\alpha_i = \alpha f_i = \alpha_i$ for any i, which means that $\alpha\beta = 1_{\varinjlim C_i}$ by the definition of direct limits, and so $\varinjlim C_i$ is a direct summand of E. Hence $\varinjlim C_i$ is FP-gr-injective, it follows that R is left gr-coherent by Theorem 3.2.

Next we prove that l.FP-gr-dim$R \leq 2$. Let M be any graded left R-module and

$$\begin{array}{ccccccccc} & & & & & & H & & \\ & & & & & \overset{\lambda}{\nearrow} & \theta \downarrow \uparrow \gamma & & \\ 0 \longrightarrow & M & \longrightarrow & E^0 & \xrightarrow{\varphi} & E^1 & \xrightarrow{\psi} & N & \longrightarrow 0 \end{array}$$

be exact with E^0 and E^1 gr-injective. Let $\theta : H \to N$ be an FP-gr-injective cover with the unique mapping property. Then there exists a graded morphism $\lambda : E^1 \to H$ such that $\psi = \theta\lambda$. Thus $\theta\lambda\varphi = \psi\varphi = 0 = \theta 0$, and so $\lambda\varphi = 0$, which implies that $\mathrm{Ker}\psi = \mathrm{Im}\varphi \subseteq \mathrm{Ker}\lambda$. Hence there is a graded morphism $\gamma : N \to H$ such that $\gamma\psi = \lambda$ by [15, Lemma I.2.1]. Therefore $\theta\gamma\psi = \psi$, and so $\theta\gamma = 1_N$ since ψ is epic. It follows that N is isomorphic to a direct summand of H, and thus N is FP-gr-injective, that is, l.FP-gr-dim$R \leq 2$. □

A graded ring R of type G is gr-regular if and only if all graded left (right) R-modules are flat by [15, Lemma I.5.4].

Proposition 3.11. *The following are equivalent for a graded ring R of type G:*

(1) R is gr-regular;

(2) Every graded left R-module is FP-gr-injective;

(3) Every finitely presented graded left R-module is projective;

(4) R is left gr-coherent and M is FP-gr-injective for any $M \in {}^{\perp}$gr-$\mathcal{FI}$;

(5) M is projective for any $M \in {}^{\perp}$gr-$\mathcal{FI}$;

(6) M is flat for any $M \in {}^{\perp}$gr-$\mathcal{FI}$;

(7) Every graded left R-module has an FP-gr-injective envelope with the unique mapping property;

(8) R is left gr-coherent and M has an FP-gr-injective envelope with the unique mapping property for any $M \in {}^{\perp}$gr-$\mathcal{FI}$.

Proof. (2) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (7) and (2) $\Rightarrow$ (4) $\Rightarrow$ (8) are obvious.

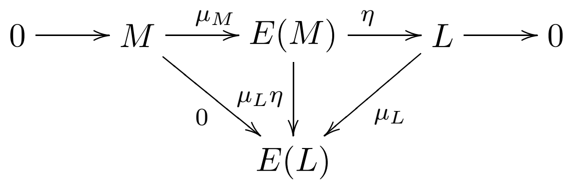
(4) $\Rightarrow$ (3) Let M be a finitely presented graded left R-module and $0 \to K \to P \to M \to 0$ be exact in R-gr with P finitely generated projective. Then K is finitely presented, and so K is FP-gr-injective by (4), which means that the sequence is split. Thus M is projective.

(1) $\Rightarrow$ (2) Since R is gr-regular, we have R is left gr-coherent by [15, Lemma I.5.4]. Let M be a graded left R-module. Then M^+ is flat, and so M is FP-gr-injective.

(2) $\Rightarrow$ (1) Let M be a graded right R-module. Then M^+ is FP-gr-injective, and so M is flat. Hence R is gr-regular.

(7) $\Rightarrow$ (2) Let M be any graded left R-module and $\mu_M : M \to E(M)$ be an FP-gr-injective envelope with the unique mapping property. Set $L =$

Cokerμ_M. Then L has an FP-gr-injective envelope $\mu_L : L \to E(L)$. Consider the following commutative diagram:



Since $\mu_L \eta \mu_M = 0 = 0\mu_M$, then $\mu_L \eta = 0$. Thus $L = \text{Im}\eta \subseteq \text{Ker}\mu_L = 0$, and so M is FP-gr-injective.

