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# A Simple Fixed Point Algorithm

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## 1. Introduction.

In Khilnani and Tse (1985), an easy and simple fixed point algorithm was introduced for a mapping that is Lipschitz-continuous and antitone. Its publication was followed by some interchanges that were concerned with the convergence of the proposed algorithm (see Marcotte (1987), Khilnani and Tse (1987, 1989), and Herceg and Cvetkovic (1989)).

So far as the contraction property of the algorithm is concerned, more general results were obtained independently by two mathematicians three decades ago: Vainberg (1960) and Zarantonello (1960). The purpose of this note is to present some theorems in the form useful to economists by considering nonnegative constraints explicitly. Some observations are also presented on the relationship of the algorithm with the weak axiom of revealed preference.

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## 2 . Main Theorems.

Let X be a Hilbert space with its inner product denoted by $(x, y)$ for $x, y$ in X. The norm is defined as $(x, x)^{1/2}$ for $x \in X$ and denoted by $\| x \|$. With this norm, X becomes a Banach space. We consider a closed, convex cone $X_+$ and a mapping $f(x)$ from $X_+$ into itself. Let us now introduce:

Definition: $f(x)$ is said to be *negatively monotone* on $X_+$ iff $(x-y, f(x)-f(y)) \leq 0$ for any $x, y \in X_+$. (See Ortega and Rheinboldt (1970, p. 141) for monotonicity.) We also introduce:

Definition: $f(x)$ is *Lipschitz-continuous* on $X_+$ iff there exists a scalar $k>0$ such that $\| f(x)-f(y) \| \leq k \| x-y \|$ for any $x, y \in X_+$.

Now, for a positive scalar $c \in (0, 1)$, let us consider a mapping on $X_+$ defined by

$$F(x) \equiv (1-c)x+cf(x).$$

Since $X_+$ is a convex cone, $F(x)$ is also a transformation from $X_+$ into itself. This type of 'averaging' mapping was studied by Browder and Petryshyn (1966) for the case of $f(x)$ being non-expansive. Earlier, Mann (1953) had considered general mean value methods in the context of iterative solutions. More recent discussions on the topic appear in Ishikawa (1976) and Thorlund-Petersen (1985). This brings us to:

*Theorem 1*: If a mapping $f(x)$ is Lipschitz-continuous with the constant $k>0$, and for some scalar $h \in (0, 1)$, satisfies

$$(x-y, f(x)-f(y)) \leq h \| x-y \|^2, \text{ for any } x, y \in X_+,$$

then $F(x)$ is contractive on $X_+$ when $0<c<c^*$. Here, $c^*=1$ when $1-2h+k^2=0$, and otherwise $c^*=\min(1, \max(0, 2(1-h)/(1-2h+k^2)))$.

*Proof*: By virtue of Zarantonello's result (1960) (see Ortega and

Rheinboldt (1970, p. 408)), the result is obvious. Nonetheless, for a ready reference, we give a proof.

$$\| F(x)-F(y) \|^2 = \| (1-c)(x-y)+c(f(x)-f(y)) \|^2$$
$$=(1-c)^2 \| x-y \|^2+2c(1-c)(x-y, f(x)-f(y))+c^2 \| f(x)-f(y) \|^2$$
$$\leq(1-c)^2 \| x-y \|^2+2c(1-c)h \| x-y \|^2+c^2 \| f(x)-f(y) \|^2$$
$$= \{(1-2h+k^2)c^2-2c(1-h)+1\} \| x-y \|^2$$

As is quite clear, if the constant c is such that $0<c<c^*$, then $F(x)$ turns out to be contractive. Q. E. D.

*Theorem 2*: Suppose the assumptions made in theorem 1 are satisfied. Then the equation $x=f(x)$ has a unique solution in $X_+$, and further, given an arbitrary initial vector $x(0)$ in $X_+$, the iteration

$$x(i+1)=F(x(i)) \text{ for } i=0, 1, 2, \cdots$$

produces a sequence $\{x(i)\}$ within $X_+$ and it converges to a unique solution to the equation $x=F(x)$ or equivalently $x=f(x)$ provided that the constant $c$ satisfies $0<c<c^*$.

*Proof*: Given that $X_+$ is closed and convex, the theorem is valid because of the contraction mapping theorem. Q. E. D.

## 3. Applications.

When $X=R^n$ and $X_+=R^n_+$, and some inner product is defined, for example $(x, y)=xPy$, where $P$ is an $n\times n$ positive definite matrix, then we can apply theorem 2 provided $f(x)$ is Lipschitz-continuous and satisfies the inequality

$$(x-y, f(x)-f(y))\leq h \| x-y \|^2 \text{ for any } x, y\in X_+.$$

Khilnani and Tse's theorem (1985, p. 131) is a special case when our $h$ is zero (i. e., $f(x)$ is negatively monotone) and $P$ is the identity matrix. (See also Fujimoto (1987) for another special case.) It should be noted here that our theorem 2 shows the existence and uniqueness of the nonnegative solution together with the convergence of the averaging iteration all in one stroke, thanks to a simple convex combination seen in $F(x)$. Note also that when the nonnegativity constraint is removed and we seek a solution on the whole space X, the existence of a fixed point was demonstrated by Minty (1962) without requiring Lipschitz-continuity. The assumptions made there were about f($x$) being continuous and the above inequality involving h being satisfied.