(8) $\Rightarrow$ (4) Let $M \in {}^{\perp}$gr-$\mathcal{FI}$ and $\mu_M : M \to E(M)$ an FP-gr-injective envelope with the unique mapping property. Then Coker$\mu_M \in {}^{\perp}$gr-$\mathcal{FI}$. So M is FP-gr-injective by analogy with the proof of (7) $\Rightarrow$ (2). ☐

4. Relative FP-gr-injective modules.

In this section, we prove that if R is right gr-coherent, then

1. (gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n^{\perp}$) is a perfect cotorsion theory whenever FP-gr-id$(R_R) \leq n$,
2. (${}^{\perp}$gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n$) is a cotorsion theory, where gr-$\mathcal{FI}_n$ is the class of all graded right R-modules of FP-gr-injective dimension at most n.

Lemma 4.1. *Let R be a graded ring and M a graded left R-module. Then fd$M$ = gr-idM^+ = FP-gr-idM^+.*

Proof. By $\text{EXT}^i_R(N, M^+) \cong \text{Tor}^R_i(N, M)^+$ for all $i \geq 1$ and any graded right R-module N. ☐

Lemma 4.2. *Let R be right gr-coherent and M a graded right R-module. Then fdM^+ = FP-gr-id$M$.*

Proof. By $\text{EXT}^i_R(N, M)^+ \cong \text{Tor}^R_i(N, M^+)$ for all $i \geq 1$ and any finitely presented graded right R-module N. ☐

For a fixed non-negative integer n, let gr-$\mathcal{FI}_n$ (gr-$\mathcal{F}_n$) be the class of all graded right (left) R-modules of FP-gr-injective (flat) dimension at most n. Now we have the following result.

Theorem 4.3. *Let n be a fixed non-negative integer. Then the following hold:*

(1) If R is right gr-coherent with FP-gr-id$(R_R) \leq n$, then (gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n^{\perp}$) is a perfect cotorsion theory.

(2) For any graded ring R, (gr-$\mathcal{F}_n$, gr-$\mathcal{F}_n^{\perp}$) is a perfect hereditary cotorsion theory.

Proof. (1) Let $0 \to A \to B \to C \to 0$ be gr-pure in gr-R with $B \in$ gr-$\mathcal{FI}_n$. Then $0 \to C^+ \to B^+ \to A^+ \to 0$ splits by [9, Proposition 3.1], and hence $A^+, C^+ \in$ gr-$\mathcal{F}_n$, which implies that $A, C \in$ gr-$\mathcal{FI}_n$. Therefore, by [9, Lemma 3.2], if $L \in$ gr-$\mathcal{FI}_n$, then L can be written as the direct union of a continuous chain of graded submodules $(L_\alpha)_{\alpha<\lambda}$ with λ an ordinal number such that $L_0 \in$ gr-$\mathcal{FI}_n$, $L_{\alpha+1}/L_\alpha \in$ gr-$\mathcal{FI}_n$ when $\alpha + 1 < \lambda$ with $\mathrm{Card}(L_0)$, $\mathrm{Card}(L_{\alpha+1}/L_\alpha) \leq \mathrm{Card}(R)\mathrm{Card}(G)$. If N is a graded right R-module such that $\mathrm{Ext}^1_{R\text{-gr}}(L_0, N) = 0$ and $\mathrm{Ext}^1_{R\text{-gr}}(L_{\alpha+1}/L_\alpha, N) = 0$ whenever $\alpha + 1 < \lambda$, then $\mathrm{Ext}^1_{R\text{-gr}}(L, N) = 0$ by the proof of [9, Proposition 3.3]. Thus gr-$\mathcal{FI}_n^\perp = X^\perp$, where X is a set of representatives of all graded modules $H \in$ gr-$\mathcal{FI}_n$ with $\mathrm{Card}(H) \leq \mathrm{Card}(R)\mathrm{Card}(G)$. We note that gr-$\mathcal{FI}_n$ is closed under direct sums, extensions, direct limits since R is right gr-coherent, and contains all gr-projective modules since FP-gr-id$(R_R) \leq n$. Hence (gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n^\perp$) is a cotorsion theory by [1, Corollary 2.13]. Since (gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n^\perp$) is cogenerated by the set X, (gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n^\perp$) is a complete cotorsion theory by [1, Corollary 2.7]. Moreover, (gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n^\perp$) is a perfect cotorsion theory since gr-$\mathcal{FI}_n$ is closed under direct limits by Lemma 3.1.

(2) Note that gr-$\mathcal{F}_n$ is closed under direct sums, extensions, direct limits, gr-pure submodules, cokernels of gr-pure monomorphisms and contains all gr-projective modules. An argument similar to that of (1) shows that (gr-$\mathcal{F}_n$, gr-$\mathcal{F}_n^\perp$) is a perfect cotorsion theory. On the other hand, let $0 \to A \to B \to C \to 0$ be exact in R-gr with B, $C \in$ gr-$\mathcal{F}_n$, then $A \in$ gr-$\mathcal{F}_n$. So (gr-$\mathcal{F}_n$, gr-$\mathcal{F}_n^\perp$) is hereditary. □

Lemma 4.4. *Let R be a graded ring of type G. Then M is an FP-gr-injective right R-module if and only if $EXT^1_R(R(\sigma)/A, M) = 0$ for all finitely generated graded submodules A of $R(\sigma)_R$ and all $\sigma \in G$.*

Proof. "⇒" is obvious.

"⇐" Let N be a finitely presented graded right R-module. Then there is an exact sequence $0 \to A \to \bigoplus_{\sigma\in G_0} R(\sigma) \to N \to 0$, where G_0 is a finite subset of G and A is finitely generated. So

$$N \cong (\oplus_{\sigma\in G_0} R(\sigma))/A \cong \oplus_{\sigma\in G_0}(R(\sigma) + A/A) \cong \oplus_{\sigma\in G_0}(R(\sigma)/R(\sigma) \cap A).$$

Consider the sequence $0 \to A \to R(\sigma) + A \to (R(\sigma) + A)/A \to 0$. Since $A, R(\sigma) + A$ are finitely generated, we have $R(\sigma)/R(\sigma) \cap A \cong (R(\sigma) + A)/A$ is finitely presented, and so $R(\sigma) \cap A$ is finitely generated. Thus $\mathrm{EXT}^1_R(N, M) \cong \mathrm{EXT}^1_R(\bigoplus_{\sigma\in G_0}(R(\sigma)/R(\sigma) \cap A), M) = 0$, which implies that M is FP-gr-injective. □

Theorem 4.5. *The following hold for a right gr-coherent ring R of type G and a fixed integer $n \geq 0$:*

(1) Every graded left R-module has a gr-$\mathcal{F}_n$-preenvelope.
(2) $({}^{\perp}$gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n)$ is a cotorsion theory.

Proof. (1) Analogous to the ungraded case.

(2) Let M be a graded right R-module. M admits a gr-injective resolution

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow E^n \longrightarrow \cdots .$$

Write $L^n = \mathrm{Im}(E^{n-1} \to E^n)$, $L^0 = M$. Then $M \in$ gr-$\mathcal{FI}_n$ if and only if L^n is FP-gr-injective if and only if $\mathrm{EXT}^1_R(R(\sigma)/A, L^n) = 0$ for all finitely generated graded submodules A of $R(\sigma)_R$ and all $\sigma \in G$ by Lemma 4.4. This means that $\mathrm{EXT}^{n+1}_R(R(\sigma)/A, M) = 0$ for all finitely generated graded submodules A of $R(\sigma)_R$ and all $\sigma \in G$ by dimension shifting. Denote by K_A the n-th syzygy module of the finitely presented graded right R-module $R(\sigma)/A$. Then $\mathrm{EXT}^{n+1}_R(R(\sigma)/A, M) = 0$ if and only if $\mathrm{EXT}^1_R(K_A, M) = 0$. Set $X_\sigma = \bigoplus K_A$, where the sum is over all finitely generated graded submodules A of $R(\sigma)_R$. Let

$$X = \{\oplus_{\sigma \in G_0} X_\sigma \mid G_0 \text{ is a finite subset of } G\}.$$

Then X is a set and gr-$\mathcal{FI}_n = X^{\perp}$. Thus $({}^{\perp}$gr-$\mathcal{FI}_n$, gr-$\mathcal{FI}_n)$ is a cotorsion theory. □

Proposition 4.6. *Let R be a right gr-coherent ring of type G and n a fixed non-negative integer. Then the following are equivalent:*
(1) FP-gr-id$(R_R) \leq n$;
(2) Every graded left R-module has a monic gr-$\mathcal{F}_n$-preenvelope;
(3) Every (FP-) gr-injective left R-module belongs to gr-$\mathcal{F}_n$;
(4) Every graded right R-module has an epic gr-$\mathcal{FI}_n$-cover;
(5) Every gr-flat right R-module belongs to gr-$\mathcal{FI}_n$.