We now take up an example from the cobweb model following Khilnani and Tse (1985). Our theorem 2 shows that an equilibrium price can be approximated by the above simple iteration not only for the case where the demand curve is downward sloping and the supply curve upward sloping, but also for the case in which the contrary is true to a certain degree.

The implications of the 'averaging' iteration are not restricted to the computational purpose. We can imagine markets where the auctioneer revises the prices, say $x$, at each period in a cautious conservative way represented by the term cx in the definition of $F(x)$. In this example, $f(x)$ may be taken as $(x+E(x))$, where $E(x)$ is the excess demand function. Then $F(x)$ becomes $(1-c)x+c(x+E(x))$ or $x+cE(x)$, that is to say, a smaller speed of adjustment. The condition of theorem 1 may be rewritten as

$$(x-y,\ E(x)-E(y))\leq m\, \| x-y \|^2 \text{ for some } m\in(0,\ 1).$$

If this is weakened to

$$(x-y,\ E(x)-E(y))<0$$

then by virtue of the Walras law it follows that $-xE(y)-yE(x)<0$. If $x$

is an equilibrium price vector, the inequality becomes $xE(y)>0$, the weak axiom of revealed preference. It has been shown in the general equilibrium theory that this axiom is enough to have the system stability of the equilibrium set, which is known to be convex (see Mckenzie (1960), and Arrow and Hurwicz (1960)).

With these remarks on the applications we go back to the general setting with an abstract space and in the next section, we show the variants of the above theorems under weaker conditions while assuming the existence of a fixed point.

## 4. More Theorems.

First we write $G(x)=f(x)-x$, and denote by $S$ the set of solutions to $G(x)=0$.

That is to say, $S=\{x \mid G(x)=0,\ x\in X_+\}$. Next, we introduce

Definition: $G(x)$ is called *strictly negatively w-monotone* iff

$$(x-y,\ G(y))>0,\ \forall x\in S,\ y\notin S.$$

It may be emphasized here that the condition requires exactly one of the two vectors to be a solution and hence is clearly weaker than requiring

$$(x-y,\ G(x)-G(y))<0,\ \forall x,\ y\in X_+ (x\neq y).$$

*Theorem 3*: Suppose $S$ is not empty. If $G$ is strictly negatively w-monotone, the solution set $S$ is convex.

*Proof*: Let $x$ and $z$ be two vectors in $S$. Consider $y=cx+(1-c)z$ for an arbitrary scalar $c\in(0,\ 1)$. If $y\notin S$ then by assumption

$$(x-y,\ G(y))>0 \text{ and } (z-y,\ G(y))>0.$$

The convex combination of the left sides of the two inequalities with respective weights of $c$ and $(1-c)$ yields $(y-y,\ G(y))>0$, a contradiction.

Q. E. D.

(See theorem 5. 4. 7 in Ortega and Rheinboldt (1970, p144) in which a similar result is proved when $G$ is continuous, under a usual monotonicity assumption. See also Minty (1962).) Next, in order to show the convergence of the averaging iteration, we need to introduce

Definition: $G(x)$ is said to be *strongly negatively w-monotone,* iff $\exists$ a scalar $m \in (0, 1)$ and $(x-y, G(y)) > m \| x-y \|^2$ $\forall x, y$ such that $x \in S$, $y \notin S$.

*Theorem 4*: Suppose $S$ is not empty. If $G$ is strongly negatively w-monotone and Lipschitz-continuous, then given any initial vector $x(0)$ in $X_+$ the iteration

$$x(i+1) = x(i) + cG(x(i)) \text{ for } i = 0, 1, 2, \cdots$$

converges to a vector in $S$ for a sufficiently small $c$.

*Proof*: We write $f(x) \equiv x + G(x)$ and proceed backwards through the argument of section 2. In the inequality of theorem 1, that is

$$(x-y, f(x)-f(y)) \leq h \| x-y \|^2 \text{ for any } x, y \in X_+,$$

we can take $x \equiv x^* \in S$ and $y \equiv x(i) \notin S$ and get

$$(x^*-x(i), x^*-x(i)+G(x^*)-G(x(i))) \leq h \| x^*-x(i) \|^2,$$ provided the constant $c$ satisfies the inequality with $k$ now being $(1-m)$. Since the rate of contraction is uniform, the convergence is guaranteed. Q. E. D.

Unfortunately, we have been unable to prove the convergence under strict negative w-monotonicity in place of the strong one. Nor have we been able to demonstrate a counter-example of non-convergence in the former case. The difficulty seems to arise from the subtle differences between the systems of nonlinear difference and differential equations.

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