Proof. (1) ⇒ (2) Let M be a graded left R-module. Then M has a gr-$\mathcal{F}_n$-preenvelope $f : M \to L$ by Theorem 4.5. Since there is an exact sequence $0 \to M \to (\bigoplus_{\sigma \in G} R(\sigma))^+$ and $\mathrm{fd}(\bigoplus_{\sigma \in G} R(\sigma))^+ = \text{FP-gr-id}\bigoplus_{\sigma \in G} R(\sigma) \leq n$ by Proposition 2.1 and Lemma 3.7, we see that f is monic.

(2) ⇒ (3) Let M be an FP-gr-injective left R-module. Then there exists a gr-pure exact sequence $0 \to M \to L$ with $L \in$ gr-$\mathcal{F}_n$ by (2) and Proposition 2.1, and hence $L^+ \to M^+ \to 0$ splits. So $M \in$ gr-$\mathcal{F}_n$ by Lemma 4.1.

(3) ⇒ (1) Since $(R_R)^+$ is gr-injective, $\mathrm{fd}(R_R)^+ \leq n$ by (3). Thus FP-gr-id$(R_R) = \mathrm{fd}(R_R)^+ \leq n$.

(1) ⇒ (4) By Theorem 4.3. (4) ⇒ (1) and (5) ⇒ (1) are obvious.

(3) ⇒ (5) Let M be a gr-flat right R-module. Then FP-gr-id$M = \mathrm{fd}M^+ \leq n$ by (3). □

Acknowledgment: The authors wish to express their sincere thanks to the referee for his/her valuable suggestions.

References

[1] S.T. Aldrich, E.E. Enochs, J.R. García Rozas and L. Oyonarte, *Covers and envelopes in Grothendieck categories: Flat covers of complexes with applications*, J. Algebra, 2001, **243**: 615–630.

[2] M.J. Asensio, J.A. López-Ramos and B. Torrecillas, *Gorenstein gr-flat modules*, Comm. Algebra, 1998, **26**: 3195–3209.

[3] M.J. Asensio, J.A. López-Ramos and B. Torrecillas, *Covers and envelopes over gr-Gorenstein rings*, J. Algebra, 1999, **215**: 437–459.

[4] M.J. Asensio, J.A. López-Ramos and B. Torrecillas, *FP-gr-injective modules and gr-FC rings*, Algebra and Number Theory, Marceldekker, 1999, 1–11.

[5] R.E. Bashir, *Covers and directed colimits*, Algebr. Represent. Theory, 2006, **9**: 423–430.

[6] N. Ding, *On envelope with the unique mapping property*, Comm. Algebra, 1996, **24**: 1458–1470.

[7] E.E. Enochs and O.M.G. Jenda, *Relative Homological Algebra*, de Gruyter Expositions in Math., Berlin-New York: Walter de Gruyter 2000.

[8] J.R. García Rozas and B. Torrecillas, *Preserving and reflecting covers by functors: Applications to graded modules*, J. Pure Appl. Algebra, 1996, **112**: 91–107.

[9] J.R. García Rozas, J.A. López-Ramos and B. Torrecillas, *On the existence of flat covers in R-gr*, Comm. Algebra, 2001, **29**: 3341–3349.

[10] L. Mao and N. Ding, *Relative FP-projective modules*, Comm. Algebra, 2005, **33**: 1587–1602.

[11] L. Mao and N. Ding, *Envelopes and covers by modules of finite FP-injective and flat dimensions*, Comm. Algebra, 2007, **35**: 833–849.

[12] L. Mao and N. Ding, *FI-injective and FI-flat modules*, J. Algebra, 2007, **309**: 367–385.

[13] C. Nastasescu and F. Van Oystaeyen, *Graded Ring Theory*, North-Holland Math. Library, 1982.

[14] K. Pinzon, *Absolutely pure covers*, Comm. Algebra, 2008, **36**: 2186–2194.

[15] S. Stenström, *Rings of Quotients*, Springer-Verlar, New York, 1975.

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(*Received May 8, 2009*)